# CONCENTRATION ESTIMATES FOR EMDEN-FOWLER EQUATIONS WITH BOUNDARY SINGULARITIES AND CRITICAL GROWTH

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Abstract. We establish –among other things– existence and multiplicity of solutions for the Dirichlet problem $\sum_i \partial_{ii} u + \frac{|u|^{2^\star-2}u}{|x|^s} = 0$ on smooth bounded domains $\Omega$ of $\mathbb{R}^n$ ($n \geq 3$) involving the critical Hardy-Sobolev exponent $2^\star = \frac{2(n-s)}{n-2}$ where $0 < s < 2$, and in the case where zero (the point of singularity) is on the boundary $\partial\Omega$. Just as in the Yamabe-type non-singular framework (i.e., when $s = 0$), there is no nontrivial solution under global convexity assumption (e.g., when $\Omega$ is star-shaped around 0). However, in contrast to the non-satisfactory situation of the non-singular case, we show the existence of an infinite number of solutions under an assumption of local strict concavity of $\partial\Omega$ at 0 in at least one direction. More precisely, we need the principal curvatures of $\partial\Omega$ at 0 to be non-positive but not all vanishing. We also show that the best constant in the Hardy-Sobolev inequality is attained as long as the mean curvature of $\partial\Omega$ at 0 is negative, extending the results of [20] and completing our result of [21] to include dimension 3. The key ingredients in our proof are refined concentration estimates which yield compactness for certain Palais-Smale sequences which do not hold in the non-singular case.

## 1. Introduction and statement of the results

We address the problem of existence and multiplicity of possibly sign-changing solutions of the following Emden-Fowler boundary value problem on a smooth domain $\Omega$ of $\mathbb{R}^n$, $n \geq 3$:

$$\left\{ \begin{array}{ll} \Delta u = \frac{|u|^{2^\star-2}u}{|x|^s} & \text{in } \mathcal{D}'(\Omega) \\ u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1}$$

where here and throughout the paper, $\Delta = -\sum_i \partial_{ii}$ is the Laplacian with minus sign convention, and $2^\star := 2^*(s) = \frac{2(n-s)}{n-2}$ with $s \in [0,2]$. The non-singular case, i.e., when $s = 0$, is the Euclidean version of the celebrated Yamabe problem considered first by Brezis and Nirenberg [6] followed by a large number of authors. Here again the situation is interesting since we are dealing with the corresponding critical exponent in the Hardy-Sobolev embedding $H_{1,0}^2(\Omega) \to L^p(\Omega; |x|^{-s}dx)$ which is not compact when $p = 2^*(s)$. We recall that $H_{1,0}^2(\Omega)$ is the completion of $C_c^\infty(\Omega)$, the set of smooth functions compactly supported in $\Omega$, for the norm

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$\|u\|_{H^2_{1,0}(\Omega)} = \sqrt{\int_\Omega |\nabla u|^2\,dx}$, and that the above embedding follows from the Hardy-Sobolev inequality ([9], [10], [23]) which states that the constant defined as

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx;\ u\in H^2_{1,0}(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^\star}}{|x|^s}\,dx = 1\right\} \tag{2}$$

satisfies $0<\mu_s(\Omega)<+\infty$. This in turn allows for a variational approach for the problem of finding solutions in $H^2_{1,0}(\Omega)\cap C^0(\overline{\Omega})$ for the Dirichlet problem (1).

Now the story of the state of the art in the non-singular case is quite extensive (see for instance Struwe [37]), but for our purpose we single out the following highlights:

1) For any domain $\Omega$, the best constant $\mu_0(\Omega)$ is the same as $\mu_0(\mathbb{R}^n)$ and it is never attained unless $\Omega$ is essentially $\mathbb{R}^n$ (i.e., $\text{cap}(\mathbb{R}^n\setminus\Omega)=0$), in which case there is an infinite number of sign-changing solutions for

$$\left\{\begin{array}{ll}\Delta u = |u|^{2^\star-2}u & \text{in } \mathcal{D}'(\Omega)\\ u=0 & \text{on } \partial\Omega.\end{array}\right. \tag{3}$$

Moreover, there are no solution for (3) whenever $\Omega$ is bounded convex or star-shaped. On the other hand, there are solutions if $\Omega$ is not contractible (in dimension 3) and an infinite number of them [3], if the domain $\Omega$ has non-trivial homology (i.e., $H_d(\Omega,\mathbf{Z}_2)\neq 0$ for some $d>0$). Unfortunately, these topological conditions are far from being optimal and no geometric condition that would guarantee the existence of one or more solutions, have so far been isolated.

2) On the other hand, the addition of a linear term to the equation, such as

$$\left\{\begin{array}{ll}\Delta u = |u|^{2^\star-2}u+\lambda u & \text{in } \mathcal{D}'(\Omega)\\ u=0 & \text{on } \partial\Omega.\end{array}\right. \tag{4}$$

improves the situation dramatically, especially when $0<\lambda<\lambda_1$, since there is then a positive solution for any smooth bounded domain $\Omega$ in $\mathbb{R}^n$ as long as $n\geq 4$ (See Brezis-Nirenberg [6]). The case $n=3$ is more delicate and was dealt by Druet [13]. Most relevant to our work, are the recent results by Devillanova and Solimini who managed in [11], to establish the existence of an infinite number of solutions for (4) in dimension $n\geq 7$.

The situation for the Emden-Fowler equations (i.e., when $s>0$) turned out to be at least as interesting, and somewhat more satisfactory. Actually, the case when 0 belongs to the interior of the domain $\Omega$ is almost identical to the non-singular case [23] as one can prove essentially the same results with a suitable adaptation of the same techniques. However, the situation is much different when $0\in\partial\Omega$.

1) Indeed, Egnell showed in [17] that for open cones of the form $C=\{x\in\mathbb{R}^n; x=r\theta, \theta\in D \text{ and } r>0\}$ where the base $D$ is a connected domain of the unit sphere $S^{n-1}$ of $\mathbb{R}^n$, the best constant $\mu_s(C)$ is attained for $0<s<2$ even when $\bar{C}\neq\mathbb{R}^n$. The case where $\partial\Omega$ is smooth at 0 was tackled in [20] and it turned out to be also quite interesting since the curvature of the boundary at 0 gets to play an important role. It was shown there that in dimension $n\geq 4$, the negativity of all principal curvatures[1] at 0 –which is essentially a condition of *"local strict concavity"* – leads to attainability of the best constant for problems with Dirichlet

[1] In our context, we specify the orientation of $\partial\Omega$ in such a way that the normal vectors of $\partial\Omega$ are pointing outward from the domain $\Omega$.

boundary conditions, while the Neumann problems required the positivity of the mean curvature at 0.

More recently, we show in [21] that for dimension $n \geq 4$, the negativity of the mean curvature of $\partial\Omega$ at 0 is sufficient to ensure the attainability of $\mu_s(\Omega)$. This result is quite satisfactory, since standard Pohozaev type arguments show non-attainability in the case where $\Omega$ is convex or star-shaped at 0. One of the results of this paper is the extension of this attainability result to cover all dimensions (greater than 3) including the more subtle context of dimension 3. We shall establish the following

**Theorem 1.1.** *Let $\Omega$ be a smooth bounded oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ and assume $s \in (0,2)$. If the mean curvature of $\partial\Omega$ at 0 is negative, then $\mu_s(\Omega)$ is achieved by a positive function which is –a positive multiple of– a solution for*

$$\begin{cases} \Delta u = \frac{|u|^{2^\star-2}u}{|x|^s} & \text{in } \mathcal{D}'(\Omega) \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

2) As to the question of multiplicity of solutions for (1), we note that Ghoussoub-Kang had shown in [20] the existence of two solutions under the same negativity condition on all of the principal curvatures at 0. More precisely, assuming that the principal curvatures $\alpha_1, ..., \alpha_{n-1}$ of $\partial\Omega$ at 0 are finite, the oriented boundary $\partial\Omega$ near the origin can then be represented (up to rotating the coordinates if necessary) by $x_1 = \varphi_0(x') = -\frac{1}{2}\sum_{i=2}^n \alpha_{i-1}x_i^2 + o(|x'|^2)$, where $x' = (x_2, ..., x_n) \in B_\delta(0) \cap \{x_1 = 0\}$ for some $\delta > 0$ where $B_\delta(0)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius $\delta$. If the principal curvatures at 0 are all negative, i.e., if

$$\max_{1\leq i\leq n-1} \alpha_i < 0, \tag{6}$$

then the sectional curvature at 0 is negative and therefore $\partial\Omega$ –viewed as an $(n-1)$-Riemannian submanifold of $\mathbb{R}^n$– is strictly convex at 0 (see for instance [18]). The latter property means that there exists a neighborhood $U$ of 0 in $\partial\Omega$, such that the whole of $U$ lies on one side of a hyperplane $H$ that is tangent to $\partial\Omega$ at 0 and $U \cap H = \{0\}$, and so does the complement $\mathbb{R}^n \setminus \Omega$, at least locally. In other words, the above curvature condition then amounts to a notion of strict local convexity of $\mathbb{R}^n \setminus \Omega$ at 0. In this paper, we complete and extend these results in many ways, since we establish the existence of infinitely many solutions under the following much weaker assumption:

$$\max_{1\leq i\leq n-1} \alpha_i \leq 0 \quad \text{and} \quad \min_{1\leq i\leq n-1} \alpha_i < 0. \tag{7}$$

which is a condition of *"local concavity at 0"* that is "strict" in at least one direction.

**Theorem 1.2.** *Let $\Omega$ be a smooth bounded oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. Let $s \in (0,2)$ and $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in $\Omega$. If the principal curvatures of $\partial\Omega$ at 0 are non-positive, but not all vanishing, then there exists an infinite number of solutions $u \in H_{1,0}^2(\Omega) \cap C^1(\overline{\Omega})$ for*

$$\begin{cases} \Delta u + au = \frac{|u|^{2^\star-2}u}{|x|^s} & \text{in } \mathcal{D}'(\Omega) \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now we do not know if the negativity of the mean curvature at 0 is sufficient for the above result, however it is a remarkably satisfactory once compared to what is known in the nonsingular case and since –as mentioned above– we have no solution when $\Omega$ is convex or star shaped at 0.

3) Now all these results rely on blow-up analysis techniques where the limiting spaces (i.e., where the blown-up solutions of corresponding Euler-Lagrange equations eventually live) play an important role. In the non-singular case, the limiting space is $\mathbb{R}^n$ while in our framework, the limiting cases occur on half-spaces of the form $\mathbb{R}^n_- = \{x \in \mathbb{R}^n_- / \, x_1 < 0\}$, where $x_1$ denotes the first coordinate of a generic point $x \in \mathbb{R}^n$ in the canonical basis of $\mathbb{R}^n$. The above theorem is a corollary of a more powerful result established below about the asymptotic behaviour of a family of solutions to elliptic pde's, which are not necessarily minimizing sequences. We actually study families of solutions to related subcritical problems, and we completely describe their asymptotic behaviour –potentially developing a singularity at zero– as we approach the critical exponent.

More precisely, we say that a function is in $C^1(\overline{\Omega})$ if it can be extended to a $C^1-$function in a open neighborhood of $\overline{\Omega}$, and consider a family $(a_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ and a function $a \in C^1(\overline{\Omega})$ such that there exists an open subset $\mathcal{U} \subset \mathbb{R}^n$ such that $a_\epsilon, a$ can be extended to $\mathcal{U}$ by $C^1-$functions that we still denote by $a_\epsilon, a$. We assume that they satisfy

$$\overline{\Omega} \subset\subset \mathcal{U} \text{ and } \lim_{\epsilon\to 0} a_\epsilon = a \text{ in } C^1(\mathcal{U}). \tag{8}$$

Here is the main result of this paper.

**Theorem 1.3.** *Let $\Omega$ be a smooth bounded oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. Assume $s \in (0,2)$ and consider $(a_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ such that (8) hold. We let $(p_\epsilon)_{\epsilon>0}$ such that $p_\epsilon \in [0, 2^\star - 2)$ for all $\epsilon > 0$ and $\lim_{\epsilon\to 0} p_\epsilon = 0$. We assume that the principal curvatures of $\partial\Omega$ at 0 are non-positive but do not all vanish. We consider a family of functions $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$ such that*

$$\left\{ \begin{array}{ll} \Delta u_\epsilon + a_\epsilon u_\epsilon = \frac{|u_\epsilon|^{2^\star-2-p_\epsilon}}{|x|^s} u_\epsilon & \text{in } \mathcal{D}'(\Omega) \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{array} \right.$$

*1) If the family $(u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $H^2_{1,0}(\Omega)$, then $(u_\epsilon)_{\epsilon>0}$ is precompact in the $C^1-$topology. In particular, there exists $u_0 \in H^2_{1,0}(\Omega)$ such that, up to a subsequence, we have that $\lim_{\epsilon\to 0} u_\epsilon = u_0$ in $C^1(\overline{\Omega})$.*

*2) Moreover, if the $u_\epsilon$'s are nonnegative for all $\epsilon > 0$, then the same conclusion holds under the sole hypothesis that the mean curvature of $\partial\Omega$ at 0 is negative and that the family $(u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $H^2_{1,0}(\Omega)$.*

The proof of this theorem uses the machinery developed in Druet-Hebey-Robert [15] and is in the spirit of Druet [14], where the concentration analysis is studied in the intricate Riemannian setting. The study of the asymptotic for elliptic nonlinear pde's was initiated by Atkinson-Peletier [1], see also Brézis-Peletier [7]. In the Riemannian context, the asymptotics have first been studied by Schoen [38] and Hebey-Vaugon [31]. This tool has happened to be a very powerful tool for the study of best constant problems in Sobolev inequalities, see for instance Druet [12], Hebey-Vaugon [31], [32] and Robert [36]). Let us also mention the study of the asymptotics for solutions to nonlinear pde's (Han [26], Hebey [28], Druet-Robert

[16] and Robert [35]). In the case of arbitrary large energies, the compactness issues become quite intricate, especially in the Riemannian context, see for instance the pioneer work of Schoen [38]. We also refer to the recent Druet [14] and Marques [34]. Compactness results for fourth order equations are in Hebey-Robert [29] and Hebey-Robert-Wen [30]. In a forthcoming paper [21], we tackle similar questions for various critical equations involving a whole affine subspace of singularities on the boundary.

The paper is organized as follows. In Section 2, we state general facts and two lemmae that will be useful throughout the paper. In Section 3, we construct the different scales of blow-up. In Sections 4 and 5, we prove strong pointwise estimates for sequences of solutions to our problem. In Section 6, we use the Pohozaev identity to describe precisely the asymptotics related to our problem and we prove theorem 1.3. Section 7 contains the proofs of Theorems 1.1 and 1.2. Finally, we give in the Appendix a regularity result for solutions to a critical PDE, some useful properties of the Green's function and a symmetry property of solutions to some nonlinear elliptic equations on the half-plane.

## 2. Basic facts and preliminary Lemmae

From now on, we let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $s \in (0,2)$. For any $\epsilon > 0$, we let $p_\epsilon \in [0, 2^\star - 2)$ such that

$$\lim_{\epsilon \to 0} p_\epsilon = 0. \tag{9}$$

We let $a \in C^1(\overline{\Omega})$ and a family $(a_\epsilon)_{\epsilon>0} \in C^1(\overline{\Omega})$ such that (8) holds. For any $\epsilon > 0$, we consider $u_\epsilon \in H_{1,0}^2(\Omega)$ a solution to the system

$$\begin{cases} \Delta u_\epsilon + a_\epsilon u_\epsilon = \frac{|u_\epsilon|^{2^\star - 2 - p_\epsilon}}{|x|^s} u_\epsilon & \text{in } \mathcal{D}'(\Omega) \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{$E_\epsilon$}$$

for all $\epsilon > 0$. Note that it follows from Proposition 8.1 of the Appendix that

$$u_\epsilon \in C^{1,\theta}(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$$

for all $\theta \in (0, \min\{1, 2^\star - s\})$. In addition, we assume that there exists $\Lambda > 0$ such that

$$\|u_\epsilon\|_{H_{1,0}^2(\Omega)} \leq \Lambda \tag{10}$$

for all $\epsilon > 0$. It then follows from the weak compactness of the unit ball of $H_{1,0}^2(\Omega)$ that there exists $u_0 \in H_{1,0}^2(\Omega)$ such that

$$u_\epsilon \rightharpoonup u_0 \tag{11}$$

weakly in $H_{1,0}^2(\Omega)$ when $\epsilon \to 0$. Note that $u_0$ verifies

$$\Delta u_0 + a u_0 = \frac{|u_0|^{2^\star - 2}}{|x|^s} u_0 \text{ in } \mathcal{D}'(\Omega).$$

It follows from the Appendix that

$$u_0 \in C^{1,\theta}(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$$

for all $\theta \in (0, \min\{1, 2^\star - s\})$. The following Proposition addresses the case when $u_\epsilon$ is uniformly bounded in $L^\infty$. Note that here and in the sequel, all the convergence results are up to the extraction of a subsequence.

**Proposition 2.1.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8) and (9) hold. We assume that there exists $C > 0$ such that $|u_\epsilon(x)| \leq C$ for all $x \in \Omega$. Then up to a subsequence, $\lim_{\epsilon\to 0} u_\epsilon = u_0$ in $C^1(\overline{\Omega})$, where $u_0$ is as in (11).*

*Proof:* It follows from the proof of Proposition 8.1 of the Appendix that for any $\theta \in (0, \min\{1, 2^\star - s\})$, there exists $C > 0$ such that $\|u_\epsilon\|_{C^{1,\theta}(\overline{\Omega})} \leq C$ for all $\epsilon > 0$. The conclusion of the Proposition then follows. We refer to the Appendix for the details. □

From now on, we assume that

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^\infty(\Omega)} = +\infty. \tag{12}$$

Throughout the paper, we shall say that blow-up occurs whenever (12) holds. We define

$$\mathbb{R}^n_- = \{x \in \mathbb{R}^n / \, x_1 < 0\}$$

where $x_1$ is the first coordinate of a generic point of $\mathbb{R}^n$. This space will be the limit space after blow-up. In the sequel of this section, we give some useful tools for the blow-up analysis. We let $y_0 \in \partial\Omega$. Since $\partial\Omega$ is smooth and $y_0 \in \partial\Omega$, there exist $U, V$ open subsets of $\mathbb{R}^n$, there exists $I$ an open intervall of $\mathbb{R}$, there exists $U'$ an open subset of $\mathbb{R}^{n-1}$ such that $0 \in U = I \times U'$ and $y_0 \in V$. There exist $\varphi \in C^\infty(U, V)$ and $\varphi_0 \in C^\infty(U')$ such that, up to rotating the coordinates if necessary,

$$\begin{array}{ll} (i) & \varphi : U \to V \text{ is a } C^\infty-\text{diffeomorphism} \\ (ii) & \varphi(0) = y_0 \\ (iii) & D_0\varphi = Id_{\mathbb{R}^n} \\ (iv) & \varphi(U \cap \{x_1 < 0\}) = \varphi(U) \cap \Omega \text{ and } \varphi(U \cap \{x_1 = 0\}) = \varphi(U) \cap \partial\Omega. \\ (v) & \varphi(x_1, y) = y_0 + (x_1 + \varphi_0(y), y) \text{ for all } (x_1, y) \in I \times U' = U \\ (vi) & \varphi_0(0) = 0 \text{ and } \nabla\varphi_0(0) = 0. \end{array} \tag{13}$$

Here $D_x\varphi$ denotes the differential of $\varphi$ at $x$. This chart will be useful throughout all the paper.

We prove two useful blow-up lemmae:

**Lemma 2.1.** *We let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. We assume that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. We let $(y_\epsilon)_{\epsilon>0} \in \Omega$. Let*

$$\nu_\epsilon := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } \beta_\epsilon := |y_\epsilon|^{\frac{s}{2}} |u_\epsilon(y_\epsilon)|^{\frac{2+p_\epsilon-2^\star}{2}}.$$

*We assume that $\lim_{\epsilon\to 0} \nu_\epsilon = 0$. In particular, $\lim_{\epsilon\to 0} \beta_\epsilon = 0$. We assume that for any $R > 0$, there exists $C(R) > 0$ such that*

$$|u_\epsilon(x)| \leq C(R)|u_\epsilon(y_\epsilon)| \tag{14}$$

*for all $x \in B_{R\beta_\epsilon}(y_\epsilon) \cap \Omega$ and all $\epsilon > 0$. Then we have that*

$$y_\epsilon = O\left(\nu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}}\right)$$

*when $\epsilon \to 0$. In particular, $\lim_{\epsilon\to 0} y_\epsilon = 0$.*

*Proof of Lemma 2.1:* We proceed by contradiction and assume that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon}=+\infty. \tag{15}$$

where $\ell_\epsilon:=\nu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}}$ for all $\epsilon>0$. In particular, it follows from the definition of $\beta_\epsilon$ and (15) that

$$\lim_{\epsilon\to 0}\beta_\epsilon=0,\ \lim_{\epsilon\to 0}\frac{\beta_\epsilon}{\ell_\epsilon}=+\infty \text{ and } \lim_{\epsilon\to 0}\frac{\beta_\epsilon}{|y_\epsilon|}=0. \tag{16}$$

*Case 1:* We assume that there exists $\rho>0$ such that

$$\frac{d(y_\epsilon,\partial\Omega)}{\beta_\epsilon}\geq 3\rho$$

for all $\epsilon>0$. For $x\in B_{2\rho}(0)$ and $\epsilon>0$, we define

$$v_\epsilon(x):=\frac{u_\epsilon(y_\epsilon+\beta_\epsilon x)}{u_\epsilon(y_\epsilon)}.$$

Note that this is well defined since $y_\epsilon+\beta_\epsilon x\in\Omega$ for all $x\in B_{2\rho}(0)$. It follows from (14) that there exists $C(\rho)>0$ such that

$$|v_\epsilon(x)|\leq C(\rho) \tag{17}$$

for all $\epsilon>0$ and all $x\in B_{2\rho}(0)$. As easily checked, we have that

$$\Delta v_\epsilon+\beta_\epsilon^2 a_\epsilon(y_\epsilon+\beta_\epsilon x)v_\epsilon=\frac{|v_\epsilon|^{2^\star-2-p_\epsilon}v_\epsilon}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\beta_\epsilon}{|y_\epsilon|}x\right|^s}$$

weakly in $B_{2\rho}(0)$. Since (16) holds, we have that

$$\Delta v_\epsilon+\beta_\epsilon^2 a_\epsilon(y_\epsilon+\beta_\epsilon x)v_\epsilon=(1+o(1))|v_\epsilon|^{2^\star-2-p_\epsilon}v_\epsilon \tag{18}$$

weakly in $B_{2\rho}(0)$, where $\lim_{\epsilon\to 0}o(1)=0$ in $C^0_{loc}(B_{2\rho}(0))$. It follows from (17), (18) and standard elliptic theory that there exists $v\in C^1(B_{2\rho}(0))$ such that

$$v_\epsilon\to v$$

in $C^1_{loc}(B_{2\rho}(0))$ when $\epsilon\to 0$. In particular,

$$v(0)=\lim_{\epsilon\to 0}v_\epsilon(0)=1 \tag{19}$$

and $v\not\equiv 0$. With a change of variables and the definition of $\beta_\epsilon$, we get that

$$\int_{\Omega\cap B_{\rho\beta_\epsilon}(y_\epsilon)}\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\,dx=\frac{|u_\epsilon(y_\epsilon)|^{2^\star-p_\epsilon}\beta_\epsilon^n}{|y_\epsilon|^s}\int_{B_\rho(0)}\frac{|v_\epsilon|^{2^\star-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\beta_\epsilon}{|y_\epsilon|}x\right|^s}\,dx$$

$$\geq\left(\frac{|y_\epsilon|}{\ell_\epsilon}\right)^{s\frac{n-2}{2}}\int_{B_\rho(0)}\frac{|v_\epsilon|^{2^\star-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\beta_\epsilon}{|y_\epsilon|}x\right|^s}\,dx.$$

Using the equation $(E_\epsilon)$, (10), (15) and (16) and passing to the limit $\epsilon\to 0$, we get that
$$\int_{B_\rho(0)}|v|^{2^\star}\,dx=0,$$
and then $v\equiv 0$ in $B_\rho(0)$. A contradiction with (19). Then (15) does not hold in Case 1.

