ELLIPTIC EQUATIONS WITH CRITICAL GROWTH AND A LARGE SET OF BOUNDARY SINGULARITIES

NASSIF GHOUSSOUB AND FRÉDÉRIC ROBERT

Abstract. We solve variationally certain equations of stellar dynamics of the form $-\sum_i \partial_{ii} u(x) = \frac{|u|^{p-2}u(x)}{\text{dist}(x,\mathcal{A})^s}$ in a domain Ω of $\mathbb{R}^n$, where $\mathcal{A}$ is a proper linear subspace of $\mathbb{R}^n$. Existence problems are related to the question of attainability of the best constant in the following inequality due to Maz'ya [20]:

$$0 < \mu_{s,\mathcal{P}}(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^2\, dx \;\middle|\; u \in H^2_{1,0}(\Omega) \text{ and } \int_\Omega \frac{|u(x)|^{2^\star(s)}}{|\pi(x)|^s}\, dx = 1 \right\}$$

where $0 < s < 2$, $2^\star(s) = \frac{2(n-s)}{n-2}$ and where π is the orthogonal projection on a linear space $\mathcal{P}$, where $\dim_{\mathbb{R}} \mathcal{P} \geq 2$ (see also Badiale-Tarantello [1]). We investigate this question and how it depends on the relative position of the subspace $\mathcal{P}^\perp$, the orthogonal of $\mathcal{P}$, with respect to the domain Ω as well as on the curvature of the boundary $\partial\Omega$ at its points of intersection with $\mathcal{P}^\perp$.

1. Introduction

Let Ω be a smooth domain of $\mathbb{R}^n$, where $n \geq 3$, and denote by $H^2_{1,0}(\Omega)$ the completion of $C^\infty_c(\Omega)$, the set of smooth functions compactly supported in Ω, for the norm $\|u\|_{H^2_{1,0}(\Omega)} = \sqrt{\int_\Omega |\nabla u|^2\, dx}$. In [20] (Corollary 2 in 2.1.6.), Maz'ya proved that if $\mathcal{P}$ is a linear subspace of $\mathbb{R}^n$ such that $2 \leq \dim_{\mathbb{R}} \mathcal{P} \leq n$, then there exists $C > 0$ such that for all $u \in H^2_{1,0}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star}}{|\pi(x)|^s}\, dx \right)^{\frac{2}{2^\star}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2\, dx, \tag{1}$$

where here $2^\star := \frac{2(n-s)}{n-2}$, $s \in (0,2)$ and π is the orthogonal projection on $\mathcal{P}$ with respect to the Euclidean structure. Recently, an alternative proof of this inequality was given by Badiale and Tarantello [1]. Define

$$\mu_{s,\mathcal{P}}(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2^\star}}{|\pi(x)|^s}\, dx \right)^{\frac{2}{2^\star}}} \;\middle|\; u \in H^2_{1,0}(\Omega) \setminus \{0\} \right\} \tag{2}$$

and note that (1) and (2) give that for all smooth domain $\Omega \subset \mathbb{R}^n$, we have

$$\mu_{s,\mathcal{P}}(\Omega) \geq \mu_{s,\mathcal{P}}(\mathbb{R}^n) > 0. \tag{3}$$

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In this article, we adress the question of the value of the best constant $\mu_{s,\mathcal{P}}(\Omega)$ as well as the issue of its attainability. As we will see, both questions are closely related to the relative positions of $\mathcal{P}^\perp$ and Ω, and to the geometry of the boundary $\partial\Omega$ at the points of $\mathcal{P}^\perp \cap \partial\Omega$.

The case when $s = 0$ (i.e., the non-singular case) is the well known Sobolev inequality. In this situation the infimum $\mu_{s,\mathcal{P}}(\Omega) = \mu_{0,\mathcal{P}}(\mathbb{R}^n)$ is not attained unless Ω is essentially the whole of $\mathbb{R}^n$.

The case $s \in (0,2)$ and $\dim_{\mathbb{R}}\mathcal{P} = n$ (that is $\mathcal{P} = \mathbb{R}^n$) was tackled in [11], [12], [13]. It was proved that when $0 \in \partial\Omega$, the infimum in (2) is then attained as soon as the mean curvature of $\partial\Omega$ (oriented with outward pointing normal vectors) at 0 is negative. The proof of this result required refined asymptotics for blown-up solutions of associated second order elliptic equations, the difficult case being when these solutions develop a "bubble" located precisely at the point 0. However, the bubble inherits the symmetry properties of the problem, and this allowed us to show in [12] that mean curvature conditions –as opposed to sectional curvature– suffice to eliminate the possibility of a bubbling-off phenomenon.

In the present paper, we tackle the case of a larger affine subspace of singularities ($1 \leq \dim_{\mathbb{R}}\mathcal{P} \leq n-1$) and in particular when $\mathcal{P}^\perp$ contains at least a line. The situation here closely depends on the relative positions of $\mathcal{P}^\perp$ and Ω, the most interesting case being when the subspace $\mathcal{P}^\perp$ does not touch the domain Ω but does touch its boundary (i.e., when $\mathcal{P}^\perp \cap \Omega = \emptyset$ and $\mathcal{P}^\perp \cap \partial\Omega \neq \emptyset$). A large part of the analysis is similar to what we have done in [12, 13] for the case of a single point of singularity on the boundary of Ω. However, a new set of difficulties arise in this situation: for one, the centers of the appearing bubbles are not bound to any particular location and may appear anywhere on $\partial\Omega$. They do eventually converge to a point in $\mathcal{P}^\perp \cap \partial\Omega \neq \emptyset$, and an important new issue becomes the precise control of the distance between the center of the bubble and this limiting point.

Another new problem related to this setting is the lack of symmetry of the bubble. As described by the next proposition, we do show that it enjoys the best symmetry possible in the $\mathcal{P}$-direction. Here and in the sequel, $\Delta = -\sum_i \partial_{ii}$ will denote the Laplacian with minus sign convention and $\mathbb{R}^n_- = \{x \in \mathbb{R}^n_- / \, x_1 < 0\}$.

Proposition 1.1. *Let π be the projection on a linear subspace $\mathcal{Q}$ of $\mathbb{R}^n$ such that $2 \leq dim_{\mathbb{R}}\mathcal{Q}$ and $\mathcal{Q}^\perp \subset \partial\mathbb{R}^n_-$. Assume $s \in (0,2)$ and consider $u \in C^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ such that*

$$\left\{\begin{array}{ll} \Delta u = \frac{u^{2^\star-1}}{|\pi(x)|^s} & \text{in } \mathbb{R}^n_- \\ u > 0 & \text{in } \mathbb{R}^n_- \\ u = 0 & \text{on } \partial\mathbb{R}^n_-. \end{array}\right. \tag{4}$$

and for some $C > 0$,

$$u(x) \leq C(1+|x|)^{1-n} \text{ for all } x \in \mathbb{R}^n_-. \tag{5}$$

Then there exists $v \in C^2(\mathbb{R}^\star_- \times \mathbb{R} \times \mathcal{Q}^\perp) \cap C^1(\mathbb{R}_- \times \mathbb{R} \times \mathcal{Q}^\perp)$ such that for all $z \in \mathcal{Q}^\perp$, and all $x_1 < 0$ and $y \in \mathbb{R}^n$ with $(x_1, y) \in \mathcal{Q}$, we have that $u(x_1, y, z) = v(x_1, |y|, z)$.

But this is not sufficient since the behavior of the bubble in the $\mathcal{P}$-direction and the $\mathcal{P}^\perp$-direction often cannot be related. Overcoming these difficulties, we prove the following theorem. In the sequel, $T_x\partial\Omega$ denotes the tangent space of $\partial\Omega$ at the point x.

Theorem 1.1. *Let Ω be a smooth bounded oriented domain of $\mathbb{R}^n$, $n \geq 3$, and let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ such that $2 \leq dim_{\mathbb{R}}\mathcal{P}$. Assume $s \in (0,2)$.*
(A) *If $\mathcal{P}^\perp \cap \Omega \neq \emptyset$, then $\mu_{s,\mathcal{P}}(\Omega) = \mu_{s,\mathcal{P}}(\mathbb{R}^n)$ and the infimum in (2) is not achieved.*

(B) *If $\mathcal{P}^\perp \cap \overline{\Omega} = \emptyset$, then the infimum in (2) is achieved.*

(C) *If $\mathcal{P}^\perp \cap \Omega = \emptyset$ and $\mathcal{P}^\perp \cap \partial\Omega \neq \emptyset$, then the infimum in (2) is achieved and the set of minimizers is pre-compact in $H^2_{1,0}(\Omega)$, provided that at any point $x \in \mathcal{P}^\perp \cap \partial\Omega$ the principal curvatures of $\partial\Omega$ at x are non-positive, but do not all vanish. Moreover, at those points $x \in \mathcal{P}^\perp \cap \partial\Omega$ where $\mathcal{P} \cap T_x\partial\Omega$ and $\mathcal{P}^\perp$ are orthogonal with respect to the second fundamental form of $\partial\Omega$ at x, it suffices that the mean curvature vector of $\partial\Omega \cap (x + (\mathcal{P}^\perp + (T_x\partial\Omega)^\perp))$ at x be null, while the mean curvature of $\partial\Omega$ at x is negative.*

The second part in (C) makes connection with the case where $\mathcal{P} = \mathbb{R}^n$ (i.e., $\mathcal{P}^\perp = \{0\}$ studied in [12]. Then the negativity of the mean curvature of $\partial\Omega$ at that point is sufficient for $\mu_{s,\mathcal{P}}(\Omega)$ to be attained. One may ask what happens in the case $\dim_{\mathbb{R}}\mathcal{P} \in \{0,1\}$. In the case when $\mathcal{P} = \{0\}$, inequality (1) is clearly irrelevant, however the case $\dim_{\mathbb{R}}\mathcal{P} = 1$ presents some interest, and this is the object of the following proposition:

Proposition 1.2. *Let Ω be a smooth bounded oriented domain of $\mathbb{R}^n$, $n \geq 3$, and let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ such that $dim_{\mathbb{R}}\mathcal{P} = 1$. Assume $s \in (0,2)$.*
(A) *If $\mathcal{P}^\perp \cap \Omega \neq \emptyset$, then the infimum in (2) is not achieved.*

(B) *If $\mathcal{P}^\perp \cap \overline{\Omega} = \emptyset$, then the infimum $\mu_{s,\mathcal{P}}(\Omega)$ in (2) is positive and is achieved.*

(C) *If $\mathcal{P}^\perp \cap \Omega = \emptyset$ while $\mathcal{P}^\perp \cap \partial\Omega \neq \emptyset$, then $\mu_{s,\mathcal{P}}(\Omega) > 0$ and the infimum is not achieved.*

Actually, when dealing with case (C) of Theorem 1.1 and Proposition 1.2, the crucial point is to have negative principal curvatures at each point of $\mathcal{P}^\perp \cap \partial\Omega$. But the fact that $\mathcal{P}^\perp$ only touches $\overline{\Omega}$ at its boundary means that the principal curvatures in the $\mathcal{P}^\perp-$direction are all nonnegative at these points –at least for those where $\mathcal{P}^\perp$ and $\mathcal{P} \cap T_x\partial\Omega$ are orthogonal for the fundamental form of $\partial\Omega$: therefore, for $\mu_{s,\mathcal{P}}(\Omega)$ to be achieved, one needs the negativity of the principal curvatures in some of the orthogonal directions, which is obviously impossible if $\mathcal{P}^\perp$ is $(n-1)-$dimensional and therefore the best constant is never achieved in this case. This means that the dimension restriction on the linear subspace in Theorem 1.1 is optimal. As a consequence of the techniques developed for the proof of Theorem 1.1, we get the following corollary.

Corollary 1.1. *Let Ω be a smooth bounded oriented domain of $\mathbb{R}^n$ and let π be the orthogonal projection onto a linear subspace $\mathcal{Q} \subset \mathbb{R}^n$ such that $2 \leq dim_{\mathbb{R}}\mathcal{Q}$. We assume that $\mathcal{Q}^\perp \cap \Omega = \emptyset$ and $\mathcal{Q}^\perp \cap \partial\Omega \neq \emptyset$. Assume $s \in (0,2)$ and consider $a \in C^1(\overline{\Omega})$ such that the operator $\Delta + a$ is coercive on Ω. Then there exists a solution $u \in H^2_{1,0}(\Omega) \cap C^1(\overline{\Omega})$ for*

$$\begin{cases} \Delta u + au = \frac{u^{2^\star-1}}{|\pi(x)|^s} & \text{in } \mathcal{D}'(\Omega) \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

provided that at any point $x \in \mathcal{Q}^\perp \cap \partial\Omega$ the principal curvatures of $\partial\Omega$ at x are non-positive, but do not all vanish.
Moreover, at those points $x \in \partial\Omega \cap \mathcal{Q}^\perp$ where $\mathcal{Q}^\perp$ and $\mathcal{Q} \cap T_x\partial\Omega$ are orthogonal with respect to the second fundamental form of $\partial\Omega$ at x, it suffices to assume that the mean curvature vector of $\partial\Omega \cap (x + (\mathcal{Q}^\perp + (T_x\partial\Omega)^\perp))$ at x is null, while the mean curvature of $\partial\Omega$ at x is negative.

Related references for best constant problems in Sobolev inequalities are Druet [5], Hebey-Vaugon [18, 19] and Egnell [10]. Concerning asymptotics for blown-up sequences of solutions to elliptic equations, we also refer to Atkinson-Peletier [2], Brézis-Peletier [3], Han [17], Druet [6], Druet-Hebey [7], Druet-Hebey-Robert [8] and Schoen-Zhang [23].

The rest of the paper is devoted to the proof of these results. As mentioned above, a significant part of the analysis was developed in [12, 13] for the case of a unique singular point at the boundary, and to which we shall refer frequently. On the other hand, we shall give all the details relating to the new difficulties arising in this new setting of large set of singularities. The paper is organized as follows. In section 2, we deal with points (A) and (B) of Theorem 1.1 and prove a symmetry result. Sections 3 to 5 are devoted to the proof of point (C) of Theorem 1.1 which is much more intricate, as it will require the full range of modern techniques for blow-up analysis and strong pointwise estimates for minimizers of the subcritical functional associated to (2). In section 6, we prove Proposition 1.2, while the appendix in section 7 provides a required regularity result for the family of elliptic pde's with singularities that we are dealing with in this paper. As a last remark, note that all the statements can be straightforwardly adapted to the case when $\mathcal{P}$ is an affine subspace of $\mathbb{R}^n$, and not only a linear space.

