Multiplicity and stability of the Pohozaev obstruction for Hardy-Schrödinger equations with boundary singularity

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Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ such that $0 \in \partial\Omega$. We consider issues of non-existence, existence, and multiplicity of variational solutions in $H_{1,0}^2(\Omega)$ for the borderline Dirichlet problem,

$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} - h(x)u &= \frac{|u|^{2^\star(s)-2}u}{|x|^s} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{E}$$

where $0 < s < 2$, $2^\star(s) := \frac{2(n-s)}{n-2}$, $\gamma \in \mathbb{R}$ and $h \in C^0(\overline{\Omega})$. We use sharp blow-up analysis on –possibly high energy– solutions of corresponding subcritical problems to establish, for example, that if $\gamma < \frac{n^2}{4} - 1$ and the principal curvatures of $\partial\Omega$ at 0 are non-positive but not all of them vanishing, then Equation (E) has an infinite number of high energy (possibly sign-changing) solutions in $H_{1,0}^2(\Omega)$. This complements results of the first and third authors, who showed in [21] that if $\gamma \leq \frac{n^2}{4} - \frac{1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative, then (E) has a positive least energy solution.

On the other hand, the sharp blow-up analysis also allows us to show that if the mean curvature at 0 is nonzero and the mass, when defined, is also nonzero, then there is a surprising stability of regimes where there are no variational positive solutions under C^1-perturbations of the potential h. In particular, and in sharp contrast with the non-singular case (i.e., when $\gamma = s = 0$), we prove non-existence of such solutions for (E) in any dimension, whenever Ω is star-shaped and h is close to 0, which include situations not covered by the classical Pohozaev obstruction.

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1. Introduction

This manuscript is the continuation of a long-time project initiated by the first and the third author in [19] about nonlinear critical equations involving the Hardy potential when the singularity is located on the boundary of the domain under study. Let Ω be such a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, with $0 \in \partial\Omega$. We fix $s \in (0,2)$ and define the critical Sobolev exponent $2^\star(s) := \frac{2(n-s)}{n-2}$. For $\gamma \in \mathbb{R}$ and $h_0 \in C^1(\overline{\Omega})$, we consider in the sequel issues of non-existence, existence, and multiplicity of variational solutions in $H^2_{1,0}(\Omega)$ for the borderline Dirichlet problem,

$$\left\{\begin{array}{rcll} -\Delta u - \gamma\frac{u}{|x|^2} - h_0(x)u &=& \frac{|u|^{2^\star(s)-2}u}{|x|^s} & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega \setminus \{0\}. \end{array}\right. \tag{1}$$

By solutions, we mean here functions $u \in H^2_{1,0}(\Omega)$, i.e., the completion of $C^\infty_c(\Omega)$ for the L_2-norm of the gradient $\|\nabla u\|_2$. This problem has by now a long history starting with the fact that when $\gamma = s = 0$ and h_0 is a constant, it is the counterpart of the Yamabe problem [1, 26, 32] in Euclidian space, as initiated by Brezis-Nirenberg [5], with important contributions in the critical dimension $n = 3$, by Druet [8], and for multiplicity results for $n \geq 7$, by Devillanova-Solimini [7], among many others.

The case dealing with least energy solutions for $s > 0$ but $\gamma = 0$, when the singularity 0 is on the boundary of the domain was initiated by Ghoussoub-Kang [18] and developed by Ghoussoub-Robert [19]. The case involving the Hardy potential, i.e., when $\gamma > 0$, was introduced by Lin-Wadade [27] with a follow-up contribution by Ghoussoub-Robert [21]. This paper addresses remaining issues about the multiplicity of solutions, but also about obstructions to the existence of solutions and their stability under small perturbations.

The existence of solutions is related to the coercivity of the operator$-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$. It is clear that the operator $-\Delta - \frac{\gamma}{|x|^2}$ is coercive on $H^2_{1,0}(\Omega)$ whenever $\gamma < \gamma_H(\Omega)$, where $\gamma_H(\Omega)$ is the Hardy constant associated to the domain Ω, that is

$$\gamma_H(\Omega) := \inf_{u \in H^2_{1,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega \frac{u^2}{|x|^2}\, dx}, \tag{2}$$

which has been extensively studied (see for example [17] and [21]). We recall that if $0 \in \Omega$, then

$$\gamma_H(\Omega) = \gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}. \tag{3}$$

When $0 \in \partial\Omega$, the situation is extremely different. For non-smooth domains modeled on cones, we refer to Egnell [13], and the more recent works of Cheikh-Ali [23, 24]. If Ω is smooth, then, around 0, the domain is modeled on the half-space $\mathbb{R}^n_- := \{x \in \mathbb{R}^n;\, x_1 < 0\}$. We then get that (see [21])

$$\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \gamma_H(\mathbb{R}^n_-) = \frac{n^2}{4}. \tag{4}$$

Note that when $h_0 \equiv 0$, (1) is the Euler-Lagrange equation for the following Hardy-Sobolev variational problem: For $\gamma < \gamma_H(\Omega)$ and $0 \leq s \leq 2$, there exists

$\mu_{\gamma,s}(\Omega) > 0$ such that

$$\mu_{\gamma,s}(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2\, dx - \gamma \int_\Omega \frac{u^2}{|x|^2}\, dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\, dx \right)^{\frac{2}{2^\star(s)}}};\ u \in H^2_{1,0}(\Omega) \setminus \{0\} \right\}. \tag{5}$$

Note that when $s = 2$ and $\gamma = 0$, this is the Hardy inequality mentioned above, while if $s = 0$ and $\gamma = 0$, it is the Sobolev inequality. If $\Omega = \mathbb{R}^n$, $s \in [0, 2]$ and $\gamma \in (-\infty, \frac{(n-2)^2}{4})$, (5) contains – after a suitable change of variables – the Caffarelli-Kohn-Nirenberg inequalities [6]. The latter state that there is a constant $C := C(a, b, n) > 0$ such that,

$$\left(\int_{\mathbb{R}^n} |x|^{-bq} |u|^q \right)^{\frac{2}{q}} \leq C \int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 dx \text{ for all } u \in C_c^\infty(\mathbb{R}^n), \tag{6}$$

where

$$-\infty < a < \frac{n-2}{2}, \quad 0 \leq b - a \leq 1, \quad \text{and} \quad q = \frac{2n}{n - 2 + 2(b-a)}. \tag{7}$$

The first difficulty in these problems is due to the fact that $2^\star(s)$ is critical from the viewpoint of the Sobolev embeddings, in such a way that if Ω is bounded, then $H^2_{1,0}(\Omega)$ is embedded in the weighted space $L^p(\Omega, |x|^{-s})$ for $1 \leq p \leq 2^\star(s)$, and the embedding is compact if and only if $p < 2^\star(s)$. This lack of compactness defeats the classical minimization strategy to get extremals for (5). In fact, when $s = 0$ and $\gamma = 0$, this is the setting of the critical case in the classical Sobolev inequalities, which started this whole line of inquiry, due to its connection with the Yamabe problem on compact Riemannian manifolds [1], [32], [26]. Another complicating feature of the problem is that the term $\frac{u}{|x|^2}$ is as critical as $\frac{u^{2^*(s)-1}}{|x|^s}$ in the sense that they have the same homogeneity as the Laplacian. These difficulties are summarized by the invariance of the problem under conformal transformation. Indeed, for a function $u : \Omega \to \mathbb{R}$ and $r > 0$, let

$$u_r : x \mapsto r^{\frac{n-2}{2}} u(r \cdot x) \tag{8}$$

and note that Equation (1) is then "essentially" invariant under the transformation $u \mapsto u_r$ in the sense that

$$\begin{cases} -\Delta u_r - \gamma \frac{u_r}{|x|^2} - r^2 h_0(rx) u_r &= \frac{|u_r|^{2^\star(s)-2} u_r}{|x|^s} \quad \text{in } r^{-1}\Omega, \\ u_r &= 0 \quad \text{on } r^{-1}\partial\Omega \setminus \{0\}. \end{cases} \tag{9}$$

This "invariance" is behind the lack of compactness in the embeddings associated to the variational formulation of (1), which prohibits the use of general abstract topological or variational methods. However, as one notices, the invariance is not complete, since the potential h has changed, and the domain itself was transformed. As we shall see, both the geometry of the domain and -to a lesser extent- the potential h break the invariance enough that one will be able to recover compactness for (1).

Another important aspect of this problem is the singularity at 0 and its location within the domain since the Hardy potential does not belong to the Kato class. Elliptic problems with singular potential arise in quantum mechanics, astrophysics, as well as in Riemannian geometry, in particular in the study of the scalar curvature problem on the standard sphere. Indeed, if the latter is equipped with its standard metric whose scalar curvature is singular at the north and south poles, then by considering its stereographic projection of $\mathbb{R}^n$, the problem of finding a conformal metric with prescribed scalar curvature $K(x)$ leads to finding solutions of the form $-\Delta u - \gamma\frac{u}{|x|^2} = K(x)u^{2^\star(0)-1}$ on $\mathbb{R}^n$. The latter is a simplified version of the nonlinear Wheeler-DeWitt equation, which appears in quantum cosmology (see [2, 3, 28, 34] and the references cited therein).

This paper deals specifically with the case where 0 belongs to the boundary of a smooth domain Ω. We shall see that the boundary at 0 plays an important role, and our starting point is the existence Theorem 1 below for least energy solutions. It was first established by Ghoussoub-Robert [19] when $\gamma = 0$, then by Lin-Wadade [27] when $0 < \gamma < \frac{(n-2)^2}{4}$ under the assumption that the mean curvature at 0 is negative. The result was extended to the range $\gamma \leq \frac{n^2-1}{4}$ in [21], but more importantly, it was shown there that in the remaining range $(\frac{n^2-1}{4}, \frac{n^2}{4})$, the curvature condition does not suffice anymore and a more global condition is needed: the boundary mass $m_{\gamma,h}(\Omega)$ of a domain associated to γ and h, that we now recall.

1.1. The models and the definition of the mass. Letting formally $r \to 0$ in (9), we get that u should behave like solutions to

$$\begin{cases} -\Delta U - \gamma\frac{U}{|x|^2} = \frac{|U|^{2^\star(s)-2}U}{|x|^s} & \text{in } \mathbb{R}^n_-, \\ U = 0 & \text{on } \partial\mathbb{R}^n_-. \end{cases} \tag{10}$$

To the best of our knowledge, no explicit positive solution of (10) is known. This was the reason why a specific blowup analysis was carried out in [19], which relied on the symmetry properties and a precise description of the asymptotic behavior of such solutions –also established in [19]. On the other hand, the asymptotic behavior of such nonlinear problems is governed by the solutions to the linear problem

$$\begin{cases} -\Delta U - \gamma\frac{U}{|x|^2} = 0 & \text{in } \mathbb{R}^n_-, \\ U = 0 & \text{on } \partial\mathbb{R}^n_-. \end{cases} \tag{11}$$

One can then easily see that a function of the form $u(x) = x_1|x|^{-\beta}$ is a solution to (11) if and only if $\beta \in \{\beta_-(\gamma), \beta_+(\gamma)\}$, where

$$\beta_\pm(\gamma) := \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - \gamma} \quad \text{for } \gamma < \frac{n^2}{4}. \tag{12}$$

THEOREM-DEFINITION 1 ([21]). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ ($n \geq 3$) such that $0 \in \partial\Omega$. Suppose $\gamma < \frac{n^2}{4}$ and let $h \in C^1(\overline{\Omega})$ be such that the operator $-\Delta - \gamma|x|^{-2} - h$ is coercive. Assuming that*

$$\gamma > \frac{n^2-1}{4},$$

then there exists $\mathcal{H} \in C^2(\overline{\Omega} \setminus \{0\})$ such that

$$\begin{cases} -\Delta\mathcal{H} - \frac{\gamma}{|x|^2}\mathcal{H} + h(x)\mathcal{H} = 0 & \text{in } \Omega \\ \mathcal{H} > 0 & \text{in } \Omega \\ \mathcal{H} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

Then, there exist constants $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ such that

$$\mathcal{H}(x) = c_1 \frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}} + c_2 \frac{d(x,\partial\Omega)}{|x|^{\beta_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\beta_-(\gamma)}}\right)$$

as $x \to 0$. In the spirit of Schoen-Yau [33], we define the boundary mass as

$$m_{\gamma,h}(\Omega) := \frac{c_2}{c_1},$$

which is independent of the choice of $\mathcal{H}$.

The problem of existence of least energy solutions can now be summarized in the following theorem, whose proof can also be deduced from the refined blow-up techniques developed in this paper.

Theorem 1 (G.-R.[19], Lin-Wadade [27], G.-R. [21]). *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ ($n \geq 3$) such that the singularity 0 belongs to the boundary $\partial\Omega$. Suppose that $0 < s < 2$ and fix $h_0 \in C^1(\overline{\Omega})$ such that $-\Delta - \gamma|x|^{-2} - h_0$ is coercive. Assume one of the following two conditions:*

- *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
- *$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$ and the boundary mass $m_{\gamma,h_0}(\Omega)$ is positive.*

Then, there is a positive solution to (1) that is a minimizer for the associated variational problem,

$$\inf\left\{\frac{\int_\Omega |\nabla u|^2\,dx - \gamma\int_\Omega \frac{u^2}{|x|^2}\,dx - \int_\Omega h_0(x)u^2\,dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}};\, u \in H^2_{1,0}(\Omega)\setminus\{0\}\right\}. \tag{13}$$

Our focus in this project, is to investigate the extent to which the above local curvature condition at 0 and the global (mass) condition on the domain are necessary for the existence of positive solutions. Most importantly, we give results pertaining to the persistence of the lack of positive solutions for (1) under C^1-perturbations of the potential h. We will also show that, under suitable curvature conditions, this equation has an infinite number of non-necessarily positive solutions.

Both existence and non-existence results will follow from a sharp blow-up analysis of solutions to perturbations of Equation (1). More precisely, we consider

$$p_\epsilon \in [0, 2^\star(s) - 2) \text{ such that } \lim_{\epsilon\to 0} p_\epsilon = 0, \tag{14}$$

and a family $(h_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ such that

$$\lim_{\epsilon\to 0} h_\epsilon = h_0 \text{ in } C^1(\overline{\Omega}) \text{ and } -\Delta - \frac{\gamma}{|x|^2} - h_0 \text{ is coercive in } \Omega. \tag{15}$$

We then perform a blow-up analysis, as ϵ go to zero, on a sequence of functions $(u_\epsilon)_{\epsilon>0}$ in $H^2_{1,0}(\Omega)$ such that u_ϵ is a solution to the Dirichlet boundary value problems:

$$\left\{\begin{array}{rcll} -\Delta u_\epsilon - \gamma\frac{u_\epsilon}{|x|^2} - h_\epsilon u_\epsilon &=& \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}u_\epsilon}{|x|^s} & \text{in } \Omega,\\ u_\epsilon &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{E_ϵ}$$

The novelty and delicacy of our analysis stem from the fact that the sequence $(u_\epsilon)_{\epsilon>0}$ might blow up along excited states, as opposed to a unique ground state in [19]. Moreover, the sequence $(u_\epsilon)_{\epsilon>0}$ is not assumed to have a fixed sign.

1.2. Non-existence: stability of the Pohozaev obstruction. Starting with issues of non-existence of solutions, we shall prove the following surprising stability of regimes where variational positive solutions do not exist.

Theorem 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ such that the singularity 0 belongs to the boundary $\partial\Omega$. Assume that $0 < s < 2$ and $\gamma < n^2/4$. Fix $h_0 \in C^1(\overline{\Omega})$ such that $-\Delta - \gamma|x|^{-2} - h_0$ is coercive, and assume that one of the following conditions hold:*

- *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature at 0 is non-zero;*
- *$\gamma > \frac{n^2-1}{4}$ and the boundary mass $m_{\gamma,h_0}(\Omega)$ is non-zero.*

If there is no positive variational solution to (1) with $h = h_0$, then for all $\Lambda > 0$, there exists $\epsilon := \epsilon(\Lambda, h_0) > 0$ such that for any $h \in C^1(\overline{\Omega})$ with $\|h - h_0\|_{C^1(\Omega)} < \epsilon$, there is no positive solution to (1) such that $\|\nabla u\|_2 \leq \Lambda$.

The above result is surprising for the following reason: Assuming Ω is star-shaped with respect to 0, then the classical Pohozaev obstruction (see Section 11) yields that (1) has no positive variational solution whenever

$$h_0(x) + \frac{1}{2}(\nabla h_0(x), x) \leq 0 \text{ for all } x \in \Omega. \tag{16}$$

We then get the following corollaries.

Corollary 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ such that $0 \in \partial\Omega$. Assume Ω is starshaped with respect to 0, $0 < s < 2$ and $\gamma < \gamma_H(\Omega)$. If $\gamma \leq \frac{n^2-1}{4}$, we shall also assume that the mean curvature at 0 is non-vanishing. If h_0 is a potential satisfying (16), then for all $\Lambda > 0$, there exists $\epsilon(\Lambda, h_0) > 0$ such that for all $h \in C^1(\overline{\Omega})$ satisfying $\|h - h_0\|_{C^1(\Omega)} < \epsilon(\Lambda, h_0)$, there is no positive solution to (1) such that $\|\nabla u\|_2 \leq \Lambda$.*

Corollary 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$, such that $0 \in \partial\Omega$. We fix $0 < s < 2$ and $\gamma < \gamma_H(\Omega)$, the Hardy constant defined in (2). Assume that*

$$\Omega \text{ is starshaped with respect to } 0.$$

When $\gamma \leq \frac{n^2-1}{4}$, we assume that the mean curvature at 0 is positive. Then for all $\Lambda > 0$, there exists $\epsilon(\Lambda) > 0$ such that for all $\lambda \in [0, \epsilon(\Lambda))$, there is no positive solution to

$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} - \lambda u = \frac{u^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{0\} \end{cases} \tag{17}$$

with $\|\nabla u\|_2 \leq \Lambda$.

It is worth comparing these results to what happens in the nonsingular case. Indeed, in contrast to the singular case, a celebrated result of Brezis-Nirenberg [5] shows that, for $\gamma = s = 0$, a variational solution to (17) always exists whenever $n \geq 4$ and $0 < \lambda < \lambda_1(\Omega)$, with the geometry of the domain playing no role whatsoever. On the other hand, Druet-Laurain [12] showed that the geometry plays a role in dimension $n = 3$, still for $\gamma = s = 0$, by proving that when Ω is star-shaped, then there is no solution to (17) for all small values of $\lambda > 0$ (with no apriori bound on $\|\nabla u\|_2$). Another point of view is that for $n = 3$, the nonexistence of solutions persists under small perturbations, but it does not for $n \geq 4$: the Pohozaev obstruction is stable only for $n = 3$ in the nonsingular case.

This is in stark contrast with the situation here, i.e. when $0 \in \partial\Omega$ and $s > 0$. In this case, for both the existence and non-existence results, the geometry plays a role in all dimensions: it is either the local geometry at 0 (i.e., depending on whether the mean curvature at 0 is vanishing or not) in high dimensions, or the global geometry of the domain (i.e., depending on whether the mass is positive or the domain is star-shaped) in low dimensions. Corollaries 1 and 2 show that the Pohozaev obstruction is stable in all dimensions in the singular case.

Let us discuss some extensions related to this absence or not of low/large dimension phenomenon.

- Our stability result still holds under an additional smooth perturbation of the domain Ω, as was done by Druet-Hebey-Laurain [10] when $n = 3$, $\gamma = s = 0$.
- In the forthcoming paper [16], we tackle the case of the interior singularity $0 \in \Omega$, where the results are much more in the spirit of Brezis-Nirenberg and Druet-Laurain concerning the dichotomy between low and high dimensions.
- On of the main features of the stability result of Druet-Laurain [12] is the absence of any apriori control on $\|\nabla u\|_2$. In the interor case $0 \in \Omega$, we expect to get rid also of the apriori bound in the singular case $s > 0$. In the boundary case $0 \in \partial\Omega$, bypassing the apriori bound by Λ is more delicate and will require extra care. These issues are projects in progress.

The proof of Theorem 2 (and Corollaries 1 and 2) relies on the blow-up analysis. Namely, arguing by contradiction, we assume the existence of solutions $(u_\epsilon)_\epsilon$ to (17) with $p_\epsilon \equiv 0$ and $(h_\epsilon)_\epsilon \to h_0$ in C^1 with a control on the Dirichlet energy. Due to the "invariance" under the conformal transformation (8), the u_ϵ's might concentrate on some peaks at 0. The formation of these peaks is described via blow-up analysis in Proposition 3. Then Proposition 6 applies which yields vanishing of the mean curvature or the mass, depending on the dimension, contradicting the hypothesis of Theorem 2. Concerning Corollaries 1 and 2, the hypothesis imply that the mass is negative when defined.

1.3. Multiplicity of sign-changing solutions. As to the question of multiplicity, we shall prove the following result, which uses that in the subcritical case, i.e., when $p_\epsilon > 0$, there is an infinite number of higher energy solutions for such ϵ. Again, the core of the proof is a sharp blow-up analysis of such solutions as $p_\epsilon \to 0$.

THEOREM 3 (The general case). *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$. Let $h_0 \in C^1(\overline{\Omega})$ and $(h_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ be such that (15) holds, and let $(p_\epsilon)_{\epsilon>0}$ be such that (14) holds. Consider a sequence of functions $(u_\epsilon)_{\epsilon>0}$ that is uniformly bounded in $H^2_{1,0}(\Omega)$ such that for each $\epsilon > 0$, u_ϵ satisfies Equation (E_ϵ). Then,*

1. *If $\gamma < \frac{n^2}{4} - 1$ and the principal curvatures of $\partial\Omega$ at 0 are non-positive but not all of them vanish, then the sequence $(u_\epsilon)_{\epsilon>0}$ is pre-compact in $H^2_{1,0}(\Omega)$.*
2. *In particular, Equation (1) has an infinite number of (possibly sign-changing) solutions in $H^2_{1,0}(\Omega)$.*

The above result was established by Ghoussoub-Robert [20] in the case when $\gamma = 0$. The main challenge here is to prove the compactness of the subcritical solutions at high energy levels, as the nonlinearities approach the critical exponent. The multiplicity result then follows from standard min-max methods. The proof relies heavily on pointwise blow-up analysis techniques in the spirit of Druet-Hebey-Robert [11] and Druet [9], though our situation adds considerable difficulties to carrying out the program.

1.4. Compactness Theorems and blow-up analysis. As mentioned above, the central tool is an analysis of the formation of peaks on families $(u_\epsilon)_\epsilon$ of solutions to equations like (1) when blow-up occurs. This long analysis yields Propositions 5 and 6 that describe the blow-up rate. When blowup does not occur, there is compactness. The following theorems are immediate consequences of these propositions.

We note that the restrictions on both γ and on the curvature at 0 are more stringent than for the existence of a ground state solution in Theorem 1. The stronger assumptions turned out to be due to the potentially sign-changing approximate solutions -actually solutions of subcritical problems- and not because they are not necessarily minimizing. Indeed, the following theorem does not assume any smallness of the energy bound as long as the approximate solutions are positive. It therefore yields another proof for Theorem 1, which does not rely on the existence of minimizing sequence below the energy level of a single bubble.

Theorem 4 (The non-changing sign case). *Assume in addition to the hypothesis of Theorem 3, that the solutions $(u_\epsilon)_{\epsilon>0}$ satisfy for all $\epsilon > 0$,*

$$u_\epsilon > 0 \quad \text{on } \Omega. \tag{18}$$

Then, the sequence $(u_\epsilon)_{\epsilon>0}$ is pre-compact in $H^2_{1,0}(\Omega)$, provided one of the following conditions is satisfied:

- *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
- *$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$ and the boundary mass $m_{\gamma,h_0}(\Omega)$ is positive.*

Our method also shows that if the –possibly sign-changing– sequence is weakly null, then the compactness result in Theorem 3 will still hold for γ up to $\frac{n^2}{4} - \frac{1}{4}$:

Theorem 5 (The case of a weak null limit). *Assume in addition to the hypothesis of Theorem 3, that the solutions $(u_\epsilon)_{\epsilon>0}$ satisfy,*

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_2 = 0. \tag{19}$$

If $\gamma < \frac{n^2-1}{4}$ and the principal curvatures of $\partial\Omega$ at 0 are non-positive but not all of them vanishing, then the sequence $(u_\epsilon)_{\epsilon>0}$ converges strongly to 0 in $H^2_{1,0}(\Omega)$.

1.5. Structure of the manuscript. This paper is organized as follows. Section 2 consists in preliminary material in order to introduce the sequence of functions that will be thoroughly analyzed in Sections 3 to 8 in the case where they "blow-up". Section 9 contains the proof of the multiplicity result and Section 10 will have the applications to non-existence regimes and their stability under perturbations. We then have five relevant appendices. The first (Appendix A, Section 11) introduces the Pohozaev identity in our setting. The second (Appendix B, Section 12) contains a technical lemma on the continuity of the first eigenvalue $\lambda_1(\Delta + V)$ with respect to variations of the potential V. Appendix C (Section 13) recalls regularity results established in [21] about the regularity and behavior at 0 of solutions of equations involving the Hardy-Schrödinger operator on bounded domains having

0 on their boundary. In Appendix D (Section 14), we construct the Green functions associated to the operators $-\Delta - \frac{\gamma}{|x|^2} - h$ on such domains, and exhibit some of their properties needed throughout the memoir. The last Appendix E (Section 15) does the same but for the Hardy-Schrödinger operator $-\Delta - \frac{\gamma}{|x|^2}$ on $\mathbb{R}^n_-$.

2. Setting up the blow-up

Throughout this paper, Ω will always be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We will always assume that $\gamma < \frac{n^2}{4}$ and $s \in (0,2)$. We set $2^\star(s) := \frac{2(n-s)}{n-2}$. When $\gamma < \gamma_H(\Omega)$, then the following Hardy-Sobolev inequality holds on Ω: there exists $C > 0$ such that

$$C\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{2/2^\star(s)} \leq \int_\Omega |\nabla u|^2\,dx - \gamma\int_\Omega \frac{u^2}{|x|^2}\,dx \text{ for all } u \in H^2_{1,0}(\Omega). \quad (20)$$

For each $\epsilon > 0$, we consider $p_\epsilon \in [0, 2^\star(s) - 2)$ such that

$$\lim_{\epsilon\to 0} p_\epsilon = 0. \quad (21)$$

Let $h_0 \in C^1(\overline{\Omega})$ and consider a family $(h_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ such that (15) holds. Consider a sequence of functions $(u_\epsilon)_{\epsilon>0}$ in $H^2_{1,0}(\Omega)$ such that for all $\epsilon > 0$ the function u_ϵ is a solution to the Dirichlet boundary value problem:

$$\left\{\begin{array}{rcll} -\Delta u_\epsilon - \gamma\frac{u_\epsilon}{|x|^2} - h_\epsilon u_\epsilon &=& \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}u_\epsilon}{|x|^s} & \text{in } H^2_{1,0}(\Omega),\\ u_\epsilon &=& 0 & \text{on } \partial\Omega.\end{array}\right. \quad (E_\epsilon)$$

By the regularity result Theorem 6 in Appendix B, we have $u_\epsilon \in C^2(\overline{\Omega}\setminus\{0\})$ and there exists $K_\epsilon \in \mathbb{R}$ such that $\lim_{x\to 0}\frac{|x|^{\beta_-(\gamma)}u_\epsilon(x)}{d(x,\partial\Omega)} = K_\epsilon$. In addition, we assume that the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $H^2_{1,0}(\Omega)$ and we let $\Lambda > 0$ be such that

$$\int_\Omega \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}dx \leq \Lambda \qquad \text{for all } \epsilon > 0. \quad (22)$$

It then follows from the weak compactness of the unit ball of $H^2_{1,0}(\Omega)$ that there exists $u_0 \in H^2_{1,0}(\Omega)$ such that as $\epsilon \to 0$

$$u_\epsilon \rightharpoonup u_0 \qquad \text{weakly in } H^2_{1,0}(\Omega). \quad (23)$$

Note that u_0 is a solution to the Dirichlet boundary value problem:

$$\left\{\begin{array}{rcll} -\Delta u - \gamma\frac{u}{|x|^2} - h_0 u &=& \frac{|u|^{2^\star(s)-2-p_\epsilon}u}{|x|^s} & \text{in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega\setminus\{0\}.\end{array}\right.$$

From the regularity Theorem 6 we have $u_0 \in C^2(\overline{\Omega}\setminus\{0\})$ and $\lim_{x\to 0}\frac{|x|^{\beta_-(\gamma)}u_0(x)}{d(x,\partial\Omega)} = K_0 \in \mathbb{R}$. It then follows that $\sup_\Omega \frac{|x|^{\beta_-(\gamma)}u_0(x)}{d(x,\partial\Omega)}$ and hence $\||x|^{\beta_-(\gamma)-1}u_0(x)\|_{L^\infty(\Omega)}$ is finite.

We fix $\tau \in \mathbb{R}$ such that

$$\beta_-(\gamma) - 1 < \tau < \frac{n-2}{2}. \quad (24)$$

The following proposition shows that the sequence $(u_\epsilon)_\epsilon$ is pre-compact in $H_{1,0}^2(\Omega)$ if $(|x|^\tau u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $L^\infty(\Omega)$.

Proposition 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$, $\gamma < \frac{n^2}{4}$. We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15) and (21) holds. Suppose that there exists $C > 0$ such that $|x|^\tau |u_\epsilon(x)| \leq C$ for all $x \in \Omega$ and for all $\epsilon > 0$. Then up to a subsequence, $\lim\limits_{\epsilon\to 0} u_\epsilon = u_0$ in $H_{1,0}^2(\Omega)$, where u_0 is as in (23).*

Proof of Proposition 1: The sequence (u_ϵ) is clearly uniformly bounded in $L^\infty(\Omega')$ for any $\Omega' \subset\subset \overline{\Omega}\setminus\{0\}$. Then by standard elliptic estimates and from (23) it follows that $u_\epsilon \to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. Now since $|x|^\tau |u_\epsilon(x)| \leq C$ for all $x \in \Omega$ and for all $\epsilon > 0$, and since $\tau < \frac{n-2}{2}$, we have

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \int\limits_{\Omega\cap B_\delta(0)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}dx = 0 \quad \text{and} \quad \lim_{\delta\to 0}\lim_{\epsilon\to 0} \int\limits_{\Omega\cap B_\delta(0)} \frac{|u_\epsilon|^2}{|x|^2}dx = 0. \tag{25}$$

Therefore

$$\lim_{\epsilon\to 0}\int\limits_\Omega \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}dx = \int\limits_\Omega \frac{|u_0|^{2^\star(s)}}{|x|^s}dx \quad \text{and} \quad \lim_{\epsilon\to 0}\int\limits_\Omega \frac{|u_\epsilon|^2}{|x|^2}dx = \int\limits_\Omega \frac{|u_0|^2}{|x|^2}dx.$$

From (E_ϵ) and (23) we then obtain

$$\lim_{\epsilon\to 0}\int\limits_\Omega \left(|\nabla u_\epsilon|^2 - \gamma\frac{u_\epsilon^2}{|x|^2} - h_\epsilon u_\epsilon^2\right)dx = \int\limits_\Omega \left(|\nabla u_0|^2 - \gamma\frac{u_0^2}{|x|^2} - h_0 u_0^2\right)dx$$

$$\text{so then } \lim_{\epsilon\to 0}\int\limits_\Omega |\nabla u_\epsilon|^2 = \lim_{\epsilon\to 0}\int\limits_\Omega |\nabla u_0|^2.$$

And hence $\lim\limits_{\epsilon\to 0} u_\epsilon = u_0$ in $H_{1,0}^2(\Omega)$. This proves Proposition 1. □

From now on, we assume that

$$\lim_{\epsilon\to 0} \||x|^\tau u_\epsilon\|_{L^\infty(\Omega)} = +\infty. \tag{26}$$

We shall say that blow-up occurs whenever (26) holds.

3. Scaling Lemmas

In this section we state and prove two scaling lemmas which we shall use many times in our analysis. We start by describing a parametrization around a point of the boundary $\partial\Omega$. Let $p \in \partial\Omega$. Then there exists U,V open in $\mathbb{R}^n$, there exists $I \subset \mathbb{R}$ an open interval, there exists $U' \subset \mathbb{R}^{n-1}$ an open subset, and there exist a smooth diffeomorphism $\mathcal{T} : U \longrightarrow V$ and $\mathcal{T}_0 \in C^\infty(U')$, such that upto a rotation of coordinates if necessary

$$\left\{\begin{array}{l}\bullet\quad 0\in U=I\times U' \text{ and } p\in V.\\ \bullet\quad \mathcal{T}(0)=p.\\ \bullet\quad \mathcal{T}\left(U\cap\{x_1<0\}\right)=V\cap\Omega \;\text{ and } \mathcal{T}\left(U\cap\{x_1=0\}\right)=V\cap\partial\Omega.\\ \bullet\quad D_0\mathcal{T}=\mathbb{I}_{\mathbb{R}^n}. \text{ Here } D_x\mathcal{T} \text{ denotes the differential of } \mathcal{T} \text{ at the point } x\\ \quad\;\; \text{and } \mathbb{I}_{\mathbb{R}^n} \text{ is the identity map on } \mathbb{R}^n.\\ \bullet\quad \mathcal{T}_*(0)\,(e_1)=\nu_p \;\text{ where } \nu_p \text{ denotes the outer unit normal vector to}\\ \quad\;\; \partial\Omega \text{ at the point } p.\\ \bullet\quad \{\mathcal{T}_*(0)(e_2),\cdots,\mathcal{T}_*(0)(e_n)\} \text{ forms an orthonormal basis of}\\ \quad\;\; T_p\partial\Omega.\\ \bullet\quad \mathcal{T}(x_1,y)=p+(x_1+\mathcal{T}_0(y),y) \text{ for all } (x_1,y)\in I\times U'=U\\ \bullet\quad \mathcal{T}_0(0)=0 \text{ and } \nabla\mathcal{T}_0(0)=0.\end{array}\right. \tag{27}$$

This boundary parametrization will be throughout useful during our analysis. An important remark is that

$$(\mathcal{T}(x_1,y),\partial\Omega)=(1+o(1))|x_1| \qquad \text{for all } (x_1,y)\in I\times U'=U \;\text{ close to } 0. \tag{28}$$

Lemma 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, such that $0\in\partial\Omega$ and assume that $0<s<2$, $\gamma<\frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) holds. Let $(y_\epsilon)_\epsilon\in\Omega$ and let*

$$\nu_\epsilon^{-\frac{n-2}{2}}:=|u_\epsilon(y_\epsilon)|,\quad \ell_\epsilon:=\nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}} \quad \textit{and} \quad \kappa_\epsilon:=|y_\epsilon|^{s/2}\,\ell_\epsilon^{\frac{2-s}{2}} \qquad \textit{for } \epsilon>0$$

Suppose $\lim_{\epsilon\to 0}y_\epsilon=0$ and $\lim_{\epsilon\to 0}\nu_\epsilon=0$. Assume that for any $R>0$ there exists $C(R)>0$ such that for all $\epsilon>0$

$$|u_\epsilon(x)|\leq C(R)\,\frac{|y_\epsilon|^\tau}{|x|^\tau}|u_\epsilon(y_\epsilon)| \qquad \textit{for all } x\in B_{R\kappa_\epsilon}(y_\epsilon)\cap\Omega. \tag{29}$$

Then

$$|y_\epsilon|=O(\ell_\epsilon) \qquad \textit{as } \epsilon\to 0.$$

Proof of Lemma 1: We proceed by contradiction and assume that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon}=+\infty. \tag{30}$$

Then it follows from the definition of κ_ϵ that

$$\lim_{\epsilon\to 0}\kappa_\epsilon=0,\;\lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{\ell_\epsilon}=+\infty \text{ and } \lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{|y_\epsilon|}=0. \tag{31}$$

Case 1: We assume that there exists $\rho>0$ such for all $\epsilon>0$ that

$$\frac{d(y_\epsilon,\partial\Omega)}{\kappa_\epsilon}\geq 3\rho.$$

We define for all $\epsilon>0$

$$v_\epsilon(x):=\nu_\epsilon^{\frac{n-2}{2}}u_\epsilon(y_\epsilon+\kappa_\epsilon x) \qquad \text{for } x\in B_{2\rho}(0)$$

Note that this is well defined for $\epsilon>0$ small enough. It follows from (29) that there exists $C(\rho)>0$ such that all $\epsilon>0$

$$|v_\epsilon(x)|\leq C(\rho)\frac{1}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^\tau} \qquad \forall x\in B_{2\rho}(0) \tag{32}$$

using (31) we then get as $\epsilon \to 0$

$$|v_\epsilon(x)| \le C(\rho)\,(1+o(1)) \qquad \forall x \in B_{2\rho}(0).$$

From equation (E_ϵ) we obtain that v_ϵ satisfies

$$-\Delta v_\epsilon - \frac{\kappa_\epsilon^2}{|y_\epsilon|^2} \frac{\gamma}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^2}\, v_\epsilon - \kappa_\epsilon^2\, h_\epsilon(y_\epsilon + \kappa_\epsilon x)\, v_\epsilon = \frac{|v_\epsilon|^{2^\star(s)-2-p_\epsilon} v_\epsilon}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}$$

weakly in $B_{2\rho}(0)$ for all $\epsilon > 0$. With the help of (31) and standard elliptic theory it then follows that there exists $v \in C^1(B_{2\rho}(0))$ such that

$$\lim_{\epsilon\to 0} v_\epsilon = v \qquad \text{in } C^1(B_\rho(0)).$$

In particular,

$$|v(0)| = \lim_{\epsilon\to 0} |v_\epsilon(0)| = 1 \tag{33}$$

and therefore $v \not\equiv 0$.

On the other hand, a change of variables and the definition of κ_ϵ yields

$$\begin{aligned}
\int_{B_{\rho\kappa_\epsilon}(y_\epsilon)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx &= \frac{|u_\epsilon(y_\epsilon)|^{2^\star(s)-p_\epsilon}\kappa_\epsilon^n}{|y_\epsilon|^s} \int_{B_\rho(0)} \frac{|v_\epsilon|^{2^\star(s)-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}\,dx \\
&= \ell_\epsilon^{-\left(1+\frac{2(2-s)}{2^\star(s)-2-p_\epsilon}\right)} \left(\frac{|y_\epsilon|}{\ell_\epsilon}\right)^{s\left(\frac{n-2}{2}\right)} \int_{B_\rho(0)} \frac{|v_\epsilon|^{2^\star(s)-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}\,dx \\
&\ge \left(\frac{|y_\epsilon|}{\ell_\epsilon}\right)^{s\left(\frac{n-2}{2}\right)} \int_{B_\rho(0)} \frac{|v_\epsilon|^{2^\star(s)-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}\,dx.
\end{aligned}$$

Using the equation (E_ϵ), (22), (30), (31) and passing to the limit $\epsilon \to 0$ we get that

$$\int_{B_\rho(0)} |v|^{2^\star(s)}\,dx = 0$$

and so then $v \equiv 0$ in $B_\rho(0)$, a contradiction with (33). Thus (30) cannot hold in that case.

Case 2: We assume that, up to a subsequence,

$$\lim_{\epsilon\to 0} \frac{d(y_\epsilon, \partial\Omega)}{\kappa_\epsilon} = 0. \tag{34}$$

Note that $\lim_{\epsilon\to 0} y_\epsilon = 0$. Consider the boundary map $\mathcal{T} : U \to V$ as in (27), where U, V are both open neighbourhoods of 0. We let $\tilde{u}_\epsilon = u_\epsilon \circ \mathcal{T}$, which is defined in $U \cap \mathbb{R}^n_-$. For any $i, j = 1, ..., n$, we let $g_{ij} = (\partial_i \mathcal{T}, \partial_j \mathcal{T})$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$, and we consider g as a metric on $\mathbb{R}^n$. We let $\Delta_g = div_g(\nabla)$ the Laplace-Beltrami operator with respect to the metric g. As easily checked, using (E_ϵ) we get that for all $\epsilon > 0$

$$-\Delta_g \tilde{u}_\epsilon - \frac{\gamma}{|\mathcal{T}(x)|^2}\tilde{u}_\epsilon - h_\epsilon \circ \mathcal{T}(x) \cdot \tilde{u}_\epsilon = \frac{|\tilde{u}_\epsilon|^{2^\star(s)-2-p_\epsilon}\tilde{u}_\epsilon}{|\mathcal{T}(x)|^s}$$

weakly in $U \cap \mathbb{R}^n_-$. We let $z_\epsilon \in \partial\Omega$ be such that

$$|z_\epsilon - y_\epsilon| = d(y_\epsilon, \partial\Omega). \tag{35}$$

We let $\tilde{y}_\epsilon, \tilde{z}_\epsilon \in U$ such that

$$\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon \text{ and } \mathcal{T}(\tilde{z}_\epsilon) = z_\epsilon. \tag{36}$$

It follows from the properties (27) of the boundary map $\mathcal{T}$ that

$$\lim_{\epsilon\to 0} \tilde{y}_\epsilon = \lim_{\epsilon\to 0} \tilde{z}_\epsilon = 0, \ (\tilde{y}_\epsilon)_1 < 0 \text{ and } (\tilde{z}_\epsilon)_1 = 0 \tag{37}$$

We rescale and define for all $\epsilon > 0$

$$\tilde{v}_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} \tilde{u}_\epsilon(\tilde{z}_\epsilon + \kappa_\epsilon x) \qquad \text{for } x \in \frac{U - \tilde{z}_\epsilon}{\kappa_\epsilon} \cap \mathbb{R}^n_-.$$

With 37), we get that $\tilde{v}_\epsilon$ is defined on $B_R(0) \cap \{x_1 < 0\}$ for all $R > 0$, for ϵ is small enough. Then for all $\epsilon > 0$ the functions $\tilde{v}_\epsilon$ satisfies the equation:

$$-\Delta_{\tilde{g}_\epsilon} \tilde{v}_\epsilon - \frac{\kappa_\epsilon^2}{|y_\epsilon|^2} \frac{\gamma}{\left|\frac{\mathcal{T}(\tilde{z}_\epsilon + \kappa_\epsilon x)}{|y_\epsilon|}\right|^2} - \kappa_\epsilon^2 h_\epsilon \circ \mathcal{T}(\tilde{z}_\epsilon + \kappa_\epsilon x)\tilde{v}_\epsilon = \frac{|\tilde{v}_\epsilon|^{2^\star(s)-2-p_\epsilon}\tilde{v}_\epsilon}{\left|\frac{\mathcal{T}(\tilde{z}_\epsilon + \kappa_\epsilon x)}{|y_\epsilon|}\right|^s}$$

weakly in $B_R(0) \cap \{x_1 < 0\}$. In this expression, $\tilde{g}_\epsilon = g(\tilde{z}_\epsilon + \kappa_\epsilon x)$ and $\Delta_{\tilde{g}_\epsilon}$ is the Laplace-Beltrami operator with respect to the metric $\tilde{g}_\epsilon$. With (34), (35) and (36), we get for all $\epsilon > 0$

$$\mathcal{T}(\tilde{z}_\epsilon + \kappa_\epsilon x) = y_\epsilon + O_R(1)\kappa_\epsilon \qquad \text{for all } x \in B_R(0) \cap \{x_1 \le 0\}$$

where, there exists $C_R > 0$ such that $|O_R(1)| \le C_R$ for all $x \in B_R(0) \cap \{x_1 \le 0\}$. With (31), we then get that

$$\lim_{\epsilon\to 0} \frac{|\mathcal{T}(\tilde{z}_\epsilon + \kappa_\epsilon x)|}{|y_\epsilon|} = 1 \qquad \text{in } C^0(B_R(0) \cap \{x_1 \le 0\}).$$

It follows from (29) that there exists $C'(R) > 0$ such that all $\epsilon > 0$

$$|\tilde{v}_\epsilon(x)| \le C(R) \frac{1}{\left|\frac{\mathcal{T}(\tilde{z}_\epsilon + \kappa_\epsilon x)}{|y_\epsilon|}\right|^\tau} \qquad \forall x \in B_R(0) \cap \{x_1 \le 0\}. \tag{38}$$

Using (31) and the propoerties of the boundary map $\mathcal{T}$ we then get as $\epsilon \to 0$

$$|\tilde{v}_\epsilon(x)| \le C(R)\,(1 + o(1)) \qquad \forall x \in B_R(0) \cap \{x_1 \le 0\}.$$

With the help of (31) and standard elliptic theory it then follows that there exists $\tilde{v} \in C^1(B_R(0) \cap \{x_1 \le 0\})$ such that

$$\lim_{\epsilon\to 0} \tilde{v}_\epsilon = \tilde{v} \qquad \text{in } C^0(B_{R/2}(0) \cap \{x_1 \le 0\}).$$

Since $\tilde{v}_\epsilon$ vanishes on $B_R(0) \cap \{x_1 = 0\}$ and (38) holds, it follows that

$$\tilde{v} \equiv 0 \text{ on } B_{R/2}(0) \cap \{x_1 = 0\}. \tag{39}$$

Moreover, from (34), (35) and (36) we have that

$$\left|\tilde{v}_\epsilon\left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\kappa_\epsilon}\right)\right| = 1 \text{ and } \lim_{\epsilon\to 0} \frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\kappa_\epsilon} = 0.$$

In particular, $\tilde{v}(0) = 1$, contradiction with (39). Thus (30) cannot hold in *Case 2* also.

In both cases, we have contradicted (30) . This proves that $y_\epsilon = O(\ell_\epsilon)$ when $\epsilon \to 0$, which proves the Lemma. □

LEMMA 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$, $\gamma < \frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) such that (E_ϵ), (15), (21) and (22) holds. Let $(y_\epsilon)_\epsilon \in \Omega$ and let*

$$\nu_\epsilon^{-\frac{n-2}{2}} := |u_\epsilon(y_\epsilon)| \quad \text{and} \quad \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}} \qquad \text{for } \epsilon > 0$$

Suppose $\nu_\epsilon \to 0$ and $|y_\epsilon| = O(\ell_\epsilon)$ as $\epsilon \to 0$.

Since $0 \in \partial\Omega$, we let $\mathcal{T} : U \to V$ as in (27) with $y_0 = 0$, where U, V are open neighborhoods of 0. For $\epsilon > 0$ we rescale and define

$$\tilde{w}_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} u_\epsilon \circ \mathcal{T}(\ell_\epsilon x) \qquad \text{for } x \in \ell_\epsilon^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\}.$$

Assume that for any $R > \delta > 0$ there exists $C(R, \delta) > 0$ such that for all $\epsilon > 0$

$$|\tilde{w}_\epsilon(x)| \leq C(R, \delta) \qquad \text{for all } x \in B_R(0) \setminus \overline{B_\delta(0)} \cap \mathbb{R}^n_-. \tag{40}$$

Then there exists $\tilde{w} \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ such that

$$\tilde{w}_\epsilon \rightharpoonup \tilde{w} \qquad \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \qquad \text{as } \epsilon \to 0$$
$$\tilde{w}_\epsilon \to \tilde{w} \qquad \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) \qquad \text{as } \epsilon \to 0$$

And $\tilde{w}$ satisfies weakly the equation

$$-\Delta \tilde{w} - \frac{\gamma}{|x|^2}\tilde{w} = \frac{|\tilde{w}|^{2^\star(s)-2}\tilde{w}}{|x|^s} \text{ in } \mathbb{R}^n_-.$$

Moreover if $\tilde{w} \not\equiv 0$, then

$$\int_{\mathbb{R}^n_-} \frac{|\tilde{w}|^{2^\star(s)}}{|x|^s} \geq \mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t \in (0, 1]$ such that $\lim_{\epsilon \to 0} \nu_\epsilon^{p_\epsilon} = t$, where $\mu_{\gamma,s}(\mathbb{R}^n_-)$ is as in (5).

Proof of Lemma 2: The proof proceeds in four steps.

Step 2.1: Let $\eta \in C_c^\infty(\mathbb{R}^n)$. One has that $\eta\tilde{w}_\epsilon \in H^2_{0,1}(\mathbb{R}^n_-)$ for $\epsilon > 0$ sufficiently small. We claim that there exists $\tilde{w}_\eta \in H^2_{1,0}(\mathbb{R}^n_-)$ such that upto a subsequence

$$\begin{cases} \eta\tilde{w}_\epsilon \rightharpoonup \tilde{w}_\eta & \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0, \\ \eta\tilde{w}_\epsilon \to \tilde{w}_\eta(x) & a.e \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0. \end{cases}$$

We prove the claim. Let $x \in \mathbb{R}^n_-$, then

$$\nabla(\eta\tilde{w}_\epsilon)(x) = \tilde{w}_\epsilon(x)\nabla\eta(x) + \nu_\epsilon^{\frac{n-2}{2}} \ell_\epsilon \, \eta(x) D_{(\ell_\epsilon x)}\mathcal{T}\left[\nabla u_\epsilon\left(\mathcal{T}(\ell_\epsilon x)\right)\right]$$

In this expression, $D_x\mathcal{T}$ is the differential of the function $\mathcal{T}$ at x.

Now for any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$

$$(a+b)^2 \leq C(\theta)a^2 + (1+\theta)b^2$$

With this inequality we then obtain

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2\ dx \leq C(\theta)\int_{\mathbb{R}^n_-} |\nabla\eta|^2\tilde{w}_\epsilon^2\ dx + (1+\theta)\nu_\epsilon^{\frac{n-2}{2}}\ell_\epsilon \int_{\mathbb{R}^n_-} \eta^2 \left|D_{(\ell_\epsilon x)}\mathcal{T}\left[\nabla u_\epsilon\left(\mathcal{T}(\ell_\epsilon x)\right)\right]\right|^2\ dx$$

Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$ we have as $\epsilon \to 0$

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2\ dx \leq C(\theta)\int_{\mathbb{R}^n_-} |\nabla\eta|^2\tilde{w}_\epsilon^2\ dx + (1+\theta)\left(1+O(\ell_\epsilon)\right)\nu_\epsilon^{\frac{n-2}{2}}\ell_\epsilon \int_{\mathbb{R}^n_-} \eta^2 \left|\nabla u_\epsilon\left(\mathcal{T}(\ell_\epsilon x)\right)\right|^2 (1+o(1))\ dx$$

With Hölder inequality and a change of variables this becomes

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2\ dx \leq C(\theta)\,\|\nabla\eta\|^2_{L^n}\left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2}\left(\int_{\Omega}|u_\epsilon|^{2^\star(s)}\ dx\right)^{\frac{n-2}{n}} + (1+\theta)\left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2}\int_{\Omega}|\nabla u_\epsilon|^2\ dx \tag{41}$$

Since $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$, so for $\epsilon > 0$ small enough

$$\|\eta\tilde{w}_\epsilon\|_{H^2_{1,0}(\mathbb{R}^n_-)} \leq C_\eta$$

Where C_η is a constant depending on the function η. The claim then follows from the reflexivity of $H^2_{1,0}(\mathbb{R}^n_-)$.