*Case 2:* We assume that, up to a subsequence,
$$\lim_{\epsilon\to 0}\frac{d(y_\epsilon,\partial\Omega)}{\beta_\epsilon}=0. \tag{20}$$
In this case,
$$\lim_{\epsilon\to 0}y_\epsilon=y_0\in\partial\Omega.$$
Since $y_0\in\partial\Omega$, we let $\varphi:U\to V$ as in (13), where $U,V$ are open neighborhoods of 0 and $y_0$ respectively. We let $\tilde{u}_\epsilon=u_\epsilon\circ\varphi$, which is defined on $U\cap\{x_1\leq 0\}$. For any $i,j=1,...,n$, we let $g_{ij}=(\partial_i\varphi,\partial_j\varphi)$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$, and we consider $g$ as a metric on $\mathbb{R}^n$. We let $\Delta_g=-div_g(\nabla)$ the Laplace-Beltrami operator with respect to the metric $g$. In our basis, we have that
$$\Delta_g=-g^{ij}\left(\partial_{ij}-\Gamma_{ij}^k\partial_k\right),$$
where $g^{ij}=(g^{-1})_{ij}$ are the coordinates of the inverse of the tensor $g$ and the $\Gamma_{ij}^k$'s are the Christoffel symbols of the metric $g$. As easily checked, we have that
$$\Delta_g\tilde{u}_\epsilon+a_\epsilon\circ\varphi(x)\cdot\tilde{u}_\epsilon=\frac{|\tilde{u}_\epsilon|^{2^\star-2-p_\epsilon}\tilde{u}_\epsilon}{|\varphi(x)|^s}$$
weakly in $U\cap\{x_1<0\}$. We let $z_\epsilon\in\partial\Omega$ such that
$$|z_\epsilon-y_\epsilon|=d(y_\epsilon,\partial\Omega). \tag{21}$$
We let $\tilde{y}_\epsilon,\tilde{z}_\epsilon\in U$ such that
$$\varphi(\tilde{y}_\epsilon)=y_\epsilon\text{ and }\varphi(\tilde{z}_\epsilon)=z_\epsilon. \tag{22}$$
It follows from the properties (13) of $\varphi$ that
$$\lim_{\epsilon\to 0}\tilde{y}_\epsilon=\lim_{\epsilon\to 0}\tilde{z}_\epsilon=0,\ (\tilde{y}_\epsilon)_1<0\text{ and }(\tilde{z}_\epsilon)_1=0. \tag{23}$$
At last, we let
$$\tilde{v}_\epsilon(x):=\frac{\tilde{u}_\epsilon(\tilde{z}_\epsilon+\beta_\epsilon x)}{\tilde{u}_\epsilon(\tilde{y}_\epsilon)}$$
for all $x\in\frac{U-\tilde{z}_\epsilon}{\beta_\epsilon}\cap\{x_1<0\}$. With (23), we get that $\tilde{v}_\epsilon$ is defined on $B_R(0)\cap\{x_1<0\}$ for all $R>0$, as soon as $\epsilon$ is small enough. It follows from (14) that there exists $C'(R)>0$ such that
$$|\tilde{v}_\epsilon(x)|\leq C'(R) \tag{24}$$
for all $\epsilon>0$ and all $x\in B_R(0)\cap\{x_1\leq 0\}$. The function $\tilde{v}_\epsilon$ verifies
$$\Delta_{\tilde{g}_\epsilon}\tilde{v}_\epsilon+\beta_\epsilon^2a_\epsilon\circ\varphi(\tilde{z}_\epsilon+\beta_\epsilon x)\tilde{v}_\epsilon=\frac{|\tilde{v}_\epsilon|^{2^\star-2-p_\epsilon}\tilde{v}_\epsilon}{\left|\frac{\varphi(\tilde{z}_\epsilon+\beta_\epsilon x)}{|y_\epsilon|}\right|^s}$$

weakly in $B_R(0)\cap\{x_1<0\}$. In this expression, $\tilde{g}_\epsilon = g(\tilde{z}_\epsilon+\beta_\epsilon x)$ and $\Delta_{\tilde{g}_\epsilon}$ is the Laplace-Beltrami operator with respect to the metric $\tilde{g}_\epsilon$. With (20), (21) and (22), we get that

$$\varphi(\tilde{z}_\epsilon+\beta_\epsilon x) = y_\epsilon + O_R(1)\beta_\epsilon,$$

for all $x\in B_R(0)\cap\{x_1\leq 0\}$ and all $\epsilon>0$, where there exists $C_R>0$ such that $|O_R(1)|\leq C_R$ for all $x\in B_R(0)\cap\{x_1<0\}$. With (16), we then get that

$$\lim_{\epsilon\to 0}\frac{|\varphi(\tilde{z}_\epsilon+\beta_\epsilon x)|}{|y_\epsilon|}=1$$

in $C^0(B_R(0)\cap\{x_1\leq 0\})$. It then follows that

$$\Delta_{\tilde{g}_\epsilon}\tilde{v}_\epsilon + \beta_\epsilon^2 a_\epsilon\circ\varphi(\tilde{z}_\epsilon+\beta_\epsilon x)\tilde{v}_\epsilon = (1+o(1))|\tilde{v}_\epsilon|^{2^\star-2-p_\epsilon}\tilde{v}_\epsilon$$

weakly in $B_R(0)\cap\{x_1<0\}$, where $\lim_{\epsilon\to 0} o(1)=0$ in $C^0(B_R(0)\cap\{x_1\leq 0\})$. Since $\tilde{v}_\epsilon$ vanishes on $B_R(0)\cap\{x_1=0\}$ and (24) holds, it follows from standard elliptic theory that there exists $\tilde{v}\in C^1(B_R(0)\cap\{x_1\leq 0\})$ such that

$$\lim_{\epsilon\to 0}\tilde{v}_\epsilon = \tilde{v}$$

in $C^0(B_{\frac{R}{2}}(0)\cap\{x_1\leq 0\})$. In particular,

$$\tilde{v}\equiv 0 \text{ on } B_{\frac{R}{2}}(0)\cap\{x_1=0\}. \tag{25}$$

Moreover, it follows from (20), (21) and (22) that

$$\tilde{v}_\epsilon\left(\frac{\tilde{y}_\epsilon-\tilde{z}_\epsilon}{\beta_\epsilon}\right)=1 \text{ and } \lim_{\epsilon\to 0}\frac{\tilde{y}_\epsilon-\tilde{z}_\epsilon}{\beta_\epsilon}=0.$$

In particular, $\tilde{v}(0)=1$. A contradiction with (25). Then (15) does not hold in Case 2.

In both cases, we have contradicted (15). This proves that $y_\epsilon = O(\ell_\epsilon)$ when $\epsilon\to 0$, which proves the Lemma. □

**Lemma 2.2.** *We let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$. We assume that $0\in\partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. We let $(\nu_\epsilon)_{\epsilon>0}$ and $(\ell_\epsilon)_{\epsilon>0}$ such that $\nu_\epsilon,\ell_\epsilon>0$ for all $\epsilon>0$ and*

$$\ell_\epsilon = \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}} \text{ and } \lim_{\epsilon\to 0}\nu_\epsilon = 0.$$

*Since $0\in\partial\Omega$, we let $\varphi: U\to V$ as in (13) with $y_0=0$, where $U,V$ are open neighborhoods of $0$. We let*

$$\tilde{u}_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} u_\epsilon\circ\varphi(\ell_\epsilon x)$$

*for all $x\in\frac{U}{\ell_\epsilon}\cap\{x_1\leq 0\}$ and all $\epsilon>0$. We assume that either*

*(L1) for all $R>0$, there exists $C(R)>0$ such that*

$$|\tilde{u}_\epsilon(x)|\leq C(R)$$

*for all $x\in B_R(0)\cap\{x_1<0\}$, or*

*(L2) for all $R>\delta>0$, there exists $C(R,\delta)>0$ such that*

$$|\tilde{u}_\epsilon(x)|\leq C(R,\delta)$$

*for all* $x \in \left(B_R(0) \setminus \overline{B}_\delta(0)\right) \cap \{x_1 < 0\}$.
*Then there exists* $\tilde{u} \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ *such that*

$$\Delta \tilde{u} = \frac{|\tilde{u}|^{2^\star-2}\tilde{u}}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-)$$

*and*

$$\lim_{\epsilon\to 0} \tilde{u}_\epsilon = \tilde{u} \text{ in } \left\{ \begin{array}{ll} C^1_{loc}(\overline{\mathbb{R}^n_-}) & \text{if (L1) holds} \\ C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\}) & \text{if (L2) holds} \end{array}\right.$$

*Proof of Lemma 2.2:* Let $\eta \in C^\infty(\mathbb{R}^n)$. As easily checked, we have that

$$\eta \tilde{u}_\epsilon \in H^2_{1,0}(\mathbb{R}^n_-)$$

for all $\epsilon > 0$ small enough, and

$$\nabla(\eta\tilde{u}_\epsilon)(x) = \tilde{u}_\epsilon \nabla\eta + \eta \ell_\epsilon \nu_\epsilon^{\frac{n-2}{2}} D_{(\ell_\epsilon x)}\varphi[(\nabla u_\epsilon)(\varphi(\ell_\epsilon x))],$$

for all $\epsilon > 0$ and all $x \in \mathbb{R}^n_-$. In this expression, $D_x\varphi$ is the differential of the function $\varphi$ at $x$. We get that

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{u}_\epsilon)|^2\, dx \leq 2\int_{\mathbb{R}^n_-} |\nabla\eta|^2 \tilde{u}_\epsilon^2\, dx$$
$$+2\ell_\epsilon^2\nu_\epsilon^{n-2} \int_{\mathbb{R}^n_- \cap \text{Supp } \eta} |D_{(\ell_\epsilon x)}\varphi[(\nabla u_\epsilon)(\varphi(\ell_\epsilon x))]|^2\, dx.$$

With Hölder's inequality and a change of variables, we get that

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{u}_\epsilon)|^2\, dx \leq 2\left(\int_{\mathbb{R}^n_-} |\nabla\eta|^n\, dx\right)^{\frac{2}{n}} \cdot \left(\int_{\mathbb{R}^n_- \cap \text{Supp } \nabla\eta} |\tilde{u}_\epsilon|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}$$
$$+4\ell_\epsilon^n \left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2} \int_{\mathbb{R}^n_- \cap \text{Supp } \eta} |\nabla u_\epsilon|^2(\varphi(\ell_\epsilon x))\, dx$$
$$\leq 2\|\nabla\eta\|_n^2 \|\tilde{u}_\epsilon\|^2_{L^{\frac{2n}{n-2}}(\text{Supp } \nabla\eta)}$$
$$+C\nu_\epsilon^{\frac{p_\epsilon(n-2)}{2^\star-2}} \int_\Omega |\nabla u_\epsilon|^2\, dx \tag{26}$$

With another change of variables, we get that

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{u}_\epsilon)|^2\, dx \leq C\nu_\epsilon^{\frac{(n-2)p_\epsilon}{2^\star-2}} \|\nabla\eta\|_n^2 \|u_\epsilon\|^2_{L^{\frac{2n}{n-2}}(\Omega)}$$
$$+C\nu_\epsilon^{\frac{p_\epsilon(n-2)}{2^\star-2}} \int_\Omega |\nabla u_\epsilon|^2\, dx \tag{27}$$

for all $\epsilon > 0$, where $C$ is independant of $\epsilon$. With (10), Sobolev's inequality and since $\nu_\epsilon^{p_\epsilon} \leq 1$ for all $\epsilon > 0$ small enough, we get with (27) that

$$\|\eta\tilde{u}_\epsilon\|_{H^2_{1,0}(\mathbb{R}^n_-)} = O(1)$$

when $\epsilon \to 0$. It then follows that there exists $\tilde{u}_\eta \in H^2_{1,0}(\mathbb{R}^n_-)$ such that, up to a subsequence,

$$\eta\tilde{u}_\epsilon \rightharpoonup \tilde{u}_\eta$$

weakly in $H^2_{1,0}(\mathbb{R}^n_-)$ when $\epsilon \to 0$. We let $\eta_1 \in C^\infty_c(\mathbb{R}^n)$ such that $\eta_1 \equiv 1$ in $B_1(0)$ and $\eta_1 \equiv 0$ in $\mathbb{R}^n \setminus \overline{B}_2(0)$. For any $R \in \mathbb{N}^\star$, we let $\eta_R(x) = \eta_1(\frac{x}{R})$ for all $x \in \mathbb{R}^n$. With a diagonal argument, we can assume that, up to a subsequence, for any $R > 0$, there exists $\tilde{u}_R \in H^2_{1,0}(\mathbb{R}^n_-)$ such that

$$\eta_R\tilde{u}_\epsilon \rightharpoonup \tilde{u}_R$$

weakly in $H^2_{1,0}(\mathbb{R}^n_-)$ when $\epsilon \to 0$. Letting $\epsilon \to 0$ in (27), with (10), Sobolev's inequality and since $\nu_\epsilon^{p_\epsilon} \leq 1$ for all $\epsilon > 0$ small enough, we get that there exists a constant $C > 0$ independant of $R$ such that

$$\int_{\mathbb{R}^n_-} |\nabla\tilde{u}_R|^2\,dx \leq C\|\nabla\eta_R\|_n^2 + C$$

for all $R > 0$. Since $\|\nabla\eta_R\|_n^2 = \|\nabla\eta_1\|_n^2$ for all $R > 0$, we get that there exists $C > 0$ independant of $R$ such that

$$\int_{\mathbb{R}^n_-} |\nabla\tilde{u}_R|^2\,dx \leq C$$

for all $R > 0$. It then follows that there exists $\tilde{u} \in H^2_{1,0}(\mathbb{R}^n_-)$ such that $\tilde{u}_R \rightharpoonup \tilde{u}$ weakly in $H^2_{1,0}(\mathbb{R}^n_-)$ when $R \to +\infty$. As easily checked, we then obtain that $\tilde{u}_\eta = \eta\tilde{u}$ (we omit the proof of this fact. It is straightforward).

For any $i, j = 1, ..., n$, we let $(\tilde{g}_\epsilon)_{ij} = (\partial_i\varphi(\ell_\epsilon x), \partial_j\varphi(\ell_\epsilon x))$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^n$. We consider $\tilde{g}_\epsilon$ as a metric on $\mathbb{R}^n$. We let

$$\Delta_{\tilde{g}_\epsilon} = -\tilde{g}_\epsilon^{ij}\left(\partial_{ij} - \Gamma^k_{ij}(\tilde{g}_\epsilon)\partial_k\right),$$

where $\tilde{g}_\epsilon^{ij} := (\tilde{g}_\epsilon^{-1})_{ij}$ are the coordinates of the inverse of the tensor $\tilde{g}_\epsilon$ and the $\Gamma^k_{ij}(\tilde{g}_\epsilon)$'s are the Christoffel symbols of the metric $\tilde{g}_\epsilon$. With a change of variable, equation $(E_\epsilon)$ rewrites as

$$\Delta_{\tilde{g}_\epsilon}(\eta_R\tilde{u}_\epsilon) + \ell_\epsilon^2 a_\epsilon\circ\varphi(\ell_\epsilon x)\eta_R\tilde{u}_\epsilon = \frac{|\eta_R\tilde{u}_\epsilon|^{2^\star-2-p_\epsilon}\eta_R\tilde{u}_\epsilon}{\left|\frac{\varphi(\ell_\epsilon x)}{\ell_\epsilon}\right|^s} \text{ in } \mathcal{D}'(B_R(0)\cap\{x_1 < 0\}) \quad (28)$$

for all $\epsilon > 0$. Passing to the weak limit $\epsilon \to 0$ and then $R \to +\infty$ in this equation, we get that

$$\Delta\tilde{u} = \frac{|\tilde{u}|^{2^\star-2}\tilde{u}}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-).$$

Since $\tilde{u} \in H^2_{1,0}(\mathbb{R}^n_-)$, it follows from Proposition 8.1 of the Appendix that $\tilde{u} \in C^{1,\theta}(\overline{\mathbb{R}^n_-})$ for all $\theta \in (0, \min\{1, 2^\star - s\})$.

We deal with case (L1). Since $s \in (0,2)$, (L1) and (28) hold and $\tilde{u}_\epsilon \equiv 0$ on $\{x_1 = 0\}$, it follows from arguments similar to the ones developed in the Appendix that for any $\theta \in (0, \min\{1, 2^\star - s\})$ and any $R > 0$, there exists $C(\theta, R) > 0$ independant of $\epsilon > 0$ small such that

$$\|\tilde{u}_\epsilon\|_{C^{1,\theta}(B_R(0)\cap\{x_1\leq 0\})} \leq C(\theta, R)$$

for all $\epsilon > 0$ small. It then follows from Ascoli's theorem that for any $\theta \in (0, \min\{1, 2^\star - s\})$,

$$\lim_{\epsilon\to 0} \tilde{u}_\epsilon = \tilde{u}$$

in $C^{1,\theta}_{loc}(\overline{\mathbb{R}^n_-})$. The proof proceeds similarly in Case (L2). This ends the proof of the Lemma. □

## 3. Construction and exhaustion of the blow-up scales

This section is devoted to the proof of the following proposition:

**Proposition 3.1.** *We let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. We assume that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. We assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^\infty(\Omega)} = +\infty.$$

*Then there exists $N \in \mathbb{N}^\star$, there exists $N$ families of points $(\mu_{\epsilon,i})_{\epsilon>0}$ such that we have that*

**(A1)** $\lim_{\epsilon\to 0} u_\epsilon = u_0$ *in $C^2_{loc}(\overline{\Omega} \setminus \{0\})$ where $u_0$ is as in (11),*

**(A2)** $0 < \mu_{\epsilon,1} < ... < \mu_{\epsilon,N}$ *for all $\epsilon > 0$,*

**(A3)**

$$\lim_{\epsilon\to 0} \mu_{\epsilon,N} = 0 \text{ and } \lim_{\epsilon\to 0} \frac{\mu_{\epsilon,i+1}}{\mu_{\epsilon,i}} = +\infty \text{ for all } i = 1...N-1$$

**(A4)** *For all $i = 1...N$, there exists $\tilde{u}_i \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-}) \setminus \{0\}$ such that*

$$\Delta \tilde{u}_i = \frac{|\tilde{u}_i|^{2^\star-2}\tilde{u}_i}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-)$$

*and*

$$\lim_{\epsilon\to 0} \tilde{u}_{\epsilon,i} = \tilde{u}_i$$

*in $C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$, where*

$$\tilde{u}_{\epsilon,i}(x) := \mu_{\epsilon,i}^{\frac{n-2}{2}} u_\epsilon(\varphi(k_{\epsilon,i}x))$$

*for all $x \in \frac{U}{k_{\epsilon,i}} \cap \{x_1 \leq 0\}$ and $k_{\epsilon,i} = \mu_{\epsilon,i}^{1-\frac{p_\epsilon}{2^\star-2}}$. Moreover, $\lim_{\epsilon\to 0} \tilde{u}_{\epsilon,1} = \tilde{u}_1$ in $C^1_{loc}(\overline{\mathbb{R}^n_-})$.*

**(A5)**

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{|x|\geq Rk_{\epsilon,N}} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star-2}} = 0$$

**(A6)** *For any $\delta > 0$ and any $i = 1...N-1$, we have that*

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\delta k_{\epsilon,i+1}\geq|x|\geq Rk_{\epsilon,i}} |x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{\epsilon,i+1}^{-\frac{n-2}{2}} u_{i+1}\left(\frac{\varphi^{-1}(x)}{k_{\epsilon,i+1}}\right)\right|^{1-\frac{p_\epsilon}{2^\star-2}} = 0.$$

**(A7)** *For any $i \in \{1, ..., N\}$, there exists $\alpha_i \in (0,1]$ such that*

$$\lim_{\epsilon\to 0}\mu_{\epsilon,i}^{p_\epsilon}=\alpha_i.$$

The proof of this proposition proceeds in seven steps.

**Step 3.1:** We let $x_{\epsilon,1}\in\Omega$ and $\mu_{\epsilon,1},k_{\epsilon,1}>0$ such that

$$\max_\Omega|u_\epsilon|=|u_\epsilon(x_{\epsilon,1})|=\mu_{\epsilon,1}^{-\frac{n-2}{2}}\text{ and }k_{\epsilon,1}=\mu_{\epsilon,1}^{1-\frac{p_\epsilon}{2^\star-2}}. \tag{29}$$

We claim that

$$|x_{\epsilon,1}|=O(k_{\epsilon,1}) \tag{30}$$

when $\epsilon\to 0$, and in particular that $\lim_{\epsilon\to 0}x_\epsilon=0$. Indeed, we use Lemma 2.1 with $y_\epsilon=x_{\epsilon,1}$, $\nu_\epsilon=\mu_{\epsilon,1}$ and $C(R)=1$. We then immediately get that $|x_{\epsilon,1}|=O(k_{\epsilon,1})$ when $\epsilon\to 0$.

From now on, we let $\varphi:U\to V$ as in (13) with $y_0=0$ and $U,V$ are open neighborhoods of 0 in $\mathbb{R}^n$. We then let

$$x_{\epsilon,1}=\varphi(a_\epsilon,b_\epsilon), \tag{31}$$

where $a_\epsilon\in\{x_1<0\}$, $b_\epsilon\in\mathbb{R}^{n-1}$ and $(a_\epsilon,b_\epsilon)\in U$. Note that $\lim_{\epsilon\to 0}(a_\epsilon,b_\epsilon)=(0,0)$.

**Step 3.2:** We claim that

$$d(x_{\epsilon,1},\partial\Omega)=(1+o(1))|a_\epsilon|=O(k_{\epsilon,1}) \tag{32}$$

where $\lim_{\epsilon\to 0}o(1)=0$.

*Proof of the Claim:* Indeed, since $0\in\partial\Omega$, we get with (30) that

$$d(x_{\epsilon,1},\partial\Omega)\leq|x_{\epsilon,1}-0|=O(k_{\epsilon,1}) \tag{33}$$

when $\epsilon\to 0$. We first remark that

$$d(x_{\epsilon,1},\partial\Omega)\leq d(x_{\epsilon,1},\varphi(0,b_\epsilon))=|a_\epsilon|.$$

We let $\gamma_\epsilon\in\mathbb{R}^{n-1}$ such that $(0,\gamma_\epsilon)\in U\cap\{x_1=0\}$ and $Y_\epsilon=\varphi(0,\gamma_\epsilon)\in\partial\Omega$ such that $d(x_{\epsilon,1},\partial\Omega)=|x_{\epsilon,1}-Y_\epsilon|$. Since $d(x_{\epsilon,1},\partial\Omega)\leq|a_\epsilon|$, we get that

$$b_\epsilon-\gamma_\epsilon=O(|a_\epsilon|),$$

when $\epsilon\to 0$. Since $\nabla\varphi_0(0)=0$ (where $\varphi_0$ is as in (13)), we get that

$$\varphi_0(b_\epsilon)=\varphi_0(\gamma_\epsilon)+o(|b_\epsilon-\gamma_\epsilon|)=\varphi_0(\gamma_\epsilon)+o(|a_\epsilon|)$$

when $\epsilon\to 0$. Moreover,

$$\begin{aligned} d(x_{\epsilon,1},\partial\Omega) &= |x_{\epsilon,1}-Y_\epsilon|\\ &= |(a_\epsilon+\varphi_0(b_\epsilon)-\varphi_0(\gamma_\epsilon),b_\epsilon-\gamma_\epsilon)|\\ &= |(a_\epsilon+o(a_\epsilon),b_\epsilon-\gamma_\epsilon)|\leq|a_\epsilon|\end{aligned}$$

when $\epsilon\to 0$. It then follows that $b_\epsilon-\gamma_\epsilon=o(|a_\epsilon|)$ and $d(x_{\epsilon,1},\partial\Omega)=(1+o(1))|a_\epsilon|$ when $\epsilon\to 0$. This prove (32). □

The classical Hardy-Sobolev inequality asserts that there exists $C>0$ such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star}}{|x|^s}\,dx\right)^{\frac{2}{2^\star}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2\,dx \tag{34}$$

for all $u \in H_{1,0}^2(\mathbb{R}^n)$. We define

$$\mu_s(\mathbb{R}^n_-) := \inf \frac{\int_{\mathbb{R}^n_-} |\nabla u|^2\,dx}{\left(\int_{\mathbb{R}^n_-} \frac{|u|^{2^\star}}{|x|^s}\,dx\right)^{\frac{2}{2^\star}}} \tag{35}$$

where the infimum is taken over functions $u \in H_{1,0}^2(\mathbb{R}^n_-) \setminus \{0\}$. The existence of $\mu_s(\mathbb{R}^n_-) > 0$ is a consequence of (34).

**Step 3.3:** The construction of the $(\mu_{\epsilon,i})$'s proceeds by induction. This step is the initiation.

**Lemma 3.1.** *We let*

$$\tilde{u}_{\epsilon,1}(x) := \mu_{\epsilon,1}^{\frac{n-2}{2}} u_\epsilon \circ \varphi(k_{\epsilon,1} x)$$

*for all $\epsilon > 0$ and all $x \in \frac{U}{k_{\epsilon,1}} \cap \{x_1 \leq 0\}$. Then, there exists $\tilde{u}_1 \in H_{1,0}^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ such that*

**(B1)** $\lim_{\epsilon\to 0} \tilde{u}_{\epsilon,1} = \tilde{u}_1$ *in* $C^1_{loc}(\overline{\mathbb{R}^n_-})$,

**(B2)**

$$\Delta \tilde{u}_1 = \frac{|\tilde{u}_1|^{2^\star-2}\tilde{u}_1}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-),$$

**(B3)**

$$\int_{\mathbb{R}^n_-} |\nabla \tilde{u}_1|^2\,dx \geq \mu_s(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}.$$

*Moreover, there exists $\alpha_1 \in (0,1]$ such that* $\lim_{\epsilon\to 0} \mu_{\epsilon,1}^{p_\epsilon} = \alpha_1$.