2. Partial symmetry of bubbles and Part (A), (B) of Theorem 1.1

We let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ with $2 \leq \dim_{\mathbb{R}}\mathcal{P} \leq n-1$. We shall denote by π the orthogonal projection on $\mathcal{P}$, and

$$\mu_{s,\mathcal{P}}(\Omega) := \inf\left\{ \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2^\star}}{|\pi(x)|^s}\, dx\right)^{\frac{2}{2^\star}}} \;\middle|\; u \in H_{1,0}^2(\Omega)\setminus\{0\}\right\} \tag{6}$$

Proof of Proposition 1.1. We first prove the partial symmetry property for the positive solutions to the limit equation on $\mathbb{R}^n_-$. For that, we consider $u \in C^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ that verifies the system (4) while verifying for some $C > 0$ the bound $u(x) \leq \frac{C}{(1+|x|)^{n-1}}$. We follow the proof of [12] to which we refer for details. For simplicity, up to a change of coordinates, we write any point $x \in \mathbb{R}^n$ as $x = (x_1, y, z)$, where $(x_1, y) \in \mathcal{Q} = \mathbb{R}^k$ and $z \in \mathcal{Q}^\perp = \mathbb{R}^{n-k}$. Therefore $\pi(x) = (x_1, y, 0)$. We let $\vec{e}_1$ be the first vector of the canonical basis of $\mathbb{R}^n$ and consider the open ball

$$D := B_{1/2}\left(-\frac{1}{2}\vec{e}_1\right).$$

We define

$$v(x) := |x|^{2-n} u\left(\vec{e}_1 + \frac{x}{|x|^2}\right) \tag{7}$$

for all $x \in \overline{D} \setminus \{0\}$. We extend v by 0 at 0. This is then well-defined and $v \in C^2(D) \cap C^1(\overline{D} \setminus \{0\})$. Moreover, $v(x) > 0$ for all $x \in D$ and $v(x) = 0$ for all $x \in \partial D \setminus \{0\}$. The function v verifies the equation

$$\Delta v = \frac{v^{2^\star - 1}}{|\pi(x + |x|^2 \vec{e}_1)|^s} \tag{8}$$

in D. Since $v > 0$ in D, it follows from Hopf's Lemma that $\frac{\partial v}{\partial \nu} < 0$ on $\partial D \setminus \{0\}$.

We prove the symmetry of u by proving a symmetry property of v, which is defined on a ball. Our proof uses the moving plane method. We take largely inspiration in [4] and [15]. We let $i \in \{2, ..., k\}$. For any $\mu \geq 0$ and $x \in \mathbb{R}^n$, we let

$$x_\mu = (x_1, ..., 2\mu - x_i, ..., x_n) \text{ and } D_\mu = \{x \in D / \, x_\mu \in D\}.$$

It follows from Hopf's Lemma that there exists $\epsilon_0 > 0$ such that for any $\mu \in (\frac{1}{2} - \epsilon_0, \frac{1}{2})$, we have that $D_\mu \neq \emptyset$ and $v(x) \geq v(x_\mu)$ for all $x \in D_\mu$ such that $x_i \leq \mu$. We let $\mu \geq 0$. We say that (P_μ) holds if $D_\mu \neq \emptyset$ and $v(x) \geq v(x_\mu)$ for all $x \in D_\mu$ such that $x_i \leq \mu$. We let

$$\lambda := \min \left\{ \mu \geq 0 \;\middle|\; (P_\nu) \text{ holds for all } \nu \in \left(\mu, \frac{1}{2}\right) \right\}. \tag{9}$$

We claim that $\lambda = 0$. Indeed we proceed by contradiction and assume that $\lambda > 0$. We then get that $D_\lambda \neq \emptyset$ and that (P_λ) holds. We let $w(x) := v(x) - v(x_\lambda)$ for all $x \in D_\lambda \cap \{x_n < \lambda\}$. Since (P_λ) holds, we have that $w(x) \geq 0$ for all $x \in D_\lambda \cap \{x_i < \lambda\}$. With the equation (8) of v and (P_λ), we get that

$$\begin{aligned} \Delta w &= \frac{v(x)^{2^\star - 1}}{|\pi(x + |x|^2 \vec{e}_1)|^s} - \frac{v(x_\lambda)^{2^\star - 1}}{|\pi(x_\lambda + |x_\lambda|^2 \vec{e}_1)|^s} \\ &\geq v(x_\lambda)^{2^\star - 1} \left(\frac{1}{|\pi(x + |x|^2 \vec{e}_1)|^s} - \frac{1}{|\pi(x_\lambda + |x_\lambda|^2 \vec{e}_1)|^s} \right) \end{aligned}$$

for all $x \in D_\lambda \cap \{x_i < \lambda\}$. Since $2 \leq i \leq k$, we get that the RHS is positive (see [12]), and then $\Delta w(x) > 0$ for all $x \in D_\lambda \cap \{x_i < \lambda\}$. It then follows from Hopf's Lemma and the strong comparison principle that

$$w > 0 \text{ in } D_\lambda \cap \{x_i < \lambda\} \text{ and } \frac{\partial w}{\partial \nu} < 0 \text{ on } D_\lambda \cap \{x_i = \lambda\}.$$

The contradiction then follows from standard arguments, we refer to [12, 13] for details. This yields $\lambda = 0$.

Here goes the final argument. Since $\lambda = 0$, it follows from the definition (9) of λ that $v(x) \geq v(x_1, ..., -x_i, ..., x_n)$ for all $x \in D$ such that $x_i \leq 0$. With the same technique, we get the reverse inequality, and then, we get that $v(x) = v(x_1, ..., -x_i, ..., x_n)$ for all $x = (x', x_n) \in D$. In other words, v is symmetric with respect to the hyperplane $\{x_i = 0\}$. The same analysis holds for any hyperplane containing $\text{Span}\{\vec{e_1}, \vec{e_{k+1}}, ..., \vec{e_n}\}$. Coming back to the initial function u, this proves Proposition 1.1 and the symmetry property.

The object of the following proposition is to deal with case (A) of Theorem 1.1 that is when $\mathcal{P}^\perp \cap \Omega \neq \emptyset$.

Proposition 2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $\mathcal{P} \subset \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$, where $2 \leq dim_{\mathbb{R}} \mathcal{P} \leq n - 1$. Let $s \in (0, 2)$ and assume that $\mathcal{P}^\perp \cap \Omega \neq \emptyset$, then $\mu_{s,\mathcal{P}}(\Omega) = \mu_{s,\mathcal{P}}(\mathbb{R}^n)$ and the infimum $\mu_{s,\mathcal{P}}(\Omega)$ is not achieved.*

Proof: Fix $x_0 \in \mathcal{P}^\perp \cap \Omega$, and let $\delta > 0$ such that $B_\delta(x_0) \subset \Omega$. Let $\alpha > 0$ and $u \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$\frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star}}{|\pi(x)|^s}\, dx\right)^{\frac{2}{2^\star}}} \leq \mu_{s,\mathcal{P}}(\mathbb{R}^n) + \alpha.$$

For $\epsilon > 0$, we let $u_\epsilon(x) := \epsilon^{-\frac{n-2}{2}} u\left(\frac{x-x_0}{\epsilon}\right)$ for all $x \in \Omega$. As is easily checked, $u_\epsilon \in C_c^\infty(\Omega)$ for $\epsilon > 0$ small and

$$\frac{\int_{\Omega} |\nabla u_\epsilon|^2\, dx}{\left(\int_{\Omega} \frac{|u_\epsilon|^{2^\star}}{|\pi(x)|^s}\, dx\right)^{\frac{2}{2^\star}}} = \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star}}{|\pi(x)|^s}\, dx\right)^{\frac{2}{2^\star}}} \leq \mu_{s,\mathcal{P}}(\mathbb{R}^n) + \alpha.$$

Here, we have used that $x_0 \in \mathcal{P}^\perp$, that is $\pi(x_0) = 0$. Coming back to the definition (6) of $\mu_{s,\mathcal{P}}(\Omega)$ letting $\alpha \to 0$ and using (3), we get that $\mu_{s,\mathcal{P}}(\Omega) = \mu_{s,\mathcal{P}}(\mathbb{R}^n)$.

We claim that $\mu_{s,\mathcal{P}}(\Omega)$ is not achieved. Indeed, assuming it is achieved by a function $u \in H_{1,0}^2(\Omega) \setminus \{0\}$, we can assume without loss that $u \geq 0$. Since $\mu_{s,\mathcal{P}}(\Omega) = \mu_{s,\mathcal{P}}(\mathbb{R}^n)$, we get that $\mu_{s,\mathcal{P}}(\mathbb{R}^n)$ is also attained by u which then verifies $\Delta u = \frac{u^{2^\star-1}}{|\pi(x)|^s}$ in $\mathcal{D}'(\mathbb{R}^n)$. Since $u \geq 0$, it follows from the regularity results of section 7 and the maximum principle that $u > 0$ on $\mathbb{R}^n \setminus \mathcal{P}$, a contradiction since $u \in H_{1,0}^2(\Omega)$. □

The case where $\mathcal{P}^\perp \cap \overline{\Omega} = \emptyset$ is dealt with in the following proposition.

Proposition 2.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, and let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ such that $2 \leq dim_{\mathbb{R}}\mathcal{P} \leq n-1$. Assume $s \in (0,2)$ and that $\mathcal{P}^\perp \cap \overline{\Omega} = \emptyset$, then the infimum $\mu_{s,\mathcal{P}}(\Omega)$ is attained.*

Proof: Since $\mathcal{P}^\perp \cap \overline{\Omega} = \emptyset$, there exists $c, C > 0$ such that $c \leq |\pi(x)| \leq C$ for all $x \in \Omega$. In particular, since $2^\star < \frac{2n}{n-2}$, we have compactness of the embedding of $H_{1,0}^2(\Omega)$ in $L^{2^\star}(\Omega, |\pi(x)|^{-s})$ and therefore the existence of minimizers. This ends the proof of the Proposition. □

3. Blow-up analysis, Part I

Throughout this section, we let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, and $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ such that $2 \leq \dim_{\mathbb{R}}\mathcal{P} \leq n-1$. Let $s \in (0,2)$ and assume that

$$\mathcal{P}^\perp \cap \Omega = \emptyset \text{ and } \mathcal{P}^\perp \cap \partial\Omega \neq \emptyset. \tag{10}$$

Here and in the sequel, we let π be the orthogonal projection on $\mathcal{P}$. This is the most intricate case to which the rest of the paper is essentially devoted.

Proposition 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, and let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$, such that $2 \leq dim_{\mathbb{R}}\mathcal{P} \leq n-1$. Let $s \in (0,2)$ and assume that $\mathcal{P}^\perp \cap \Omega = \emptyset$ and $\mathcal{P}^\perp \cap \partial\Omega \neq \emptyset$, then $\mu_{s,\mathcal{P}}(\Omega) \leq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)$.*

Proof: Let $x_0 \in \mathcal{P}^\perp \cap \partial\Omega$. Since $\mathcal{P}^\perp \cap \Omega = \emptyset$, we have that

$$\mathcal{P}^\perp \subset T_{x_0}\partial\Omega, \tag{11}$$

where $T_{x_0}\partial\Omega$ is the tangent space at x_0 of the smooth manifold $\partial\Omega$. It follows from (11) that $(T_{x_0}\partial\Omega)^\perp \subset \mathcal{P}$. We choose a direct orthonormal basis $(\vec{e}_1, ..., \vec{e}_n)$ of $\mathbb{R}^n$ such that

$$\begin{array}{l} \vec{e}_1 = \vec{n}_{x_0} \text{ is the normal outward vector at } x_0 \text{ of } \partial\Omega \\ (\vec{e}_1, ..., \vec{e}_k) \text{ is an orthonormal basis of } \mathcal{P} \\ (\vec{e}_{k+1}, ..., \vec{e}_n) \text{ is an orthonormal basis of } \mathcal{P}^\perp. \end{array} \tag{12}$$

Here and in what follows, $k = \dim_{\mathbb{R}}\mathcal{P}$, so that $2 \leq k \leq n-1$. In particular, $(\vec{e}_2, ..., \vec{e}_n)$ is an orthonormal basis of $T_{x_0}\partial\Omega$. For the rest of this section, we shall be refering to this particular basis. In particular, we adopt the following notation: we write any element $x \in \mathbb{R}^n$ as $x = (x_1, y, z)$, with $x_1 \in \mathbb{R}$, $y \in \text{span}(\vec{e}_2, ..., \vec{e}_k)$ and $z \in \text{span}(\vec{e}_{k+1}, ..., \vec{e}_n) = \mathcal{P}^\perp$.

Since $\partial\Omega$ is smooth, there exist U, V open subsets of $\mathbb{R}^n$, such that $0 \in U$ and $x_0 \in V$, there exists $\varphi \in C^\infty(U, V)$ and $\varphi_0 \in C^\infty(U')$ (where $U' = \{(y,z)/ \exists x_1 \in \mathbb{R} \text{ s.t. } (x_1, y, z) \in U\}$) such that

$$\begin{array}{ll} (i) & \varphi : U \to V \text{ is a } C^\infty - \text{diffeomorphism} \\ (ii) & \varphi(0) = x_0 \\ (iii) & \varphi(U \cap \{x_1 < 0\}) = \varphi(U) \cap \Omega \text{ and } \varphi(U \cap \{x_1 = 0\}) = \varphi(U) \cap \partial\Omega. \\ (iv) & \varphi_0(0) = 0 \text{ and } \nabla\varphi_0(0) = 0 \\ (v) & \varphi(x_1, y, z) = (x_1 + \varphi_0(y,z), y, z) + x_0 \text{ for all } (x_1, y, z) \in U \end{array} \tag{13}$$

where $D_x\varphi_0$ denotes the differential of φ_0 at x. Let $\alpha > 0$ and $u \in C_c^\infty(\mathbb{R}^n_-) \setminus \{0\}$ such that

$$\frac{\int_{\mathbb{R}^n_-} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n_-} \frac{|u|^{2^\star}}{|\pi(x)|^s} \, dx\right)^{\frac{2}{2^\star}}} \leq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-) + \alpha.$$

Define $u_\epsilon(x) := \epsilon^{-\frac{n-2}{2}} u\left(\frac{\varphi^{-1}(x)}{\epsilon}\right)$ for all $x \in \Omega$ and all $\epsilon > 0$. As easily checked, for $\epsilon > 0$ small enough, we have that $u_\epsilon \in C_c^\infty(\Omega)$. Standard computations yield that

$$\mu_{s,\mathcal{P}}(\Omega) \leq \frac{\int_\Omega |\nabla u_\epsilon|^2 \, dx}{\left(\int_\Omega \frac{|u_\epsilon|^{2^\star}}{|\pi(x)|^s} \, dx\right)^{\frac{2}{2^\star}}} = \frac{\int_{\mathbb{R}^n_-} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n_-} \frac{|u|^{2^\star}}{|\pi(x)|^s} \, dx\right)^{\frac{2}{2^\star}}} + o(1) \leq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-) + \alpha + o(1)$$

where $\lim_{\epsilon \to 0} o(1) = 0$. Letting $\epsilon \to 0$ and $\alpha \to 0$, we get the claimed result. □

In order to construct minimizers for $\mu_{s,\mathcal{P}}(\Omega)$, we consider a subcritical minimization problem for which we have compactness. The proof of this result is standard and we refer to [12] for details.