Step 2.2: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \eta_1 \leq 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for} \quad x \in B_1(0) \\ 0 & \text{for} \quad x \in \mathbb{R}^n\backslash B_2(0) \end{cases} \tag{42}$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that upto a subsequence for any $R > 0$ there exists $\tilde{w}_R \in H^2_{1,0}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \eta_R\tilde{w}_\epsilon \rightharpoonup \tilde{w}_R & \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \ \text{ as } \epsilon \to 0 \\ \eta_R\tilde{w}_\epsilon(x) \to \tilde{w}_R(x) & a.e\ x \ \text{ in } \mathbb{R}^n_- \ \text{ as } \epsilon \to 0 \end{cases}$$

Since $\|\nabla\eta_R\|_n^2 = \|\nabla\eta_1\|_n^2$ for all $R > 0$, letting $\epsilon \to 0$ in (41) we obtain that

$$\int_{\mathbb{R}^n_-} |\nabla w_R|^2\, dx \leq C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $\tilde{w} \in H^2_{1,0}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \tilde{w}_R \rightharpoonup \tilde{w} & \text{weakly in } D^{1,2}(\mathbb{R}^n) \ \text{ as } R \to +\infty \\ \tilde{w}_R(x) \to \tilde{w}(x) & a.e\ x \ \text{ in } \mathbb{R}^n_- \ \text{ as } R \to +\infty \end{cases}$$

Step 2.3: We claim that $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ and it satisfies weakly the equation

$$\left\{ \begin{array}{rcll} -\Delta \tilde{w} - \frac{\gamma}{|x|^2}\tilde{w} & = & \frac{|\tilde{w}|^{2^\star(s)-2}\tilde{w}}{|x|^s} & \text{in } \mathbb{R}^n_- \\ \tilde{w} & = & 0 & \text{on } \partial\mathbb{R}^n_- \setminus \{0\}. \end{array} \right.$$

We prove the claim. For any $i,j = 1,...,n$, we let $(\tilde{g}_\epsilon)_{ij} = (\partial_i \mathcal{T}(\ell_\epsilon x), \partial_j \mathcal{T}(\ell_\epsilon x))$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$. We consider $\tilde{g}_\epsilon$ as a metric on $\mathbb{R}^n$. We let $\Delta_g = div_g(\nabla)$ the Laplace-Beltrami operator with respect to the metric g. From (E_ϵ) it follows that for any $\epsilon > 0$ and $R > 0$, $\eta_R \tilde{w}_\epsilon$ satisfies weakly the equation

$$-\Delta_{\tilde{g}_\epsilon}(\eta_R \tilde{w}_\epsilon) - \frac{\gamma}{\left|\frac{\mathcal{T}(\ell_\epsilon x)}{\ell_\epsilon}\right|^2}\eta_R \tilde{w}_\epsilon - \ell_\epsilon^2 \; h_\epsilon \circ \mathcal{T}(\ell_\epsilon x)\eta_R \tilde{w}_\epsilon = \frac{|(\eta_R \tilde{w}_\epsilon)|^{2^\star(s)-2-p_\epsilon}(\eta_R \tilde{w}_\epsilon)}{\left|\frac{\mathcal{T}(\ell_\epsilon x)}{\ell_\epsilon}\right|^s}. \tag{43}$$

and note that $\eta_R \tilde{w}_\epsilon \equiv 0$ on $B_R(0) \setminus \{0\} \cap \partial\mathbb{R}^n_-$. From (27), (40) and using the standard elliptic estimates it follows that $\tilde{w}_R \in C^1\left(B_R(0) \setminus \{0\} \cap \overline{\mathbb{R}^n_-}\right)$ and that up to a subsequence

$$\lim_{\epsilon \to 0} \eta_R \tilde{w}_\epsilon = \tilde{w}_R \qquad \text{in } C^1_{loc}\left(B_{R/2}(0) \setminus \{0\} \cap \overline{\mathbb{R}^n_-}\right).$$

Letting $\epsilon \to 0$ in eqn (43) gives that w_R satisfies weakly the equation

$$-\Delta \tilde{w}_R - \frac{\gamma}{|x|^2}\tilde{w}_R = \frac{|\tilde{w}_R|^{2^\star(s)-2-p_\epsilon}\tilde{w}_R}{|x|^s}.$$

Again we have that $|\tilde{w}_R(x)| \leq C(R,\delta)$ for all $x \in \overline{B_{R/2}(0)} \setminus \overline{B_{2\delta}(0)}$ and then again from standard elliptic estimates it follows that $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ and $\lim_{R \to +\infty} \tilde{w}_R = \tilde{w}$ in $C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$, up to a subsequence. Letting $R \to +\infty$ we obtain that $\tilde{w}$ satisfies weakly the equation

$$\left\{ \begin{array}{rcll} -\Delta \tilde{w} - \frac{\gamma}{|x|^2}\tilde{w} & = & \frac{|\tilde{w}|^{2^\star(s)-2}\tilde{w}}{|x|^s} & \text{in } \mathbb{R}^n_- \\ \tilde{w} & = & 0 & \text{on } \partial\mathbb{R}^n_- \setminus \{0\}. \end{array} \right.$$

This proves our claim.

Step 2.4: Coming back to equation (41) we have for $R > 0$

$$\int_{\mathbb{R}^n_-} |\nabla(\eta_R \tilde{w}_\epsilon)|^2 \; dx \leq \; C(\theta) \left(\int_{\{x \in \mathbb{R}^n_- : R < |x| < 2R\}} (\eta_{2R} \tilde{w}_\epsilon)^{2^*} \; dx \right)^{\frac{n-2}{n}} + (1+\theta) \left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2} \int_\Omega |\nabla u_\epsilon|^2 \; dx. \tag{44}$$

Since the sequence $(u_\epsilon)_\epsilon$ is bounded in $H^2_{1,0}(\Omega)$, letting $\epsilon \to 0$ and then $R \to +\infty$ we obtain for some constant C

$$\int_{\mathbb{R}^n_-} |\nabla \tilde{w}|^2 \; dx \leq C \left(\lim_{\epsilon \to 0} \left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)\right)^{n-2}.$$

Now if $w \not\equiv 0$ weakly satisfies the equation

$$\left\{ \begin{array}{rcll} -\Delta \tilde{w} - \frac{\gamma}{|x|^2} \tilde{w} & = & \frac{|\tilde{w}|^{2^\star(s)-2} \tilde{w}}{|x|^s} & \text{in } \mathbb{R}^n_- \\ \tilde{w} & = & 0 & \text{on } \partial \mathbb{R}^n_- \setminus \{0\}. \end{array} \right.$$

using the definition of $\mu_{\gamma,s}(\mathbb{R}^n_-)$ it then follows that

$$\int_{\mathbb{R}^n_-} \frac{|w|^{2^\star(s)}}{|x|^s} \geq \mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}}.$$

Hence $\lim_{\epsilon \to 0} \left(\frac{\nu_\epsilon}{\ell_\epsilon} \right) > 0$ which implies that

$$t := \lim_{\epsilon \to 0} \nu_\epsilon^{p_\epsilon} > 0.$$

Since $\lim_{\epsilon \to 0} \nu_\epsilon = 0$, therefore we have that $0 < t \leq 1$. This completes the lemma. $\square$

4. Construction and exhaustion of the blow-up scales

In this section we prove the following proposition in the spirit of Druet-Hebey-Robert [11]:

Proposition 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$, $\gamma < \frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon \to 0} \| |x|^\tau u_\epsilon \|_{L^\infty(\Omega)} = +\infty \quad \text{where} \quad \beta_-(\gamma) - 1 < \tau < \frac{n-2}{2}.$$

Then there exists $N \in \mathbb{N}^\star$ families of scales $(\mu_{i,\epsilon})_{\epsilon > 0}$ such that we have:

- **(A1)** $\lim_{\epsilon \to 0} u_\epsilon = u_0$ *in $C^2_{loc}(\overline{\Omega} \setminus \{0\})$ where u_0 is as in*(23).
- **(A2)** $0 < \mu_{1,\epsilon} < ... < \mu_{N,\epsilon}$, *for all $\epsilon > 0$.*
- **(A3)** $\lim_{\epsilon \to 0} \mu_{N,\epsilon} = 0$ *and* $\lim_{\epsilon \to 0} \frac{\mu_{i+1,\epsilon}}{\mu_{i,\epsilon}} = +\infty$ *for all $1 \leq i \leq N-1$.*
- **(A4)** *For any $1 \leq i \leq N$ and for $\epsilon > 0$ we rescale and define*

$$\tilde{u}_{i,\epsilon}(x) := \mu_{i,\epsilon}^{\frac{n-2}{2}} u_\epsilon(\mathcal{T}(k_{i,\epsilon} x)) \qquad \text{for } x \in k_{i,\epsilon}^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\},$$

where $k_{i,\epsilon} = \mu_{i,\epsilon}^{1 - \frac{p_\epsilon}{2^\star(s)-2}}$. Then there exists $\tilde{u}_i \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$, $\tilde{u}_i \not\equiv 0$ such that $\tilde{u}_i$ weakly solves the equation

$$\left\{ \begin{array}{rcll} -\Delta \tilde{u}_i - \frac{\gamma}{|x|^2} \tilde{u}_i & = & \frac{|\tilde{u}_i|^{2^\star(s)-2} \tilde{u}_i}{|x|^s} & \text{in } \mathbb{R}^n_- \\ \tilde{u}_i & = & 0 & \text{on } \partial \mathbb{R}^n_- \setminus \{0\}. \end{array} \right.$$

and

$$\begin{array}{llll} \tilde{u}_{i,\epsilon} \longrightarrow \tilde{u}_i & \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) & \text{as } \epsilon \to 0, \\ \tilde{u}_{i,\epsilon} \rightharpoonup \tilde{u}_i & \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) & \text{as } \epsilon \to 0. \end{array}$$

- **(A5)** *There exists $C > 0$ such that*

$$|x|^{\frac{n-2}{2}} |u_\epsilon(x)|^{1 - \frac{p_\epsilon}{2^\star(s)-2}} \leq C \quad \text{for all } \epsilon > 0 \text{ and all } x \in \Omega.$$

(A6) $\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{\Omega\setminus B_{Rk_{N,\epsilon}}(0)} |x|^{\frac{n-2}{2}}|u_\epsilon(x)-u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=0.$

(A7) $\lim_{\delta\to 0}\lim_{\epsilon\to 0}\sup_{B_{\delta k_{1,\epsilon}}(0)\cap\Omega} |x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{\mathcal{T}^{-1}(x)}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=0.$

(A8) *For any $\delta>0$ and any $1\le i\le N-1$, we have*

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{\delta k_{i+1,\epsilon}\ge|x|\ge Rk_{i,\epsilon}} |x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=0.$$

(A9) *For any $i\in\{1,...,N\}$, there exists $t_i\in(0,1]$ such that $\lim_{\epsilon\to 0}\mu_{i,\epsilon}^{p_\epsilon}=t_i$.*

The proof of this proposition is inspired by [11] and proceeds in five steps.

Since $s>0$, the subcriticality $2^\star(s)<2^\star(s)$ of equations (E_ϵ) along with (23) yields that $u_\epsilon\to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. So the only blow-up point is the origin.

Step 4.1: The construction of the $\mu_{i,\epsilon}$'s proceeds by induction. This step is the initiation.

By the regularity Theorem 6 and the definition of τ in (24) it follows that for any $\epsilon>0$ there exists $x_{1,\epsilon}\in\overline{\Omega}\setminus\{0\}$ such that

$$\sup_{x\in\Omega}|x|^\tau|u_\epsilon(x)|=|x_{1,\epsilon}|^\tau|u_\epsilon(x_{1,\epsilon})|. \tag{45}$$

We define $\mu_{1,\epsilon}$ and $k_{1,\epsilon}>0$ as follows

$$\mu_{1,\epsilon}^{-\frac{n-2}{2}}:=|u_\epsilon(x_{1,\epsilon})| \quad\text{and}\quad k_{1,\epsilon}:=\mu_{1,\epsilon}^{1-\frac{p_\epsilon}{2^\star(s)-2}}. \tag{46}$$

Since blow-up occurs, that is (26) holds and since $u_\epsilon\to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$, we have that

$$\lim_{\epsilon\to 0}x_{1,\epsilon}=0\in\partial\Omega \qquad\text{and}\qquad \lim_{\epsilon\to 0}\mu_{1,\epsilon}=0.$$

It follows that u_ϵ satisfies the hypothesis (29) of Lemma 1 with $y_\epsilon=x_{1,\epsilon}$, $\nu_\epsilon=\mu_{1,\epsilon}$. Therefore

$$|x_{1,\epsilon}|=O\left(k_{1,\epsilon}\right)\quad\text{as }\epsilon\to 0.$$

In fact, we claim that there exists $c_1>0$ such that

$$\lim_{\epsilon\to 0}\frac{|x_{1,\epsilon}|}{k_{1,\epsilon}}=c_1. \tag{47}$$

We argue by contradiction and we assume that $|x_{1,\epsilon}|=o(k_{1,\epsilon})$ as $\epsilon\to 0$. Let $\tilde{x}_{1,\epsilon}:=\mathcal{T}^{-1}(x_{1,\epsilon})\in\mathbb{R}^n_-$. Since $|x_{1,\epsilon}|=o(k_{1,\epsilon})$ as $\epsilon\to 0$, so also $|\tilde{x}_{1,\epsilon}|=o(k_{1,\epsilon})$ as $\epsilon\to 0$.

We define for $\epsilon>0$

$$\tilde{v}_\epsilon(x):=\mu_{1,\epsilon}^{\frac{n-2}{2}}u_\epsilon(\mathcal{T}(|\tilde{x}_{1,\epsilon}|\,x)) \qquad \text{for } x\in\frac{U}{|\tilde{x}_{1,\epsilon}|}\cap\overline{\mathbb{R}^n_-}\setminus\{0\}$$

Using (E_ϵ) we obtain that $\tilde{v}_\epsilon$ satisfies the equation

$$-\Delta\tilde{v}_\epsilon-\frac{\gamma}{\left|\frac{\mathcal{T}(|\tilde{x}_{1,\epsilon}|x)}{|\tilde{x}_{1,\epsilon}|}\right|^2}\tilde{v}_\epsilon+|x_{1,\epsilon}|^2h_\epsilon\circ\mathcal{T}(|\tilde{x}_{1,\epsilon}|x)\;\tilde{v}_\epsilon=\left(\frac{|\tilde{x}_{1,\epsilon}|}{k_{1,\epsilon}}\right)^{2-s-p_\epsilon}\frac{|\tilde{v}_\epsilon|^{2^\star(s)-2-p_\epsilon}\tilde{v}_\epsilon}{\left|\frac{\mathcal{T}(|\tilde{x}_{1,\epsilon}|x)}{|\tilde{x}_{1,\epsilon}|}\right|^s}$$

The definition (45) yields as $\epsilon \to 0$, $|x|^{\tau}\left|\tilde{v}_\epsilon(x)\right| \leq 2$ for all $x \in \mathbb{R}^n_-$.
Standard elliptic theory then yields the existence of $\tilde{v} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0\})$ such that $\tilde{v}_\epsilon \to \tilde{v}$ in $C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$ where

$$\begin{cases} -\Delta \tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} &= 0 \quad \text{in } \mathbb{R}^n_- \\ \tilde{v} &= 0 \quad \text{on } \partial\mathbb{R}^n_- \setminus \{0\}. \end{cases}$$

In addition, we have that $\left|\tilde{v}_\epsilon\left(|\tilde{x}_{1,\epsilon}|^{-1}\tilde{x}_{1,\epsilon}\right)\right| = 1$ and so $\tilde{v} \not\equiv 0$. Also since $|x|^\tau|\tilde{v}(x)| \leq 2$ in $\overline{\mathbb{R}^n_-} \setminus \{0\}$, we have the bound that

$$|x|^{\tau+1}|\tilde{v}(x)| \leq 2|x_1| \qquad \text{for all} \;\; x = (x_1, \tilde{x}) \text{ in } \mathbb{R}^n_-, \tag{48}$$

which implies that

$$|\tilde{v}(x)| < 4\frac{|x_1|}{|x|^{\beta_+(\gamma)}} + 4\frac{|x_1|}{|x|^{\beta_-(\gamma)}} \qquad \text{for all} \;\; x = (x_1, \tilde{x}) \text{ in } \mathbb{R}^n_-.$$

Therefore $x \mapsto \tilde{V}(x) := 4\frac{|x_1|}{|x|^{\beta_+(\gamma)}} + 4\frac{|x_1|}{|x|^{\beta_-(\gamma)}} - \tilde{v}(x)$ is a positive solution to $-\Delta\tilde{V} - \frac{\gamma}{|x|^2}\tilde{V} = 0$ in $\mathbb{R}^n_-$. Proposition 9 yields the existence of $A, B \in \mathbb{R}$ such that

$$\tilde{v}(x) = A\frac{|x_1|}{|x|^{\beta_+(\gamma)}} + B\frac{|x_1|}{|x|^{\beta_-(\gamma)}} \qquad \text{for all} \;\; x \text{ in } \mathbb{R}^n_-.$$

But the pointwise control (48) then implies $A = B = 0$ by letting $|x| \to 0$ and $\to \infty$. This contradicts $\tilde{v} \not\equiv 0$. This proves Claim (47).

We rescale and define for all $\epsilon > 0$

$$\tilde{u}_{1,\epsilon}(x) := \mu_{1,\epsilon}^{\frac{n-2}{2}}\, u_\epsilon(\mathcal{T}(k_{1,\epsilon}\, x)) \qquad \text{for } x \in k_{1,\epsilon}^{-1}U \cap \overline{\mathbb{R}^n_-} \setminus \{0\}$$

It follows from (45) and (47) that $\tilde{u}_{1,\epsilon}$ satisfies the hypothesis (40) of Lemma 2 with $y_\epsilon = x_{1,\epsilon}$, $\nu_\epsilon = \mu_{1,\epsilon}$. Then using Lemma 2 we get that there exists $\tilde{u}_1 \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ weakly satisfying the equation:

$$\begin{cases} -\Delta\tilde{u}_1 - \frac{\gamma}{|x|^2}\tilde{u}_1 &= \frac{|\tilde{u}_1|^{2^\star(s)-2}\tilde{u}_i}{|x|^s} \quad \text{in } \mathbb{R}^n_- \\ \tilde{u}_1 &= 0 \quad \text{on } \partial\mathbb{R}^n_- \setminus \{0\}. \end{cases}$$

and

$$\tilde{u}_{1,\epsilon} \longrightarrow \tilde{u}_1 \qquad \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) \qquad \text{as } \epsilon \to 0,$$
$$\tilde{u}_{1,\epsilon} \rightharpoonup \tilde{u}_1 \qquad \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \qquad \text{as } \epsilon \to 0.$$

It follows from the definition that $\left|\tilde{u}_{i_o,\epsilon}\left(\frac{\tilde{x}_{1,\epsilon}}{k_{1,\epsilon}}\right)\right| = 1$. From (47) we therefore have that $\tilde{u}_1 \not\equiv 0$. And hence again from Lemma 2 we get that

$$\int\limits_{\mathbb{R}^n_-} \frac{|\tilde{u}_1|^{2^\star(s)}}{|x|^s} \geq \mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}}.$$

Moreover, there exists $t_1 \in (0,1]$ such that $\lim\limits_{\epsilon\to 0}\mu_{1,\epsilon}^{p_\epsilon} = t_1$. Since $\frac{|x|^{\beta_-(\gamma)}}{|x_1|}\tilde{u}_1 \in C^0(\mathbb{R}^n)$, we get as $\epsilon \to 0$

$$|y_\epsilon|^{\frac{n-2}{2}}\left|\mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{\mathcal{T}^{-1}(y_\epsilon)}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = O\left(|\tilde{y}_\epsilon|\right)^{\frac{n}{2}-\beta_-(\gamma)} = o(1),$$

and

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\sup_{B_{\delta k_{1,\epsilon}}(0)\cap\Omega}|x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{\mathcal{T}^{-1}(x)}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=0.$$

□

Step 4.2: We claim that there exists $C>0$ such that

$$|x|^{\frac{n-2}{2}}|u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}\leq C \quad \text{for all } \epsilon>0 \text{ and all } x\in\Omega. \tag{49}$$

We argue by contradiction and let $(y_\epsilon)_{\epsilon>0}\in\Omega$ be such that

$$\sup_{x\in\Omega}|x|^{\frac{n-2}{2}}|u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=|y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}\to+\infty \quad \text{as } \epsilon\to 0. \tag{50}$$

By the regularity Theorem 6, it follows that the sequence $(y_\epsilon)_{\epsilon>0}$ is well-defined and moreover $\lim_{\epsilon\to 0}y_\epsilon=0$, since $u_\epsilon\to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. For $\epsilon>0$ we let

$$\nu_\epsilon:=|u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}},\ \ell_\epsilon:=\nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}} \text{ and } \kappa_\epsilon:=|y_\epsilon|^{s/2}\,\ell_\epsilon^{\frac{2-s}{2}}.$$

Then it follows from (50) that

$$\lim_{\epsilon\to 0}\nu_\epsilon=0,\ \lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon}=+\infty \text{ and } \lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{|y_\epsilon|}=0. \tag{51}$$

Let $R>0$ and let $x\in B_R(0)$ be such that $y_\epsilon+\kappa_\epsilon x\in\Omega$. It follows from the definition (50) of y_ϵ that for all $\epsilon>0$

$$|y_\epsilon+\kappa_\epsilon x|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon+\kappa_\epsilon x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}\leq|y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}$$

and then, for all $\epsilon>0$

$$\left(\frac{|u_\epsilon(y_\epsilon+\kappa_\epsilon x)|}{|u_\epsilon(y_\epsilon)|}\right)^{1-\frac{p_\epsilon}{2^\star(s)-2}}\leq\left(\frac{1}{1-\frac{\kappa_\epsilon}{|y_\epsilon|}R}\right)^{\frac{n-2}{2}}$$

for all $x\in B_R(0)$ such that $y_\epsilon+\kappa_\epsilon x\in\Omega$. Using (51), we get that there exists $C(R)>0$ such that the hypothesis (29) of Lemma 1 is satisfied and therefore one has $|y_\epsilon|=O(\ell_\epsilon)$ when $\epsilon\to 0$, contradiction to (51). This proves (49). □

Let $\mathcal{I}\in\mathbb{N}^\star$. We consider the following assertions:

(B1) $0<\mu_{1,\epsilon}<\ldots<\mu_{\mathcal{I},\epsilon}$.

(B2) $\lim_{\epsilon\to 0}\mu_{\epsilon,\mathcal{I}}=0$ and $\lim_{\epsilon\to 0}\frac{\mu_{i+1,\epsilon}}{\mu_{i,\epsilon}}=+\infty$ for all $1\leq i\leq\mathcal{I}-1$

(B3) For all $1\leq i\leq\mathcal{I}$, there exists $\tilde{u}_i\in H^2_{1,0}(\mathbb{R}^n_-)\cap C^2(\overline{\mathbb{R}^n_-}\setminus\{0\})$ such that $\tilde{u}_i$ weakly solves the equation

$$\begin{cases} -\Delta\tilde{u}_i-\frac{\gamma}{|x|^2}\tilde{u}_i &= \frac{|\tilde{u}_i|^{2^\star(s)-2}\tilde{u}_i}{|x|^s} & \text{in } \mathbb{R}^n_- \\ \tilde{u}_i &= 0 & \text{on } \partial\mathbb{R}^n_-\setminus\{0\}, \end{cases}$$

with

$$\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\geq\mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}},$$

and

$$\tilde{u}_{i,\epsilon} \longrightarrow \tilde{u}_i \qquad \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) \qquad \text{as } \epsilon \to 0,$$
$$\tilde{u}_{i,\epsilon} \rightharpoonup \tilde{u}_i \qquad \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \quad \text{as } \epsilon \to 0,$$

where for $\epsilon > 0$, we have set $k_{i,\epsilon} = \mu_{i,\epsilon}^{1-\frac{p_\epsilon}{2^\star(s)-2}}$ and

$$\tilde{u}_{i,\epsilon}(x) := \mu_{1,\epsilon}^{\frac{n-2}{2}} u_\epsilon(\mathcal{T}(k_{i,\epsilon}\ x)) \qquad \text{for } x \in k_{i,\epsilon}^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\}.$$

(B4) For all $1 \leq i \leq \mathcal{I}$, there exists $t_i \in (0,1]$ such that $\lim_{\epsilon\to 0} \mu_{i,\epsilon}^{p_\epsilon} = t_i$.

We shall then say that $(\mathcal{H}_\mathcal{I})$ holds if there exists $\mathcal{I}$ sequences $(\mu_{i,\epsilon})_{\epsilon>0}$, $i = 1, ..., \mathcal{I}$ such that items (B1), (B2) (B3) and (B4) holds. Note that it follows from Step 4.1 that $(\mathcal{H}_1)$ holds. Next we show the following:

Step 4.3 Let $I \geq 1$. We assume that $(\mathcal{H}_\mathcal{I})$ holds. Then, either

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\Omega\setminus B_{Rk_{\mathcal{I},\epsilon}}(0)} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0,$$

or $\mathcal{H}_{\mathcal{I}+1}$ holds.

Proof of Step 4.3: Suppose $\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\Omega\setminus B_{Rk_{\mathcal{I},\epsilon}}(0)} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \neq 0$. Then, there exists a sequence of points $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{\mathcal{I},\epsilon}} = +\infty \text{ and } \lim_{\epsilon\to 0} |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon) - u_0(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0. \tag{52}$$

Since $u_\epsilon \to u_0$ in $C^2_{loc}(\overline{\Omega} \setminus \{0\})$ it follows that $\lim_{\epsilon\to 0} y_\epsilon = 0$. Then by the regularity Theorem 6 and since $\beta_-(\gamma) < \frac{n-2}{2}$, we get

$$\lim_{\epsilon\to 0} |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0 \tag{53}$$

for some positive constant a. In particular, $\lim_{\epsilon\to 0} |u_\epsilon(y_\epsilon)| = +\infty$. Let

$$\mu_{\mathcal{I}+1,\epsilon} := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } \ k_{\mathcal{I}+1,\epsilon} := \mu_{\mathcal{I}+1,\epsilon}^{1-\frac{p_\epsilon}{2^\star(s)-2}}.$$

As a consequence we have

$$\lim_{\epsilon\to 0} \mu_{\mathcal{I}+1,\epsilon} = 0 \qquad \text{and} \qquad \lim_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{\mathcal{I}+1,\epsilon}} = a > 0. \tag{54}$$

We rescale and define

$$\tilde{u}_{\mathcal{I}+1,\epsilon}(x) := \mu_{\mathcal{I}+1,\epsilon}^{\frac{n-2}{2}} u_\epsilon(\mathcal{T}(k_{I+1,\epsilon}\ x)) \qquad \text{for } x \in k_{\mathcal{I}+1,\epsilon}^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\}.$$

It follows from (49) that for all $\epsilon > 0$

$$\left| \frac{\mathcal{T}(k_{I+1,\epsilon}\ x)}{k_{\mathcal{I}+1,\epsilon}} \right|^{\frac{n-2}{2}} |\tilde{u}_{\mathcal{I}+1,\epsilon}(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq C \qquad \text{for } x \in k_{\mathcal{I}+1,\epsilon}^{-1} \Omega \setminus \{0\},$$

and so hypothesis (40) of Lemma 2 is satisfied. Using Lemma 2, we then get that there exists $\tilde{u}_{\mathcal{I}+1} \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ that satisfies weakly the equation:

$$-\Delta \tilde{u}_{\mathcal{I}+1} - \frac{\gamma}{|x|^2} \tilde{u}_{\mathcal{I}+1} = \frac{|\tilde{u}_{\mathcal{I}+1}|^{2^\star(s)-2} \tilde{u}_{\mathcal{I}+1}}{|x|^s} \text{ in } \mathbb{R}^n_-.$$

while

$$\tilde{u}_{\mathcal{I}+1,\epsilon} \rightharpoonup \tilde{u}_{\mathcal{I}+1} \quad \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \quad \text{and} \quad \tilde{u}_{\mathcal{I}+1,\epsilon} \to \tilde{u}_{\mathcal{I}+1} \quad \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}),$$

as $\epsilon \to 0$.

We denote $\tilde{y}_\epsilon := \dfrac{\mathcal{T}^{-1}(y_\epsilon)}{k_{\mathcal{I}+1,\epsilon}} \in \mathbb{R}^n_-$. From (54) it follows that that $\lim_{\epsilon\to 0} |\tilde{y}_\epsilon| := |\tilde{y}_0| > a/2 \neq 0$. Therefore

$$|\tilde{u}_{\mathcal{I}+1}(\tilde{y}_0)| = \lim_{\epsilon\to 0} |\tilde{u}_{\mathcal{I}+1,\epsilon}(\tilde{y}_\epsilon)| = 1.$$

Since $\tilde{u}_{\mathcal{I}+1} \equiv 0$ on $\partial\mathbb{R}^n_- \setminus \{0\}$ so $\tilde{y}_\epsilon \notin \partial\mathbb{R}^n_-$ and hence $\tilde{u}_{\mathcal{I}+1} \not\equiv 0$. Hence again from Lemma 2, we get

$$\int_{\mathbb{R}^n_-} \frac{|\tilde{u}_{\mathcal{I}+1}|^{2^\star(s)}}{|x|^s} \geq \mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t_{\mathcal{I}+1} \in (0,1]$ such that $\lim_{\epsilon\to 0} \mu_{\mathcal{I}+1,\epsilon}^{p_\epsilon} = t_{\mathcal{I}+1}$. Moreover, it follows from (52) and (54) that

$$\lim_{\epsilon\to 0} \frac{\mu_{\mathcal{I}+1,\epsilon}}{\mu_{\mathcal{I},\epsilon}} = +\infty \text{ and } \lim_{\epsilon\to 0} \mu_{\mathcal{I}+1,\epsilon} = 0.$$

Hence the families $(\mu_{i,\epsilon})_{\epsilon>0}$, $1 \leq i \leq \mathcal{I}+1$ satisfy $\mathcal{H}_{\mathcal{I}+1}$. □

The next step is equivalent to step 4.3 at intermediate scales.

Step 4.4 Let $I \geq 1$. We assume that $(\mathcal{H}_{\mathcal{I}})$ holds. Then, for any $1 \leq i \leq \mathcal{I}-1$ and for any $\delta > 0$, either

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\Omega\cap B_{\delta k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} |x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0$$

or $(\mathcal{H}_{\mathcal{I}+1})$ holds.

Proof of Step 4.4: We assume that there exist an $i \leq \mathcal{I}-1$ and $\delta > 0$ such that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\Omega\cap B_{\delta k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} |x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} > 0.$$

It then follows that there exists a sequence $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{i,\epsilon}} = +\infty, \qquad |y_\epsilon| \leq \delta k_{i+1,\epsilon} \text{ for all } \epsilon > 0 \tag{55}$$

$$|y_\epsilon|^{\frac{n-2}{2}} \left| u_\epsilon(y_\epsilon) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(y_\epsilon)}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0, \tag{56}$$

for some positive constant a. Note that $a < +\infty$ since

$$|x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}}$$

is uniformly bounded for all $x \in \Omega \cap B_{\delta k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$.

Let $\tilde{y}^*_\epsilon \in \mathbb{R}^n_-$ be such that $\mathcal{T}^{-1}(y_\epsilon) = k_{i+1,\epsilon}\, \tilde{y}^*_\epsilon$. It follows that $|\tilde{y}^*_\epsilon| \leq \delta$ for all $\epsilon > 0$. We rewrite (56) as

$$\lim_{\epsilon\to 0} |\tilde{y}^*_\epsilon|^{\frac{n-2}{2}} \left|\tilde{u}_{i+1,\epsilon}(\tilde{y}^*_\epsilon) - \tilde{u}_{i+1}(\tilde{y}^*_\epsilon)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0.$$

Then from point (B3) of $\mathcal{H}_{\mathcal{I}}$ it follows that $\tilde{y}_\epsilon^* \to 0$ as $\epsilon \to 0$. Since $\frac{|x|^{\beta_-(\gamma)}}{|x_1|}\tilde{u}_{i+1} \in C^0(\mathbb{R}^n)$, we get as $\epsilon \to 0$

$$|y_\epsilon|^{\frac{n-2}{2}} \left|\mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{y_\epsilon}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = O\left(\frac{|y_\epsilon|}{k_{i+1,\epsilon}}\right)^{\frac{n}{2}-\beta_-(\gamma)} = o(1)$$

Then (56) becomes

$$\lim_{\epsilon\to 0} |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0. \tag{57}$$

In particular, $\lim_{\epsilon\to 0} |u_\epsilon(y_\epsilon)| = +\infty$. We let

$$\nu_\epsilon := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}}.$$

Then we have

$$\lim_{\epsilon\to 0} \nu_\epsilon = 0 \qquad \text{and} \qquad \lim_{\epsilon\to 0} \frac{|y_\epsilon|}{\ell_\epsilon} = a > 0. \tag{58}$$

We rescale and define

$$\tilde{u}_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} u_\epsilon(\mathcal{T}(\ell_\epsilon\, x)) \qquad \text{for } x \in \ell_\epsilon^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\}.$$

It follows from (49) that for all $\epsilon > 0$

$$|x|^{\frac{n-2}{2}} |\tilde{u}_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \le C \qquad \text{for } x \in \ell_\epsilon^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\},$$

so that hypothesis (40) of Lemma 2 is satisfied. We can then use it to get that there exists $\tilde{u} \in D^{1.2}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ that satisfies weakly the equation:

$$-\Delta \tilde{u} - \frac{\gamma}{|x|^2}\tilde{u} = \frac{|\tilde{u}|^{2^\star(s)-2}\tilde{u}}{|x|^s} \text{ in } \mathbb{R}^n_-,$$

while

$$\begin{aligned} \tilde{u}_\epsilon &\rightharpoonup \tilde{u} \qquad \text{weakly in } H^2_{1,0}(\mathbb{R}^n_-) \quad \text{as } \epsilon \to 0 \\ \tilde{u}_\epsilon &\to \tilde{u} \qquad \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) \qquad \text{as } \epsilon \to 0. \end{aligned}$$

We denote $\tilde{y}_\epsilon := \dfrac{\mathcal{T}^{-1}(y_\epsilon)}{\ell_\epsilon} \in \mathbb{R}^n_-$. From (57) it follows that that $\lim_{\epsilon\to 0} |\tilde{y}_\epsilon| := |\tilde{y}_0| > a/2 \neq 0$. Therefore

$$|\tilde{u}(\tilde{y}_0)| = \lim_{\epsilon\to 0} |\tilde{u}_\epsilon(\tilde{y}_\epsilon)| = 1.$$

Since $\tilde{u} \equiv 0$ on $\partial\mathbb{R}^n_- \setminus \{0\}$ so $\tilde{y}_\epsilon \notin \partial\mathbb{R}^n_-$ and hence $\tilde{u} \not\equiv 0$. Hence again from Lemma 2 we get

$$\int_{\mathbb{R}^n_-} \frac{|\tilde{u}|^{2^\star(s)}}{|x|^s} \ge \mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}},$$

and there exists $t \in (0, 1]$ such that $\lim_{\epsilon\to 0} \nu_\epsilon^{p_\epsilon} = t$. Moreover, from (57), (55), and since $\lim_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{i+1,\epsilon}} = 0$, it follows that

$$\lim_{\epsilon\to 0} \frac{\nu_\epsilon}{\mu_{i,\epsilon}} = +\infty \text{ and } \lim_{\epsilon\to 0} \frac{\mu_{i+1,\epsilon}}{\nu_\epsilon} = +\infty.$$

Hence the families $(\mu_{1,\epsilon})$,..., $(\mu_{i,\epsilon})$, (ν_ϵ), $(\mu_{i+1,\epsilon})$,..., $(\mu_{\mathcal{I},\epsilon})$ satisfy $(\mathcal{H}_{\mathcal{I}+1})$. $\square$

The last step tells us that the process of constructing $\{\mathcal{H}_{\mathcal{I}}\}$ stops after a finite number of steps.

Step 4.5: Let $N_0 = \max\{\mathcal{I} : (\mathcal{H}_{\mathcal{I}}) \text{ holds }\}$. Then $N_0 < +\infty$ and the conclusion of Proposition 2 holds with $N = N_0$.

Proof of Step 4.5: Indeed, assume that $(\mathcal{H}_{\mathcal{I}})$ holds. Since $\mu_{i,\epsilon} = o(\mu_{i+1,\epsilon})$ for all $1 \leq i \leq N-1$, we get with a change of variable and the definition of $\tilde{u}_{i,\epsilon}$ that for any $R > \delta > 0$

$$\int_{\Omega} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} dx \geq \sum_{i=1}^{\mathcal{I}} \int_{\mathcal{T}\left(B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\delta k_{i,\epsilon}}(0)\cap \mathbb{R}^n_-\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} dx$$
$$\geq \sum_{i=1}^{\mathcal{I}} \int_{B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\delta k_{i,\epsilon}}(0)\cap \mathbb{R}^n_-} \frac{|\tilde{u}_{i,\epsilon}|^{2^\star(s)-p_\epsilon}}{|x|^s} dv_{g_{i,\epsilon}}.$$

Here $g_{i,\epsilon}$ is the metric such that $(g_{\epsilon,i})_{qr} = (\partial_q \mathcal{T}(k_{i,\epsilon}x), \partial_r \mathcal{T}(k_{i,\epsilon}x))$ for all $q, r \in \{1, ..., n\}$. Then from (22) we have

$$\Lambda \geq \sum_{i=1}^{\mathcal{I}} \int_{B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\delta k_{i,\epsilon}}(0)\cap \mathbb{R}^n_-} \frac{|\tilde{u}_{i,\epsilon}|^{2^\star(s)-p_\epsilon}}{|x|^s} dv_{g_{i,\epsilon}}. \tag{59}$$

Passing to the limit $\epsilon \to 0$ and then $\delta \to 0$, $R \to +\infty$ we obtain using point (B3) of $\mathcal{H}_{\mathcal{I}}$, that

$$\Lambda \geq \mathcal{I}\mu_{\gamma,s}(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}},$$

from which it follows that $N_0 < +\infty$. □

To complete the proof, we let families $(\mu_{1,\epsilon})_{\epsilon>0}$,..., $(\mu_{N_0,\epsilon})_{\epsilon>0}$ be such that $\mathcal{H}_{N_0}$ holds. We argue by contradiction and assume that the conclusion of Proposition 2 does not hold with $N = N_0$. Assertions (A1), (A2), (A3),(A4), (A5) , (A7) and (A9) hold. Assume that (A6) or (A8) does not hold. It then follows from Steps 4.3, 4.4 and 4.5 that $\mathcal{H}_{N+1}$ holds. A contradiction with the choice of $N = N_0$. Hence the proposition is proved. □

5. Strong pointwise estimates

The objective of this section is to obtain pointwise controls on u_ϵ and ∇u_ϵ. The core is the proof of the following proposition in the spirit of Druet-Hebey-Robert [11]:

PROPOSITION 3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$, $\gamma < \frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0} \||x|^\tau u_\epsilon\|_{L^\infty(\Omega)} = +\infty \quad \textit{where} \quad \beta_-(\gamma) - 1 < \tau < \frac{n-2}{2}.$$

Consider $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ from Proposition 2. Then, there exists $C > 0$ such that for all $\epsilon > 0$

$$|u_\epsilon(x)| \leq \; C \left(\sum_{i=1}^{N} \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |x|}{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} |x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} + \frac{\| |x|^{\beta_-(\gamma)-1} u_0 \|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}} |x| \right). \tag{60}$$

for all $x \in \Omega$.

The proof of this estimate, inspired by the methodology of [11], proceeds in seven steps.

Step 5.1: We claim that for any $\alpha > 0$ small and any $R > 0$, there exists $C(\alpha, R) > 0$ such that for all $\epsilon > 0$ sufficiently small, we have for all $x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0)$,

$$|u_\epsilon(x)| \leq C(\alpha, R) \left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha} |x|}{|x|^{\beta_+(\gamma)-\alpha}} + \frac{\| |x|^{\beta_-(\gamma)-1} u_0 \|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)+\alpha}} |x| \right). \tag{61}$$

Proof of Step 5.1: We fix γ' such that $\gamma < \gamma' < \frac{n^2}{4}$. Since the operator $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$ is coercive, taking γ' close to γ it follows that the operator $-\Delta - \frac{\gamma'}{|x|^2} - h_0$ is also coercive in Ω. From Theorem 7, there exists $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that

$$\begin{cases} -\Delta H - \frac{\gamma'}{|x|^2} H - h_0(x) H = 0 & \text{in } \Omega \\ H > 0 & \text{in } \Omega \\ H = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{62}$$

And we have the following bound on H, that there exists $C_1 > 0$ such that

$$\frac{1}{C_1} \frac{d(x, \partial\Omega)}{|x|^{\beta_+(\gamma')}} \leq H(x) \leq C_1 \frac{d(x, \partial\Omega)}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in \Omega. \tag{63}$$

We let $\lambda_1^{\gamma'} > 0$ be the first eigenvalue of the coercive operator $-\Delta - \frac{\gamma'}{|x|^2} - h_0$ on Ω and we let $\varphi \in C^2(\overline{\Omega} \setminus \{0\}) \cap H^2_{1,0}(\Omega)$ be the unique eigenfunction such that

$$\begin{cases} -\Delta \varphi - \frac{\gamma'}{|x|^2} \varphi - h_0(x) \varphi &= \lambda_1^{\gamma'} \varphi & \text{in } \Omega \\ \varphi &> 0 & \text{in } \Omega \\ \varphi &= 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{64}$$

It follows from the regularity result, Theorem 6 that there exists $C_2 > 0$ such that

$$\frac{1}{C_2} \frac{d(x, \partial\Omega)}{|x|^{\beta_-(\gamma')}} \leq \varphi(x) \leq C_2 \frac{d(x, \partial\Omega)}{|x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \Omega. \tag{65}$$

We define the operator

$$\mathcal{L}_\epsilon := -\Delta - \left(\frac{\gamma}{|x|^2} + h_\epsilon \right) - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}. \tag{66}$$

Step 5.1.1: We claim that given any $\gamma < \gamma' < \frac{n^2}{4}$ there exist $\delta_0 > 0$ and $R_0 > 0$ such that for any $0 < \delta < \delta_0$ and $R > R_0$, we have for $\epsilon > 0$ sufficiently small

$$\begin{aligned} \mathcal{L}_\epsilon H(x) > 0 \text{ and } \mathcal{L}_\epsilon \varphi(x) &> 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega. \\ \mathcal{L}_\epsilon H(x) &> 0 \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \equiv 0. \end{aligned} \tag{67}$$

We prove the claim. As one checks for all $\epsilon > 0$ and $x \in \Omega$

$$\frac{\mathcal{L}_\epsilon H(x)}{H(x)} = \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}.$$

and

$$\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} = \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s} + \lambda_1^{\gamma'}.$$

One has for $\epsilon > 0$ sufficiently small $\|h_0 - h_\epsilon\|_\infty \leq \frac{\gamma'-\gamma}{4(1+\sup\limits_{\Omega}|x|^2)}$ and we choose $0 < \delta_0 < 1$ such that

$$\delta_0^{(2^\star(s)-2)\left(\frac{n}{2}-\beta_-(\gamma)\right)} \| |x|^{\beta_-(\gamma)-1} u_0 ||_{L^\infty(\Omega)}^{2^\star(s)-2} \leq \frac{\gamma' - \gamma}{2^{2^\star(s)+3}}. \tag{68}$$

This choice is possible thanks to (15) and the regularity Theorem 6 respectively. It follows from point (A6) of Proposition 2 that, there exists $R_0 > 0$ such that for any $R > R_0$, we have for all $\epsilon > 0$ sufficiently small

$$|x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma' - \gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}} \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0)$$

With this choice of δ_0 and R_0 we get that for any $0 < \delta < \delta_0$ and $R > R_0$, we have for $\epsilon > 0$ small enough

$$\begin{aligned}
|x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}|u_\epsilon(x) - u_0(x)|^{2^\star(s)-2-p_\epsilon} \\
&\quad + 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}|u_0(x)|^{2^\star(s)-2-p_\epsilon} \\
&\leq 2^{-p_\epsilon}\frac{\gamma' - \gamma}{4} \leq \frac{\gamma' - \gamma}{4}
\end{aligned}$$

for all $x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega$, if $u_0 \not\equiv 0$, and

$$|x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} \leq \frac{\gamma' - \gamma}{4} \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \equiv 0.$$

Hence we obtain that for $\epsilon > 0$ small enough

$$\begin{aligned}
\frac{\mathcal{L}_\epsilon H(x)}{H(x)} &= \frac{\gamma' - \gamma}{|x|^2} + h_0 - h_\epsilon - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s} \\
&\geq \frac{\gamma' - \gamma}{|x|^2} + h_0 - h_\epsilon - \frac{\gamma' - \gamma}{4|x|^2} \\
&\geq \frac{\gamma' - \gamma}{|x|^2} - \frac{\gamma' - \gamma}{4|x|^2} - \frac{\gamma' - \gamma}{4|x|^2} = \frac{\gamma' - \gamma}{2|x|^2} \\
&> 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega \text{ if } u_0 \not\equiv 0 \\
\text{and} \quad \frac{\mathcal{L}_\epsilon H(x)}{H(x)} &> 0 \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \equiv 0.
\end{aligned}$$

Similarly we have

$$\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} > 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega.$$

□

Step 5.1.2: It follows from point (A4) of Proposition 2 that there exists $C_1'(R) > 0$ such that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_1'(R) \frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} d(x,\partial\Omega)}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in \Omega \cap \partial B_{Rk_{N,\epsilon}}(0).$$

By estimate (63) on H, we then have for some constant $C_1(R) > 0$

$$|u_\epsilon(x)| \leq C_1(R) \mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) \qquad \text{for all } x \in \Omega \cap \partial B_{Rk_{N,\epsilon}}(0). \tag{69}$$

It follows from point (A1) of Proposition 2 and the regularity Theorem 6, that there exists $C_2'(\delta) > 0$ such that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_2'(\delta) \| |x|^{\beta_-(\gamma)-1} u_0 \|_{L^\infty(\Omega)} \frac{d(x,\partial\Omega)}{|x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \Omega \cap \partial B_\delta(0), \text{ if } u_0 \not\equiv 0. \tag{70}$$

And then by the estimate (65) on φ we have for some constant $C_2(\delta) > 0$

$$|u_\epsilon(x)| \leq C_2(\delta) \| |x|^{\beta_-(\gamma)-1} u_0 \|_{L^\infty(\Omega)} \; \varphi(x) \qquad \text{for all } x \in \Omega \cap \partial B_\delta(0), \text{ if } u_0 \not\equiv 0. \tag{71}$$

We now let for all $\epsilon > 0$

$$\Psi_\epsilon(x) := C_1(R) \mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\delta) \| |x|^{\beta_-(\gamma)-1} u_0 \|_{L^\infty(\Omega)} \; \varphi(x) \;\; \text{for } x \in \Omega.$$

Then (70) and (69) imply that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq \Psi_\epsilon(x) \qquad \text{for all } x \in \partial\left(B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega\right), \text{ if } u_0 \not\equiv 0 \tag{72}$$

and

$$|u_\epsilon(x)| \leq \Psi_\epsilon(x) \qquad \text{for all } x \in \partial(\Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0)), \text{ if } u_0 \equiv 0. \tag{73}$$

Therefore when $u_0 \not\equiv 0$ it follows from (67) and (72) that for all $\epsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\epsilon \Psi_\epsilon \geq 0 = \mathcal{L}_\epsilon u_\epsilon & \text{in } B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega \\ \Psi_\epsilon \geq u_\epsilon & \text{on } \partial\left(B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega\right) \\ \mathcal{L}_\epsilon \Psi_\epsilon \geq 0 = -\mathcal{L}_\epsilon u_\epsilon & \text{in } B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega \\ \Psi_\epsilon \geq -u_\epsilon & \text{on } \partial\left(B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega\right). \end{cases}$$

and from (67) and (73), in case $u_0 \equiv 0$, we have for $\epsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\epsilon \Psi_\epsilon \geq 0 = \mathcal{L}_\epsilon u_\epsilon & \text{in } \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \\ \Psi_\epsilon \geq u_\epsilon & \text{on } \partial(\Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0)) \\ \mathcal{L}_\epsilon \Psi_\epsilon \geq 0 = -\mathcal{L}_\epsilon u_\epsilon & \text{in } \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \\ \Psi_\epsilon \geq -u_\epsilon & \text{on } \partial(\Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0)). \end{cases}$$

Since $\Psi_\epsilon > 0$ and $\mathcal{L}_\epsilon \Psi_\epsilon > 0$, it follows from the comparison principle of Berestycki-Nirenberg-Varadhan [4] that the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle on $B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega$. Therefore

$$|u_\epsilon(x)| \leq \Psi_\epsilon(x) \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega,$$
$$\text{and } \;\; |u_\epsilon(x)| \leq \Psi_\epsilon(x) \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \;\; \text{if } u_0 \equiv 0.$$

Therefore when $u_0 \not\equiv 0$ we have for for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_1(R)\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\delta)\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}\ \varphi(x)$$

for all $x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega$, for R large and δ small.

Then, when $u_0 \not\equiv 0$, using the estimates (63) and (65), we have or all $\epsilon > 0$ small

$$\begin{aligned} |u_\epsilon(x)| &\leq C_1(R)\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} d(x,\partial\Omega)}{|x|^{\beta_+(\gamma')}} + C_2(\delta)\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}\frac{d(x,\partial\Omega)}{|x|^{\beta_-(\gamma')}} \\ &\leq C_1(R)\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|}{|x|^{\beta_+(\gamma')}} + C_2(\delta)\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma')}}|x|. \end{aligned}$$

for all $x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0) \cap \Omega$, for R large and δ small.