*Proof of Lemma 3.1:* Indeed, since $|\tilde{u}_{\epsilon,1}(x)| \leq 1$ for all $x \in \frac{U}{k_{\epsilon,1}} \cap \{x_1 \leq 0\}$, hypothesis (L1) of Lemma 2.2 is satisfied and it follows from Lemma 2.2 that points (B1) and (B2) hold. We let $\lambda_\epsilon = -\frac{a_\epsilon}{k_{\epsilon,1}} > 0$ and $\theta_\epsilon = \frac{b_\epsilon}{k_{\epsilon,1}} \in \mathbb{R}^{n-1}$, where $a_\epsilon, b_\epsilon$ are defined in (31). It follows from Steps 3.1 and 3.2 that there exists $\lambda_0 \geq 0$ and $\theta_0 \in \mathbb{R}^{n-1}$ such that $\lim_{\epsilon\to 0}(\lambda_\epsilon, \theta_\epsilon) = (\lambda_0, \theta_0)$. It then follows from the definition of $\tilde{u}_{\epsilon,1}$ and (29) that

$$|\tilde{u}_{\epsilon,1}(-\lambda_\epsilon, \theta_\epsilon)| = 1.$$

Passing to the limit $\epsilon \to 0$ and using point (B1), we get that $|\tilde{u}_1(-\lambda_0, \theta_0)| = 1$. In particular $\tilde{u}_1 \not\equiv 0$ and $\lambda_0 \neq 0$. Multiplying (B2) by $\tilde{u}_1$ and integrating by parts over $\mathbb{R}^n_-$, we get that

$$\int_{\mathbb{R}^n_-} |\nabla \tilde{u}_1|^2\,dx = \int_{\mathbb{R}^n_-} \frac{|u|^{2^\star}}{|x|^s}\,dx.$$

Using the Hardy-Sobolev inequality (35) and that $\tilde{u}_1 \not\equiv 0$, we get (B3). At last, with (10), (27) and Sobolev's inequality, we get that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^n_-} |\nabla(\eta \tilde{u}_{\epsilon,1})|^2\,dx \leq C \mu_{\epsilon,1}^{\frac{(n-2)p_\epsilon}{2^\star-2}}$$

for all $\epsilon > 0$. Letting $\epsilon \to 0$ and using that $\tilde{u}_1 \not\equiv 0$, we get that $\lim_{\epsilon\to 0} \mu_{\epsilon,1}^{p_\epsilon} > 0$. □

**Step 3.4:** We claim that there exists $C > 0$ such that

$$|x|^{\frac{n-2}{2}} |u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star-2}} \leq C \tag{36}$$

for all $\epsilon > 0$ and all $x \in \Omega$.

*Proof of the Claim:* We argue by contradiction and we let $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that

$$\sup_{x\in\Omega} |x|^{\frac{n-2}{2}} |u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star-2}} = |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star-2}} \to +\infty \tag{37}$$

when $\epsilon \to 0$. We let

$$\nu_\epsilon := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}}$$

for all $\epsilon > 0$. It follows from (37) that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon|}{\ell_\epsilon} = +\infty \text{ and } \lim_{\epsilon\to 0} \nu_\epsilon = 0. \tag{38}$$

We let

$$\beta_\epsilon := |y_\epsilon|^{\frac{s}{2}} |u_\epsilon(y_\epsilon)|^{\frac{2+p_\epsilon-2^\star}{2}}.$$

It follows from (37) that

$$\lim_{\epsilon\to 0} \frac{\beta_\epsilon}{|y_\epsilon|} = 0. \tag{39}$$

We let $R > 0$. We let $x \in B_R(0)$ such that $y_\epsilon + \beta_\epsilon x \in \Omega$. It follows from the definition (37) of $y_\epsilon$ that

$$|y_\epsilon + \beta_\epsilon x|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon + \beta_\epsilon x)| \leq |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon)|,$$

and then

$$\left(\frac{|u_\epsilon(y_\epsilon + \beta_\epsilon x)|}{|u_\epsilon(y_\epsilon)|}\right)^{1-\frac{p_\epsilon}{2^\star-2}} \leq \left(\frac{1}{1-\frac{\beta_\epsilon}{|y_\epsilon|}R}\right)^{\frac{n-2}{2}}$$

for all $\epsilon > 0$ and all $x \in B_R(0)$ such that $y_\epsilon + \beta_\epsilon x \in \Omega$. With (39), we get that there exists $\epsilon(R) > 0$ such that

$$|u_\epsilon(y_\epsilon + \beta_\epsilon x)| \leq 2|u_\epsilon(y_\epsilon)|$$

for all $x \in B_R(0)$ such that $y_\epsilon + \beta_\epsilon x \in \Omega$ and all $0 < \epsilon < \epsilon(R)$. It then follows from Lemma 2.1 that $y_\epsilon = O(\ell_\epsilon)$ when $\epsilon \to 0$. A contradiction with (38). This proves (36). □

As a remark, it follows from $(E_\epsilon)$, (11), (36) and standard elliptic theory that

$$\lim_{\epsilon\to 0} u_\epsilon = u_0 \text{ in } C^2_{loc}(\overline{\Omega} \setminus \{0\}). \tag{40}$$

We let $p \in \mathbb{N}^\star$. We consider the following assertions:

**(C1)** $0 < \mu_{\epsilon,1} < ... < \mu_{\epsilon,p}$

**(C2)**

$$\lim_{\epsilon\to 0} \mu_{\epsilon,p} = 0 \text{ and } \lim_{\epsilon\to 0} \frac{\mu_{\epsilon,i+1}}{\mu_{\epsilon,i}} = +\infty \text{ for all } i = 1...p-1$$

**(C3)** For all $i = 1...p$, there exists $\tilde{u}_i \in H^2_{1,0}(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-}) \setminus \{0\}$ such that

$$\Delta \tilde{u}_i = \frac{|\tilde{u}_i|^{2^\star-2}\tilde{u}_i}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-), \qquad \int_{\mathbb{R}^n_-} |\nabla \tilde{u}_i|^2\, dx \geq \mu_s(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}$$

and

$$\lim_{\epsilon\to 0} \tilde{u}_{\epsilon,i} = \tilde{u}_i$$

in $C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$, where

$$\tilde{u}_{\epsilon,i}(x) := \mu_{\epsilon,i}^{\frac{n-2}{2}} u_\epsilon(\varphi(k_{\epsilon,i}x))$$

for all $x \in \frac{U}{k_{\epsilon,i}} \cap \{x_1 \leq 0\}$ and $k_{\epsilon,i} := \mu_{\epsilon,i}^{1-\frac{p_\epsilon}{2^\star-2}}$.

**(C4)** For any $i \in \{1, ..., p\}$, there exists $\alpha_i \in (0, 1]$ such that

$$\lim_{\epsilon\to 0} \mu_{\epsilon,i}^{p_\epsilon} = \alpha_i.$$

We say that $\mathcal{H}_p$ holds if there exists $p$ families of points $(\mu_{\epsilon,i})_{\epsilon>0}$, $i = 1, ..., p$ such that $(\mu_{\epsilon,1})_{\epsilon>0}$ is as in (29) and points (C1), (C2) (C3) and (C4) hold. Note that it follows from Step 3.4 that $\mathcal{H}_1$ holds with the improvement that the convergence in (C3) holds in $C^1_{loc}(\overline{\mathbb{R}^n_-})$.

**Step 3.5:** We prove the following proposition:

**Proposition 3.2.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. Let $p \geq 1$. We assume that $\mathcal{H}_p$ holds. Then either*

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{|x|\geq Rk_{\epsilon,p}} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star-2}} = 0$$

*or $\mathcal{H}_{p+1}$ holds.*

*Proof of Proposition 3.2:* We assume that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{|x|\geq Rk_{\epsilon,p}} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star-2}} \neq 0.$$

It then follows that there exists a family $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{\epsilon,p}} = +\infty \text{ and } \lim_{\epsilon\to 0} |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon) - u_0(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star-2}} = \alpha > 0. \tag{41}$$

We claim that $\lim_{\epsilon\to 0} y_\epsilon = 0$. Otherwise, it follows from (40) that $\lim_{\epsilon\to 0} |u_\epsilon(y_\epsilon) - u_0(y_\epsilon)| = 0$. A contradiction.

Since $u_0 \in C^0(\overline{\Omega})$ and $\lim_{\epsilon\to 0} y_\epsilon = 0$, we get that

$$\lim_{\epsilon\to 0} |y_\epsilon|^{\frac{n-2}{2}} |u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star-2}} = \alpha > 0. \tag{42}$$

In particular, $\lim_{\epsilon\to 0} |u_\epsilon(y_\epsilon)| = +\infty$. We let

$$\mu_{\epsilon,p+1} := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } k_{\epsilon,p+1} := \mu_{\epsilon,p+1}^{1-\frac{p_\epsilon}{2^\star-2}}.$$

As a consequence, $\lim_{\epsilon\to 0} \mu_{\epsilon,p+1} = 0$. We define

$$\tilde{u}_{\epsilon,p+1}(x) := \mu_{\epsilon,p+1}^{\frac{n-2}{2}} u_\epsilon(\varphi(k_{\epsilon,p+1}x))$$

for all $x \in \frac{U}{k_{\epsilon,p+1}} \cap \{x_1 \leq 0\}$. It follows from (36) that

$$|\varphi(k_{\epsilon,p+1}x)|^{\frac{n-2}{2}} |u_\epsilon(\varphi(k_{\epsilon,p+1}x))|^{1-\frac{p_\epsilon}{2^\star-2}} \leq C$$

for all $x \in \frac{U}{k_{\epsilon,p+1}} \cap \{x_1 \leq 0\}$. With the definition of $\tilde{u}_{\epsilon,p+1}$ and the properties (13) of $\varphi$, we get that there exists $C > 0$ such that

$$|x|^{\frac{n-2}{2}} |\tilde{u}_{\epsilon,p+1}(x)|^{1-\frac{p_\epsilon}{2^\star-2}} \leq C$$

for all $x \in \frac{U}{k_{\epsilon,p+1}} \cap \{x_1 \leq 0\}$. It then follows that hypothesis (L2) of Lemma 2.2 is satisfied. It then follows from Lemma 2.2 that there exists $\tilde{u}_{p+1} \in H_{1,0}^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ such that

$$\Delta \tilde{u}_{p+1} = \frac{|\tilde{u}_{p+1}|^{2^\star-2}\tilde{u}_{p+1}}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-),$$

and

$$\lim_{\epsilon\to 0} \tilde{u}_{\epsilon,p+1} = \tilde{u}_{\epsilon,p+1} \tag{43}$$

in $C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$. It follows from (42) and the definition of $k_{\epsilon,p+1}$ that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{\epsilon,p+1}} = \alpha > 0.$$

We let $\tilde{y}_\epsilon \in \{x_1 < 0\}$ such that $y_\epsilon = \varphi(k_{\epsilon,p+1}\tilde{y}_\epsilon)$. It then exists $\tilde{y}_0 \in \overline{\mathbb{R}^n_-}$ such that $\lim_{\epsilon\to 0} \tilde{y}_\epsilon = \tilde{y}_0 \neq 0$. It then follows from (43) that

$$|\tilde{u}_{p+1}(y_0)| = \lim_{\epsilon\to 0} |\tilde{u}_{\epsilon,p+1}(\tilde{y}_\epsilon)| = 1,$$

and then $\tilde{u}_{p+1} \not\equiv 0$. With arguments similar to the ones developed in the proof of Lemma 3.1, we then get that

$$\int_{\mathbb{R}^n_-} |\nabla \tilde{u}_{p+1}|^2\, dx \geq \mu_s(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}$$

and there exists $\alpha_{p+1} \in (0,1]$ such that $\lim_{\epsilon\to 0} \mu_{\epsilon,p+1}^{p_\epsilon} = \alpha_{p+1}$. Moreover, it follows from (42), (41) and the definition of $\mu_{\epsilon,p+1}$ that

$$\lim_{\epsilon\to 0} \frac{\mu_{\epsilon,p+1}}{\mu_{\epsilon,p}} = +\infty \text{ and } \lim_{\epsilon\to 0} \mu_{\epsilon,p+1} = 0.$$

As easily checked, the families $(\mu_{\epsilon,i})_{\epsilon>0}$, $i \in \{1,...,p+1\}$ satisfy $\mathcal{H}_{p+1}$. □

**Step 3.6:** Next proposition is the equivalent of Proposition 3.2 at smaller scales.

**Proposition 3.3.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. Let $p \geq 1$. We assume that $\mathcal{H}_p$ holds. Then either for any $i \in \{1,...,p-1\}$ and for any $\delta > 0$*

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{x\in B_{\delta k_{\epsilon,i+1}}(0)\setminus \overline{B}_{Rk_{\epsilon,i}(0)}}|x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{\epsilon,i+1}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\varphi^{-1}(x)}{k_{\epsilon,i+1}}\right)\right|^{1-\frac{p_\epsilon}{2^\star-2}}=0$$

*or $\mathcal{H}_{p+1}$ holds.*

*Proof of Proposition 3.3:* We assume that there exist $i\leq p-1$, $\delta>0$ such that

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{x\in B_{\delta k_{\epsilon,i+1}}(0)\setminus \overline{B}_{Rk_{\epsilon,i}(0)}}|x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{\epsilon,i+1}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\varphi^{-1}(x)}{k_{\epsilon,i+1}}\right)\right|^{1-\frac{p_\epsilon}{2^\star-2}}>0.$$

It then follows that there exists a family $(y_\epsilon)_{\epsilon>0}\in\Omega$ such that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{k_{\epsilon,i}}=+\infty,\qquad |y_\epsilon|\leq\delta k_{\epsilon,i+1}\text{ for all }\epsilon>0 \tag{44}$$

$$\lim_{\epsilon\to 0}|y_\epsilon|^{\frac{n-2}{2}}\left|u_\epsilon(y_\epsilon)-\mu_{\epsilon,i+1}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\varphi^{-1}(y_\epsilon)}{k_{\epsilon,i+1}}\right)\right|^{1-\frac{p_\epsilon}{2^\star-2}}=\alpha>0. \tag{45}$$

We let $\tilde{y}_\epsilon\in\mathbb{R}^n_-$ such that $y_\epsilon=\varphi(k_{\epsilon,i+1}\tilde{y}_\epsilon)$. It follows from (44) that $|\tilde{y}_\epsilon|\leq 2\delta$ for all $\epsilon>0$. We claim that $\lim_{\epsilon\to 0}\tilde{y}_\epsilon=0$. Indeed, we rewrite (45) as

$$\lim_{\epsilon\to 0}|\tilde{y}_\epsilon|^{\frac{n-2}{2}}\left|\tilde{u}_{\epsilon,i+1}(\tilde{y}_\epsilon)-\tilde{u}_{i+1}(\tilde{y}_\epsilon)\right|^{1-\frac{p_\epsilon}{2^\star-2}}=\alpha>0.$$

A contradiction with point (C3) of $\mathcal{H}_p$ in case $\tilde{y}_\epsilon\not\to 0$ when $\epsilon\to 0$. Since $\tilde{u}_{i+1}\in C^0(\overline{\mathbb{R}^n_-})$, we then get that

$$|y_\epsilon|^{\frac{n-2}{2}}\left|\mu_{\epsilon,i+1}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\varphi^{-1}(y_\epsilon)}{k_{\epsilon,i+1}}\right)\right|^{1-\frac{p_\epsilon}{2^\star-2}}=O\left(\frac{|y_\epsilon|}{k_{\epsilon,i+1}}\right)^{\frac{n-2}{2}}=o(1)$$

when $\epsilon\to 0$. We rewrite (45) as

$$\lim_{\epsilon\to 0}|y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star-2}}=\alpha>0. \tag{46}$$

We let

$$\nu_\epsilon:=|u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}}\text{ and }\ell_\epsilon:=\nu_\epsilon^{1-\frac{p_\epsilon}{2^\star-2}}.$$

We define

$$\tilde{u}_\epsilon(x):=\nu_\epsilon^{\frac{n-2}{2}}u_\epsilon(\varphi(\ell_\epsilon x))$$

for all $x\in\frac{U}{\ell_\epsilon}\cap\{x_1\leq 0\}$. It follows from (36) that

$$|\varphi(\ell_\epsilon x)|^{\frac{n-2}{2}}|u_\epsilon(\varphi(\ell_\epsilon x))|^{1-\frac{p_\epsilon}{2^\star-2}}\leq C$$

for all $x\in\frac{U}{\ell_\epsilon}\cap\{x_1\leq 0\}$. With the definition of $\tilde{u}_\epsilon$ and the properties (13) of $\varphi$, we get that there exists $C>0$ such that

$$|x|^{\frac{n-2}{2}}|\tilde{u}_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star-2}}\leq C$$

for all $x\in\frac{U}{\ell_\epsilon}\cap\{x_1\leq 0\}$. It then follows that hypothesis (L2) of Lemma 2.2 is satisfied. It then follows from Lemma 2.2 that there exists $\tilde{u}\in H^2_{1,0}(\mathbb{R}^n_-)\cap C^1(\overline{\mathbb{R}^n_-})$ such that

$$\Delta \tilde{u} = \frac{|\tilde{u}|^{2^\star-2}\tilde{u}}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-),$$

and

$$\lim_{\epsilon\to 0} \tilde{u}_\epsilon = \tilde{u} \tag{47}$$

in $C^1_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0\})$. It follows from (46) and the definition of $\ell_\epsilon$ that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon} = \alpha > 0.$$

We let $\bar{y}_\epsilon \in \{x_1 < 0\}$ such that $y_\epsilon = \varphi(\ell_\epsilon \bar{y}_\epsilon)$. It then exists $\bar{y}_0 \in \overline{\mathbb{R}^n_-}$ such that $\lim_{\epsilon\to 0}\bar{y}_\epsilon = \bar{y}_0 \neq 0$. It follows from (47) and the definition of $\tilde{u}_\epsilon$ and $\tilde{y}_\epsilon$ that

$$|\tilde{u}(\bar{y}_0)| = \lim_{\epsilon\to 0}|\tilde{u}_\epsilon(\bar{y}_\epsilon)| = 1,$$

and then $\tilde{u} \not\equiv 0$. With arguments similar to the ones developed in the proof of Lemma 3.1, we then get that

$$\int_{\mathbb{R}^n_-} |\nabla \tilde{u}|^2\, dx \geq \mu_s(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}}$$

and there exists $\alpha \in (0,1]$ such that $\lim_{\epsilon\to 0}\nu_\epsilon^{p_\epsilon} = \alpha$. Moreover, it follows from (46), (44) and the definition of $\nu_\epsilon$ that

$$\lim_{\epsilon\to 0}\frac{\nu_\epsilon}{\mu_{\epsilon,i}} = +\infty \text{ and } \lim_{\epsilon\to 0}\frac{\mu_{\epsilon,i+1}}{\nu_\epsilon} = +\infty.$$

As easily checked, the families $(\mu_{\epsilon,1})$,..., $(\mu_{\epsilon,i})$, $(\nu_\epsilon)$, $(\mu_{\epsilon,i+1})$,..., $(\mu_{\epsilon,N})_{\epsilon>0}$ satisfy $\mathcal{H}_{p+1}$. □

**Step 3.7:** This last Step is the proof of Proposition 3.1.

**Proposition 3.4.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. We let $N_0 = \max\{p/\,\mathcal{H}_p$ holds$\}$. Then $N_0 < +\infty$ and the conclusion of Proposition 3.1 holds with $N = N_0$.*

*Proof of Proposition 3.4:* Indeed, assume that $\mathcal{H}_p$ holds. Let $\delta, R > 0$. Since $\mu_{\epsilon,i} = o(\mu_{\epsilon,i+1})$ for all $i \in \{1,...,N-1\}$, we then get with a change of variable and the definition of $\tilde{u}_{\epsilon,i}$ (see (C3)) that

$$\begin{aligned}
\int_\Omega |\nabla u_\epsilon|^2\, dx &\geq \sum_{i=1}^N \int_{\varphi(B_{Rk_{\epsilon,i}}(0)\setminus \overline{B}_{\delta k_{\epsilon,i}}(0))} |\nabla u_\epsilon|^2\, dx\\
&\geq \sum_{i=1}^N \mu_{\epsilon,i}^{-\frac{n-2}{2^\star-2}p_\epsilon}\int_{B_R(0)\setminus\overline{B}_\delta(0)} |\nabla \tilde{u}_{\epsilon,i}|^2_{g_{\epsilon,i}}\, dv_{g_{\epsilon,i}}\\
&\geq \sum_{i=1}^N \int_{B_R(0)\setminus\overline{B}_\delta(0)} |\nabla \tilde{u}_{\epsilon,i}|^2_{g_{\epsilon,i}}\, dv_{g_{\epsilon,i}}
\end{aligned}$$

where $g_{\epsilon,i}$ is the metric such that $(g_{\epsilon,i})_{qr} = (\partial_q\varphi(k_{\epsilon,i}x), \partial_r\varphi(k_{\epsilon,i}x))$ for all $q, r \in \{1,...,p\}$. Passing to the limit $\epsilon \to 0$ and using point (C3) of $\mathcal{H}_p$, we get that

$$\int_\Omega |\nabla u_\epsilon|^2\,dx \geq p\mu_s(\mathbb{R}^n_-)^{\frac{2^\star}{2^\star-2}} + o(1)$$

when $\epsilon\to 0$. With (10), we get that there exists $C>0$ such that

$$p \leq \Lambda^2 \mu_s(\mathbb{R}^n_-)^{-\frac{2^\star}{2^\star-2}}.$$

It then follows that $N_0 < +\infty$ exists.

We let families $(\mu_{\epsilon,1})_{\epsilon>0}$,..., $(\mu_{\epsilon,N_0})_{\epsilon>0}$ such that $\mathcal{H}_{N_0}$ holds. We argue by contradiction and assume that the conclusion of Proposition 3.1 does not hold with $N=N_0$. Assertions (A1), (A2), (A3) (A4) and (A7) hold. Assume that (A5) or (A6) does not hold. It then follows from Propositions 3.2 and 3.3 that $\mathcal{H}_{N+1}$ holds. A contradiction with the choice of $N=N_0$, and the proposition is proved. □

## 4. Strong pointwise estimates, Part 1

The objective of this section is the proof of the following strong pointwise estimate:

**Proposition 4.1.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$. We let $s\in(0,2)$. We let $(p_\epsilon)_{\epsilon>0}$ such that $p_\epsilon\in[0,2^\star-2)$ for all $\epsilon>0$ and (9) holds. We consider $(u_\epsilon)_{\epsilon>0}\in H^2_{1,0}(\Omega)$ such that (8), $(E_\epsilon)$ and (10) hold. We assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0}\|u_\epsilon\|_{L^\infty(\Omega)} = +\infty.$$

*We let $\mu_{\epsilon,1},...,\mu_{\epsilon,N}$ as in Proposition 3.1. Then, there exists $C>0$ such that*

$$|u_\epsilon(x)| \leq C\sum_{i=1}^N \frac{\mu_{\epsilon,i}^{\frac{n}{2}}|x|}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}}} + C|x| \tag{48}$$

*for all $\epsilon>0$ and all $x\in\Omega$.*

The proof of this estimate goes through seven steps. We let $s\in(0,2)$. We let $(p_\epsilon)_{\epsilon>0}$ such that $p_\epsilon\in[0,2^\star-2)$ for all $\epsilon>0$ and (9) holds. We consider $(u_\epsilon)_{\epsilon>0}\in H^2_{1,0}(\Omega)$ that satisfies the hypothesis of Proposition 4.1. We let $\mu_{\epsilon,1},...,\mu_{\epsilon,N}$ as in Proposition 3.1.

**Step 4.1:** We claim that for any $\nu\in(0,1)$ and any $R>0$, there exists $C(\nu,R)>0$ such that

$$|u_\epsilon(x)| \leq C(\nu,R)\cdot\left(\frac{\mu_{\epsilon,N}^{\frac{n}{2}-\nu(n-1)}d(x,\partial\Omega)^{1-\nu}}{|x|^{n(1-\nu)}} + d(x,\partial\Omega)^{1-\nu}\right) \tag{49}$$

for all $x\in\Omega\setminus\overline{B}_{Rk_{\epsilon,N}}(0)$ and all $\epsilon>0$.

*Proof of the Claim:* Since $\Delta$ is coercive on $\Omega$, we let $G$ be the Green's function for $\Delta$ in $\Omega$ with Dirichlet boundary condition. We let

$$H(x) = -\partial_\nu G(x,0)$$

for all $x\in\overline{\Omega}\setminus\{0\}$. Here $\nu$ denotes the outward normal vector at $\partial\Omega$. It follows from Theorem 9.2 of the Appendix that $H\in C^2(\overline{\Omega}\setminus\{0\})$, that

$$\Delta H = 0 \tag{50}$$

in $\Omega$ and that there exist $\delta_1, C_1>0$ such that

$$\frac{d(x,\partial\Omega)}{C_1|x|^n} \leq H(x) \leq \frac{C_1 d(x,\partial\Omega)}{|x|^n} \tag{51}$$

and

$$\frac{|\nabla H(x)|}{H(x)} \geq \frac{1}{C_1 d(x,\partial\Omega)} \geq \frac{1}{C_1|x|} \tag{52}$$

for all $x \in \Omega \cap B_{2\delta_1}(0)$.