Proposition 3.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, and let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ such that $2 \leq \dim_{\mathbb{R}}\mathcal{P} \leq n-1$. Let $s \in (0,2)$ and assume that (10) holds, then for any $\epsilon \in (0, 2^\star - 2)$, the infimum*

$$\mu^\epsilon_{s,\mathcal{P}}(\Omega) := \inf_{u \in H^2_{1,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega \frac{|u|^{2^\star - \epsilon}}{|\pi(x)|^s} \, dx\right)^{\frac{2}{2^\star - \epsilon}}},$$

is achieved by a function $u_\epsilon \in H_{1,0}^2(\Omega)$. *Moreover,* $u_\epsilon \in C^\infty(\overline{\Omega} \setminus \mathcal{P}^\perp)$ *and can be assumed to satisfy the system*

$$\begin{cases} \Delta u_\epsilon = \frac{u_\epsilon^{2^\star-1-\epsilon}}{|\pi(x)|^s} & \text{in } \mathcal{D}'(\Omega) \\ u_\epsilon > 0 & \text{in } \Omega \\ \int_\Omega \frac{u_\epsilon^{2^\star-\epsilon}}{|\pi(x)|^s}\, dx = (\mu_{s,\mathcal{P}}^\epsilon(\Omega))^{\frac{2^\star-\epsilon}{2^\star-2-\epsilon}} & \end{cases}$$

Moreover, we have that $\lim_{\epsilon\to 0} \mu_{s,\mathcal{P}}^\epsilon(\Omega) = \mu_{s,\mathcal{P}}(\Omega)$, *and there exists* $u_0 \in H_{1,0}^2(\Omega)$ *such that, up to a subsequence,* $u_\epsilon \rightharpoonup u_0$ *weakly in* $H_{1,0}^2(\Omega)$ *when* $\epsilon \to 0$. *If* $u_0 \not\equiv 0$, *then* $\lim_{\epsilon\to 0} u_\epsilon = u_0$ *strongly in* $H_{1,0}^2(\Omega)$ *and* u_0 *is a minimizer for* $\mu_{s,\mathcal{P}}(\Omega)$. *In particular,* $\mu_{s,\mathcal{P}}(\Omega)$ *is attained.*

We now start the blow-up analysis for minimizing sequences. Actually, we consider a more general case. Here and in the sequel, we let $p_\epsilon \in [0, 2^\star - 2)$ such that

$$\lim_{\epsilon\to 0} p_\epsilon = 0.$$

We assume that (10) holds. We consider a family $(a_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ such that there exists $\lambda, C > 0$ such that

$$\|a_\epsilon\|_{C^1(\overline{\Omega})} \le C \text{ and } \int_\Omega (|\nabla\varphi|^2 + a_\epsilon \varphi^2)\, dx \ge \lambda \int_\Omega \varphi^2\, dx \tag{14}$$

for all $\epsilon \to 0$ and all $\varphi \in C_c^\infty(\Omega)$. For any $\epsilon > 0$, we consider $u_\epsilon \in H_{1,0}^2(\Omega) \cap C^2(\overline{\Omega} \setminus \mathcal{P}^\perp)$ a solution to the system

$$\begin{cases} \Delta u_\epsilon + a_\epsilon u_\epsilon = \frac{u_\epsilon^{2^\star-1-p_\epsilon}}{|\pi(x)|^s} & \text{in } \mathcal{D}'(\Omega) \\ u_\epsilon > 0 & \text{in } \Omega \end{cases} \tag{15}$$

We assume that u_ϵ is of minimal energy type, that is

$$\int_\Omega \frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|\pi(x)|^s}\, dx = \mu_{s,\mathcal{P}}(\Omega)^{\frac{2^\star}{2^\star-2}} + o(1) \tag{16}$$

where $\lim_{\epsilon\to 0} o(1) = 0$. We also assume that blow-up occurs, that is

$$u_\epsilon \rightharpoonup 0 \tag{17}$$

weakly in $H_{1,0}^2(\Omega)$ when $\epsilon \to 0$. Such a family arises naturally when $u_0 \equiv 0$ in Proposition 3.2. It follows from Proposition 7.1 of the Appendix that $u_\epsilon \in C^0(\overline{\Omega})$. We let $x_\epsilon \in \Omega$ and $\mu_\epsilon, k_\epsilon > 0$ such that

$$\max_\Omega u_\epsilon = u_\epsilon(x_\epsilon) = \mu_\epsilon^{-\frac{n-2}{2}} \text{ and } k_\epsilon := \mu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}}. \tag{18}$$

Our goal in this section is to prove the following:

Proposition 3.3. *Under the above assumption, there exists* $x_0 \in \mathcal{P}^\perp \cap \partial\Omega$, *a chart* φ *as in (13), there exists* $(\bar{z}_\epsilon)_{\epsilon>0} \in \partial\mathbb{R}_-^n$ *such that* $\lim_{\epsilon\to 0} \bar{z}_\epsilon = 0$ *and such that the function*

$$v_\epsilon(x) := \mu_\epsilon^{\frac{n-2}{2}} u_\epsilon \circ \varphi(\bar{z}_\epsilon + k_\epsilon x)$$

defined for $x \in \frac{U-\bar{z}_\epsilon}{k_\epsilon}$ *and* $\epsilon > 0$ *verifies that there exists* $v \in H_{1,0}^2(\mathbb{R}_-^n) \setminus \{0\}$ *such that for any* $\eta \in C_c^\infty(\mathbb{R}^n)$, $\eta v_\epsilon \rightharpoonup \eta v$ *in* $H_{1,0}^2(\mathbb{R}_-^n)$ *weakly in* $\mathcal{D}'(\mathbb{R}_-^n)$ *when* $\epsilon \to 0$.

The function v verifies that

$$\Delta v = \frac{v^{2^\star-1}}{|\pi(x)|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-)$$

and $\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx = \mu_{s,\mathcal{P}}(\Omega)^{\frac{2^\star}{2^\star-2}} = \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}$. *In addition,* $v \in C^1(\overline{\mathbb{R}^n_-})$ *and*

$$\lim_{\epsilon\to 0} v_\epsilon = v \text{ in } C^1_{loc}(\overline{\mathbb{R}^n_-}). \tag{19}$$

Moreover,

$$\lim_{\epsilon\to 0} \mu_\epsilon^{p_\epsilon} = 1.$$

Proof: The proof goes in five steps.

Step 3.1: We claim that

$$\mu_\epsilon = o(1) \text{ and } \pi(x_\epsilon) = O(k_\epsilon) \tag{20}$$

when $\epsilon \to 0$.

Indeed assume that $\lim_{\epsilon\to 0} \mu_\epsilon \neq 0$, then up to a subsequence, there exists $C > 0$ such that $|u_\epsilon(x)| \leq C$ for all $x \in \Omega$ and all $\epsilon > 0$. Mimicking the proof of the Appendix, we get that there exists $C > 0$ such that $\|u_\epsilon\|_{C^1(\overline{\Omega})} \leq C$. Since (17) holds, it follows from Ascoli's theorem that, up to a subsequence, $\lim_{\epsilon\to 0} u_\epsilon = 0$ in $C^0(\overline{\Omega})$. A contradiction with (16). This proves that $\lim_{\epsilon\to 0} \mu_\epsilon = 0$.

To prove the second part of the claim assume that

$$\lim_{\epsilon\to 0} \frac{|\pi(x_\epsilon)|}{k_\epsilon} = +\infty. \tag{21}$$

For any $\epsilon > 0$, set

$$\beta_\epsilon = |\pi(x_\epsilon)|^{\frac{s}{2}} u_\epsilon(x_\epsilon)^{\frac{2+p_\epsilon-2^\star}{2}} = |\pi(x_\epsilon)|^{\frac{s}{2}} k_\epsilon^{\frac{2-s}{2}}. \tag{22}$$

It follows from the definition (22) of β_ϵ and (21) that

$$\lim_{\epsilon\to 0} \beta_\epsilon = 0,\ \lim_{\epsilon\to 0} \left(\frac{\beta_\epsilon}{k_\epsilon}\right)^2 = +\infty \text{ and } \lim_{\epsilon\to 0} \left(\frac{\beta_\epsilon}{|\pi(x_\epsilon)|}\right)^2 = 0 \tag{23}$$

when $\epsilon \to 0$.

Case 3.1.1: Assume first there exists $\rho > 0$ such that $\frac{d(x_\epsilon,\partial\Omega)}{\beta_\epsilon} \geq 2\rho$ for all $\epsilon > 0$. For $x \in B_{2\rho}(0)$ and $\epsilon > 0$, define

$$v_\epsilon(x) := \frac{u_\epsilon(x_\epsilon + \beta_\epsilon x)}{u_\epsilon(x_\epsilon)}.$$

This is well defined since $x_\epsilon + \beta_\epsilon x \in \Omega$ for all $x \in B_{2\rho}(0)$. As easily checked, with (15), we have that

$$\Delta v_\epsilon + k_\epsilon^2 a_\epsilon(x_\epsilon + \beta_\epsilon x) v_\epsilon = \frac{v_\epsilon^{2^\star-1-p_\epsilon}}{\left|\frac{\pi(x_\epsilon)}{|\pi(x_\epsilon)|} + \frac{\beta_\epsilon}{|\pi(x_\epsilon)|}\pi(x)\right|^s}$$

weakly in $B_{2\rho}(0)$. Since $0 \leq v_\epsilon(x) \leq v_\epsilon(0) = 1$ for all $x \in B_\rho(0)$, it follows from standard elliptic theory and (23) that there exists $v \in C^1(B_{2\rho}(0))$ such that $v_\epsilon \to v$ in $C^1_{loc}(B_{2\rho}(0))$ as $\epsilon \to 0$. In particular,

$$v(0) = \lim_{\epsilon\to 0} v_\epsilon(0) = 1 \tag{24}$$

With a change of variables and the definition (22) of β_ϵ, we get that

$$\int_{\Omega\cap B_{\rho\beta_\epsilon}(x_\epsilon)}\frac{u_\epsilon^{2^\star-p_\epsilon}}{|\pi(x)|^s}\,dx=\frac{u_\epsilon(x_\epsilon)^{2^\star-p_\epsilon}\beta_\epsilon^n}{|\pi(x_\epsilon)|^s}\int_{B_\rho(0)}\frac{v_\epsilon^{2^\star-p_\epsilon}}{\left|\frac{\pi(x_\epsilon)}{|\pi(x_\epsilon)|}+\frac{\beta_\epsilon}{|\pi(x_\epsilon)|}\pi(x)\right|^s}\,dx$$

$$\geq\left(\frac{\beta_\epsilon}{k_\epsilon}\right)^{n-2}\mu_\epsilon^{-p_\epsilon\frac{n-2}{2^\star-2}}\int_{B_\rho(0)}\frac{v_\epsilon^{2^\star-p_\epsilon}}{\left|\frac{\pi(x_\epsilon)}{|\pi(x_\epsilon)|}+\frac{\beta_\epsilon}{|\pi(x_\epsilon)|}\pi(x)\right|^s}\,dx.$$

Using (16), (23) and passing to the limit $\epsilon\to 0$ (note that $\mu_\epsilon^{-1}\geq 1$ for $\epsilon>0$ small), we get that $\int_{B_\rho(0)}v^{2^\star}\,dx=0$, and then $v\equiv 0$. This contradicts (24) and therefore (21) does not hold, which proves the claim in Case 3.1.1.

Case 3.1.2: Now assume that, up to a subsequence, $\lim_{\epsilon\to 0}\frac{d(x_\epsilon,\partial\Omega)}{\beta_\epsilon}=0$. We then get a contradiction by a rescaling of u_ϵ as in [12]. The proof uses the techniques of Case 3.1.1 and is rather similar to [12] to which we refer for the details.

In both cases, we have obtained a contradiction and Step 3.1 is established. □

Step 3.2: Up to a subsequence, we claim that x_0 defined as

$$x_0:=\lim_{\epsilon\to 0}x_\epsilon \tag{25}$$

belongs to $\mathcal{P}^\perp\cap\partial\Omega$.

Indeed, it follows from (20) and (18) that $\pi(x_0)=0$, that is $x_0\in\mathcal{P}^\perp$. Since $x_0\in\overline{\Omega}$, it follows from (10) that $x_0\in\mathcal{P}^\perp\cap\partial\Omega$.

Since (10) holds, we have that (11) holds. We choose a basis as in (12) and we choose a chart φ as in (13). In particular, here again, we let $k=\dim_{\mathbb{R}}\mathcal{P}\in\{2,...,n-1\}$.

Step 3.3: Setting

$$x_\epsilon=\varphi(x_{1,\epsilon},y_\epsilon,z_\epsilon), \tag{26}$$

where $x_{1,\epsilon}<0$, $y_\epsilon\in\mathrm{span}(\vec{e}_2,...,\vec{e}_k)$ and $z_\epsilon\in\mathrm{span}(\vec{e}_{k+1},...,\vec{e}_n)=\mathcal{P}^\perp$, we claim that

$$d(x_\epsilon,\partial\Omega)=(1+o(1))|x_{1,\epsilon}|=O(k_\epsilon),\ y_\epsilon=O(k_\epsilon)\text{ and }\varphi_0(0,z_\epsilon)=O(k_\epsilon), \tag{27}$$

when $\epsilon\to 0$. Here φ_0 is as in (13).