Similarly if $u_0 \equiv 0$, then all $\epsilon > 0$ small and $R > 0$ large

$$|u_\epsilon(x)| \leq C_1(R)\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|}{|x|^{\beta_+(\gamma')}} \quad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0).$$

Taking γ' close to γ, along with points (A1) and (A4) of Proposition 2, it then follows that estimate (61) holds on $\Omega \setminus \overline{B}_{Rk_{\epsilon,N}}(0)$ for all $R > 0$. □

Step 5.2: Let $1 \leq i \leq N-1$. We claim that for any $\alpha > 0$ small and any $R, \rho > 0$, there exists $C(\alpha, R, \rho) > 0$ such that all $\epsilon > 0$.

$$|u_\epsilon(x)| \leq C(\alpha,R,\rho)\left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}|x|}{|x|^{\beta_+(\gamma)-\alpha}} + \frac{|x|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}|x|^{\beta_-(\gamma)+\alpha}}\right) \tag{74}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega$.

Proof of Step 5.2: We let $i \in \{1, ..., N-1\}$. We emulate the proof of Step 5.1. Fix γ' such that $\gamma < \gamma' < \frac{n^2}{4}$. Consider the functions H and φ defined in Step 5.1 satisfying (62) and (62) respectively.

Step 5.2.1: We claim that given any $\gamma < \gamma' < \frac{n^2}{4}$ there exist $\rho_0 > 0$ and $R_0 > 0$ such that for any $0 < \rho < \rho_0$ and $R > R_0$, we have for $\epsilon > 0$ sufficiently small

$$\mathcal{L}_\epsilon H(x) > 0 \text{ and } \mathcal{L}_\epsilon\varphi(x) > 0 \qquad \text{for all } x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega \tag{75}$$

where $\mathcal{L}\epsilon$ is as in (66).

We prove the claim. As one checks for all $\epsilon > 0$ and $x \in \Omega$

$$\begin{aligned} \frac{\mathcal{L}_\epsilon H(x)}{H(x)} &= \frac{\gamma'-\gamma}{|x|^2} + h_0 - h_\epsilon - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}, \\ \frac{\mathcal{L}_\epsilon\varphi(x)}{\varphi(x)} &\geq \frac{\gamma'-\gamma}{|x|^2} + h_0 - h_\epsilon - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}. \end{aligned}$$

We choose $0 < \rho_0 < 1$ such that

$$\rho_0^2 \sup_\Omega |h_0 - h_\epsilon| \leq \frac{\gamma' - \gamma}{4} \qquad \text{for all } \epsilon > 0 \text{ small and}$$
$$\rho_0^{(2^\star(s)-2)\left(\frac{n}{2}-\beta_-(\gamma)\right)} \| |x|^{\beta_-(\gamma)-1} \tilde{u}_{i+1} \|_{L^\infty(B_2(0)\cap\mathbb{R}^n_-)}^{2^\star(s)-2} \leq \frac{\gamma'-\gamma}{2^{2^\star(s)+3}} \tag{76}$$

It follows from point (A8) of Proposition 2 that there exists $R_0 > 0$ such that for any $R > R_0$ and any $0 < \rho < \rho_0$, we have for all $\epsilon > 0$ sufficiently small

$$|x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right) \right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma'-\gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega$.

With this choice of ρ_0 and R_0 we get that for any $0 < \rho < \rho_0$ and $R > R_0$, we have for $\epsilon > 0$ small enough

$$\begin{aligned} |x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}\left| u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right)\right|^{2^\star(s)-2-p_\epsilon} \\ &\quad + 2^{2^\star(s)-1-p_\epsilon}\left(\frac{|x|}{k_{i+1,\epsilon}}\right)^{2-s}\left|\tilde{u}_{i+1}\left(\frac{\mathcal{T}^{-1}(x)}{k_{i+1,\epsilon}}\right)\right|^{2^\star(s)-2-p_\epsilon} \\ &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0). \end{aligned}$$

Hence as in Step 5.1 we have that for $\epsilon > 0$ small enough

$$\frac{\mathcal{L}_\epsilon H(x)}{H(x)} > 0 \text{ and } \frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} > 0 \text{ for all } x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega.$$

Step 5.2.2: Let $i \in \{1, ..., N-1\}$. It follows from point (A4) of Proposition 2 that there exists $C_1'(R) > 0$ such that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_1'(R) \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} d(x,\partial\Omega)}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in \Omega \cap \partial B_{Rk_{i,\epsilon}}(0),$$

And then by the estimate (63) on H we have for some constant $C_1(R) > 0$

$$|u_\epsilon(x)| \leq C_1(R) \mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) \qquad \text{for all } x \in \Omega \cap \partial B_{Rk_{i,\epsilon}}(0). \tag{77}$$

Again from point (A4) of Proposition 2 it follows that there exists $C_2'(\rho) > 0$ such that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_2'(\rho) \frac{d(x,\partial\Omega)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \Omega \cap \partial B_{\rho k_{i+1,\epsilon}}(0),$$

and then by the estimate (65) on φ we have for some constant $C_2(\delta) > 0$

$$|u_\epsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for all } x \in \Omega \cap \partial B_{\rho k_{i+1,\epsilon}}(0). \tag{78}$$

We let for all $\epsilon > 0$

$$\tilde{\Psi}_\epsilon(x) := C_1(R)\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\rho)\frac{\varphi(x)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for } x \in \Omega.$$

Then (77) and (78) implies that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq \tilde{\Psi}_\epsilon(x) \qquad \text{for all } x \in \partial\left(B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega\right). \tag{79}$$

Therefore it follows from (75) and (79) that $\epsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\epsilon \tilde{\Psi}_\epsilon \geq 0 = \mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega \\ \tilde{\Psi}_\epsilon \geq u_\epsilon & \text{on } \partial\left(B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega\right) \\ \mathcal{L}_\epsilon \tilde{\Psi}_\epsilon \geq 0 = -\mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega \\ \tilde{\Psi}_\epsilon \geq -u_\epsilon & \text{on } \partial\left(B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega\right). \end{cases}$$

Since $\tilde{\Psi}_\epsilon > 0$ and $\mathcal{L}_\epsilon \tilde{\Psi}_\epsilon > 0$, it follows from the comparison principle of Berestycki-Nirenberg-Varadhan [4] that the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle on $B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega$. Therefore

$$|u_\epsilon(x)| \leq \tilde{\Psi}_\epsilon(x) \qquad \text{for all } x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega.$$

So for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_1(R)\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\rho)\frac{\varphi(x)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega$, for R large and ρ small. Then using the estimates (63) and (65) we have or all $\epsilon > 0$ small

$$\begin{aligned} |u_\epsilon(x)| &\leq C_1(R)\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} d\left(x, \partial\Omega\right)}{|x|^{\beta_+(\gamma')}} + C_2(\rho)\frac{d(x,\partial\Omega)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|^{\beta_-(\gamma')}} \\ &\leq C_1(R)\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|}{|x|^{\beta_+(\gamma')}} + C_2(\rho)\frac{|x|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|^{\beta_-(\gamma')}}. \end{aligned}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega$, for R large and ρ small.

Taking γ' close to γ, along with point (A4) of Proposition 2 it then follows that estimate (74) holds on $B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \cap \Omega$, for all $R\rho > 0$. □

Step 5.3: We claim that for any $\alpha > 0$ small and any $\rho > 0$, there exists $C(\alpha,\rho) > 0$ such that all $\epsilon > 0$.

$$|u_\epsilon(x)| \leq C(\alpha,\rho)\frac{|x|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}|x|^{\beta_-(\gamma)+\alpha}} \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega. \tag{80}$$

Proof of Step 5.3: Fix γ' such that $\gamma < \gamma' < \frac{n^2}{4}$. Consider the function φ defined in Step 5.1 satisfying (62).

Step 5.3.1: We claim that given any $\gamma < \gamma' < \frac{n^2}{4}$ there exist $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$ we have for $\epsilon > 0$ sufficiently small

$$\mathcal{L}_\epsilon \varphi(x) > 0 \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega, \tag{81}$$

where $\mathcal{L}\epsilon$ is as in (66).

Indeed, for all $\epsilon > 0$ and $x \in \Omega$

$$\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} \geq \frac{\gamma' - \gamma}{|x|^2} - h_\epsilon - \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}.$$

We choose $0 < \rho_0 < 1$ such that

$$\rho_0^2 \sup_\Omega |h_\epsilon| \leq \frac{\gamma' - \gamma}{4} \qquad \text{for all } \epsilon > 0 \text{ small and}$$
$$\rho_0^{(2^\star(s)-2)\left(\frac{n}{2}-\beta_-(\gamma)\right)} \| |x|^{\beta_-(\gamma)-1} \tilde{u}_1 \|_{L^\infty(B_2(0)\cap\mathbb{R}^n_-)}^{2^\star(s)-2} \leq \frac{\gamma' - \gamma}{2^{2^\star(s)+3}}$$

It follows from point (A7) of Proposition 2 that for any $0 < \rho < \rho_0$, we have for all $\epsilon > 0$ sufficiently small

$$|x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_1 \left(\frac{\mathcal{T}^{-1}(x)}{k_{1,\epsilon}} \right) \right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma' - \gamma}{2^{2^\star(s)+2}} \right)^{\frac{1}{2^\star(s)-2}}$$

for all $x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega$.

With this choice of ρ_0 we get that for any $0 < \rho < \rho_0$ we have for $\epsilon > 0$ small enough

$$\begin{aligned} |x|^{2-s} |u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq 2^{2^\star(s)-1-p_\epsilon} |x|^{2-s} \left| u_\epsilon(x) - \mu_{1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_1 \left(\frac{\mathcal{T}^{-1}(x)}{k_{1,\epsilon}} \right) \right|^{2^\star(s)-2-p_\epsilon} \\ &\quad + 2^{2^\star(s)-1-p_\epsilon} \left(\frac{|x|}{k_{1,\epsilon}} \right)^{2-s} \left| \tilde{u}_1 \left(\frac{\mathcal{T}^{-1}(x)}{k_{1,\epsilon}} \right) \right|^{2^\star(s)-2-p_\epsilon} \\ &\leq \frac{\gamma' - \gamma}{4} \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega. \end{aligned}$$

Hence as in Step 5.1 we have that for $\epsilon > 0$ small enough

$$\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} > 0 \quad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega.$$

□

Step 5.3.2: It follows from point (A4) of Proposition 2 that there exists $C_2'(\rho) > 0$ such that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_2'(\rho) \frac{d(x, \partial\Omega)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \partial B_{\rho k_{1,\epsilon}}(0) \cap \Omega$$

and then by the estimate (65) on φ we have for some constant $C_2(\delta) > 0$

$$|u_\epsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for all } x \in \partial B_{\rho k_{1,\epsilon}}(0) \cap \Omega. \tag{82}$$

We let for all $\epsilon > 0$

$$\Psi_\epsilon^0(x) := C_2(\rho) \frac{\varphi(x)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for } x \in \Omega.$$

Then (82) implies that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq \Psi_\epsilon^0(x) \qquad \text{for all } x \in \partial \left(B_{\rho k_{1,\epsilon}}(0) \cap \Omega \setminus \{0\} \right). \tag{83}$$

Therefore it follows from (81) and (83) that $\epsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\epsilon \Psi_\epsilon^0 \geq 0 = \mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{1,\epsilon}}(0) \cap \Omega \\ \Psi_\epsilon^0 \geq u_\epsilon & \text{on } \partial\left(B_{\rho k_{1,\epsilon}}(0) \cap \Omega \setminus \{0\}\right) \\ \mathcal{L}_\epsilon \Psi_\epsilon^0 \geq 0 = -\mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{1,\epsilon}}(0) \cap \Omega \\ \Psi_\epsilon^0 \geq -u_\epsilon & \text{on } \partial\left(B_{\rho k_{1,\epsilon}}(0) \cap \Omega \setminus \{0\}\right). \end{cases}$$

Since the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle on $B_{\rho k_{1,\epsilon}}(0)$. Therefore

$$|u_\epsilon(x)| \leq \Psi_\epsilon^0(x) \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega.$$

And so for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \quad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega.$$

for ρ small. Using the estimate (65) we have or all $\epsilon > 0$ small

$$\begin{aligned} |u_\epsilon(x)| &\leq C_2(\rho) \frac{d(x, \partial\Omega)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \\ &\leq C_2(\rho) \frac{|x|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}}. \end{aligned}$$

for ρ small. It then follows from point (A4) of Proposition 2 that estimate (80) holds on $x \in B_{\rho k_{1,\epsilon}}(0) \cap \Omega$ for all $\rho > 0$. □

Step 5.4: Combining the previous three steps, it follows from (61), (74), (80) and Proposition 2 that for any $\alpha > 0$ small, there exists $C(\alpha) > 0$ such that for all $\epsilon > 0$ we have for all $x \in \Omega$,

$$\begin{aligned} |u_\epsilon(x)| \quad &\leq \quad C(\alpha) \left(\sum_{i=1}^{N} \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha} |x|}{\mu_{i,\epsilon}^{(\beta_+(\gamma)-\beta_-(\gamma))-2\alpha} |x|^{\beta_-(\gamma)+\alpha} + |x|^{\beta_+(\gamma)-\alpha}} \right. \\ &\left. + \frac{\||x|^{\beta_-(\gamma)-1} u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)+\alpha}} |x| \right). \end{aligned} \tag{84}$$

Next we improve the above estimate and show that one can take $\alpha = 0$ in (84).

We let G_ϵ be the Green's function for the coercive operator $-\Delta - \frac{\gamma}{|x|^2} - h_\epsilon$ on Ω with Dirichlet boundary condition. Green's representation formula, the pointwise bounds on the Green's function (207) and the regularity Theorem 6, yields for any $z \in \Omega$,

$$u_\epsilon(z) = \int_\Omega G_\epsilon(z, x) \left(\frac{|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} u_\epsilon(x)}{|x|^s} \right) dx,$$

and therefore,

$$|u_\epsilon(z)| \leq \int_\Omega G_\epsilon(z,x)\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx$$
$$\leq C\int_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)d(z,\partial\Omega)}{|x-z|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx. \quad (85)$$

To estimate the above integral we break it into three parts.

Step 5.5: There exist $C>0$ such that for any sequence (z_ϵ) with $z_\epsilon \in \Omega\setminus\overline{B}_{k_{N,\epsilon}}(0)$, we have

$$\int_\Omega G_\epsilon(z_\epsilon,x)\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx \leq C\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\right). \quad (86)$$

Proof of Step 5.5: To estimate the right-hand-side of (85) in this case, we split Ω into four subdomains as: $\Omega=\bigcup_{i=1}^{4} D_{i,\epsilon}^N$ where

- $D_{1,\epsilon}^N := B_{k_{N,\epsilon}}(0)\cap\Omega,\ D_{2,\epsilon}^N := \{k_{N,\epsilon}<|x|<\frac{1}{2}|z_\epsilon|\}\cap\Omega,$
- $D_{3,\epsilon}^N := \{\frac{1}{2}|z_\epsilon|<|x|<2|z_\epsilon|\}\cap\Omega,\ D_{4,\epsilon}^N := \{2|z_\epsilon|<|x|\}\cap\Omega.$

Note that one has $\frac{1}{2}|z_\epsilon|<|x-z_\epsilon|$ in $D_{2,\epsilon}^N$ and $\frac{1}{2}|x|<|x-z_\epsilon|$ in $D_{4,\epsilon}^N$.
Using point (A5) of Proposition 2 and a change of variable, we get

$$I_1^N := C\int_{D_{1,\epsilon}^N}\left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx$$
$$\leq C\,d(z_\epsilon,\partial\Omega)\int_{D_{1,\epsilon}^N}\frac{|z_\epsilon|^{\beta_-(\gamma)}}{|x|^{\beta_-(\gamma)-1}}|x-z_\epsilon|^{-n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx$$
$$\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int_{D_{1,\epsilon}^N}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)-1+s}}\,dx$$
$$\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int_{B_{k_{N,\epsilon}}(0)}\frac{1}{|x|^{\beta_-(\gamma)-1+s+(2^\star(s)-1-p_\epsilon)\frac{n-2}{2}}}\,dx$$
$$\leq C\,\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int_{B_1(0)}\frac{1}{|x|^{n-\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-p_\epsilon\frac{n-2}{2}}}\,dx$$
$$\leq C\,\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}. \quad (87)$$

Now we estimate

$$
\begin{aligned}
I_2^N &:= C \int\limits_{D_{2,\epsilon}^N} \left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx\\
&\leq C\ d(z_\epsilon,\partial\Omega)\frac{|z_\epsilon|^{\beta_-(\gamma)}}{|z_\epsilon|^n}\int\limits_{D_{2,\epsilon}^N}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)-1+s}}\,dx.
\end{aligned}
$$

Using (61) we get for $0<\alpha<\frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$
\begin{aligned}
I_2^N &\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{D_{2,\epsilon}^N}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)-1+s}}\,dx\\
&\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\mu_{N,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}\int\limits_{D_{2,\epsilon}^N}\frac{|x|^{-\beta_-(\gamma)+1-s}}{|x|^{(2^\star(s)-1-p_\epsilon)(\beta_+(\gamma)-1-\alpha)}}\,dx\\
&\quad + C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{D_{2,\epsilon}^N}\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|x|^{(2^\star(s)-1-p_\epsilon)(\beta_-(\gamma)-1+\alpha)+\beta_-(\gamma)-1+s}}\,dx\\
&\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}\int\limits_{1\leq|x|}\frac{1}{|x|^{n+(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}}\,dx\\
&\quad + C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{|x|\leq\frac{1}{2}|z_\epsilon|}\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|x|^{(2^\star(s)-p_\epsilon)(\beta_-(\gamma)-1)+s+\alpha(2^\star(s)-1-p_\epsilon)}}\,dx\\
&\leq C\ \frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{1\leq|x|}\frac{1}{|x|^{n+(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}}\,dx\\
&\quad + C\ \frac{|z_\epsilon|^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\\
&\leq C\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\right). \qquad (88)
\end{aligned}
$$

For the next integral

$$
\begin{aligned}
I_3^N &:= C \int\limits_{D_{3,\epsilon}^N} \left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx\\
&\leq C\ d(z_\epsilon,\partial\Omega)\int\limits_{D_{3,\epsilon}^N}\frac{|x|}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx.
\end{aligned}
$$

From (61) we get for $0 < \alpha < \frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$
\begin{aligned}
I_3^N \leq\; & C\, d(z_\epsilon,\partial\Omega)\mu_{N,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)} \int\limits_{D_{3,\epsilon}^N} \frac{|x|^{1-s}|x-z_\epsilon|^{-n}}{|x|^{(\beta_+(\gamma)-1-\alpha)(2^\star(s)-1-p_\epsilon)}}\, dx \\
& + C d(z_\epsilon,\partial\Omega) \int\limits_{D_{3,\epsilon}^N} \frac{|x|^{1-s}|x-z_\epsilon|^{-n}}{|x|^{(\beta_-(\gamma)-1+\alpha)(2^\star(s)-1-p_\epsilon)}} \||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}\, dx \\
\leq\; & C\, \frac{\mu_{N,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{(\beta_+(\gamma)-1-\alpha)(2^\star(s)-1-p_\epsilon)+s-1}} \int\limits_{D_{3,\epsilon}^N} |x-z_\epsilon|^{-n}\, dx \\
& + C \frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{(\beta_-(\gamma)-1+\alpha)(2^\star(s)-1-p_\epsilon)+s-1}} \int\limits_{D_{3,\epsilon}^N} |x-z_\epsilon|^{-n}\, dx \\
\leq\; & C\, \frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \left(\frac{\mu_{N,\epsilon}}{|z_\epsilon|}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)} \\
& + C\, \frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} |z_\epsilon|^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)} \\
\leq\; & C\, \left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|z_\epsilon|^{\beta_-(\gamma)-1}}|z_\epsilon|\right). \qquad (89)
\end{aligned}
$$

Finally we estimate

$$
\begin{aligned}
I_4^N :=\; & C \int\limits_{D_{4,\epsilon}^N} \left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \\
\leq\; & C\, \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} \int\limits_{2|z_\epsilon|\leq|x|} |x|^{\beta_-(\gamma)+1-n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \\
\leq\; & C\, \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} \int\limits_{2|z_\epsilon|\leq|x|} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_+(\gamma)+s-1}}\, dx.
\end{aligned}
$$

Then from (61) we get for $0<\alpha<\frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$
\begin{aligned}
I_4^N \leq\; & C\,\frac{\mu_{N,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{2|z_\epsilon|\leq|x|}\frac{|x|^{\alpha(2^\star(s)-1-p_\epsilon)}}{|x|^{(2^\star(s)-p_\epsilon)(\beta_+(\gamma)-1)+s}}\,dx\\
&+C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{2|z_\epsilon|\leq|x|}\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_+(\gamma)+s+\beta_-(\gamma)(2^\star(s)-1-p_\epsilon)+\alpha(2^\star(s)-1-p_\epsilon)}}\,dx\\
\leq\; & C\,\frac{\mu_{N,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{2|z_\epsilon|\leq|x|}\frac{|x|^{\alpha(2^\star(s)-1-p_\epsilon)}}{|x|^{n+(2^\star(s)-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)}}\,dx\\
&+C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{2|z_\epsilon|\leq|x|}\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|x|^{n-\left[(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)\right]}}\,dx\\
\leq\; & C\,\frac{\mu_{N,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}}{|z_\epsilon|^{\beta_-(\gamma)}}\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{(2^\star(s)-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}}\\
&+C\,\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\\
\leq\; & C\,\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\left(\frac{\mu_{N,\epsilon}}{|z_\epsilon|}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}\\
&+C\,\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\\
\leq\; & C\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\right).
\end{aligned}\tag{90}
$$

Combining (87), (88), (89) and (90), we then obtain for some constant $C>0$

$$
\begin{aligned}
\int\limits_{\Omega}G_\epsilon(z_\epsilon,x)\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx \quad\leq\quad & C\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}\right.\\
&\left.+\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\right),
\end{aligned}
$$

which we write as

$$
\begin{aligned}
&\int\limits_{\Omega}G_\epsilon(z_\epsilon,x)\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx\\
&\leq\; C\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\right)
\end{aligned}
$$

for some $C > 0$. This proves (86). $\square$

Step 5.6: There exists $C > 0$ such that for sequence of points (z_ϵ) in $B_{k_{1,\epsilon}}(0) \cap \Omega$ we have

$$\int_\Omega G_\epsilon(z_\epsilon, x) \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \leq C\, \frac{|z_\epsilon|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |z_\epsilon|^{\beta_-(\gamma)}}. \tag{91}$$

Proof of Step 5.6: Here again, to estimate the right-hand-side of (85) in this case, we split Ω into four subdomains as: $\Omega = \bigcup_{i=1}^{4} D^1_{i,\epsilon}(R)$ where

- $D^1_{1,\epsilon} := \{|x| < \frac{1}{2}|z_\epsilon|\} \cap \Omega$,
- $D^1_{2,\epsilon} := \{\frac{1}{2}|z_\epsilon| < |x| < 2|z_\epsilon|\} \cap \Omega$,
- $D^1_{3,\epsilon} := \{2|z_\epsilon| < |x| \leq k_{1,\epsilon}\} \cap \Omega$,
- $D^1_{4,\epsilon} := \{k_{1,\epsilon} < |x|\} \cap \Omega$.

Note that one has $\frac{1}{2}|z_\epsilon| < |x - z_\epsilon|$ in $D^1_{1,\epsilon}$ and $\frac{1}{2}|x| < |x - z_\epsilon|$ in $D^1_{3,\epsilon}$. We then have

$$\begin{aligned} I_1^1 &:= C \int_{D^1_{1,\epsilon}} \left(\frac{\max\{|z_\epsilon|, |x|\}}{\min\{|z_\epsilon|, |x|\}} \right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega) d(z_\epsilon,\partial\Omega)}{|x - z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \\ &\leq C d(z_\epsilon, \partial\Omega) \frac{|z_\epsilon|^{\beta_-(\gamma)}}{|z_\epsilon|^{n-2}} \int_{D^1_{1,\epsilon}} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)+s-1}}\, dx. \end{aligned}$$

Using (80) we get for $0 < \alpha < \frac{2^\star(s)-2}{2^\star(s)-1} \left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} \right)$

$$\begin{aligned} I_1^1 &\leq C\, \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \int_{D^1_{1,\epsilon}} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)+s-1}}\, dx \\ &\leq C\, \frac{\mu_{1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} - \alpha\right)(2^\star(s)-1-p_\epsilon)} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \int_{D^1_{1,\epsilon}} \frac{|x|^{-\beta_-(\gamma)-s+1}}{|x|^{(2^\star(s)-1-p_\epsilon)(\beta_-(\gamma)-1+\alpha)}}\, dx \\ &\leq C\, \frac{\mu_{1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} - \alpha\right)(2^\star(s)-1-p_\epsilon)} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \int_{|x| \leq \frac{1}{2}|z_\epsilon|} \frac{|x|^{-\alpha(2^\star(s)-1-p_\epsilon)}}{|x|^{(2^\star(s)-p_\epsilon)(\beta_-(\gamma)-1)+s}}\, dx \\ &\leq C \left(\frac{|z_\epsilon|}{\mu_{1,\epsilon}} \right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right) - \alpha(2^\star(s)-1-p_\epsilon)} \frac{d(z_\epsilon,\partial\Omega)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |z_\epsilon|^{\beta_-(\gamma)}} \\ &\leq C\, \frac{|z_\epsilon|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |z_\epsilon|^{\beta_-(\gamma)}}. \end{aligned} \tag{92}$$

Next we have

$$I_2^1 := C \int_{D_{2,\epsilon}^1} \left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx$$
$$\leq C\, d(z_\epsilon,\partial\Omega) \int_{D_{2,\epsilon}^1} \frac{d(x,\partial\Omega)}{|x-z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx.$$

From (80) we get for $0<\alpha<\frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$I_2^1 \leq C\, \mu_{1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)} d(z_\epsilon,\partial\Omega) \int_{D_{2,\epsilon}^1} \frac{|x|^{-s+1}|x-z_\epsilon|^{-n}}{|x|^{(\beta_-(\gamma)-1+\alpha)(2^\star(s)-1-p_\epsilon)}}\, dx$$
$$\leq C\, \frac{\mu_{1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{(\beta_-(\gamma)-1+\alpha)(2^\star(s)-1-p_\epsilon)+s-1}} \int_{D_{2,\epsilon}^1} |x|^{-n}\, dx$$
$$\leq C\, \left(\frac{|z_\epsilon|}{\mu_{1,\epsilon}}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)} \frac{d(z_\epsilon,\partial\Omega)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}$$
$$\leq C\, \frac{|z_\epsilon|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}. \tag{93}$$

For

$$I_3^1 := C \int_{D_{3,\epsilon}^1} \left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx$$
$$\leq C\, \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} \int_{2|z_\epsilon|\leq|x|\leq k_{1,\epsilon}} |x|^{\beta_-(\gamma)+1-n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx$$
$$\leq C\, \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} \int_{2|z_\epsilon|\leq|x|\leq k_{1,\epsilon}} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_+(\gamma)+s-1}}\, dx.$$

Then from (80) we get for $0<\alpha<\frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$I_3^1 \leq C\, \frac{\mu_{1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} A_\epsilon$$

$$A_\epsilon := \int_{2|z_\epsilon|\leq|x|\leq k_{1,\epsilon}} \frac{1}{|x|^{\beta_+(\gamma)+s-1+(\beta_-(\gamma)-1-p_\epsilon)(2^\star(s)-1)+\alpha(2^\star(s)-1-p_\epsilon)}}\, dx$$

With a change of variable, we get that

$$
\begin{aligned}
A_\epsilon &\leq \int\limits_{2|z_\epsilon|\leq |x|\leq k_{1,\epsilon}} \frac{dx}{|x|^{n-\left[(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)\right]}}\\
&\leq \mu_{1,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)(2^\star(s)-2-p_\epsilon)-\alpha(2^\star(s)-1-p_\epsilon)} \int\limits_{B_1(0)} \frac{|x|^{-\alpha(2^\star(s)-1-p_\epsilon)}dx}{|x|^{n-(2^\star(s)-2)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)}}\\
&\leq C\mu_{1,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)(2^\star(s)-2-p_\epsilon)-\alpha(2^\star(s)-1-p_\epsilon)}
\end{aligned}
$$

and then

$$
I_3^1 \leq C\,\frac{d(z_\epsilon,\partial\Omega)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}} \leq C\,\frac{|z_\epsilon|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}. \tag{94}
$$

For the last integral, we use point (A5) of Proposition 2 and a change of variable to get

$$
\begin{aligned}
I_4^1 &:= C\int\limits_{D_{4,\epsilon}^1}\left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\,\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx\\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{|x|\geq k_{1,\epsilon}}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_+(\gamma)+s-1}}\,dx\\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{|x|\geq k_{1,\epsilon}}\frac{1}{|x|^{\beta_+(\gamma)+s-1+\frac{n-2}{2}(2^\star(s)-1-p_\epsilon)}}\,dx \qquad (95)\\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{|x|\geq 1}\frac{d}{|x|^{n+\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}} \leq C\,\frac{|z_\epsilon|}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}.
\end{aligned}
$$

Combining (92), (93), (94) and (95), we then obtain (91). □

Step 5.7: Let $1\leq i\leq N-1$. There exists $C>0$ such that for sequence of points (z_ϵ) in $B_{k_{i+1,\epsilon}}(0)\setminus \overline{B}_{k_{i,\epsilon}}(0)\cap\Omega$ we have

$$
\int\limits_{\Omega} G_\epsilon(z_\epsilon,x)\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx \leq C\left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\right). \tag{96}
$$

Proof of Step 5.7: To estimate the right-hand-side of (85) in this case, we split Ω into five subdomains as: $\Omega=\bigcup_{j=1}^{5} D_{j,\epsilon}^i$ where

- $D_{1,\epsilon}^i := B_{k_{i,\epsilon}}(0)\cap\Omega$,
- $D_{2,\epsilon}^i := \{k_{i,\epsilon}<|x|<\frac{1}{2}|z_\epsilon|\}\cap\Omega$,
- $D_{3,\epsilon}^i := \{\frac{1}{2}|z_\epsilon|<|x|<2|z_\epsilon|\}\cap\Omega$,

- $D^i_{4,\epsilon} := \{2|z_\epsilon| < |x| < k_{i+1,\epsilon}\} \cap \Omega$,
- $D^i_{5,\epsilon} := \{k_{i+1,\epsilon} < |x|\} \cap \Omega$.

Note that one has $\frac{1}{2}|z_\epsilon| < |x - z_\epsilon|$ in $D^i_{2,\epsilon}$ and $\frac{1}{2}|x| < |x - z_\epsilon|$ in $D^i_{4,\epsilon}$.

First we have using point (A5) of Proposition 2 and a change of variable

$$\begin{aligned}
I^i_1 &:= C \int\limits_{D^i_{1,\epsilon}} \left(\frac{\max\{|z_\epsilon|, |x|\}}{\min\{|z_\epsilon|, |x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x - z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \\
&\leq C\ d(z_\epsilon,\partial\Omega) \int\limits_{D^i_{1,\epsilon}} \frac{|z_\epsilon|^{\beta_-(\gamma)}}{|x|^{\beta_-(\gamma)-1}} |x - z_\epsilon|^{-n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \\
&\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \int\limits_{D^i_{1,\epsilon}} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)-1+s}}\, dx \\
&\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \int\limits_{B_{k_{i,\epsilon}}(0)} \frac{1}{|x|^{\beta_-(\gamma)-1+s+(2^\star(s)-1-p_\epsilon)\frac{n-2}{2}}}\, dx \\
&\leq C\ \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}} \int\limits_{B_1(0)} \frac{1}{|x|^{n-\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\, dx \leq C\ \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}.
\end{aligned} \tag{97}$$

Now we estimate

$$\begin{aligned}
I^i_2 &:= C \int\limits_{D^i_{2,\epsilon}} \left(\frac{\max\{|z_\epsilon|, |x|\}}{\min\{|z_\epsilon|, |x|\}}\right)^{\beta_-(\gamma)} \frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x - z_\epsilon|^n} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx \\
&\leq C\ d(z_\epsilon,\partial\Omega) \frac{|z_\epsilon|^{\beta_-(\gamma)}}{|z_\epsilon|^n} \int\limits_{D^i_{2,\epsilon}} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)-1+s}}\, dx.
\end{aligned}$$

Using (74) we get for $0 < \alpha < \frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$
\begin{aligned}
I_2^i &\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{D_{2,\epsilon}^i}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_-(\gamma)+s}}\,dx\\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\mu_{i,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}\int\limits_{D_{2,\epsilon}^i}\frac{|x|^{-\beta_-(\gamma)+1-s}\,dx}{|x|^{(2^\star(s)-1-p_\epsilon)(\beta_+(\gamma)-1-\alpha)}}\\
&\quad+C\,\frac{\mu_{i+1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{D_{2,\epsilon}^i}\frac{|x|^{-\beta_-(\gamma)+1-s}\,dx}{|x|^{(2^\star(s)-1-p_\epsilon)(\beta_-(\gamma)-1+\alpha)}}\\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}\int\limits_{1\leq|x|}\frac{dx}{|x|^{n+(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}}\\
&\quad+C\,\frac{\mu_{i+1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{|x|\leq\frac{1}{2}|z_\epsilon|}\frac{|x|^{-\alpha(2^\star(s)-1-p_\epsilon)}\,dx}{|x|^{2^\star(s)(\beta_-(\gamma)-1)+s}}\\
&\leq C\,\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\int\limits_{1\leq|x|}\frac{dx}{|x|^{n+(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}}\\
&\quad+C\,\left(\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}\frac{d(z_\epsilon,\partial\Omega)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\\
&\leq C\,\left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\right).
\end{aligned}
\tag{98}
$$

And next

$$
\begin{aligned}
I_3^i &:= C\int\limits_{D_{3,\epsilon}^i}\left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx\\
&\leq C\,d(z_\epsilon,\partial\Omega)\int\limits_{D_{3,\epsilon}^i}\frac{|x|}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx.
\end{aligned}
$$

From (74) we get for $0<\alpha<\frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$
\begin{aligned}
I_3^i &\leq C\ d(z_\epsilon,\partial\Omega)\mu_{i,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}\int_{D_{3,\epsilon}^i}\frac{|x|^{1-s}|x-z_\epsilon|^{-n}\,dx}{|x|^{(\beta_+(\gamma)-1-\alpha)(2^\star(s)-1-p_\epsilon)}}\\
&\quad + C\ d(z_\epsilon,\partial\Omega)\mu_{i+1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}\int_{D_{3,\epsilon}^i}\frac{|x|^{1-s}|x-z_\epsilon|^{-n}\,dx}{|x|^{(\beta_-(\gamma)-1+\alpha)(2^\star(s)-1-p_\epsilon)}}\\
&\leq C\ \frac{\mu_{i,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{(\beta_+(\gamma)-1-\alpha)(2^\star(s)-1-p_\epsilon)+s-1}}\int_{D_{3,\epsilon}^i}|x-z_\epsilon|^{-n}\ dx\\
&\quad + C\ \frac{\mu_{i+1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{(\beta_-(\gamma)-1+\alpha)(2^\star(s)-1-p_\epsilon)+s-1}}\int_{D_{3,\epsilon}^i}|x-z_\epsilon|^{-n}\ dx\\
&\leq C\ \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\left(\frac{\mu_{i,\epsilon}}{|z_\epsilon|}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}\\
&\quad + C\ \left(\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\\
&\leq C\ \left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\right).
\end{aligned}
\tag{99}
$$

The next integral becomes

$$
\begin{aligned}
I_4^i &:= C\int_{D_{4,\epsilon}^i}\left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\ dx\\
&\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int_{2|z_\epsilon|\leq|x|<k_{i+1,\epsilon}}|x|^{\beta_-(\gamma)+1-n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\ dx\\
&\leq C\ \frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int_{2|z_\epsilon|\leq|x|<k_{i+1,\epsilon}}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_+(\gamma)+s-1}}\ dx.
\end{aligned}
$$

Then from (74) we get for $0<\alpha<\frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$

$$
\begin{aligned}
I_4^i \quad\leq\quad & C\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\left(\mu_{i,\epsilon}^{\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}A_\epsilon\right.\\
&\left.+\mu_{i+1,\epsilon}^{-\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha\right)(2^\star(s)-1-p_\epsilon)}B_\epsilon\right)
\end{aligned}
$$

with

$$
\begin{aligned}
A_\epsilon &:= \int\limits_{2|z_\epsilon|\leq |x|<k_{i+1,\epsilon}} \frac{|x|^{\alpha(2^\star(s)-1-p_\epsilon)}\,dx}{|x|^{(2^\star(s)-p_\epsilon)(\beta_+(\gamma)-1)+s}} \\
&\leq \int\limits_{2|z_\epsilon|\leq |x|<k_{i+1,\epsilon}} \frac{|x|^{\alpha(2^\star(s)-1-p_\epsilon)}\,dx}{|x|^{n+(2^\star(s)-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)}}\,dx
\end{aligned}
$$

$$
\begin{aligned}
B_\epsilon &:= \int\limits_{2|z_\epsilon|\leq |x|<k_{i+1,\epsilon}} \frac{dx}{|x|^{\beta_+(\gamma)+s+\beta_-(\gamma)(2^\star(s)-1-p_\epsilon)+\alpha(2^\star(s)-1-p_\epsilon)}} \\
&\leq \int\limits_{2|z_\epsilon|\leq |x|<k_{i+1,\epsilon}} \frac{dx}{|x|^{n-\left[(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)\right]}}
\end{aligned}
$$

Arguing as in the case $i=1$, with a change of variable we get that

$$
\begin{aligned}
I_4^i &\leq C\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_+(\gamma)}}\left(\frac{\mu_{i,\epsilon}}{|z_\epsilon|}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)} \\
&\quad +C\left(\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}}\right)^{(2^\star(s)-2-p_\epsilon)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-\alpha(2^\star(s)-1-p_\epsilon)}\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}} \\
&\leq C\left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{|z_\epsilon|^{\beta_+(\gamma)}}+\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\right). \qquad (100)
\end{aligned}
$$

Finally we get for the last integral from point (A5) of Proposition 2 and a change of variable

$$
\begin{aligned}
I_5^i &:= C\int\limits_{D_{5,\epsilon}^i}\left(\frac{\max\{|z_\epsilon|,|x|\}}{\min\{|z_\epsilon|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)d(z_\epsilon,\partial\Omega)}{|x-z_\epsilon|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\,dx \\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{|x|\geq k_{i+1,\epsilon}}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^{\beta_+(\gamma)+s-1}}\,dx \\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{|x|\geq k_{i+1,\epsilon}}\frac{1}{|x|^{\beta_+(\gamma)+s-1+\frac{n-2}{2}(2^\star(s)-1-p_\epsilon)}}\,dx \\
&\leq C\,\frac{d(z_\epsilon,\partial\Omega)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}\int\limits_{|x|\geq 1}\frac{1}{|x|^{n+\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}} \\
&\leq C\,\frac{|z_\epsilon|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|^{\beta_-(\gamma)}}. \qquad (101)
\end{aligned}
$$

Then from (97), (98), (99), (100) and (101) we get the estimate (96). □

Combining the estimates (85), (86), (91) and (96) we get that, there exists a constant $C>0$ such that for any sequence of points (z_ϵ) in Ω we have

$$|u_\epsilon(z_\epsilon)| \leq C\left(\sum_{i=1}^{N}\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|z_\epsilon|}{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|z_\epsilon|^{\beta_-(\gamma)}+|z_\epsilon|^{\beta_+(\gamma)}}+\frac{\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|z_\epsilon|^{\beta_-(\gamma)}}|z_\epsilon|\right).$$

This completes the proof of Proposition 3. □

In our next result we obtain a pointwise control on the gradient.

Proposition 4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, such that $0\in\partial\Omega$ and assume that $0<s<2$, $\gamma<\frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0}\||x|^\tau u_\epsilon\|_{L^\infty(\Omega)}=+\infty \quad \textit{where} \quad \beta_-(\gamma)-1<\tau<\frac{n-2}{2}.$$

Consider the $\mu_{1,\epsilon},...,\mu_{N,\epsilon}$ from Proposition 2. Then there exists $C>0$ such that for all $\epsilon>0$

$$|\nabla u_\epsilon(x)| \leq C\left(\sum_{i=1}^{N}\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|x|^{\beta_-(\gamma)}+|x|^{\beta_+(\gamma)}}+\frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}}\right) \tag{102}$$

for all $x\in\overline{\Omega}\setminus\{0\}$.

Proof of Proposition 4: We let G_ϵ be the Green's function of the coercive operator $-\Delta-\frac{\gamma}{|x|^2}-h_\epsilon$ on Ω with Dirichlet boundary condition. Differentiating the Green's representation formula, and then using the pointwise bounds on the gradient Green's function (209) and the regularity result Theorem 6 yields for any $z\in\Omega$

$$u_\epsilon(z)=\int_\Omega G_\epsilon(z,x)\frac{|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon}\;u_\epsilon(x)}{|x|^s}dx$$

$$\begin{aligned}|\nabla u_\epsilon(z)| &\leq C\int_\Omega|\nabla_z G_\epsilon(z,x)|\,\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}dx\\ &\leq C\int_\Omega|\nabla_z G_\epsilon(z,x)|\,\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\;dx\\ &\leq \int_\Omega\left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}}\right)^{\beta_-(\gamma)}\frac{d(x,\partial\Omega)}{|x-z|^n}\frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\;dx.\end{aligned}$$

Then using the pointwise estimates (60) the proof goes exactly as in Proposition 3.
□

6. Sharp blow-up rates and the proof of Compactness

The proof of compactness rely on the following two key propositions.

PROPOSITION 5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$, $\gamma < \frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon \to 0} \| |x|^\tau u_\epsilon \|_{L^\infty(\Omega)} = +\infty \quad \text{for some } \beta_-(\gamma) - 1 < \tau < \frac{n-2}{2}. \tag{103}$$

Consider the $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Suppose that

$$\text{either } \{ \beta_+(\gamma) - \beta_-(\gamma) > 2 \} \text{ or } \{ \beta_+(\gamma) - \beta_-(\gamma) > 1 \text{ and } u_0 \equiv 0 \}. \tag{104}$$

Then, we have following blow-up rates:

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = c_{n,s,t_N} \frac{\int_{\partial \mathbb{R}^n_-} II_0(x,x) |\nabla \tilde{u}_N|^2 \, d\sigma}{\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n_-} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s} \, dx}. \tag{105}$$

Here II_0 denotes the second fundamental form of $\partial\Omega$ at $0 \in \partial\Omega$ and

$$c_{n,s,t_N} := \frac{n-s}{(n-2)^2} \frac{1}{t_N^{\frac{n-1}{2^\star(s)-2}}}.$$

PROPOSITION 6 (*The positive case*). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume that $0 < s < 2$, $\gamma < \frac{n^2}{4}$. Let (u_ϵ), (h_ϵ) and (p_ϵ) be as in Proposition 5 and let $H(0)$ denote the mean curvature of $\partial\Omega$ at 0. Assume that blow-up occurs as in (103). Consider the $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Suppose in addition that*

$$u_\epsilon > 0 \qquad \text{for all } \epsilon > 0. \tag{106}$$

We define

$$C_{n,s,(t_i),(\tilde{u})_i} := \frac{c_{n,s,t_N} \int_{\partial \mathbb{R}^n_-} |x|^2 |\nabla \tilde{u}_N|^2 \, d\sigma}{(n-1) \sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n_-} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s} \, dx} \tag{107}$$

Then, we have the following blow-up rates:

1) When $\beta_+(\gamma) - \beta_-(\gamma) \geq 2$, then

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = C_{n,s,(t_i),(\tilde{u})_i} \cdot H(0) \quad \text{if} \left\{ \begin{array}{l} \beta_+(\gamma) - \beta_-(\gamma) > 2 \\ \text{or } \beta_+(\gamma) - \beta_-(\gamma) = 2 \text{ and } u_0 \equiv 0 \end{array} \right\}.$$

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = C_{n,s,(t_i),(\tilde{u})_i} \cdot H(0) - \tilde{K} \quad \text{if } \beta_+(\gamma) - \beta_-(\gamma) = 2 \text{ and } u_0 > 0.$$

for some $\tilde{K} > 0$.

2) When $\beta_+(\gamma) - \beta_-(\gamma) < 2$, then $u_0 \equiv 0$ and

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = C_{n,s,(t_i),(\tilde{u})_i} \cdot H(0) \qquad \text{if } \beta_+(\gamma) - \beta_-(\gamma) > 1.$$

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon} \ln \frac{1}{\mu_{\epsilon,N}}} = C'_{n,s,(t_i),(\tilde{u})_i} \cdot H(0) \qquad \text{if } \beta_+(\gamma) - \beta_-(\gamma) = 1 \tag{108}$$

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}} = -\chi\cdot m_{\gamma,h}(\Omega) \quad \text{if } \beta_+(\gamma)-\beta_-(\gamma)<1 \tag{109}$$

where

$$C'_{n,s,(t_i),(\tilde{u})_i} := \frac{n-s}{(n-2)^2}\frac{K^2\omega_{n-2}}{(n-1)\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx},$$

the constant K *is as in* (169), $\chi>0$ *is a constant and* $m_{\gamma,h}(\Omega)$ *is the boundary mass defined in Theorem* 1*.*

Proof of Theorems 3*,* 5 *and* 4*:* We argue by contradiction and assume that the family is not pre-compact. Then, up to a subsequence, it blows up. We then apply Propositions 5 and 6 to get the blow-up rate (that is nonegative). However, the hypothesis of Theorems 3, 5 and 4 yield exactly negative blow-up rates. This is a contradiction, and therefore the family is pre-compact. This proves the Theorems. □

We now establish Propositions 5 and 6. The proof is divided in 13 steps in Sections 7 to 8. These steps are numbered Steps P1, P2, etc.

7. Estimates on the localized Pohozaev identity

In the sequel, we let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. Note that

$$\gamma<\frac{n^2}{4}-1 \iff \beta_+(\gamma)-\beta_-(\gamma)>2,$$

and

$$\gamma<\frac{n^2-1}{4} \iff \beta_+(\gamma)-\beta_-(\gamma)>1.$$

In the sequel, we will permanently use the following consequence of (A9) of Proposition 2: for all $i=1,...,N$, there exists $c_i>1$ such that

$$c_i^{-1}\mu_{\epsilon,i}\leq k_{\epsilon,i}\leq c_i\mu_{\epsilon,i}. \tag{110}$$

Step P1 (Pohozaev identity). *We let* (u_ϵ)*,* (h_ϵ) *and* (p_ϵ) *be such that* (E_ϵ)*,* (15)*,* (21) *and* (22) *hold. We assume that blow-up occurs. We define*

$$F_\epsilon(x) := (x,\nu)\left(\frac{|\nabla u_\epsilon|^2}{2}-\frac{\gamma}{2}\frac{u_\epsilon^2}{|x|^2}-\frac{h_\epsilon(x)}{2}u_\epsilon^2-\frac{1}{2^\star(s)-p_\epsilon}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right)$$
$$-\left(x^i\partial_i u_\epsilon+\frac{n-2}{2}u_\epsilon\right)\partial_\nu u_\epsilon \tag{111}$$

We let $\mathcal{T}$ be a chart at 0 as in (27). We define $r_\epsilon := \sqrt{\mu_{N,\epsilon}}$. Then

$$\begin{aligned}
&\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right) u_\epsilon^2 \, dx \\
&\quad + \frac{p_\epsilon}{2^\star(s)} \left(\frac{n-s}{2^\star(s) - p_\epsilon}\right) \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx \\
&= -\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{r_\epsilon}(0)\right)} F_\epsilon(x) \, d\sigma + \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{k^3_{1,\epsilon}}(0)\right)} F_\epsilon(x) \, d\sigma \\
&\quad + \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2} \, d\sigma \qquad (112)
\end{aligned}$$

and, for $\delta_0 > 0$ small enough,

$$\begin{aligned}
&\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right) u_\epsilon^2 \, dx \\
&\quad + \frac{p_\epsilon}{2^\star(s)} \left(\frac{n-s}{2^\star(s) - p_\epsilon}\right) \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx \\
&= -\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{\delta_0}(0)\right)} F_\epsilon(x) \, d\sigma + \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{k^3_{1,\epsilon}}(0)\right)} F_\epsilon(x) \, d\sigma \\
&\quad + \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2} \, d\sigma \qquad (113)
\end{aligned}$$

Proof of Step P1: We apply the Pohozaev identity (196) with $y_0 = 0$ and

$$U_\epsilon = \mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0) \setminus B_{k^3_{1,\epsilon}}(0)\right) \subset \Omega.$$

This yields

$$\begin{aligned}
-\int_{U_\epsilon} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right) u_\epsilon^2 \, dx \; - \frac{p_\epsilon}{2^\star(s)} \left(\frac{n-s}{2^\star(s) - p_\epsilon}\right) \int_{U_\epsilon} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx \\
= \int_{\partial U_\epsilon} F_\epsilon(x) \, d\sigma. \qquad (114)
\end{aligned}$$

It follows from the properties of the boundary map that

$$\begin{aligned}\partial U_\epsilon &= \partial\left(\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)\right)\\ &= \mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{r_\epsilon}(0)\right)\cup\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)\right)\cup\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)\end{aligned}$$

Since for all $\epsilon > 0$, $u_\epsilon \equiv 0$ on $\partial\Omega$, identity (114) yields (112). Concerning (113), we apply the Pohozaev identity (196) with $y_0 = 0$ and

$$V_\epsilon = \mathcal{T}\left(\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)\subset\Omega.$$

The argument is similar. This ends the proof of Step P1. □

We will estimate each of the terms in the above integral identities and calculate the limit as $\epsilon \to 0$.