Since $\Delta$ is coercive, we let $\lambda_1 > 0$ be the first eigenvalue of $\Delta$ on $\Omega$, and we let $\psi \in C^2(\overline{\Omega})$ be the unique eigenfunction such that

$$\left\{ \begin{array}{ll} \Delta\psi = \lambda_1\psi & \text{in } \Omega \\ \psi > 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \\ \int_\Omega \psi^2\, dx = 1 & \end{array} \right\}$$

It follows from standard elliptic theory and Hopf's maximum principle that there exists $C_2', \delta_2 > 0$ such that

$$\frac{1}{C_2'} d(x,\partial\Omega) \leq \psi(x) \leq C_2' d(x,\partial\Omega) \text{ and } \frac{1}{C_2'} \leq |\nabla\psi(x)| \leq C_2'$$

for all $x \in \Omega \cap B_{2\delta_2}(0)$. Consequently, there exists $C_2 > 0$ such that

$$\frac{1}{C_2} d(x,\partial\Omega) \leq \psi(x) \leq C_2 d(x,\partial\Omega) \text{ and } \frac{|\nabla\psi(x)|}{\psi(x)} \geq \frac{1}{C_2 d(x,\partial\Omega)} \geq \frac{1}{C_2|x|} \tag{53}$$

for all $x \in \Omega \cap B_{2\delta_2}(0)$. We let the operator

$$L_\epsilon = \Delta + \left( a_\epsilon - \frac{|u_\epsilon|^{2^\star-2-p_\epsilon}}{|x|^s} \right).$$

*Step 4.1.1:* We claim that there exist $\delta_0 > 0$ and $R_0 > 0$ such that for any $\nu \in (0,1)$ and any $R > R_0$, $\delta \in (0,\delta_0)$, we have that

$$L_\epsilon H^{1-\nu} > 0, \text{ and } L_\epsilon \psi^{1-\nu} > 0 \tag{54}$$

for all $x \in \Omega \cap B_\delta(0) \setminus \overline{B}_{Rk_{\epsilon,N}}(0)$ and for all $\epsilon > 0$ sufficiently small. Indeed, with (50), we get that

$$\frac{L_\epsilon H^{1-\nu}}{H^{1-\nu}}(x) = a_\epsilon(x) + \nu(1-\nu)\frac{|\nabla H|^2}{H^2}(x) - \frac{|u_\epsilon(x)|^{2^\star-2-p_\epsilon}}{|x|^s} \tag{55}$$

for all $x \in \Omega \setminus \{0\}$ and all $\epsilon > 0$. We let $0 < \delta_0 \leq \min\{\delta_1, \delta_2\}$ such that

$$\left\{ \begin{array}{l} 2\delta_0^2 \sup_\Omega |a_\epsilon| \leq \frac{\nu(1-\nu)}{2\cdot\max\{C_1^2, C_2^2\}} \\ \\ 2^{2^\star+1}\delta_0^{2-s}\|u_0\|_{L^\infty(\Omega)}^{2^\star-2} < \frac{\nu(1-\nu)}{4\cdot\max\{C_1^2, C_2^2\}} \end{array} \right\} \tag{56}$$

for all $\epsilon > 0$. This choice is possible thanks to (8). It follows from point (A5) of Proposition 3.1 that there exists $R_0 > 0$ such that for any $R > R_0$, we have that

$$|x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star-2}} \leq \left( \frac{\nu(1-\nu)}{2^{2^\star+1}\max\{C_1^2, C_2^2\}} \right)^{\frac{1}{2^\star-2}}$$

for all $x \in \Omega \setminus \overline{B}_{Rk_{\epsilon,N}}(0)$ and all $\epsilon > 0$. We then get that

$$
\begin{aligned}
|x|^{2-s}|u_\epsilon(x)|^{2^\star-2-p_\epsilon} &\leq 2^{2^\star-1-p_\epsilon}|x|^{2-s}|u_\epsilon(x)-u_0(x)|^{2^\star-2-p_\epsilon}\\
&\quad +2^{2^\star-1-p_\epsilon}|x|^{2-s}|u_0(x)|^{2^\star-2-p_\epsilon}\\
&\leq 2^{-p_\epsilon}\frac{\nu(1-\nu)}{4\cdot\max\{C_1^2,C_2^2\}}\\
&\quad +2^{2^\star-1-p_\epsilon}\delta^{2-s}\|u_0\|_{L^\infty(\Omega)}^{2^\star-2-p_\epsilon}
\end{aligned}
$$

for all $x\in\Omega\setminus\overline{B}_{Rk_{\epsilon,N}}(0)$ and all $\epsilon>0$. We get with the choice (56) of $\delta_0$ that for any $\delta\in(0,\delta_0)$ and all $R>R_0$

$$
\begin{aligned}
|x|^{2-s}|u_\epsilon(x)|^{2^\star-2-p_\epsilon} &\leq \frac{\nu(1-\nu)}{4\cdot\max\{C_1^2,C_2^2\}}+2^{2^\star-1}\delta_0^{2-s}\|u_0\|_{L^\infty(\Omega)}^{2^\star-2}\\
&< \frac{\nu(1-\nu)}{2\cdot\max\{C_1^2,C_2^2\}}
\end{aligned}
$$

for all $x\in(B_\delta(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\Omega$ and all $\epsilon>0$ small enough. With (55) and (56), we get that

$$
\begin{aligned}
\frac{L_\epsilon H^{1-\nu}}{H^{1-\nu}}(x) &\geq \frac{\nu(1-\nu)}{C_1^2|x|^2}+a_\epsilon(x)-\frac{\nu(1-\nu)}{2C_1^2|x|^2}\\
&\geq \frac{\nu(1-\nu)-2C_1^2|x|^2|a_\epsilon(x)|}{2C_1^2|x|^2}>0
\end{aligned}
$$

for all $x\in(B_\delta(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\Omega$ and all $\epsilon>0$ small enough. We deal with the second inequality of (54). We have that

$$
\frac{L_\epsilon\psi^{1-\nu}}{\psi^{1-\nu}}(x)=a_\epsilon(x)+(1-\nu)\lambda_1+\nu(1-\nu)\frac{|\nabla\psi|^2}{\psi^2}(x)-\frac{|u_\epsilon(x)|^{2^\star-2-p_\epsilon}}{|x|^s}
$$

for all $x\in\Omega$. With (53) and (56) we get that

$$
\frac{L_\epsilon\psi^{1-\nu}}{\psi^{1-\nu}}(x)\geq\frac{\nu(1-\nu)-2C_2^2|a_\epsilon(x)|\delta^2+2(1-\nu)\lambda_1|x|^2C_2^2}{2C_2^2|x|^2}>0
$$

for all $x\in(B_\delta(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\Omega$ and all $\epsilon>0$. This proves the last inequality of (54).

*Step 4.1.2:* It follows from point (A4) of Proposition 3.1 that there exists $C_1(R)>0$ such that

$$
|u_\epsilon(x)|\leq C_1(R)\mu_{\epsilon,N}^{-\frac{n}{2}}d(x,\partial\Omega) \tag{57}
$$

for all $x\in\Omega\cap\partial B_{Rk_{\epsilon,N}}(0)$ and all $\epsilon>0$. It follows from point (A1) of Proposition 3.1 that there exists $C_2(\delta)>0$ such that

$$
|u_\epsilon(x)|\leq C_2(\delta)d(x,\partial\Omega) \tag{58}
$$

for all $x\in\Omega\cap\partial B_\delta(0)$ and all $\epsilon>0$. We let

$$
D_{\epsilon,R,\delta}:=(B_\delta(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\Omega.
$$

We let
$$\alpha_\epsilon := 2C_1(R)C_1^{1-\nu}R^{n-(n-1)\nu}\alpha_N^{-\frac{n-(n-1)\nu}{2^\star-2}}\mu_{\epsilon,N}^{\frac{n}{2}-\nu(n-1)}$$
and
$$\beta_\epsilon := 2\delta^\nu C_2(\delta)C_2^{1-\nu},$$
and
$$\varphi_\epsilon(x) = \alpha_\epsilon H^{1-\nu}(x) + \beta_\epsilon \psi^{1-\nu}(x)$$
for all $x \in \overline{D_{\epsilon,R,\delta}}$ and all $\epsilon > 0$. Here, $\alpha_N$ is as in point (A7) of Proposition 3.1. We claim that
$$|u_\epsilon(x)| \leq \varphi_\epsilon(x) \tag{59}$$
for all $\epsilon > 0$ and all $x \in \partial D_{\epsilon,R,\delta}$. Indeed, with inequalities (51) and (57), we get that for any $x \in \Omega \cap \partial B_{Rk_{\epsilon,N}}(0)$,
$$\frac{|u_\epsilon(x)|}{\alpha_\epsilon H(x)^{1-\nu}} \leq \frac{\mu_{\epsilon,N}^{\nu(n-1)-n} d(x,\partial\Omega)^\nu |x|^{n-n\nu}}{2R^{n-(n-1)\nu}\alpha_N^{-\frac{n-(n-1)\nu}{2^\star-2}}} \leq \frac{\mu_{\epsilon,N}^{-\frac{n-(n-1)\nu}{2^\star-2}p_\epsilon}}{2\alpha_N^{-\frac{n-(n-1)\nu}{2^\star-2}}} \leq 1$$
when $\epsilon \to 0$ with point (A7) of Proposition 3.1. Similarly, we have with (53) and (58) that
$$\frac{|u_\epsilon(x)|}{\beta_\epsilon \psi(x)^{1-\nu}} \leq \frac{d(x,\partial\Omega)^\nu}{2\delta^\nu} < 1$$
for all $x \in \Omega \cap \partial B_\delta(0)$ and all $\epsilon > 0$. On $\partial\Omega \cap (B_\delta(0) \setminus \overline{B}_{Rk_{\epsilon,N}}(0))$, we clearly have $\varphi_\epsilon(x) > |u_\epsilon(x)| = 0$. As easily checked, these assertions prove (59).

*Step 4.1.3:* We claim that $L_\epsilon$ verifies the following comparison maximum: if $\varphi \in C^2(D_{\epsilon,R,\delta}) \cap C^0(\overline{D_{\epsilon,R,\delta}})$, then
$$\left\{\begin{array}{ll} L_\epsilon\varphi \geq 0 & \text{in } D_{\epsilon,R,\delta} \\ \varphi \geq 0 & \text{on } \partial D_{\epsilon,R,\delta} \end{array}\right\} \Rightarrow \varphi \geq 0 \text{ in } D_{\epsilon,R,\delta}.$$
Indeed, we let $U_0$ be an open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset\subset U_0$. Since the operator $\Delta$ is coercive in $U_0$ (with boundary Dirichlet condition), we let $\tilde{G} \in C^2(U_0 \times U_0 \setminus \{(x,x)/x \in U_0\})$ be the Green's function for $\Delta$ with Dirichlet condition in $U_0$. In other words, $\tilde{G}$ satisfies
$$\Delta\tilde{G}(x,\cdot) = \delta_x$$
weakly in $\mathcal{D}(U_0)$. For the existence, we refer to Theorem 9.1 of Appendix B. Moreover, since $0 \in U_0$ is in the interior of the domain, there exists $\hat{\delta}_0 > 0$ and $C_0 > 0$ such that
$$\frac{1}{C_0|x|^{n-2}} \leq \tilde{G}(0,x) \leq \frac{C_0}{|x|^{n-2}}$$
and
$$\frac{|\nabla\tilde{G}(0,x)|}{\tilde{G}(0,x)} \geq \frac{C_0'}{|x|}$$
for all $\epsilon > 0$ and all $x \in B_{2\hat{\delta}_0}(0) \setminus \{0\}$. The proof of these estimates goes as in the proof of points (G9) and (G10) of Theorem 9.2. We refer to [15] for the details. With the same techniques as in Step 4.1.1, we get that for $R > 0$ large enough and $\delta > 0$ small enough, then
$$\tilde{G}^{1-\nu} > 0 \text{ and } L_\epsilon\tilde{G}^{1-\nu} > 0$$

in $\overline{D_{\epsilon,R,\delta}}$ for all $\epsilon > 0$. It then follows from [5] that $L_\epsilon$ verifies the above mentioned comparison principle.

*Step 4.1.4:* It follows from (54) and (59) that

$$\left\{\begin{array}{ll} L_\epsilon\varphi_\epsilon \geq 0 = L_\epsilon u_\epsilon & \text{in } D_{\epsilon,R,\delta} \\ \varphi_\epsilon \geq 0 = u_\epsilon & \text{on } \partial D_{\epsilon,R,\delta} \\ L_\epsilon\varphi_\epsilon \geq 0 = -L_\epsilon u_\epsilon & \text{in } D_{\epsilon,R,\delta} \\ \varphi_\epsilon \geq 0 = -u_\epsilon & \text{on } \partial D_{\epsilon,R,\delta} \end{array}\right\}.$$

It follows from the above comparison principle that

$$|u_\epsilon(x)| \leq \varphi_\epsilon(x)$$

for all $x \in D_{\epsilon,R,\delta}$. With (51), we then get that (49) holds on $D_{\epsilon,R,\delta} = (B_\delta(0) \setminus \overline{B}_{Rk_{\epsilon,N}}(0)) \cap \Omega$ for $R$ large and $\delta$ small. It follows from this last assertion, (51) and points (A1) and (A4) of Proposition 3.1 that (49) holds on $\Omega \setminus \overline{B}_{Rk_{\epsilon,N}}(0)$ for all $R > 0$. □

**Step 4.2:** Let $i \in \{1, ..., N-1\}$. We claim that for any $\nu \in (0,1)$ and any $R, \rho > 0$, there exists $C(\nu, R, \rho) > 0$ such that

$$|u_\epsilon(x)| \leq C(\nu, R, \rho) \cdot \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2}-\nu(n-1)} d(x, \partial\Omega)^{1-\nu}}{|x|^{n(1-\nu)}} + \mu_{i+1,\epsilon}^{\nu-\frac{n}{2}} d(x, \partial\Omega)^{1-\nu} \right) \tag{60}$$

for all $x \in B_{Rk_{\epsilon,i+1}}(0) \setminus \overline{B}_{\rho k_{\epsilon,i}}(0)$ and all $\epsilon > 0$.

*Proof of the Claim:* We let $i \in \{1, ..., N-1\}$. We follow the lines of the proof of Step 4.1. We let $H$ and $\psi$ as in Step 4.1. Recall that we then get that there exists $\delta_1 > 0$ and $C_1 > 0$ such that

$$\frac{d(x, \partial\Omega)}{C_1|x|^n} \leq H(x) \leq \frac{C_1 d(x, \partial\Omega)}{|x|^n} \tag{61}$$

and

$$\frac{|\nabla H(x)|}{H(x)} \geq \frac{1}{C_1 d(x, \partial\Omega)} \geq \frac{1}{C_1|x|} \tag{62}$$

for all $x \in B_{2\delta_1}(0) \setminus \{0\}$. Moreover there exists $C_2, \delta_2 > 0$ such that $\psi$ verifies

$$\frac{1}{C_2} d(x, \partial\Omega) \leq \psi(x) \leq C_2 d(x, \partial\Omega) \text{ and } \frac{|\nabla\psi(x)|}{\psi(x)} \geq \frac{1}{C_2|x|} \tag{63}$$

for all $x \in \Omega \cap B_{2\delta_2}(0)$. We let the operator

$$L_\epsilon = \Delta + \left( a_\epsilon - \frac{|u_\epsilon|^{2^\star-2-p_\epsilon}}{|x|^s} \right).$$

*Step 4.2.1:* We claim that there exist $\rho_0 > 0$ and $R_0 > 0$ such that for any $\nu \in (0,1)$ and any $R > R_0$, $\rho \in (0, \rho_0)$, we have that

$$L_\epsilon H^{1-\nu} > 0 \text{ and } L_\epsilon \psi^{1-\nu} > 0 \tag{64}$$

for all $x \in \Omega \cap (B_{\rho k_{\epsilon,i+1}}(0) \setminus \overline{B}_{Rk_{\epsilon,i}}(0))$ and for all $\epsilon > 0$ sufficiently small. Indeed, as in Step 4.1, we get that

$$\frac{L_\epsilon H^{1-\nu}}{H^{1-\nu}}(x) = a_\epsilon(x) + \nu(1-\nu)\frac{|\nabla H|^2}{H^2}(x) - \frac{|u_\epsilon(x)|^{2^\star-2-p_\epsilon}}{|x|^s} \tag{65}$$

for all $x \in \Omega \setminus \{0\}$ and all $\epsilon > 0$. We let $0 < \rho_0 < 1$ such that

$$2^{2^\star+1}\rho_0^{2-s}\|\tilde{u}_{i+1}\|_{L^\infty(B_2(0)\cap\mathbb{R}^n_-)}^{2^\star-2} < \frac{\nu(1-\nu)}{\max\{C_1^2, C_2^2\}} \tag{66}$$

for all $\epsilon > 0$. It follows from point (A6) of Proposition 3.1 that there exists $R_0 > 0$ such that for any $R > R_0$

$$|x|^{\frac{n-2}{2}}\left|u_\epsilon(x) - \mu_{\epsilon,i+1}^{\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\varphi^{-1}(x)}{k_{\epsilon,i+1}}\right)\right|^{1-\frac{p_\epsilon}{2^\star-2}} \leq \left(\frac{\nu(1-\nu)}{2^{2^\star+1}\max\{C_1^2, C_2^2\}}\right)^{\frac{1}{2^\star-2}}$$

for all $x \in \Omega \cap \left(B_{k_{\epsilon,i+1}}(0) \setminus \overline{B}_{Rk_{\epsilon,i}}(0)\right)$ and all $\epsilon > 0$. We then get that

$$\begin{aligned}
&|x|^{2-s}|u_\epsilon(x)|^{2^\star-2-p_\epsilon} \\
&\leq 2^{2^\star-1-p_\epsilon}|x|^{2-s}\left|u_\epsilon(x) - \mu_{\epsilon,i+1}^{\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{\varphi^{-1}(x)}{k_{\epsilon,i+1}}\right)\right|^{2^\star-2-p_\epsilon} \\
&+2^{2^\star-1-p_\epsilon}|x|^{2-s}\mu_{\epsilon,i+1}^{-\frac{n-2}{2}(2^\star-2)\cdot(1-\frac{p_\epsilon}{2^\star-2})}\sup_{B_2(0)\cap\mathbb{R}^n_-}|\tilde{u}_{i+1}|^{2^\star-2-p_\epsilon} \\
&\leq 2^{-p_\epsilon}\frac{\nu(1-\nu)}{4\cdot\max\{C_1^2, C_2^2\}} + 2^{2^\star-1-p_\epsilon}\left(\frac{|x|}{k_{\epsilon,i+1}}\right)^{2-s}\|\tilde{u}_{i+1}\|_{L^\infty(B_2(0)\cap\mathbb{R}^n_-)}^{2^\star-2-p_\epsilon}
\end{aligned}$$

for all $x \in \Omega \cap \left(B_{k_{\epsilon,i+1}}(0) \setminus \overline{B}_{Rk_{\epsilon,i}}(0)\right)$ and all $\epsilon > 0$. We then get with the choice (66) of $\rho_0$ that for any $\rho \in (0, \rho_0)$ and all $R > R_0$

$$\begin{aligned}
|x|^{2-s}|u_\epsilon(x)|^{2^\star-2-p_\epsilon} &\leq \frac{\nu(1-\nu)}{4\cdot\max\{C_1^2, C_2^2\}} + 2^{2^\star-1}\rho_0^{2-s}\|\tilde{u}_{i+1}\|_{L^\infty(B_2(0)\cap\mathbb{R}^n_-)}^{2^\star-2} + o(1) \\
&< \frac{\nu(1-\nu)}{2\cdot\max\{C_1^2, C_2^2\}}
\end{aligned}$$

for all $x \in (B_{\rho k_{\epsilon,i+1}}(0) \setminus \overline{B}_{Rk_{\epsilon,i}}(0)) \cap \Omega$ and all $\epsilon > 0$ small enough. Since (8) holds, we get with (65) that

$$\begin{aligned}
\frac{L_\epsilon H^{1-\nu}}{H^{1-\nu}}(x) &\geq \frac{\nu(1-\nu)}{C_1^2|x|^2} + a_\epsilon(x) - \frac{\nu(1-\nu)}{2C_1^2|x|^2} \\
&\geq \frac{\nu(1-\nu) + 2|x|^2C_1^2a_\epsilon(x)}{2C_1^2|x|^2} > 0
\end{aligned}$$

for all $x \in \Omega \cap \left(B_{\rho k_{\epsilon,i+1}}(0) \setminus \overline{B}_{Rk_{\epsilon,i}}(0)\right)$ and all $\epsilon > 0$ small enough. The proof of the second inequality of (64) goes similarly (see Step 4.1 for details). This proves (64).

*Step 4.2.2:* It follows from point (A4) of Proposition 3.1 that there exists $C_1(R) > 0$ and $C_2(\rho)$ such that

$$|u_\epsilon(x)| \le C_1(R)\mu_{\epsilon,i}^{-\frac{n}{2}} d(x,\partial\Omega) \qquad \text{for all } x\in\Omega\cap\partial B_{Rk_{\epsilon,i}}(0)$$

$$|u_\epsilon(x)| \le C_2(\rho)\mu_{\epsilon,i+1}^{-\frac{n}{2}} d(x,\partial\Omega) \qquad \text{for all } x\in\Omega\cap\partial B_{\rho k_{\epsilon,i+1}}(0).$$

We let

$$D_{\epsilon,R,\rho} := \left(B_{\rho k_{\epsilon,i+1}}(0)\setminus \overline{B}_{Rk_{\epsilon,i}}(0)\right)\cap\Omega.$$

We let

$$\alpha_\epsilon := 2C_1(R)C_1^{1-\nu}R^{n-\nu(n-1)}\alpha_i^{-\frac{n-\nu(n-1)}{2^\star-2}}\mu_{\epsilon,i}^{\frac{n}{2}-\nu(n-1)}$$

and

$$\beta_\epsilon := 2C_2(\rho)C_2^{1-\nu}\rho^\nu\alpha_{i+1}^{-\frac{\nu}{2^\star-2}}\mu_{\epsilon,i+1}^{-\frac{n}{2}+\nu},$$

and

$$\varphi_\epsilon(x) := \alpha_\epsilon H^{1-\nu}(x)+\beta_\epsilon\psi^{1-\nu}(x)$$

for all $x\in\overline{D_{\epsilon,R,\delta}}$ and all $\epsilon>0$. Here, the $\alpha_i$'s are as in Point (A7) of Proposition 3.1. Similarly to what was done in Step 4.1, we then get that

$$|u_\epsilon(x)|\le\varphi_\epsilon(x) \tag{67}$$

for all $\epsilon>0$ and all $x\in\partial D_{\epsilon,R,\rho}$. The operator $L_\epsilon$ verifies the comparison principle on $D_{\epsilon,R,\rho}$ as in Step 4.1.3. It then follows that

$$|u_\epsilon(x)|\le\varphi_\epsilon(x)$$

for all $x\in D_{\epsilon,R,\rho}$. With (61), we then get that (60) holds on $D_{\epsilon,R,\rho}$ for $R$ large and $\rho$ small. It follows from this last assertion and point (A4) of Proposition 3.1 that (60) holds on $\left(B_{\rho k_{\epsilon,i+1}}(0)\setminus\overline{B}_{Rk_{\epsilon,i}}(0)\right)\cap\Omega$ for all $R,\rho>0$. □

**Step 4.3:** As easily checked, it follows from (49), (60) and Proposition 3.1 that for any $\nu\in(0,1)$, there exists $C_\nu>0$ such that

$$|u_\epsilon(x)|\le C_\nu\sum_{i=1}^N\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|x|^{1-\nu}}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}(1-\nu)}}+C_\nu|x|^{1-\nu} \tag{68}$$

for all $x\in\Omega$ and all $\epsilon>0$. Note that we have used that $d(x,\partial\Omega)\le|x-0|=|x|$ for all $x\in\Omega$. We let $G$ be the Green's function of $\Delta$ on $\Omega$ with Dirichlet boundary condition. It follows from Green's representation formula and (68) that

$$\begin{aligned}|u_\epsilon(x)| &= \left|\int_\Omega G(x,y)\left(\frac{|u_\epsilon(y)|^{2^\star-2-p_\epsilon}u_\epsilon(y)}{|y|^s}-a_\epsilon(y)u_\epsilon(y)\right)dy\right| &(69)\\ &\le C\int_\Omega G(x,y)\left(\frac{|u_\epsilon(y)|^{2^\star-1-p_\epsilon}}{|y|^s}+1\right)dy\\ &\le C_\nu\sum_{i=1}^N\int_\Omega\frac{G(x,y)}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}(1-\nu)}}\right)^{2^\star-1-p_\epsilon}dy\\ &\quad+C_\nu\int_\Omega G(x,y)\left(|y|^{(1-\nu)(2^\star-1-p_\epsilon)-s}+1\right)dy &(70)\end{aligned}$$

**Step 4.4:** We claim that there exists $C>0$ such that

$$\int_\Omega G(x,y)\left(|y|^{(1-\nu)(2^\star-1-p_\epsilon)-s}+1\right)dy\le C|x| \tag{71}$$

*Proof of the Claim:* Indeed, we let $\psi_\epsilon \in H_{1,0}^p(\Omega)$ $(1 < p < \frac{n}{s})$ such that

$$\Delta \psi_\epsilon = |y|^{(1-\nu)(2^\star - 1 - p_\epsilon) - s} + 1 \text{ in } \mathcal{D}'(\Omega).$$

Here, $H_{1,0}^p(\Omega)$ denote the completion of $C_c^\infty(\Omega)$ for the norm $\|\cdot\| := \|\nabla \cdot\|_p$. Since $s \in (0,2)$, it follows from standard elliptic theory that for $\nu > 0$ small, $\psi_\epsilon \in C^1(\overline{\Omega})$ and that there exists $C > 0$ such that

$$\|\psi_\epsilon\|_{C^1(\overline{\Omega})} \leq C.$$

Since $\psi_\epsilon(0) = 0$, we get that

$$|\psi_\epsilon(x)| \leq C|x|$$

for all $x \in \Omega$. Moreover, since $s \in (0,2)$, we get with Green's representation formula that

$$\psi_\epsilon(x) = \int_\Omega G(x,y) \left( |y|^{(1-\nu)(2^\star - 1 - p_\epsilon) - s} + 1 \right) \, dy$$

for all $x \in \Omega$ and all $\epsilon > 0$. Inequation (71) then follows. □

**Step 4.5:** We let $i \in \{1, ..., N\}$. We claim that there exists $C > 0$ such that

$$\begin{aligned} &\int_\Omega \frac{G(x,y)}{|y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2} - (n-1)\nu} |y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2 + |y|^2\right)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy \\ &\leq C \frac{\mu_{\epsilon,i}^{\frac{n}{2}} |x|}{\left(\mu_{\epsilon,i}^2 + |x|^2\right)^{\frac{n}{2}}} \end{aligned} \tag{72}$$

for all $x \in \Omega$ such that $|x| \geq \mu_{\epsilon,i}$.