Proof of the claim: our first remark is that

$$d(x_\epsilon,\partial\Omega)=O(k_\epsilon) \tag{28}$$

when $\epsilon\to 0$. Indeed, since $\mathcal{P}^\perp\cap\Omega=\emptyset$, we have that $x_\epsilon-\pi(x_\epsilon)\in\mathcal{P}^\perp\in\mathbb{R}^n\setminus\Omega$. Since $x_\epsilon\in\Omega$, there exists $t_\epsilon\in(0,1)$ such that $t_\epsilon x_\epsilon+(1-t_\epsilon)\cdot(x_\epsilon-\pi(x_\epsilon))\in\partial\Omega$. Consequently,

$$d(x_\epsilon,\partial\Omega)\leq|x_\epsilon-(t_\epsilon x_\epsilon+(1-t_\epsilon)\cdot(x_\epsilon-\pi(x_\epsilon)))|=(1-t_\epsilon)|\pi(x_\epsilon)|\leq|\pi(x_\epsilon)|=O(k_\epsilon)$$

when $\epsilon\to 0$. This proves (28).

As in [12], we get that

$$d(x_\epsilon,\partial\Omega)=(1+o(1))|x_{1,\epsilon}| \tag{29}$$

when $\epsilon\to 0$. We write that

$$\pi(x_\epsilon)=\pi(x_{1,\epsilon}+\varphi_0(y_\epsilon,z_\epsilon),y_\epsilon,z_\epsilon)=(x_{1,\epsilon}+\varphi_0(y_\epsilon,z_\epsilon),y_\epsilon,0).$$

With (20) and (28), we then get that

$$\varphi_0(y_\epsilon, z_\epsilon) = O(k_\epsilon) \text{ and } y_\epsilon = O(k_\epsilon) \tag{30}$$

when $\epsilon \to 0$. Noting that $\varphi_0(y_\epsilon, z_\epsilon) = \varphi_0(0, z_\epsilon) + O(|y_\epsilon|)$ when $\epsilon \to 0$, we get that $\varphi_0(0, z_\epsilon) = O(k_\epsilon)$. These last equalities, (28), (29) and (30) prove (27). ☐

We let

$$\lambda_\epsilon := -\frac{x_{1,\epsilon}}{k_\epsilon} > 0,\ \theta_\epsilon := \frac{y_\epsilon}{k_\epsilon} \in \mathcal{P} \text{ and } \rho_\epsilon := -\frac{\varphi_0(0, z_\epsilon)}{k_\epsilon}. \tag{31}$$

It follows from (27) and (29) that there exist $\lambda_0 \geq 0$, $\rho_0 \in \mathbb{R}$ and $\theta_0 \in \mathcal{P}$ such that

$$\lambda_0 := \lim_{\epsilon\to 0} \lambda_\epsilon,\ \theta_0 := \lim_{\epsilon\to 0} \theta_\epsilon \text{ and } \rho_0 := \lim_{\epsilon\to 0} \rho_\epsilon. \tag{32}$$

We claim that $\rho_\epsilon \geq 0$ for all $\epsilon > 0$. Indeed, since $\mathcal{P}^\perp \cap \Omega = \emptyset$, there exists $\delta > 0$ such that for all $z \in \text{span}\{\vec{e}_{k+1}, ..., \vec{e}_n\} \cap B_\delta(0)$

$$\varphi_0(0, z) \leq 0. \tag{33}$$

The definition (31) of ρ_ϵ yields that $\rho_\epsilon \geq 0$ for all $\epsilon > 0$. Note that it follows from (33) that there exists $C > 0$ such that

$$d(x, \partial\Omega) \leq C|\pi(x)| \tag{34}$$

for all $x \in \Omega$.

Step 3.4: From now on, we let $\bar{z}_\epsilon = (0, 0, z_\epsilon)$ for all $\epsilon > 0$ where z_ϵ is defined in (26), and for any $x \in \frac{U - \bar{z}_\epsilon}{k_\epsilon} \cap \{x_1 \leq 0\}$, we set

$$v_\epsilon(x) := \frac{u_\epsilon \circ \varphi(\bar{z}_\epsilon + k_\epsilon x)}{u_\epsilon(x_\epsilon)}, \tag{35}$$

where φ is defined in (13). It follows from (31) that

$$v_\epsilon(-\lambda_\epsilon, \theta_\epsilon, 0) = 1. \tag{36}$$

As easily checked, for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that $\eta v_\epsilon \in H_{1,0}^2(\mathbb{R}^n_-)$ for all $\epsilon > 0$.

Step 3.4.1: There exists $v \in H_{1,0}^2(\mathbb{R}^n_-)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$,

$$\eta v_\epsilon \rightharpoonup \eta v$$

weakly in $H_{1,0}^2(\mathbb{R}^n_-)$ when $\epsilon \to 0$. The proof is rather similar to what was done in [12] to which we refer for details.

Step 3.4.2: We claim that $\lim_{\epsilon\to 0} v_\epsilon = v$ in $C^1_{loc}(\overline{\mathbb{R}^n_-})$, where $v \not\equiv 0$.

Indeed, let $R > 0$ and for any $i, j = 1, ..., n$, we let $(\tilde{g}_\epsilon)_{ij} = (\partial_i\varphi(\bar{z}_\epsilon + k_\epsilon x), \partial_j\varphi(\bar{z}_\epsilon + k_\epsilon x))$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$. We consider $\tilde{g}_\epsilon$ as a metric on $\mathbb{R}^n$. We let

$$\Delta_{\tilde{g}_\epsilon} = -\tilde{g}_\epsilon^{ij}\left(\partial_{ij} - \Gamma_{ij}^k(\tilde{g}_\epsilon)\partial_k\right),$$

where $\tilde{g}_\epsilon^{ij} := (\tilde{g}_\epsilon^{-1})_{ij}$ are the coordinates of the inverse of the tensor $\tilde{g}_\epsilon$ and the $\Gamma_{ij}^k(\tilde{g}_\epsilon)$'s are the Christoffel symbols of the metric $\tilde{g}_\epsilon$. With a change of variable and the definition (35), equation (15) rewrites as

$$\Delta_{\tilde{g}_\epsilon} v_\epsilon + k_\epsilon^2 a_\epsilon \circ \varphi(\bar{z}_\epsilon + k_\epsilon x) v_\epsilon = \frac{v_\epsilon^{2^\star - 1 - p_\epsilon}}{\left|\frac{\pi(\varphi(\bar{z}_\epsilon + k_\epsilon x))}{k_\epsilon}\right|^s} \text{ in } \mathcal{D}'(\{x_1 < 0\}) \tag{37}$$

for all $\epsilon > 0$. It follows from the definition (13) of φ and (33) that there exists $C_R > 0$ such that $|\pi(\varphi(\bar{z}_\epsilon + k_\epsilon x))| \geq C_R k_\epsilon |\pi(x)|$ for all $x \in \mathbb{R}^n_- \cap B_R(0)$. With (18) and (35), we get that $0 \leq v_\epsilon \leq 1$. With the method used in the Appendix, we get that $(v_\epsilon)_{\epsilon>0}$ converges in $C^1_{loc}(\mathbb{R}^n_-)$. Since $v_\epsilon \rightharpoonup v$ weakly in $H^2_{1,0}(\mathbb{R}^n_-)$ when $k \to +\infty$, we get that $\lim_{\epsilon\to 0} v_\epsilon = v$ in $C^1_{loc}(\overline{\mathbb{R}^n_-})$. With (36) and (32), we get that $v(-\lambda_0, \theta_0, 0) = 1$, and in particular, $v \not\equiv 0$ and $\lambda_0 > 0$. □

Step 3.4.3: We claim that $\Delta v = \frac{v^{2^\star-1}}{|\pi(x)|^s}$ in $\mathcal{D}'(\mathbb{R}^n_-)$ and that

$$\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx = \mu_{s,\mathcal{P}}(\Omega)^{\frac{2^\star}{2^\star-2}} = \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}.$$

Indeed, by passing to the weak limit $\epsilon \to 0$ in (37), we get that

$$\Delta v = \frac{v^{2^\star-1}}{|\pi(x) - (\rho_0, 0, 0)|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-).$$

Testing this equality with $v \in H^2_{1,0}(\mathbb{R}^n_-)$ and using the optimal Hardy-Sobolev inequality (6), we get that

$$\begin{aligned}\left(\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx\right)^{\frac{2^\star-2}{2^\star}} &= \frac{\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx}{\left(\int_{\mathbb{R}^n_-} \frac{v^{2^\star}}{|\pi(x)-(\rho_0,0,0)|^s}\, dx\right)^{\frac{2}{2^\star}}} \\ &\geq \frac{\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx}{\left(\int_{\mathbb{R}^n_-} \frac{v^{2^\star}}{|\pi(x)|^s}\, dx\right)^{\frac{2}{2^\star}}} \geq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-). \qquad (38)\end{aligned}$$

Here, we have used that $|\pi(x) - (\rho_0, 0, 0)| \geq |\pi(x)|$ since $\rho_0 \geq 0$ and $x_1 < 0$ for all $x \in \mathbb{R}^n_-$. We then obtain that

$$\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx \geq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}. \qquad (39)$$

Moreover, see for instance [12], we have that $\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx \leq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}$. We then get that

$$\int_{\mathbb{R}^n_-} |\nabla v|^2\, dx = \mu_{s,\mathcal{P}}(\Omega)^{\frac{2^\star}{2^\star-2}} = \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}},$$

and that

$$\lim_{\epsilon\to 0} \rho_\epsilon = \rho_0 = 0 \text{ and } \lim_{\epsilon\to 0} \mu_\epsilon^{p_\epsilon} = 1. \qquad (40)$$

For this last assertion, we refer to [12]. □

Proposition 3.3 now follows from Steps 3.1 to 3.4. □

We shall also need the following two claims for the next section

Step 3.5: Under the hypothesis of Proposition 3.3, we have that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \int_{\Omega\setminus B_{Rk_\epsilon}(\varphi(\bar{z}_\epsilon))} \frac{u_\epsilon^{2^\star-p_\epsilon}}{|\pi(x)|^s}\, dx = 0. \qquad (41)$$

We omit the proof which is quite similar to [12].

Step 3.6: We also claim that

$$\lim_{\epsilon\to 0} u_\epsilon = 0 \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}). \qquad (42)$$

Indeed, for $\delta > 0$, it follows from (41) that

$$\lim_{\epsilon\to 0}\int_{\Omega\setminus B_\delta(x_0)}\frac{u_\epsilon^{2^\star-1-p_\epsilon}(x)}{|\pi(x)|^s}\,dx = 0.$$

Using the techniques in the Appendix of [12, 13], we get that

$$\lim_{\epsilon\to 0}\|u_\epsilon\|_{L^p(\Omega\setminus B_\delta(x_0))} = 0$$

for all $p \geq 1$, and the method developed in this paper's Appendix, we get (42). □

4. Blow-up analysis, Part II

This section is devoted to the proof of the following strong pointwise estimate.

Proposition 4.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$ and let $\mathcal{P}$ be a linear subspace of $\mathbb{R}^n$ such that $2 \leq dim_{\mathbb{R}}\mathcal{P} \leq n-1$. Let $s \in (0,2)$ and assume that (10) holds. For $(p_\epsilon)_{\epsilon>0}$ in $[0, 2^\star - 2)$ such that $\lim_{\epsilon\to 0} p_\epsilon = 0$ and $(a_\epsilon)_{\epsilon>0}$ as in (14), we consider $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)\cap C^2(\overline{\Omega}\setminus\mathcal{P}^\perp)$ such that (15), (16) and (17) hold. We let x_0, φ, $(\mu_\epsilon)_{\epsilon>0}$ and $(\bar{z}_\epsilon)_{\epsilon>0}$ as in Proposition 3.3. Then, there exists $C > 0$ such that*

$$u_\epsilon(x) \leq Cd(x,\partial\Omega) + C\frac{\mu_\epsilon^{\frac{n}{2}}d(x,\partial\Omega)}{(\mu_\epsilon^2 + |x-\varphi(\bar{z}_\epsilon)|^2)^{\frac{n}{2}}} \tag{43}$$

and

$$|\nabla u_\epsilon(x)| \leq C + C\frac{\mu_\epsilon^{\frac{n}{2}}}{(\mu_\epsilon^2 + |x-\varphi(\bar{z}_\epsilon)|^2)^{\frac{n}{2}}} \tag{44}$$

for all $\epsilon > 0$ and all $x \in \Omega$.

Proof: We take inspiration from [8]. We proceed in five steps.

Step 4.1: We claim that there exists $C > 0$ such that

$$|\pi(x)|^{\frac{n-2}{2}}u_\epsilon(x)^{1-\frac{p_\epsilon}{2^\star-2}} \leq C \tag{45}$$

for all $\epsilon > 0$ and all $x \in \Omega$.

Indeed if not, we let $y_\epsilon \in \Omega$ such that

$$|\pi(y_\epsilon)|^{\frac{n-2}{2}}u_\epsilon(y_\epsilon)^{1-\frac{p_\epsilon}{2^\star-2}} = \sup_{x\in\Omega}|\pi(x)|^{\frac{n-2}{2}}u_\epsilon(x)^{1-\frac{p_\epsilon}{2^\star-2}} \to +\infty \tag{46}$$

as $\epsilon \to 0$. We then let

$$\nu_\epsilon := u_\epsilon(y_\epsilon)^{-\frac{2}{n-2}} \text{ and } \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}} \tag{47}$$

for all $\epsilon > 0$. It follows from (46) and (47) that

$$\lim_{\epsilon\to 0}\nu_\epsilon = 0 \text{ and } \lim_{\epsilon\to 0}\frac{|\pi(y_\epsilon)|}{\ell_\epsilon} = +\infty, \tag{48}$$

and from (18) and (40) that

$$\lim_{\epsilon\to 0}\nu_\epsilon^{p_\epsilon} = 1. \tag{49}$$

We also let

$$\gamma_\epsilon := |\pi(y_\epsilon)|^{\frac{s}{2}}|u_\epsilon(y_\epsilon)|^{\frac{2-2^\star+p_\epsilon}{2}}, \tag{50}$$

for all $\epsilon > 0$. It follows from (48) that

$$\lim_{\epsilon\to 0}\frac{\gamma_\epsilon}{|\pi(y_\epsilon)|} = 0. \tag{51}$$

Case 4.1.1: We assume first that, up to a subsequence, there exists $\rho > 0$ such that

$$\frac{d(y_\epsilon, \partial\Omega)}{\gamma_\epsilon} \geq 3\rho \tag{52}$$

for all $\epsilon > 0$. For any $x \in B_{2\rho}(0)$ and any $\epsilon > 0$, we let

$$w_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon + \gamma_\epsilon x). \tag{53}$$

Note that w_ϵ is well defined thanks to (52). With (46) and (50), we get that

$$\left|\frac{\pi(y_\epsilon)}{|\pi(y_\epsilon)|} + \frac{\gamma_\epsilon}{|\pi(y_\epsilon)|}\pi(x)\right|^{\frac{n-2}{2}} w_\epsilon(x)^{1-\frac{p_\epsilon}{2^\star-2}} \leq 1.$$

In particular, with (51), there exists $C_0 > 0$ such that

$$0 \leq w_\epsilon(x) \leq C_0 \tag{54}$$

for all $x \in B_{2\rho}(0)$ and all $\epsilon > 0$. With (15), we get that

$$\Delta w_\epsilon + \gamma_\epsilon^2 a_\epsilon(y_\epsilon + \gamma_\epsilon x) w_\epsilon = \frac{w_\epsilon^{2^\star-1-p_\epsilon}}{\left|\frac{\pi(y_\epsilon)}{|\pi(y_\epsilon)|} + \frac{\gamma_\epsilon}{|\pi(y_\epsilon)|}\pi(x)\right|^s}$$

for all $x \in B_{2\rho}(0)$ and all $\epsilon > 0$. Since (48) and (54) hold, it follows from standard elliptic theory that there exists $w \in C^1(B_{2\rho}(0))$ such that $w(0) = 1$ and

$$\lim_{\epsilon\to 0} w_\epsilon = w \tag{55}$$

in $C^1_{loc}(B_{2\rho}(0))$. Mimicking what was done in Step 3.1, we get a contradiction.