7.1. Estimates of the $L^{2^\star(s)}$ and L^2−terms in the localized Pohozaev identity.

Step P2. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We claim that, as $\epsilon\to 0$*

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx = \sum_{i=1}^N\frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx + o(1). \quad (115)$$

and

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx = \sum_{i=1}^N\frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx + o(1) \text{ if } u_0\equiv 0. \quad (116)$$

Proof of Step P2: For any $R,\rho>0$ we decompose the above integral as

$$\begin{aligned}\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx &= \int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx\\ &+\sum_{i=1}^N\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{Rk_{i,\epsilon}}(0)\setminus\overline{B}_{\rho k_{i,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx\\ &+\sum_{i=1}^{N-1}\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx\\ &+\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\rho k_{1,\epsilon}}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx.\end{aligned}$$

We will evaluate each of the above terms and calculate the limit $\lim\limits_{R\to+\infty}\lim\limits_{\rho\to 0}\lim\limits_{\epsilon\to 0}$.

From the estimate (60), we get as $\epsilon \to 0$

$$\begin{aligned}
&\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx \\
&\leq C \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)\right)} \left[\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_+(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} + \frac{1}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} \right] dx \\
&\leq C \int_{\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} \frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_+(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} \left|\text{Jac } \mathcal{T}(x)\right|\, dx \\
&+ C \int_{\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} \frac{1}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} \left|\text{Jac } \mathcal{T}(x)\right|\, dx \\
&\leq C \int_{\mathbb{R}^n_- \cap B_{\frac{r_\epsilon}{k_{N,\epsilon}}}(0)\setminus \overline{B}_{R}(0)} \frac{1}{|x|^{n+2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_+(\gamma)-1)}} \left|\text{Jac } \mathcal{T}(k_{N,\epsilon}x)\right|\, dx \\
&+ C \int_{\mathbb{R}^n_- \cap B_1(0)\setminus \overline{B}_{\frac{Rk_{N,\epsilon}}{r_\epsilon}}(0)} \frac{1}{|x|^{n-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_-(\gamma)-1)}} \left|\text{Jac } \mathcal{T}(r_\epsilon x)\right|\, dx \\
&\leq C \left(R^{-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_+(\gamma)-1)} + r_\epsilon^{2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)+p_\epsilon(\beta_-(\gamma)-1)} \right).
\end{aligned}$$

Therefore

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx = 0. \tag{117}$$

It follows from Proposition 2 that for any $1 \leq i \leq N$

$$\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\epsilon\to 0} \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\rho k_{i,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx = \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n_-} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx. \tag{118}$$

Let $1 \leq i \leq N-1$. In Proposition 3, we had obtained the following pointwise estimates: For any $R, \rho > 0$ and all $\epsilon > 0$ we have

$$|u_\epsilon(x)| \leq C\, \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |x|}{|x|^{\beta_+(\gamma)}} + C\, \frac{|x|}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |x|^{\beta_-(\gamma)}}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$.

Then we have as $\epsilon \to 0$

$$\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx$$

$$\leq C \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)\right)} \left[\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_+(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} + \frac{\mu_{i+1,\epsilon}^{-\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} \right] dx$$

$$\leq C \int_{\mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} \left[\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_+(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} + \frac{\mu_{i+1,\epsilon}^{-\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} \right] dx$$

$$\leq C \int_{\mathbb{R}^n_- \cap B_{\frac{\rho k_{i+1,\epsilon}}{k_{i,\epsilon}}}(0)\setminus \overline{B}_R(0)} \frac{1}{|x|^{n+2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_+(\gamma)-1)}}\, dx$$

$$+ \ C \int_{\mathbb{R}^n_- \cap B_{2\rho}(0)\setminus \overline{B}_{\frac{Rk_{i,\epsilon}}{k_{i+1,\epsilon}}}(0)} \frac{1}{|x|^{n-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_-(\gamma)-1)}}\, dx$$

$$\leq C \left(R^{-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_+(\gamma)-1)} + \rho^{2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)+p_\epsilon(\beta_-(\gamma)-1)} \right).$$

And so

$$\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\epsilon\to 0} \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx = 0. \tag{119}$$

Again, from the pointwise estimates of Proposition 3, we have as $\epsilon \to 0$

$$\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx$$

$$\leq C \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \frac{\mu_{1,\epsilon}^{-\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}}\, dx$$

$$\leq C \int_{\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0)\setminus \overline{B}_{k^3_{1,\epsilon}}(0)} \frac{\mu_{1,\epsilon}^{-\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}} |\mathrm{Jac}\ \mathcal{T}(x)|\, dx$$

$$\leq C \int_{\mathbb{R}^n_- \cap B_{\rho}(0)\setminus \overline{B}_{k^2_{1,\epsilon}}(0)} \frac{1}{|x|^{n-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)-p_\epsilon(\beta_-(\gamma)-1)}} |\mathrm{Jac}\ \mathcal{T}(k_{1,\epsilon}x)|\, dx$$

$$\leq C\ \rho^{2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)+p_\epsilon(\beta_-(\gamma)-1)}.$$

Therefore

$$\lim_{\rho\to 0}\lim_{\epsilon\to 0}\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\rho k_{1,\epsilon}}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx=0. \tag{120}$$

Combining (117), (118), (119) and (120) we obtain (115).

We now prove (116) under the assumption that $u_0\equiv 0$. We decompose the integral as

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx=\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx$$

$$+\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx,$$

with $r_\epsilon:=\sqrt{\mu_{N,\epsilon}}$. From the estimate (60) and $u_0\equiv 0$, we get as $\epsilon\to 0$

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx\leq C\int_{\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)}\left[\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{(\beta_+(\gamma)-1)(2^\star(s)-p_\epsilon)+s}}\right]dx$$

Since $(\beta_+(\gamma)-1)2^\star(s)+s>n$, we then get that

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)\right)}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx\leq C\left(\frac{\mu_{N,\epsilon}}{r_\epsilon}\right)^{\frac{2^\star(s)}{2}(\beta_+(\gamma)-\beta_-(\gamma))}=o(1)$$

as $\epsilon\to 0$. Therefore, with (117), we get (116). This proves (116).

This ends the proof of Step P2. □

Step P3. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We claim that*

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx$$

$$=\begin{cases} O(\mu_{N,\epsilon}^2) & \text{if } \beta_+(\gamma)-\beta_-(\gamma)>2\\ O(\mu_{N,\epsilon}^2\ln\frac{1}{\mu_{N,\epsilon}}) & \text{if } \beta_+(\gamma)-\beta_-(\gamma)=2\\ O(\mu_{N,\epsilon}^{1+\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}) & \text{if } \beta_+(\gamma)-\beta_-(\gamma)<2.\end{cases} \tag{121}$$

And if $u_0 \equiv 0$

$$\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right) u_\epsilon^2 \, dx$$
$$= \begin{cases} O(\mu_{N,\epsilon}^2) & \text{if } \beta_+(\gamma) - \beta_-(\gamma) > 2 \\ O(\mu_{N,\epsilon}^2 \ln \frac{1}{\mu_{N,\epsilon}}) & \text{if } \beta_+(\gamma) - \beta_-(\gamma) = 2 \\ O(\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}) & \text{if } \beta_+(\gamma) - \beta_-(\gamma) < 2. \end{cases} \tag{122}$$

Proof of Step P3*:* From estimate (60) and after a change of variables, we get as $\epsilon \to 0$,

$$\begin{aligned}
&\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right) u_\epsilon^2 \, dx \\
&\le C \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} u_\epsilon^2 \, dx \\
&\le C \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} \left[\frac{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2(\beta_+(\gamma)-1)}} \, dx + \frac{1}{|x|^{2(\beta_-(\gamma)-1)}} \, dx\right] \\
&\le C \int_{\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk^3_{1,\epsilon}}(0)} \left(\sum_{i=1}^N \frac{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|x|^2}{\mu_{i,\epsilon}^{2(\beta_+(\gamma)-\beta_-(\gamma))}|x|^{2\beta_-(\gamma)} + |x|^{2\beta_+(\gamma)}} \right. \\
&\qquad \left. + \frac{1}{|x|^{2(\beta_-(\gamma)-1)}}\right) dx.
\end{aligned} \tag{123}$$

Case 1: Assuming that $\beta_+(\gamma) - \beta_-(\gamma) < 2$, we then have the following rough bound from (123),

$$\begin{aligned}
&\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} u_\epsilon^2 \, dx \\
&\le C \int_{\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk^3_{1,\epsilon}}(0)} \left(\frac{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2(\beta_+(\gamma)-1)}} + \frac{1}{|x|^{2(\beta_-(\gamma)-1)}}\right) dx \\
&\le C \mu_{N,\epsilon}^{1+\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} \quad \text{if } \beta_+(\gamma) - \beta_-(\gamma) < 2.
\end{aligned} \tag{124}$$

Case 2: Assuming $\beta_+(\gamma)-\beta_-(\gamma)\geq 2$, then via a change of variable in (123), we get

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} u_\epsilon^2\, dx \leq C\sum_{i=1}^N \mu_{i,\epsilon}^2 \int_{B_{\frac{r_\epsilon}{\mu_{i,\epsilon}}}(0)\setminus\overline{B}_{\frac{k^3_{1,\epsilon}}{\mu_{i,\epsilon}}}(0)} \frac{|x|^2\, dx}{|x|^{2\beta_-(\gamma)}+|x|^{2\beta_+(\gamma)}}$$
$$+C\int_{B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)} |x|^{2-2\beta_-(\gamma)}\, dx.$$

Therefore, if $\beta_+(\gamma)-\beta_-(\gamma)>2$, then

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} u_\epsilon^2\, dx \leq C\sum_{i=1}^N \mu_{i,\epsilon}^2 + Cr_\epsilon^{n+2-2\beta_-(\gamma)} \leq C\mu_{N,\epsilon}^2. \quad (125)$$

When $\beta_+(\gamma)-\beta_-(\gamma)=2$, we get that

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} u_\epsilon^2\, dx \leq C\sum_{i=1}^N \mu_{i,\epsilon}^2\left(1+\int_{B_{\frac{r_\epsilon}{\mu_{i,\epsilon}}}(0)\setminus\overline{B}_1(0)} |x|^{2-\beta_+(\gamma)}\, dx\right)$$
$$+Cr_\epsilon^{2+\beta_+(\gamma)-\beta_-(\gamma)}$$
$$\leq C\mu_{N,\epsilon}^2\ln\frac{1}{\mu_{N,\epsilon}} + C\sum_{i=1}^{N-1}\mu_{i,\epsilon}^2\ln\frac{1}{\mu_{i,\epsilon}}.$$

Since $\mu_{N,\epsilon}\to 0$ and $\lim_{\epsilon\to 0}\mu_{i,\epsilon}/\mu_{N,\epsilon}$ is finite for all $i=1,\dots,N-1$, we get that

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} u_\epsilon^2\, dx = O\left(\mu_{N,\epsilon}^2\ln\frac{1}{\mu_{N,\epsilon}}\right), \quad (126)$$

since $\beta_+(\gamma)-\beta_-(\gamma)=2$. Inequality (123) put together with (124), (125) and (126) yield (121).

When $u_0\equiv 0$ we decompose the integral and proceed as in the proof of (116) to obtain (122). This ends Step P3. □

7.2. Estimate of the curvature term in the Pohozaev identity when $\beta_+(\gamma)-\beta_-(\gamma)>1$.

STEP P4. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs and that $\beta_+(\gamma)-\beta_-(\gamma)>1$. We claim that, as $\epsilon\to 0$*

$$\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$
$$=\frac{\mu_{N,\epsilon}}{2}\left(\frac{1}{t_N^{\frac{n-1}{2^\star(s)-2}}}\int_{\partial\mathbb{R}^n_-} II_0(x,x)\frac{|\nabla\tilde{u}_N|^2}{2}\, d\sigma + o(1)\right). \quad (127)$$

Here, see Proposition 5, II_0 denotes the second fundamental form. Moreover, when $u_0 \equiv 0$, we claim that as $\epsilon \to 0$,

$$\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$

$$= \frac{\mu_{N,\epsilon}}{2}\left(\frac{1}{t_N^{\frac{n-1}{2^\star(s)-2}}}\int_{\partial\mathbb{R}^n_-} II_0(x,x)\frac{|\nabla \tilde{u}_N|^2}{2}\, d\sigma + o(1)\right). \tag{128}$$

Proof of Step P4: We have for any $R,\rho>0$,

$$\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \tag{129}$$

$$= \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$

$$+\sum_{i=1}^{N}\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\rho k_{i,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$

$$+\sum_{i=1}^{N-1}\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$

$$+\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma. \tag{130}$$

We consider the second fundamental form associated to $\partial\Omega$, $II_0(x,y) = (d\nu_p x, y)$ for $0\in\partial\Omega$ and all $x,y\in T_0\partial\Omega$ (ν is the outward normal vector at the hypersurface $\partial\Omega$). In the canonical basis of $\partial\mathbb{R}^n_- = T_0\partial\Omega$, the matrix of the bilinear form II_0 is $-D_0^2\mathcal{T}_0$, where $D_0^2\mathcal{T}_0$ is the Hessian matrix of $\mathcal{T}_0$ at 0. Using the expression of $\mathcal{T}$ (see (27)), we can write for all $x\in U\cap\partial\mathbb{R}^n_-$

$$\nu(\mathcal{T}(x)) = \frac{(1,-\partial_2\mathcal{T}_0(x),\dots,-\partial_n\mathcal{T}_0(x))}{\sqrt{1+\sum_{i=2}^n(\partial_i\mathcal{T}_0(x))^2}}.$$

With the expression of $\mathcal{T}$, we then get that

$$(\nu\circ\mathcal{T}(x),\mathcal{T}(x)) = \frac{\mathcal{T}_0(x)-\sum_{p=2}^n x^p\partial_p\mathcal{T}_0(x)}{\sqrt{1+\sum_{p=2}^n(\partial_p\mathcal{T}_0(x))^2}}$$

And so for all $x\in U\cap\partial\mathbb{R}^n_-$.

$$|(\mathcal{T}(x),\nu\circ\mathcal{T}(x))| \le C|x|^2 \tag{131}$$

Since $\mathcal{T}_0(0) = 0$ and $\nabla \mathcal{T}_0(0) = 0$ (see (27)), we then get as $|x| \to 0$

$$(\nu \circ \mathcal{T}(x), \mathcal{T}(x)) = -\frac{1}{2} \sum_{p,q=2}^{n} x^p x^q \partial_{pq} \mathcal{T}_0(0) + O(|x|^3) \tag{132}$$

and therefore for all $\epsilon > 0$ and all $x \in B_R(0) \cap \partial \mathbb{R}^n_-$

$$\begin{aligned}(\mathcal{T}(k_{N,\epsilon} x), \nu \ \circ T(k_{N,\epsilon} x)) &= -\frac{1}{2} k_{N,\epsilon}^2 \sum_{p,q=2}^{n} x^p x^q \partial_{pq} \mathcal{T}_0(0) + \theta_{\epsilon,R}(x) k_{N,\epsilon}^2 \\ &= \frac{1}{2} k_{N,\epsilon}^2 II_0(x,x) + \theta_{\epsilon,R}(x) k_{N,\epsilon}^2 \end{aligned} \tag{133}$$

where $\lim_{\epsilon \to 0} \sup_{B_R(0) \cap \{x_1 = 0\}} |\theta_{\epsilon,R}| = 0$ for any $R > 0$.

Step P4.*1:* Let $1 \le i \le N-1$. In Proposition 4 we have obtained the pointwise estimates, that for any $R, \rho > 0$ and all $\epsilon > 0$ we have for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{R k_{i,\epsilon}}(0)$,

$$|\nabla u_\epsilon(x)| \le C \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma) - \beta_-(\gamma)}{2}}}{|x|^{\beta_+(\gamma)}} + C \frac{1}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma) - \beta_-(\gamma)}{2}} |x|^{\beta_-(\gamma)}}.$$

For clearness, we write in this step

$$D_\epsilon := \mathcal{T}\left(\partial \mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{R k_{i,\epsilon}}(0)\right).$$

As $\epsilon \to 0$ we have that

$$\begin{aligned}
\left| \int_{D_\epsilon} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2} \, d\sigma \right| &\le C \int_{D_\epsilon} (x,\nu) \left[\frac{\mu_{i,\epsilon}^{\beta_+(\gamma) - \beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}} + \frac{1}{\mu_{i+1,\epsilon}^{\beta_+(\gamma) - \beta_-(\gamma)} |x|^{2\beta_-(\gamma)}} \right] d\sigma \\
&\le C \int_{D_\epsilon} |x|^2 \left[\frac{\mu_{i,\epsilon}^{\beta_+(\gamma) - \beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}} + \frac{1}{\mu_{i+1,\epsilon}^{\beta_+(\gamma) - \beta_-(\gamma)} |x|^{2\beta_-(\gamma)}} \right] d\sigma \\
&\le C \, \mu_{i,\epsilon} \int_{\partial \mathbb{R}^n_- \cap B_{\frac{\rho k_{i+1,\epsilon}}{k_{i,\epsilon}}}(0) \setminus \overline{B}_R(0)} \frac{d\sigma}{|x|^{(n-1) + (\beta_+(\gamma) - \beta_-(\gamma) - 1)}} \\
&\quad + C \, \mu_{i+1,\epsilon} \int_{\partial \mathbb{R}^n_- \cap B_\rho(0) \setminus \overline{B}_{\frac{R k_{i,\epsilon}}{k_{i+1,\epsilon}}}(0)} \frac{d\sigma}{|x|^{(n-1) - (\beta_+(\gamma) - \beta_-(\gamma) + 1)}} \\
&\le C \left(\mu_{i,\epsilon} R^{-(\beta_+(\gamma) - \beta_-(\gamma) - 1)} + \mu_{i+1,\epsilon} \rho^{\beta_+(\gamma) - \beta_-(\gamma) + 1} \right).
\end{aligned}$$

So then for all $1 \le i \le N-1$

$$\lim_{R \to +\infty} \lim_{\rho \to 0} \lim_{\epsilon \to 0} \left(\mu_{N,\epsilon}^{-1} \int_{\mathcal{T}\left(\partial \mathbb{R}^n_- \cap B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{R k_{i,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2} \, d\sigma \right) = 0. \tag{134}$$

This ends Step P4.1.

Step P4.2: Again from the estimates of Proposition 4, we have as $\epsilon \to 0$

$$\left| \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0) \setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \right|$$
$$\leq C \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0) \setminus B_{k^3_{1,\epsilon}}(0)\right)} \frac{|(x,\nu)|\, d\sigma}{\mu_{1,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} |x|^{2\beta_-(\gamma)}}$$
$$\leq C \int_{\partial\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0) \setminus B_{k^3_{1,\epsilon}}(0)} \frac{|x|^2\, d\sigma}{\mu_{1,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} |x|^{2\beta_-(\gamma)}}$$
$$\leq C\ k_{1,\epsilon} \int_{\partial\mathbb{R}^n_- \cap B_{\rho}(0) \setminus B_{k^2_{1,\epsilon}}(0)} \frac{d\sigma}{|x|^{2\beta_-(\gamma)-2}} \leq C\ \mu_{1,\epsilon}\ \rho^{\beta_+(\gamma)-\beta_-(\gamma)+1}.$$

Then, using again (110), we get that

$$\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\epsilon\to 0} \left(\mu_{N,\epsilon}^{-1} \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\rho k_{1,\epsilon}}(0) \setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \right) = 0. \tag{135}$$

This ends Step P4.2.

Step P4.3: With the pointwise estimates of Proposition 4, we obtain as $\epsilon \to 0$

$$\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{r_\epsilon}(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$
$$\leq C \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{r_\epsilon}(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0)\right)} |x|^2 \left[\frac{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}} + \frac{1}{|x|^{2\beta_-(\gamma)}} \right] d\sigma$$
$$\leq C\ k_{N,\epsilon} \int_{\partial\mathbb{R}^n_- \cap B_{\frac{r_\epsilon}{k_{N,\epsilon}}}(0) \setminus \overline{B}_R(0)} \frac{1}{|x|^{2\beta_+(\gamma)-2}}\, d\sigma$$
$$+ C\ r_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)+1} \int_{\partial\mathbb{R}^n_- \cap B_1(0) \setminus \overline{B}_{\frac{Rk_{N,\epsilon}}{r_\epsilon}}(0)} \frac{1}{|x|^{2\beta_-(\gamma)-2}}\, d\sigma$$
$$\leq C\ k_{N,\epsilon} \int_{\partial\mathbb{R}^n_- \cap B_{\frac{k_{N,\epsilon}}{k_{N-1,\epsilon}}}(0) \setminus \overline{B}_{R/2}(0)} \frac{1}{|x|^{(n-1)+(\beta_+(\gamma)-\beta_-(\gamma)-1)}}\, d\sigma,$$

and then

$$\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma$$
$$+ C\, r_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)+1} \int_{\partial\mathbb{R}^n_-\cap B_1(0)\setminus\overline{B}_{\frac{Rk_{N,\epsilon}}{2r_\epsilon}}(0)} \frac{1}{|x|^{(n-1)-(\beta_+(\gamma)-\beta_-(\gamma)+1)}}\, d\sigma$$
$$\leq C\, k_{N,\epsilon}\, \left(R^{-(\beta_+(\gamma)-\beta_-(\gamma)-1)} + r_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)-1}\right)\, d\sigma.$$

So if $\beta_+(\gamma)-\beta_-(\gamma)>1$

$$\lim_{R\to+\infty}\lim_{\rho\to 0}\lim_{\epsilon\to 0}\left(\mu_{N,\epsilon}^{-1}\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma\right) = 0. \tag{136}$$

This ends Step P4.3.

Step P4.4: Let $1\leq i\leq N$. When $\beta_+(\gamma)-\beta_-(\gamma)>1$, we have

$$\lim_{R\to+\infty}\lim_{\rho\to 0}\lim_{\epsilon\to 0}\left(\mu_{i,\epsilon}^{-1}\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{Rk_{i,\epsilon}}(0)\setminus\overline{B}_{\rho k_{i,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma\right)$$
$$= \frac{1}{2}\frac{1}{t_i^{\frac{n-1}{2^\star(s)-2}}}\int_{\partial\mathbb{R}^n_-} II_0(x,x)\frac{|\nabla\tilde{u}_i|^2}{2}\, d\sigma, \tag{137}$$

where $II_0(x,x)$ is the second fundamental form of the boundary $\partial\Omega$ at 0.

Proof of Step P4.4: Consider $\tilde{u}_i$ obtained in Proposition 2. It follows that for some constant $C>0$,

$$|\nabla\tilde{u}_i(x)|\leq\frac{C}{|x|^{\beta_-(\gamma)}+|x|^{\beta_+(\gamma)}}\qquad\text{for all } x\in\overline{\mathbb{R}^n_-}\setminus\{0\}.$$

So when $\beta_+(\gamma)-\beta_-(\gamma)>1$, the function $|x|^2|\nabla\tilde{u}_i|\in L^2(\mathbb{R}^{n-1})$.

With a change of variable and the definition of $\tilde{u}_{i,\epsilon}$ we then obtain

$$
\begin{aligned}
&\mu_{i,\epsilon}^{-1} \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\rho k_{i,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \\
&= \frac{k_{i,\epsilon}^{n-3}}{\mu_{i,\epsilon}^{n-1}} \int_{\partial\mathbb{R}^n_- \cap B_R(0)\setminus \overline{B}_\rho(0)} (\mathcal{T}(k_{N,\epsilon}x), \nu \ \circ T(k_{N,\epsilon}x))\frac{|\nabla \tilde{u}_{i,\epsilon}|^2}{2}\, d\sigma \\
&= -\frac{k_{i,\epsilon}^{n-3}}{\mu_{i,\epsilon}^{n-1}} \left(\int_{\partial\mathbb{R}^n_- \cap B_R(0)\setminus \overline{B}_\rho(0)} \frac{1}{2} k_{N,\epsilon}^2 \sum_{p,q=2}^n \partial_{pq}\mathcal{T}_0(0)x^p x^q \frac{|\nabla \tilde{u}_{i,\epsilon}|^2}{2}\, d\sigma + \theta_{\epsilon,R}(x) k_{N,\epsilon}^2 \right) \\
&= -\left(\frac{k_{i,\epsilon}}{\mu_{i,\epsilon}}\right)^{n-1} \left(\int_{\partial\mathbb{R}^n_- \cap B_R(0)\setminus \overline{B}_\rho(0)} \frac{1}{2} \sum_{p,q=2}^n \partial_{pq}\mathcal{T}_0(0)x^p x^q \frac{|\nabla \tilde{u}_{i,\epsilon}|^2}{2}\, d\sigma + \theta_{\epsilon,R}(x) \right).
\end{aligned}
$$

Since $|x|^2|\nabla \tilde{u}_i| \in L^2(\mathbb{R}^{n-1})$, passing to the limits it follows from the expression of the second fundamental form in (133), that

$$
\begin{aligned}
&\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\epsilon\to 0} \left(\mu_{i,\epsilon}^{-1} \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\rho k_{i,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \right) \\
&= -\frac{1}{2}\frac{1}{t^{\frac{n-1}{2^\star(s)-2}}} \int_{\partial\mathbb{R}^n_- \cap B_R(0)\setminus \overline{B}_\rho(0)} \sum_{p,q=2}^n \partial_{pq}\mathcal{T}_0(0)x^p x^q \frac{|\nabla \tilde{u}_i|^2}{2}\, d\sigma \\
&= \frac{1}{2}\frac{1}{t_i^{\frac{n-1}{2^\star(s)-2}}} \int_{\partial\mathbb{R}^n_-} II_0(x,x)\frac{|\nabla \tilde{u}_i|^2}{2}\, d\sigma.
\end{aligned}
$$

This ends Step P4.4.

Plugging (136), (137), (134) and (135) in the integral (129), we get (127). This proves the first identity of Step P4.

Step P4.5*:* We now assume that $u_0 \equiv 0$ and $\beta_+(\gamma) - \beta_-(\gamma) > 1$. We prove (128). We write

$$
\begin{aligned}
\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^3}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma = &\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_\delta(0)\setminus \overline{B}_{r_\epsilon}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \\
&+ \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap \overline{B}_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^3}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma
\end{aligned}
\tag{138}
$$

With the pointwise estimates of Proposition 4 with $u_0 \equiv 0$, and using that $\beta_+(\gamma) - \beta_-(\gamma) > 1$, we obtain as $\epsilon \to 0$

$$\begin{aligned} &\left| \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\, d\sigma \right| \\ &\leq C \int_{\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)} |x|^2 \left[\frac{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}} \right] d\sigma \\ &\leq C \frac{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{r_\epsilon^{2\beta_+(\gamma)-2-n+1}} \leq C\mu_{N,\epsilon}^{1+\frac{\beta_+(\gamma)-\beta_-(\gamma)-1}{2}} = o(\mu_{N,\epsilon}), \end{aligned}$$

since $\beta_+(\gamma) - \beta_-(\gamma) > 1$. Then, with (127), we get (128). This ends Step P4.5.

These five substeps prove Step P4. □

7.3. Estimates of the boundary terms.

STEP P5. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We fix a chart $\mathcal{T}$ as in (27) and, for any $\epsilon > 0$, we define*

$$\tilde{v}_\epsilon(x) := r_\epsilon^{\beta_-(\gamma)-1} u_\epsilon(\mathcal{T}(r_\epsilon x)) \qquad \text{for } x \in r_\epsilon^{-1} U \cap \overline{\mathbb{R}^n_-} \setminus \{0\},$$

where $r_\epsilon := \sqrt{\mu_{N,\epsilon}}$. We claim that there exists $\tilde{v} \in C^1(\overline{\mathbb{R}^n_-} \setminus \{0\})$ such that

$$\lim_{\epsilon\to 0} \tilde{v}_\epsilon(x) = \tilde{v} \quad \text{in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$$

where $\tilde{v}$ is a solution of

$$\begin{cases} -\Delta\tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} = 0 & \text{in } \mathbb{R}^n_- \\ \tilde{v} = 0 & \text{on } \partial\mathbb{R}^n_- \setminus \{0\}. \end{cases} \tag{139}$$

Proof of Step P5: For any $i,j = 1,...,n$, we let $(\tilde{g}_\epsilon)_{ij} = (\mathcal{T}^* Eucl)(r_\epsilon x)_{ij} = (\partial_i \mathcal{T}(r_\epsilon x), \partial_j \mathcal{T}(r_\epsilon x))$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$. We consider $\tilde{g}_\epsilon$ as a metric on $\mathbb{R}^n$. In the sequel, we let $\Delta_g = div_g(\nabla)$ be the Laplace-Beltrami operator with respect to a metric g. From (E_ϵ) it follows that for all $\epsilon > 0$, the rescaled functions $\tilde{v}_\epsilon$ weakly satisfies the equation

$$-\Delta_{\tilde{g}_\epsilon}\tilde{v}_\epsilon - \frac{\gamma}{\left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right|^2}\tilde{v}_\epsilon - r_\epsilon^2\ h_\epsilon \circ \mathcal{T}(r_\epsilon x)\ \tilde{v}_\epsilon = r_\epsilon^{\theta + p_\epsilon \beta_-(\gamma)} \frac{|\tilde{v}_\epsilon|^{2^\star(s)-2-p_\epsilon}\tilde{v}_\epsilon}{\left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right|^s}. \tag{140}$$

with $\theta := (2^\star(s) - 2)\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} > 0$ and $\tilde{v}_\epsilon \equiv 0$ on $\partial\mathbb{R}^n_- \setminus \{0\}$.

Using the pointwise estimates (60) we obtain the bound, that as $\epsilon \to 0$ we have for $x \in \mathbb{R}^n_-$

$$
\begin{aligned}
|\tilde{v}_\epsilon(x)| &\leq C\ r_\epsilon^{\beta_-(\gamma)-1} \sum_{i=1}^{N} \frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |\mathcal{T}(r_\epsilon x)|}{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} |\mathcal{T}(r_\epsilon x)|^{\beta_-(\gamma)} + |\mathcal{T}(r_\epsilon x)|^{\beta_+(\gamma)}} \\
&\quad + C\ r_\epsilon^{\beta_-(\gamma)-1} \frac{\||x|^{\beta_-(\gamma)-1} u_0\|_{L^\infty(\Omega)}}{|\mathcal{T}(r_\epsilon x)|^{\beta_-(\gamma)}} |\mathcal{T}(r_\epsilon x)| \\
&\leq C \sum_{i=1}^{N} \frac{\left(\frac{\mu_{i,\epsilon}}{\mu_{N,\epsilon}}\right)^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} \left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right|}{\left(\frac{\mu_{i,\epsilon}}{\sqrt{\mu_{N,\epsilon}}}\right)^{\beta_+(\gamma)-\beta_-(\gamma)} \left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right|^{\beta_-(\gamma)} + \left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right|^{\beta_+(\gamma)}} \\
&\quad + C \frac{\||x|^{\beta_-(\gamma)-1} u_0\|_{L^\infty(\Omega)}}{\left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right|^{\beta_-(\gamma)}} \left|\frac{\mathcal{T}(r_\epsilon x)}{r_\epsilon}\right| \\
&\leq C \left(\sum_{i=1}^{N} \frac{\left(\frac{\mu_{i,\epsilon}}{\mu_{N,\epsilon}}\right)^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |x|}{\left(\frac{\mu_{i,\epsilon}}{\sqrt{\mu_{N,\epsilon}}}\right)^{\beta_+(\gamma)-\beta_-(\gamma)} |x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} \right. \\
&\quad \left. + \frac{\||x|^{\beta_-(\gamma)-1} u_0\|_{L^\infty(\Omega)} +}{|x|^{\beta_-(\gamma)}} |x| \right) \\
&\leq C \left(\frac{1}{|x|^{\beta_+(\gamma)-1}} + \frac{\||x|^{\beta_-(\gamma)-1} u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)-1}} \right).
\end{aligned}
$$

Then passing to limits in the equation (140), standard elliptic theory yields the existence of $\tilde{v} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0\})$ such that $\tilde{v}_\epsilon \to \tilde{v}$ in $C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$ and $\tilde{v}$ satisfies the equation:

$$
\begin{cases}
-\Delta \tilde{v} - \frac{\gamma}{|x|^2} \tilde{v} = 0 & \text{in } \mathbb{R}^n_- \\
\tilde{v} = 0 & \text{on } \partial \mathbb{R}^n_- \setminus \{0\}.
\end{cases}
$$

and we have the following bound on $\tilde{v}$

$$
|\tilde{v}(x)| \leq C \left(\frac{|x_1|}{|x|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)-1} u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}} |x_1| \right) \qquad \text{for all } x = (x_1, \tilde{x}) \text{ in } \mathbb{R}^n_-.
$$

This ends the proof of Step P5. ☐

Step P6. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We claim that, as $\epsilon \to 0$,*

$$
\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{r_\epsilon}(0)\right)} F_\epsilon(x)\, d\sigma = \mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} (\mathcal{F}_0 + o(1)) \tag{141}
$$

with

$$
\mathcal{F}_0 := \int_{\mathbb{R}^n_- \cap \partial B_1(0)} (x, \nu) \left(\frac{|\nabla \tilde{v}|^2}{2} - \frac{\gamma}{2} \frac{\tilde{v}^2}{|x|^2} \right) - \left(x^i \partial_i \tilde{v} + \frac{n-2}{2} \tilde{v} \right) \partial_\nu \tilde{v}\, d\sigma \tag{142}
$$

and

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)\right)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\right). \tag{143}$$

Proof of Step P6: We keep the notations of Step P5. With a change of variable and the definition of $\tilde{v}_\epsilon$, and $\theta := (2^\star(s)-2)\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} > 0$, we get

$$\int_{\mathcal{T}(\mathbb{R}^n_-\cap\partial B_{r_\epsilon}(0))} F_\epsilon(x)\, d\sigma =$$

$$r_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)} \int_{\mathbb{R}^n_-\cap\partial B_1(0)} (x,\nu)_{\tilde{g}_\epsilon}\left(\frac{|\nabla_{\tilde{g}_\epsilon}\tilde{v}_\epsilon|^2}{2} - \frac{\gamma}{2}\frac{\tilde{v}_\epsilon^2}{|x|^2_{\tilde{g}_\epsilon}}\right) - \left(x^i\partial_i\tilde{v}_\epsilon + \frac{n-2}{2}\tilde{v}_\epsilon\right)\partial_\nu\tilde{v}_\epsilon\, d\sigma_{\tilde{g}_\epsilon}$$

$$- r_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)} \int_{\mathbb{R}^n_-\cap\partial B_1(0)} \left(r_\epsilon^2\frac{h_\epsilon(\mathcal{T}(r_\epsilon x))}{2}\tilde{v}_\epsilon^2 - \frac{r_\epsilon^{\theta+(\beta_-(\gamma)-1)p_\epsilon}}{2^\star(s)-p_\epsilon}\frac{|\tilde{v}_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s_{\tilde{g}_\epsilon}}\right) d\sigma_{\tilde{g}_\epsilon}.$$

From the convergence result of Step P5, we then get (141).

For the next boundary term, from the estimates (60) and (102) we obtain

$$\begin{aligned}
&\left|\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)\right)} F_\epsilon(x)\, d\sigma\right| \\
&\leq \frac{C}{\mu_{1,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}} \int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)\right)} |x|\left(\frac{1}{|x|^{2\beta_-(\gamma)}} + \frac{|x|^2}{|x|^{2\beta_-(\gamma)}}\right) dx \\
&+C \int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)\right)} |x|\frac{\mu_{1,\epsilon}^{-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)+p_\epsilon\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)}}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}}\, dx \\
&\leq \frac{C}{\mu_{1,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}} \int_{\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)} |x|\left(\frac{1}{|x|^{2\beta_-(\gamma)}} + \frac{|x|^2}{|x|^{2\beta_-(\gamma)}}\right) dx \\
&+C \int_{\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)} |x|\frac{\mu_{1,\epsilon}^{-2^\star(s)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)+p_\epsilon\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)}}{|x|^{(\beta_-(\gamma)-1)(2^\star(s)-p_\epsilon)+s}}\, dx \\
&\leq C\mu_{1,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\left(\mu_{1,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} + \mu_{1,\epsilon}^{(\beta_+(\gamma)-\beta_-(\gamma))\left(\frac{2-s}{n-2}\right)+p_\epsilon\left(\frac{n-2}{2}\right)}\right).
\end{aligned}$$

And so

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^2_{1,\epsilon}}(0)\right)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\right). \tag{144}$$

This ends Step P6. □

Step P7. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $u_0 \equiv 0$. We define*

$$\bar{u}_\epsilon := \frac{u_\epsilon}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}. \tag{145}$$

We claim that there exists $\bar{u} \in C^2(\overline{\Omega}\setminus\{0\})$ such that

$$\lim_{\epsilon\to 0}\bar{u}_\epsilon = \bar{u} \text{ in } C^2_{loc}(\overline{\Omega}\setminus\{0\}) \text{ with } \left\{\begin{array}{ll} -\Delta\bar{u} - \left(\frac{\gamma}{|x|^2}+h_0\right)\bar{u} = 0 & \text{in } \Omega \\ \bar{u} = 0 & \text{in } \partial\Omega\setminus\{0\} \end{array}\right. \tag{146}$$

Proof of Step P7: Since $u_0 \equiv 0$, it follows from (60) that there exists $C > 0$ such that

$$|\bar{u}_\epsilon(x)| \leq C|x|^{1-\beta_+(\gamma)} \text{ for all } x \in \Omega \text{ and } \epsilon > 0. \tag{147}$$

Moreover, equation (E_ϵ) rewrites

$$-\Delta\bar{u}_\epsilon - \left(\frac{\gamma}{|x|^2}+h_\epsilon\right)\bar{u}_\epsilon = \mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-2-p_\epsilon)}\frac{|\bar{u}_\epsilon|^{2^\star(s)-2-p_\epsilon}\bar{u}_\epsilon}{|x|^s} \text{ in } \Omega,$$

and $\bar{u}_\epsilon = 0$ on $\partial\Omega$. It then follows from standard elliptic theory that the claim holds. This ends Step P7. □

Step P8. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $u_0 \equiv 0$. We claim that*

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{\delta_0}(0)\right)} F_\epsilon(x)\, d\sigma = \left(\mathcal{F}_{\delta_0}+o(1)\right)\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}, \tag{148}$$

and

$$\int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{k^3_{1,\epsilon}}(0)\right)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\right), \tag{149}$$

where

$$\mathcal{F}_{\delta_0} := \int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{\delta_0}(0)\right)} (x,\nu)\left(\frac{|\nabla\bar{u}|^2}{2} - \left(\frac{\gamma}{|x|^2}+h_0\right)\frac{\bar{u}^2}{2}\right) - \left(x^i\partial_i\bar{u} + \frac{n-2}{2}\bar{u}\right)\partial_\nu\bar{u}\, d\sigma. \tag{150}$$

Proof of Step P8: The second term has already been estimated in (143). We are left with the first term. With a change of variable, the definition of $\bar{u}_\epsilon$ and the convergence (146), we get

$$\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{\delta_0}(0)\right)} F_\epsilon(x)\, d\sigma \tag{151}$$

$$= \mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{\delta_0}(0)\right)} (x,\nu)\left(\frac{|\nabla \bar{u}_\epsilon|^2}{2} - \left(\frac{\gamma}{|x|^2} + h_\epsilon\right)\frac{\bar{u}_\epsilon^2}{2}\right) d\sigma$$

$$-\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-2-\epsilon)} \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{\delta_0}(0)\right)} \frac{|\bar{u}_\epsilon|^{2^\star(s)-2-\epsilon}\bar{u}_\epsilon}{|x|^2}\, d\sigma$$

$$- \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{\delta_0}(0)\right)} \left(x^i\partial_i \bar{u}_\epsilon + \frac{n-2}{2}\bar{u}_\epsilon\right)\partial_\nu \bar{u}_\epsilon\, d\sigma$$

$$= \mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\left(\mathcal{F}_{\delta_0} + o(1)\right). \tag{152}$$

where $\mathcal{F}_{\delta_0}$ is as above. Arguing as in the proof of (144), we get that

$$\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap \partial B_{k^3_{1,\epsilon}}(0)\right)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\right) \text{ as } \epsilon \to 0. \tag{153}$$

This ends Step P8. □

STEP P9. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $u_\epsilon > 0$ for all $\epsilon > 0$. Then $\mathcal{F}_0 \geq 0$ and*

$$\mathcal{F}_0 > 0 \iff u_0 > 0.$$

where $\mathcal{F}_0$ is as in (142).

Proof of Step P9: We let $\tilde{v}$ be defined as in Step P5. It follows from Step P5 that $\tilde{v}$ satisfies (139) and we have the following bound on $\tilde{v}$

$$|\tilde{v}(x)| \leq C\left(\frac{|x_1|}{|x|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)-1}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}}|x_1|\right) \qquad \text{for all } x = (x_1, \tilde{x}) \text{ in } \mathbb{R}^n_-. \tag{154}$$

Given $\alpha \in \mathbb{R}$, we define $v_\alpha(x) := x_1|x|^{-\alpha}$ for all $x \in \mathbb{R}^n_-$. Since $\tilde{v} \geq 0$, it follows from Proposition 6.4 in Ghoussoub-Robert [21] that there exists $A, B \geq 0$ such that

$$\tilde{v} := Av_{\beta_+(\gamma)} + Bv_{\beta_-(\gamma)}. \tag{155}$$

Step P9.1: We claim that $B = 0$ when $u_0 \equiv 0$.

This is a direct consequence of controling (155) with (154) when $u_0 \equiv 0$ and letting $|x| \to \infty$.

Step P9.2: We claim that $B > 0$ when $u_0 > 0$.

We prove the claim. We fix $x \in \mathbb{R}^n_-$. Green's representation formula yields

$$\tilde{v}_\epsilon(x) \quad = \quad \int_\Omega r_\epsilon^{\beta_-(\gamma)-1} G_\epsilon(\mathcal{T}(r_\epsilon x), y)\frac{u_\epsilon^{2^\star(s)-1}(y)}{|y|^s}\, dy.$$

We fix $\omega \subset\subset \Omega$. Then there exists $c(\omega) > 0$ such that $|y| \geq d(y, \partial\Omega) \geq c(\omega)$ for all $y \in \omega$. Moreover, the control (207) of the Green's function yields

$$\tilde{v}_\epsilon(x) \quad \geq \quad c \int_\omega r_\epsilon^{\beta_-(\gamma)-1} \frac{r_\epsilon x_1}{r_\epsilon^{\beta_-(\gamma)}|x|^{\beta_-(\gamma)}} |c(\omega) - r_\epsilon|x||^{-n} \frac{u_\epsilon^{2^\star(s)-1}(y)}{|y|^s}\, dy,$$

and then, passing to the limit $\epsilon \to 0$, we get that

$$\tilde{v}(x) \quad \geq \quad \frac{cx_1}{|x|^{\beta_-(\gamma)}} \int_\omega \frac{u_0^{2^\star(s)-1}(y)}{|y|^s}\, dy,$$

for all $x \in \mathbb{R}^n_-$. As one checks, this yields $B \geq c\int_\omega \frac{u_0^{2^\star(s)-1}(y)}{|y|^s}\, dy > 0$ when $u_0 > 0$. This ends Step P9.2.

Step P9.3: We claim that $A > 0$.

The proof is similar to Step P9.2. We fix $x \in \mathbb{R}^n_-$ and $\omega \subset\subset \mathbb{R}^n_-$. Green's representation formula and the pointwise control (207) yield

$$\begin{aligned}
\tilde{v}_\epsilon(x) \quad &\geq \quad \int_{\mathcal{T}(\mu_{N,\epsilon}\omega)} r_\epsilon^{\beta_-(\gamma)-1} G_\epsilon(\mathcal{T}(r_\epsilon x), y) \frac{u_\epsilon^{2^\star(s)-1}(y)}{|y|^s}\, dy \\
&\geq \quad \int_\omega r_\epsilon^{\beta_-(\gamma)-1} G_\epsilon(\mathcal{T}(r_\epsilon x), \mathcal{T}(\mu_{N,\epsilon} y))\mu_{N,\epsilon}^n \frac{u_\epsilon(\mathcal{T}(\mu_{N,\epsilon} y))^{2^\star(s)-1}}{|\mu_{N,\epsilon} y|^s}\, dy \\
&\geq \quad \int_\omega r_\epsilon^{\beta_-(\gamma)-1} \left(\frac{r_\epsilon|x|}{\mu_{N,\epsilon}|y|}\right)^{\beta_-(\gamma)} K_\epsilon(x,y)\mu_{N,\epsilon}^{\frac{n-2}{2}} \frac{\tilde{u}_{i,\epsilon}(y)^{2^\star(s)-1}}{|y|^s}\, dy \\
&\geq \quad \int_\omega r_\epsilon^{2\beta_-(\gamma)-n}|x|^{\beta_-(\gamma)} \left| x - \frac{\mu_{N,\epsilon}}{r_\epsilon} y\right|^{-n} x_1 y_1 \mu_{N,\epsilon}^{\frac{n}{2}-\beta_-(\gamma)} \frac{\tilde{u}_{i,\epsilon}(y)^{2^\star(s)-1}}{|y|^s}\, dy
\end{aligned}$$

with

$$K_\epsilon(x,y) = |r_\epsilon x - \mu_{N,\epsilon} y|^{2-n} \min\left\{1, \frac{\mu_{N,\epsilon}}{r_\epsilon} \frac{x_1 y_1}{|x - \frac{\mu_{N,\epsilon}}{r_\epsilon} y|^2}\right\}$$

Since $r_\epsilon := \sqrt{\mu_{N,\epsilon}}$, letting $\epsilon \to 0$, we get with the convergence (A4) of Proposition 2 that

$$\begin{aligned}
\tilde{v}_\epsilon(x) \quad &\geq \quad \int_\omega r_\epsilon^{2\beta_-(\gamma)-n}|x|^{\beta_-(\gamma)} \left| x - \frac{\mu_{N,\epsilon}}{r_\epsilon} y\right|^{-n} x_1 y_1 \mu_{N,\epsilon}^{\frac{n}{2}-\beta_-(\gamma)} \frac{\tilde{u}_{i,\epsilon}(y)^{2^\star(s)-1}}{|y|^s}\, dy \\
&\geq \quad \frac{x_1}{|x|^{\beta_+(\gamma)}} \int_\omega \frac{\tilde{u}_i(y)^{2^\star(s)-1}}{|y|^s}\, dy
\end{aligned}$$

for all $x \in \mathbb{R}^n_-$. Therefore, as one checks, $A \geq \int_\omega \frac{\tilde{u}_i(y)^{2^\star(s)-1}}{|y|^s}\, dy > 0$. This ends Step P9.3.

Step P9.4: We claim that

$$\mathcal{F}_0 = \frac{\omega_{n-1}}{n}\left(\frac{n^2}{4} - \gamma\right) \cdot AB. \tag{156}$$

We prove the claim. The definition (142) reads

$$\mathcal{F}_0 := \int\limits_{\mathbb{R}^n_- \cap \partial B_1(0)} (x,\nu)\left(\frac{|\nabla \tilde{v}|^2}{2} - \frac{\gamma}{2}\frac{\tilde{v}^2}{|x|^2}\right) - \left(x^i \partial_i \tilde{v} + \frac{n-2}{2}\tilde{v}\right)\partial_\nu \tilde{v}\, d\sigma \tag{157}$$

For simplicity, we define the bilinear form

$$\mathcal{H}_\delta(u,v) = \int_{\mathbb{R}^n_-\cap\partial B_\delta(0)} \left[(x,\nu)\left((\nabla u,\nabla v)-\gamma\frac{uv}{|x|^2}\right)-\left(x^i\partial_i u+\frac{n-2}{2}u\right)\partial_\nu v -\left(x^i\partial_i v+\frac{n-2}{2}v\right)\partial_\nu u\right]\,d\sigma$$

As one checks,

$$\begin{aligned}\mathcal{F}_0 &= \frac{1}{2}\mathcal{H}_1(Av_{\beta_+(\gamma)}+Bv_{\beta_-(\gamma)},Av_{\beta_+(\gamma)}+Bv_{\beta_-(\gamma)})\\ &= \frac{A^2}{2}\mathcal{H}_1(v_{\beta_+(\gamma)},v_{\beta_+(\gamma)})+AB\mathcal{H}_1(v_{\beta_+(\gamma)},v_{\beta_-(\gamma)})\\ &\quad+\frac{B^2}{2}\mathcal{H}_1(v_{\beta_-(\gamma)},v_{\beta_-(\gamma)})\end{aligned}$$

In full generality, we compute $\mathcal{H}_\delta(v_\alpha,v_\beta)$ for all $\alpha,\beta\in\mathbb{R}$ and all $\delta>0$. As one checks, for any $i=1,...,n$, we have that $\partial_i v_\alpha=\left(\delta_{i,1}-\alpha\frac{x_1x_i}{|x|^2}\right)|x|^{-\alpha}$ for all $x\in\mathbb{R}^n_-$. Moreover, for $x\in\partial B_\delta(0)$, we have that $\partial_\nu v_\alpha=\frac{x^i}{|x|}\partial_i v_\alpha$. Consequently, straightforward computations yield

$$\left(x^i\partial_i v_\alpha+\frac{n-2}{2}v_\alpha\right)\partial_\nu v_\beta=-(\beta-1)\left(\frac{n}{2}-\alpha\right)\frac{v_\alpha v_\beta}{|x|}$$

and

$$(x,\nu)\left((\nabla v_\alpha,\nabla v_\beta)-\frac{\gamma}{|x|^2}v_\alpha v_\beta\right)=|x|^{1-\alpha-\beta}+(\alpha\beta-\alpha-\beta-\gamma)\frac{v_\alpha v_\beta}{|x|}$$

and then

$$\mathcal{H}_\delta(v_\alpha,v_\beta) = \int_{\mathbb{R}^n_-\cap\partial B_\delta(0)}\left(|x|^{1-\alpha-\beta}+\left(\frac{n}{2}(\alpha+\beta)-n-\alpha\beta-\gamma\right)\frac{v_\alpha v_\beta}{|x|}\right)\,d\sigma$$

We have that

$$\int_{\mathbb{R}^n_-\cap\partial B_\delta(0)}|x|^{1-\alpha-\beta}\,d\sigma=\frac{1}{2}\int_{B_\delta(0)}|x|^{1-\alpha-\beta}\,d\sigma=\frac{\omega_{n-1}}{2}\delta^{n-\alpha-\beta}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n_-\cap\partial B_\delta(0)}\frac{v_\alpha v_\beta}{|x|}\,d\sigma &= \frac{1}{2}\int_{B_\delta(0)}x_1^2|x|^{-\alpha-\beta-1}\,d\sigma\\ &= \frac{1}{2n}\int_{B_\delta(0)}|x|^{-\alpha-\beta+1}\,d\sigma=\frac{\omega_{n-1}}{2n}\delta^{n-\alpha-\beta}\end{aligned}$$

Plugging all these identities together yields

$$\mathcal{H}_\delta(v_\alpha,v_\beta)=\frac{\omega_{n-1}}{2n}\delta^{n-\alpha-\beta}\left(\frac{n}{2}(\alpha+\beta)-\alpha\beta-\gamma\right).$$

Since $\beta_+(\gamma),\beta_-(\gamma)$ are solutions to $X^2-nX+\gamma=0$, we get that

$$\mathcal{H}_\delta(v_{\beta_-(\gamma)},v_{\beta_-(\gamma)})=\mathcal{H}_\delta(v_{\beta_+(\gamma)},v_{\beta_+(\gamma)})=0.$$

Since $\beta_+(\gamma)+\beta_-(\gamma)=n$ and $\beta_+(\gamma)\beta_-(\gamma)=\gamma$, we get that

$$\mathcal{H}_\delta(v_{\beta_-(\gamma)},v_{\beta_+(\gamma)})=\frac{\omega_{n-1}}{n}\left(\frac{n^2}{4}-\gamma\right).$$

Plugging all these results together yields (156). This ends Step P9.4.