*Proof of the Claim:* Indeed, with point (G6) of Theorem 9.1 on the Green's function, we get that

$$\begin{aligned} &\int_\Omega \frac{G(x,y)}{|y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2} - (n-1)\nu} |y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2 + |y|^2\right)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy \\ &\leq C \int_\Omega \frac{|y|}{|x-y|^{n-1} |y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2} - (n-1)\nu} |y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2 + |y|^2\right)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy \\ &\leq I_{1,\epsilon}(x) + I_{2,\epsilon}(x). \end{aligned}$$

Here,

$$I_{i,\epsilon}(x) := C \int_{\Omega_i(x)} \frac{|y|}{|x-y|^{n-1} |y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2} - (n-1)\nu} |y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2 + |y|^2\right)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy$$

where

$$\Omega_1(x) = \Omega \cap \{|x-y| > |x|/2\} \text{ and } \Omega_2(x) = \Omega \cap \{|x-y| < |x|/2\}.$$

We compute these two integrals separately. We let $R > 0$ such that $\Omega \subset B_R(0)$. We have that

$$\begin{aligned} I_{1,\epsilon}(x) &\leq C|x|^{1-n}\int_{B_R(0)}|y|^{1-s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}(1-\nu)}}\right)^{2^\star-1-p_\epsilon}dy\\ &\leq C|x|^{1-n}\mu_{\epsilon,i}^{\frac{n}{2}}\int_{B_{\frac{R}{\mu_{\epsilon,i}}}(0)}|y|^{1-s}\left(\frac{|y|^{1-\nu}}{\left(1+|y|^2\right)^{\frac{n}{2}(1-\nu)}}\right)^{2^\star-1-p_\epsilon}dy\\ &\leq C'|x|^{1-n}\mu_{\epsilon,i}^{\frac{n}{2}} \end{aligned} \tag{73}$$

since $s\in(0,2)$ and up to taking $\nu>0$ small enough. Note that we have used here point (A7) of Proposition 3.1.

We deal with the second integral. Note that when $|x-y|\leq|x|/2$, we have that

$$\frac{|x|}{2}\leq|y|\leq\frac{3|x|}{2}.$$

Taking $\nu>0$ small enough, we then get that

$$\begin{aligned} I_{2,\epsilon}(x) &\leq C|x|^{1-s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|x|^{1-\nu}}{|x|^{n(1-\nu)}}\right)^{2^\star-1-p_\epsilon}\int_{\{|x-y|\leq|x|/2\}}|x-y|^{1-n}\,dy\\ &\leq C|x|^{1-n}\mu_{\epsilon,i}^{\frac{n}{2}}\cdot|x|^{n+1-s}\mu_{\epsilon,i}^{-\frac{n}{2}}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|x|^{1-\nu}}{|x|^{n(1-\nu)}}\right)^{2^\star-1-p_\epsilon}\\ &\leq C'|x|^{1-n}\mu_{\epsilon,i}^{\frac{n}{2}} \end{aligned} \tag{74}$$

since $|x|\geq\mu_{\epsilon,i}$. Plugging together (73) and (74), we get that

$$\int_\Omega\frac{G(x,y)}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}(1-\nu)}}\right)^{2^\star-1-p_\epsilon}dy\leq C|x|^{1-n}\mu_{\epsilon,i}^{\frac{n}{2}}$$

Since $|x|\geq\mu_{\epsilon,i}$, we get (72). □

**Step 4.6:** We let $i\in\{1,...,N\}$. We claim that there exists $C>0$ such that

$$\begin{aligned} &\int_\Omega\frac{G(x,y)}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}(1-\nu)}}\right)^{2^\star-1-p_\epsilon}dy\\ &\leq C\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|x|}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}}} \end{aligned} \tag{75}$$

for all $x\in\Omega$ such that $|x|\leq\mu_{\epsilon,i}$.

*Proof of the Claim:* Indeed, let $p\in(1,n/s)$. We let $\varphi_{\epsilon,i}\in H_{1,0}^p(\Omega)$ such that

$$\Delta\varphi_{\epsilon,i}=\frac{1}{|x|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu}|x|^{1-\nu}}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}(1-\nu)}}\right)^{2^\star-1-p_\epsilon}\quad\text{in }\mathcal{D}'(\Omega). \tag{76}$$

We let $\varphi : U \to V$ defined in (13) with $y_0 = 0$. We let

$$\tilde{\varphi}_{\epsilon,i}(x) = \mu_{\epsilon,i}^{\frac{n-2}{2}} \varphi_{\epsilon,i} \circ \varphi(\mu_{\epsilon,i} x)$$

for all $x \in \frac{U}{\mu_{\epsilon,i}} \cap \mathbb{R}^n_-$. We let $R > 0$ such that $\Omega \subset B_R(0)$. It follows from Green's representation formula and the estimate (G5) on the Green's function that for any $x \in \frac{U}{\mu_{\epsilon,i}} \cap \mathbb{R}^n_-$, we have that

$$\begin{aligned} |\tilde{\varphi}_{\epsilon,i}(x)| &\leq \mu_{\epsilon,i}^{\frac{n-2}{2}} \int_\Omega \frac{G(\varphi(\mu_{\epsilon,i}x), y)}{|y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu} |y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2 + |y|^2\right)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy \\ &\leq C \mu_{\epsilon,i}^{\frac{n-2}{2}} \int_\Omega \frac{1}{|\varphi(\mu_{\epsilon,i}x) - y|^{n-2} |y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2}-(n-1)\nu} |y|^{1-\nu}}{\left(\mu_{\epsilon,i}^2 + |y|^2\right)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy \\ &\leq C \int_{B_{R/\mu_{\epsilon,i}}(0)} \frac{1}{\left| \frac{\varphi(\mu_{\epsilon,i}x)}{\mu_{\epsilon,i}} - y \right|^{n-2} |y|^s} \left( \frac{|y|^{1-\nu}}{(1 + |y|^2)^{\frac{n}{2}(1-\nu)}} \right)^{2^\star - 1 - p_\epsilon} dy. \end{aligned}$$

Since $s \in (0, 2)$ and with the properties (13) of $\varphi$, we get that there exists $C > 0$ such that

$$|\tilde{\varphi}_{\epsilon,i}(x)| \leq C \tag{77}$$

for all $x \in B_3(0) \cap \mathbb{R}^n_-$ and all $\epsilon > 0$. We let the metric $(\tilde{g}_\epsilon)_{kl} = (\partial_k \varphi, \partial_l \varphi)(\mu_{\epsilon,i} x)$ for $k, l = 1, ..., n$. Equation (76) rewrites as

$$\Delta_{\tilde{g}_\epsilon} \tilde{\varphi}_{\epsilon,i} = c_{i,\epsilon} \frac{\beta_{\epsilon,i}(x)^{(1-\nu)(2^\star - 1 - p_\epsilon) - s}}{(1 + \beta_{\epsilon,i}(x)^2)^{\frac{n}{2}(1-\nu)(2^\star - 1 - p_\epsilon)}} \text{ in } \mathcal{D}'(B_3(0) \cap \mathbb{R}^n_-),$$

where $c_{i,\epsilon} \in \mathbb{R}$ for all $\epsilon > 0$ and $\lim_{\epsilon \to 0} c_{i,\epsilon} = c_i > 0$.

$$\beta_{\epsilon,i}(x) := \left| \frac{\varphi(\mu_{\epsilon,i} x)}{\mu_{\epsilon,i}} \right|$$

for all $x \in B_3(0) \cap \mathbb{R}^n_-$. In particular, there exists $C > 0$ such that

$$\frac{|x|}{C} \leq \beta_{\epsilon,i}(x) \leq C|x|$$

for all $x \in B_3(0) \cap \mathbb{R}^n_-$. Since (77) holds, $s \in (0, 2)$ and $\tilde{\varphi}_{\epsilon,i} \equiv 0$ on $\{x_1 = 0\}$, it follows from standard elliptic theory and the equation satisfies by $\tilde{\varphi}_{\epsilon,i}$ that there exists $C > 0$ such that

$$\|\tilde{\varphi}_{\epsilon,i}\|_{C^1(B_2(0) \cap \overline{\mathbb{R}^n_-})} \leq C$$

for all $\epsilon > 0$. Since $\tilde{\varphi}_{\epsilon,i}(0) = 0$, we get that

$$|\tilde{\varphi}^i_\epsilon(x)| \leq C|x|$$

for all $x \in B_2(0) \cap \mathbb{R}^n_-$ and all $\epsilon > 0$. Coming back to the definition of $\tilde{\varphi}_{\epsilon,i}$, we then get that there exists $C > 0$ such that

$$|\varphi_{\epsilon,i}(x)| \leq C \frac{\mu_{\epsilon,i}^{\frac{n}{2}} |x|}{\left(\mu_{\epsilon,i}^2 + |x|^2\right)^{\frac{n}{2}}}$$

for all $x \in \Omega \cap B_{\mu_{\epsilon,i}}(0)$. Inequality (75) then follows from Green's representation formula. $\square$

**Step 4.7:** Plugging together (71), (72) and (75) into (70), we get that

$$|u_\epsilon(x)| \leq C \sum_{i=1}^N \frac{\mu_{\epsilon,i}^{\frac{n}{2}}|x|}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}}} + C|x|$$

for all $x \in \Omega$ and all $\epsilon > 0$. This proves (48). □

## 5. Strong pointwise estimates, Part 2

This section is devoted to a refinement and a derivation of Proposition 4.1:

**Proposition 5.1.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. We let $s \in (0,2)$. We let $(p_\epsilon)_{\epsilon>0}$ such that $p_\epsilon \in [0, 2^\star - 2)$ for all $\epsilon > 0$ and (9) holds. We consider $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$ such that (8), $(E_\epsilon)$ and (10) hold. We assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^\infty(\Omega)} = +\infty.$$

*We let $\mu_{\epsilon,1}, ..., \mu_{\epsilon,N}$ as in Proposition 3.1. Then, there exists $C > 0$ such that*

$$|u_\epsilon(x)| \leq C \sum_{i=1}^N \frac{\mu_{\epsilon,i}^{\frac{n}{2}}|x|}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}}} + C|x| \tag{78}$$

$$|\nabla u_\epsilon(x)| \leq C \sum_{i=1}^N \frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{\left(\mu_{\epsilon,i}^2+|x|^2\right)^{\frac{n}{2}}} + C \tag{79}$$

*for all $\epsilon > 0$ and all $x \in \Omega$.*

Inequality (78) was proved in Proposition 4.1. We prove inequality (79). We let $G$ be the Green's function for the operator $\Delta$ on $\Omega$ with Dirichlet boundary condiction. Derivating Green's representation formula (69) that

$$\nabla u_\epsilon(x) = \int_\Omega \nabla_x G(x,y) \left( \frac{|u_\epsilon(y)|^{2^\star-2-p_\epsilon}u_\epsilon(y)}{|y|^s} - a_\epsilon(y)u_\epsilon(y)\right) dy$$

for all $x \in \Omega$ and all $\epsilon > 0$. It then follows from (78) that

$$\begin{aligned} |\nabla u_\epsilon(x)| &\leq C \int_\Omega |\nabla_x G(x,y)| \left( \frac{|u_\epsilon(y)|^{2^\star-1-p_\epsilon}}{|y|^s} + 1\right) dy \\ &\leq C \sum_{i=1}^N \int_\Omega \frac{|\nabla_x G(x,y)|}{|y|^s} \left( \frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy \\ &+ C \int_\Omega |\nabla_x G(x,y)| \cdot (|y|^{2^\star-1-s-p_\epsilon}+1)\, dy \end{aligned} \tag{80}$$

for all $x \in \Omega$ and all $\epsilon > 0$.

**Step 5.1:** We claim that there exists $C > 0$ such that

$$\int_\Omega |\nabla_x G(x,y)| \cdot \left(|y|^{2^\star-1-s-p_\epsilon}+1\right) dy \leq C \tag{81}$$

for all $x \in \Omega$ and all $\epsilon > 0$.

*Proof of the Claim:* Indeed, it follows from property (G7) of Theorem 9.1 that there exists $C>0$ such that

$$|\nabla_x G(x,y)| \leq C|x-y|^{1-n} \tag{82}$$

for all $x,y\in\Omega$ such that $x\neq y$. Since $s\in(0,2)$, we then obtain that there exists $C>0$ such that

$$\int_\Omega |\nabla_x G(x,y)|\cdot\left(|y|^{2^\star-1-s-p_\epsilon}+1\right)\,dy \leq C\int_\Omega |x-y|^{1-n}\cdot\left(|y|^{2^\star-1-s-p_\epsilon}+1\right)\,dy\leq C$$

for all $x\in\Omega$ and all $\epsilon>0$. This proves (81). □

**Step 5.2:** We let $i\in\{1,...,N\}$. We claim that there exists $C>0$ such that

$$\int_\Omega \frac{|\nabla_x G(x,y)|}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy \leq C\frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{(\mu_{\epsilon,i}^2+|x|^2)^{\frac{n}{2}}} \tag{83}$$

for all $x\in\Omega$ such that $|x|\leq\mu_{\epsilon,i}$ and all $\epsilon>0$.

*Proof of the Claim:* We let $\theta_\epsilon:=\frac{x}{\mu_{\epsilon,i}}$. Note that with our assumption, we have that $|\theta_\epsilon|\leq 1$. We let $R>0$ such that $\Omega\subset B_R(0)$. With (82), and a change of variables, we get that

$$\begin{aligned}&\int_\Omega \frac{|\nabla_x G(x,y)|}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy\\ &\leq C\int_{B_R(0)} |x-y|^{1-n}|y|^{-s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy\\ &\leq C\mu_{\epsilon,i}^{-\frac{n}{2}}\int_{B_{\frac{R}{\mu_{\epsilon,i}}}(0)} \frac{|z|^{2^\star-1-s-p_\epsilon}}{|\theta_\epsilon-z|^{n-1}(1+|z|^2)^{\frac{n}{2}(2^\star-1-p_\epsilon)}}\,dz\end{aligned}$$

Since $s\in(0,2)$ and $|\theta_\epsilon|\leq1$, we get that there exists $C>0$ such that

$$\int_\Omega \frac{|\nabla_x G(x,y)|}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy\leq C\mu_{\epsilon,i}^{-\frac{n}{2}}.$$

Since $|x|\leq\mu_{\epsilon,i}$, inequality (83) follows. □

**Step 5.3:** We let $i\in\{1,...,N\}$. We claim that there exists $C>0$ such that

$$\int_\Omega \frac{|\nabla_x G(x,y)|}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy \leq C\frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{(\mu_{\epsilon,i}^2+|x|^2)^{\frac{n}{2}}} \tag{84}$$

for all $x\in\Omega$ such that $|x|\geq\mu_{\epsilon,i}$ and all $\epsilon>0$.

*Proof of the Claim:* We split the integral in two parts:

$$\int_\Omega \frac{|\nabla_x G(x,y)|}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon} dy = I_{\epsilon,1}(x)+I_{\epsilon,2}(x) \tag{85}$$

where

$$I_{\epsilon,j}(x)=\int_{\Omega_{\epsilon,j}(x)}\frac{|\nabla_x G(x,y)|}{|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon}\,dy$$

and

$$\Omega_{\epsilon,1}(x)=\Omega\cap\left\{|x-y|\geq\frac{|x|}{2}\right\}\text{ and }\Omega_{\epsilon,1}(x)=\Omega\cap\left\{|x-y|<\frac{|x|}{2}\right\}$$

*Step 5.3.1:* We deal with $I_{\epsilon,1}(x)$. It follows from point (G8) of Theorem 9.1 that there exists $C>0$ such that

$$|\nabla_x G(x,y)|\leq C\frac{d(y,\partial\Omega)}{|x-y|^n}\leq C\frac{|y|}{|x-y|^n}$$

for all $x,y\in\Omega$, $x\neq y$. We let $R>0$ such that $\Omega\subset B_R(0)$. With a change of variable, we get that

$$\begin{aligned}
I_{\epsilon,1}(x) &\leq C\int_{\Omega\cap\left\{|x-y|\geq\frac{|x|}{2}\right\}}\frac{|y|}{|x-y|^n|y|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon}\,dy\\
&\leq C|x|^{-n}\int_{B_R(0)}|y|^{1-s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}|y|}{\left(\mu_{\epsilon,i}^2+|y|^2\right)^{\frac{n}{2}}}\right)^{2^\star-1-p_\epsilon}\,dy\\
&\leq C|x|^{-n}\mu_{\epsilon,i}^{\frac{n}{2}}\int_{B_{\frac{R}{\mu_{\epsilon,i}}}(0)}\frac{|z|^{2^\star-s-p_\epsilon}}{(1+|z|^2)^{\frac{n}{2}(2^\star-1-p_\epsilon)}}\,dz.
\end{aligned}$$

Since $|x|\geq\mu_{\epsilon,i}$ and $s\in(0,2)$, we then get that

$$I_{\epsilon,1}(x)\leq C|x|^{-n}\mu_{\epsilon,i}^{\frac{n}{2}}\leq C'\frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{(\mu_{\epsilon,i}^2+|x|^2)^{\frac{n}{2}}}. \tag{86}$$

*Step 5.3.2:* We deal with $I_{\epsilon,2}(x)$. As easily checked, we have that

$$\frac{|x|}{2}\leq|y|\leq\frac{3|x|}{2} \tag{87}$$

for all $y\in\Omega_{\epsilon,2}(x)$. With (82) and (87), we get that

$$\begin{aligned}
I_{\epsilon,2}(x) &\leq \frac{C}{|x|^s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{|x|^{n-1}}\right)^{2^\star-1-p_\epsilon}\int_{\left\{|x-y|<\frac{|x|}{2}\right\}}|x-y|^{1-n}\,dy\\
&\leq C|x|^{1-s}\left(\frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{|x|^{n-1}}\right)^{2^\star-1-p_\epsilon}\\
&\leq C|x|^{-n}\mu_{\epsilon,i}^{\frac{n}{2}}\frac{\mu_{\epsilon,i}^{\frac{n}{2}(2^\star-2-p_\epsilon)}}{|x|^{n(2^\star-1-p_\epsilon)-(2^\star-1-p_\epsilon)-n-1+s}}.
\end{aligned}$$

Since $|x|\geq\mu_{\epsilon,i}$ and $s\in(0,2)$, we then get that

$$I_{\epsilon,2}(x) \leq C|x|^{-n}\mu_{\epsilon,i}^{\frac{n}{2}} \leq C' \frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{(\mu_{\epsilon,i}^2 + |x|^2)^{\frac{n}{2}}}. \tag{88}$$

Plugging (86) and (88) into (85), we get (84). □

**Step 5.4:** Plugging (81), (83) and (84) into (80), we get inequality (79).

## 6. Pohozaev identity and proof of compactness

This section is mainly devoted to the proof of the following proposition:

**Proposition 6.1.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. We assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^\infty(\Omega)} = +\infty.$$

*Then we have that*

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{\epsilon,N}} = \frac{(n-s)\displaystyle\int_{\partial\mathbb{R}^n_-} II_0(x,x)|\nabla \tilde{u}_N|^2\,dx}{(n-2)^2 \alpha_N^{\frac{(n-1)(n-2)}{2(2-s)}} \displaystyle\sum_{i=1}^N \alpha_i^{-\frac{(n-2)^2}{2(2-s)}} \int_{\mathbb{R}^n_-} |\nabla \tilde{u}_i|^2\,dx}$$

*when $n \geq 3$. In this expression, $II_0$ is the second fondamental form at $0$ of the oriented boundary $\partial\Omega$ and $\partial\mathbb{R}^n_-$ is the oriented tangent space of $\partial\Omega$ at $0$. The sequences and families $\mu_{\epsilon,N} > 0$, $\alpha_i$, $\tilde{u}_i$, $i \in \{1,...,N\}$ are as in Proposition 3.1. In addition, if $u_\epsilon \geq 0$ for all $\epsilon \geq 0$, we have that*

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{\epsilon,N}} = \frac{(n-s)\displaystyle\int_{\partial\mathbb{R}^n_-} |x|^2|\nabla \tilde{u}_N|^2\,dx}{n(n-2)^2 \alpha_N^{\frac{(n-1)(n-2)}{2(2-s)}} \displaystyle\sum_{i=1}^N \alpha_i^{-\frac{(n-2)^2}{2(2-s)}} \int_{\mathbb{R}^n_-} |\nabla \tilde{u}_i|^2\,dx} \cdot H(0)$$

*when $n \geq 3$. In this expression, $H(0)$ is the mean curvature at $0$ of the oriented boundary $\partial\Omega$.*

We prove the proposition in Steps 6.1 to 6.3. We prove Theorem 1.3 in Step 6.4.

**Step 6.1:** We provide a Pohozaev-type identity for $u_\epsilon$. It follows from Proposition 8.1 that $u_\epsilon \in C^1(\overline{\Omega})$ and that $\Delta u_\epsilon \in L^p(\Omega)$ for all $p \in (1, \frac{n}{s})$. We let

$$W_\epsilon := \Omega \cap \varphi(B_{r_\epsilon}(0)), \text{ where } r_\epsilon = \sqrt{\mu_{\epsilon,N}}. \tag{89}$$

In the sequel, we denote by $\nu(x)$ the outward normal vector at $x \in \partial W_\epsilon$ of the oriented hypersurface $\partial W_\epsilon$ (oriented as the boundary of $W_\epsilon$). Integrating by parts, we get that

$$\begin{aligned}
&\int_{W_\epsilon} x^i\partial_i u_\epsilon \Delta u_\epsilon\, dx\\
&= -\int_{\partial W_\epsilon} x^i\partial_i u_\epsilon \partial_\nu u_\epsilon\, d\sigma + \int_{W_\epsilon} \partial_j(x^i\partial_i u_\epsilon)\partial_j u_\epsilon\, dx\\
&= -\int_{\partial W_\epsilon} x^i\partial_i u_\epsilon \partial_\nu u_\epsilon\, d\sigma + \int_{W_\epsilon} |\nabla u_\epsilon|^2\, dx + \int_{W_\epsilon} x^i\partial_i \frac{|\nabla u_\epsilon|^2}{2}\, dx\\
&= \left(1-\frac{n}{2}\right)\int_{W_\epsilon} |\nabla u_\epsilon|^2\, dx + \int_{\partial W_\epsilon}\left((x,\nu)\frac{|\nabla u_\epsilon|^2}{2} - x^i\partial_i u_\epsilon\partial_\nu u_\epsilon\right) d\sigma\\
&= \left(1-\frac{n}{2}\right)\left(\int_{\partial W_\epsilon} u_\epsilon\partial_\nu u_\epsilon\, d\sigma + \int_{W_\epsilon} u_\epsilon \Delta u_\epsilon\, dx\right)\\
&+\int_{\partial W_\epsilon}\left((x,\nu)\frac{|\nabla u_\epsilon|^2}{2} - x^i\partial_i u_\epsilon\partial_\nu u_\epsilon\right) d\sigma.
\end{aligned}$$

Using the equation $(E_\epsilon)$ in the RHS, we get that

$$\begin{aligned}
&\int_{W_\epsilon} x^i\partial_i u_\epsilon \Delta u_\epsilon\, dx = \left(1-\frac{n}{2}\right)\left(\int_{W_\epsilon} \frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\, dx - \int_{W_\epsilon} a_\epsilon u_\epsilon^2\, dx\right)\\
&+\int_{\partial W_\epsilon}\left(\left(1-\frac{n}{2}\right)u_\epsilon\partial_\nu u_\epsilon + (x,\nu)\frac{|\nabla u_\epsilon|^2}{2} - x^i\partial_i u_\epsilon\partial_\nu u_\epsilon\right) d\sigma. \qquad (90)
\end{aligned}$$

On the other hand, using the equation $(E_\epsilon)$ satisfied by $u_\epsilon$, we get that

$$\begin{aligned}
&\int_{W_\epsilon} x^i\partial_i u_\epsilon \Delta u_\epsilon\, dx = \int_{W_\epsilon} x^i\partial_i u_\epsilon \frac{|u_\epsilon|^{2^\star-2-\epsilon}u_\epsilon}{|x|^s}\, dx - \int_{W_\epsilon} x^i\partial_i u_\epsilon a_\epsilon u_\epsilon\, dx\\
&= \int_{W_\epsilon} x^i|x|^{-s}\partial_i\left(\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\right) dx - \int_{W_\epsilon} x^i\partial_i u_\epsilon a_\epsilon u_\epsilon\, dx\\
&= -\int_{W_\epsilon}\left(\partial_i(x^i|x|^{-s})\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{2^\star-p_\epsilon} + x^i\partial_i u_\epsilon a_\epsilon u_\epsilon\right) dx\\
&+\int_{\partial W_\epsilon} \frac{(x,\nu)}{2^\star-p_\epsilon}\cdot\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\, d\sigma\\
&= -\int_{W_\epsilon} \frac{n-s}{|x|^s}\cdot\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{2^\star-p_\epsilon}\, dx + \frac{1}{2}\int_{W_\epsilon}(na_\epsilon + x^i\partial_i a_\epsilon)u_\epsilon^2\, dx\\
&+\int_{\partial W_\epsilon} \frac{(x,\nu)}{2^\star-p_\epsilon}\cdot\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\, d\sigma - \int_{\partial W_\epsilon}\frac{(x,\nu)}{2}a_\epsilon u_\epsilon^2\, d\sigma. \qquad (91)
\end{aligned}$$