Case 4.1.2: We assume that

$$\lim_{\epsilon\to 0} \frac{d(y_\epsilon, \partial\Omega)}{\gamma_\epsilon} = 0. \tag{56}$$

As in Step 3.1, we get a contradiction. We refer to [12] for proof in a similar context.

In both cases, we have contradicted (46). This proves (45). □

Step 4.2: This step is a slight improvement of (45). We claim that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{x\in\Omega\setminus B_{Rk_\epsilon}(\varphi(\bar{z}_\epsilon))} |\pi(x)|^{\frac{n-2}{2}} u_\epsilon(x)^{1-\frac{p_\epsilon}{2^\star-2}} = 0. \tag{57}$$

The proof is similar to Step 4.1 and uses the techniques developed in [12]. We refer to Step 4.1 and [12] for the details.

Step 4.3: We claim that for any $\nu \in (0,1)$ and any $R > 0$, there exists $C(\nu, R) > 0$ such that

$$u_\epsilon(x) \leq C(\nu, R) \cdot \left(\frac{\mu_\epsilon^{\frac{n}{2}-\nu(n-1)} d(x, \partial\Omega)^{1-\nu}}{(\mu_\epsilon^2 + |x - \varphi(\bar{z}_\epsilon)|^2)^{\frac{n(1-\nu)}{2}}} + d(x, \partial\Omega)^{1-\nu} \right) \tag{58}$$

for all $x \in \Omega$ and all $\epsilon > 0$.

Indeed, let G be the Green's function for Δ in Ω with Dirichlet boundary condition, and set $H_\epsilon(x) = -\partial_{\vec{n}} G(x, \varphi(\bar{z}_\epsilon))$ for all $x \in \overline{\Omega} \setminus \{\varphi(\bar{z}_\epsilon)\}$, where here $\vec{n}$ denotes the outward normal vector at $\partial\Omega$. It follows from Theorem 9.2 of [13] that $H_\epsilon \in C^2(\overline{\Omega} \setminus \{\varphi(\bar{z}_\epsilon)\})$, that

$$\Delta H_\epsilon = 0 \tag{59}$$

in Ω and that there exist $\delta_1, C_1 > 0$ such that

$$\frac{d(x,\partial\Omega)}{C_1|x-\varphi(\bar z_\epsilon)|^n} \le H_\epsilon(x) \le \frac{C_1 d(x,\partial\Omega)}{|x-\varphi(\bar z_\epsilon)|^n} \tag{60}$$

and –using (34)– that

$$\frac{|\nabla H_\epsilon(x)|}{H_\epsilon(x)} \ge \frac{1}{C_1' d(x,\partial\Omega)} \ge \frac{1}{C_1|\pi(x)|} \tag{61}$$

for all $x \in \Omega \cap B_{2\delta_1}(0)$. Let $\lambda_1 > 0$ be the first eigenvalue of Δ on Ω, and let $\psi \in C^2(\overline{\Omega})$ be "the first eigenfunction" in such a way that

$$\begin{cases} \Delta\psi = \lambda_1\psi & \text{in } \Omega \\ \psi > 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from standard elliptic theory, Hopf's maximum principle and again (34) that there exists $C_2, \delta_2 > 0$ such that

$$\frac{1}{C_2} d(x,\partial\Omega) \le \psi(x) \le C_2 d(x,\partial\Omega) \text{ and } \frac{|\nabla\psi(x)|}{\psi(x)} \ge \frac{1}{C_2' d(x,\partial\Omega)} \ge \frac{1}{C_2|\pi(x)|} \tag{62}$$

for all $x \in \Omega \cap B_{2\delta_2}(\varphi(\bar z_\epsilon))$. We now consider the operator

$$L_\epsilon = \Delta + \left(a_\epsilon - \frac{u_\epsilon^{2^\star-2-p_\epsilon}}{|\pi(x)|^s}\right).$$

Step 4.3.1: We claim that there exist $\delta_0 > 0$ and $R_0 > 0$ such that for any $\nu \in (0,1)$ and any $R > R_0$, $\delta \in (0,\delta_0)$, we have that

$$L_\epsilon H_\epsilon^{1-\nu} > 0, \text{ and } L_\epsilon \psi^{1-\nu} > 0 \tag{63}$$

for all $x \in \Omega \cap B_\delta(\varphi(\bar z_\epsilon)) \setminus \overline{B}_{Rk_\epsilon}(\varphi(\bar z_\epsilon))$ and for all $\epsilon > 0$ sufficiently small.
Indeed, with (59), we get that

$$\frac{L_\epsilon H_\epsilon^{1-\nu}}{H_\epsilon^{1-\nu}}(x) = a_\epsilon(x) + \nu(1-\nu)\frac{|\nabla H_\epsilon|^2}{H_\epsilon^2}(x) - \frac{u_\epsilon(x)^{2^\star-2-p_\epsilon}}{|\pi(x)|^s} \tag{64}$$

for all $x \in \Omega$ and all $\epsilon > 0$. We let $\alpha > 0$. It follows from (57) that there exists $R_0 > 0$ such that for any $R > R_0$, we have that

$$|\pi(x)|^{2-s}|u_\epsilon(x)|^{2^\star-2-p_\epsilon} < \alpha$$

for all $x \in (B_\delta(\varphi(\bar z_\epsilon)) \setminus \overline{B}_{Rk_\epsilon}(\varphi(\bar z_\epsilon))) \cap \Omega$ and all $\epsilon > 0$ small enough. With (14), (64) and (61), we get that for $\alpha > 0$ and $\delta > 0$ small enough, we have that

$$\frac{L_\epsilon H_\epsilon^{1-\nu}}{H_\epsilon^{1-\nu}}(x) > \frac{\nu(1-\nu) - \alpha C_1^2 - C_1^2|\pi(x)|^2|a_\epsilon(x)|}{C_1^2|\pi(x)|^2} > 0$$

for all $x \in (B_\delta(\varphi(\bar z_\epsilon)) \setminus \overline{B}_{Rk_\epsilon}(\varphi(\bar z_\epsilon))) \cap \Omega$ and all $\epsilon > 0$ small enough. The proof of the second inequality of (63) goes the same way.

Step 4.3.2: It follows from (19) in Proposition 3.3 that there exists $C_1(R) > 0$ such that

$$u_\epsilon(x) \le C_1(R)\mu_\epsilon^{-\frac{n}{2}} d(x,\partial\Omega)$$

for all $x \in \Omega \cap \partial B_{Rk_\epsilon}(\varphi(\bar z_\epsilon))$ and all $\epsilon > 0$. In particular, there exists $C(R) > 0$ such that

$$u_\epsilon(x) \le C(R)\mu_\epsilon^{\frac{n}{2}-\nu(n-1)} H_\epsilon^{1-\nu}(x) \tag{65}$$

for all $x \in \Omega \cap \partial B_{Rk_\epsilon}(\varphi(\bar z_\epsilon))$ and all $\epsilon > 0$.

It follows from (42) there exists $C_1(\delta) > 0$ such that

$$u_\epsilon(x) \leq C_1(\delta) d(x, \partial\Omega) \tag{66}$$

for all $x \in \Omega \cap \partial B_\delta(\varphi(\bar{z}_\epsilon))$ and all $\epsilon > 0$. In particular, there exists $C(\delta) > 0$ such that $u_\epsilon(x) \leq C(\delta)\psi(x)^{1-\nu}$ for all $x \in \Omega \cap \partial B_\delta(\varphi(\bar{z}_\epsilon))$ and all $\epsilon > 0$. We let

$$D_{\epsilon,R,\delta} := (B_\delta(\varphi(\bar{z}_\epsilon)) \setminus \overline{B}_{Rk_\epsilon}(\varphi(\bar{z}_\epsilon))) \cap \Omega.$$

It follows from (65) and (66) that

$$u_\epsilon(x) \leq C(R)\mu_\epsilon^{\frac{n}{2}-\nu(n-1)} H_\epsilon^{1-\nu}(x) + C(\delta)\psi(x)^{1-\nu} \tag{67}$$

for all $\epsilon > 0$ and all $x \in \partial D_{\epsilon,R,\delta}$.

Step 4.3.3: We claim that L_ϵ is coercive and therefore verifies the comparison principle on $D_{\epsilon,R,\delta}$.

Indeed, with (41), we get that for any $\alpha > 0$, there exists $\tilde{R}_0 > 0$ such that for any $R > \tilde{R}_0$, we have that

$$\int_{\Omega \setminus B_{Rk_\epsilon}(\varphi(\bar{z}_\epsilon))} \frac{u_\epsilon^{2^\star - p_\epsilon}(x)}{|\pi(x)|^s}\, dx \leq \alpha.$$

Since $\Delta + a_\epsilon$ is uniformly coercive, we get that L_ϵ is coercive on $\Omega \setminus B_{Rk_\epsilon}(\varphi(\bar{z}_\epsilon))$ for R large enough. We refer to Lemma 3.4 of [21] for details on this assertion.

Step 4.3.4: Since

$$L_\epsilon(C(R)\mu_\epsilon^{\frac{n}{2}-\nu(n-1)} H_\epsilon^{1-\nu}(x) + C(\delta)\psi(x)^{1-\nu}) > 0 = L_\epsilon u_\epsilon$$

in $D_{\epsilon,R,\delta}$ and (67) holds, we get from Step 4.3.3 that

$$u_\epsilon(x) \leq C(R)\mu_\epsilon^{\frac{n}{2}-\nu(n-1)} H_\epsilon^{1-\nu}(x) + C(\delta)\psi(x)^{1-\nu}$$

for all $x \in D_{\epsilon,R,\delta}$. With (60) and (62), we then get that (58) holds on $D_{\epsilon,R,\delta} = (B_\delta(\varphi(\bar{z}_\epsilon)) \setminus \overline{B}_{Rk_\epsilon}(\varphi(\bar{z}_\epsilon))) \cap \Omega$ for R large and δ small. It follows from this last assertion, (19) in Proposition 3.3 and (42) that (58) holds on Ω. □

Step 4.4: We claim that there exists $C > 0$ such that

$$u_\epsilon(x) \leq C d(x, \partial\Omega) + C \frac{\mu_\epsilon^{\frac{n}{2}} d(x, \partial\Omega)}{(\mu_\epsilon^2 + |x - \varphi(\bar{z}_\epsilon)|^2)^{\frac{n}{2}}} \tag{68}$$

for all $x \in \Omega$ and all $\epsilon > 0$.

Indeed, it follows from (19) in Proposition 3.3 and (42) that for any $\delta, R > 0$, inequality (68) holds for all $x \in (\Omega \setminus B_\delta(\varphi(\bar{z}_\epsilon))) \cup (\Omega \cap B_{R\mu_\epsilon}(\varphi(\bar{z}_\epsilon)))$ for all $\epsilon > 0$. What is left is to prove (68) for any sequence $(y_\epsilon)_{\epsilon > 0} \in \Omega$ such that

$$\lim_{\epsilon \to 0} y_\epsilon = x_0 \text{ and } \lim_{\epsilon \to 0} \frac{|y_\epsilon - \varphi(\bar{z}_\epsilon)|}{k_\epsilon} = +\infty. \tag{69}$$

We show that (68) holds for $x = y_\epsilon$. With Green's representation formula, we get that

$$u_\epsilon(y_\epsilon) = \int_\Omega G_\epsilon(y_\epsilon, y) \frac{u_\epsilon(y)^{2^\star - 1 - p_\epsilon}}{|\pi(y)|^s}\, dy,$$

where G_ϵ is the Green's function for the uniformly coercive operator $\Delta + a_\epsilon$. For $\nu \in (0,1)$, we use (58) and (34) to get that

$$\begin{aligned} u_\epsilon(y_\epsilon) &\leq C\int_\Omega G_\epsilon(y_\epsilon, y)\frac{d(y,\partial\Omega)^{(1-\nu)(2^\star-1-p_\epsilon)}}{|\pi(y)|^s}\,dy \\ &\quad +C\int_\Omega \frac{G_\epsilon(y_\epsilon,y)}{|\pi(y)|^s}\left(\frac{\mu_\epsilon^{\frac{n}{2}-(n-1)\nu}d(y,\partial\Omega)^{1-\nu}}{(\mu_\epsilon^2+|y-\varphi(\bar z_\epsilon)|^2)^{\frac{n(1-\nu)}{2}}}\right)^{2^\star-1-p_\epsilon} dy \\ &\leq I_{\epsilon,1}+I_{\epsilon,2}+I_{\epsilon,3} \end{aligned} \tag{70}$$

where

$$I_{\epsilon,1} := \int_\Omega G_\epsilon(y_\epsilon,y)|\pi(y)|^{(1-\nu)(2^\star-1-p_\epsilon)-s}\,dy,$$

$$I_{\epsilon,2} := \int_{D_{\epsilon,2}} \frac{G_\epsilon(y_\epsilon,y)}{|\pi(y)|^s}\left(\frac{\mu_\epsilon^{\frac{n}{2}-(n-1)\nu}d(y,\partial\Omega)^{1-\nu}}{(\mu_\epsilon^2+|y-\varphi(\bar z_\epsilon)|^2)^{\frac{n(1-\nu)}{2}}}\right)^{2^\star-1-p_\epsilon} dy,$$

and

$$I_{\epsilon,3} := \int_{D_{\epsilon,3}} \frac{G_\epsilon(y_\epsilon,y)}{|\pi(y)|^s}\left(\frac{\mu_\epsilon^{\frac{n}{2}-(n-1)\nu}d(y,\partial\Omega)^{1-\nu}}{(\mu_\epsilon^2+|y-\varphi(\bar z_\epsilon)|^2)^{\frac{n(1-\nu)}{2}}}\right)^{2^\star-1-p_\epsilon} dy$$

for all $\epsilon > 0$, where

$$D_{\epsilon,2} := \left\{|y_\epsilon - y| > \frac{1}{2}|y_\epsilon - \varphi(\bar z_\epsilon)|\right\} \text{ and } D_{\epsilon,3} := \left\{|y_\epsilon - y| < \frac{1}{2}|y_\epsilon - \varphi(\bar z_\epsilon)|\right\}$$

We first deal with $I_{\epsilon,1}$. The Green's function verifies

$$G_\epsilon(y_\epsilon,y) \leq C\frac{d(y_\epsilon,\partial\Omega)}{|y_\epsilon - y|^{n-1}}$$

for all $y \in \Omega\setminus\{y_\epsilon\}$ and all $\epsilon > 0$. We refer to [13] for the proof of this assertion. Since $s \in (0,2)$ and $\varphi(\bar z_\epsilon) \in \partial\Omega$, we then get that

$$I_{\epsilon,1} \leq Cd(y_\epsilon,\partial\Omega)\int_\Omega \frac{|\pi(y)|^{(1-\nu)(2^\star-1-p_\epsilon)-s}}{|y_\epsilon-y|^{n-1}} \leq Cd(y_\epsilon,\partial\Omega) \tag{71}$$

for all $\epsilon > 0$.