These substeps end the proof of Step P9. ☐

STEP P10. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $\beta_+(\gamma)-\beta_-(\gamma)<2$ and $u_\epsilon>0$ for all $\epsilon>0$. Then $u_0\equiv 0$.*

Proof of Step P10: We claim that, as $\epsilon\to 0$,

$$\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}(x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma=o\left(\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}\right)\text{ when }\beta_+(\gamma)-\beta_-(\gamma)<2 \tag{158}$$

Indeed, if $\beta_+(\gamma)-\beta_-(\gamma)>1$, the claim follows from (127) and $1>\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}$. If now $\beta_+(\gamma)-\beta_-(\gamma)<1$, then (131) and the control (102) yield that

$$\begin{aligned}&\left|\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^2_{1,\epsilon}}(0)\right)}(x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma\right|\\&\le C\int_{\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)}|x|^2\left(\sum_{i=1}^N\frac{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}}\,dx+\frac{dx}{|x|^{2\beta_-(\gamma)}}\right)\,d\sigma\\&\le C\sum_{i=1}^N\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}r_\epsilon^{n-1-2(\beta_+(\gamma)-1)}+Cr_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)+1}=o(\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}})\end{aligned}$$

as $\epsilon\to 0$. The limit case $\beta_+(\gamma)-\beta_-(\gamma)=1$ is similar. This proves the claim.

Plugging (115), (121), (141), (143) and (158) into the Pohozaev identity (112), we get

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^N\frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)=-\left(\mathcal{F}_0+o(1)\right)\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} \tag{159}$$

as $\epsilon\to 0$, where $\mathcal{F}_0$ is as in (157). Therefore $\mathcal{F}_0\le 0$. Since $u_\epsilon>0$, it then follows from (156) of Step P9 that $u_0\equiv 0$. This proves Step P10. ☐

8. Proof of the sharp blow-up rates

We now prove the sharp blow-up rates claimed in Propositions 5 and 6. We start with the case when $\beta_+(\gamma)-\beta_-(\gamma)\neq 1$. As a preliminary estimate, we claim that

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p_\epsilon}\right)\left(\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)$$
$$=\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{r_\epsilon}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}(x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma-(\mathcal{F}_0+o(1))\,\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} \tag{160}$$

as $\epsilon\to 0$, where $\mathcal{F}_0$ is as in (142); and, when $u_0\equiv 0$, we claim that

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p_\epsilon}\right)\left(\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)$$
$$=\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)}(x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma-(\mathcal{F}_{\delta_0}+o(1))\,\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}$$
$$+\underbrace{o(\mu_{N,\epsilon})}_{\text{when }\beta_+(\gamma)-\beta_-(\gamma)\geq 2}+\underbrace{O(\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)})}_{\text{when }\beta_+(\gamma)-\beta_-(\gamma)<2}, \tag{161}$$

where $\mathcal{F}_{\delta_0}$ is as in (150).

We prove the claim. Collecting the first estimate of Step P2, (121), (141) and (143) of the terms of the Pohozaev identity (112) gives (160). Similarly, the second estimate of Step P2, (122), (148) and (149) of the terms of the Pohozaev identity (113) gives (161).

8.1. Proof of the sharp blow-up rates when $\beta_+(\gamma)-\beta_-(\gamma)\neq 1$. We first assume $u_\epsilon>0$ and $\beta_+(\gamma)-\beta_-(\gamma)<1$.

STEP P11. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $u_\epsilon>0$ and $\beta_+(\gamma)-\beta_-(\gamma)<1$. Then (109) holds, that is*

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}=-\frac{\frac{\omega_{n-1}2^\star(s)^2}{n}\left(\frac{n^2}{4}-\gamma\right)A^2}{(n-s)\sum_{i=1}^N\frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx}\cdot m_{\gamma,h}(\Omega) \tag{162}$$

for some $A>0$, where $m_{\gamma,h}(\Omega)$ is the boundary mass.

Proof of Step P11: It follows from Step P10 that $u_0\equiv 0$.

Step P11:1: We now claim that

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)=\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}(M_{\delta_0}+o(1))$$

where

$$\begin{aligned} M_{\delta_0} \quad := \quad & -\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\delta_0}(0)\right)} \left(h_0(x) + \frac{(\nabla h_0, x)}{2}\right) \bar{u}^2 \, dx - \mathcal{F}_{\delta_0} \\ & + \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\right)} (x,\nu) \frac{|\nabla \bar{u}|^2}{2} \, d\sigma, \end{aligned} \tag{163}$$

and $\mathcal{F}_{\delta_0}$ is as in (150) and $\bar{u}$ is as in (146).

Indeed, the Pohozaev identity (112), the convergence (145), (147), (146) and $\beta_+(\gamma) - \beta_-(\gamma) < 1$ yield

$$\begin{aligned} & \int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\delta_0}(0) \setminus B_{k^3_{1,\epsilon}}(0)\right)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right) u_\epsilon^2 \, dx \\ & = \mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} \left(\int_{\mathcal{T}\left(\mathbb{R}^n_- \cap B_{\delta_0}(0)\right)} \left(h_0(x) + \frac{(\nabla h_0, x)}{2}\right) \bar{u}^2 \, dx + o(1) \right) \end{aligned} \tag{164}$$

With $u_0 \equiv 0$ and the control (102), we get that $|\nabla u_\epsilon(x)| \leq C\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}} |x|^{-\beta_+(\gamma)}$ for all $\epsilon > 0$ and $x \in \Omega$. Therefore, with (145) and (146), we get that

$$\begin{aligned} & \int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0) \setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu) \frac{|\nabla u_\epsilon|^2}{2} \, d\sigma \\ & = \mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)} \left(\int_{\mathcal{T}\left(\partial\mathbb{R}^n_- \cap B_{\delta_0}(0)\right)} (x,\nu) \frac{|\nabla \bar{u}|^2}{2} \, d\sigma + o(1) \right) \end{aligned} \tag{165}$$

as $\epsilon \to 0$. Plugging (143), (164) and (165) into (113), we get (163). This proves the claim and ends Step P11.1.

We fix $\delta < \delta'$. Taking $U := \mathcal{T}(\mathbb{R}^n_- \cap B_{\delta'}(0) \setminus B_\delta(0))$, $K = 0$ and $u = \bar{u}$ in (196), and using (146), we get that M_δ is independent of the choice of $\delta > 0$ small enough.

Step P11*.2:* We claim that $\bar{u} > 0$.

We prove the claim. Since $\bar{u} \geq 0$ is a solution to (146), it is enough to prove that $\bar{u} \not\equiv 0$. We argue as in the proof of Step P9. We fix $x \in \Omega$. Green's identity and$u_\epsilon > 0$ yield

$$\begin{aligned} \overline{u}_\epsilon(x) &= \mu_{N,\epsilon}^{-(\beta_+(\gamma)-\beta_-(\gamma))/2} \int_\Omega G_\epsilon(x,y) \frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s} \, dy \\ &\geq \mu_{N,\epsilon}^{-(\beta_+(\gamma)-\beta_-(\gamma))/2} \int_{A_\epsilon} G_\epsilon(x,y) \frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s} \, dy \\ &\geq C\mu_{N,\epsilon}^{n-s-(\beta_+(\gamma)-\beta_-(\gamma))/2} \int_A G_\epsilon(x, \mathcal{T}(\mu_{N,\epsilon} y)) \frac{u_\epsilon(\mathcal{T}(\mu_{N,\epsilon} y))^{2^\star(s)-1-p_\epsilon}}{|y|^s} \, dy, \end{aligned}$$

where $A_\epsilon := \mathcal{T}(\mathbb{R}^n_- \cap B_{2\mu_{N,\epsilon}}(0) \setminus B_{\mu_{N,\epsilon}}(0))$, $A := \mathbb{R}^n_- \cap B_2(0) \setminus B_1(0)$. With the pointwise control (207), we get

$$\begin{aligned} & \overline{u}_\epsilon(x) \geq \\ & C \int_A \left(\frac{|x|}{|y|}\right)^{\beta_-(\gamma)} |x - \mathcal{T}(\mu_{N,\epsilon} y)|^{2-n} \left(\frac{d(x,\partial\Omega)|y_1|}{|x - \mathcal{T}(\mu_{N,\epsilon} y)|^2}\right) \frac{u_{\epsilon,i}(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s} \, dy \end{aligned}$$

where $u_{\epsilon,i}$ is as in Proposition 2. Letting $\epsilon \to 0$ and using the convergence (A4) of Proposition 2, we get that

$$\bar{u}(x) \geq C\frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}} \text{ for all } x \in \Omega.$$

And then $\bar{u} > 0$ in Ω. This proves the claim and Step P11.2.

We fix $r_0 > 0$ and $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ in $B_{r_0}(0)$ and $\eta(x) = 0$ in $\mathbb{R}^n \setminus B_{2r_0}(0)$. It then follows from [21, 22] that, for $r_0 > 0$ small enough, there exists $A > 0$ and $\beta \in H_0^1(\Omega)$ such that

$$\bar{u}(x) = A\left(\frac{\eta(x)d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}} + \beta(x)\right) \text{ for all } x \in \Omega$$

with

$$\beta(x) = m_{\gamma,h}(\Omega)\frac{\eta(x)d(x,\partial\Omega)}{|x|^{\beta_-(\gamma)}} + o\left(\frac{\eta(x)d(x,\partial\Omega)}{|x|^{\beta_-(\gamma)}}\right)$$

as $\epsilon \to 0$. Here, $m_{\gamma,h}(\Omega)$ is the boundary mass.

Step P11.3: We claim that

$$\lim_{\delta\to 0} M_\delta = -\frac{\omega_{n-1}}{n}\left(\frac{n^2}{4} - \gamma\right)A^2 \cdot m_{\gamma,h}(\Omega) \tag{166}$$

We prove the claim. Since $\bar{u}$ is a solution to (146), it follows from standard elliptic theory that there exists $C > 0$ such that $\bar{u}(x) + |x||\nabla\bar{u}(x)| \leq C|x|^{1-\beta_+(\gamma)}$ for all $x \in \Omega$. Therefore, since $\beta_+(\gamma) - \beta_-(\gamma) < 1$, we get that

$$\lim_{\delta\to 0}\int_{\mathcal{T}(\mathbb{R}^n_-\cap B_\delta(0))}\bar{u}^2\,dx + \int_{\mathcal{T}(\mathbb{R}^n_-\cap\partial B_\delta(0))}\bar{u}^2\,d\sigma + \int_{\mathcal{T}(\partial\mathbb{R}^n_-\cap B_\delta(0))}|x|^2|\nabla\bar{u}|^2\,d\sigma = 0.$$

Therefore,

$$M_\delta = -\frac{A^2}{2}\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_+(\gamma)} + \bar{v}_{\beta_-(\gamma)}, \bar{v}_{\beta_+(\gamma)} + \bar{v}_{\beta_-(\gamma)}) + o(1)$$

as $\delta \to 0$, where

$$\begin{aligned}\bar{\mathcal{H}}_\delta(u,v) \quad := \quad & \int_{\mathcal{T}\left(\mathbb{R}^n_-\cap\partial B_{\delta_0}(0)\right)}\Bigg[(x,\nu)\left((\nabla u,\nabla v) - \frac{\gamma}{|x|^2}uv\right) - \left(x^i\partial_i u + \frac{n-2}{2}u\right)\partial_\nu v \\ & - \left(x^i\partial_i v + \frac{n-2}{2}v\right)\partial_\nu u\Bigg]\,d\sigma\end{aligned}$$

and

$$\bar{v}_{\beta_+(\gamma)}(x) := \frac{\eta(x)d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}} \text{ and } \bar{v}_{\beta_-(\gamma)}(x) = \beta(x) \text{ for all } x \in \Omega.$$

We then get that

$$\begin{aligned}M_\delta \quad = \quad & -\frac{A^2}{2}\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_+(\gamma)}, \bar{v}_{\beta_+(\gamma)}) - A^2\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_+(\gamma)}, \bar{v}_{\beta_-(\gamma)}) \\ & -\frac{A^2}{2}\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_-(\gamma)}, \bar{v}_{\beta_-(\gamma)}) + o(1)\end{aligned}$$

as $\delta \to 0$. For any $x \in \mathbb{R}^n_- \cap B_\delta(0)$, with the chart $\mathcal{T}$ and the definition of β, we get

$$\bar{v}_{\beta_+(\gamma)}(\mathcal{T}(x)) := \frac{|x_1|}{|x|^{\beta_+(\gamma)}} + O(|x|^{2-\beta_+(\gamma)}) = v_{\beta_+(\gamma)} + O(|x|^{2-\beta_+(\gamma)})$$

and $\bar{v}_{\beta_-(\gamma)}(\mathcal{T}(x)) = m_{\gamma,h}(\Omega)\dfrac{|x_1|}{|x|^{\beta_-(\gamma)}} + O(|x|^{2-\beta_-(\gamma)}) = m\cdot v_{\beta_-(\gamma)} + O(|x|^{2-\beta_-(\gamma)})$.

Moreover, elliptic theory yields

$$\nabla(\bar{v}_{\beta_+(\gamma)}\circ\mathcal{T}(x)) := \nabla v_{\beta_+(\gamma)} + O(|x|^{1-\beta_+(\gamma)}).$$

and $\nabla(\bar{v}_{\beta_-(\gamma)}\circ\mathcal{T}(x)) = m_{\gamma,h}(\Omega)\cdot\nabla v_{\beta_-(\gamma)} + O(|x|^{1-\beta_-(\gamma)})$ for all $x\in\mathbb{R}^n_-\cap B_\delta(0)$, where v_β is defined in the proof of Step P9. Since $\beta_+(\gamma)-\beta_-(\gamma)<1$ and $\beta_+(\gamma)+\beta_-(\gamma)=n$, we get with a change of variable that as $\delta\to 0$,

$$\begin{aligned}
\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_+(\gamma)},\bar{v}_{\beta_+(\gamma)}) &= \mathcal{H}_\delta(v_{\beta_+(\gamma)},v_{\beta_+(\gamma)}) + O(\delta^{1-(\beta_+(\gamma)-\beta_-(\gamma))})\\
\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_+(\gamma)},\bar{v}_{\beta_-(\gamma)}) &= m_{\gamma,h}(\Omega)\cdot\mathcal{H}_\delta(v_{\beta_+(\gamma)},v_{\beta_-(\gamma)}) + O(\delta^{1-(\beta_+(\gamma)-\beta_-(\gamma))})\\
\bar{\mathcal{H}}_\delta(\bar{v}_{\beta_-(\gamma)},\bar{v}_{\beta_-(\gamma)}) &= O(\delta^{n-2\beta_-(\gamma)}).
\end{aligned}$$

Using the computations performed in the proof of Step P9, we then get (166). This proves the claim and ends Step P11.3.

End of the proof of Step P11: Since M_δ is independent of δ small, we then get that $M_{\delta_0} = -\frac{\omega_{n-1}}{n}\left(\frac{n^2}{4}-\gamma\right)A^2 m_{\gamma,h}(\Omega)$. Putting this estimate in (163), we then get (162). This end Step P11. □

Proof of Proposition 5 when $\beta_+(\gamma)-\beta_-(\gamma)>2$: Plugging (127) into (160) and using that $\beta_+(\gamma)-\beta_-(\gamma)>2$, we obtain

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}} = \frac{n-s}{(n-2)^2}\frac{1}{t_N^{\frac{n-1}{2^\star(s)-2}}}\frac{\int\limits_{\partial\mathbb{R}^n_-} II_0(x,x)|\nabla\tilde{u}_N|^2\,d\sigma}{\sum\limits_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx}.$$

This yields (105) when $\beta_+(\gamma)-\beta_-(\gamma)>2$.

Proof of Proposition 5 when $\beta_+(\gamma)-\beta_-(\gamma)>1$ and $u_0\equiv 0$. Plugging (128) into (161) and using that $\beta_+(\gamma)-\beta_-(\gamma)>1$, we obtain also (105).

Proof of Proposition 6 when $\beta_+(\gamma)-\beta_-(\gamma)>1$. Since $u_\epsilon>0$, we get that $\tilde{u}_N>0$. Therefore, it follows from Ghoussoub-Robert [21] that $\bar{u}_N(x_1,x') = \bar{U}_N(x_1,|x'|)$ for all $(x_1,x')\in(0,+\infty)\times\mathbb{R}^{n-1}$. Due to this symmetry, when $\beta_+(\gamma)-\beta_-(\gamma)>1$, we get that

$$\int_{\partial\mathbb{R}^n_-} II_0(x,x)|\nabla\tilde{u}_N|^2\,d\sigma = \sum_{i,j=1}^{n-1}\int_{\partial\mathbb{R}^n_-} II_{0,ij}x^ix^j|\nabla\tilde{u}_N|^2\,d\sigma \tag{167}$$
$$= \frac{\sum_{i=1}^{n-1} II_{0,ii}}{n-1}\int_{\partial\mathbb{R}^n_-}|x|^2|\nabla\tilde{u}_N|^2\,d\sigma = \frac{\int_{\partial\mathbb{R}^n_-}|x|^2|\nabla\tilde{u}_N|^2\,d\sigma}{n-1}H(0).$$

When $\beta_+(\gamma)-\beta_-(\gamma)>2$ or $\{\beta_+(\gamma)-\beta_-(\gamma)=2$ and $u_0\equiv 0\}$, Proposition 6 follows from (105) and (167). When $\{\beta_+(\gamma)-\beta_-(\gamma)=2$ and $u_0>0\}$, Proposition 6 follows from (160), (156) of Step P9, (127) and (167). When $1<\beta_+(\gamma)-\beta_-(\gamma)<2$, Proposition 6 follows from Step P10, (161), (128) and (167).

Proof of Proposition 6 when $\beta_+(\gamma)-\beta_-(\gamma)<1$: This is a direct consequence of Steps P10 and P11.

8.2. Proof of the sharp blow-up rates when $\beta_+(\gamma)-\beta_-(\gamma)=1$. We start with the following refined asymptotics when $u_\epsilon>0$, $\beta_+(\gamma)-\beta_-(\gamma)=1$ and $u_0\equiv 0$.

STEP P12. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $u_\epsilon>0$ and $u_0\equiv 0$. We fix a family of parameters $(\lambda_\epsilon)_{\epsilon>0}\in(0,+\infty)$ such that*

$$\lim_{\epsilon\to 0}\lambda_\epsilon=0 \text{ and } \lim_{\epsilon\to 0}\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}=0. \tag{168}$$

Then, for all $x\in\overline{\mathbb{R}^n_-}$, $x\neq 0$, we have that

$$\lim_{\epsilon\to 0}\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}u_\epsilon(\mathcal{T}(\lambda_\epsilon x))=K\cdot\frac{|x_1|}{|x|^{\beta_+(\gamma)}},$$

where $\mathcal{T}$ is as in (27),

$$K:=t_N^{-\frac{\beta_+(\gamma)-1}{2^\star(s)-2}}L_{\gamma,\Omega}\int_{\mathbb{R}^n_-}\frac{|y_1|}{|y|^{\beta_-(\gamma)}}\frac{\tilde{u}_N^{2^\star(s)-1}(y)}{|y|^s}\,dy>0 \tag{169}$$

and $L_{\gamma,\Omega}>0$ is given by (210). Moreover, this limit holds in $C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0\})$.

Proof of Step P12: We define

$$w_\epsilon(x):=\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}u_\epsilon(\mathcal{T}(\lambda_\epsilon x))$$

for all $x\in\mathbb{R}^n_-\cap\lambda_\epsilon^{-1}U$. As in the proof of (141), for any $i,j=1,...,n$, we let $(\tilde{g}_\epsilon)_{ij}=(\partial_i\mathcal{T}(r_\epsilon x),\partial_j\mathcal{T}(r_\epsilon x))$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$. We consider $\tilde{g}_\epsilon$ as a metric on $\mathbb{R}^n$. We let $\Delta_g=div_g(\nabla)$, the Laplace-Beltrami operator with respect to the metric g. From (E_ϵ) it follows that for all $\epsilon>0$, we have that

$$\begin{cases}-\Delta_{\tilde{g}_\epsilon}w_\epsilon-\frac{\gamma}{\left|\frac{\mathcal{T}(\lambda_\epsilon x)}{\lambda_\epsilon}\right|^2}w_\epsilon-\lambda_\epsilon^2\,h_\epsilon\circ\mathcal{T}(\lambda_\epsilon x)w_\epsilon=s_\epsilon\frac{w_\epsilon^{2^\star(s)-1-p_\epsilon}}{\left|\frac{\mathcal{T}(\lambda_\epsilon x)}{\lambda_\epsilon}\right|^s} & \text{in } \mathbb{R}^n_-\cap\lambda_\epsilon^{-1}U\\ w_\epsilon>0 & \text{in } \mathbb{R}^n_-\cap\lambda_\epsilon^{-1}U\\ w_\epsilon=0 & \text{on } (\partial\mathbb{R}^n_-\setminus\{0\})\cap\lambda_\epsilon^{-1}U.\end{cases}$$

With

$$s_\epsilon:=\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\lambda_\epsilon^{\beta_+(\gamma)-1}}\right)^{2^\star(s)-2-p_\epsilon}\lambda_\epsilon^{2-s}.$$

Since $\mu_{N,\epsilon}^{p_\epsilon}\to t_N>0$ (see (A9) of Proposition 2) and

$$(\beta_+(\gamma)-1)(2^\star(s)-2)-(2-s)=(2^\star(s)-2)\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2},$$

then using the hypothesis (168), we get that

$$\left(\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\lambda_\epsilon^{\beta_+(\gamma)-1}}\right)^{2^\star(s)-2-p_\epsilon}\lambda_\epsilon^{2-s}\leq C\left(\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}\right)^{(\beta_+(\gamma)-1)(2^\star(s)-2-p_\epsilon)-(2-s)}=o(1)$$

as $\epsilon\to 0$. Since $u_0\equiv 0$, it follows from the pointwise control (60) that there exists $C>0$ such that $0<w_\epsilon(x)\leq C|x_1|\cdot|x|^{-\beta_+(\gamma)}$ for all $x\in\mathbb{R}^n_-\cap\lambda_\epsilon^{-1}U$. It then follows from standard elliptic theory that there exists $w\in C^2(\overline{\mathbb{R}^n_-}\setminus\{0\})$ such that

$$\lim_{\epsilon\to 0} w_\epsilon = w \text{ in } C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0\}) \tag{170}$$

with

$$\begin{cases} -\Delta w - \frac{\gamma}{|x|^2}w = 0 & \text{in } \mathbb{R}^n_- \\ 0\leq w(x)\leq C|x_1|\cdot|x|^{-\beta_+(\gamma)} & \text{in } \mathbb{R}^n_- \\ w=0 & \text{on } \partial\mathbb{R}^n_-\setminus\{0\}. \end{cases}$$

It follows from Lemma 4.2 in Ghoussoub-Robert [21] (see also Pinchover-Tintarev [29]) that there exists $\Lambda\geq 0$ such that $w(x)=\Lambda|x_1|\cdot|x|^{-\beta_+(\gamma)}$ for all $x\in\mathbb{R}^n_-$. We are left with proving that $\Lambda=K$ defined in (169). We fix $x\in\mathbb{R}^n_-$. Green's representation formula yields

$$\begin{aligned} w_\epsilon(x) &= \int_\Omega \frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}} G_\epsilon(\mathcal{T}(\lambda_\epsilon x),y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy \\ &= \int_{\mathcal{T}(\mathbb{R}^n_-\cap(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0))} + \int_{\Omega\setminus\mathcal{T}(\mathbb{R}^n_-\cap(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0))} \end{aligned} \tag{171}$$

Step P12*.1:* We estimate the first term of the right-hand-side. Since $D_0\mathcal{T}=\mathbb{I}_{\mathbb{R}^n}$, a change of variable yields

$$\begin{aligned} &\int_{\mathcal{T}(\mathbb{R}^n_-\cap(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0))} \frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}} G_\epsilon(\mathcal{T}(\lambda_\epsilon x),y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy \\ &= s_\epsilon^{(1)}\int_{\mathbb{R}^n_-\cap(B_R(0)\setminus B_\delta(0))} G_\epsilon(\mathcal{T}(\lambda_\epsilon x),\mathcal{T}(k_{N,\epsilon}z))\frac{\tilde{u}_{N,\epsilon}(z)^{2^\star(s)-1-p_\epsilon}}{|z|^s}(1+o(1))\,dz \end{aligned}$$

with

$$s_\epsilon^{(1)} := \frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}} k_{N,\epsilon}^{n-s}\mu_{N,\epsilon}^{-\frac{n-2}{2}(2^\star(s)-1-p_\epsilon)}$$

It follows from (210) that for any $z\in\mathbb{R}^n_-$, we have that

$$G_\epsilon(\mathcal{T}(\lambda_\epsilon x),\mathcal{T}(k_{N,\epsilon}z)) = (L_{\gamma,\Omega}+o(1))\frac{\lambda_\epsilon|x_1|}{\lambda_\epsilon^{\beta_+(\gamma)}|x|^{\beta_+(\gamma)}}\cdot\frac{k_{N,\epsilon}|y_1|}{k_{N,\epsilon}^{\beta_-(\gamma)}|z|^{\beta_-(\gamma)}},$$

and that the convergence is uniform with repect to $z\in\mathbb{R}^n_-\cap(B_R(0)\setminus B_\delta(0))$. Plugging this estimate in the above equality, using that $k_{N,\epsilon}=\mu_{N,\epsilon}^{1-p_\epsilon/(2^\star(s)-2)}$, $\mu_{N,\epsilon}^{p_\epsilon}\to t_N>0$ and the convergence of $\tilde{u}_{N,\epsilon}$ to $\tilde{u}_N$ (see Proposition 2), we get that

$$\begin{aligned} &\int_{\mathcal{T}(\mathbb{R}^n_-\cap(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0))} \frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}} G_\epsilon(\mathcal{T}(\lambda_\epsilon x),y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy \\ &= L_{\gamma,\Omega}\frac{|x_1|}{|x|^{\beta_+(\gamma)}}t_N^{-\frac{\beta_+(\gamma)-1}{2^\star(s)-2}}\int_{\mathbb{R}^n_-\cap(B_R(0)\setminus B_\delta(0))}\frac{|y_1|}{|y|^{\beta_-(\gamma)}}\frac{\tilde{u}_N(z)^{2^\star(s)-1}}{|z|^s}\,dz+o(1) \end{aligned}$$

as $\epsilon \to 0$. Therefore,

$$\lim_{R\to+\infty,\delta\to 0}\lim_{\epsilon\to 0}\int_{\mathcal{T}(\mathbb{R}^n_-\cap(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0))}\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}G_\epsilon(\mathcal{T}(\lambda_\epsilon x),y)\frac{u_\epsilon(y)^{2^\star(s)-1}}{|y|^s}\,dy$$

$$= K\frac{|x_1|}{|x|^{\beta_+(\gamma)}} \tag{172}$$

where K is as in (169).

Step P12.2: With the control (207) on the Green's function and the pointwise control (60) on u_ϵ, we get that

$$\int_{\Omega\setminus\mathcal{T}(\mathbb{R}^n_-\cap(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0))}\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}G_\epsilon(\mathcal{T}(\lambda_\epsilon x),y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy$$

$$\leq \sum_{i=1}^{N-1}A_{i,\epsilon}+B_\epsilon(R)+C_\epsilon(\delta) \tag{173}$$

where

$$A_{i,\epsilon}:=C\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\int_{B_{R_0}(0)}\frac{\ell_\epsilon(x,y)^{\beta_-(\gamma)}r_\epsilon(x,y)}{|\mathcal{T}(\lambda_\epsilon x)-y|^{n-2}|y|^s}\left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|y|}{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|y|^{\beta_-(\gamma)}+|y|^{\beta_+(\gamma)}}\right)^{2^\star(s)-1}dy$$

$$B_\epsilon(R):=C\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\int_{B_{R_0}(0)\setminus B_{Rk_{N,\epsilon}}(0)}\frac{\ell_\epsilon(x,y)^{\beta_-(\gamma)}r_\epsilon(x,y)}{|\mathcal{T}(\lambda_\epsilon x)-y|^{n-2}}\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-1)}}{|y|^{(\beta_+(\gamma)-1)(2^\star(s)-1)+s}}\,dy$$

$$C_\epsilon(\delta):=C(x)\frac{\lambda_\epsilon^{\beta_+(\gamma)-1+2-n+\beta_-(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\cdot 2^\star(s)}}\int_{B_{\delta k_{N,\epsilon}}(0)}\frac{dy}{|y|^{(\beta_-(\gamma)-1)(2^\star(s)-1)+s+\beta_-(\gamma)-1}}$$

where $\ell_\epsilon(x,y):=\frac{\max\{\lambda_\epsilon|x|,|y|\}}{\min\{\lambda_\epsilon|x|,|y|\}}$, and $r_\epsilon(x,y)=\min\left\{1,\frac{\lambda_\epsilon|x_1|\cdot|y|}{|\mathcal{T}(\lambda_\epsilon x)-y|^2}\right\}$.

Step P12.3. We first estimate $C_\epsilon(\delta)$. Since $n>s+2^\star(s)(\beta_-(\gamma)-1)$ (this is a consequence of $\beta_-(\gamma)<n/2$), straightforward computations yield

$$C_\epsilon(\delta)\leq C(x)\delta^{\frac{2^\star(s)}{2}(\beta_+(\gamma)-\beta_-(\gamma))},$$

and therefore

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}C_\epsilon(\delta)=0. \tag{174}$$

Step P12.4. We estimate $B_\epsilon(R)$. We split the integral as

$$B_\epsilon(R)=\int_{Rk_{\epsilon,N}<|y|<\frac{\lambda_\epsilon|x|}{2}}I_\epsilon(y)\,dy+\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|}I_\epsilon(y)\,dy+\int_{|y|>2\lambda_\epsilon|x|}I_\epsilon(y)\,dy$$

where $I_\epsilon(y)$ is the integrand. Since

$$n-(s+(\beta_+(\gamma)-1)(2^\star(s)-1)+\beta_-(\gamma)-1)=-\frac{2^\star(s)-2}{2}(\beta_+(\gamma)-\beta_-(\gamma))<0,$$

straightforward computations yield

$$\begin{aligned}
&\int_{Rk_{N,\epsilon}<|y|<\frac{\lambda_\epsilon|x|}{2}} I_\epsilon(y)\,dy\\
&\leq C(x)\frac{\lambda_\epsilon^{\beta_+(\gamma)-1+\beta_-(\gamma)-1+2-n}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\int_{Rk_{N,\epsilon}<|y|<\frac{\lambda_\epsilon|x|}{2}}\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-1)}}{|y|^{(\beta_+(\gamma)-1)(2^\star(s)-1)+s+\beta_-(\gamma)-1}}\,dy\\
&\leq C(x)R^{-\frac{2^\star(s)-2}{2}(\beta_+(\gamma)-\beta_-(\gamma))},
\end{aligned}$$

For the next term, a change of variable yields

$$\begin{aligned}
&\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|} I_\epsilon(y)\,dy\\
&\leq C(x)\frac{\lambda_\epsilon^{\beta_+(\gamma)-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|}|\mathcal{T}(\lambda_\epsilon x)-y|^{2-n}\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}(2^\star(s)-1)}}{|y|^{(\beta_+(\gamma)-1)(2^\star(s)-1)+s}}\,dy\\
&\leq C(x)\left(\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}\right)^{\frac{2^\star(s)-2}{2}(\beta_+(\gamma)-\beta_-(\gamma))}\int_{\frac{|x|}{2}<|z|<2|x|}|x-z|^{2-n}\,dz=o(1)
\end{aligned}$$

as $\epsilon\to 0$. Finally, since $\beta_+(\gamma)+\beta_-(\gamma)=n$ and $n-s-(\beta_+(\gamma)-1)2^\star(s)=\frac{2^\star(s)}{2}(\beta_+(\gamma)-\beta_-(\gamma))$, we estimate the last term

$$\begin{aligned}
&\int_{|y|>2\lambda_\epsilon|x|} I_\epsilon(y)\,dy\\
&\leq C(x)\mu_{N,\epsilon}^{\frac{2^\star(s)-2}{2}(\beta_+(\gamma)-\beta_-(\gamma))}\lambda_\epsilon^{\beta_+(\gamma)-\beta_-(\gamma)}\int_{|y|>2\lambda_\epsilon|x|}\frac{|y|^{\beta_-(\gamma)+1-n-s}\,dy}{|y|^{(\beta_+(\gamma)-1)(2^\star(s)-1)}}\\
&\leq C(x)\left(\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}\right)^{\frac{2^\star(s)-2}{2}(\beta_+(\gamma)-\beta_-(\gamma))}=o(1)
\end{aligned}$$

as $\epsilon\to 0$. All these inequalities yield

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}B_\epsilon(R)=0. \tag{175}$$

Step P12*.5.* We fix $i\in\{1,...,N-1\}$ and estimate $A_{i,\epsilon}$. As above, we split the integral as

$$A_{i,\epsilon}=\int_{|y|<\frac{\lambda_\epsilon|x|}{2}}J_{i,\epsilon}(y)\,dy+\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|}J_{i,\epsilon}(y)\,dy+\int_{|y|>2\lambda_\epsilon|x|}J_{i,\epsilon}(y)\,dy,$$

where $J_{i,\epsilon}$ is the integrand. Since $\mu_{i,\epsilon}\leq\mu_{N,\epsilon}$, as one checks, the second and the third integral of the right-hand-side are controled from above respectively by $\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|}I_\epsilon(y)\,dy$ and $\int_{|y|>2\lambda_\epsilon|x|}I_\epsilon(y)\,dy$ that have been computed just above and go to 0 as $\epsilon\to 0$. We are then left with the first term. With a change of variables, we have that

$$\begin{aligned}
&\int_{|y|<\frac{\lambda_\epsilon|x|}{2}} J_{i,\epsilon}(y)\,dy\\
&\leq C(x)\frac{\lambda_\epsilon^{\beta_+(\gamma)-1+\beta_-(\gamma)+2-n-1}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\\
&\times\int_{|y|<\frac{\lambda_\epsilon|x|}{2}}\left(\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}|y|}{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|y|^{\beta_-(\gamma)}+|y|^{\beta_+(\gamma)}}\right)^{2^\star(s)-1}\frac{dy}{|y|^{s+\beta_-(\gamma)-1}}\\
&\leq C(x)\frac{\mu_{i,\epsilon}^{1+n-s-\beta_-(\gamma)-\frac{n-2}{2}(2^\star(s)-1)}}{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}\\
&\times\int_{|z|<\frac{\lambda_\epsilon|x|}{2\mu_{i,\epsilon}}}\frac{1}{|z|^{(\beta_-(\gamma)-1)+s}}\left(\frac{|z|}{|z|^{\beta_-(\gamma)}+|z|^{\beta_+(\gamma)}}\right)^{2^\star(s)-1}dz\\
&\leq C(x)\left(\frac{\mu_{i,\epsilon}}{\mu_{N,\epsilon}}\right)^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}
\end{aligned}$$

since $n > s + (2^\star(s)(\beta_-(\gamma)-1))$ and $n < (\beta_-(\gamma)-1) + s + (2^\star(s)-1)(\beta_+(\gamma)-1)$. Since $\mu_{i,\epsilon} = o(\mu_{N,\epsilon})$ as $\epsilon \to 0$, we get that

$$\lim_{\epsilon\to 0} A_{i,\epsilon} = 0. \tag{176}$$

Step P12.*6:* Plugging (172), (174), (175) and (176) into (171) and (173) yields $\lim_{\epsilon\to 0} w_\epsilon(x) = K\frac{|x_1|}{|x|^{\beta_+(\gamma)}}$ for all $x \in \mathbb{R}^n_-$. With (170), we then get that $\Lambda = K$. This proves Step P12.

Now we can prove Proposition 6 when $\beta_+(\gamma) - \beta_-(\gamma) = 1$ in the case when $u_\epsilon > 0$.

Step P13. *We let (u_ϵ), (h_ϵ) and (p_ϵ) be such that (E_ϵ), (15), (21) and (22) hold. We assume that blow-up occurs. We assume that $u_\epsilon > 0$ and $\beta_+(\gamma) - \beta_-(\gamma) = 1$. Then $u_0 \equiv 0$ and*

$$\begin{aligned}
&\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx + o(1)\right)\\
&= \frac{K^2\omega_{n-2}H(0)}{4(n-1)}\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}} + o\left(\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}}\right).
\end{aligned} \tag{177}$$

The case $\beta_+(\gamma) - \beta_-(\gamma) = 1$ of Proposition 6 is a consequence of Step P13.

Proof of Step P13*:* First remark that since $\beta_+(\gamma) + \beta_-(\gamma) = n$, we then have that

$$\beta_+(\gamma) = \frac{n+1}{2} \text{ and } \beta_-(\gamma) = \frac{n-1}{2}.$$

It follows from Step P10 that $u_0 \equiv 0$. We use (161) that writes

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p_\epsilon}\right)\left(\sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)=\int_{\mathcal{T}_\epsilon}(x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma+O\left(\mu_{N,\epsilon}\right). \tag{178}$$

where $\mathcal{T}_\epsilon := \mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^3}(0)\right)$. It follows from (132) that

$$\begin{aligned}\int_{\mathcal{T}_\epsilon}(x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma &= -\frac{1}{4}\int_{\mathcal{T}_\epsilon}\sum_{p,q=2}^n x^p x^q \partial_{pq}\mathcal{T}_0(0)|\nabla(u_\epsilon\circ\mathcal{T})|^2_{\mathcal{T}^\star\mathrm{Eucl}}(1+O(|x|)\,d\sigma\\ &+O\left(\int_{\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)}|x|^3|\nabla(u_\epsilon\circ\mathcal{T})|^2_{\mathcal{T}^\star\mathrm{Eucl}}\,d\sigma\right)\\ &= -\frac{1}{4}\int_{\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^3}(0)}\sum_{p,q=2}^n x^p x^q\partial_{pq}\mathcal{T}_0(0)|\nabla(u_\epsilon\circ\mathcal{T})|^2\,d\sigma\\ &+O\left(\int_{\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)}|x|^3|\nabla(u_\epsilon\circ\mathcal{T})|^2\,d\sigma\right).\end{aligned} \tag{179}$$

With the control (102) and $\beta_+(\gamma)-\beta_-(\gamma)=1$, we get that

$$\begin{aligned}\int_{\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)}|x|^3|\nabla(u_\epsilon\circ\mathcal{T})|^2\,d\sigma &\leq C\sum_{i=1}^N\int_{\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)}|x|^3\frac{\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}}\,d\sigma\\ &\leq C\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}=C\mu_{N,\epsilon}\end{aligned} \tag{180}$$

We need an intermediate result. We let $(s_\epsilon)_\epsilon,(t_\epsilon)_\epsilon\in[0,+\infty)$ such that $0\leq s_\epsilon\leq t_\epsilon$, and $\mu_{\epsilon,N}=o(t_\epsilon)$ as $\epsilon\to 0$. We claim that

$$\int_{\partial\mathbb{R}^n_-\cap(B_{t_\epsilon}(0)\setminus B_{s_\epsilon}(0))}|x|^2|\nabla(u_\epsilon\circ\mathcal{T})|^2\,d\sigma \leq C\sum_i \mu_{i,\epsilon}\ln\left(\frac{t_\epsilon}{\max\{s_\epsilon,\mu_{i,\epsilon}\}}\right) \tag{181}$$

Indeed, with the pointwise control (102), $u_0\equiv 0$ and $2\beta_+(\gamma)=n+1$, we get that

$$\begin{aligned}&\int_{\partial\mathbb{R}^n_-\cap(B_{t_\epsilon}(0)\setminus B_{s_\epsilon}(0))}|x|^2|\nabla(u_\epsilon\circ\mathcal{T})|^2\,d\sigma\\ &\leq C\sum_{i=1,\dots,N}\mu_{i,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\int_{s_\epsilon}^{t_\epsilon}\frac{r^{2+(n-1)-1}\,dr}{\mu_{i,\epsilon}^{2(\beta_+(\gamma)-\beta_-(\gamma))}r^{2\beta_-(\gamma)}+r^{2\beta_+(\gamma)}}\\ &\leq C\sum_{i=1,\dots,N}\mu_{i,\epsilon}\int_{\frac{s_\epsilon}{\mu_{i,\epsilon}}}^{\frac{t_\epsilon}{\mu_{i,\epsilon}}}\frac{r^{2\beta_+(\gamma)-1}\,dr}{r^{2\beta_-(\gamma)}+r^{2\beta_+(\gamma)}}\end{aligned}$$

Distinguishing the cases $s_\epsilon\leq\mu_{i,\epsilon}$ and $s_\epsilon\geq\mu_{i,\epsilon}$, we get (181). This proves the claim.

We define $\theta_\epsilon := \frac{1}{\sqrt{|\ln \mu_{N,\epsilon}|}}$, $\alpha_\epsilon := \mu_{N,\epsilon}^{\theta_\epsilon}$ and $\beta_\epsilon := \mu_{N,\epsilon}^{1-\theta_\epsilon}$. As one checks, we have that

$$\left\{\begin{array}{ccc} \mu_{\epsilon,N} = o(\beta_\epsilon) & \beta_\epsilon = o(\alpha_\epsilon) & \alpha_\epsilon = o(1) \\ \ln \frac{\alpha_\epsilon}{\beta_\epsilon} \simeq \ln \frac{1}{\mu_{N,\epsilon}} & \ln \frac{\beta_\epsilon}{\mu_{N,\epsilon}} = o\left(\ln \frac{1}{\mu_{N,\epsilon}}\right) & \ln \alpha_\epsilon = o(\ln \mu_{N,\epsilon}) \end{array}\right\} \tag{182}$$

as $\epsilon \to 0$. It then follows from (181) and the properties (182) that

$$\left\{\begin{array}{c} \displaystyle\int_{\partial\mathbb{R}^n_- \cap \left(B_{\delta_0}(0)\setminus B_{\alpha_\epsilon}(0)\right)} |x|^2 |\nabla(u_\epsilon \circ \mathcal{T})|^2 = o\left(\mu_{N,\epsilon} \ln \frac{1}{\mu_{N,\epsilon}}\right); \\ \displaystyle\int_{\partial\mathbb{R}^n_- \cap B_{\beta_\epsilon}(0)} |x|^2 |\nabla(u_\epsilon \circ \mathcal{T})|^2 = o\left(\mu_{N,\epsilon} \ln \frac{1}{\mu_{N,\epsilon}}\right) \end{array}\right\} \tag{183}$$

Since $\mu_{N,\epsilon} = o(\beta_\epsilon)$ and $\alpha_\epsilon = o(1)$ as $\epsilon \to 0$, it follows from Proposition P12 that

$$\lim_{\epsilon\to 0} \sup_{x \in \partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \left| \frac{|x|^{2\beta_+(\gamma)} |\nabla(u_\epsilon \circ \mathcal{T})|^2(x)}{\mu_{N,\epsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}} - K^2 \right| = 0 \tag{184}$$

We fix $i, j \in \{2, ..., n\}$. It follows from (184) and $\beta_+(\gamma) - \beta_-(\gamma) = 1$ that

$$\begin{aligned} &\int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} x^i x^j \partial_{ij}\mathcal{T}_0(0) |\nabla(u_\epsilon \circ \mathcal{T})|^2 \, dx \\ &= \int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \mu_{N,\epsilon} \frac{x^i x^j \partial_{ij}\mathcal{T}_0(0)}{|x|^{2\beta_+(\gamma)}} K^2 \, dx \\ &+ \int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \mu_{N,\epsilon} \frac{x^i x^j \partial_{ij}\mathcal{T}_0(0)}{|x|^{2\beta_+(\gamma)}} \left(\frac{|x|^{2\beta_+(\gamma)} |\nabla(u_\epsilon \circ \mathcal{T})|^2}{\mu_{N,\epsilon}} - K^2 \right) dx \\ &= \int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \mu_{N,\epsilon} \frac{x^i x^j \partial_{ij}\mathcal{T}_0(0)}{|x|^{2\beta_+(\gamma)}} K^2 \, dx \\ &+ o\left(\int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \mu_{N,\epsilon} \frac{|x|^2}{|x|^{2\beta_+(\gamma)}} \, dx \right) \end{aligned} \tag{185}$$

Independently, with a change of variable and $2\beta_+(\gamma) = n+1$, we get that

$$\begin{aligned} \int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \frac{x^i x^j \partial_{ij}\mathcal{T}_0(0)}{|x|^{2\beta_+(\gamma)}} \, dx &= \partial_{ij}\mathcal{T}_0(0) \left(\int_{\beta_\epsilon}^{\alpha_\epsilon} \frac{dr}{r} \right) \left(\int_{\mathbb{S}^{n-2}} \sigma^i \sigma^j \, d\sigma \right) \\ &= \delta_{ij} \partial_{ij}\mathcal{T}_0(0) \frac{\omega_{n-2}}{n-1} \ln \frac{\alpha_\epsilon}{\beta_\epsilon}, \end{aligned}$$

where ω_{n-2} is the volume of the round $(n-2)-$unit sphere. This equality, (185) and the properties (182) yield

$$\begin{aligned} &\int_{\partial\mathbb{R}^n_- \cap B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} x^i x^j \delta_{ij} \partial_{ij}\mathcal{T}_0(0) |\nabla(u_\epsilon \circ \mathcal{T})|^2 \, dx \\ &= \delta_{ij} \partial_{ij}\mathcal{T}_0(0) \frac{K^2 \omega_{n-2}}{n-1} \mu_{N,\epsilon} \ln \frac{1}{\mu_{N,\epsilon}} + o\left(\mu_{N,\epsilon} \ln \frac{1}{\mu_{N,\epsilon}} \right). \end{aligned} \tag{186}$$

Therefore, plugging (180), (183) and (186) into (179) yields

$$\begin{aligned}
&\int_{\mathcal{T}\left(\partial\mathbb{R}^n_-\cap B_{\delta_0}(0)\setminus B_{k^3_{1,\epsilon}}(0)\right)} (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\,d\sigma\\
&= -\frac{K^2\omega_{n-2}\sum_{i=2}^n \partial_{ii}\mathcal{T}_0(0)}{4(n-1)}\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}} + o\left(\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}}\right)\\
&= \frac{K^2\omega_{n-2}\sum_{i=2}^n II_{0,ii}}{4(n-1)}\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}} + o\left(\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}}\right)\\
&= \frac{K^2\omega_{n-2}H(0)}{4(n-1)}\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}} + o\left(\mu_{N,\epsilon}\ln\frac{1}{\mu_{N,\epsilon}}\right).
\end{aligned}$$

Plugging this latest estimate into (178) yields (177). This ends the proof of Step P13. □

9. Proof of multiplicity

Proof of Theorem 3: We fix $\gamma < n^2/4$ and $h \in C^1(\overline{\Omega})$ such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. For each $2 < p \leq 2^\star(s)$, we consider the C^2-functional

$$I_{p,\gamma}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2\,dx - \frac{\gamma}{2}|u|^2|x|^2 - hu^2\right)dx - \frac{1}{p}\int_\Omega \frac{|u|^p}{|x|^s}\,dx$$

on $H^2_{1,0}(\Omega)$, whose critical points are the weak solutions of

$$\left\{\begin{array}{rll} -\Delta u - \frac{\gamma}{|x|^2}u - hu &= \frac{|u|^{p-2}u}{|x|^s} & \text{on } \Omega\\ u &= 0 & \text{on } \partial\Omega.\end{array}\right. \tag{187}$$

For a fixed $u \in H^2_{1,0}(\Omega)$, $u \not\equiv 0$, we have that

$$I_{p,\gamma}(\lambda u) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\gamma\lambda^2}{2}\int_\Omega \frac{|u|^2}{|x|^2}\,dx - \lambda^2\int_\Omega hu^2\,dx - \frac{\lambda^p}{p}\int_\Omega \frac{|u|^p}{|x|^s}\,dx$$

Then, since coercivity holds, we have that that $\lim_{\lambda\to\infty} I_{p,\gamma}(\lambda u) = -\infty$, which means that for each finite dimensional subspace $E_k \subset E := H^2_{1,0}(\Omega)$, there exists $R_k > 0$ such that

$$\sup\{I_{p,\gamma}(u); u \in E_k, \|u\|_{H^2_1} > R_k\} < 0 \tag{188}$$

when $p \to 2^\star(s)(s)$. Let $(E_k)_{k=1}^\infty$ be an increasing sequence of subspaces of $H^2_{1,0}(\Omega)$ such that $\dim E_k = k$ and $\overline{\cup_{k=1}^\infty E_k} = E := H^2_{1,0}(\Omega)$ and define the min-max values:

$$c_{p,k} = \inf_{g\in\mathbf{H}_k}\sup_{x\in E_k} I_{p,\gamma}(g(x)),$$

where

$$\mathbf{H}_k = \{g \in C(E,E);\ g \text{ is odd and } g(v) = v \text{ for } \|v\| > R_k \text{ for some } R_k > 0\}.$$

Proposition 7. *With the above notation and assuming $n \geq 3$, we have:*

1. *For each $k \in \mathbb{N}$, $c_{p,k} > 0$ and $\lim_{p\to 2^\star(s)} c_{p,k} = c_{2^\star(s),k} := c_k$.*
2. *If $2 < p < 2^\star(s)$, there exists for each k, functions $u_{p,k} \in H^2_{1,0}(\Omega)$ such that $I'_{p,\gamma}(u_{p,k}) = 0$, and $I_{p,\gamma}(u_{p,k}) = c_{p,k}$.*
3. *For each $2 < p < 2^\star(s)$, we have $c_{p,k} \geq D_{n,p}k^{\frac{p+1}{p-1}\frac{2}{n}}$ where $D_{n,p} > 0$ is such that $\lim_{p\to 2^\star(s)} D_{n,p} = 0$.*

(4) $\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_{2^\star(s),k} = +\infty$.