Plugging together (90) and (91), we get that

$$\begin{aligned}
&\left(\frac{n-2}{2}-\frac{n-s}{2^\star-p_\epsilon}\right)\int_{W_\epsilon}\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\, dx + \int_{W_\epsilon}\left(a_\epsilon + \frac{(x,\nabla a_\epsilon)}{2}\right)u_\epsilon^2\, dx\\
&= \int_{\partial W_\epsilon}\left(-\frac{n-2}{2}u_\epsilon\partial_\nu u_\epsilon + (x,\nu)\frac{|\nabla u_\epsilon|^2}{2}\right.\\
&\left.-x^i\partial_i u_\epsilon\partial_\nu u_\epsilon - \frac{(x,\nu)}{2^\star-p_\epsilon}\cdot\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\right) d\sigma + \int_{\partial W_\epsilon}\frac{(x,\nu)}{2}a_\epsilon u_\epsilon^2\, dx
\end{aligned}$$

for all $\epsilon > 0$. Since

$$\partial W_\epsilon = [\varphi(B_{r_\epsilon}(0)) \cap \partial\Omega] \cup [\Omega \cap \varphi(\partial B_{r_\epsilon}(0))]$$

and since $u_\epsilon \equiv 0$ on $\partial\Omega$, we get that

$$\begin{aligned}
&\frac{(n-2)p_\epsilon}{2\cdot(2^\star - p_\epsilon)}\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega} \frac{|u_\epsilon|^{2^\star - p_\epsilon}}{|x|^s}\, dx - \int_{\varphi(B_{r_\epsilon}(0))\cap\Omega}\left(a_\epsilon + \frac{(x,\nabla a_\epsilon)}{2}\right) u_\epsilon^2\, dx\\
&= \frac{1}{2}\int_{\varphi(B_{r_\epsilon}(0))\cap\partial\Omega} (x,\nu)|\nabla u_\epsilon|^2\, d\sigma \qquad (92)\\
&- \int_{\Omega\cap\varphi(\partial B_{r_\epsilon}(0))}\left(-\frac{n-2}{2}u_\epsilon\partial_\nu u_\epsilon + (x,\nu)\frac{|\nabla u_\epsilon|^2}{2} - x^i\partial_i u_\epsilon \partial_\nu u_\epsilon\right.\\
&\left. - \frac{(x,\nu)}{2^\star - p_\epsilon}\cdot\frac{|u_\epsilon|^{2^\star - p_\epsilon}}{|x|^s} + \frac{(x,\nu)}{2}a_\epsilon u_\epsilon^2\right) d\sigma.
\end{aligned}$$

It follows from (78) and (79) that there exists $C > 0$ such that

$$|u_\epsilon(x)| \le C r_\epsilon \text{ and } |\nabla u_\epsilon(x)| \le C$$

for all $x \in \Omega\cap\varphi(\partial B_{r_\epsilon}(0))$ (recall that $r_\epsilon = \sqrt{\mu_{\epsilon,N}}$). We then get that

$$\begin{aligned}
&\int_{\Omega\cap\varphi(\partial B_{r_\epsilon}(0))}\left(-\frac{n-2}{2}u_\epsilon\partial_\nu u_\epsilon + (x,\nu)\frac{|\nabla u_\epsilon|^2}{2} - x^i\partial_i u_\epsilon\partial_\nu u_\epsilon\right.\\
&\left. - \frac{(x,\nu)}{2^\star - p_\epsilon}\cdot\frac{|u_\epsilon|^{2^\star - p_\epsilon}}{|x|^s} - \frac{(x,\nu)}{2}a_\epsilon u_\epsilon^2\right) d\sigma = O(\mu_{\epsilon,N}^{\frac{n}{2}}) = o(\mu_{\epsilon,N}) \qquad (93)
\end{aligned}$$

when $\epsilon \to 0$ since $n \ge 3$. With (78) and Proposition 3.1, we get that

$$\begin{aligned}
&\left|\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega}\left(a_\epsilon + \frac{(x,\nabla a_\epsilon)}{2}\right)u_\epsilon^2\, dx\right| \le C\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega} u_\epsilon^2\, dx\\
&\le C\sum_{i=1}^N \int_{\varphi(B_{r_\epsilon}(0))\cap\Omega}\frac{\mu_{\epsilon,i}^n}{(\mu_{\epsilon,i}^2 + |x|^2)^{n-1}}\, dx + C\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega}|x|^2\, dx\\
&\le C\sum_{1=1}^N \mu_{\epsilon,i}^2\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{n-1}}\, dx + C r_\epsilon^{n+2}\\
&= o(\mu_{\epsilon,N}) \qquad (94)
\end{aligned}$$

when $\epsilon \to 0$ since $n \ge 3$. Plugging (93) and (94) in (92), we get that

$$\frac{(n-2)p_\epsilon}{2\cdot(2^\star - p_\epsilon)}\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega}\frac{|u_\epsilon|^{2^\star - p_\epsilon}}{|x|^s}\, dx = \frac{1}{2}\int_{\varphi(B_{r_\epsilon}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\, d\sigma + o(\mu_{\epsilon,N}) \qquad (95)$$

when $\epsilon \to 0$ and $n \ge 3$.

**Step 6.2:** We deal with the LHS of (95). We let $\varphi$ as in (13). Since

$$\lim_{\epsilon\to 0}\frac{r_\epsilon}{\mu_{\epsilon,N}} = +\infty$$

(see (89)), with a change of variables, we get for any $R > \alpha > 0$ that

$$\begin{aligned}
&\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega} \frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\,dx = \int_{\varphi(B_{r_\epsilon}(0)\cap\mathbb{R}^n_-)} \frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\,dx \qquad (96)\\
&= \int_{B_{r_\epsilon}(0)\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx\\
&= \int_{B_{Rk_{\epsilon,1}}(0)\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx\\
&+\sum_{i=1}^{N-1}\left[\int_{(B_{Rk_{\epsilon,i+1}}(0)\setminus\overline{B}_{\alpha k_{\epsilon,i+1}}(0))\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx\right.\\
&\qquad\left.+\int_{(B_{\alpha k_{\epsilon,i+1}}(0)\setminus\overline{B}_{Rk_{\epsilon,i}}(0))\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx\right]\\
&+\int_{(B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx
\end{aligned}$$

It follows from Proposition 3.1 that

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\int_{B_{Rk_{\epsilon,1}}(0)\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx = \alpha_1^{-\frac{(n-2)^2}{2(2-s)}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_1|^{2^\star}}{|x|^s}\,dx \tag{97}$$

and for any $i\in\{1,...,N-1\}$ that

$$\begin{aligned}
&\lim_{R\to+\infty}\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\int_{(B_{Rk_{\epsilon,i+1}}(0)\setminus\overline{B}_{\alpha k_{\epsilon,i+1}}(0))\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx\\
&= \alpha_{i+1}^{-\frac{(n-2)^2}{2(2-s)}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_{i+1}|^{2^\star}}{|x|^s}\,dx. \qquad (98)
\end{aligned}$$

It follows from the pointwise estimate (78) that there exists $C>0$ such that

$$|u_\epsilon(x)|\leq C\mu_{\epsilon,N}^{\frac{n}{2}}|x|^{1-n}+C|x|$$

for all $x\in\Omega$. It then follows that there exists $C>0$ independant of $R>1$ such that

$$\begin{aligned}
&\int_{(B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\mathbb{R}^n_-} \frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\mathrm{Jac}\ \varphi(x)|\,dx\\
&\leq C\int_{B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0)}\frac{1}{|y|^s}\left(|y|+\frac{\mu_{\epsilon,N}^{\frac{n}{2}}}{|y|^{n-1}}\right)^{2^\star-p_\epsilon}dy\\
&\leq C\int_{B_{r_\epsilon}(0)}|y|^{2^\star-s-p_\epsilon}\,dy + C\mu_{\epsilon,N}^{\frac{n}{2}2^\star}\int_{B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0)}|y|^{-(n-1)(2^\star-p_\epsilon)-s}\,dy\\
&\leq Cr_\epsilon^n+\frac{C}{R^{(n-1)(2^\star-p_\epsilon)-n+s}}.
\end{aligned}$$

Since $\lim_{\epsilon\to 0}r_\epsilon=0$, we get that

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\int_{(B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\mathbb{R}^n_-}\frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\text{Jac }\varphi(x)|\,dx=0. \tag{99}$$

We let $i\in\{1,...,N-1\}$. Using the pointwise estimate (78), we get that

$$|u_\epsilon(x)|\le C\frac{\mu_{\epsilon,i}^{\frac{n}{2}}}{|x|^{n-1}}+C\mu_{\epsilon,i+1}^{1-\frac{n}{2}}$$

for all $x\in\Omega$ and all $\epsilon>0$. With computations similar to the ones provided for the proof of (99), we get that

$$\lim_{R\to+\infty}\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\int_{(B_{\alpha k_{\epsilon,i+1}}(0)\setminus\overline{B}_{Rk_{\epsilon,i}}(0))\cap\mathbb{R}^n_-}\frac{|u_\epsilon\circ\varphi(x)|^{2^\star-p_\epsilon}}{|\varphi(x)|^s}\cdot|\text{Jac }\varphi(x)|\,dx=0. \tag{100}$$

Plugging together (97), (98), (99) and (100) in (96), using point (A4) of Proposition 3.1, we get that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\varphi(B_{r_\epsilon}(0))\cap\Omega}\frac{|u_\epsilon|^{2^\star-p_\epsilon}}{|x|^s}\,dx &= \sum_{i=1}^N\alpha_i^{-\frac{(n-2)^2}{2(2-s)}}\int_{\mathbb{R}^n_-}\frac{|\tilde{u}_i|^{2^\star}}{|x|^s}\,dx\\ &= \sum_{i=1}^N\alpha_i^{-\frac{(n-2)^2}{2(2-s)}}\int_{\mathbb{R}^n_-}|\nabla\tilde{u}_i|^2\,dx\end{aligned} \tag{101}$$

**Step 6.3:** We deal with the RHS of (95). We have that

$$\begin{aligned}&\int_{\varphi(B_{r_\epsilon}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma=\int_{\varphi(B_{Rk_{\epsilon,1}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\ &+\sum_{i=1}^{N-2}\int_{\varphi(B_{Rk_{\epsilon,i+1}}(0)\setminus\overline{B}_{Rk_{\epsilon,i}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\ &+\int_{\varphi(B_{\alpha k_{\epsilon,N}}(0)\setminus\overline{B}_{Rk_{\epsilon,N-1}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\ &+\int_{\varphi(B_{Rk_{\epsilon,N}}(0)\setminus\overline{B}_{\alpha k_{\epsilon,N}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\ &+\int_{\varphi(B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\end{aligned} \tag{102}$$

Using the expression of $\varphi$ (see (13)), we get that

$$\nu(\varphi(x))=\frac{(1,-\partial_2\varphi_0(x),...,-\partial_n\varphi_0(x))}{\sqrt{1+\sum_{i=2}^n(\partial_i\varphi_0(x))^2}}$$

for all $x\in U\cap\{x_1=0\}$. With the expression of $\varphi$, we then get that

$$(\nu\circ\varphi(x),\varphi(x))=(1+O(1)|x|^2)\cdot\left(\varphi_0(x)-\sum_{i=2}^n x^i\partial_i\varphi_0(x)\right) \tag{103}$$

for all $x\in U\cap\{x_1=0\}$. In this expression, there exists $C>0$ such that $|O(1)|\leq C$ for all $x\in U\cap\{x_1=0\}$. Since $\varphi_0(0)=0$ and $\nabla\varphi_0(0)=0$ (see (13)), we then get that there exists $C>0$ such that

$$|(\varphi(x),\nu\circ\varphi(x))|\leq C|x|^2 \tag{104}$$

for all $x\in U\cap\{x_1=0\}$.

*Step 6.3.1:* We deal with the second term in the RHS of (102). We let $i\in\{1,...,N-2\}$. It follows from the pointwise estimate (79) that

$$|\nabla u_\epsilon(x)|\leq C\mu_{\epsilon,i}^{\frac{n}{2}}|x|^{-n}+C\mu_{\epsilon,i+1}^{-\frac{n}{2}} \tag{105}$$

for all $x\in\Omega$. With (104) and (105), we get that

$$\begin{aligned}
&\int_{\varphi(B_{Rk_{\epsilon,i+1}}(0)\setminus\overline{B}_{Rk_{\epsilon,i}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\
&\leq C\int_{B_{2Rk_{\epsilon,i+1}}(0)\setminus\overline{B}_{Rk_{\epsilon,i}/2}(0)\cap\{x_1=0\}}|x|^2\left(\mu_{\epsilon,i}^n|x|^{-2n}+\mu_{\epsilon,i+1}^{-n}\right)\,dx\\
&\leq C\mu_{\epsilon,i}+C\mu_{\epsilon,i+1}=o(\mu_{\epsilon,N})
\end{aligned}\tag{106}$$

when $\epsilon\to0$ when $n\geq3$. Here, we have used that $i+1<N$ and point (A3) of Proposition 3.1. With the same type of arguments, we get that

$$\int_{\varphi(B_{Rk_{\epsilon,1}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma=o(\mu_{\epsilon,N}) \tag{107}$$

when $\epsilon\to0$ as soon as $N\geq2$.

*Step 6.3.2:* We deal with the third term of the RHS of (102). It follows from the pointwise estimate (79) that

$$|\nabla u_\epsilon(x)|\leq C\mu_{\epsilon,N-1}^{\frac{n}{2}}|x|^{-n}+C\mu_{\epsilon,N}^{-\frac{n}{2}} \tag{108}$$

for all $x\in\Omega$. With (104) and (108), we get that

$$\begin{aligned}
&\int_{\varphi(B_{\alpha k_{\epsilon,N}}(0)\setminus\overline{B}_{Rk_{\epsilon,N-1}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\
&\leq C\int_{B_{2\alpha k_{\epsilon,N}}(0)\setminus\overline{B}_{Rk_{\epsilon,N-1}/2}(0)\cap\{x_1=0\}}|x|^2\left(\mu_{\epsilon,N-1}^n|x|^{-2n}+\mu_{\epsilon,N}^{-n}\right)\,dx\\
&\leq C\mu_{\epsilon,N-1}+C\alpha^{n+1}\mu_{\epsilon,N}
\end{aligned}$$

since $n\geq3$ and where $C>0$ is independant of $\alpha$ and $\epsilon>0$. With point (A3) of Proposition 3.1, we get that

$$\lim_{\alpha\to0}\lim_{\epsilon\to0}\mu_{\epsilon,N}^{-1}\int_{\varphi(B_{\alpha k_{\epsilon,N}}(0)\setminus\overline{B}_{Rk_{\epsilon,N-1}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma=0. \tag{109}$$

*Step 6.3.3:* We deal with the fifth term of the RHS of (102). It follows from the pointwise estimate (79) that

$$|\nabla u_\epsilon(x)|\leq C\mu_{\epsilon,N}^{\frac{n}{2}}|x|^{-n}+C \tag{110}$$

for all $x \in \Omega$. With (104) and (110), we get that

$$\begin{aligned}&\int_{\varphi(B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{\epsilon,N}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\&\leq C\int_{B_{2r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}/2}(0)\cap\{x_1=0\}}|x|^2\left(\mu_{\epsilon,N}^n|x|^{-2n}+C\right)\,dx\\&\leq CR^{1-n}\mu_{\epsilon,N}+Cr_\epsilon^{n+1}\end{aligned}$$

since $n \geq 3$ and where $C > 0$ is independant of $R$ and $\epsilon > 0$. With the definition (89) of $r_\epsilon$, we get that $r_\epsilon^{n+1} = o(\mu_{\epsilon,N})$ when $\epsilon \to 0$. It then follows from point (A3) of Proposition 3.1 that

$$\lim_{R\to+\infty}\lim_{\epsilon\to0}\mu_{\epsilon,N}^{-1}\int_{\varphi(B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{\epsilon,N}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma=0 \tag{111}$$

when $n \geq 3$.

*Step 6.3.4:* We deal with the fourth term of the RHS of (102). Since $\varphi_0(0) = 0$ and $\nabla\varphi_0(0) = 0$, it follows from the definition (13) of $\varphi$ and (103) that

$$\begin{aligned}&(\varphi(k_{\epsilon,N}x),\nu\circ\varphi(k_{\epsilon,N}x))\\&=(1+O(k_{\epsilon,N}^2|x|^2))\left(\varphi_0(k_{\epsilon,N}x)-k_\epsilon\sum_{i=2}^n x^i\partial_i\varphi_0(k_{\epsilon,N}x)\right)\\&=-\frac{1}{2}k_{\epsilon,N}^2\sum_{i,j=2}^n\partial_{ij}\varphi_0(0)x^ix^j+\theta_{\epsilon,R}(x)k_{\epsilon,N}^2,\end{aligned} \tag{112}$$

for all $\epsilon > 0$ and all $x \in B_R(0)\cap\{x_1 = 0\}$ and where $\lim_{\epsilon\to0}\sup_{B_R(0)\cap\{x_1=0\}}|\theta_{\epsilon,R}| = 0$ for any $R > 0$. With a change of variable, (112) and the definition of $\tilde{u}_{\epsilon,N}$ (see Proposition 3.1), we have that

$$\begin{aligned}&\mu_{\epsilon,N}^{-1}\int_{\varphi(B_{Rk_{\epsilon,N}}(0)\setminus\overline{B}_{\alpha k_{\epsilon,N}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma\\&=\left(\frac{k_{\epsilon,N}}{\mu_{\epsilon,N}}\right)^{n-1}\left(-\frac{1}{2}\int_{(B_R(0)\setminus\overline{B}_\alpha(0))\cap\{x_1=0\}}\sum_{i,j=2}^n x^ix^j\partial_{ij}\varphi_0(x)|\nabla\tilde{u}_{\epsilon,N}|_{\tilde{g}_\epsilon}^2\,dv_{\tilde{g}_\epsilon}\right)\\&+o(1)\end{aligned}$$

when $\epsilon \to 0$. In this expression, $(\tilde{g}_\epsilon)_{ij} = (\partial_i\varphi,\partial_j\varphi)(k_{\epsilon,N}x)$ for all $i, j = 2, ..., n$. It follows from (110) and the definition of $\tilde{u}_{\epsilon,N}$ that there exists $C > 0$ such that

$$|\nabla\tilde{u}_N(x)| \leq \frac{C}{1+|x|^n} \tag{113}$$

for all $x \in \mathbb{R}^n_-$. With points (A4) and (A7) of Proposition 3.1 and inequality (113), we get that

$$\lim_{R\to+\infty}\lim_{\alpha\to 0}\lim_{\epsilon\to 0}\mu_{\epsilon,N}^{-1}\int_{\varphi(B_{Rk_{\epsilon,N}}(0)\setminus\overline{B}_{\alpha k_{\epsilon,N}}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma$$
$$= -\frac{\alpha_N^{-\frac{n-1}{2^\star-2}}}{2}\int_{\partial\mathbb{R}^n_-}\sum_{i,j=2}^n x^i x^j\partial_{ij}\varphi_0(x)|\nabla\tilde{u}_N|^2\,dx \tag{114}$$

when $n\geq 3$. Plugging (106), (107), (109), (111) and (114) in (102), we get that

$$\lim_{\epsilon\to 0}\mu_{\epsilon,N}^{-1}\int_{\varphi(B_{r_\epsilon}(0))\cap\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\,d\sigma = -\frac{\alpha_N^{-\frac{n-1}{2^\star-2}}}{2}\int_{\partial\mathbb{R}^n_-}\sum_{i,j=2}^n x^i x^j\partial_{ij}\varphi_0(x)|\nabla\tilde{u}_N|^2\,dx \tag{115}$$

We consider the second fondamental form associated to $\partial\Omega$, namely

$$II_p(x,y) = (d\nu_p x, y)$$

for all $p\in\partial\Omega$ and all $x,y\in T_p\partial\Omega$ (recall that $\nu$ is the outward normal vector at the hypersurface $\partial\Omega$). In the canonical basis of $\partial\mathbb{R}^n_- = T_0\partial\Omega$, the matrix of the bilinear form $II_0$ is $-D_0^2\varphi_0$, where $D_0^2\varphi_0$ is the Hessian matrix of $\varphi_0$ at 0. With this remark, plugging (101) and (115) into (95), we get that

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{\epsilon,N}} = \frac{n-s}{(n-2)^2}\cdot\alpha_N^{-\frac{(n-1)(n-2)}{2(2-s)}}\cdot\frac{\int_{\partial\mathbb{R}^n_-}II_0(x,x)|\nabla\tilde{u}_N|^2\,dx}{\sum_{i=1}^N\alpha_i^{-\frac{(n-2)^2}{2(2-s)}}\int_{\mathbb{R}^n_-}|\nabla\tilde{u}_i|^2\,dx} \tag{116}$$

when $n\geq 3$. This proves the first part of Proposition 6.1.

We prove the second part of the Proposition and assume that $u_\epsilon\geq 0$ for all $\epsilon$. It follows that the limit function $\tilde{u}_N$ is nonnegative, and then positive on $\mathbb{R}^n_-$. Moreover, we have that

$$\Delta\tilde{u}_N = \frac{\tilde{u}_N^{2^\star-1}}{|x|^s}$$

in $\mathbb{R}^n_-$. It follows from (78) that there exists $C>0$ such that

$$|\tilde{u}_N(x)|\leq\frac{C}{1+|x|^{n-1}}$$

for all $x\in\mathbb{R}^n_-$. It then follows from Proposition 10.1 of Appendix C that there exists $v\in C^2(\mathbb{R}^\star_-\times\mathbb{R})$ such that $\tilde{u}_N(x_1,x') = v(x_1,|x'|)$ for all $(x_1,x')\in\mathbb{R}^\star_-\times\mathbb{R}^{n-1}$. In particular, $|\nabla\tilde{u}_N|(0,x')$ is radially symmetrical wrt $x'\in\partial\mathbb{R}^n_-$. Since we have chosen a chart $\varphi$ that is Euclidean at 0, we get that

$$\begin{aligned}\int_{\partial\mathbb{R}^n_-}II_0(x,x)|\nabla\tilde{u}_N|^2\,dx &= \frac{\sum_{i=2}^n(II_0)^{ii}}{n}\int_{\partial\mathbb{R}^n_-}|x|^2|\nabla\tilde{u}_N|^2\,dx\\ &= \frac{H(0)}{n}\int_{\partial\mathbb{R}^n_-}|x|^2|\nabla\tilde{u}_N|^2\,dx.\end{aligned}$$

Note that we have used here that in the chart $\varphi$ defined in (13), the matrix of the first fundamental form at 0 is the identity. The second part of the Proposition then follows.

**Step 6.4: Proof of Theorem 1.3:** We let $(u_\epsilon)$, $(a_\epsilon)$ and $(p_\epsilon)$ such that $(E_\epsilon)$, (8), (9) and (10) hold. Assume that

$$\lim_{\epsilon\to 0}\|u_\epsilon\|_{L^\infty(\Omega)} = +\infty. \tag{117}$$

Then we can apply Proposition 6.1, and (116) holds. Since the principal curvatures of $\partial\Omega$ at 0 are nonpositive, but do not all vanish, we have that $II_0(x,x)\leq 0$ for all $x\in\partial\mathbb{R}^n_-$, but $II_0\not\equiv 0$. In particular, the RHS of (116) is negative. A contradiction since $p_\epsilon\geq 0$, and then the LHS of (116) is nonnegative. Then (117) does not hold, and there exists $C>0$ such that $|u_\epsilon(x)|\leq C$ for all $\epsilon>0$ and all $x\in\Omega$. The first part of Theorem 1.3 then follows from Proposition 2.1. In the case $u_\epsilon\geq 0$ for all $\epsilon>0$, we apply the second part of Proposition 6.1 to recover compactness as soon as $H(0)<0$, and the second part of Theorem 1.3 is proved.

## 7. Proof of existence and multiplicity

7.1. **Proof of Theorem 1.1.** For any subcritical $p$, i.e., $2<p<2^\star(s)$ we define the corresponding best constant

$$\mu_{s,p}(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2dx;\ u\in H^2_{1,0}(\Omega)\text{ and }\int_\Omega \frac{|u|^p}{|x|^s}\,dx = 1\right\}. \tag{118}$$

Because of the compactness of the embedding $H^2_{1,0}(\Omega)$ into $L^p(\Omega;|x|^{-s}dx)$, the infimum $\mu_{s,p}(\Omega)$ is attained at a positive extremal $v_p$ satisfying

$$\begin{cases}\Delta u = \frac{u^{p-1}}{|x|^s} & \text{in }\mathcal{D}'(\Omega)\\ u>0 & \text{in }\Omega\\ u=0 & \text{on }\partial\Omega.\end{cases} \tag{119}$$

Moreover, the family $(v_p)$ is uniformly bounded in $H^2_{1,0}(\Omega)$ when $p\to 2^\star$. Part 2 of the main compactness Theorem 1.3 for positive sequences now yields a nontrivial limit $v$ that is an extremal for $\mu_s(\Omega)$.