For $I_{\epsilon,2}$, we note that the Green's function verifies

$$G_\epsilon(y_\epsilon,y) \leq C\frac{d(y_\epsilon,\partial\Omega)d(y,\partial\Omega)}{|y_\epsilon-y|^n} \tag{72}$$

for all $y \in \Omega\setminus\{y_\epsilon\}$ and all $\epsilon > 0$. We again refer to [13] for the proof of this assertion. We then get with (34) and a change of variables that

$$\begin{aligned} &I_{\epsilon,2} \\ &\leq C\int_{D_{\epsilon,2}} \frac{d(y_\epsilon,\partial\Omega)}{|y_\epsilon-y|^n}\frac{\mu_\epsilon^{(\frac{n}{2}-(n-1)\nu)(2^\star-1-p_\epsilon)}d(y,\partial\Omega)^{(1-\nu)(2^\star-1-p_\epsilon)+1-s}}{(\mu_\epsilon^2+|x-\varphi(\bar z_\epsilon)|^2)^{\frac{n(1-\nu)}{2}(2^\star-1-p_\epsilon)}}\,dy \\ &\leq C\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{(\frac{n}{2}-(n-1)\nu)(2^\star-1-p_\epsilon)}}{|y_\epsilon-\varphi(\bar z_\epsilon)|^n}\int_\Omega \frac{|y-\varphi(\bar z_\epsilon)|^{(1-\nu)(2^\star-1-p_\epsilon)+1-s}}{(\mu_\epsilon^2+|x-\varphi(\bar z_\epsilon)|^2)^{\frac{n(1-\nu)}{2}(2^\star-1-p_\epsilon)}}\,dy \\ &\leq C\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{\frac{n}{2}}}{|y_\epsilon-\varphi(\bar z_\epsilon)|^n}\int_{\mathbb{R}^n}\frac{|z|^{(1-\nu)(2^\star-1-p_\epsilon)+1-s}}{(1+|z|^2)^{\frac{n(1-\nu)}{2}(2^\star-1-p_\epsilon)}}\,dy \leq C\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{\frac{n}{2}}}{|y_\epsilon-\varphi(\bar z_\epsilon)|^n}. \end{aligned} \tag{73}$$

To deal with $I_{\epsilon,3}$, we first note that for any $y\in D_{\epsilon,3}$, we have that

$$\frac{1}{2}|y_\epsilon-\varphi(\bar{z}_\epsilon)|\leq |y-\varphi(\bar{z}_\epsilon)|\leq \frac{3}{2}|y_\epsilon-\varphi(\bar{z}_\epsilon)|. \tag{74}$$

With inequality (101) (with $\theta=1$) on the Green's function, we then get that

$$I_{\epsilon,3} \leq C\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{(\frac{n}{2}-(n-1)\nu)(2^\star-1-p_\epsilon)}}{|y_\epsilon-\varphi(\bar{z}_\epsilon)|^{n(1-\nu)(2^\star-1-p_\epsilon)}}\int_{D_{\epsilon,3}}\frac{dy}{|y_\epsilon-y|^{n-1}|\pi(y)|^{s-(1-\nu)(2^\star-1-p_\epsilon)}}.$$

We let

$$\theta_\epsilon=\frac{y_\epsilon-\varphi(\bar{z}_\epsilon)}{|y_\epsilon-\varphi(\bar{z}_\epsilon)|}+\frac{(\varphi_0(0,\bar{z}_\epsilon),0,0)}{|y_\epsilon-\varphi(\bar{z}_\epsilon)|}.$$

With (31), (40) and (69), we get that there exists $\theta_0\in\mathbb{R}^n$ such that $|\theta_0|=1$ and $\lim_{\epsilon\to 0}\theta_\epsilon=\theta_0$. With the change of variables $y=y_\epsilon+|y_\epsilon-\varphi(\bar{z}_\epsilon)|z$ and using (69), we get that

$$I_{\epsilon,3}\leq \tag{75}$$

$$C\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{(\frac{n}{2}-(n-1)\nu)(2^\star-1-p_\epsilon)}}{|y_\epsilon-\varphi(\bar{z}_\epsilon)|^{(n-1)(1-\nu)(2^\star-1-p_\epsilon)+s-1}}\int_{|z|<\frac{1}{2}}\frac{dz}{|z|^{n-1}|\pi(\theta_\epsilon+z)|^{s-(1-\nu)(2^\star-1-p_\epsilon)}}$$

$$\leq C\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{(\frac{n}{2}-(n-1)\nu)(2^\star-1-p_\epsilon)}}{|y_\epsilon-\varphi(\bar{z}_\epsilon)|^{(n-1)(1-\nu)(2^\star-1-p_\epsilon)+s-1}}$$

$$=o\left(\frac{d(y_\epsilon,\partial\Omega)\mu_\epsilon^{\frac{n}{2}}}{|y_\epsilon-\varphi(\bar{z}_\epsilon)|^n}\right) \tag{76}$$

when $\epsilon\to 0$. Plugging (71), (73) and (76) in (70) and using again (69), we get that

$$u_\epsilon(y_\epsilon)\leq Cd(y_\epsilon,\partial\Omega)+C\frac{\mu_\epsilon^{\frac{n}{2}}d(y_\epsilon,\partial\Omega)}{(\mu_\epsilon^2+|y_\epsilon-\varphi(\bar{z}_\epsilon)|^2)^{\frac{n}{2}}}$$

when $\epsilon\to 0$. This ends the proof of (68).

Step 4.5: We claim that there exists $C>0$ such that

$$|\nabla u_\epsilon(x)|\leq C+C\frac{\mu_\epsilon^{\frac{n}{2}}}{(\mu_\epsilon^2+|x-\varphi(\bar{z}_\epsilon)|^2)^{\frac{n}{2}}} \tag{77}$$

for all $x\in\Omega$.

To prove the claim, as in Step 4.4, we just need to consider $(y_\epsilon)_{\epsilon>0}\in\Omega$ as in (69). We use Green's representation formula to write

$$\nabla u_\epsilon(y_\epsilon)=\int_\Omega \nabla_x G_\epsilon(y_\epsilon,y)\frac{u_\epsilon(y)^{2^\star-1-p_\epsilon}}{|\pi(y)|^s}\,dy.$$

With (68), we get that

$$|\nabla u_\epsilon(y_\epsilon)|\leq J_{\epsilon,1}+J_{\epsilon,2}+J_{\epsilon,3}, \tag{78}$$

where

$$J_{\epsilon,1}:=C\int_\Omega|\nabla_x G_\epsilon(y_\epsilon,y)|\frac{d(y,\partial\Omega)^{2^\star-1-p_\epsilon}}{|\pi(y)|^s}\,dy,$$

$$J_{\epsilon,2}:=C\int_{|y_\epsilon-y|>\frac{1}{2}|y_\epsilon-\varphi(\bar{z}_\epsilon)|}|\nabla_x G_\epsilon(y_\epsilon,y)|\frac{\mu_\epsilon^{\frac{n}{2}(2^\star-1-p_\epsilon)}d(y,\partial\Omega)^{2^\star-1-p_\epsilon}}{|\pi(y)|^s\,(\mu_\epsilon^2+|y-\varphi(\bar{z}_\epsilon)|^2)^{\frac{n}{2}(2^\star-1-p_\epsilon)}}dy$$

and

$$J_{\epsilon,3} := C \int_{|y_\epsilon - y| < \frac{1}{2}|y_\epsilon - \varphi(\bar{z}_\epsilon)|} |\nabla_x G_\epsilon(y_\epsilon, y)| \frac{\mu_\epsilon^{\frac{n}{2}(2^\star - 1 - p_\epsilon)} d(y, \partial\Omega)^{2^\star - 1 - p_\epsilon}}{|\pi(y)|^s \left(\mu_\epsilon^2 + |y - \varphi(\bar{z}_\epsilon)|^2\right)^{\frac{n}{2}(2^\star - 1 - p_\epsilon)}} dy.$$

To estimate $J_{\epsilon,1}$, use that the Green's function satisfies

$$|\nabla_x G(y_\epsilon, y)| \leq \frac{C}{|y_\epsilon - y|^{n-1}} \tag{79}$$

for all $y \in \Omega \setminus \{y_\epsilon\}$ and all $\epsilon > 0$. We refer to [13] for the proof of this inequality. With (34), we then get that

$$J_{\epsilon,1} \leq C \int_\Omega \frac{dy}{|y_\epsilon - y|^{n-1} |\pi(y)|^{s - (2^\star - 1 - p_\epsilon)}} \leq C \tag{80}$$

For $J_{\epsilon,2}$, we use that (see [13])

$$|\nabla_x G(y_\epsilon, y)| \leq \frac{C d(y, \partial\Omega)}{|y_\epsilon - y|^n}$$

for all $y \in \Omega \setminus \{y_\epsilon\}$ and all $\epsilon > 0$. Plugging this inequality in $J_{\epsilon,2}$ and performing computations similar to what was done in the proof of (73), we get that

$$J_{\epsilon,2} \leq C \frac{\mu_\epsilon^{\frac{n}{2}}}{\left(\mu_\epsilon^2 + |x - \varphi(\bar{z}_\epsilon)|^2\right)^{\frac{n}{2}}} \tag{81}$$

To deal finally with $J_{\epsilon,3}$, we again use estimate (79) on the Green's function combined with the same techniques as in the proof of (76), to obtain

$$J_{\epsilon,3} \leq C \frac{\mu_\epsilon^{\frac{n}{2}}}{\left(\mu_\epsilon^2 + |x - \varphi(\bar{z}_\epsilon)|^2\right)^{\frac{n}{2}}} \tag{82}$$

Plugging (80), (81) and (82) in (78), we get (77) and Proposition 4.1. □

5. Pohozaev identity and proof of Theorem 1.1

We first prove the following

Proposition 5.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$ and let $\mathcal{P}$ be a linear vector subspace of $\mathbb{R}^n$ such that $2 \leq dim_{\mathbb{R}}\mathcal{P} \leq n-1$. Assume that $s \in (0,2)$ and that (10) holds. For $(p_\epsilon)_{\epsilon>0} \in [0, 2^\star - 2)$ and $(a_\epsilon)_{\epsilon>0}$ as in (14), we consider $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega) \cap C^2(\overline{\Omega} \setminus \mathcal{P}^\perp)$ such that (15), (16) and (17) hold. Then there exist $x_0 \in \partial\Omega \cap \mathcal{P}^\perp$, $\gamma_0 \geq 0$ and a family $(\mu_\epsilon)_{\epsilon>0} \in \mathbb{R}_+$ such that $\lim_{\epsilon \to 0} \mu_\epsilon = 0$ and*

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_\epsilon} = \frac{2(n-s)}{(n-2)^2} \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{-\frac{n-s}{2-s}} \int_{\partial\mathbb{R}^n_-} \left(\frac{1}{2} II_{x_0}(x,x) - \gamma_0\right) |\nabla v|^2 \, dx, \tag{83}$$

where II_{x_0} is the second fundamental form of $\partial\Omega$ at x_0.

Sections 5.1 to 5.3 below are devoted to the proof of Proposition 5.1, while Theorem 1.1 and Corollary 1.1 are proved in Step 5.4.