Proof: (1) Coercivity yields the existence of $a_0 > 0$ such that

$$\int_\Omega \left(|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2 - hu^2 \right) dx \geq a_0 \int_\Omega |\nabla u|^2 \, dx \text{ for all } u \in H_{1,0}^2(\Omega). \quad (189)$$

With (189), the Hardy and the Hardy-Sobolev inequality (20), there exists $C > 0$ and $\alpha > 0$ such that

$$I_{p,\gamma}(u) \geq \frac{a_0}{2}\|\nabla u\|_2^2 - C\|\nabla u\|_2^p = \|\nabla u\|_2^2 \left(\frac{a_0}{2} - C\|\nabla u\|_2^{p-2} \right) \geq \alpha > 0$$

for all $u \in H_{1,0}^2(\Omega)$ such that provided $\|\nabla u\|_2 = \rho$ for some $\rho > 0$ small enough. Then the sphere $S_\rho = \{u \in E; \|u\|_{H_{1,0}^2(\Omega)} = \rho\}$ intersects every image $g(E_k)$ by an odd continuous function g. It follows that

$$c_{p,k} \geq \inf\{I_{p,\gamma}(u); u \in S_\rho\} \geq \alpha > 0.$$

In view of (188), it follows that for each $g \in \mathbf{H}_k$, we have that

$$\sup_{x\in E_k} I_{p_i,\gamma}(g(x)) = \sup_{x\in D_k} I_{p,\gamma}(g(x))$$

where D_k denotes the ball in E_k of radius R_k. Consider now a sequence $p_i \to 2^\star(s)$ and note first that for each $u \in E$, we have that $I_{p_i,\gamma}(u) \to I_{2^\star(s),\gamma}(u)$. Since $g(D_k)$ is compact and the family of functionals $(I_{p,\gamma})_p$ is equicontinuous, it follows that $\sup_{x\in E_k} I_{p,\gamma}(g(x)) \to \sup_{x\in E_k} I_{2^\star(s),\gamma}(g(x))$, from which follows that $\limsup_{i\in\mathbb{N}} c_{p_i,k} \leq \sup_{x\in E_k} I_{2^\star(s),\gamma}(g(x))$. Since this holds for any $g \in \mathbf{H}_k$, it follows that

$$\limsup_{i\in\mathbb{N}} c_{p_i,k} \leq c_{2^\star(s),k} = c_k.$$

On the other hand, the function $f(r) = \frac{1}{p}r^p - \frac{1}{2^\star(s)}r^{2^\star(s)}$ attains its maximum on $[0, +\infty)$ at $r = 1$ and therefore $f(r) \leq \frac{1}{p} - \frac{1}{2^\star(s)}$ for all $r > 0$. It follows

$$\begin{aligned} I_{2^\star(s),\gamma}(u) &= I_{p,\gamma}(u) + \int_\Omega \frac{1}{|x|^s}\left(\frac{1}{p}|u(x)|^p - \frac{1}{2^\star(s)}|u(x)|^{2^\star(s)} \right) dx \\ &\leq I_{p,\gamma}(u) + \int_\Omega \frac{1}{|x|^s}\left(\frac{1}{p} - \frac{1}{2^\star(s)} \right) dx \end{aligned}$$

from which follows that $c_k \leq \liminf_{i\in\mathbb{N}} c_{p_i,k}$, and claim (1) is proved.

If now $p < 2^\star(s)$, we are in the subcritical case, that is we have compactness in the Sobolev embedding $H_{1,0}^2(\Omega) \to L^p(\Omega; |x|^{-s}dx)$ and therefore $I_{p,\gamma}$ has the Palais-Smale condition. It is then standard to find critical points $u_{p,k}$ for $I_{p,\gamma}$ at each level $c_{p,k}$ (see for example the book [15]). Consider now the functional

$$I_{p,0}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p}\int_\Omega \frac{|u|^p}{|x|^s} \, dx$$

and its critical values

$$c_{p,k}^0 = \inf_{g\in\mathbf{H}_k} \sup_{x\in E_k} I_{p,0}(g(x)).$$

It has been shown in [20] that (1), (2) and (3) of Proposition 7 hold, with $c_{p,k}^0$ and c_k^0 replacing $c_{p,k}$ and c_k respectively. In particular, $\lim_{k\to\infty} c_k^0 = \lim_{k\to\infty} c_{2^\star(s),k}^0 = +\infty$.

On the other hand, with the coercivity (189), we have that

$$I_{p,\gamma}(u) \geq a_0^{\frac{p}{p-2}} I_{p,0}(v) \quad \text{for every } u \in H_{1,0}^2(\Omega),$$

where $v = a_0^{-\frac{1}{p-2}} u$. It then follows that $\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_{2^\star(s),k} = +\infty$.

To complete the proof of Theorem 3, notice that since for each k, we have

$$\lim_{p_i \to 2^\star(s)} I_{p_i,\gamma}(u_{p_i,k}) = \lim_{p_i\to 2^\star(s)} c_{p_i,k} = c_k,$$

it follows that the sequence $(u_{p_i,k})_i$ is uniformly bounded in $H_{1,0}^2(\Omega)$. Moreover, since $I'_{p_i}(u_{p_i,k}) = 0$, it follows from the compactness result that by letting $p_i \to 2^\star(s)$, we get a solution u_k of (187) in such a way that $I_{2^\star(s)(s),\gamma}(u_k) = \lim_{p\to 2^\star(s)} I_{p,\gamma}(u_{p,k}) = \lim_{p\to 2^\star(s)} c_{p,k} = c_k$. Since the latter sequence goes to infinity, it follows that (187) has an infinite number of critical levels.

10. Proof of the non-existence result

Proof of Theorem 2: We argue by contradiction. We fix $\gamma < \gamma_H(\Omega) \leq \frac{n^2}{4}$ and $\Lambda > 0$. We assume that there is a family $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$ of solutions to

$$\begin{cases} -\Delta u_\epsilon - \gamma \frac{u_\epsilon}{|x|^2} - h_\epsilon u_\epsilon = \frac{u_\epsilon^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega, \\ u_\epsilon > 0 & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \setminus \{0\} \end{cases} \tag{190}$$

with $\|\nabla u_\epsilon\|_2 \leq \Lambda$ and $\lim_{\epsilon\to 0} h_\epsilon = h_0$ in $C^1(\overline{\Omega})$.

We claim that $(u_\epsilon)_{\epsilon>0}$ is not pre-compact in $H_{1,0}^2(\Omega)$. Otherwise, up to extraction, there would be $u_0 \in H_{1,0}^2(\Omega)$, $u_0 \geq 0$, such that $u_\epsilon \to u_0$ in $H_{1,0}^2(\Omega)$ as $\epsilon \to 0$. Passing to the limit in the equation, we get that $u_0 \geq 0$ and

$$\begin{cases} -\Delta u_0 - \gamma \frac{u_0}{|x|^2} - h_0 u_0 = \frac{u_0^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega, \\ u_0 \geq 0 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{191}$$

The coercivity of $-\Delta u_0 - \gamma|x|^{-2} - h_0$ and the convergence of $(h_\epsilon)_\epsilon$ yield

$$C\left(\int_\Omega \frac{u_\epsilon^{2^\star(s)}}{|x|^s} dx\right)^{2/2^\star(s)} \leq \int_\Omega |\nabla u_\epsilon|^2\, dx - \int_\Omega \left(\frac{\gamma}{|x|^2} + h_\epsilon\right) u_\epsilon^2\, dx \leq \int_\Omega \frac{u_\epsilon^{2^\star(s)}}{|x|^s} dx,$$

for small $\epsilon > 0$, and then, since $u_\epsilon > 0$, there exists $c_0 > 0$ such that

$$\int_\Omega \frac{u_\epsilon^{2^\star(s)}}{|x|^s} dx \geq c_0$$

for all $\epsilon > 0$. Passing to the limit yields $u_0 \not\equiv 0$. Therefore, $u_0 > 0$ is a solution to (190) with $\epsilon = 0$. This is not possible simply by the hypothesis.

The family $(u_\epsilon)_\epsilon$ is not pre-compact and it therefore blows-up with bounded energy. Let $u_0 \in H_{1,0}^2(\Omega)$ be its weak limit, which is necessarily a solution to (191), and hence must be the trivial solution $u_0 \equiv 0$. Proposition 6 then yields that either

$$\beta_+(\gamma) - \beta_-(\gamma) \geq 1 \text{ and therefore } H(0) = 0, \tag{192}$$

or

$$\beta_+(\gamma) - \beta_-(\gamma) < 1 \text{ and therefore } m_{\gamma,h_0}(\Omega) = 0. \tag{193}$$

It now suffices to note that when $\gamma \leq (n^2-1)/4$ then $\beta_+(\gamma) - \beta_-(\gamma) \geq 1$ and the above contradicts our assumption that $H(0) \neq 0$. Similarly, if $\gamma > (n^2-1)/4$, then $\beta_+(\gamma) - \beta_-(\gamma) < 1$ and the above contradicts our assumption that the mass is non-zero. In either case, this means that no such a family of positive solutions $(u_\epsilon)_{\epsilon>0}$ exist. $\square$

Proof of Corollary 1: First note that if h_0 satisfies

$$h_0(x) + \frac{1}{2}(\nabla h_0(x), x) \leq 0 \text{ for all } x \in \Omega, \tag{194}$$

then by differentiating for any $x \in \Omega$, the function $t \mapsto t^2 h_0(tx)$ (which is well defined for $t \in [0,1]$ since Ω is starshaped), we get that $h_0 \leq 0$. Therefore $-\Delta - \gamma|x|^{-2} - h_0$ is coercive.
Assume now there is a positive variational solution u_0 corresponding to h_0, the Pohozaev identity (196) then gives

$$\int_{\partial\Omega} (x,\nu)\frac{(\partial_\nu u_0)^2}{2}\, d\sigma - \int_\Omega \left(h_0 + \frac{1}{2}(\nabla h_0, x)\right) u_0^2\, dx = 0.$$

Hopf's strong comparison principle yields $\partial_\nu u_0 < 0$. Since Ω is starshaped with respect to 0, we get that $(x,\nu) \geq 0$ on $\partial\Omega$. Therefore, with (194), we get that $(x,\nu) = 0$ for all $x \in \Omega$, which is a contradiction since Ω is smooth and bounded.
If now $\gamma \leq (n^2-1)/4$, the result follows from Theorem 2 since we have assumed that $H(0) \neq 0$.
If $\gamma > (n^2-1)/4$, we use Theorem 7.1 in Ghoussoub-Robert [21] to find $\mathcal{K} \in C^2(\overline{\Omega} \setminus \{0\})$ and $A > 0$ such that

$$\begin{cases} -\Delta\mathcal{K} - \frac{\gamma}{|x|^2}\mathcal{K} - h_0\mathcal{K} = 0 & \text{in } \Omega \\ \mathcal{K} > 0 & \text{in } \Omega \\ \mathcal{K} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

and such that

$$\mathcal{K}(x) = A\left(\frac{\eta(x) d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}} + \beta(x)\right) \text{ for all } x \in \Omega,$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$ and $\beta \in H_{1,0}^2(\Omega)$ are as in Step P11. We now apply the Pohozaev identity (196) to $\mathcal{K}$ on the domain $U := \Omega \setminus \mathcal{T}(B_\delta(0))$ for $\mathcal{T}$ as in (27): using that $\mathcal{K}^2 \in L^1(\Omega)$ and $(\cdot,\nu)(\partial_\nu\mathcal{K})^2 \in L^1(\partial\Omega)$ when $\beta_+(\gamma) - \beta_-(\gamma) < 1$, we get that

$$\int_{\partial\Omega} (x,\nu)\frac{(\partial_\nu \mathcal{K})^2}{2}\, d\sigma - \int_\Omega \left(h_0 + \frac{1}{2}(\nabla h_0, x)\right) \mathcal{K}^2\, dx = M_\delta$$

where M_δ is defined in (163). With (166), we then get

$$\int_{\partial\Omega} (x,\nu)\frac{(\partial_\nu \mathcal{K})^2}{2}\, d\sigma - \int_\Omega \left(h_0 + \frac{1}{2}(\nabla h_0, x)\right) \mathcal{K}^2\, dx = -\frac{\omega_{n-1}}{n}\left(\frac{n^2}{4} - \gamma\right) A^2 \cdot m_{\gamma,h_0}(\Omega).$$

Since Ω is star-shaped and h_0 satisfies (194), it follows that $m_{\gamma,h_0}(\Omega) < 0$ and Theorem 2 then applies to complete our corollary.

11. Appendix A: The Pohozaev identity

Proposition 8. *Let $U \subset \mathbb{R}^n$ be a smooth bounded domain and let $u \in C^2(\overline{U})$ be a solution of*

$$-\Delta u - \gamma \frac{u}{|x|^2} - hu = K \frac{|u|^{2^\star(s)-2-p}}{|x|^s} u \quad \text{on } U. \tag{195}$$

Then, we have

$$-\int_U \left(h(x) + \frac{(\nabla h, x)}{2}\right) u^2 \, dx \; - \frac{p}{2^\star(s)} \left(\frac{n-s}{2^\star(s)-p}\right) \int_U K \frac{|u|^{2^\star(s)-p}}{|x|^s} \, dx$$
$$= \int_{\partial U} F(x) \, d\sigma,$$

where

$$\begin{aligned} F(x) \quad := \quad & (x,\nu) \left(\frac{|\nabla u|^2}{2} - \frac{\gamma}{2} \frac{u^2}{|x|^2} - \frac{h(x)}{2} u^2 - \frac{K}{2^\star(s)-p} \frac{|u|^{2^\star(s)-p}}{|x|^s}\right) \\ & - \left(x^i \partial_i u + \frac{n-2}{2} u\right) \partial_\nu u. \end{aligned}$$

Proof: For any $y_0 \in \mathbb{R}^n$, the classical Pohozaev identity yields

$$\begin{aligned} & -\int_U \left((x-y_0)^i \partial_i u + \frac{n-2}{2} u\right) \Delta u \, dx \\ & = \int_{\partial U} \left[(x-y_0,\nu) \frac{|\nabla u|^2}{2} - \left((x-y_0)^i \partial_i u + \frac{n-2}{2} u\right) \partial_\nu u\right] d\sigma, \end{aligned}$$

where ν is the outer normal to the boundary ∂U.

One has for $1 \leq j \leq n$

$$\partial_j \left(\frac{|u|^{2^\star(s)-p}}{|x|^s}\right) = -s \frac{x^j}{|x|^{s+2}} |u|^{2^\star(s)-p} + (2^\star(s)-p) \frac{|u|^{2^\star(s)-2-p}}{|x|^s} u \partial_j u$$

So

$$\begin{aligned} & (x-y_0, \nabla u) \frac{|u|^{2^\star(s)-2-p}}{|x|^s} u = \frac{1}{2^\star(s)-p} (x-y_0)^j \partial_j \left(\frac{|u|^{2^\star(s)-p}}{|x|^s}\right) \\ & + \frac{s}{2^\star(s)-p} \frac{|u|^{2^\star(s)-p}}{|x|^s} - \frac{s}{2^\star(s)-p} \frac{(x,y_0)}{|x|^{s+2}} |u|^{2^\star(s)-p}. \end{aligned}$$

Then integration by parts yields

$$\begin{aligned} \int_U (x-y_0, \nabla u) \frac{|u|^{2^\star(s)-2-p}}{|x|^s} u \, dx = & \frac{1}{2^\star(s)-p} \int_U (x-y_0)^j \partial_j \left(\frac{|u|^{2^\star(s)-p}}{|x|^s}\right) dx \\ & + \frac{s}{2^\star(s)-p} \int_U \frac{|u|^{2^\star(s)-p}}{|x|^s} dx - \frac{s}{2^\star(s)-p} \int_U \frac{(x,y_0)}{|x|^{s+2}} |u|^{2^\star(s)-p} dx \end{aligned}$$

$$= -\frac{n-s}{2^\star(s)-p}\int_U \frac{|u|^{2^\star(s)-p}}{|x|^s}\,dx - \frac{s}{2^\star(s)-p}\int_U \frac{(x,y_0)}{|x|^{s+2}}|u|^{2^\star(s)-p}dx$$
$$+\frac{1}{2^\star(s)-p}\int_{\partial U}(x-y_0,\nu)\frac{|u|^{2^\star(s)-p}}{|x|^s}\,d\sigma.$$

Similarly,

$$(x-y_0,\nabla u)\,\frac{u}{|x|^2} = \frac{1}{2}(x-y_0)^j\partial_j\left(\frac{u^2}{|x|^2}\right)+\frac{u^2}{|x|^2}-\frac{(x,y_0)}{|x|^4}u^2$$

$$\int_U (x-y_0,\nabla u)\,\frac{u}{|x|^2}\,dx = -\frac{n-2}{2}\int_U \frac{u^2}{|x|^2}\,dx - \int_U \frac{(x,y_0)}{|x|^4}u^2dx$$
$$+\frac{1}{2}\int_{\partial U}(x-y_0,\nu)\frac{u^2}{|x|^2}\,d\sigma$$

and

$$\int_U (x-y_0,\nabla u)\,h(x)u\,dx = -\frac{n}{2}\int_U h(x)u^2\,dx - \frac{1}{2}\int_U (\nabla h, x-y_0)\,u^2\,dx$$
$$+\frac{1}{2}\int_{\partial U}(x-y_0,\nu)h(x)u^2\,d\sigma$$

Combining the above, we obtain for any K and any $y_0\in\mathbb{R}^n$,

$$\int_U \left((x-y_0)^i\partial_i u+\frac{n-2}{2}u\right)\left(-\Delta u-\gamma\frac{u}{|x|^2}-hu-K\frac{|u|^{2^\star(s)-2-p}}{|x|^s}u\right)\,dx$$
$$-\int_U h(x)u^2\,dx-\frac{1}{2}\int_U(\nabla h,x-y_0)\,u^2\,dx$$
$$-\frac{p}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p}\right)\int_U K\frac{|u|^{2^\star(s)-p}}{|x|^s}\,dx$$
$$-\gamma\int_U\frac{(x,y_0)}{|x|^4}u^2dx-\frac{s}{2^\star(s)-p}\int_U\frac{(x,y_0)}{|x|^{s+2}}K|u|^{2^\star(s)-p}dx$$
$$=\int_{\partial U}\left[(x-y_0,\nu)\left(\frac{|\nabla u|^2}{2}-\frac{\gamma}{2}\frac{u^2}{|x|^2}-\frac{h(x)}{2}u^2-\frac{K}{2^\star(s)-p}\frac{|u|^{2^\star(s)-p}}{|x|^s}\right)\right]d\sigma$$
$$-\int_{\partial U}\left[\left((x-y_0)^i\partial_i u+\frac{n-2}{2}u\right)\partial_\nu u\right]d\sigma. \tag{196}$$

We conclude by taking $y_0=0$ and using that u satisfies (195) on U.

12. Appendix B: A continuity property of the first eigenvalue of Schrödinger operators

LEMMA 3. *Let $\Omega\subset\mathbb{R}^n$, $n\geq 3$, be a smooth bounded domain. Let $(V_k)_k:\Omega\to\mathbb{R}$ and $V_\infty:\Omega\to\mathbb{R}$ be measurable functions and let $(x_k)_k\in\overline{\Omega}$ be a sequence of points. We assume that* *i)* $\lim_{k\to+\infty} V_k(x) = V_\infty(x)$ *for a.e.* $x \in \Omega$,

ii) There exists $C > 0$ *such that* $|V_k(x)| \leq C|x - x_k|^{-2}$ *for all* $k \in \mathbb{N}$ *and* $x \in \Omega$.

iii) $\lim_{k\to+\infty} x_k = 0 \in \partial\Omega$.

iv) For some $\gamma_0 < n^2/4$, *there exists* $\delta > 0$ *such that* $|V_k(x)| \leq \gamma_0|x - x_k|^{-2}$ *for all* $k \in \mathbb{N}$ *and* $x \in B_\delta(0) \cap \Omega$.

v) The first eigenvalue $\lambda_1(-\Delta + V_k)$ *is achieved for all* $k \in \mathbb{N}$.

Then,

$$\lim_{k\to+\infty} \lambda_1(-\Delta + V_k) = \lambda_1(-\Delta + V_\infty). \tag{197}$$

Proof: We first claim that $(\lambda_1(-\Delta+V_k))_k$ is bounded. Indeed, fix $\varphi \in H^2_{1,0}(\Omega)\setminus\{0\}$ and use the Hardy inequality to write for all $k \in \mathbb{N}$,

$$\lambda_1(-\Delta+V_k) \leq \frac{\int_\Omega (|\nabla\varphi|^2 + V_k\varphi^2)\,dx}{\int_\Omega \varphi^2\,dx} \leq \frac{\int_\Omega (|\nabla\varphi|^2 + C|x - x_k|^{-2}\varphi^2)\,dx}{\int_\Omega \varphi^2\,dx} := M < +\infty$$

For the lower bound, we have for any $\varphi \in H^2_{1,0}(\Omega)$,

$$\begin{aligned}\int_\Omega (|\nabla\varphi|^2 + V_k\varphi^2)\,dx &= \int_\Omega |\nabla\varphi|^2\,dx + \int_{B_\delta(0)} V_k\varphi^2\,dx + \int_{\Omega\setminus B_\delta(0)} V_k\varphi^2\,dx \\ &\geq \int_\Omega |\nabla\varphi|^2\,dx - \gamma_0 \int_{B_\delta(0)} |x - x_k|^{-2}\varphi^2\,dx \\ &\quad -4C\delta^{-2}\int_{\Omega\setminus B_\delta(0)} \varphi^2\,dx \\ &\geq \left(1 - 4\gamma_0/n^2\right)\int_\Omega |\nabla\varphi|^2\,dx - 4C\delta^{-2}\int_\Omega \varphi^2\,dx. \end{aligned} \tag{198}$$

Since $\gamma_0 < n^2/4$, we then get that $\lambda_1(-\Delta+V_k) \geq -4C\delta^{-2}$ for large k, which proves the lower bound.

Up to a subsequence, we can now assume that $(\lambda_1(-\Delta + V_k))_k$ converges as $k \to +\infty$. We now show that

$$\liminf_{k\to+\infty} \lambda_1(-\Delta + V_k) \geq \lambda_1(-\Delta + V_\infty). \tag{199}$$

For $k \in \mathbb{N}$, we let $\varphi_k \in H^2_{1,0}(\Omega)$ be a minimizer of $\lambda_1(-\Delta+V_k)$ such that $\int_\Omega \varphi_k^2\,dx = 1$. In particular,

$$-\Delta\varphi_k + V_k\varphi_k = \lambda_1(-\Delta + V_k)\varphi_k \text{ weakly in } H^2_{1,0}(\Omega). \tag{200}$$

Inequality (198) above yields the boundedness of $(\varphi_k)_k$ in $H^2_{1,0}(\Omega)$. Up to a subsequence, we let $\varphi \in H^2_{1,0}(\Omega)$ such that, as $k \to +\infty$, $\varphi_k \rightharpoonup \varphi$ weakly in $H^2_{1,0}(\Omega)$, $\varphi_k \to \varphi$ strongly in $L^2(\Omega)$ (then $\int_\Omega \varphi^2\,dx = 1$) and $\varphi_k(x) \to \varphi(x)$ for a.e. $x \in \Omega$. Letting $k \to +\infty$ in (200), the hypothesis on (V_k) allow us to conclude that

$$-\Delta\varphi + V_\infty\varphi = \lim_{k\to+\infty} \lambda_1(-\Delta + V_k)\varphi \text{ weakly in } H^2_{1,0}(\Omega).$$

Since $\int_\Omega \varphi^2\,dx = 1$ and we have extracted subsequences, we then get (199).

Finally, we prove the reverse inequality. For $\epsilon > 0$, let $\varphi \in H^2_{1,0}(\Omega)$ be such that

$$\frac{\int_\Omega (|\nabla\varphi|^2 + V_\infty\varphi^2)\,dx}{\int_\Omega \varphi^2\,dx} \leq \lambda_1(-\Delta + V_\infty) + \epsilon.$$

We have

$$\lambda_1(-\Delta + V_k) \leq \lambda_1(-\Delta + V_\infty) + \epsilon + \frac{\int_\Omega |V_k - V_\infty|\varphi^2\, dx}{\int_\Omega \varphi^2\, dx}.$$

The hypothesis of Lemma 3 allow us to conclude that $\int_\Omega |V_k - V_\infty|\varphi^2\, dx \to 0$ as $k \to +\infty$. Therefore $\limsup_{k\to+\infty} \lambda_1(-\Delta + V_k) \leq \lambda_1(-\Delta + V_\infty) + \epsilon$ for all $\epsilon > 0$. Letting $\epsilon \to 0$, we get the reverse inequality and the conclusion of Lemma 3. □

13. Appendix C: Regularity and the Hardy-Schrödinger operator on $\mathbb{R}^n_-$

In this section, we collect some important results from the paper [21] used in the proof of the compactness theorems. First we state the following regularity result:

Theorem 6 ([21], see also [14]). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ ($n \geq 3$) such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that*

$$|f(x, v)| \leq C|v| \left(1 + \frac{|v|^{2^\star(s)-2}}{|x|^s}\right) \text{ for all } x \in \Omega \text{ and } v \in \mathbb{R}.$$

Let $u \in H^2_{1,0}(\Omega)$ be a weak solution of

$$-\Delta u - \frac{\gamma + O(|x|^\theta)}{|x|^2} u = f(x, u) \qquad \text{in } \left(H^2_{1,0}(\Omega)\right)' \tag{201}$$

for some $\theta > 0$. Then there exists $K \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{u(x)}{d(x, \partial\Omega)|x|^{-\beta_-(\gamma)}} = K. \tag{202}$$

Moreover, if $u \geq 0$ and $u \not\equiv 0$, we have that $K > 0$.

The following result characterizes the positive solution to the singular global equation

Proposition 9 ([21]). *Let $\gamma < \frac{n^2}{4}$ and let $u \in C^2(\mathbb{R}^n \setminus \{0\})$ be a nonnegative function such that*

$$\begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u &= 0 \quad \text{in } \mathbb{R}^n_- \\ u &= 0 \quad \text{on } \partial\mathbb{R}^n_- \setminus \{0\}. \end{cases}$$

Then there exist $C_-, C_+ \geq 0$ such that

$$u(x) = C_- \frac{|x_1|}{|x|^{\beta_+(\gamma)}} + C_+ \frac{|x_1|}{|x|^{\beta_-(\gamma)}} \qquad \text{for all } x \in \mathbb{R}^n_-.$$

Next, we recall the existence and behaviour of the singular solution to the homogeneous equation.

Theorem 7 ([21]). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ ($n \geq 3$) such that $0 \in \partial\Omega$. Fix $\gamma < \frac{n^2}{4}$ and $h \in C^1(\overline{\Omega})$ be such that the operator $\Delta - \gamma|x|^{-2} - h$ is coercive. There exists then $\mathcal{H} \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$\begin{cases} -\Delta\mathcal{H} - \frac{\gamma}{|x|^2}\mathcal{H} + h(x)\mathcal{H} = 0 & \text{in } \Omega \\ \mathcal{H} > 0 & \text{in } \Omega \\ \mathcal{H} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

These solutions are unique up to a positive multiplicative constant, and there exists $c>0$ *such that* $\mathcal{H}(x)\simeq_{x\to 0} c\frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}}$.

14. Appendix D: Green's function for the Hardy-Schrödinger operator with boundary singularity on a bounded domain

Definition 1. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^n$, $n\geq 3$, *such that* $0\in\partial\Omega$. *We fix* $\gamma<n^2/4$ *and* $h\in C^{0,\theta}(\overline{\Omega})$, $\theta\in(0,1)$ *such that* $-\Delta-(\gamma|x|^{-2}+h)$ *is coercive. We say that* $G:\Omega\times\Omega\setminus\{(x,x)/\,x\in\Omega\}$ *is a Green's function for* $-\Delta-\gamma|x|^{-2}-h$ *if*

- *For any* $p\in\Omega$, $G_p:=G(p,\cdot)\in L^1(\Omega)$.
- *For all* $f\in C_c^\infty(\Omega)$ *and all* $p\in\Omega$, *then*

$$\varphi(p)=\int_\Omega G_p(x)f(x)\,dx.$$

where $\varphi\in H_{1,0}^2(\Omega)\cap C^0(\Omega)$ *is the unique solution to*

$$-\Delta\varphi-\left(\frac{\gamma}{|x|^2}+h(x)\right)\varphi=f \text{ in }\Omega\,;\ \varphi_{|\partial\Omega}=0.$$

This appendix is devoted to the proof of the following result.

Theorem 8. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^n$ *such that* $0\in\partial\Omega$. *We fix* $\gamma<\frac{n^2}{4}$. *We let* $h\in C^{0,\theta}(\overline{\Omega})$ *be such that* $-\Delta-\gamma|x|^{-2}-h$ *is coercive. Then there exists a unique Green's function* G *for* $-\Delta-\gamma|x|^{-2}-h$. *Moreover:*

I. Properties of G. *The Green's function* G *is such that*

(a) $G_p\in C^{2,\theta}(\overline{\Omega}\setminus\{0,p\})$ *and* $G_p>0$ *for all* $p\in\Omega$.

(b) *For all* $p\in\Omega$ *and all* $\eta\in C_c^\infty(\mathbb{R}^n\setminus\{p\})$, *we have that* $\eta G_p\in H_{1,0}^2(\Omega)$.

(c) *For all* $f\in L^{\frac{2n}{n+2}}(\Omega)\cap L^q(\Omega\setminus B_\delta(0))$, *for all* $\delta>0$ *and some* $q>n/2$, *we have for any* $p\in\Omega$

$$\varphi(p)=\int_\Omega G_p(x)f(x)\,dx. \tag{203}$$

where $\varphi\in H_{1,0}^2(\Omega)\cap C^0(\Omega)$ *is the unique solution to*

$$-\Delta\varphi-\left(\frac{\gamma}{|x|^2}+h(x)\right)\varphi=f \text{ in }\Omega\,;\ \varphi_{|\partial\Omega}=0, \tag{204}$$

In particular,

$$\begin{cases} -\Delta G_p-\left(\frac{\gamma}{|x|^2}+h(x)\right)G_p=0 & \text{in }\Omega\setminus\{p\},\\ G_p>0 & \text{in }\Omega\setminus\{p\},\\ G_p=0 & \text{in }\partial\Omega\setminus\{0\}.\end{cases} \tag{205}$$

II. Asymptotics. G *satisfies the following properties:*

(d) *For all* $p\in\Omega\setminus\{0\}$, *there exists* $c_0(p)>0$ *such that*

$$G_p(x)\sim_{x\to 0} c_0(p)\frac{d(x,\partial\Omega)}{|x|^{\beta_-(\gamma)}} \text{ and } G_p(x)\sim_{x\to p}\frac{1}{(n-2)\omega_{n-1}|x-p|^{n-2}} \tag{206}$$

where

$$\beta_-(\gamma):=\frac{n}{2}-\sqrt{\frac{n^2}{4}-\gamma} \text{ and } \beta_+(\gamma):=\frac{n}{2}+\sqrt{\frac{n^2}{4}-\gamma}.$$

(e) *There exists $c>0$ depending only on γ, the coercivity constant and an upper-bound for $\|h\|_{C^{0,\theta}}$ such that*

$$c^{-1}H_p(x) < G_p(x) < cH_p(x) \text{ for } x\in\Omega-\{0,p\}, \tag{207}$$

where

$$H_p(x) := \left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}}\right)^{\beta_-(\gamma)} |x-p|^{2-n}\min\left\{1,\frac{d(x,\partial\Omega)d(p,\partial\Omega)}{|x-p|^2}\right\}. \tag{208}$$

And

$$|\nabla G_p(x)| \leq c\left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}}\right)^{\beta_-(\gamma)} |x-p|^{1-n}\min\left\{1,\frac{d(p,\partial\Omega)}{|x-p|}\right\} \text{ for } x\in\Omega-\{0,p\}. \tag{209}$$

(f) *There exists $L_{\gamma,\Omega}>0$ such that for any $(h_i)_i\in C^{0,\theta}(\Omega)$ such that $\lim_{i\to+\infty} h_i = h$ in $C^{0,\theta}$, then for any sequences $(x_i)_i,(y_i)_i\in\Omega$ such that*

$$y_i = o(|x_i|) \text{ and } x_i = o(1) \text{ as } i\to+\infty,$$

then, as $i\to+\infty$ we have that

$$G_{h_i}(x_i,y_i) = (L_{\gamma,\Omega}+o(1))\frac{d(x_i,\partial\Omega)}{|x_i|^{\beta_+(\gamma)}}\frac{d(y_i,\partial\Omega)}{|y_i|^{\beta_-(\gamma)}} \tag{210}$$

Notations: In order to simplify notations, we will often drop the dependence in the domain Ω and the dimension $n\geq 3$. If $F:A\times B\to\mathbb{R}$ is a function, then for any $x\in A$, we define $F_x:B\to\mathbb{R}$ by $F_x(y):=F(x,y)$ for all $y\in B$. Finally, we will write $\text{Diag}(A):=\{(x,x)/\,x\in A\}$ for any set A.

We split the proof into several parts.

14.1. Proof of existence and uniqueness of the Green function. We let $\eta_\epsilon(x):=\tilde\eta(\epsilon^{-1}|x|)$ for all $x\in\mathbb{R}^n$ and $\epsilon>0$, where $\tilde\eta\in C^\infty(\mathbb{R})$ is nondecreasing and such that $\tilde\eta(t)=0$ for $t<1$ and $\tilde\eta(t)=1$ for $t>1$. It follows from Lemma 3 (see Appendix B) and the coercivity of $-\Delta-\left(\gamma|x|^{-2}+h\right)$ that there exists $\epsilon_0>0$ and $c>0$ such that such that for all $\varphi\in H^2_{1,0}(\Omega)$ and $\epsilon\in(0,\epsilon_0)$,

$$\int_\Omega\left(|\nabla\varphi|^2-\left(\frac{\gamma\eta_\epsilon}{|x|^2}+h(x)\right)\varphi^2\right)dx \geq c\int_\Omega\varphi^2\,dx.$$

As a consequence, there exists $c>0$ such that for all $\varphi\in H^2_{1,0}(\Omega)$ and $\epsilon\in(0,\epsilon_0)$,

$$\int_\Omega\left(|\nabla\varphi|^2-\left(\frac{\gamma\eta_\epsilon}{|x|^2}+h(x)\right)\varphi^2\right)dx \geq c\|\varphi\|^2_{H_1^2}. \tag{211}$$

Let $G_\epsilon>0$ be the Green's function of $-\Delta-\left(\gamma\eta_\epsilon|x|^{-2}+h\right)$ on Ω with Dirichlet boundary condition. The existence follows from the coercivity and the $C^{0,\theta}$ regularity of the potential for any $\epsilon>0$ (see Robert [30]). In particular, we have that

$$\left\{\begin{array}{ll} -\Delta G_\epsilon(x,\cdot)-\left(\frac{\gamma\eta_\epsilon}{|\cdot|^2}+h\right)G_\epsilon(x,\cdot)=0 & \text{in } \Omega\setminus\{x\}\\ G_\epsilon(x,\cdot)=0 & \text{on } \partial\Omega\end{array}\right. \tag{212}$$

Step 14.1: Integral bounds for G_ϵ. We claim that for all $\delta>0$ and $1<q<\frac{n}{n-2}$ and $\delta'\in(0,\delta)$, there exists $C(\delta,q)>0$ and $C(\delta,\delta')>0$ such that

$$\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)}\leq C(\delta,q) \text{ and } \|G_\epsilon(x,\cdot)\|_{L^{\frac{2n}{n-2}}(\Omega\setminus B_{\delta'}(x))}\leq C(\delta,\delta') \tag{213}$$

for all $x \in \Omega$, $|x| > \delta$. We prove the claim. We fix $f \in C_c^\infty(\Omega)$ and let $\varphi_\epsilon \in C^{2,\theta}(\overline{\Omega})$ be the solution to the boundary value problem

$$\left\{ \begin{array}{ll} -\Delta\varphi_\epsilon - \left(\frac{\gamma\eta_\epsilon}{|x|^2} + h(x)\right)\varphi_\epsilon = f & \text{in } \Omega \\ \varphi_\epsilon = 0 & \text{on } \partial\Omega \end{array} \right. \tag{214}$$

Multiplying the equation by φ_ϵ, integrating by parts on Ω, using (211) and Hölder's inequality, we get that

$$\int_\Omega |\nabla\varphi_\epsilon|^2\, dx \leq C\|f\|_{\frac{2n}{n+2}}\|\varphi_\epsilon\|_{\frac{2n}{n-2}}$$

where $C > 0$ is independent of ϵ, f and φ_ϵ. The Sobolev inequality $\|\varphi\|_{\frac{2n}{n-2}} \leq C\|\nabla\varphi\|_2$ for $\varphi \in H_{1,0}^2(\Omega)$ then yields

$$\|\varphi_\epsilon\|_{\frac{2n}{n-2}} \leq C\|f\|_{\frac{2n}{n+2}}$$

where $C > 0$ is independent of ϵ, f and φ_ϵ. Fix $p > n/2$ and $\delta \in (0, \delta_0)$ and $\delta_1, \delta_2 > 0$ such that $\delta_1 + \delta_2 < \delta$, and $x \in \Omega$ such that $|x| > \delta$. It follows from standard elliptic theory that

$$\begin{array}{rcl} |\varphi_\epsilon(x)| & \leq & \|\varphi_\epsilon\|_{C^0(B_{\delta_1}(x))} \\ & \leq & C\left(\|\varphi_\epsilon\|_{L^{\frac{2n}{n-2}}(B_{\delta_1+\delta_2}(x))} + \|f\|_{L^p(B_{\delta_1+\delta_2}(x))}\right) \\ & \leq & C\left(\|f\|_{L^{\frac{2n}{n+2}}(\Omega)} + \|f\|_{L^p(B_{\delta_1+\delta_2}(x))}\right) \end{array}$$

where $C > 0$ depends on $p, \delta, \delta_1, \delta_2, \gamma$ and $\|h\|_\infty$. Therefore, Green's representation formula yields

$$\left|\int_\Omega G_\epsilon(x,\cdot) f\, dy\right| \leq C\left(\|f\|_{L^{\frac{2n}{n+2}}(\Omega)} + \|f\|_{L^p(B_{\delta_1+\delta_2}(x))}\right) \tag{215}$$

for all $f \in C_c^\infty(\Omega)$. It follows from (215) that

$$\left|\int_\Omega G_\epsilon(x,\cdot) f\, dy\right| \leq C \cdot \|f\|_{L^p(\Omega)}$$

for all $f \in C_c^\infty(\Omega)$ where $p > n/2$. It then follows from duality arguments that for any $q \in (1, n/(n-2))$ and any $\delta > 0$, there exists $C(\delta, q) > 0$ such that $\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta, q)$ for all $\epsilon < \epsilon_0$ and $x \in \Omega \setminus B_\delta(0)$.

Let $\delta' \in (0, \delta)$ and $\delta_1, \delta_2 > 0$ such that $\delta_1 + \delta_2 < \delta'$. We get from (215) that

$$\left|\int_\Omega G_\epsilon(x,\cdot) f\, dy\right| \leq C\|f\|_{L^{\frac{2n}{n+2}}(\Omega\setminus B_{\delta'}(x))} \tag{216}$$

for all $f \in C_c^\infty(\Omega \setminus B_{\delta'}(x))$. Here again, a duality argument yields (213), which proves the claim in Step 14.1.

Using the same method, we can get an improvement of the control, the cost being the integrability exponent q. When $q \in (1, n/(n-1))$, we get that $p > n$. Then, $\|\varphi_\epsilon\|_{C^1(B_{\delta_1}(x)\cap\Omega)}$ is controled by the L^p and $L^{\frac{2n}{n+2}}$ norms. Moreover, $|\varphi_\epsilon(x)| \leq \|\varphi_\epsilon\|_{C^0(B_{\delta_1}(x)\cap\Omega)} d(x, \partial\Omega)$. The argument above then yields

$$\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta, q) d(x, \partial\Omega) \text{ for } q \in \left(1, \frac{n}{n-1}\right). \tag{217}$$

Step 14.2: Convergence of G_ϵ. Fix $x \in \Omega \setminus \{0\}$. For $0 < \epsilon < \epsilon'$, since $G_\epsilon(x,\cdot)$, $G_{\epsilon'}(x,\cdot)$ are C^2 outside x, (212) yields

$$-\Delta(G_\epsilon(x,\cdot) - G_{\epsilon'}(x,\cdot)) - \left(\frac{\gamma\eta_\epsilon}{|\cdot|^2} + h\right)(G_\epsilon(x,\cdot) - G_{\epsilon'}(x,\cdot)) = \frac{\gamma(\eta_\epsilon - \eta_{\epsilon'})}{|\cdot|^2} G_{\epsilon'}(x,\cdot)$$

in the strong sense. The coercivity (211) then yields $G_\epsilon(x,\cdot) \geq G_{\epsilon'}(x,\cdot)$ for $0 < \epsilon < \epsilon'$ if $\gamma \geq 0$, and the reverse inequality if $\gamma < 0$. It then follows from the integral bound (213) and elliptic regularity that there exists $G(x,\cdot) \in C^{2,\theta}(\overline{\Omega} \setminus \{0,x\})$ such that

$$\lim_{\epsilon \to 0} G_\epsilon(x,\cdot) = G(x,\cdot) \geq 0 \text{ in } C^{2,\theta}_{loc}(\overline{\Omega} - \{0,x\}). \tag{218}$$

In particular, G is symmetric and

$$-\Delta G(x,\cdot) - \left(\frac{\gamma}{|\cdot|^2} + h\right) G(x,\cdot) = 0 \text{ in } \Omega \setminus \{x\} \text{ and } G(x,\cdot) = 0 \text{ on } \partial\Omega. \tag{219}$$

Moreover, passing to the limit $\epsilon \to 0$ in (213), (217) and using elliptic regularity, we get that for all $\delta > 0$, $1 < q < \frac{n}{n-2}$ and $\delta' \in (0,\delta)$, there exist $C(\delta,q) > 0$ and $C(\delta,\delta') > 0$ such that for all $x \in \Omega$, $|x| > \delta$,

$$\|G(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta,q) \text{ and } \|G(x,\cdot)\|_{L^{\frac{2n}{n-2}}(\Omega \setminus B_{\delta'}(x))} \leq C(\delta,\delta') \tag{220}$$

and

$$\|G(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta,q) d(x,\partial\Omega) \text{ for } q \in \left(1, \frac{n}{n-1}\right). \tag{221}$$

In particular, for any $x \in \Omega \setminus \{0\}$, $G(x,\cdot) \in L^k(\Omega)$ for all $1 < k < n/(n-2)$ and $G(x,\cdot) \in L^{2n/(n-2)}(\Omega \setminus B_\delta(x))$ for all $\delta > 0$. Moreover, for any $f \in L^{\frac{2n}{n+2}}(\Omega) \cap L^q(\Omega \setminus B_\delta(0))$ for all $\delta > 0$ with $q > n/2$, let $\varphi_\epsilon \in H^2_{1,0}(\Omega)$ be such that (214) holds. It follows from elliptic theory that $\varphi_\epsilon \in C^{0,\tau}(\overline{\Omega} \setminus \{0\})$ for some $\tau \in (0,1)$ and that for all $\delta_1 > 0$, there exists $C(\delta_1) > 0$ such that $\|\varphi_\epsilon\|_{C^{0,\tau}(\overline{\Omega} \setminus B_{\delta_1}(0))} \leq C(\delta_1)$. We fix $x \in \Omega \setminus \{0\}$. Passing to the limit $\epsilon \to 0$ in the Green identity $\varphi_\epsilon(x) = \int_\Omega G_\epsilon(x,\cdot) f\, dy$ yields

$$\varphi(x) = \int_\Omega G(x,\cdot) f\, dy \text{ for all } x \in \Omega \setminus \{0\} \tag{222}$$

where $\varphi \in H^2_{1,0}(\Omega) \cap C^0(\overline{\Omega} \setminus \{0\})$ is the only weak solution to

$$\begin{cases} -\Delta\varphi - \left(\frac{\gamma}{|x|^2} + h(x)\right)\varphi = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

Since $G(x,\cdot) \geq 0$, (219) and the strong comparison principle yield $G(x,\cdot) > 0$. These points prove that G is a Green's function for the operator and that (c) holds.

We now prove point (b). We fix $\eta \in C^\infty_c(\mathbb{R}^n - \{x\})$ such that $\eta(y) = 1$ when $y \in B_\delta(0)$ for some $\delta > 0$. Then $\eta G_\epsilon(x,\cdot) \in C^{2,\theta}(\overline{\Omega}) \cap H^2_{1,0}(\Omega)$. It follows from (212) and (218) that

$$-\Delta(\eta G_\epsilon(x,\cdot)) - \left(\frac{\gamma\eta_\epsilon}{|\cdot|^2} + h\right)(\eta G_\epsilon(x,\cdot)) = \mathbf{1}_{B_\delta(0)^c} f_\epsilon \text{ in } \Omega$$

where $\|f_\epsilon\|_{C^0(\overline{\Omega})} \leq C$ for some $C > 0$ and all $\epsilon > 0$. Therefore, with the coercivity (211) and the convergence (218), we get that

$$c\|\eta G_\epsilon(x,\cdot)\|^2_{H^2_1} \leq \int_{\Omega \setminus B_\delta(0)} f_\epsilon \eta G_\epsilon(x,\cdot)\, dy \leq C$$

for all $\epsilon > 0$. Reflexivity yields convergence of $(\eta G_\epsilon(x,\cdot))$ in $H^2_{1,0}(\Omega)\cap L^2(\Omega)$ as $\epsilon \to 0$ up to extraction. The convergence in C^2 and uniqueness then yields $\eta G(x,\cdot) \in H^2_{1,0}(\Omega)$ and $\eta G_\epsilon(x,\cdot) \to \eta G(x,\cdot)$ in $H^2_{1,0}(\Omega)$ as $\epsilon \to 0$. The case of a general η is a direct consequence. This proves point (b).

For the uniqueness, we suppose G' be another Green's function. We fix $x \in \Omega$ and we define $H_x := G_x - G'_x$. Then $H_x \in L^1(\Omega)$ and for any $f \in C_c^\infty(\Omega)$, we have that $\int_\Omega H_x f\, dy = 0$. Approximating a compactly supported function by smooth fonctions with compact support, we get that this equality holds for all $f \in C^0_c(\Omega)$. Integration theory then yields $H_x \equiv 0$, and then $G'_x \equiv G_x$. This proves uniqueness. This finishes the proof of (a).

This proves existence and uniqueness of the Green's function in Theorem 8(I).

14.2. Proof of the upper bound. The behavior (206) is a consequence of the classification of solutions to harmonic equations and Theorem 4.1 in Ghoussoub-Robert [21].

In the proof, we will often use sub- and super-solutions to the linear problem. The following existence result is contained in Proposition 4.3 of [21]:

PROPOSITION 10. *Let Ω be a smooth domain and $h \in C^0(\overline{\Omega})$ be a continuous fonction. We fix $\gamma < \frac{n^2}{4}$ and $\beta \in \{\beta_-(\gamma), \beta_+(\gamma)\}$. Then, there exist $r > 0$, and $\overline{u}_\beta, \underline{u}_\beta \in C^\infty(\overline{\Omega}\setminus\{0\})$ such that*

$$\begin{cases} \overline{u}_\beta, \underline{u}_\beta = 0 & \text{on } \partial\Omega\cap B_r(0)\\ -\Delta \overline{u}_\beta - \left(\frac{\gamma}{|x|^2}+h\right)\overline{u}_\beta > 0 & \text{in } \Omega\cap B_r(0)\\ -\Delta \underline{u}_\beta - \left(\frac{\gamma}{|x|^2}+h\right)\underline{u}_\beta < 0 & \text{in } \Omega\cap B_r(0). \end{cases} \tag{223}$$

Moreover, for some $\tau > 0$, we have that, as $x \to 0$, $x \in \Omega$,

$$\overline{u}_\beta(x) = \underline{u}_\beta(x)(1+O(|x|^\tau)) = \frac{d(x,\partial\Omega)}{|x|^\beta}(1+O(|x|^\tau)). \tag{224}$$

Step 14.3: Upper bound for $G(x,y)$ when one variable is far from 0.

Step 14.3.1: It follows from (219), elliptic theory, (221) and (220) that for any $\delta > 0$, there exists $C(\delta) > 0$ such that

$$0 < G(x,y) \le C(\delta) d(y,\partial\Omega) d(x,\partial\Omega) \text{ for } x,y \in \Omega \text{ s.t. } |x|,|y| > \delta,\ |x-y| > \delta. \tag{225}$$

Step 14.3.2: We claim that for any $\delta > 0$, there exists $C(\delta) > 0$ such that

$$|x-y|^{n-2} G(x,y) \le C(\delta)\min\left\{1, \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\} \text{ for } x,y \in \Omega \text{ s.t. } |x|,|y| > \delta. \tag{226}$$

Indeed, with no loss of generality, we can assume that $\delta \in (0,\delta_0)$. Let Ω_δ be a smooth domain of $\mathbb{R}^n$ be such that $\Omega\setminus B_{3\delta/4}(0) \subset \Omega_\delta \subset \Omega \setminus B_{\delta/2}(0)$. We fix $x \in \Omega$ such that $|x| > \delta$. Let H_x be the Green's function for $-\Delta - \left(\frac{\gamma}{|x|^2}+h(x)\right)$ in Ω_δ with Dirichlet boundary condition. Classical estimates (see [30]) yield the existence of $C(\delta) > 0$ such that

$$|x-y|^{n-2} H_x(y) \le C(\delta)\min\left\{1, \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\} \text{ for all } x,y \in \Omega_\delta.$$

It is easy to check that

$$\left\{\begin{array}{ll} -\Delta(G_x-H_x)-\left(\frac{\gamma}{|\cdot|^2}+h\right)(G_x-H_x)=0 & \text{weakly in } \Omega_\delta \\ G_x-H_x=0 & \text{on } (\partial\Omega_\delta)\setminus B_{3\delta/4}(0) \\ G_x-H_x=G_x & \text{on } (\partial\Omega_\delta)\cap B_{3\delta/4}(0). \end{array}\right.$$

Regularity theory then yields that $G_x-H_x\in C^{2,\theta}(\overline{\Omega_\delta})$. It follows from (225) that $G_x(y)\leq C_1(\delta)d(y,\partial\Omega)d(x,\partial\Omega)$ on $(\partial\Omega_\delta)\cap B_{3\delta/4}(0)$ for $|x|>\delta$. The comparison principle then yields $G_x(y)-H_x(y)\leq C_1(\delta)d(y,\partial\Omega)d(x,\partial\Omega)$ for $y\in\Omega_\delta$ and $|x|>\delta$. The above bound for H_x and (225) then yields (226).