7.2. **Proof of Theorem 1.2.** For each $2<p\leq 2^\star(s)$, consider the $C^2$-functional

$$I_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega \frac{|u|^p}{|x|^s}\,dx \tag{120}$$

on $H^2_{1,0}(\Omega)$ whose critical points are the weak solutions of

$$\begin{cases}\Delta u = \frac{|u|^{p-2}u}{|x|^s} & \text{on }\Omega\\ u=0 & \text{on }\partial\Omega.\end{cases} \tag{121}$$

First note that for a fixed $u\in H^2_{1,0}(\Omega)$, we have since

$$I_p(\lambda u) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda^p}{p}\int_\Omega \frac{|u|^p}{|x|^s}\,dx$$

that $\text{limit}_{\lambda\to\infty} I(\lambda u) = -\infty$, which means that for each finite dimensional subspace $E_k\subset E := H^2_{1,0}(\Omega)$, there exists $R_k>0$ such that

$$\sup\{I_p(u);u\in E_k, \|u\|>R_k\}<0 \tag{122}$$

when $p \to 2^\star$. Let $(E_k)_{k=1}^\infty$ be an increasing sequence of subspaces of $H_{1,0}^2(\Omega)$ such that $\dim E_k = k$ and $\overline{\cup_{k=1}^\infty E_k} = E := H_{1,0}^2(\Omega)$ and define the min-max values:

$$c_{p,k} = \inf_{h\in \mathbf{H}_k} \sup_{x\in E_k} I_p(h(x)),$$

where

$$\mathbf{H}_k = \{h \in C(E,E);\ h \text{ is odd and } h(v) = v \text{ for } \|v\| > R_k \text{ for some } R_k > 0\}.$$

**Proposition 7.1.** *With the above notation and assuming $n \geq 3$, we have:*

1. *For each $k \in \mathbb{N}$, $c_{p,k} > 0$ and $\lim\limits_{p\to 2^\star} c_{p,k} = c_{2^\star,k} := c_k$.*
2. *If $2 < p < 2^\star$, there exists for each $k$, functions $u_{p,k} \in H_{1,0}^2(\Omega)$ such that $I_p'(u_{p,k}) = 0$, and $I_p(u_{p,k}) = c_{p,k}$.*
3. *For each $2 < p < 2^\star$, we have $c_{p,k}$ satisfy $c_{p,k} \geq D_{n,p} k^{\frac{p+1}{p-1}\frac{2}{n}}$ where $D_{n,p} > 0$ is such that $\lim_{p\to 2^\star} D_{n,p} = 0$.*
4. $\lim\limits_{k\to\infty} c_k = \lim\limits_{k\to\infty} c_{2^\star,k} = +\infty$.

**Proof:** (1) First note that in view of the Hardy-Sobolev inequality, we have

$$I_p(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - C\|\nabla u\|_2^p = \|\nabla u\|_2^2\left(\frac{1}{2} - C\|\nabla u\|_2^{p-2}\right) \geq \alpha > 0$$

provided $\|u\|_{H_{1,0}^2(\Omega)} = \rho$ for some $\rho > 0$ small enough. A standard intersection lemma gives that the sphere $S_\rho = \{u \in E; \|u\|_{H_{1,0}^2(\Omega)} = \rho\}$ must intersect every image $h(E_k)$ by an odd continuous function $h$. It follows that

$$c_{p,k} \geq \inf\{I_p(u); u \in S_\rho\} \geq \alpha > 0.$$

In view of (122), it follows that for each $h \in \mathbf{H}_k$, we have that

$$\sup_{x\in E_k} I_{p_i}(h(x)) = \sup_{x\in D_k} I_p(h(x))$$

where $D_k$ denotes the ball in $E_k$ of radius $R_k$. Consider now a sequence $p_i \to 2^\star$ and note first that for each $u \in E$, we have that $I_{p_i}(u) \to I_{2^\star}(u)$. Since $h(D_k)$ is compact and the family of functionals $(I_p)_p$ is equicontinuous, it follows that $\sup\limits_{x\in E_k} I_p(h(x)) \to \sup\limits_{x\in E_k} I_{2^\star}(h(x))$, from which follows that $\limsup\limits_{i\in\mathbb{N}} c_{p_i,k} \leq \sup\limits_{x\in E_k} I_{2^\star}(h(x))$. Since this holds for any $h \in \mathbf{H}_k$, it follows that

$$\limsup_{i\in\mathbb{N}} c_{p_i,k} \leq c_{2^\star,k} = c_k.$$

On the other hand, the function $f(r) = \frac{1}{p}r^p - \frac{1}{2^*}r^{2^*}$ attains its maximum on $[0,+\infty)$ at $r = 1$ and therefore $h(r) \leq \frac{1}{p} - \frac{1}{2^*}$ for all $r > 0$. It follows

$$I_{2^\star}(u) = I_p(u) + \int_\Omega \frac{1}{|x|^s}\left(\frac{1}{p}|u(x)|^p - \frac{1}{2^\star}|u(x)|^{2^\star}\right)dx \leq I_p(u) + \int_\Omega \frac{1}{|x|^s}\left(\frac{1}{p} - \frac{1}{2^\star}\right)dx$$

from which follows that $c_k \leq \liminf\limits_{i\in\mathbb{N}} c_{p_i,k}$, and claim (1) is proved.

If now $p < 2^\star$, we are in the subcritical case, that is we have compactness in the Sobolev embedding $H_{1,0}^2(\Omega) \to L^p(\Omega; |x|^{-s}dx)$ and therefore $I_p$ has the Palais-Smale condition. It is then standard to find critical points $u_{p,k}$ for $I_p$ at each level $c_{p,k}$ (see for example [19]). Now there are many ways to establish growth estimates for $c_{p,k}$ as $k \to +\infty$, and we shall use here the one based on the Morse indices of these variationally obtained solutions, a method first used by by Bahri-Lions [4] and independently by Tanaka [40]. We need the following key estimate of Li-Yau [33].

**Lemma 7.1.** *Let $V \in L^{n/2}(\Omega)$ and denote by $m^*(V)$ the number of non-positive eigenvalues of the following eigenvalue problem:*

$$\begin{cases} \Delta u - Vu = \lambda u & \text{on}\Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

*If $n \geq 3$, then there is a constant $C_n > 0$ such that $m^*(V) \leq C_n \|V\|_{n/2}^{n/2}$.*

To prove the growth estimates on the critical values $c_{p,k}$, one can follow [40] (see also [19]) and identify a cohomotopic family of sets $\mathbf{F_k}$ of dimension $k$ in such a way that if $D_k$ denotes the ball in $E_k$ of radius $R_k$ and if $\gamma \in \mathbf{H}_k$, then $\gamma(D_k) \in \mathbf{F_k}$. It then follows that there exists $v_{p,k} \in H^2_{1,0}(\Omega)$ such that $I_p(v_{p,k}) \leq c_{p,k}$, $I'(v_{p,k}) = 0$ and $m^*(v_{p,k}) \geq k$, where $m^*(v_{p,k})$ is the augmented Morse index of $I_p$ at $v_{p,k}$. In other words, since

$$\begin{aligned} I_p''(v)(h,h) &= \int_\Omega |\nabla h|^2\,dx - (p-1)\int_\Omega \frac{|v|^{p-2}}{|x|^s} h^2\,dx \\ &= \langle (\Delta - (p-1)\tfrac{|v|^{p-2}}{|x|^s})h, h\rangle, \end{aligned}$$

in $H^{-1}(\Omega)$, this means that the operator $(\Delta - (p-1)\frac{|v_{p,k}|^{p-2}}{|x|^s})$ possesses at least $k$ non-positive eigenvalues. Applying the above lemma, we get that the number of these non-positive eigenvalues is bounded above by $C_n \int_\Omega [(p-1)\frac{|v_{p,k}|^{p-2}}{|x|^s}]^{\frac{n}{2}}\,dx)$. Since $p < \frac{2n}{n-2}$, we have $q := \frac{2p}{n(p-2)} > 1$, as well as its conjugate $q'$. Moreover, since $p < \frac{2(n-s)}{n-2}$, we have that $\frac{2sn}{2p-np+2n} < n$. It then follows from Holder's inequality that:

$$\begin{aligned} k &\leq C_n \int_\Omega |p-1|^{\frac{n}{2}} \frac{|v_{p,k}|^{(p-2)\frac{n}{2}}}{|x|^{s\frac{n}{2}}}\,dx \qquad (123)\\ &\leq C_n|p-1|^{\frac{n}{2}} \left(\int_\Omega \frac{1}{|x|^{\frac{2sn}{2p-np+2n}}}\,dx\right)^{\frac{1}{q'}} \left(\int_\Omega \frac{|v_{p,k}|^p}{|x|^s}\,dx\right)^{\frac{n(p-2)}{2p}} \\ &\leq C_{n,p} \left(\int_\Omega \frac{|v_{p,k}|^p}{|x|^s}\,dx\right)^{\frac{n(p-2)}{2p}} \end{aligned}$$

where $C_{n,p} = C_n|p-1|^{\frac{n}{2}} \left(\int_\Omega \frac{1}{|x|^{\frac{2sn}{2p-np+2n}}}\,dx\right)^{\frac{1}{q'}}$.

Since $\langle I'(v_{p,k}), v_{p,k}\rangle = 0$, it follows that $\int_\Omega |\nabla v_{p,k}|^2\,dx = \int_\Omega \frac{|v_{p,k}|^p}{|x|^s}\,dx$, which finally implies that

$$\begin{aligned} c_{p,k} \geq I(v_{p,k}) &= \frac{1}{2}\int_\Omega |\nabla v_{p,k}|^2\,dx - \frac{1}{p}\int_\Omega \frac{|v_{p,k}|^p}{|x|^s}\,dx \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega \frac{|v_{p,k}|^p}{|x|^s}\,dx \\ &\geq D_{n,p}k^{\frac{2p}{n(p-2)}}, \end{aligned}$$

where $D_{n,p} = (\frac{1}{2}-\frac{1}{p})C_{n,p}^{-\frac{2p}{n(p-2)}}$.

To prove 4) we proceed by contradiction and assume that $(c_k)_k$ is bounded so that a subsequence of which converges to some real number $c$. Using the first claim of the proposition, there exists for each $k \in \mathbb{N}$, $2 < p_k < 2^\star$ such that $|c_{p_k,k} - c_k| < \frac{1}{k}$ in such a way that $\lim_{k\to+\infty} p_k = 2^\star$ and

$$\lim_{k\to+\infty} c_{p_k,k} = \lim_{k\to+\infty} c_k = c. \tag{124}$$

As above, there exists $v_{p_k,k} \in H_{1,0}^2(\Omega)$ such that $I_{p_k}(v_{p_k,k}) \leq c_{p_k,k}$, $I'_{p_k}(v_{p_k,k}) = 0$ and $m^*(v_{p_k,k}) \geq k$, where $m^*(v_{p_k,k})$ is the augmented Morse index of $I_{p_k}$ at $v_{p_k,k}$.

But (124) gives that the energies of $(v_{p_k,k})_k$ are uniformly bounded and therefore $(v_{p_k,k})_k$ is bounded in $H_{1,0}^2(\Omega)$. It follows from Proposition 8.1 and the compactness Theorem 1.3 that they converge to a solution $v$ of (121) with energy below level $c$. In particular, there exists $C > 0$ such that

$$|v_{p_k,k}(x)| \leq C \tag{125}$$

for all $x \in \Omega$ and all $k \in \mathbb{N}$. With (123) applied to $v_{p_k,k}$, we get that

$$k \leq C_n \int_\Omega |p_k - 1|^{\frac{n}{2}} \frac{|v_{p_k,k}|^{(p-2)\frac{n}{2}}}{|x|^{s\frac{n}{2}}}\,dx.$$

With (125), we get that there exists a constant $C > 0$ independant of $k$ such that

$$k \leq C \int_\Omega \frac{dx}{|x|^{\frac{sn}{2}}}.$$

In particular, since $s \in (0,2)$, the integral is finite and there existe $C > 0$ such that $k \leq C$ for all $k \in \mathbb{N}$. A contradiction, and we are done with the proposition.

To complete the proof of Theorem 1.3, notice that since for each $k$, we have $\lim_{p_i\to 2^\star} I_{p_i}(u_{p_i,k}) = \lim_{p_i\to 2^\star} c_{p_i,k} = c_k$, it follows that the sequence $(u_{p_i,k})_i$ is uniformly bounded in $H_{1,0}^2(\Omega)$. Moreover, since $I'_{p_i}(u_{p_i,k}) = 0$, it follows from Proposition 8.1 and the compactness Theorem 1.3 that by letting $p_i \to 2^\star$, we get a solution $u_k$ of (121) in such a way that $I_{2^\star}(u_k) = \lim_{p\to 2^\star} I_p(u_{p,k}) = \lim_{p\to 2^\star} c_{p,k} = c_k$. Since the latter sequence goes to infinity, it follows that (121) has an infinite number of critical levels. The result for the equation $\Delta u + au = \frac{|u|^{2^\star-2}u}{|x|^s}$ when $\Delta + a$ is coercive goes the same way, and Theorem 1.2 is proved.

## 8. Appendix A: Regularity of weak solutions

In this appendix, we prove the following regularity result. Note that such a $C^1-$regularity was first proved out by Egnell [17]. We include the proof for completeness.

**Proposition 8.1.** *Let $\Omega$ be a smooth domain of $\mathbb{R}^n$, $n\geq 3$. We assume that either $\Omega$ is bounded, or $\Omega=\mathbb{R}^n_-$. We let $s\in(0,2)$ and $a\in C^0(\overline{\Omega})$. We let $\epsilon\in[0,2^\star-2)$ and consider $u\in H^2_{1,0}(\Omega)$ a weak solution of*

$$\Delta u+au=\frac{|u|^{2^\star-2-\epsilon}u}{|x|^s}\text{ in }\mathcal{D}'(\Omega).$$

*Then $u\in C^{1,\theta}(\overline{\Omega})$ for all $\theta\in(0,\min\{1,2^\star-\epsilon-s\})$ if $\Omega$ is bounded, and $u\in C^{1,\theta}_{loc}(\overline{\mathbb{R}^n_-})$ for all $\theta\in(0,\min\{1,2^\star-\epsilon-s\})$ if $\Omega=\mathbb{R}^n_-$. In addition, in all the cases, we have that $u\in C^2(\overline{\Omega}\setminus\{0\})$ if $a\in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$.*

*Proof.* We prove the result when $\Omega$ is bounded. The arguments and the results are basically local, and the proof goes the same way when $\Omega=\mathbb{R}^n_-$.

**Step 8.1:** We follow the strategy developed by Trudinger ([41], and [27] for an exposition in book form). Let $\beta\geq 1$, and $L>0$. We let

$$G_L(t)=\left\{\begin{array}{ll}|t|^{\beta-1}t & \text{if } |t|\leq L\\ \beta L^{\beta-1}(t-L)+L^\beta & \text{if } t\geq L\\ \beta L^{\beta-1}(t+L)-L^\beta & \text{if } t\leq -L\end{array}\right.$$

and

$$H_L(t)=\left\{\begin{array}{ll}|t|^{\frac{\beta-1}{2}}t & \text{if } |t|\leq L\\ \frac{\beta+1}{2}L^{\frac{\beta-1}{2}}(t-L)+L^{\frac{\beta+1}{2}} & \text{if } t\geq L\\ \frac{\beta+1}{2}L^{\frac{\beta-1}{2}}(t+L)-L^{\frac{\beta+1}{2}} & \text{if } t\leq -L\end{array}\right.$$

As easily checked,

$$0\leq tG_L(t)\leq H_L(t)^2\text{ and }G_L'(t)=\frac{4\beta}{(\beta+1)^2}(H_L'(t))^2$$

for all $t\in\mathbb{R}$ and all $L>0$. Let $\eta\in C^\infty_c(\mathbb{R}^n)$. As easily checked, $\eta^2G_L(u),\eta H_L(u)\in H^2_{1,0}(\Omega)$. With the equation verified by $u$, we get that

$$\int_\Omega\nabla u\nabla(\eta^2G_L(u))\,dx=\int_\Omega\frac{|u|^{2^\star-2-\epsilon}}{|x|^s}\eta^2uG_L(u)\,dx-\int_\Omega a\eta^2uG_L(u)\,dx. \tag{126}$$

We let $J_L(t)=\int_0^tG_L(\tau)\,d\tau$ for all $t\in\mathbb{R}$. Integrating by parts, we get that

$$\begin{aligned}&\int_\Omega\nabla u\nabla(\eta^2G_L(u))\,dx=\int_\Omega\eta^2G_L'(u)|\nabla u|^2\,dx+\int_\Omega\nabla\eta^2\nabla J_L(u)\,dx\\ &=\frac{4\beta}{(\beta+1)^2}\int_\Omega\eta^2|\nabla H_L(u)|^2\,dx+\int_\Omega(\Delta\eta^2)J_L(u)\,dx\\ &=\frac{4\beta}{(\beta+1)^2}\int_\Omega|\nabla(\eta H_L(u))|^2\,dx-\frac{4\beta}{(\beta+1)^2}\int_\Omega\eta\Delta\eta|H_L(u)|^2\,dx\\ &+\int_\Omega(\Delta\eta^2)J_L(u)\,dx\end{aligned} \tag{127}$$

On the other hand, with Hölder's inequality and the definition of $\mu_s(\mathbb{R}^n)$, we get that

$$\begin{aligned}
&\int_\Omega \left(\frac{|u|^{2^\star-2-\epsilon}}{|x|^s} - a\right)\cdot \eta^2 u G_L(u)\,dx \le \int_\Omega \left(|a| + \frac{|u|^{2^\star-2-\epsilon}}{|x|^s}\right)\cdot (\eta H_L(u))^2\,dx\\
&\le \left(\int_{\Omega\cap \text{Supp }\eta} \frac{(|a|\cdot|x|^s + |u|^{2^\star-2-\epsilon})^{\frac{2^\star-\epsilon}{2^\star-2-\epsilon}}}{|x|^s}\,dx\right)^{1-\frac{2}{2^\star-\epsilon}}\\
&\times \left(\int_\Omega \frac{|\eta H_L(u)|^{2^\star}}{|x|^s}\,dx\right)^{\frac{2}{2^\star}} \times \left(\int_{\Omega\cap\text{Supp }\eta} \frac{dx}{|x|^s}\right)^{\frac{2\epsilon}{2^\star(2^\star-\epsilon)}}\\
&\le \alpha \int_\Omega |\nabla(\eta H_L(u))|^2\,dx \qquad (128)
\end{aligned}$$

where

$$\begin{aligned}
\alpha \;:=\; &\left(\int_{\Omega\cap \text{Supp }\eta} \frac{(|a|\cdot|x|^s + |u|^{2^\star-2-\epsilon})^{\frac{2^\star-\epsilon}{2^\star-2-\epsilon}}}{|x|^s}\,dx\right)^{1-\frac{2}{2^\star-\epsilon}}\\
&\times \mu_s(\mathbb{R}^n)^{-1}\left(\int_{\Omega\cap\text{Supp }\eta} \frac{dx}{|x|^s}\right)^{\frac{2\epsilon}{2^\star(2^\star-\epsilon)}}
\end{aligned}$$

Plugging (127) and (128) into (126), we get that

$$A\cdot \int_\Omega |\nabla(\eta H_L(u))|^2\,dx \le \frac{4\beta}{(\beta+1)^2}\int_\Omega |\eta\Delta\eta||H_L(u)|^2\,dx + \int_\Omega |\Delta(\eta^2)J_L(u)|\,dx \qquad (129)$$

where

$$\begin{aligned}
A \;:=\; &\frac{4\beta}{(\beta+1)^2} - \left(\int_{\Omega\cap \text{Supp }\eta} \frac{(|a|\cdot|x|^s + |u|^{2^\star-2-\epsilon})^{\frac{2^\star-\epsilon}{2^\star-2-\epsilon}}}{|x|^s}\,dx\right)^{1-\frac{2}{2^\star-\epsilon}}\\
&\times \mu_s(\mathbb{R}^n)^{-1}\left(\int_{\Omega\cap\text{Supp }\eta} \frac{dx}{|x|^s}\right)^{\frac{2\epsilon}{2^\star(2^\star-\epsilon)}}
\end{aligned}$$

**Step 8.2:** We let

$$p_0 = \sup\{p \ge 1/\, u \in L^p(\Omega)\}.$$

It follows from Sobolev's embedding theorem that $p_0 \ge \frac{2n}{n-2}$. We claim that

$$p_0 = +\infty.$$

We proceed by contradiction and assume that

$$p_0 < \infty.$$

Let $p \in (2, p_0)$. It follows from the definition of $p_0$ that $u \in L^p(\Omega)$. Let $\beta = p-1 > 1$. For any $x \in \overline{\Omega}$, we let $\delta_x > 0$ such that

$$\left(\int_{\Omega\cap B_{2\delta_x}(x)}\frac{(|a|\cdot|y|^s+|u|^{2^\star-2-\epsilon})^{\frac{2^\star-\epsilon}{2^\star-2-\epsilon}}}{|y|^s}\,dy\right)^{1-\frac{2}{2^\star-\epsilon}}\mu_s(\mathbb{R}^n)^{-1}$$
$$\times\left(\int_{\Omega\cap B_{2\delta_x}(x)}\frac{dy}{|y|^s}\right)^{\frac{2\epsilon}{2^\star(2^\star-\epsilon)}}\leq\frac{2\beta}{(\beta+1)^2}. \tag{130}$$

Since $\overline{\Omega}$ is compact, we get that there exists $x_1,...,x_N\in\overline{\Omega}$ such that

$$\Omega\subset\bigcup_{i=1}^N B_{\delta_{x_i}}(x_i).$$

We fix $i\in\{1,...,N\}$ and let $\eta\in C^\infty(B_{2\delta_{x_i}}(x_i))$ such that $\eta(x)=1$ for all $x\in B_{\delta_{x_i}}(x_i)$. We then get with (129) and (130) that

$$\frac{2\beta}{(\beta+1)^2}\int_\Omega|\nabla(\eta H_L(u))|^2\,dx$$
$$\leq\frac{4\beta}{(\beta+1)^2}\int_\Omega|\eta\Delta\eta||H_L(u)|^2\,dx+\int_\Omega|\Delta\eta^2||J_L(u)|\,dx. \tag{131}$$

Recall that it follows from Sobolev's inequality that there exists $K(n,2)>0$ that depends only on $n$ such that

$$\left(\int_{\mathbb{R}^n}|f|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\leq K(n,2)\int_{\mathbb{R}^n}|\nabla f|^2\,dx \tag{132}$$

for all $f\in H_{1,0}^2(\mathbb{R}^n)$. It follows from (131) and (132) that

$$\frac{2\beta}{(\beta+1)^2}K(n,2)^{-1}\left(\int_\Omega|\eta H_L(u)|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}$$
$$\leq\frac{4\beta}{(\beta+1)^2}\int_\Omega|\eta\Delta\eta||H_L(u)|^2\,dx+\int_\Omega|\Delta\eta^2|\cdot|J_L(u)|\,dx$$

for all $L>0$. As easily checked, there exists $C_0>0$ such that $|J_L(t)|\leq C_0|t|^{\beta+1}$ for all $t\in\mathbb{R}$ and all $L>0$. Since $u\in L^{\beta+1}(\Omega)$, we get that there exists a constant $C=C(\eta,u,\beta,\Omega)$ independant of $L$ such that

$$\int_{\Omega\cap B_{\delta_{x_i}}(x_i)}|H_L(u)|^{\frac{2n}{n-2}}\,dx\leq\int_\Omega|\eta H_L(u))|^{\frac{2n}{n-2}}\,dx\leq C$$

for all $L>0$. Letting $L\to+\infty$, we get that

$$\int_{\Omega\cap B_{\delta_{x_i}}(x_i)}|u|^{\frac{n}{n-2}(\beta+1)}\,dx<+\infty,$$

for all $i=1...N$. We then get that $u\in L^{\frac{n}{n-2}(\beta+1)}(\Omega)=L^{\frac{n}{n-2}p}(\Omega)$. And then, $\frac{n}{n-2}p\leq p_0$ for all $p\in(2,p_0)$. Letting $p\to p_0$, we get a contradiction. Then $p_0=+\infty$ and $u\in L^p(\Omega)$ for all $p\geq1$. This ends Step 8.2.

**Step 8.3:** We claim that

$$u\in C^{0,\alpha}(\overline{\Omega})$$

for all $\alpha \in (0,1)$. Indeed, it follows from Step 8.2 and the assumption $0 < s < 2$ that there exists $p > \frac{n}{2}$ such that

$$f_\epsilon := \frac{|u|^{2^\star-2-\epsilon}u}{|x|^s} - au \in L^p(\Omega).$$

It follows from standard elliptic theory that, in this case, $u \in C^{0,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, \min\{2-s,1\})$. We let

$$\alpha_0 = \sup\{\alpha \in (0,1)/\, u \in C^{0,\alpha}(\overline{\Omega})\}.$$

Note that it follows from the preceding remark that $\alpha_0 > 0$. We let $\alpha \in (0,\alpha_0)$. Then $u \in C^{0,\alpha}(\overline{\Omega})$. Since $u(0) = 0$, we then get that

$$|u(x)| \leq |u(x) - u(0)| \leq C|x|^\alpha. \tag{133}$$

We then get with (133) that

$$|f_\epsilon(x)| = \left|\frac{|u(x)|^{2^\star-1-\epsilon}u}{|x|^s} - au\right| \leq \frac{C}{|x|^{s-(2^\star-1-\epsilon)\alpha}} + C$$

for all $x \in \Omega$. We distinguish 2 cases:

*Case 8.3.1:* $s - (2^\star - 1 - \epsilon)\alpha_0 \leq 0$. In this case, for any $p > 1$, up to taking $\alpha$ close enough to $\alpha_0$, we get that

$$f_\epsilon \in L^p(\Omega).$$

Since $\Delta u = f_\epsilon$ and $u \in H_{1,0}^2(\Omega)$, it follows from standard elliptic theory that for any $\theta \in (0,1)$, we have that $u \in C^{1,\theta}(\overline{\Omega})$. It follows that $\alpha_0 = 1$. This proves the claim in Case 8.3.1.