Step 5.1: We establish a Pohozaev-type identity for u_ϵ. In the sequel, we let $(\bar{z}_\epsilon)_{\epsilon>0}$, $(\mu_\epsilon)_{\epsilon>0}$, $(k_\epsilon)_{\epsilon>0}$ and $x_0 \in \mathcal{P}^\perp \cap \partial\Omega$ as in Proposition 4.1. We also consider the chart φ defined in (13). We let

$$V_\epsilon = \Omega \cap \varphi(B_{\sqrt{\mu_\epsilon}}(\bar{z}_\epsilon)) = \varphi(\mathbb{R}^n_- \cap B_{\sqrt{\mu_\epsilon}}(\bar{z}_\epsilon)).$$

In particular,

$$\partial V_\epsilon = \varphi(\mathbb{R}^n_- \cap \partial B_{\sqrt{\mu_\epsilon}}(\bar z_\epsilon)) \cup \varphi(B_{\sqrt{\mu_\epsilon}}(\bar z_\epsilon)\cap \partial\mathbb{R}^n_-) = V_\epsilon^1 \cup V_\epsilon^2.$$

In the sequel, we denote by $\nu(x)$ the outward normal vector at $x\in\partial V_\epsilon$ of the oriented hypersurface ∂V_ϵ (this is defined outside a null measure set). Let $\tilde x_0\in\mathbb{R}^n$. After integrations by parts (for instance, we refer to [12, 13]), we get that

$$\begin{aligned}&\left(\frac{n-2}{2}-\frac{n-s}{2^\star-p_\epsilon}\right)\int_{V_\epsilon}\frac{u_\epsilon^{2^\star-p_\epsilon}}{|\pi(x)|^s}\,dx - s\int_{V_\epsilon}\frac{(\tilde x_0,\pi(x))}{|\pi(x)|^{s+2}}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\\ &+\int_{V_\epsilon}\left(a_\epsilon+\frac{(x-\tilde x_0)^i\partial_i a_\epsilon}{2}\right)u_\epsilon^2\,dx\\ &=\int_{\partial V_\epsilon}\left(-\frac{n-2}{2}u_\epsilon\partial_\nu u_\epsilon+(x-\tilde x_0,\nu)\frac{|\nabla u_\epsilon|^2}{2}-(x-\tilde x_0)^i\partial_i u_\epsilon\partial_\nu u_\epsilon\right.\\ &\left.-\frac{(x-\tilde x_0,\nu)}{2^\star-p_\epsilon}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{|\pi(x)|^s}+\frac{a_\epsilon(x-\tilde x_0,\nu)}{2}u_\epsilon^2\right)d\sigma \qquad (84)\end{aligned}$$

for all $\epsilon>0$. Since $u_\epsilon\equiv 0$ on $\partial\Omega$, taking $\tilde x_0=\varphi(\bar z_\epsilon)$ in (84), we get that

$$\begin{aligned}&\left(\frac{n-2}{2}-\frac{n-s}{2^\star-p_\epsilon}\right)\int_{V_\epsilon}\frac{u_\epsilon^{2^\star-p_\epsilon}}{|\pi(x)|^s}\,dx - s\int_{V_\epsilon}\frac{(\varphi(\bar z_\epsilon),\pi(x))}{|\pi(x)|^{s+2}}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\\ &+\int_{V_\epsilon}\left(a_\epsilon+\frac{(x-\varphi(\bar z_\epsilon))^i\partial_i a_\epsilon}{2}\right)u_\epsilon^2\,dx\\ &=\int_{V_\epsilon^1}\left(-\frac{n-2}{2}u_\epsilon\partial_\nu u_\epsilon+(x-\varphi(\bar z_\epsilon),\nu)\frac{|\nabla u_\epsilon|^2}{2}\right.\\ &\left.-(x-\varphi(\bar z_\epsilon))^i\partial_i u_\epsilon\partial_\nu u_\epsilon-\frac{(x-\varphi(\bar z_\epsilon),\nu)}{2^\star-p_\epsilon}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{|\pi(x)|^s}+\frac{a_\epsilon(x-\varphi(\bar z_\epsilon),\nu)}{2}u_\epsilon^2\right)d\sigma\\ &-\frac12\int_{V_\epsilon^2}(x-\varphi(\bar z_\epsilon),\nu)|\nabla u_\epsilon|^2\,d\sigma. \qquad (85)\end{aligned}$$

With (16), (41), (43), (44) and Proposition 3.3, we get that

$$\begin{aligned}&\left(\frac{(n-2)^2}{4(n-s)}\mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}+o(1)\right)p_\epsilon + s\int_{V_\epsilon}\frac{(\varphi(\bar z_\epsilon),\pi(x))}{|\pi(x)|^{s+2}}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\\ &=\frac12\int_{V_\epsilon^2}(x-\varphi(\bar z_\epsilon),\nu)|\nabla u_\epsilon|^2\,d\sigma+o(\mu_\epsilon). \qquad (86)\end{aligned}$$

Step 5.2: We deal with the RHS of (86). With a change of variable, we get that

$$\begin{aligned}&\int_{\varphi(B_{\sqrt{\mu_\epsilon}}(\bar z_\epsilon)\cap\partial\mathbb{R}^n_-)}(x-\varphi(\bar z_\epsilon),\nu)|\nabla u_\epsilon|^2\,d\sigma = \qquad (87)\\ &(1+o(1))\mu_\epsilon\int_{D_\epsilon}\left(\frac{\varphi(\bar z_\epsilon+k_\epsilon x)-\varphi(\bar z_\epsilon)}{k_\epsilon^2},\nu\circ\varphi(\bar z_\epsilon+k_\epsilon x)\right)|\nabla v_\epsilon|^2_{\tilde g_\epsilon}\sqrt{|\tilde g_\epsilon|}\,dx\end{aligned}$$

where the metric $\tilde g_\epsilon$ is such that $(\tilde g_\epsilon)_{ij}=(\partial_i\varphi,\partial_j\varphi)(\bar z_\epsilon+k_\epsilon x)$ for all $i,j=2,...,n$, v_ϵ is as in Proposition 3.3 and

$$D_\epsilon = B_{\frac{\sqrt{\mu_\epsilon}}{k_\epsilon}}(0)\cap\{x_1=0\}.$$

Using the expression of φ (see (13)), we get (see [12, 13] for details) that

$$\begin{aligned}&\left(\frac{\varphi(\bar{z}_\epsilon+k_\epsilon x)-\varphi(\bar{z}_\epsilon)}{k_\epsilon^2}, \nu\circ\varphi(\bar{z}_\epsilon+k_\epsilon x)\right)\\ &=\frac{1+o(1)}{k_\epsilon^2}\left(\varphi_0(\bar{z}_\epsilon+k_\epsilon x)-\varphi_0(\bar{z}_\epsilon)-k_\epsilon\sum_{i=2}^n x^i\partial_i\varphi(\bar{z}_\epsilon+k_\epsilon x)\right)\\ &=-\frac{1}{2}\sum_{i,j=2}^n\partial_{ij}\varphi(\bar{z}_\epsilon)x^ix^j+o_\epsilon(1)|x|^2 \end{aligned}\tag{88}$$

for $\epsilon>0$ and $x\in D_\epsilon$. In this expression, $\lim_{\epsilon\to 0}o_\epsilon(1)=0$ uniformly in D_ϵ. Plugging (88) into (87), using the estimate (44), Lebesgue's convergence theorem and (19), we get that

$$\lim_{\epsilon\to 0}\frac{1}{\mu_\epsilon}\int_{\varphi(B_{\sqrt{\mu_\epsilon}}(\bar{z}_\epsilon)\cap\partial\mathbb{R}^n_-)}(x-\varphi(\bar{z}_\epsilon),\nu)|\nabla u_\epsilon|^2\,d\sigma=-\frac{1}{2}\int_{\partial\mathbb{R}^n_-}\partial_{ij}\varphi_0(0)x^ix^j|\nabla v|^2\,dx.\tag{89}$$

Step 5.3: We deal with the second term of the LHS of (86). With the pointwise estimate (43) and a change of variables, we get that

$$\begin{aligned}&\int_{V_\epsilon}\frac{(\varphi(\bar{z}_\epsilon),\pi(x))}{|\pi(x)|^{s+2}}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\\ &=\frac{1+o(1)}{\mu_\epsilon^2}\int_{D'_\epsilon}\frac{(\pi\circ\varphi(\bar{z}_\epsilon),\pi\circ\varphi(\bar{z}_\epsilon+k_\epsilon x))}{\left|\frac{\pi\circ\varphi(\bar{z}_\epsilon+k_\epsilon x)}{k_\epsilon}\right|^{s+2}}\cdot\frac{v_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\end{aligned}$$

when $\epsilon\to 0$, where

$$D'_\epsilon:=B_{R\frac{\sqrt{\mu_\epsilon}}{k_\epsilon}}(0)\cap\{x_1<0\}.$$

With the explicit expression of φ (see (13)) and noting $x=(x_1,y,z)$ as in (13), we get that

$$\begin{aligned}&\int_{V_\epsilon}\frac{(\varphi(\bar{z}_\epsilon),\pi(x))}{|\pi(x)|^{s+2}}\cdot\frac{u_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\\ &=(1+o(1))\frac{\varphi_0(0,z_\epsilon)}{k_\epsilon}\int_{D'_\epsilon}\frac{x_1+\frac{\varphi_0(k_\epsilon y,z_\epsilon+k_\epsilon z))}{k_\epsilon}}{\left|\pi\left(x_1+\frac{\varphi_0(k_\epsilon y,z_\epsilon+k_\epsilon z)}{k_\epsilon},y,z\right)\right|^{s+2}}\frac{v_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\end{aligned}$$

when $\epsilon\to 0$. With point (iii) of (13), the estimate (43) and Lebesgue's convergence theorem, we get that

$$\begin{aligned}&\int_{V_\epsilon}\frac{(\varphi(\bar{z}_\epsilon),\pi(x))}{|\pi(x)|^{s+2}}\frac{u_\epsilon^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\,dx\\ &=\frac{\varphi_0(0,z_\epsilon)}{\mu_\epsilon}\left(\int_{\mathbb{R}^n_-}\frac{x_1v^{2^\star}}{2^\star|\pi(x)|^{s+2}}\,dx+o(1)\right)\end{aligned}\tag{90}$$

where $\lim_{\epsilon\to 0} o(1)=0$. Plugging (89) and (90) into (86) and noting that $\varphi_0(0,z_\epsilon)\leq 0$ (see (33)), we get that

$$\left(\frac{(n-2)^2}{4(n-s)}\mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{n-s}{2-s}}+o(1)\right)p_\epsilon+\left(\int_{\mathbb{R}^n_-}\frac{s|x_1|v^{2^\star}}{2^\star|\pi(x)|^{s+2}}\,dx+o(1)\right)\frac{|\varphi_0(0,z_\epsilon)|}{\mu_\epsilon}$$
$$=\left(-\frac{1}{4}\int_{\partial\mathbb{R}^n_-}\partial_{ij}\varphi_0(0)x^ix^j|\nabla v|^2\,dx+o(1)\right)\cdot\mu_\epsilon \tag{91}$$

where $\lim_{\epsilon\to 0} o(1)=0$. In particular, we get that $|\varphi_0(0,z_\epsilon)|=O(\mu_\epsilon^2)$ when $\epsilon\to 0$. We let

$$\gamma_0:=-\lim_{\epsilon\to 0}\frac{\varphi_0(0,z_\epsilon)}{\mu_\epsilon^2}\geq 0.$$

With (91), we get that

$$\lim_{\epsilon\to 0}\frac{(n-2)^2}{4(n-s)}\mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{\frac{n-s}{2-s}}\frac{p_\epsilon}{\mu_\epsilon}$$
$$=-\frac{1}{4}\int_{\partial\mathbb{R}^n_-}\partial_{ij}\varphi_0(0)x^ix^j|\nabla v|^2\,dx-\gamma_0\frac{s}{2^\star}\int_{\mathbb{R}^n_-}\frac{|x_1|\cdot v^{2^\star}}{|\pi(x)|^{s+2}}\,dx.$$

Taking $\tilde{x_0}=\vec{e}_1$ in (84), using a change of variable and the arguments used to prove (90), we get that

$$\frac{s}{2^\star}\int_{\mathbb{R}^n_-}\frac{|x_1|v^{2^\star}}{|\pi(x)|^{s+2}}\,dx=\frac{1}{2}\int_{\partial\mathbb{R}^n_-}|\nabla v|^2\,dx. \tag{92}$$

We consider the second fondamental form associated to $\partial\Omega$, namely

$$II_p(x,y)=(d\nu_p x,y)$$

for all $p\in\partial\Omega$ and all $x,y\in T_{x_0}\partial\Omega$ (recall that ν is the outward normal vector at the hypersurface $\partial\Omega$). In the basis $(\vec{e}_1,...,\vec{e}_n)$, the matrix of the bilinear form II_{x_0} is $-D_0^2\varphi_0$, where $D_0^2\varphi_0$ is the Hessian matrix of φ_0 at 0. With this remark, (91) and (92), we get that

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_\epsilon}=\frac{2(n-s)}{(n-2)^2}\mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{-\frac{n-s}{2-s}}\int_{\partial\mathbb{R}^n_-}\left(\frac{1}{2}II_{x_0}(x,x)-\gamma_0\right)|\nabla v|^2\,dx,$$

where $\gamma_0\geq 0$. This ends the proof of Proposition 5.1.

Step 5.4: We are now in position to prove Theorem 1.1. Points (A) and (B) of Theorem 1.1 are direct consequences of Propositions 2.1 and 2.2. To establish Part (C) of Theorem 1.1, assume that (10) holds and let us suppose that there are no extremals for (6). It follows from Proposition 3.2 that there exists $(u_\epsilon)_{\epsilon>0}\in H_{1,0}^2(\Omega)$ such that (15) and (16) hold with $p_\epsilon=\epsilon$ and $a_\epsilon\equiv 0$. Since there are no extremals, it follows from Proposition 3.2 that (17) holds. We apply Proposition 5.1 and we get that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\mu_\epsilon}=\frac{2(n-s)}{(n-2)^2}\mu_{s,\mathcal{P}}(\mathbb{R}^n_-)^{-\frac{n-s}{2-s}}\int_{\partial\mathbb{R}^n_-}\left(\frac{1}{2}II_{x_0}(x,x)-\gamma_0\right)|\nabla v|^2\,dx$$

where $x_0\in\mathcal{P}^\perp\cap\partial\Omega$ and $\gamma_0\geq 0$. We then get that

$$\int_{\partial\mathbb{R}^n_-}II_{x_0}(x,x)|\nabla v|^2\,dx\geq 0 \tag{93}$$

Assume that we are in the first case of point (C) of Theorem 1.1. We then get that $II_{x_0}(x,x) \leq 0$ for all $x \in \partial\mathbb{R}^n_-$, but $II_{x_0}(x,x) \not\equiv 0$, contradicting (93).