Step 14.3.3: We now claim that for any $0<\delta'<\delta$, there exists $C(\delta,\delta')>0$ such that

$$|y|^{\beta_-(\gamma)}G(x,y)\leq C(\delta,\delta')d(y,\partial\Omega)d(x,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |x|>\delta>\delta'>|y|. \tag{227}$$

We let $\delta_1\in(0,\delta')$ that will be fixed later. We use (225) to deduce that $G_x(y)\leq C(\delta,\delta_1)d(x,\partial\Omega)d(y,\partial\Omega)$ for all $x\in\Omega\setminus B_\delta(0)$ and $y\in\partial B_{\delta_1}(0)\cap\Omega$. Since $\delta_1<|x|$, we have that

$$\left\{\begin{array}{ll} -\Delta G_x-\left(\frac{\gamma}{|x|^2}+h\right)G_x=0 & \text{in } \Omega\cap B_{\delta_1}(0) \\ 0\leq G_x\leq C(\delta,\delta_1)d(y,\partial\Omega)d(x,\partial\Omega) & \text{on } \partial(\Omega\cap B_{\delta_1}(0))\setminus\{0\}. \end{array}\right.$$

We choose a supersolution $\overline{u}_{\beta_-(\gamma)}$ as in (223) of Proposition 10. It follows from (224) and (225) that for $\delta_1>0$, there exists $C(\delta,\delta_1)>0$ such that $G_x(z)\leq C(\delta,\delta_1)d(x,\partial\Omega)u_{\beta_-}(z)$ for all $z\in\partial(\Omega\cap B_{\delta_1}(0))$. It then follows from the comparison principle that $G_x(y)\leq C(\delta,\delta_1)d(x,\partial\Omega)u_{\beta_-}(y)$ for all $y\in(\Omega\cap B_{\delta_1}(0))\setminus\{0\}$. Combining this with (225) and (223), we obtain (227).

Note that by symmetry, we also get that for any $0<\delta'<\delta$, there exists $C(\delta,\delta')>0$ such that

$$|x|^{\beta_-(\gamma)}G(x,y)\leq C(\delta,\delta')d(x,\partial\Omega)d(y,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |y|>\delta>\delta'>|x|. \tag{228}$$

Step 14.4: Upper bound for $G(x,y)$ when both variables approach 0.

We claim first that for all $c_1,c_2,c_3>0$, there exists $C(c_1,c_2,c_3)>0$ such that for $x,y\in\Omega$ such that $c_1|x|<|y|<c_2|x|$ and $|x-y|>c_3|x|$, we have

$$|x-y|^{n-2}G(x,y)\leq C(c_1,c_2,c_3)\frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x|^2}. \tag{229}$$

When one of the variables stays far from 0, (229) is a consequence of (225). We now consider a chart $\mathcal{T}$ at 0 as in (27). In particular, there is $\delta_0>0$, $0\in V\subset\mathbb{R}^n$ and $\mathcal{T}:B_{2\delta_0}(0)\to V$ a smooth diffeomorphism such that $\mathcal{T}(0)=0$ and

$$\mathcal{T}(B_{2\delta_0}(0)\cap\mathbb{R}^n_-)=\mathcal{T}(U)\cap\Omega \text{ and } \mathcal{T}(B_{2\delta_0}(0)\cap\partial\mathbb{R}^n_-)=\mathcal{T}(U)\cap\partial\Omega. \tag{230}$$

Moreover, $D_0\mathcal{T}=\mathbb{I}_{\mathbb{R}^n}$ and

$$|\mathcal{T}(X)|=(1+O(|X|))|X| \text{ for all } X\in B_{3\delta_0/2}(0). \tag{231}$$

We fix $X\in\mathbb{R}^n_-$ such that $0<|X|<3\delta_0/2$. We define

$$H(z):=G_{\mathcal{T}(X)}(\mathcal{T}(|X|z)) \text{ for } z\in B_{\delta_0/|X|}(0)\setminus\left\{0,\frac{X}{|X|}\right\},$$

so that

$$-\Delta_{g_X} H - \left(\frac{\gamma}{\left(\frac{|\mathcal{T}(|X|z|)}{|X|}\right)^2} + |X|^2 h(\mathcal{T}(|X|z))\right) H = 0 \text{ in } B_{\delta_0/|X|}(0) \setminus \left\{0, \frac{X}{|X|}\right\}.$$

where $g_X := (\mathcal{T}^\star \text{Eucl})_X$ is the pulled-back metric of the Euclidean metric Eucl via the chart $\mathcal{T}$ at the point X. Since $H > 0$, it follows from the Harnack inequality on the boundary (see Proposition 6.3 in Ghoussoub-Robert [21]) that for all $R > 0$ large enough and $r > 0$ small enough, there exist $\delta_1 > 0$ and $C > 0$ independent of $|X| < 3\delta_0/2$ such that

$$\frac{H(z)}{|z_1|} \leq C\frac{H(z')}{|z_1'|} \text{ for all } z, z' \in (B_R(0) \cap \mathbb{R}^n_-) \setminus \left(B_r(0) \cup B_r\left(\frac{X}{|X|}\right)\right),$$

which, via the chart $\mathcal{T}$, yields

$$\frac{G_x(y)}{d(y, \partial\Omega)} \leq C\frac{G_x(y')}{d(y', \partial\Omega)} \text{ for all } y, y' \in \Omega \cap B_{R|x|/2}(0) \setminus \left(B_{2r|x|}(0) \cup B_{2r|x|}(x)\right). \tag{232}$$

for all $x \in \Omega$ such that $|x| < \delta_0$. We let W be a smooth domain of $\mathbb{R}^n$ such that for some $\lambda > 0$ small enough, we have

$$B_\lambda(0) \cap \Omega \subset W \subset B_{2\lambda}(0) \cap \Omega \text{ and } B_\lambda(0) \cap \partial W = B_\lambda(0) \cap \partial\Omega. \tag{233}$$

We choose a subsolution $\underline{u}_{\beta_+(\gamma)}$ as in (223) of Proposition 10. It follows from (224) and (225) that for $|x| < \delta_2$ small

$$G_x(z) \geq C(R)|x|^{\beta_+(\gamma)} \left(\inf_{y \in \Omega \cap \partial B_{R|x|}(0)} \frac{G_x(y)}{d(y, \partial\Omega)}\right) \underline{u}_{\beta_+(\gamma)}(z) \text{ for all } z \in W \cap \partial B_{R|x|/3}(0).$$

Since $-\Delta G_x - (\gamma|\cdot|^{-2} + h)G_x = 0$ outside 0, it follows from coercivity and the comparison principle that

$$G_x(z) \geq c|x|^{\beta_+(\gamma)} \left(\inf_{y \in \Omega \cap \partial B_{R|x|}(0)} \frac{G_x(y)}{d(y, \partial\Omega)}\right) \underline{u}_{\beta_+(\gamma)}(z) \text{ for all } z \in W \setminus B_{R|x|/3}(0).$$

We fix $z_0 \in W \setminus \{0\}$. Then for δ_3 small enough, when $|x| < \delta_3$, it follows from (228) and the Harnack inequality (232) that there exists $C > 0$ independent of x such that

$$G_x(y) \leq C|x|^{-\beta_+(\gamma) - \beta_-(\gamma)} d(x, \partial\Omega) d(y, \partial\Omega) \text{ for all } y \in B_{R|x|}(0) \setminus \left(B_{r|x|}(0) \cup B_{r|x|}(x)\right)$$

Taking $r > 0$ small enough and $R > 0$ large enough, we then get (229) for $|x| < \delta_3$. The general case for arbitrary $x \in \Omega \setminus \{0\}$ then follows from (226). This completes the proof of (229).

Step 14.4.2: We claim that for all $c_1, c_2 > 0$, there exists $C(c_1, c_2) > 0$ such that

$$|x - y|^{n-2} G(x, y) \leq C(c_1, c_2) \min\left\{1, \frac{d(x, \partial\Omega) d(y, \partial\Omega)}{|x - y|^2}\right\} \tag{234}$$

for all $x, y \in \Omega$ s.t. $c_1|x| < |y| < c_2|x|$. To prove (234), we distinguish three cases:

Case 1: We assume that

$$|x| \leq C_1 d(x, \partial\Omega) \text{ with } C_1 > 1. \tag{235}$$

We define

$$H(z) := |x|^{n-2} G_x(x + |x|z) \text{ for } z \in B_{1/C_1}(0) \setminus \{0\}.$$

Note that this definition makes sense since for such z, $x+|x|z\in\Omega$. We then have that $H\in C^2(\overline{B_{1/(2C_1)}(0)}\setminus\{0\})$ and

$$-\Delta H-\left(\frac{\gamma}{\left|\frac{x}{|x|}+z\right|^2}+|x|^2h(x+|x|z)\right)H=\delta_0 \text{ weakly in } B_{1/(2C_1)}(0).$$

We now argue as in the proof of (226). From (229), we have that $|H(z)|\leq C$ for all $z\in\partial B_{1/(2C_1)}(0)$ where C is independent of $x\in\Omega\setminus\{0\}$ satisfying (235). Let Γ_0 be the Green's function of $-\Delta-\left(\frac{\gamma}{\left|\frac{x}{|x|}+z\right|^2}+|x|^2h(x+|x|z)\right)$ at 0 on $B_{1/(2C_1)}(0)$ with Dirichlet boundary condition. Therefore, $H-\Gamma_0\in C^2(\overline{B_{1/(2C_1)}(0)})$ and, via the comparison principle, it is bounded by its supremum on the boundary. Therefore $|z|^{n-2}H(z)\leq C$ for all $B_{1/(2C_1)}(0)\setminus\{0\}$ where C is independent of $x\in\Omega\setminus\{0\}$ satisfying (235). Scaling back and using (229), we get $|x-y|^{n-2}G_x(y)\leq C$ for all $x,y\in\Omega\setminus\{0\}$ such that $c_1|x|<|y|<c_2|x|$ and (235) holds. This proves (234) if $d(x,\partial\Omega)d(y,\partial\Omega)\geq|x-y|^2$. If $d(x,\partial\Omega)d(y,\partial\Omega)<|x-y|^2$, we get that $d(x,\partial\Omega)<2|x-y|$, and then (235) yields $|x|\leq 2C_1|x-y|$, and (234) is a consequence of (229).

This ends the proof of (234) in Case 1.

Case 2: By symmetry, (234) also holds when $|y|\leq C_1d(y,\partial\Omega)$.

Case 3: We assume that $d(x,\partial\Omega)\leq C_1^{-1}|x|$ and $d(y,\partial\Omega)\leq C_1^{-1}|y|$. We consider a chart at 0, that is $\delta_0>0$, $0\in V\subset\mathbb{R}^n$ and $\mathcal{T}:B_{2\delta_0}(0)\to V$ a smooth diffeomorphism such that $\mathcal{T}(0)=0$ and that (230) and (231) hold. We fix $x'\in\mathbb{R}^{n-1}$ such that $0<|x'|<3\delta_0/2$.

We assume that $r\leq c_0|x'|$. We define

$$H_y(z):=r^{n-2}G_{\mathcal{T}((0,x')+ry)}(\mathcal{T}((0,x')+rz)) \text{ for } y,z\in B_{\delta_0/(2r)}(0)\cap\mathbb{R}^n_-\setminus\{0\}.$$

We then have that $H_y\in C^2(\overline{B_{R_0}(0)}\cap\mathbb{R}^n_-\setminus\{0,y\})$ and

$$-\Delta_{g_r}H_y-\left(\frac{\gamma}{\left(\frac{|\mathcal{T}((0,x')+rz)}{r}\right)^2}+r^2h(\mathcal{T}((0,x')+rz))\right)H_y=\delta_y \text{ weakly in } B_{R_0}(0)\cap\mathbb{R}^n_-,$$

where $g_r:=(\mathcal{T}^\star\mathrm{Eucl})_{(0,x')+rz}$ is the pulled-back metric of the Euclidean metric Eucl via the chart $\mathcal{T}$ at the point $(0,x')+rz$. We now argue as in the proof of (226). From (178), we have that $|H_y(z)|\leq C$ for all $z\in\partial B_{R_0}(0)\cap\mathbb{R}^n_-$ where C is independent of $y\in B_{R_0/2}(0)$ and $r\in(0,\delta_0/4)$. Let Γ_y be the Green's function of $-\Delta_{g_r}-\left(\frac{\gamma}{\left(\frac{|\mathcal{T}((0,x')+rz)}{r}\right)^2}+r^2h(\mathcal{T}((0,x')+rz))\right)$ at y on $B_{c_0/2}(0)\cap\mathbb{R}^n_-$ with Dirichlet boundary condition. Therefore, $H_y-\Gamma_y\in C^2(\overline{B_{c_0/2}(0)\cap\mathbb{R}^n_-})$ and, via the comparison principle, it is bounded by its supremum on the boundary. It follows from (178) and elliptic estimates for Γ_y (see for instance [30]) that $|H_y-\Gamma_y|(z)\leq C|y_1|\cdot|z_1|$ for $z\in\partial(B_{c_0/2}(0)\cap\mathbb{R}^n_-)$ and $y\in B_{c_0/4}(0)\cap\mathbb{R}^n_-$. Applying elliptic estimates, we then get that $|H_y-\Gamma_y|(z)\leq C|y_1|\cdot|z_1|$ for $z\in B_{c_0/2}(0)\cap\mathbb{R}^n_-$ and

$y \in B_{c_0/4}(0) \cap \mathbb{R}^n_-$, and since

$$\Gamma_y(z) \leq C|z-y|^{2-n} \min\left\{1, \frac{|y_1|\cdot|z_1|}{|y-z|^2}\right\} \text{ for all } y,z \in B_{c_0/2}(0)\cap\mathbb{R}^n_-$$

(see [30]), we get that

$$|z-y|^{n-2}H_y(z) \leq C \min\left\{1, \frac{|y_1|\cdot|z_1|}{|y-z|^2}\right\} \text{ for all } y,z \in B_{c_0/2}(0)\cap\mathbb{R}^n_-$$

where C is independent of $x' \in B_{\delta_0/2}(0)\setminus\{0\}$. This yields

$$|rz-ry|^{n-2}G_{\mathcal{T}((0,x')+ry)}(\mathcal{T}((0,x')+rz)) \leq C\min\{1, \frac{|y_1|\cdot|z_1|}{|y-z|^2}\} \tag{236}$$

for $|x'|<\delta_0/3$, $r \leq c_0|x'|$ and $|y|,|z|\leq c_0/4$.

We now prove (234) in the last case. We fix $x \in \Omega\setminus\{0\}$ such that $|x|<\delta_0/3$. We assume that $d(x,\partial\Omega) \leq C_1^{-1}|x|$, $d(y,\partial\Omega)\leq C_1^{-1}|y|$ and $|x-y|\leq \epsilon_0|x|$. We let $(x_1,x'),(y_1,y')\in B_{\delta_0}(0)$ be such that $x=\mathcal{T}(x_1,x')$ and $y=\mathcal{T}(y_1,y')$. Taking the norm $|(x_1,x')|=|x_1|+|x'|$, we define $r:=\max\{d(x,\partial\Omega),|x-y|\}$. Using that $|X|/2\leq|\mathcal{T}(X)|\leq 2|X|$ for $X\in B_{\delta_0}(0)$, up to taking $\epsilon_0>0$ small and $C_1,c_0>1$ large enough, we get that

$$\left|\frac{x_1}{r}\right| \leq \frac{c_0}{4}\ ,\ \left|\left(\frac{y_1}{r},\frac{y'-x'}{r}\right)\right| \leq \frac{c_0}{4} \text{ and } r\leq c_0|x'|.$$

Therefore, (236) applies and we get (234) in Case 3.

We are now in position to conclude. Inequality (234) is a consequence of Cases 1, 2, 3, (226) and (178). This ends the proof of (234).

Step 14.4.3: We now show that there exists $C>0$ such that

$$|y|^{\beta_-(\gamma)}|x|^{\beta_+(\gamma)}G(x,y) \leq Cd(x,\partial\Omega)d(y,\partial\Omega) \text{ for } x,y\in\Omega \text{ such that } |y|<\frac{1}{2}|x|. \tag{237}$$

The proof goes essentially as in (227). For $|x|<\delta$ with $\delta>0$ small, we have that

$$-\Delta G_x - \left(\frac{\gamma}{|\cdot|^2}+h\right)G_x = 0 \text{ in } H^1(\Omega\cap B_{|x|/3}(0))\cap C^2(\overline{\Omega}\cap B_{|x|/3}(0)\setminus\{0\}).$$

It follows from (178) that $G_x(y)\leq C|x|^{-n}d(x,\partial\Omega)d(y,\partial\Omega)$ in $\Omega\cap\partial B_{|x|/3}(0)$. We choose a supersolution $\overline{u}_{\beta_-(\gamma)}$ as in (223) of Proposition 10. It follows from (224) and (178) that there exists $C>0$ such that

$$G_x(y)\leq C|x|^{-\beta_+(\gamma)}d(x,\partial\Omega)\overline{u}_{\beta_-(\gamma)}(y) \text{ for all } y\in\Omega\cap\partial B_{|x|/3}(0).$$

The comparison principle yields that this inequality holds on $\Omega\cap B_{|x|/3}(0)$.

Step 14.4.4: By symmetry, we conclude that there exists $C>0$ such that

$$|x|^{\beta_-(\gamma)}|y|^{\beta_+(\gamma)}G(x,y)\leq Cd(x,\partial\Omega)d(y,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |x|<\frac{1}{2}|y|. \tag{238}$$

Step 14.5: Finally, it follows from (237), (238) and (234) that there exists $c>0$ such that

$$G(x,y)\leq c\left(\frac{\max\{|y|,|x|\}}{\min\{|y|,|x|\}}\right)^{\beta_-(\gamma)}|x-y|^{2-n}\min\left\{1,\frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\} \tag{239}$$

for all $x, y \in \Omega$, $x \neq y$. This proves the upper bound in (207) of Theorem 8. The lower-bound and the control of the gradient will be proved in Section 14.4.

14.3. Behavior at infinitesimal scale. We prove three convergence results to get a comprehensive behavior of the Green's function. Throughout this subsection, we assume Ω is a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$ and let $h \in C^{0,\theta}(\overline{\Omega})$ be such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. We consider G to be the Green's function of $-\Delta - \gamma|x|^{-2} - h$ with Dirichlet boundary condition on $\partial\Omega$.

LEMMA 4. *Let $(x_i)_i \in \Omega$ and $(r_i)_i \in (0, +\infty)$ be such that*

$$\lim_{i \to +\infty} r_i = 0 \text{ and } \lim_{i \to +\infty} \frac{d(x_i, \partial\Omega)}{r_i} = +\infty.$$

Then, for all $X, Y \in \mathbb{R}^n$ such that $X \neq Y$, we have that

$$\lim_{i \to +\infty} r_i^{n-2} G(x_i + r_i X, x_i + r_i Y) = \frac{1}{(n-2)\omega_{n-1}} |X - Y|^{2-n}$$

Moreover, the convergence holds in $C^2_{loc}((\mathbb{R}^n)^2 \setminus Diag(\mathbb{R}^n))$.

To deal with the case when the points approach the boundary, we consider a chart $\mathcal{T}$ as in (27). In particular, $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$.

LEMMA 5. *Let $(x_i)_i \in \partial\Omega$ and $(r_i)_i \in (0, +\infty)$ and $x_0 \in \partial\Omega$ be such that*

$$\lim_{i \to +\infty} r_i = 0, \lim_{i \to +\infty} x_i = x_0 \in \partial\Omega \text{ and } \lim_{i \to +\infty} \frac{|x_i|}{r_i} = +\infty.$$

We let $\mathcal{T}$ be a chart at x_0 as in (27). We define $x_i' \in \mathbb{R}^{n-1}$ such that $x_i = \mathcal{T}(0, x_i')$. Then, for all $X, Y \in \mathbb{R}^n_-$ such that $X \neq Y$, we have that

$$\begin{aligned} &\lim_{i \to +\infty} r_i^{n-2} G(\mathcal{T}\left((0, x_i') + r_i X\right), \mathcal{T}\left((0, x_i') + r_i Y\right)) \\ &= \frac{1}{(n-2)\omega_{n-1}} \left(|X - Y|^{2-n} - |X - Y^*|^{2-n}\right) \end{aligned}$$

where $(Y_1, Y')^* = (-Y_1, Y')$ for $(Y_1, Y') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Moreover, the convergence holds in $C^2_{loc}((\overline{\mathbb{R}^n_-})^2 \setminus Diag(\overline{\mathbb{R}^n_-})\})$.

LEMMA 6. *Let $(r_i)_i \in (0, +\infty)$ be such that $\lim_{i \to +\infty} r_i = 0$. We let $\mathcal{T}$ be a chart at 0 as in (27). Then, for all $X, Y \in \overline{\mathbb{R}^n_-} \setminus \{0\}$ such that $X \neq Y$, we have that*

$$\lim_{i \to +\infty} r_i^{n-2} G(\mathcal{T}\left(r_i X\right), \mathcal{T}\left(r_i Y\right)) = \mathcal{G}(X, Y)$$

where $\mathcal{G}(X, Y) = \mathcal{G}_X(Y)$ is the Green's function for $-\Delta - \gamma|x|^{-2}$ on $\mathbb{R}^n_-$ with Dirichlet boundary condition. Moreover, the convergence holds in $C^2_{loc}((\overline{\mathbb{R}^n_-} \setminus \{0\})^2 \setminus Diag(\overline{\mathbb{R}^n_-} \setminus \{0\}))$.

Proof of Lemma 4: We let $(r_i)_i \in (0, +\infty)$ and $(x_i)_i \in \Omega$ as in the statement of the lemma. For any $X, Y \in \mathbb{R}^n$, $X \neq Y$, we define

$$G_i(X, Y) := r_i^{n-2} G(x_i + r_i X, x_i + r_i Y)$$

for all $i \in \mathbb{N}$. Since $r_i = o(d(x_i, \partial\Omega))$ as $i \to +\infty$, for any $R > 0$, there exists $i_0 \in \mathbb{N}$ such that this definition makes sense for any $X, Y \in B_R(0)$. Equation (205) yields

$$- \Delta G_i(X, \cdot) - \left(\frac{\gamma}{\left| \frac{x_i}{r_i} + \cdot \right|^2} + r_i^2 h(x_i + r_i \cdot) \right) G_i(X, \cdot) = 0 \text{ in } B_R(0) \setminus \{X\}. \quad (240)$$

The pointwise control (239) writes

$$0 < G_i(X, Y) \leq c \left(\frac{\max\{|x_i + r_i X|, |x_i + r_i Y|\}}{\min\{|x_i + r_i X|, |x_i + r_i Y|\}} \right)^{\beta_-(\gamma)} |X - Y|^{2-n} \quad (241)$$

for all $X, Y \in B_R(0)$ such that $X \neq Y$. Since $0 \in \partial\Omega$, we have that $d(x_i, \partial\Omega) \leq |x_i|$, and therefore $r_i = o(|x_i|)$ as $i \to +\infty$. Equation (240) and inequality (241) yield

$$-\Delta G_i(X, \cdot) + \theta_i(X, \cdot) G_i(X, \cdot) = 0 \text{ in } B_R(0) \setminus \{X\}.$$

where $\theta_i \to 0$ uniformly in $C^0_{loc}((\mathbb{R}^n)^2)$ and $0 < G_i(X, Y) \leq c|X - Y|^{2-n}$ for all $X, Y \in B_R(0)$ such that $X \neq Y$. It then follows from standard elliptic theory that, up to a subsequence, there exists $G_\infty(X, \cdot) \in C^2(\mathbb{R}^n \setminus \{X\})$ such that $G_i(X, \cdot) \to G_\infty(X, \cdot) \geq 0$ in $C^2_{loc}(\mathbb{R}^n \setminus \{X\})$ and

$$-\Delta G_\infty(X, \cdot) = 0 \text{ in } \mathbb{R}^n \setminus \{X\} \text{ and } G_\infty(X, Y) \leq c|X - Y|^{2-n} \text{ for } X, Y \in \mathbb{R}^n, \ X \neq Y.$$

It then follows from the classification of positive harmonic functions that there exists $\lambda > 0$ such that $G_\infty(X, Y) = \lambda|X - Y|^{2-n}$ for all $X, Y \in \mathbb{R}^n$, $X \neq Y$.

We fix $\varphi \in C^\infty_c(\mathbb{R}^n)$. We define $\varphi_i(x) := \varphi(r_i^{-1}(x - x_i))$ for $x \in \Omega$ (this makes sense for i large enough). It follows from (204) that

$$\varphi_i(x_i + r_i X) = \int_\Omega G(x_i + r_i X, y) \left(-\Delta \varphi_i(y) - \left(\frac{\gamma}{|y|^2} + h(y) \right) \varphi_i(y) \right) dy.$$

Via a change of variable, and passing to the limit, we get that

$$\varphi(X) = \int_{\mathbb{R}^n} G_\infty(X, Y) \left(-\Delta \varphi(Y) \right) dy.$$

Since $G_\infty(X, Y) = \lambda|X - Y|^{2-n}$, we get that $\lambda = 1/((n-2)\omega_{n-1})$. Since the limit is unique, the convergence holds without extracting a subsequence. The convergence in $C^2_{loc}((\mathbb{R}^n)^2 \setminus \text{Diag}(\mathbb{R}^n))$ follows from the symmetry of G and elliptic theory. □

Proof of Lemma 5: The proof goes as in the proof of lemma 4, except that we have to take a chart due to the closeness of the boundary. We let $(r_i)_i \in (0, +\infty)$, $(x_i)_i \in \partial\Omega$ and $x_0 \in \partial\Omega$ as in the statement of the lemma. We let $\mathcal{T}$ be a chart at x_0 as in (27) (in particular $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$) and we set $x_i' \in \mathbb{R}^n$ such that $x_i = \mathcal{T}(0, x_i')$. In particular, $\lim_{i \to +\infty} x_i' = 0$. For any $X, Y \in \overline{\mathbb{R}^n_-}$, $X \neq Y$, we define

$$G_i(X, Y) := r_i^{n-2} G(\mathcal{T}((0, x_i') + r_i X), \mathcal{T}((0, x_i') + r_i Y))$$

for all $i \in \mathbb{N}$. Here again, provided X, Y remain in a given compact set, the definition of G_i makes sense for large i. Equation (205) then rewrites

$$- \Delta_{g_i} G_i(X, \cdot) - \hat{\theta}_i G_i(X, \cdot) = 0 \text{ in } B_R(0) \cap \mathbb{R}^n_- \setminus \{X\} \ ; \ G_i(X, \cdot) \equiv 0 \text{ on } \partial\mathbb{R}^n_- \cap B_R(0) \quad (242)$$

where

$$\hat{\theta}_i(Y) := \frac{\gamma}{\left| \frac{\mathcal{T}((0, x_i') + r_i Y)}{r_i} \right|^2} + r_i^2 h(\mathcal{T}((0, x_i') + r_i Y))$$

and $g_i = \mathcal{T}^\star \mathrm{Eucl}((0,x_i')+r_i\cdot)$ is the pull-back of the Euclidean metric. In particular, since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, we get that $g_i \to \mathrm{Eucl}$ in $C^2_{loc}(\mathbb{R}^n)$. Since $r_i = o(|x_i|)$, we get that $r_i = o(|x_i'|)$ as $i \to +\infty$, and, using again that $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, we get that $\hat{\theta}_i \to 0$ uniformly in $B_R(0)\cap\mathbb{R}^n_-$. The pointwise control (239) rewrite $G_i(X,Y) \leq c|X-Y|^{2-n}$ for all $X,Y\in\mathbb{R}^n_-$, $X\neq Y$. With the same arguments as above, we get that for any $X\in\overline{\mathbb{R}^n_-}$, there exists $G_\infty(X,\cdot)\in C^2(\overline{\mathbb{R}^n_-}\setminus\{X\})$ such that

$$\lim_{i\to+\infty} G_i(X,\cdot) = G_\infty(X,\cdot) \text{ in } C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{X\})$$

$$\text{with} \left\{ \begin{array}{ll} -\Delta G_\infty(X,\cdot) = 0 & \text{in } \mathbb{R}^n_-\setminus\{X\} \\ G_\infty(X,\cdot)\geq 0 & \\ G_\infty(X,\cdot)\equiv 0 & \text{on } \partial\mathbb{R}^n_-\setminus\{X\} \end{array} \right.$$

and

$$\varphi(X) = \int_{\mathbb{R}^n_-} G_\infty(X,\cdot)(-\Delta\varphi)\,dY \text{ for all } \varphi\in C^\infty_c(\mathbb{R}^n_-).$$

with $0\leq G_\infty(X,Y)\leq c|X-Y|^{2-n}$ for all $X,Y\in\mathbb{R}^n_-$, $X\neq Y$. Define

$$\Gamma_{\mathbb{R}^n_-}(X,Y) = \frac{1}{(n-2)\omega_{n-1}}\left(|X-Y|^{2-n} - |X-Y^*|^{2-n}\right).$$

As one checks (see for instance [30]), $\Gamma_{\mathbb{R}^n_-}$ satisfies the same properties as G_∞. We set $f := G_\infty(X,\cdot) - \Gamma_{\mathbb{R}^n_-}(X,\cdot)$. As one checks, $f\in C^\infty(\overline{\mathbb{R}^n_-}\setminus\{X\})$, $-\Delta f = 0$ in the distribution sense in $\mathbb{R}^n_-$, $|f|\leq C|X-\cdot|^{2-n}$ in $\mathbb{R}^n_-\setminus\{X\}$ and $f_{\partial\mathbb{R}^n_-} = 0$. Hypoellipticity yields $f\in C^\infty(\overline{\mathbb{R}^n_-})$. Multiplying $-\Delta f$ by f and integrating by parts, we get that $f\equiv 0$, and then $G_\infty(X,\cdot) = \Gamma_{\mathbb{R}^n_-}(X,\cdot)$. As above, this proves the convergence without any extraction. The convergence in $C^2_{loc}((\overline{\mathbb{R}^n_-})^2\setminus \mathrm{Diag}(\overline{\mathbb{R}^n_-}))$ follows from the symmetry of G and elliptic theory. □

Proof of Lemma 6: Here again, the proof is similar to the two preceding proofs. We let $(r_i)_i\in(0,+\infty)$ such that $\lim_{i\to+\infty} r_i = 0$. We let $\mathcal{T}$ be a chart at 0 as in (27) (in particular $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$). For any $X,Y\in\overline{\mathbb{R}^n_-}\setminus\{0\}$, we define

$$G_i(X,Y) := r_i^{n-2} G(\mathcal{T}(r_iX), \mathcal{T}(r_iY))$$

for all $i\in\mathbb{N}$. Equation (205) rewrites

$$-\Delta_{g_i} G_i(X,\cdot) - \left(\frac{\gamma}{\left|\frac{\mathcal{T}(r_i\cdot)}{r_i}\right|^2} + r_i^2 h(\mathcal{T}(r_i\cdot))\right) G_i(X,\cdot) = 0 \text{ in } B_R(0)\cap\mathbb{R}^n_-\setminus\{0,X\}.$$

with $G_i(X,\cdot)\equiv 0$ on $B_R(0)\cap\partial\mathbb{R}^n_-$, where $g_i = \mathcal{T}^\star\mathrm{Eucl}(r_i\cdot)$ is the pull-back of the Euclidean metric. In particular, since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, we get that $g_i\to\mathrm{Eucl}$ in $C^2_{loc}(\mathbb{R}^n)$. The pointwise control (239) writes

$$0\leq G_i(X,Y)\leq C\left(\frac{\max\{|X|,|Y|\}}{\min\{|X|,|Y|\}}\right)^{\beta_-(\gamma)} |X-Y|^{2-n} \text{ for } X,Y\in\mathbb{R}^n_-,\ X\neq Y.$$

It then follows from elliptic theory that $G_i(X,\cdot)\to G_\infty(X,\cdot)$ in $C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0,X\})$. In particular, $G_\infty(X,\cdot)$ vanishes on $\partial\mathbb{R}^n_-\setminus\{0\}$ and

$$0\leq G_\infty(X,Y)\leq C\left(\frac{\max\{|X|,|Y|\}}{\min\{|X|,|Y|\}}\right)^{\beta_-(\gamma)} |X-Y|^{2-n} \text{ for } X,Y\in\mathbb{R}^n_-,\ X\neq Y. \tag{243}$$

Moreover, passing to the limit in Green's representation formula, we get that

$$\varphi(X) = \int_{\mathbb{R}^n_-} G_\infty(X,Y)\left(-\Delta\varphi - \frac{\gamma}{|Y|^2}\varphi\right)dY \text{ for all } \varphi\in C^\infty_c(\mathbb{R}^n_-).$$

Since $G(x,\cdot)$ is locally in $H^2_{1,0}(\Omega)$ (see (b) in Theorem 8), we get that $(\eta G_i(X,\cdot))_i$ is uniformly bounded in $H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta\in C^\infty_c(\mathbb{R}^n\backslash\{X\})$. Up to another extraction, we get weak convergence in $H^2_{1,0}(\mathbb{R}^n_-)$, and then $\eta G_\infty(X,\cdot)\in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta\in C^\infty_c(\mathbb{R}^n\setminus\{X\})$. It then follows from Theorem 9 and (243) that $G_\infty(X,\cdot)=\mathcal{G}_X$ is the unique Green's function of $-\Delta-\gamma|x|^{-2}$ on $\mathbb{R}^n_-$ with Dirichlet boundary condition. Here again, the convergence in C^2 follows from elliptic theory. □

14.4. A lower bound for the Green's function. We let Ω, γ, h be as in Theorem 8. We let G be the Green's function for $-\Delta-(\gamma|x|^{-2}+h)$ on Ω with Dirichlet boundary condition. We let $(x_i),(y_i)_{i\in\mathbb{N}}$ be such that $x_i,y_i\in\Omega$ and $x_i\neq y_i$ for all $i\in\mathbb{N}$. We also assume that there exists $x_\infty,y_\infty\in\overline{\Omega}$ such that

$$\lim_{i\to+\infty}x_i=x_\infty \text{ and } \lim_{i\to+\infty}y_i=y_\infty$$

and that there exists c_1,c_2 such that

$$\lim_{i\to+\infty}\frac{G(x_i,y_i)}{H(x_i,y_i)}=c_1\in[0,+\infty] \text{ and } \lim_{i\to+\infty}\frac{|\nabla G_{x_i}(y_i)|}{\Gamma(x_i,y_i)}=c_2\in[0,+\infty]$$

where $H(x,y)$ is defined in (208) and

$$\Gamma(x,y):=\left(\frac{\max\{|x|,|y|\}}{\min\{|x|,|y|\}}\right)^{\beta_-(\gamma)}|x-y|^{1-n}\min\left\{1,\frac{d(x,\partial\Omega)}{|x-y|}\right\}$$

for $x,y\in\Omega$, $x\neq y$. Note that $c_1<+\infty$ by (239). We claim that

$$0<c_1 \text{ and } 0\leq c_2<+\infty \tag{244}$$

The lower bound in (207) and the upper bound in (209) both follow from (244).

This section is devoted to proving (244). We distinguish several cases:

Case 1: $x_\infty\neq y_\infty$, $x_\infty,y_\infty\in\Omega$. As one checks, we then have that

$$\lim_{i\to+\infty}G(x_i,y_i)=G(x_\infty,y_\infty)>0.$$

Therefore, we get that $c_1\in(0,+\infty)$. Concerning the gradient, $\lim_{i\to+\infty}|\nabla G_{x_i}(y_i)|=|\nabla G_{x_\infty}(y_\infty)|\geq 0$ and this yields $c_2<+\infty$. This proves (244) in Case 1.

Case 2: $x_\infty\in\Omega$ and $y_\infty\in\partial\Omega\setminus\{0\}$. Since x_∞,y_∞ are distinct and far from 0, we have that $G(x_i,y_i)=d(y_i,\partial\Omega)\left(-\partial_\nu G_{x_\infty}(y_\infty)+o(1)\right)$ as $i\to+\infty$, where $\partial_\nu G_{x_\infty}(y_\infty)$ is the normal derivative of $G_{x_\infty}>0$ at the boundary point y_∞. Hopf's Lemma then yields $\partial_\nu G_{x_\infty}(y_\infty)<0$. As one checks, we have that $H(x_i,y_i)=(c+o(1))d(y_i,\partial\Omega)$ as $i\to+\infty$. This then yields $0<c_1<+\infty$. Concerning the gradient, we get that $\lim_{i\to+\infty}|\nabla G_{x_i}(y_i)|=|\nabla G_{x_\infty}(y_\infty)|\geq 0$ and $\lim_{i\to+\infty}\Gamma(x_i,y_i)\in(0,+\infty)$, which yields $c_2<+\infty$. This proves (244) in Case 2.

Case 3: $x_\infty\in\Omega$ and $y_\infty=0\in\partial\Omega$. It follows from Case 2 above that there exists $c>0$ such that $G_{x_i}(y)\geq cd(y,\partial\Omega)|y|^{-\beta_-(\gamma)}$ for all $y\in\partial(\Omega\cap B_{r_0}(0))$. We take the subsolution $\underline{u}_{\beta_-(\gamma)}$ defined in Proposition 10. With (224), there exists $c'>0$ such that $G_{x_i}(y)\geq c_1\underline{u}_{\beta_-(\gamma)}(y)$ for all $y\in\partial(\Omega\cap B_{r_0}(0))$. Since G_{x_i} is locally in $H^2_{1,0}$ around 0, the comparison principle and (224) yields $G_{x_i}(y)\geq c" d(y,\partial\Omega)|y|^{-\beta_-(\gamma)}$ for all $y\in\Omega\cap B_{r_0}(0)$. This yields $c_1>0$.

We deal with the gradient. We let $\mathcal{T}$ be a chart at 0 as in (27) and we define

$$G_i(y) := r_i^{\beta_-(\gamma)-1} G_{x_i}(\mathcal{T}(r_i y)) \text{ for } y \in \mathbb{R}^n_- \cap B_2(0)$$

with $r_i \to 0$. It follows from (239) that $G_i(y) \leq C|y_1| \cdot |y|^{-\beta_-(\gamma)}$ for all $y \in \mathbb{R}^n_- \cap B_2(0)$. It follows from (205) that $-\Delta_{g_i} G_i - \left(\gamma|\cdot|^2 + o(1)\right) G_i = 0$ in $\mathbb{R}^n_- \cap B_2(0)$ where $g_i := \mathcal{T}^\star \text{Eucl}(r_i \cdot)$ and $o(1) \to 0$ in $L^\infty_{loc}(\mathbb{R}^n)$. Elliptic regularity then yields $|\nabla G_i(y)| \leq C$ for $y \in \mathbb{R}^n_- \cap B_{3/2}(0)$. We now let $r_i := |\tilde{y}_i|$ where $y_i := \mathcal{T}(\tilde{y}_i)$, so that $r_i \to 0$. We then have that $|\nabla G_i(\tilde{y}_i/r_i)| \leq C$, which rewrites $|\nabla G_{x_i}(y_i)| \leq C|y_i|^{-\beta_-(\gamma)}$. By estimating $\Gamma(x_i, y_i)$, we then get that $c_2 < +\infty$. This proves (244) in Case 3.

Case 4: $x_\infty \neq y_\infty$, $x_\infty, y_\infty \in \partial\Omega \setminus \{0\}$. Since x_∞, y_∞ are distinct and far from 0, we have that $G(x_i, y_i) = d(y_i, \partial\Omega) d(x_i, \partial\Omega) \left(\partial_{\nu_x} \partial_{\nu_y} G_{x_\infty}(y_\infty) + o(1)\right)$ as $i \to +\infty$, where ∂_{ν_x} is the normal derivative along the first coordinate, and ∂_{ν_y} is the normal derivative along the second coordinate. Since $y \mapsto G_x(y)$ is positive for $x, y \in \Omega$, $x \neq y$, and solves (205), Hopf's maximum principle yields $-\partial_{\nu_y} G(x, y_\infty) > 0$ for $x \in \Omega$. Moreover, it follows from the symmetry of G that $-\partial_{\nu_y} G(x, y_\infty) > 0$ solves also (205). Another application of Hopf's principle yields $\partial_{\nu_x} \partial_{\nu_y} G_{x_\infty}(y_\infty) > 0$. Estimating independently $H(x_i, y_i)$, we get that $0 < c_1 < +\infty$.

We deal with the gradient. We have that $|\nabla_y G_{x_i}(y_i)| = |\nabla_y (G_{x_i} - G_{\tilde{x}_i})(y_i)|$ where $\tilde{x}_i \in \partial\Omega$ is the projection of x_i on $\partial\Omega$. The C^2-control then yields $|\nabla_y G_{x_i}(y_i)| \leq C d(x_i, \partial\Omega)$. Estimating independently $\Gamma(x_i, y_i)$, we get that $c_2 < +\infty$. This proves (244) in Case 4.

Case 5: $x_\infty \neq y_\infty$, $x_\infty \in \partial\Omega \setminus \{0\}$ and $y_\infty = 0$. It follows from Cases 2 and 4 that $G_{x_i}(y) \geq C d(x_i, \partial\Omega) d(y_i, \partial\Omega)$ for all $y \in \partial(B_{|x_\infty|/2}(0) \cap \Omega)$. Using a sub-solution as in Case 3, we get that $G_{x_i}(y) \geq c d(x_i, \partial\Omega) d(y, \partial\Omega) |y|^{-\beta_-(\gamma)}$ for all $y \in \partial(B_{|x_\infty|/2}(0) \cap \Omega)$. This yields $0 < c_1$.

For the gradient estimate, we choose a chart $\mathcal{T}$ around $y_\infty = 0$ as in (27), and we let $r_i := |\tilde{y}_i| \to 0$ where $y_i = \mathcal{T}(\tilde{y}_i)$we define $G_i(y) := r_i^{\beta_-(\gamma)-1} G_{x_i}(\mathcal{T}(r_i y))/d(x_i, \partial\Omega)$ for $y \in \mathbb{R}^n_- \cap B_2(0)$ where $r_i \to 0$. The pointwise control (239) and equation (205) yields the convergence of (G_i) in $C^1_{loc}(\overline{\mathbb{R}^n_-} \cap B_2(0) \setminus \{0\})$ as $i \to +\infty$. The boundedness of $|\nabla G_i|$ yields $c_2 < +\infty$. This proves (244) in Case 5.

Since G is symmetric, it follows from Cases 1 to 5 that (244) holds when $x_\infty \neq y_\infty$.

We now deal with the case $x_\infty = y_\infty$, which rewrites $\lim_{i \to +\infty} |x_i - y_i| = 0$. Via a rescaling, we are essentially back to the case $x_\infty \neq y_\infty$ via the convergence Theorems 4, 5 and 6.

Case 6: $|x_i - y_i| = o(d(x_i, \partial\Omega))$ as $i \to +\infty$. We set $r_i := |x_i - y_i| \to 0$ as $i \to +\infty$ and we define

$$G_i(Y) := r_i^{n-2} G(x_i, x_i + r_i Y) \text{ for } Y \in \frac{\Omega - x_i}{r_i} \setminus \{0\}.$$

It follows from Theorem 4 that $G_i \to c_n |\cdot|^{2-n}$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ as $i \to +\infty$, with $c_n := ((n-2)\omega_{n-1})^{-1}$. We define $Y_i := \frac{y_i - x_i}{|y_i - x_i|}$, and we then get that $|y_i - x_i|^{n-2} G(x_i, y_i) = G_i(Y_i) \to c_n$ as $i \to +\infty$. Estimating $H(x_i, y_i)$ (and noting that $d(x_i, \partial\Omega) \leq |x_i - 0| = |x_i|$), we get that $0 < c_1 < +\infty$.

The convergence of the gradient yields $|\nabla G_i(Y_i)| \leq C$ for all i. With the original function G and points x_i, y_i, this yields $c_2 < +\infty$. This proves (244) in Case 6.

Case 7: $d(x_i,\partial\Omega)=O(|x_i-y_i|)$ and $|x_i-y_i|=o(|x_i|)$ as $i\to+\infty$. Then $\lim_{i\to+\infty}x_i=x_\infty\in\partial\Omega$. We let $\mathcal{T}$ be a chart at x_∞ as in (27), in particular $D_0\mathcal{T}=\mathbb{I}_{\mathbb{R}^n}$. We let $x_i=\mathcal{T}(x_{i,1},x_i')$ and $y_i=\mathcal{T}(y_{i,1},y_i')$ where $(x_{i,1},x_i'),(y_{i,1},y_i')\in(-\infty,0)\times\mathbb{R}^{n-1}$ are going to 0 as $i\to+\infty$. In particular $d(x_i,\partial\Omega)=(1+o(1))|x_{i,1}|$ and $d(y_i,\partial\Omega)=(1+o(1))|y_{i,1}|$ as $i\to+\infty$. We define $r_i:=|(y_{i,1},y_i')-(x_{i,1},x_i')|$. In particular $r_i=(1+o(1))|x_i-y_i|$ as $i\to+\infty$. The hypothesis of Case 7 rewrite $x_{i,1}=O(r_i)$ and $r_i=o(|(x_{i,1},x_i')|)$. Consequently, we have that $y_{i,1}=O(r_i)$ and $r_i=o(|x_i'|)$ as $i\to+\infty$. We define

$$G_i(X,Y):=r_i^{n-2}G(\mathcal{T}\left((0,x_i')+r_iX\right),\mathcal{T}\left((0,x_i')+r_iY\right))$$

for $X,Y\in\mathbb{R}^n_-$ such that $X\neq Y$. It follows from Theorem 5 that

$$\lim_{i\to+\infty}G_i(X,Y)=c_n\left(|X-Y|^{2-n}-|X-Y^*|^{2-n}\right):=\Psi(X,Y)$$

for all $X,Y\in\overline{\mathbb{R}^n_-}$, $X\neq Y$, and this convergence holds in C^2_{loc}. We define $X_i:=(r_i^{-1}x_{i,1},0)$ and $Y_i:=(r_i^{-1}y_{i,1},r_i^{-1}(y_i'-x_i'))$: the definition of r_i yields $X_i\to X_\infty\in\overline{\mathbb{R}^n_-}$ and $Y_i\to Y_\infty\in\overline{\mathbb{R}^n_-}$ as $i\to+\infty$. Therefore, we get that

$$|x_i-y_i|^{n-2}G(x_i,y_i)=(1+o(1))G_i(X_i,Y_i)\to\Psi(X_\infty,Y_\infty)$$

as $i\to+\infty$, and

$$|X_{\infty,1}|=\lim_{i\to+\infty}\frac{|x_{i,1}|}{r_i}=\lim_{i\to+\infty}\frac{d(x_i,\partial\Omega)}{r_i}. \tag{245}$$

Case 7.1: $X_{\infty,1}\neq0$ and $Y_{\infty,1}\neq0$. We then get that $\lim_{i\to+\infty}|x_i-y_i|^{n-2}G(x_i,y_i)=\Psi(X_\infty,Y_\infty)>0$. Moreover, it follows from (245) that $d(x_i,\partial\Omega)d(y_i,\partial\Omega)=(c+o(1))|x_i-y_i|^2$ as $i\to+\infty$ for some $c>0$. Since $|x_i|=(1+o(1))|y_i|$ as $i\to+\infty$ (this follows from the assumption of Case 7), we get that $\lim_{i\to+\infty}|x_i-y_i|^{n-2}H(x_i,y_i)\in(0,+\infty)$. Then $0<c_1<+\infty$.
Case 7.2: $X_{\infty,1}\neq0$ and $Y_{\infty,1}=0$. Then $Y_{i,1}\to0$ as $i\to+\infty$, and then, there exists $(\tau_i)_i\in(0,1)$ such that $G_i(X_i,Y_i)=Y_{i,1}\partial_{Y_1}G_i(X_i,(\tau_iY_{i,1},Y_i'))$. Letting $i\to+\infty$ and using the convergence of G_i in C^1, we get that

$$\begin{aligned}|x_i-y_i|^{n-2}G(x_i,y_i) &= (1+o(1))G_i(X_i,Y_i)=Y_{i,1}\partial_{Y_1}G_i(X_i,\tau_iY_i)\\ &= \frac{d(y_i,\partial\Omega)}{|x_i-y_i|}\left(-\partial_{Y_1}\Psi(X_\infty,Y_\infty)+o(1)\right)\end{aligned}$$

as $i\to+\infty$. As one checks, $\partial_{Y_1}\Psi(X_\infty,Y_\infty)<0$. Arguing as in Case 7.1, we get that $0<c_1<+\infty$.
Case 7.3: $X_{\infty,1}=Y_{\infty,1}=0$. As in Case 7.2, there exists $(\tau_i)_i,(\sigma_i)_i\in(0,1)$ such that $G_i(X_i,Y_i)=Y_{i,1}X_{i,1}\partial_{Y_1}\partial_{X_1}G_i((\sigma_iX_{i,1},X_i')X_i,(\tau_iY_{i,1},Y_i'))$. We conclude as above, noting that $\partial_{Y_1}\partial_{X_1}\Psi(X_\infty,Y_\infty)>0$. Then $0<c_1<+\infty$.

The gradient estimate is proved as in Cases 1 to 6. This proves (244) in Case 7.