To relate our main result to conditions on the mean curvature, we now assume that $\mathcal{P} \cap T_x\partial\Omega$ and $\mathcal{P}^\perp$ are orthogonal with respect to the bilinear form II_{x_0}, we get in the coordinates (12) and the chart (13) that $(II_{x_0})_{ij} = 0$ when $i \in \{2,...,k\}$ and $j \in \{k+1,n\}$. In particular, we have with (93) that

$$\left(\sum_{i,j=2}^{k} (II_{x_0})_{ij} \int_{\partial\mathbb{R}^n_-} x^i x^j |\nabla v|^2\, dx\right) + \left(\sum_{i,j=k+1}^{n} (II_{x_0})_{ij} \int_{\partial\mathbb{R}^n_-} x^i x^j |\nabla v|^2\, dx\right) \geq 0. \tag{94}$$

The matrix of the second fundamental form of $\partial\Omega \cap (\mathcal{P}^\perp + (T_{x_0}\partial\Omega)^\perp)$ at x_0 with respect to a given vector $\vec{X}$ is $\left((II_{x_0}(\vec{X}))_{ij}\right)_{i,j\geq k+1} = -\left(\partial_{ij}\varphi_0(0)X^1\right)_{i,j\geq k+1}$. Since $\nabla\varphi_0(0) = 0$ and $\varphi_0(0,z) \leq 0$ for z close to 0, we get that for any direction $\vec{X}$, the principal curvatures of $\partial\Omega \cap (x_0 + (\mathcal{P}^\perp + (T_{x_0}\partial\Omega)^\perp))$ at x_0 have a sign. If the mean curvature vector of $\partial\Omega \cap (x_0 + (\mathcal{P}^\perp + (T_{x_0}\partial\Omega)^\perp))$ at x_0 is assumed to be null, it then follows that the second fundamental form of $\partial\Omega \cap (\mathcal{P}^\perp + (T_{x_0}\partial\Omega)^\perp)$ at x_0 is null, and we then get from (94) that

$$\sum_{i,j=2}^{k} (II_{x_0})_{ij} \int_{\partial\mathbb{R}^n_-} x^i x^j |\nabla v|^2\, dx \geq 0. \tag{95}$$

Here, $v \in H^2_{1,0}(\mathbb{R}^n_-)$ is positive, verifies $\Delta v = \frac{v^{2^\star-1}}{|\pi(x)|^s}$ weakly, and $v(x) \leq C(1+|x|^2)^{-n}$ for all $x \in \mathbb{R}^n_-$ (this last statement is a consequence of (19) and (43)). It follows from Proposition 1.1 that there exists $\tilde{v}$ such that $v(x_1,y,z) = \tilde{v}(x_1,|y|,z)$. With this symmetry property, with (95) we get that $\sum_{i=1}^k (II_{x_0})_{ii} \geq 0$, and then the mean curvature at x_0 of $\partial\Omega$ is nonnegative. This contradicts assumption (2) of case (C) of Theorem 1.1. This ends the proof of the Theorem.

Concerning Corollary 1.1, the subcritical problem yields families of positive solutions to (15) and (16) with $a_\epsilon \equiv a$ and $p_\epsilon = \epsilon$. The proof of Corollary 1.1 then goes as in the Proof of Theorem 1.1.

6. Proof of Proposition 1.2

We let Ω and $\mathcal{P}$ be as in Proposition 1.2. In particular $\dim_{\mathbb{R}} \mathcal{P} = 1$. The proof of case (B) of Proposition 1.2 goes exactly as the proof of Proposition 2.2. Concerning case (C), we claim that $\mu_{s,\mathcal{P}}(\Omega) = 0$ when $s \in [1,2)$. Indeed, taking $u \in C^\infty_c(\Omega)$ such that $u(x_0) = 1$, where $x_0 \in \mathcal{P}^\perp \cap \Omega$, it is easily checked that $\int_\Omega \frac{u^{2^\star}}{|\pi(x)|^s}\, dx = +\infty$, and then $\mu_{s,\mathcal{P}}(\Omega) = 0$ is not achieved. When $s \in (0,1)$ in case (A), the proof of non-achievement goes as the proof of Proposition 2.1.

We are left with case (C) of Proposition 1.2, that is $\mathcal{P}^\perp \cap \Omega = \emptyset$ and $\mathcal{P}^\perp \cap \partial\Omega \neq \emptyset$. Up to a change of coordinates, we assume that $\mathcal{P}^\perp = \{x_1 = 0\}$, $\Omega \subset \mathbb{R}^n_-$ and $0 \in \mathcal{P}^\perp \cap \partial\Omega$ and $|\pi(x)| = |x_1|$. In particular, it follows from the Sobolev inequality and the Hardy inequality that

$$\mu_{s,\mathcal{P}}(\mathbb{R}^n_-) := \inf\left\{ \frac{\int_{\mathbb{R}^n_-} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n_-} \frac{|u|^{2^\star}}{|x_1|^s}\, dx\right)^{\frac{2}{2^\star}}} \;\middle|\; u \in H^2_{1,0}(\mathbb{R}^n_-) \setminus \{0\} \right\} > 0.$$

Since $\Omega \subset \mathbb{R}^n_-$, we get that $\mu_{s,\mathcal{P}}(\Omega) \geq \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)$. With arguments similar to the proof of Proposition 3.1, we also get the reverse inequality, and then $\mu_{s,\mathcal{P}}(\Omega) = \mu_{s,\mathcal{P}}(\mathbb{R}^n_-)$. In particular, an extremal for $\mu_{s,\mathcal{P}}(\Omega)$ is an extremal for $\mu_{s,\mathcal{P}}(\mathbb{R}^n_-)$ and vice-versa. As in the proof of Proposition 2.1, the maximum principle yields a contradiction.

7. Appendix: Regularity of weak solutions

In this appendix, we prove the following regularity result:

Proposition 7.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $\mathcal{P} \subset \mathbb{R}^n$ be a $k-$dimensional linear subspace of $\mathbb{R}^n$, where $2 \leq k \leq n-1$. We assume that*

$$\mathcal{P}^\perp \cap \Omega = \emptyset \text{ and } \mathcal{P}^\perp \cap \partial\Omega \neq \emptyset.$$

We let $s \in (0,2)$ and $a \in C^{0,\alpha}(\overline{\Omega})$, where $\alpha \in (0,1)$. We let $\epsilon \in [0, 2^\star - 2)$ and consider $u \in H^2_{1,0}(\Omega)$ a weak solution of

$$\Delta u + au = \frac{|u|^{2^\star-2-\epsilon}u}{|\pi(x)|^s} \text{ in } \mathcal{D}'(\Omega). \tag{96}$$

Then $u \in C^1(\overline{\Omega}) \cap C^{2,\alpha}(\overline{\Omega} \setminus \mathcal{P}^\perp)$.

Proof of Proposition 7.1: Note that since $2^\star < \frac{2n}{n-2}$, it follows from standard elliptic theory that $u \in C^{2,\alpha}(\overline{\Omega} \setminus \mathcal{P}^\perp)$. In particular, $u \in C^{2,\alpha}(\Omega)$.

Step 7.1: We claim that

$$u \in L^p(\Omega) \tag{97}$$

for all $p \geq 1$. Indeed, the proof is similar to the case $\mathcal{P} = \mathbb{R}^n$ provided in [12, 13]. We omit the proof and refer to [12, 13] for the details.

In particular, we get that $\frac{|u|^{2^\star-2-\epsilon}u}{|\pi(x)|^s} \in L^p(\Omega)$ for all $1 \leq p < \frac{k}{s}$. In the case $k = n$, we take $p > \frac{n}{2}$, and then $u \in L^\infty(\Omega)$. A bootstrap argument (see also [10]) then yields that $u \in C^1(\overline{\Omega})$. However, in the general case $2 \leq k \leq n-1$, such an argument using standard elliptic theory does not hold, and we have to use the Green's function to prove the proposition.

Step 7.2: We let $\theta \in (0, \min\{2-s, 1\})$. We claim that there exists $C > 0$ such that

$$|u(x)| \leq C d(x, \partial\Omega)^\theta \tag{98}$$

for a.e. $x \in \Omega$.

Proof of the claim: We let $(\eta_k)_{k\in\mathbb{N}} \in C^\infty_c(\Omega)$ such that $0 \leq \eta_k \leq 1$ for all k and $\eta_k(x) = 1$ for $d(x, \partial\Omega) \geq 2k^{-1}$. We let $(u_k)_{k\in\mathbb{N}} \in H^2_{1,0}(\Omega)$ such that

$$\Delta u_k = \eta_k \left(\frac{|u|^{2^\star-2-\epsilon}u}{|\pi(x)|^s} - au \right). \tag{99}$$

Since $u \in C^2(\Omega)$ and $\Omega \cap \mathcal{P}^\perp = \emptyset$, we get that $u_k \in C^2(\overline{\Omega})$ for all $k \in \mathbb{N}$. We let G be the Green's function for Δ with Dirichlet boundary condition. It follows from Green's representation formula that

$$u_k(x) = \int_\Omega G(x,y)\eta_k(y) \left(\frac{|u|^{2^\star-1-\epsilon}(y)}{|\pi(y)|^s} - au \right) dy \tag{100}$$

for a.e. $x \in \Omega$. It follows from Theorem 9.1 of [13] that there exists $C > 0$ such that

$$0 < G(x,y) \leq C\frac{d(x,\partial\Omega)^\theta}{|x-y|^{n-2+\theta}} \tag{101}$$

for all $x, y \in \Omega$, $x \neq y$. Plugging this inequality in (100) and using Hölder's inequality, we get that

$$\begin{aligned} |u_k(x)| &\leq Cd(x,\partial\Omega)^\theta \int_\Omega \frac{1}{|x-y|^{n-2+\theta}}\left(\frac{|u(y)|^{2^\star-1-\epsilon}}{|\pi(y)|^s} + |u(y)|\right) dy \\ &\leq Cd(x,\partial\Omega)^\theta \||u|^{2^\star-1-\epsilon}\|_q \left(\int_\Omega \frac{dy}{|x-y|^{p(n-2+\theta)}|\pi(y)|^{sp}}\right)^{\frac{1}{p}} \\ &\quad + Cd(x,\partial\Omega)^\theta \|u\|_{q'} \left(\int_\Omega \frac{dy}{|x-y|^{p'(n-2+\theta)}}\right)^{\frac{1}{p'}} \end{aligned} \tag{102}$$

where $p, q, p', q' > 1$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} = 1$. Since $\theta \in (0,1)$ and (97) holds, we get that there exists $C > 0$ such that for $p, p' > 1$ sufficiently close to 1, we have that

$$|u(x)| \leq Cd(x,\partial\Omega)^\theta \left(\int_\Omega \frac{dy}{|x-y|^{p(n-2+\theta)}|\pi(y)|^{sp}}\right)^{\frac{1}{p}} + Cd(x,\partial\Omega)^\theta$$

for all $x \in \Omega$. For simplicity, up to a change of coordinates, we write any $y \in \mathbb{R}^n$ as $y = (y', y'')$, where $y' = \pi(y) \in \mathbb{R}^k = \mathcal{P}$ and $y'' \in \mathbb{R}^{n-k} = \mathcal{P}^\perp$. We let $R > 0$ such that $\Omega \subset B_R^k(0) \times B_R^{n-k}(0)$ (the product of the ball of radius R in $\mathbb{R}^k$ and the ball of radius R in $\mathbb{R}^{n-k}$). We then get with a change of variable that

$$\begin{aligned} &\int_\Omega \frac{dy}{|x-y|^{p(n-2+\theta)}|\pi(y)|^{sp}} \\ &\leq C\int_{B_R^k(0)} \frac{1}{|y'|^{ps}} \int_{B_R(0)^{n-k}} \left(\frac{dy''}{|x'-y'|^{p(n-2+\theta)} + |x''-y''|^{p(n-2+\theta)}}\right) dy' \\ &\leq C\int_{B_R^k(0)} \frac{1}{|y'|^{ps}|x'-y'|^{p(n-2+\theta)+k-n}} \int_{B_{\frac{2R}{|x'-y'|}}(0)^{n-k}} \left(\frac{dz''}{1+|z''|^{p(n-2+\theta)}}\right) dy' \\ &\leq C\int_{B_R^k(0)} \frac{dy'}{|y'|^{ps}|x'-y'|^{p(n-2+\theta)+k-n}} \leq C \end{aligned}$$

for all $(x', x'') \in \Omega$. Here, we have taken $p > 1$ close to 1 and we have used that $s \in (0,2)$. Plugging this inequality in (102), we get that there exists $C > 0$ such that

$$|u_k(x)| \leq Cd(x,\partial\Omega)^\theta \tag{103}$$

for all $x \in \Omega$ and all $k \in \mathbb{N}$. Multiplying (99) by u_k, integrating over Ω, using that $u \in H_{1,0}^2(\Omega)$, the inequality (1) and (103), we get that there exists $C > 0$ such that $\|u_k\|_{H_{1,0}^2(\Omega)} \leq C$ for all $k \in \mathbb{N}$. It then follows that there exists $\tilde{u} \in H_{1,0}^2(\Omega)$ such that $u_k \rightharpoonup \tilde{u}$ weakly in $H_{1,0}^2(\Omega)$ when $k \to +\infty$ and $\lim_{k\to+\infty} u_k(x) = \tilde{u}(x)$ for a.e. $x \in \Omega$. The function $\tilde{u}$ verifies $\Delta\tilde{u} = \frac{|u|^{2^\star-2-\epsilon}}{|\pi(x)|^s} - au$ in $\mathcal{D}'(\Omega)$. Since Δ is coercive, it then follows from (96) that $\tilde{u} = u$. With (103), we then get (98). ☐

Step 7.3: We claim that there exists $C > 0$ such that

$$|u(x)| \leq Cd(x,\partial\Omega) \tag{104}$$

for a.e. $x \in \Omega$.

Proof of the claim: Indeed, we let $\theta_0 \in (0,1)$ such that there exists $C>0$ such that $|u(x)| \leq Cd(x,\partial\Omega)^{\theta_0}$. With (34), we get that there exists $C>0$ such that $|u(x)| \leq C|\pi(x)|^{\theta_0}$ for all $x \in \Omega$. We let $\theta \in (0,1)$. It follows from Green's representation formula and (101) that there exists $C>0$ such that

$$\begin{aligned} |u_k(x)| &= \left|\int_\Omega G(x,y)\eta_k(y)\left(\frac{|u|^{2^\star-1-\epsilon}(y)}{|\pi(y)|^s} - au\right) dy\right| \\ &\leq Cd(x,\partial\Omega)^\theta + C\int_\Omega \frac{d(x,\partial\Omega)^\theta}{|x-y|^{n-2+\theta}|\pi(y)|^{s-\theta_0(2^\star-1-\epsilon)}}\, dy. \end{aligned}$$

We proceed as in Step 7.3 and get that $|u(x)| \leq Cd(x,\partial\Omega)^\theta$ for some $\theta > \theta_0$. The claim follows by induction. □

Step 7.4: We claim that $u \in C^1(\overline{\Omega})$.

Proof of the claim: With inequality (104), and the method used in Step 7.2, we get that

$$\lim_{k\to+\infty} u_k(x) = \int_\Omega G(x,y)\left(\frac{|u|^{2^\star-2-\epsilon}u(y)}{|\pi(y)|^s} - au\right) dy$$

and

$$\lim_{k\to+\infty} \nabla u_k(x) = \int_\Omega \nabla_x G(x,y)\left(\frac{|u|^{2^\star-2-\epsilon}u(y)}{|\pi(y)|^s} - au\right) dy$$

uniformly for $x \in \overline{\Omega}$. Since $u_k \rightharpoonup u$ in $H_{1,0}^2(\Omega)$ when $k \to +\infty$, we get that $u \in C^1(\overline{\Omega})$. □

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