Case 8: $d(x_i,\partial\Omega)=O(|x_i-y_i|)$, $|x_i|=O(|x_i-y_i|)$ and $|y_i|=O(|x_i-y_i|)$ as $i\to+\infty$. In particular, $x_\infty=y_\infty=0$. We take a chart at 0 as in Case 7, and we define $(x_{i,1},x_i'),(y_{i,1},y_i')$ similarly. We define $r_i:=|(y_{i,1},y_i')-(x_{i,1},x_i')|=(1+o(1))|x_i-y_i|$ as $i\to+\infty$. We define

$$G_i(X,Y):=r_i^{n-2}G(\mathcal{T}\left(r_iX\right),\mathcal{T}\left(r_iY\right))$$

for $X, Y \in \mathbb{R}^n_-$. It follows from Theorem 6 that $G_i \to \mathcal{G}$ in $C^2_{loc}((\overline{\mathbb{R}^n_-} \setminus \{0\})^2 \setminus \text{Diag}(\overline{\mathbb{R}^n_-} \setminus \{0\}))$, where $\mathcal{G}$ is the Green's function for $-\Delta - \gamma|\cdot|^{-2}$ in $\mathbb{R}^n_-$. Then

$$|x_i - y_i|^{n-2} G(x_i, y_i) = (1 + o(1)) G_i(X_i, Y_i) = \mathcal{G}(X_\infty, Y_\infty) + o(1)$$

as $i \to +\infty$.

Case 8.1: We assume that $X_{\infty,1} \neq 0$ *and* $Y_{\infty,1} \neq 0$. Then we get $0 < c_1 < +\infty$ as in Case 7.1.

Case 8.2: We assume that $X_\infty \in \mathbb{R}^n_-$ *and* $Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$ *or* $X_\infty, Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$. Then we argue as in Cases 7.2 and 7.3 to get $0 < c_1 < +\infty$ provided $\{\partial_{Y_1}\mathcal{G}(X_\infty, Y_\infty) < 0 \text{ if } X_\infty \in \mathbb{R}^n_- \text{ and } Y_\infty \in \partial\mathbb{R}^n_-\}$ and $\{\partial_{Y_1}\partial_{X_1}\mathcal{G}(X_\infty, Y_\infty) > 0 \text{ if } X_\infty, Y_\infty \in \partial\mathbb{R}^n_-\}$. So we are just left with proving these two inequalities.

We assume that $X_\infty \in \mathbb{R}^n_-$. It follows from Theorem 9 below that $\mathcal{G}(X_\infty, \cdot) > 0$ is a solution to $(-\Delta - \gamma|\cdot|^{-2})\mathcal{G}(X_\infty, \cdot) = 0$ in $\mathbb{R}^n_- - \{X_\infty\}$, vanishing on $\partial\mathbb{R}^n_- \setminus \{0\}$. Hopf's maximum principle then yields $-\partial_{Y_1}\mathcal{G}(X_\infty, Y_\infty) > 0$ for $Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$.

We fix $Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$. For $X \in \mathbb{R}^n_-$, we then define $H(X) := -\partial_{Y_1}\mathcal{G}(X, Y_\infty) > 0$ by the above argument. Moreover, $(-\Delta - \gamma|\cdot|^{-2})H = 0$ in $\mathbb{R}^n_-$, vanishing on $\partial\mathbb{R}^n_- \setminus \{0, Y_\infty\}$. Hopf's maximum principle yields $-\partial_{X_1}H(X_\infty) = \partial_{Y_1}\partial_{X_1}\mathcal{G}(X_\infty, Y_\infty) > 0$ for $X_\infty, Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$

Case 8.3: we assume that $X_\infty = 0$ *or* $Y_\infty = 0$. Since $|X_\infty - Y_\infty| = 1$, without loss of generality, we can assume that $X_\infty \neq 0$. It follows from Cases 8.1 and 8.2 that there exists $C > 0$ such that

$$C^{-1} \frac{d(x_i, \partial\Omega)}{|x_i|^{n-\beta_-(\gamma)}} \frac{d(y, \partial\Omega)}{|y|^{\beta_-(\gamma)}} \leq G_{x_i}(y) \leq C \frac{d(x_i, \partial\Omega)}{|x_i|^{n-\beta_-(\gamma)}} \frac{d(y, \partial\Omega)}{|y|^{\beta_-(\gamma)}} \tag{246}$$

for all $y \in \partial(B_{|x_i|/2}(0) \cap \Omega)$. We let $\underline{u}_{\beta_-(\gamma)}$ be the sub-solution given by Proposition 10. Arguing as in Case 3, it then follows from the comparison principle that (246) holds for $y \in B_{|x_i|/2}(0) \cap \Omega$. Since $|y_i| = o(|x_i|)$, we then get that (246) holds with $y := y_i$. Estimating $H(x_i, y_i)$, we then get that $0 < c_1 < +\infty$.

The gradient estimate is proved as in Cases 1 to 6. This proves (244) in Case 8.

Since G is symmetric, it follows from Cases 7 and 8 that (244) holds when $x_\infty = y_\infty$.

In conclusion, we get that (244) holds, which proves the initial claim. As noted previously, both the lower bound in (207) and the upper bound in (209) follow from these results.

We are now left with proving (210). We let $(\tilde{x}_i)_i, (\tilde{y}_i)_i \in \Omega$ be such that

$$\tilde{y}_i = o(|\tilde{x}_i|) \text{ and } \tilde{x}_i = o(1) \text{ as } i \to +\infty,$$

and $(h_i)_i \in C^{0,\theta}(\Omega)$ such that $\lim\limits_{i\to+\infty} h_i = h$ in $C^{0,\theta}$. It follows from (207) that, up to extraction, there exists $l > 0$ such that

$$G_{h_i}(\tilde{x}_i, \tilde{y}_i) = (l + o(1)) \frac{d(\tilde{x}_i, \partial\Omega)}{|\tilde{x}_i|^{\beta_+(\gamma)}} \frac{d(\tilde{y}_i, \partial\Omega)}{|\tilde{y}_i|^{\beta_-(\gamma)}} \tag{247}$$

From now on, to avoid unnecessary notations, the extraction is <u>fixed</u>. We define

$$r_i := |\tilde{x}_i|\,;\, s_i := |\tilde{y}_i|\,;\, \tau_i := s_i^{-1}\mathcal{T}^{-1}(\tilde{y}_i) \in \mathbb{R}^n_- \text{ and } \theta_i := r_i^{-1}\mathcal{T}^{-1}(\tilde{x}_i) \in \mathbb{R}^n_-,$$

and $\theta_\infty, \tau_\infty \in \overline{\mathbb{R}^n_-}$ such that

$$\tilde{x}_i = \mathcal{T}(r_i\theta_i)\,;\, \tilde{y}_i = \mathcal{T}(s_i\tau_i)\,;\, \theta_i \to \theta_\infty \neq 0 \text{ and } \tau_i \to \tau_\infty \neq 0 \text{ as } i \to +\infty. \tag{248}$$

STEP P14. *We fix $R>0$. We claim that*

$$G_{h_i}(\tilde{x}_i, y) = (l+o(1))\frac{d(\tilde{x}_i,\partial\Omega)}{|\tilde{x}_i|^{\beta_+(\gamma)}}\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}} \text{ as } i\to+\infty \tag{249}$$

uniformly for $y\in\Omega\cap\mathcal{T}(B_{Rs_i}\setminus B_{R^{-1}s_i})$.

Proof of Step P14: For $z\in B_{2R}\setminus B_{(2R)^{-1}}$, we define

$$G_i(z) := \frac{s_i^{\beta_-(\gamma)-1}|\tilde{x}_i|^{\beta_+(\gamma)}}{d(\tilde{x}_i,\partial\Omega)}G_{h_i}(\tilde{x}_i,\mathcal{T}(s_i z)).$$

As one checks, (249) is equivalent to prove that

$$G_i(y) = (l+o(1))\frac{|y_1|}{|y|^{\beta_-(\gamma)}} \text{ uniformly for } y\in B_R(0)\setminus B_{R^{-1}}(0) \tag{250}$$

Since $s_i = o(|\tilde{x}_i|)$ and (28) holds, it follows from the control (207) that there exists $C>0$ such that

$$\frac{1}{C}\cdot\frac{|z_1|}{|z|^{\beta_-(\gamma)}} \leq G_i(z) \leq C\cdot\frac{|z_1|}{|z|^{\beta_-(\gamma)}} \text{ for all } z\in\mathbb{R}^n_-\cap B_{2R}(0)\setminus B_{(2R)^{-1}}(0). \tag{251}$$

As for (242), it follows from (205) that

$$-\Delta_{g_i}G_i - \left(\frac{\gamma s_i^2}{|\mathcal{T}(s_i\cdot)|^2}+O(s_i^2)\right)G_i = 0 \text{ in } B_R(0)\cap\mathbb{R}^n_- \text{ ; } G_i\equiv 0 \text{ on } \partial\mathbb{R}^n_-\cap B_R(0)\setminus\{0\}. \tag{252}$$

It follows from (251), (252) and standard elliptic theory that there exists $G\in C^2(\overline{\mathbb{R}^n_-}\setminus\{0\})$ such that, up to a subsequence,

$$\lim_{i\to+\infty} G_i = G \text{ in } C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0\}) \tag{253}$$

with

$$-\Delta G - \frac{\gamma}{|x|^2}G = 0 \text{ in } \overline{\mathbb{R}^n_-}\setminus\{0\}\,;\, G=0 \text{ on } \partial\mathbb{R}^n_-\setminus\{0\}\,;$$
$$\frac{1}{C}\cdot\frac{|z_1|}{|z|^{\beta_-(\gamma)}} \leq G(z) \leq C\cdot\frac{|z_1|}{|z|^{\beta_-(\gamma)}} \text{ for all } z\in\mathbb{R}^n_-\setminus\{0\}.$$

It the follows from Proposition 6.4 in [21] that there exists $\lambda>0$ such that

$$G(z) = \lambda\cdot\frac{|z_1|}{|z|^{\beta_-(\gamma)}} \text{ for all } z\in\mathbb{R}^n_-. \tag{254}$$

We claim that $\lambda = l$. We prove the claim. It follows from (247) and the definition (248) of τ_i that

$$G_i(\tau_i) = (l+o(1))\frac{|\tau_{i,1}|}{|\tau_i|^{\beta_-(\gamma)}} \text{ and } \tau_i\to\tau_\infty\neq 0 \text{ as } i\to+\infty. \tag{255}$$

Case 1: we assume that $\tau_\infty\in\mathbb{R}^n_-\setminus\{0\}$, that is $\tau_{\infty,1}\neq 0$. Passing to the limit in (255), using the convergence (253) and the explicit form (254), we get that

$$l\frac{|\tau_{\infty,1}|}{|\tau_\infty|^{\beta_-(\gamma)}} = \lambda\frac{|\tau_{\infty,1}|}{|\tau_\infty|^{\beta_-(\gamma)}},$$

and therefore, since $\tau_{\infty,1}\neq 0$, we get that $\lambda = l$.

Case 2: we assume that $\tau_\infty\in\partial\mathbb{R}^n_-\setminus\{0\}$, that is $\tau_{i,1}\to 0$ as $i\to+\infty$. With a Taylor expansion, we get that there exists a sequence $(t_i)_{i\in\mathbb{N}}\in(0,1)$ such that

$G_i(\tau_i) = \partial_1 G_i(t_i\tau_{i,1}, \theta_i')\tau_{i,1}$ for all $i \in \mathbb{N}$. With the convergence (253) of G_i to G in C^1, we get that

$$G_i(\tau_i) = (\partial_1 G(\tau_\infty) + o(1)) \cdot \tau_{i,1} = \left(\frac{\lambda}{|\tau_\infty|^{\beta_-(\gamma)}} + o(1)\right) \cdot |\tau_{i,1}|.$$

Since $\tau_{i,1} \neq 0$ for all $i \in \mathbb{N}$, it follows form (255) that $\lambda = l$.

Therefore, in both cases, we have proved that $\lambda = l$. It follows from this uniqueness that the convergence of G_i holds with no extraction.

We now prove (250). We let $(z_i)_i \in \mathbb{R}^n_- \setminus \{0\}$ be such that $z_i \to z_\infty \in \overline{\mathbb{R}^n_-} \setminus \{0\}$. Then $G_i(z_i) \to G(z_\infty)$ as $i \to +\infty$. Therefore, if $z_{\infty,1} \neq 0$, we get that $G_i(z_i) = (1+o(1))G(z_i)$ as $i \to +\infty$. We now assume that $z_{\infty,1} = 0$, that is $z_{i,1} \to 0$ as $i \to +\infty$. We use the C^1-convergence of (G_i) and argue as in Case 2 above to get that $\lim_{i\to+\infty} |z_{i,1}|^{-1} G_i(z_i) = -\partial_1 G(z_\infty) \neq 0$. As one checks, this yields also $G_i(z_i) = (1+o(1))G(z_i)$ as $i \to +\infty$. As noticed above, this proves (249) and ends Step P14. □

STEP P15. *We fix $R > 0$. We claim that*

$$G_{h_i}(\tilde{x}_i, y) = (l + o(1)) \frac{d(\tilde{x}_i, \partial\Omega)}{|\tilde{x}_i|^{\beta_+(\gamma)}} \frac{d(y, \partial\Omega)}{|y|^{\beta_-(\gamma)}} \text{ as } i \to +\infty \tag{256}$$

uniformly for $y \in \Omega \cap \mathcal{T}(B_{Rs_i}(0))$.

Proof of Step P15: For $r > 0$ small, we choose $\bar{u}_{\beta_-(\gamma)} \in C^2(\Omega \cap B_r(0))$ a supersolution to $-\Delta \bar{u}_{\beta_-(\gamma)} - (\gamma|x|^{-2} + h_i)\bar{u}_{\beta_-(\gamma)} > 0$ as in (223) and (224). Note that, due to the convergence of (h_i) to h in C^0, the choice of $\bar{u}_{\beta_-(\gamma)}$ can be made independently of i. We fix $\epsilon > 0$. It follows from the convergence (249) of Step P14 and (224) that there exists $i_0 \in \mathbb{N}$

$$G_{h_i}(\tilde{x}_i, y) \leq (l + \epsilon) \frac{d(\tilde{x}_i, \partial\Omega)}{|\tilde{x}_i|^{\beta_+(\gamma)}} \bar{u}_{\beta_-(\gamma)}(y) \text{ for all } y \in \partial\left(\Omega \cap \mathcal{T}(B_{Rs_i}(0))\right) \text{ for all } i \geq i_0. \tag{257}$$

Note that $G_{h_i}(\tilde{x}_i, \cdot), \bar{u}_{\beta_-(\gamma)} \in H_1^2\left(\Omega \cap \mathcal{T}(B_{Rs_i}(0))\right)$ (these are variational super- or sub-solutions) and that the operator $-\Delta - (\gamma|x|^{-2} + h_i)$ is coercive. Since $G_{h_i}(\tilde{x}_i, \cdot)$ is a solution and $\bar{u}_{\beta_-(\gamma)}$ is a supersolution to $-\Delta u - (\gamma|x|^{-2} + h_i)u = 0$, it follows from the comparison principle that (257) holds for $y \in \Omega \cap \mathcal{T}(B_{Rs_i}(0))$. With (224), we get that there exists $i_1 \in \mathbb{N}$ such that

$$G_{h_i}(\tilde{x}_i, y) \leq (l + 2\epsilon) \frac{d(\tilde{x}_i, \partial\Omega)}{|\tilde{x}_i|^{\beta_+(\gamma)}} \frac{d(y, \partial\Omega)}{|y|^{\beta_-(\gamma)}} \text{ for all } y \in \Omega \cap \mathcal{T}(B_{Rs_i}(0)) \text{ for all } i \geq i_1. \tag{258}$$

Using a subsolution $\underline{u}_{\beta_-(\gamma)}$ as in (223) and (224) and arguing as above, we get that

$$G_{h_i}(\tilde{x}_i, y) \geq (l - 2\epsilon) \frac{d(\tilde{x}_i, \partial\Omega)}{|\tilde{x}_i|^{\beta_+(\gamma)}} \frac{d(y, \partial\Omega)}{|y|^{\beta_-(\gamma)}} \text{ for all } y \in \Omega \cap \mathcal{T}(B_{Rs_i}(0)) \text{ for all } i \geq i_2. \tag{259}$$

The inequalities (258) and (259) put together yield (256). This ends Step P15. □

We now vary the $x-$variable.

STEP P16. *We fix $R, R' > 0$. We claim that*

$$G_{h_i}(\tilde{x}_i, y) = (l + o(1))\frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}}\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}} \text{ as } i \to +\infty \tag{260}$$
$$\text{uniformly for } y \in \Omega \cap \mathcal{T}(B_{Rs_i}(0)) \text{ and } x \in \Omega \cap \mathcal{T}(B_{R'r_i}(0) \setminus B_{(R')^{-1}r_i}(0)).$$

Proof of Step P16: We fix a sequence $(y_i)_i \in \Omega$ such that $y_i \in \mathcal{T}(B_{Rs_i}(0))$ for all $i \in \mathbb{N}$. For $z \in B_{2R'} \setminus B_{(2R')^{-1}}$, we define

$$\tilde{G}_i(z) := \frac{|y_i|^{\beta_-(\gamma)} r_i^{\beta_+(\gamma)-1}}{d(y_i, \partial\Omega)} G_{h_i}(\mathcal{T}(s_i z), y_i).$$

As one checks, (260) is equivalent to prove that

$$\tilde{G}_i(x) = (l + o(1))\frac{|x_1|}{|x|^{\beta_+(\gamma)}} \text{ uniformly for } x \in B_{R'}(0) \setminus B_{(R')^{-1}}(0) \tag{261}$$

Since $|y_i| = o(r_i)$ as $i \to +\infty$ and (28) holds, it follows from the control (207) that there exists $C > 0$ such that

$$\frac{1}{C} \cdot \frac{|z_1|}{|z|^{\beta_+(\gamma)}} \le \tilde{G}_i(z) \le C \cdot \frac{|z_1|}{|z|^{\beta_+(\gamma)}} \text{ for all } z \in \mathbb{R}^n_- \cap B_{2R'} \setminus B_{(2R')^{-1}}. \tag{262}$$

As for (242), it follows from (205) that

$$-\Delta_{g_i}\tilde{G}_i - \left(\frac{\gamma r_i^2}{|\mathcal{T}(r_i \cdot)|^2} + O(r_i^2)\right)\tilde{G}_i = 0 \text{ in } B_{2R'}(0) \cap \mathbb{R}^n_- \;;\; \tilde{G}_i \equiv 0 \text{ on } \partial\mathbb{R}^n_- \cap B_{2R'}(0) \setminus \{0\}. \tag{263}$$

It follows from (262), (263) and standard elliptic theory that there exists $\tilde{G} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0\})$ such that, up to a subsequence,

$$\lim_{i \to +\infty} \tilde{G}_i = \tilde{G} \text{ in } C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) \tag{264}$$

with

$$-\Delta\tilde{G} - \frac{\gamma}{|x|^2}\tilde{G} = 0 \text{ in } \overline{\mathbb{R}^n_-} \setminus \{0\} \,;\, \tilde{G} = 0 \text{ on } \partial\mathbb{R}^n_- \setminus \{0\} \,;$$
$$\frac{1}{C} \cdot \frac{|z_1|}{|z|^{\beta_+(\gamma)}} \le \tilde{G}(z) \le C \cdot \frac{|z_1|}{|z|^{\beta_+(\gamma)}} \text{ for all } z \in \mathbb{R}^n_- \setminus \{0\}.$$

It the follows from Proposition 6.4 in [21] that there exists $\mu > 0$ such that

$$\tilde{G}(z) = \mu \cdot \frac{|z_1|}{|z|^{\beta_+(\gamma)}} \text{ for all } z \in \mathbb{R}^n_-. \tag{265}$$

We claim that $\mu = l$. We prove the claim. It follows from (256) and the definition (248) of θ_i that

$$\tilde{G}_i(\theta_i) = (l + o(1))\frac{|\theta_{i,1}|}{|\theta_i|^{\beta_+(\gamma)}} \text{ and } \theta_i \to \theta_\infty \neq 0 \text{ as } i \to +\infty. \tag{266}$$

Case 1: we assume that $\theta_\infty \in \mathbb{R}^n_- \setminus \{0\}$, that is $\theta_{\infty,1} \neq 0$. Passing to the limit in (266), using the convergence (264) and the explicit form (265), as in Case 1 of Step P14, we get that $l|\theta_{\infty,1}| \cdot |\theta_\infty|^{-\beta_-(\gamma)} = \mu|\theta_{\infty,1}| \cdot |\theta_\infty|^{-\beta_-(\gamma)}$, and therefore, since $\theta_{\infty,1} \neq 0$, we get that $\mu = l$.

Case 2: we assume that $\theta_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$, that is $\theta_{i,1} \to 0$ as $i \to +\infty$. With a Taylor expansion, we get that there exists a sequence $(\tilde{t}_i)_{i\in\mathbb{N}} \in (0,1)$ such that

$\tilde{G}_i(\theta_i) = \partial_1\tilde{G}_i(\tilde{t}_i\theta_{i,1}, \theta_i')\theta_{i,1}$ for all $i \in \mathbb{N}$. With the convergence (264) of $\tilde{G}_i$ to $\tilde{G}$ in C^1, we get that

$$\tilde{G}_i(\theta_i) = \Big(\partial_1\tilde{G}(\theta_\infty) + o(1)\Big) \cdot \theta_{i,1} = \left(\frac{\mu}{|\theta_\infty|^{\beta_+(\gamma)}} + o(1)\right) \cdot |\theta_{i,1}|.$$

Since $\theta_{i,1} \neq 0$ for all $i \in \mathbb{N}$, it follows form (266) that $\mu = l$.

Therefore, in both cases, we have proved that $\mu = l$. It follows from this uniqueness that the convergence of $\tilde{G}_i$ holds with no extraction. As for Step P14, we get (249). This ends Step P16. □

Step P17. *We fix $R, R' > 0$. We claim that*

$$G_{h_i}(x,y) = (l + o(1) + O(|x|^{\beta_+(\gamma)-\beta_-(\gamma)}))\frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}}\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}} \text{ as } i \to +\infty \tag{267}$$

uniformly for $y \in \Omega \cap \mathcal{T}(B_{Rs_i}(0))$ and $x \in \Omega \setminus \mathcal{T}(B_{(R')^{-1}r_i}(0))$.

Proof of Step P17: The differs from Step P15 since one works on domains exteriors to the ball of radius r_i. Here again, we choose $(y_i)_i$ such that $y_i \in \mathcal{T}(B_{Rs_i}(0))$. For $r > 0$ small, we choose $\bar{u}_{\beta_+(\gamma)} \in C^2(\Omega \cap B_r(0))$ a supersolution to $-\Delta\bar{u}_{\beta_+(\gamma)} - (\gamma|x|^{-2} + h_i)\bar{u}_{\beta_+(\gamma)} > 0$ as in (223) and (224). Note that, due to the convergence of (h_i) to h in C^0, the choice of $\bar{u}_{\beta_-(\gamma)}$ can be made independently of i. We fix $\epsilon > 0$. It follows from the convergence (260) of Step P16 and (224) that there exists $i_0 \in \mathbb{N}$

$$G_{h_i}(x,y_i) \leq (l+\epsilon)\frac{d(y_i,\partial\Omega)}{|y_i|^{\beta_-(\gamma)}}\bar{u}_{\beta_+(\gamma)}(x) \text{ for all } x \in \Omega \cap \partial\mathcal{T}(B_{R'r_i}(0)) \text{ for all } i \geq i_0. \tag{268}$$

We fix $\delta > 0$ such that $\delta < r$. We choose a supersolution $\bar{u}_{\beta_-(\gamma)}$ as in (223) and (224). It follows from the upper bound (207) that for some $i_1 \in \mathbb{N}$, there exists $C > 0$ such that

$$G_{h_i}(x,y_i) \leq C\frac{d(y_i,\partial\Omega)}{|y_i|^{\beta_-(\gamma)}}\bar{u}_{\beta_-(\gamma)}(x) \text{ for all } x \in \Omega \cap \partial B_\delta(0) \text{ for all } i \geq i_1. \tag{269}$$

Therefore,

$$G_{h_i}(x,y_i) \leq w_i(x) \text{ for all } x \in \partial\left(\Omega \cap \mathcal{T}(B_\delta(0) \setminus B_{(R')^{-1}r_i}(0))\right) \tag{270}$$

where

$$w_i := \frac{d(y_i,\partial\Omega)}{|y_i|^{\beta_-(\gamma)}}\left((l+\epsilon)\bar{u}_{\beta_+(\gamma)} + C\bar{u}_{\beta_-(\gamma)}\right)$$

and, since $\bar{u}_{\beta_+(\gamma)}, \bar{u}_{\beta_-(\gamma)}$ are supersolution,

$$-\Delta w_i - \left(\frac{\gamma}{|x|^2} + h_i\right)w_i \geq 0 \text{ in } \Omega \cap \mathcal{T}(B_\delta(0) \setminus B_{(R')^{-1}r_i}(0)).$$

Since $-\Delta - \left(\gamma|x|^{-2} + h_i\right)$ is coercive, the maximum principle holds and (270) holds on $\Omega \cap \mathcal{T}(B_\delta(0) \setminus B_{(R')^{-1}r_i}(0))$. With (224), we get that there exists $i_2 \in \mathbb{N}$ such that

$$G_{h_i}(x,y_i) \leq \left(l + 2\epsilon + C|x|^{\beta_+(\gamma)-\beta_-(\gamma)}\right)\frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}}\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}} \tag{271}$$

for all $x \in \Omega \cap \mathcal{T}(B_\delta(0) \setminus B_{(R')^{-1}r_i}(0))$ for all $i \geq i_2$. Using subsolutions and arguing as above, we get that for some $i_3 \in \mathbb{N}$

$$G_{h_i}(x,y_i) \geq \left(l - 2\epsilon - C|x|^{\beta_+(\gamma)-\beta_-(\gamma)}\right)\frac{d(x,\partial\Omega)}{|x|^{\beta_+(\gamma)}}\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}} \tag{272}$$

for all $x \in \Omega \cap \mathcal{T}(B_\delta(0) \setminus B_{(R')^{-1}r_i}(0))$ for all $i \geq i_3$. The inequalities (271) and (272) put together yield (267). This ends Step P17. $\square$

Step P18. *We let $(X_i)_i, (Y_i)_i \in \Omega$ such that $|Y_i| = o(|X_i|)$ and $X_i = o(1)$ as $i \to +\infty$. We assume that there exists $l' > 0$ such that*

$$G_{h_i}(X_i, Y_i) = (l' + o(1)) \frac{d(X_i, \partial\Omega)}{|X_i|^{\beta_+(\gamma)}} \frac{d(Y_i, \partial\Omega)}{|Y_i|^{\beta_-(\gamma)}} \text{ as } i \to +\infty.$$

Then $l' = l$.

Proof of Step P18: We define

$$\sigma_i := \min\{|\tilde{y}_i|, |Y_i|\} \text{ and } \rho_i := \max\{|\tilde{x}_i|, |X_i|\}.$$

We let $(z_i)_i, (t_i)_i \in \Omega$ such that $c_1 \sigma_i \leq |z_i| \leq c_2 \sigma_i$ and $c_1 \rho_i \leq |t_i| \leq c_2 \rho_i$ for all $i \in \mathbb{N}$. Since $|z_i| = O(s_i)$, $r_i = O(|t_i|)$ and $t_i \to 0$ as $i \to +\infty$, it follows from (267) that

$$G_{h_i}(z_i, t_i) = (l + o(1)) \frac{d(z_i, \partial\Omega)}{|z_i|^{\beta_-(\gamma)}} \frac{d(t_i, \partial\Omega)}{|t_i|^{\beta_+(\gamma)}} \text{ as } i \to +\infty.$$

In addition, since $|z_i| = O(|Y|_i)$, $|X_i| = O(|t_i|)$ and $t_i \to 0$ as $i \to +\infty$, it follows from (267) that

$$G_{h_i}(z_i, t_i) = (l' + o(1)) \frac{d(z_i, \partial\Omega)}{|z_i|^{\beta_-(\gamma)}} \frac{d(t_i, \partial\Omega)}{|t_i|^{\beta_+(\gamma)}} \text{ as } i \to +\infty.$$

Therefore, we get that $l' = l$. This ends Step P18. $\square$

Step P19. *We let $(X_i)_i, (Y_i)_i \in \Omega$ such that $|Y_i| = o(|X_i|)$ and $X_i = o(1)$ as $i \to +\infty$. Then*

$$G_{h_i}(X_i, Y_i) = (l + o(1)) \frac{d(X_i, \partial\Omega)}{|X_i|^{\beta_+(\gamma)}} \frac{d(Y_i, \partial\Omega)}{|Y_i|^{\beta_-(\gamma)}} \text{ as } i \to +\infty.$$

Proof of Step P19: We argue by contradiction and we assume that there exists $\epsilon_0 > 0$ and a subsequences $(\varphi(i))_i$ such that $|U_{\varphi(i)} - l| \geq \epsilon_0$ for all $i \in \mathbb{N}$ where

$$U_i := \frac{G_{h_i}(X_i, Y_i)|Y_i|^{\beta_-(\gamma)}|X_i|^{\beta_+(\gamma)}}{d(X_i, \partial\Omega) d(Y_i, \partial\Omega)}.$$

Since $(U_{\varphi(i)})$ is bounded, up to another extraction, there exists $l'' > 0$ such that $U_{\varphi(i)} \to l''$ as $i \to +\infty$. Therefore, $|l - l''| \geq \epsilon_0$ and $l'' \neq l$. Since (247) holds for the subfamily $(\varphi(i))$, it then follows from Step P18 that $l'' = l$, contradicting $l'' \neq l$. This ends Step P19.

We are now in position to prove (210), that is the convergence with no extraction of subsequence. It follows from (247) and Step P18 applied to $(h_i)_i$ and to the null function that there exists a subsequence $(h_{\varphi(i)})$ and $l, L_{\gamma,\Omega} > 0$ such that for any $(x_i)_i, (y_i)_i \in \Omega$ such that $|y_i| = o(|x_i|)$ and $x_i = o(1)$ as $i \to +\infty$, then

$$G_{h_{\varphi(i)}}(x_i, y_i) = (l + o(1)) \frac{d(x_i, \partial\Omega)}{|x_i|^{\beta_+(\gamma)}} \frac{d(y_i, \partial\Omega)}{|y_i|^{\beta_-(\gamma)}}, \tag{273}$$

and

$$G_0(x_i, y_i) = (L_{\gamma,\Omega} + o(1)) \frac{d(x_i, \partial\Omega)}{|x_i|^{\beta_+(\gamma)}} \frac{d(y_i, \partial\Omega)}{|y_i|^{\beta_-(\gamma)}} \tag{274}$$

as $i \to +\infty$. We fix a sequence $(x_i)_i \in \Omega$ such that $x_i \to 0$ and $d(x_i, \partial\Omega) \geq |x_i|/2$ as $i \to +\infty$. In the distribution sense, we have that

$$-\Delta(G_{h_{\varphi(i)}}(x_i,\cdot)-G_0(x_i,\cdot))+h_{\varphi(i)}(G_{h_{\varphi(i)}}(x_i,\cdot)-G_0(x_i,\cdot)) = (0-h_{\varphi(i)})G_0(x_i,\cdot) \text{ in } \Omega$$

in the distribution sense and $G_{h_{\varphi(i)}}(x_i,\cdot) - G_0(x_i,\cdot) = 0$ on $\partial\Omega$. It follows from (207) that for any $1 < p < \frac{n}{n-2}$, we have that $\|G_0(x_i,\cdot)\|_p \leq C(p)$ for all $i \in \mathbb{N}$. It then follows from elliptic theory that $G_{h_{\varphi(i)}}(x_i,\cdot) - G_h(x_i,\cdot) \in W^{2,p}(\Omega)$ and that

$$\|G_{h_{\varphi(i)}}(x_i,\cdot) - G_0(x_i,\cdot)\|_{W^{2,p}} \leq C\|h_{\varphi(i)}\|_\infty$$

For $1 < p < \min\{n/2; n/(n-2)\}$, we define $q := \frac{np}{n-2p}$. Sobolev embeddings then yield

$$\|G_{h_{\varphi(i)}}(x_i,\cdot) - G_0(x_i,\cdot)\|_{L^q(\Omega)} \leq C\|h_{\varphi(i)}\|_\infty.$$

We let $(\epsilon_i)_>0$ such that $\epsilon_i \to 0$ as $i \to +\infty$. We define $\alpha_i := \epsilon_i|x_i|$ so that $\alpha_i = o(|x_i|)$ as $i \to +\infty$. We have that

$$\int_{B_{\alpha_i}(0)} \left|G_{h_{\varphi(i)}}(x_i,y) - G_0(x_i,y)\right|^q \, dy \leq C\|h_{\varphi(i)}\|_\infty^q.$$

It then follows from (273), (274) and the boundedness of (h_i) in C^0 that

$$\int_{B_{\alpha_i}(0)} \left|(l - L_{\gamma,\Omega} + o(1))\frac{d(x_i,\partial\Omega)}{|x_i|^{\beta_+(\gamma)}}\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}}\right|^q \, dy \leq C.$$

We assume by contradiction that $l \neq L_{\gamma,\Omega}$, so that

$$\frac{d(x_i,\partial\Omega)}{|x_i|^{\beta_+(\gamma)}}\left(\int_{B_{\alpha_i}(0)}\left|\frac{d(y,\partial\Omega)}{|y|^{\beta_-(\gamma)}}\right|^q \, dy\right)^{1/q} \leq C.$$

If $n \leq q(1-\beta_-(\gamma))$, then the integral is infinite. This is a contradiction. Therefore $n > q(1-\beta_-(\gamma))$. Estimating the integral and using that $|x_i| \leq 2d(x_i,\partial\Omega)$, we get that

$$|x_i|^{1-\beta_+(\gamma)}\alpha_i^{1-\beta_-(\gamma)+\frac{n}{q}} \leq C.$$

With $\alpha_i = \epsilon_i|x|_i$, $\beta_-(\gamma)+\beta_+(\gamma) = n$ and the definition of q, we get that

$$|x_i|^{-n\left(1-\frac{1}{p}\right)}\epsilon_i^{1-\beta_-(\gamma)+\frac{n}{q}} \leq C.$$

Since $|x_i| \to 0$, with a suitable choice of $\epsilon_i \to 0$, we get a contradiction.

Therefore $l = L_{\gamma,\Omega}$ that is independent of the choice of the sequence (h_i). This proves (210) and ends the proof of Theorem 8.

15. Appendix E: Green's function for the Hardy-Schrödinger operator on $\mathbb{R}^n_-$

In this section, we prove the following:

THEOREM 9. *Fix $\gamma < \frac{n^2}{4}$. For all $p \in \mathbb{R}^n_- \setminus \{0\}$, there exists $G_p \in L^1(\mathbb{R}^n_-)$ such that*

(i) *$\eta G_p \in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n - \{p\})$,*

(ii) *For all $\varphi \in C^\infty_c(\mathbb{R}^n_-)$, we have that*

$$\varphi(p) = \int_{\mathbb{R}^n_-} G_p(x)\left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right) dx, \tag{275}$$

Moreover, if G_p, G'_p satisfy (i) and (ii) and are positive, then there exists $C \in \mathbb{R}$

such that $G_p(x) - G'_p(x) = C|x_1|\cdot|x|^{-\beta_-(\gamma)}$ *for all* $x\in\mathbb{R}^n_-\setminus\{0,p\}$.

In particular, there exists one and only one function $\mathcal{G}_p=\mathcal{G}(p,\cdot)>0$ *such that (i) and (ii) hold with* $G_p=\mathcal{G}_p$ *and*

(iii) $\mathcal{G}_p(x) = O\left(\frac{|x_1|}{|x|^{\beta_+(\gamma)}}\right)$ *as* $|x|\to+\infty$.

We then say that $\mathcal{G}$ *is the Green's function for* $-\Delta-\gamma|x|^{-2}$ *on* $\mathbb{R}^n_-$ *with Dirichlet boundary condition.*

In addition, $\mathcal{G}$ *satisfies the following properties:*

(iv) *For all* $p\in\mathbb{R}^n\setminus\{0\}$, *there exists* $c_0(p), c_\infty(p)>0$ *such that*

$$\mathcal{G}_p(x)\sim_{x\to 0}\frac{c_0(p)|x_1|}{|x|^{\beta_-(\gamma)}} \text{ and } \mathcal{G}_p(x)\sim_{x\to\infty}\frac{c_\infty(p)|x_1|}{|x|^{\beta_+(\gamma)}} \tag{276}$$

and

$$\mathcal{G}_p(x)\sim_{x\to p}\frac{1}{(n-2)\omega_{n-1}|x-p|^{n-2}}. \tag{277}$$

(v) *There exists* $c>0$ *independent of* p *such that*

$$c^{-1}\mathcal{H}_p(x)\le\mathcal{G}_p(x)\le c\mathcal{H}_p(x) \tag{278}$$

where

$$\mathcal{H}_p(x) := \left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}}\right)^{\beta_-(\gamma)}|x-p|^{2-n}\min\left\{1,\frac{|x_1|\cdot|p_1|}{|x-p|^2}\right\} \tag{279}$$

Proof of Theorem 9: We shall again proceed with several steps.

Step 15.1: Construction of a positive kernel at a given point: For a fixed $p_0\in\mathbb{R}^n\setminus\{0\}$, we show that there exists $G_{p_0}\in C^2(\overline{\mathbb{R}^n_-}\setminus\{0,p_0\})$ such that

$$\left\{\begin{array}{ll} -\Delta G_{p_0}-\frac{\gamma}{|x|^2}G_{p_0}=0 & \text{in } \mathbb{R}^n_-\setminus\{0,p_0\}\\ G_{p_0}>0 & \\ G_{p_0}\in L^{\frac{2n}{n-2}}(B_\delta(0)\cap\mathbb{R}^n_-) & \text{with } \delta:=|p_0|/4\\ G_{p_0} \text{ satisfies } (ii) \text{ with } p=p_0. & \end{array}\right. \tag{280}$$

Indeed, let $\tilde\eta\in C^\infty(\mathbb{R})$ be a nondecreasing function such that $0\le\tilde\eta\le 1$, $\tilde\eta(t)=0$ for all $t\le 1$ and $\tilde\eta(t)=1$ for all $t\ge 2$. For $\epsilon>0$, set $\eta_\epsilon(x):=\tilde\eta\left(\frac{|x|}{\epsilon}\right)$ for all $x\in\mathbb{R}^n$.

We let Ω_1 be a smooth bounded domain of $\mathbb{R}^n$ such that $\mathbb{R}^n_-\cap B_1(0)\subset\Omega_1\subset\mathbb{R}^n_-\cap B_3(0)$. We define $\Omega_R:=R\cdot\Omega_1$ so that $\mathbb{R}^n_-\cap B_R(0)\subset\Omega_R\subset\mathbb{R}^n_-\cap B_{3R}(0)$. We argue as in the proof of (211) to deduce that the operator $-\Delta-\frac{\gamma\eta_\epsilon}{|x|^2}$ is coercive on Ω_R and that there exists $c>0$ independent of $R,\epsilon>0$ such that

$$\int_{\Omega_R}\left(|\nabla\varphi|^2-\frac{\gamma\eta_\epsilon}{|x|^2}\varphi^2\right)dx\ge c\int_{\Omega_R}|\nabla\varphi|^2\,dx \quad\text{for all } \varphi\in C_c^\infty(\Omega_R).$$

Consider $R,\epsilon>0$ such that $R>2|p_0|$ and $\epsilon<\frac{|p_0|}{6}$, and let $G_{R,\epsilon}$ be the Green's function of $-\Delta-\frac{\gamma\eta_\epsilon}{|x|^2}$ in Ω_R with Dirichlet boundary condition. We have that $G_{R,\epsilon}>0$ since the operator is coercive.

Fix $R_0>0$ and $q'\in(1,\frac{n}{n-2})$, then by arguing as in the proof of (213), we get that there exists $C=C(\gamma,p_0,q',R_0)$ such that

$$\|G_{R,\epsilon}(p_0,\cdot)\|_{L^{q'}(B_{R_0}(0)\cap\mathbb{R}^n_-)}\le C \text{ for all } R>R_0 \text{ and } 0<\epsilon<\frac{|p_0|}{6}, \tag{281}$$

and

$$\|G_{R,\epsilon}(p_0,\cdot)\|_{L^{\frac{2n}{n-2}}(B_{\delta_0}(0)\cap\mathbb{R}^n_-)} \leq C \text{ for all } R > R_0 \text{ and } 0 < \epsilon < \frac{|p_0|}{6}, \tag{282}$$

where $\delta := |p_0|/4$. Arguing again as in Step 14.2 of the proof of Theorem 8, there exists $G_{p_0} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0, p_0\})$ such that

$$\left\{\begin{array}{ll} G_{R,\epsilon}(p_0,\cdot) \to G_{p_0} \geq 0 & \text{in } C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0,p_0\}) \text{ as } R \to +\infty,\ \epsilon \to 0 \\ -\Delta G_{p_0} - \frac{\gamma}{|x|^2} G_{p_0} = 0 & \text{in } \mathbb{R}^n_- \setminus \{0, p_0\} \\ G_{p_0} \equiv 0 \text{ on } \partial\mathbb{R}^n_- \setminus \{0\} & \\ G_{p_0} \in L^{\frac{2n}{n-2}}(B_\delta(0) \cap \mathbb{R}^n_-) & \end{array}\right. \tag{283}$$

and $\eta G_{p_0} \in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n \setminus \{p_0\})$. Fix $\varphi \in C^\infty_c(\mathbb{R}^n_-)$. For $R > 0$ large enough, we have that $\varphi(p_0) = \int_{\mathbb{R}^n_-} G_{R,\epsilon}(p_0,\cdot)(-\Delta\varphi - \gamma\eta_\epsilon|x|^{-2}\varphi)\,dx$. The integral bounds above yield $x \mapsto G_{p_0}(x)|x|^{-2} \in L^1_{loc}(\mathbb{R}^n_-)$. Therefore, we get

$$\varphi(p_0) = \int_{\mathbb{R}^n_-} G_{p_0}(x)\left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right)dx \text{ for all } \varphi \in C^\infty_c(\mathbb{R}^n_-). \tag{284}$$

As a consequence, $G_{p_0} > 0$.

Step 15.2: Asymptotic behavior at 0 and p_0 for solutions to (280). It follows from Theorem 6.1 in Ghoussoub-Robert [21] that either G_{p_0} behaves like $|x_1|\cdot|x|^{-\beta_-(\gamma)}$ or $|x_1|\cdot|x|^{-\beta_+(\gamma)}$ at 0. Since $G_{p_0} \in L^{\frac{2n}{n-2}}(B_\delta(0)\cap\mathbb{R}^n_-)$ for some small $\delta > 0$ and $\beta_-(\gamma) < \frac{n}{2} < \beta_+(\gamma)$, we get that there exists $c_0 > 0$ such that

$$\lim_{x\to 0} \frac{G_{p_0}(x)}{|x_1|\cdot|x|^{-\beta_-(\gamma)}} = c_0. \tag{285}$$

Since G_{p_0} is positive and smooth in a neighborhood of p_0, it follows from (284) and the classification of solutions to harmonic equations that

$$G_{p_0}(x) \sim_{x\to p_0} \frac{1}{(n-2)\omega_{n-1}|x-p_0|^{n-2}}. \tag{286}$$

Step 15.3: Asymptotic behavior at ∞ for solutions to (280): We let

$$\tilde{G}_{p_0}(x) := \frac{1}{|x|^{n-2}} G_{p_0}\left(\frac{x}{|x|^2}\right) \text{ for all } x \in \mathbb{R}^n_- \setminus \left\{0, \frac{p_0}{|p_0|^2}\right\},$$

be the Kelvin's transform of G. We have that

$$-\Delta\tilde{G}_{p_0} - \frac{\gamma}{|x|^2}\tilde{G}_{p_0} = 0 \text{ in } \mathbb{R}^n_- \setminus \left\{0, \frac{p_0}{|p_0|^2}\right\}\ ;\ \tilde{G} \equiv 0 \text{ on } \partial\mathbb{R}^n_- \setminus \{p_0\}.$$

Since $\tilde{G}_{p_0} > 0$, it follows from Theorem 6.1 in [21] that there exists $c_1 > 0$ such that

$$\text{either } \tilde{G}_{p_0}(x) \sim_{x\to 0} c_1\frac{|x_1|}{|x|^{\beta_-(\gamma)}} \text{ or } \tilde{G}_{p_0}(x) \sim_{x\to 0} c_1\frac{|x_1|}{|x|^{\beta_+(\gamma)}}.$$

Coming back to G_{p_0}, we get that

$$\text{either } G_{p_0}(x) \sim_{|x|\to\infty} c_1\frac{|x_1|}{|x|^{\beta_+(\gamma)}} \text{ or } G_{p_0}(x) \sim_{|x|\to\infty} c_1\frac{|x_1|}{|x|^{\beta_-(\gamma)}}. \tag{287}$$

Assuming we are in the second case, for any $c \leq c_1$, we define

$$\bar{G}_c(x) := G_{p_0}(x) - c\frac{|x_1|}{|x|^{\beta_-(\gamma)}} \text{ in } \mathbb{R}^n_- \setminus \{0, p_0\},$$

which satisfy $-\Delta\bar{G}_c - \frac{\gamma}{|x|^2}\bar{G}_c = 0$ in $\mathbb{R}^n_- \setminus \{0, p_0\}$. It follows from (287) and (286) that for $c < c_1$, $\bar{G}_c > 0$ around p_0 and ∞. Using that $\eta\bar{G}_c \in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n \setminus \{p_0\})$, it follows from the coercivity of $-\Delta - \gamma|x|^{-2}$ that $\bar{G}_c > 0$ in $\mathbb{R}^n_- \setminus \{0, p_0\}$ for $c < c_1$. Letting $c \to c_1$ yields $\bar{G}_{c_1} \geq 0$, and then $\bar{G}_{c_1} > 0$. Since $\bar{G}_{c_1}(x) = o(|x_1| \cdot |x|^{-\beta_-(\gamma)})$ as $|x| \to \infty$, another Kelvin transform and Theorem 6.1 in [21] yield $|x_1|^{-1}|x|^{\beta_+(\gamma)}\bar{G}_{c_1}(x) \to c_2 > 0$ as $|x| \to \infty$ for some $c_2 > 0$. Then there exists $c_3 > 0$ such that

$$\lim_{x\to 0} \frac{\bar{G}_{c_1}(x)}{|x_1| \cdot |x|^{-\beta_-(\gamma)}} = c_3 > 0 \text{ and } \lim_{x\to\infty} \frac{\bar{G}_{c_1}(x)}{|x_1| \cdot |x|^{-\beta_+(\gamma)}} = c_2. \tag{288}$$

Since $x \mapsto |x_1| \cdot |x|^{-\beta_-(\gamma)} \in H^2_{1,loc}(\mathbb{R}^n)$, we get that

$$\varphi(p) = \int_{\mathbb{R}^n_-} \bar{G}_{c_1}(x) \left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right) dx \text{ for all } \varphi \in C^\infty_c(\mathbb{R}^n_-).$$

Step 15.4: Uniqueness: Let $G_1, G_2 > 0$ be 2 functions such that $(i), (ii)$ hold for $p := p_0$, and set $H := G_1 - G_2$. It follows from Steps 2 and 3 that there exists $c \in \mathbb{R}$ such that $H'(x) := H(x) - c|x_1| \cdot |x|^{-\beta_-(\gamma)}$ satisfies

$$H'(x) =_{x\to 0} O\left(|x_1| \cdot |x|^{-\beta_-(\gamma)}\right) \text{ and } H'(x) =_{|x|\to\infty} O\left(|x_1| \cdot |x|^{-\beta_+(\gamma)}\right). \tag{289}$$

We then have that $\eta H' \in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n \setminus \{p_0\})$ and

$$\int_{\mathbb{R}^n_-} H'(x) \left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right) dx = 0 \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n_-).$$

The ellipticity of the Laplacian then yields $H' \in C^\infty(\overline{\mathbb{R}^n_-} \setminus \{0\})$. The pointwise bounds (289) yield that $H' \in H^2_{1,0}(\mathbb{R}^n_-)$. Multiplying $-\Delta H' - \frac{\gamma}{|x|^2}H' = 0$ by H', integrating by parts and the coercivity yield $H' \equiv 0$, and therefore, $(G_1 - G_2)(x) = c|x_1| \cdot |x|^{-\beta_-(\gamma)}$ for all $x \in \mathbb{R}^n_-$. This proves uniqueness.

Step 15.5: Existence. It follows from Steps 2 and 3 that, up to substracting a multiple of $x \mapsto |x_1| \cdot |x|^{-\beta_-(\gamma)}$, there exists a unique function $\mathcal{G}_{p_0} > 0$ satisfying (i), (ii) and the pointwise control (iii). Moreover, (285), (286) and (288) yield (276) and (277). As a consequence, (278) holds with $p = p_0$.

For $p \in \mathbb{R}^n \setminus \{0\}$, consider $\rho_p : \mathbb{R}^n_- \to \mathbb{R}^n_-$ a linear isometry fixing $\mathbb{R}^n_-$ such that $\rho_p(\frac{p_0}{|p_0|}) = \frac{p}{|p|}$, and define

$$\mathcal{G}_p(x) := \left(\frac{|p_0|}{|p|}\right)^{n-2} \mathcal{G}_{p_0}\left(\left(\rho_p^{-1}\left(\frac{|p_0|}{|p|}x\right)\right)\right) \text{ for all } x \in \mathbb{R}^n \setminus \{0, p\}.$$

As one checks, $\mathcal{G}_p > 0$ satisfies (i), (ii), (iii), (276), (277) and (278).

The definition of $\mathcal{G}_p$ is independent of the choice of ρ_p. Indeed, for any linear isometry $\rho_{p_0} : \mathbb{R}^n_- \to \mathbb{R}^n_-$ fixing p_0 and $\mathbb{R}^n_-$, $\mathcal{G}_{p_0} \circ \rho_{p_0}^{-1}$ satisfies (i), (ii), (iii), and therefore $\mathcal{G}_{p_0} \circ \rho_{p_0}^{-1} = \mathcal{G}_{p_0}$. The argument goes similarly of any isometry fixing p.

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