*Case 8.3.2:* $s - (2^\star - 1 - \epsilon)\alpha_0 > 0$. In this case, for any $p < \frac{n}{s-(2^\star-1-\epsilon)\alpha_0}$, up to taking $\alpha$ close enough to $\alpha_0$, we get that

$$f_\epsilon \in L^p(\Omega).$$

We distinguish 3 subcases.

*Case 8.3.2.1:* $s - (2^\star - 1 - \epsilon)\alpha_0 < 1$. In this case, up to taking $\alpha$ close enough to $\alpha_0$, there exists $p > n$ such that

$$f_\epsilon \in L^p(\Omega).$$

Since $\Delta u = f_\epsilon$ and $u \in H_{1,0}^2(\Omega)$, it follows from standard elliptic theory that there exist exists $\theta \in (0,1)$ such that $u \in C^{1,\theta}(\overline{\Omega})$. It follows that $\alpha_0 = 1$. This proves the claim in Case 8.3.2.1.

*Case 8.3.2.2:* $s - (2^\star - 1 - \epsilon)\alpha_0 = 1$. In this case, for any $p < n$, up to taking $\alpha$ close enough to $\alpha_0$, we get that

$$f_\epsilon \in L^p(\Omega).$$

Since $\Delta u = f_\epsilon$ and $u \in H_{1,0}^2(\Omega)$, it follows from standard elliptic theory that $u \in C^{0,\tilde{\alpha}}(\overline{\Omega})$ for all $\tilde{\alpha} \in (0,1)$. It follows that $\alpha_0 = 1$. This proves the claim in Case 8.3.2.2.

*Case 8.3.2.3:* $s - (2^\star - 1 - \epsilon)\alpha_0 > 1$. In this case, it follows from standard elliptic theory that $u \in C^{0,\tilde{\alpha}}(\overline{\Omega})$ for all

$$\tilde{\alpha} \leq 2 - (s - (2^\star - 1 - \epsilon)\alpha_0).$$

It follows from the definition of $\alpha_0$ that

$$\alpha_0 \geq 2 - (s - (2^\star - 1 - \epsilon)\alpha_0),$$

and then

$$0 \geq 2 - s + (2^\star - 2 - \epsilon)\,\alpha_0 > 0,$$

a contradiction since $s < 2$ and $\epsilon < 2^\star - 2$. This proves that Case 8.3.2.3 does not occur, and we are back to the other cases.

Clearly, theses cases end Step 8.3.

**Step 8.4:** We claim that

$$u \in C^{1,\theta}(\overline{\Omega})$$

for all $\theta \in (0, \min\{1, 2^\star - \epsilon - s\})$. We proceed as in Step 8.3. We let $\alpha \in (0,1)$ (note that $\alpha_0 = 1$). We then get that

$$|f_\epsilon(x)| = \left| \frac{|u|^{2^\star - 1 - \epsilon} u}{|x|^s} - au \right| \leq \frac{C}{|x|^{s - (2^\star - 1 - \epsilon)\alpha}} + C$$

for all $x \in \Omega$. We distinguish 2 cases:

*Case 8.4.1:* $s - (2^\star - 1 - \epsilon) \leq 0$. In this case, for any $p > 1$, up to taking $\alpha$ close enough to $\alpha_0$, we get that

$$f_\epsilon \in L^p(\Omega).$$

Since $\Delta u = f_\epsilon$ and $u \in H_{1,0}^2(\Omega)$, it follows from standard elliptic theory that $u \in C^{1,\theta}(\overline{\Omega})$ for all $\theta \in (0,1)$. It follows that $\alpha_0 = 1$. This proves the claim in Case 8.4.1.

*Case 8.4.2:* $s - (2^\star - 1 - \epsilon) > 0$. In this case, for any $p < \frac{n}{s - (2^\star - 1 - \epsilon)}$, up to taking $\alpha$ close enough to 1, we get that

$$f_\epsilon \in L^p(\Omega).$$

As easily checked,

$$1 - (s - (2^\star - 1 - \epsilon)) = 2 - s + (2^\star - 1 - \epsilon) - 1 > 2^\star - 2 - \epsilon > 0$$

We then get that there exists $p > n$ such that $f_\epsilon \in L^p(\Omega)$. Since $\Delta u = f_\epsilon$ and $u \in H_{1,0}^2(\Omega)$, it follows from standard elliptic theory that $u \in C^{1,\theta}(\overline{\Omega})$ for all $\theta \in (0, \min\{1, 2^\star - \epsilon - s\})$. This proves the claim in Case 8.4.2.

Combining Case 8.4.1 and Case 8.4.2, we obtain Step 8.4. Proposition 8.1 follows from Step 8.4. ☐

## 9. Appendix B: Properties of the Green's function

This section is devoted to the proof of some useful properties of the Green's function for a coercive operator. Concerning notations, for any function $F : X \times Y \to \mathbb{R}$ and any $x \in X$, we let $F_x : Y \to \mathbb{R}$ such that $F_x(y) = F(x,y)$ for all $y \in Y$. We prove the following:

**Theorem 9.1.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $K, \lambda > 0$. Let $\theta \in (0,1)$ and $a \in C^{0,\theta}(\overline{\Omega})$ such that*

$$|a(x)| \leq K \text{ and } |a(x) - a(y)| \leq K|x-y|^\theta \tag{134}$$

*for all $x, y \in \overline{\Omega}$ and*

$$\int_\Omega (|\nabla\varphi|^2 + a\varphi^2)\, dx \geq \lambda \int_\Omega \varphi^2\, dx \tag{135}$$

*for all $\varphi \in C_c^\infty(\Omega)$. Then there exists $G : \overline{\Omega} \times \overline{\Omega} \setminus \{(x,x)/\, x \in \overline{\Omega}\} \to \mathbb{R}$ such that*

**(G1)** *For any $x \in \Omega$, $G_x \in L^1(\Omega)$ and $G_x \in C^{2,\theta}(\overline{\Omega} \setminus \{x\})$.*

**(G2)** *For any $x \in \Omega$, $G_x > 0$ in $\Omega \setminus \{x\}$ and $G_x = 0$ on $\partial\Omega$.*

**(G3)** *For any $\varphi \in C^2(\overline{\Omega})$ such that $\varphi \equiv 0$ on $\partial\Omega$, we have that*

$$\varphi(x) = \int_\Omega G(x,y)(\Delta\varphi + a\varphi)(y)\, dy$$

*for all $x \in \Omega$.*

**(G4)** *$G(x,y) = G(y,x)$ for all $x, y \in \Omega$, $x \neq y$.*

**(G5)** *There exists $C = C(\Omega, K, \lambda) > 0$ such that*

$$|x-y|^{n-2}|G(x,y)| \leq C(\Omega, K, \lambda)$$

*for all $x, y \in \Omega$, $x \neq y$.*

**(G6)** *There exists $C = C(\Omega, K, \lambda) > 0$ such that*

$$|x-y|^{n-1}|G(x,y)| \leq C(\Omega, K, \lambda) d(y, \partial\Omega)$$

*for all $x, y \in \Omega$, $x \neq y$.*

**(G7)** *There exists $C = C(\Omega, K, \lambda) > 0$ such that*

$$|x-y|^{n-1}|\nabla G_x(y)| \leq C(\Omega, K, \lambda)$$

*for all $x, y \in \Omega$, $x \neq y$.*

**(G8)** *There exists $C = C(\Omega, K, \lambda) > 0$ such that*

$$|x-y|^{n}|\nabla_y G_x(y)| \leq C(\Omega, K, \lambda) d(x, \partial\Omega)$$

*for all $x, y \in \Omega$, $x \neq y$.*

Some similar properties are available for the normal derivative of $G$ at the boundary. Namely,

**Theorem 9.2.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 3$. We assume $0 \in \partial\Omega$. Let $K, \lambda > 0$. Let $\theta \in (0,1)$ and $a \in C^{0,\theta}(\overline{\Omega})$ such that (134) and (135) hold. We let $G$ as in Theorem 9.1. We let $H(x) = -\partial_\nu G_x(0)$ for all $x \in \overline{\Omega} \setminus \{x\}$. Then the following assertions hold:*

**(G9)** $H \in C^2(\overline{\Omega} \setminus \{0\})$, $H > 0$ *in* $\Omega$ *and* $H \equiv 0$ *on* $\partial\Omega \setminus \{0\}$,

**(G10)** $\Delta H + aH = 0$ *in* $\Omega$,

**(G11)** *There exists* $C = C(\Omega, K, \lambda) > 0$ *such that*

$$\frac{d(x, \partial\Omega)}{C|x|^n} \leq H(x) \leq \frac{Cd(x, \partial\Omega)}{|x|^n}$$

*for all* $x \in \Omega$.

**(G12)** *There exists* $C = C(\Omega, K, \lambda) > 0$ *and* $\delta = \delta(\Omega, K, \lambda) > 0$ *such that*

$$\frac{1}{C|x|^n} \leq |\nabla H(x)| \leq \frac{C}{|x|^n}$$

*for all* $x \in B_\delta(0) \cap \Omega$.

The proof of Theorem 9.1 is very close to the proof of the existence of the Green's function on a compact manifold without boundary provided in [15]. We just give the main steps of the proof and outline the difference with [15] when necessary. We prove Theorem 9.2 in details.

**Step 9.1:** This Step is devoted to the proof of points (G1)-(G5) of Theorem 9.1. We only sketch the proof. Details are available in [15]. We define

$$\mathcal{H}(x, y) = \frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}$$

for all $x, y \in \mathbb{R}^n$ such that $x \neq y$. In this expression, $\omega_{n-1}$ denotes the volume of the standard $(n-1)-$sphere. The function $\mathcal{H}$ is the standard Green kernel of the Laplacian in $\mathbb{R}^n$. We define the functions $\Gamma_i$'s by induction. Given $x, y \in \overline{\Omega}$, $x \neq y$, we let

$$\begin{aligned} &\Gamma_1(x, y) = -a(y)\mathcal{H}(x, y) \\ &\Gamma_{i+1}(x, y) = \int_\Omega \Gamma_i(x, z)\Gamma_1(z, y)\, dz \quad \text{for all } i \geq 1. \end{aligned}$$

As easily checked, $\Gamma_i \in C^0(\overline{\Omega} \times \overline{\Omega} \setminus \{(x, x)/\, x \in \overline{\Omega}\})$ for all $i \geq 1$. Standard computations yield that there exists $C(\Omega, n, K) > 0$ such that

$$\begin{aligned} &|\Gamma_i(x, y)| \leq C(\Omega, n, K)|x-y|^{2i-n} && \text{if } 2i < n \\ &|\Gamma_i(x, y)| \leq C(\Omega, n, K)\left(1 + |\ln|x-y||\right) && \text{if } 2i = n \\ &|\Gamma_i(x, y)| \leq C(\Omega, n, K) && \text{if } 2i > n,\ i \leq n. \end{aligned}$$

for all $x, y \in \overline{\Omega}$, $x \neq y$. In addition, $\Gamma_i$ can be extended to a continuous function in $\overline{\Omega} \times \overline{\Omega}$ for all $i > n/2$. We let $x \in \Omega$. We let $U_x \in H^2_{1,0}(\Omega)$ such that

$$\Delta U_x + aU_x = \Gamma_{n+1}(x, \cdot) \text{ in } \mathcal{D}'(\Omega).$$

Since $\Gamma_{n+1}$ is uniformly bounded in $L^\infty$, it follows from standard elliptic theory that $U_x \in H^p_2(\Omega)$ for all $p > 1$ and that there exists $C(\Omega, K, \lambda) > 0$ such that

$$\|U_x\|_{C^1(\overline{\Omega})} \leq C(\Omega, K, \lambda)$$

for all $x \in \Omega$. We let $V_x \in H_1^2(\Omega)$ such that

$$\begin{cases} \Delta V_x + aV_x = 0 & \text{in } \mathcal{D}'(\Omega) \\ V_x(y) = -\mathcal{H}(x,y) - \sum_{i=1}^n \int_\Omega \Gamma_i(x,z)\mathcal{H}(z,y)\,dz & \text{for all } y \in \partial\Omega. \end{cases}$$

It follows from standard elliptic theory that for any $x \in \Omega$, $V_x \in C^1(\Omega)$. Moreover, it follows from the explicit expression of $\mathcal{H}$ and the $\Gamma_i$'s that there exists $C(\Omega, K, \lambda)' > 0$ such that $V_x(y) \leq C(\Omega, K, \lambda)'$ for all $x \in \Omega$ and all $y \in \partial\Omega$. Since $\Delta + a$ is coercive, it follows from the comparison principle that there exists $C(\Omega, K, \lambda) > 0$ such that

$$V_x(y) \leq C(\Omega, K, \lambda)$$

for all $x \in \Omega$ and all $y \in \Omega$. We let

$$G_x(y) := \mathcal{H}(x,y) + \sum_{i=1}^n \int_\Omega \Gamma_i(x,z)\mathcal{H}(z,y)\,dz + U_x(y) + V_x(y)$$

for all $y \in \Omega$. It follows from the construction of $G$ that there exists $C(\Omega, K, \lambda) > 0$ such that

$$G(x,y) \leq C(\Omega, K, \lambda) \cdot |x-y|^{2-n}$$

for all $x, y \in \Omega$, $x \neq y$ and that $G_x$ vanishes on $\partial\Omega$ for all $x \in \Omega$. This prove point (G5). We let $\varphi \in C^2(\overline{\Omega})$ such that $\varphi \equiv 0$ on $\partial\Omega$. Noting that

$$\varphi(z) = \int_\Omega \mathcal{H}(z,y)\Delta\varphi(y)\,dy + \int_{\partial\Omega} \mathcal{H}(x,y)\partial_\nu\varphi(y)\,d\sigma(y)$$

for all $z \in \Omega$, we get with some integrations by parts that

$$\varphi(x) = \int_\Omega G(x,y)(\Delta\varphi + a\varphi)(y)\,dy.$$

This proves point (G3). It then follows that

$$\Delta G_x + aG_x = 0 \text{ in } \mathcal{D}'(\Omega \setminus \{x\}).$$

Since $G_x \equiv 0$ on $\partial\Omega$, we get that $G_x \in C^{2,\theta}_{loc}(\overline{\Omega} \setminus \{x\})$. It the follows from the construction and the maximum principle that $G_x > 0$ in $\Omega \setminus \{x\}$. This proves points (G2) and (G1). Point (G4) is standard, we refer to [2] or [15].

**Step 9.2:** We prove points (G6) and (G7) of Theorem 9.1. We proceed by contradiction and assume that there exists a sequence $(a_k)_{k\in\mathbb{N}} \in C^{0,\theta}(\overline{\Omega})$ and sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \in \Omega$ such that (134) and (135) hold and

$$\lim_{k\to+\infty} \left[ |x_k - y_k|^{n-1}|\nabla G_{x_k}(y_k)| + \frac{|x_k - y_k|^{n-1}G_{x_k}(y_k)}{d(y_k, \partial\Omega)} \right] = +\infty \tag{136}$$

where $G_{x_k}$ is the Green's function for $\Delta + a_k$ at $x_k$. We let $x_\infty = \lim_{k\to+\infty} x_k$ and $y_\infty = \lim_{k\to+\infty} y_k$ (these limits exist up to a subsequence).

*Case 1: $x_\infty \neq y_\infty$.* We let $0 < \delta < |x_\infty - y_\infty|/4$. It follows from point (G5) that there exists $C > 0$ independant of $k$ such that $|G_{x_k}(y)| \leq C$ for all $y \in \Omega \cap B_{y_\infty}(2\delta)$. Since $\Delta G_{x_k} + a_k G_{x_k} = 0$ and $G_{x_k} = 0$ on $\partial\Omega$, it follows from standard elliptic theory that

$$\|G_{x_k}\|_{C^1(\overline{\Omega}\cap B_{y_\infty}(\delta))} = O(1)$$

when $k \to +\infty$. Since $G_{x_k}$ vanishes on $\partial\Omega$, we get that there exists $C > 0$ such that

$$|G_{x_k}(y)| \leq Cd(y, \partial\Omega) \text{ and } |\nabla G_{x_k}(y)| \leq C$$

for all $y \in \overline{\Omega} \cap B_{y_\infty}(\delta)$ and all $\epsilon > 0$. A contradiction with (136).

*Case 2:* $x_\infty = y_\infty$.

*Case 2.1:* We assume that

$$d(x_k, \partial\Omega) \geq 2|y_k - x_k| \tag{137}$$

up to a subsequence. We let

$$\tilde{G}_k(z) = |y_k - x_k|^{n-2} G(x_k, x_k + |y_k - x_k| z)$$

for all $z \in B_{3/2}(0)$. With our assumption, this is well defined. It follows from (G5) that there exists $C > 0$ such that

$$|\tilde{G}_k(z)| \leq C$$

for all $z \in B_{3/2}(0) \setminus \overline{B}_{1/4}(0)$. Moreover, $\tilde{G}_k$ verifies the equation

$$\Delta\tilde{G}_k + |y_k - x_k|^2 a_k(x_k + |y_k - x_k| z) \tilde{G}_k(z) = 0$$

in $B_{3/2}(0) \setminus \overline{B}_{1/4}(0)$. It follows from standard elliptic theory that

$$\|\tilde{G}_k\|_{C^1(B_{5/4}(0) \setminus \overline{B}_{1/2}(0))} = O(1)$$

when $k \to +\infty$. Taking $z = \frac{y_k - x_k}{|y_k - x_k|}$ and coming back to $G_{x_k}$, we get that

$$|x_k - y_k|^{n-1} |\nabla G_{x_k}(y_k)| = O(1) \tag{138}$$

when $k \to +\infty$. Moreover, it follows from point (G5) of Theorem 9.1 and (137) that there exists $C > 0$ such that

$$|x_k - y_k|^{n-1} G_{x_k}(y_k) \leq Cd(y_k, \partial\Omega) \tag{139}$$

when $k \to +\infty$. Inequations (138) and (139) contradict (136).

*Case 2.2:* We assume that

$$d(x_k, \partial\Omega) \leq 2|y_k - x_k| \tag{140}$$

up to a subsequence. In particular, $x_\infty \in \partial\Omega$. We let a chart $\varphi : U \to V$ as in (13) with $y_0 = x_\infty$ and where $U, V$ are open neighborhoods of 0 and $x_\infty$ respectively. We let $\tilde{x}_k, \tilde{y}_k \in U \cap \{x_1 < 0\}$ such that $x_k = \varphi(\tilde{x}_k)$ and $y_k = \varphi(\tilde{y}_k)$. As a remark, $\lim_{k \to +\infty} \tilde{x}_k = \lim_{k \to +\infty} \tilde{y}_k = 0$. We let $\tilde{x}_{k,1} < 0$ be the first coordinate of $\tilde{x}_k$. As in Step 3.2, we have that $d(x_k, \partial\Omega) = (1 + o(1))|\tilde{x}_{k,1}|$ when $k \to +\infty$. We then get with (140) that $\tilde{x}_{k,1} = O(|\tilde{y}_k - \tilde{x}_k|)$ when $k \to +\infty$. We let

$$\rho_k = \frac{\tilde{x}_{k,1}}{|\tilde{y}_k - \tilde{x}_k|} \text{ and } \rho_\infty = \lim_{k \to +\infty} \rho_k$$

(this limit exists up to a subsequence). We let $R > 0$ and we let

$$\tilde{G}_k(z) = |\tilde{y}_k - \tilde{x}_k|^{n-2} G(x_k, \varphi(\tilde{x}_k + |\tilde{y}_k - \tilde{x}_k| (z - \rho_k \vec{e}_1)))$$

for all $k$ and all $z \in B_R(0) \cap \{z_1 \leq 0\}$. Here $\vec{e}_1$ denotes the first vector of the canonical basis of $\mathbb{R}^n$. Note that $\tilde{G}_k$ vanishes on $B_R(0) \cap \{z_1 = 0\}$. It follows of the pointwise estimate (G5) that for any $R, \delta > 0$, there exists $C(R, \delta) > 0$ such that

$$|\tilde{G}_k(z)| \leq C(R,\delta)$$

for all $z \in [B_R(0) \setminus \overline{B}_\delta((\rho_\infty, 0, ..., 0))] \cap \{z_1 \leq 0\}$. The function $\tilde{G}_k$ verifies the equation

$$\Delta_{g_k}\tilde{G}_k + |\tilde{y}_k - \tilde{x}_k|^2 a_k(\varphi(\tilde{x}_k + |\tilde{y}_k - \tilde{x}_k|(z - (\rho_k, 0, .., 0))))\tilde{G}_k = 0$$

in $[B_R(0) \setminus \overline{B}_\delta((\rho_\infty, 0, ..., 0))] \cap \{z_1 \leq 0\}$. It then follows from standard elliptic theory that $\|\tilde{G}_k\|_{C^1([B_{R/2}(0)\setminus\overline{B}_{2\delta}(\rho_\infty,...,0)]\cap\{z_1\leq 0\})} = O(1)$ when $k \to +\infty$. As in Case 2.1, we get that

$$|x_k - y_k|^{n-1}|\nabla G_{x_k}(y_k)| = O(1) \tag{141}$$

when $k \to +\infty$. Moreover, since $\tilde{G}_k$ vanishes on $\partial\mathbb{R}^n_-$, there exists $C > 0$ such that

$$|\tilde{G}_k(z)| \leq C|z_1|$$

for all $z \in [B_{R/2}(0) \setminus \overline{B}_{2\delta}(\rho_\infty, ..., 0)] \cap \{z_1 \leq 0\}$. Taking $z = (\rho_k, ..., 0) + \frac{\tilde{y}_k - \tilde{x}_k}{|\tilde{y}_k - \tilde{x}_k|}$, we get that

$$|x_k - y_k|^{n-1}G_{x_k}(y_k) \leq Cd(y_k, \partial\Omega) \tag{142}$$

for all $k$ large enough. A contradiction with (136).

In all the cases, we have contradicted (136). This proves points (G6) and (G7) of Theorem 9.1.

**Step 9.3:** We prove point (G8) of Theorem 9.1. More precisely, we claim that there exists $C = C(\Omega, K, \lambda) > 0$ such that

$$|x - y|^n G(x, y) \leq Cd(y, \partial\Omega)d(x, \partial\Omega) \tag{143}$$

and

$$|x - y|^n|\nabla_y G(x, y)| \leq Cd(x, \partial\Omega)$$

for all $x, y \in \Omega$, $x \neq \Omega$. Indeed we proceed as in the proof of points (G6) and (G7). We proceed by contradiction and assume that there exist a sequence $(a_k)_{k\in\mathbb{N}} \in C^{0,\theta}(\overline{\Omega})$ and sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \in \Omega$ such that (134) and (135) hold and

$$\lim_{k\to+\infty} |x_k - y_k|^n \frac{|G(x_k, y_k)|}{d(x_k, \partial\Omega)d(y_k, \partial\Omega)} + |x_k - y_k|^n \frac{|\nabla G_{x_k}(y_k)|}{d(x_k, \partial\Omega)} = +\infty \tag{144}$$

where $G_{x_k}$ is the Green's function for $\Delta + a_k$ at $x_k$. We let $x_\infty = \lim_{k\to+\infty} x_k$ and $y_\infty = \lim_{k\to+\infty} y_k$ (these limits exist up to a subsequence).

*Case 1:* $x_\infty \neq y_\infty$. We let $0 < \delta < |x_\infty - y_\infty|/4$. We let

$$\tilde{G}_k(z) = \frac{G_k(x_k, z)}{d(x_k, \partial\Omega)}$$

for all $z \in \Omega$. As in Case 1 of the proof of (G6)-(G7), using (G6), we get that

$$\|\tilde{G}_k\|_{C^1(\overline{\Omega}\cap B_{y_\infty}(\delta))} = O(1)$$

when $k \to +\infty$. It then follows that

$$\tilde{G}_k(y_k) \leq Cd(y_k, \partial\Omega) \text{ and } |\nabla\tilde{G}_k(y_k)| \leq C$$

when $k \to +\infty$. A contradiction with (144).

*Case 2:* $x_\infty = y_\infty$.

*Case 2.1:* We assume that

$$d(x_k, \partial\Omega) \geq 2|y_k - x_k|$$

up to a subsequence. We then obtain that $|x_k - y_k| \leq d(y_k, \partial\Omega)$. This inequality and (G6)-(G7) yield to a contradiction with (144).

*Case 2.2:* We assume that

$$d(x_k, \partial\Omega) \leq 2|y_k - x_k|$$

up to a subsequence. In particular, $x_\infty \in \partial\Omega$. We let a chart $\varphi : U \to V$ as in (13) with $y_0 = x_\infty$ and where $U, V$ are open neighborhoods of 0 and $x_\infty$ respectively. We let $\tilde{x}_k, \tilde{y}_k \in U \cap \{x_1 < 0\}$ such that $x_k = \varphi(\tilde{x}_k)$ and $y_k = \varphi(\tilde{y}_k)$. We let

$$\tilde{G}_k(z) = |\tilde{y}_k - \tilde{x}_k|^{n-1} \frac{G\left[x_k, \varphi\left(\tilde{x}_k + |\tilde{y}_k - \tilde{x}_k|\left(z - (\frac{\tilde{x}_{k,1}}{|\tilde{y}_k - \tilde{x}_k|}, 0, .., 0)\right)\right)\right]}{d(x_k, \partial\Omega)}$$

for all $z \in [B_R(0) \setminus \overline{B}_\delta(\rho_\infty, 0, ..., 0)] \cap \{z_1 \leq 0\}$. As in Case 2.2 of the proof of (G6)-(G7), we get with (G6) that for any $R > 4\delta > 0$, we have that

$$\|\tilde{G}_k\|_{C^1([B_{R/2}(0) \setminus \overline{B}_{2\delta}(\rho_\infty, 0, ..., 0)] \cap \{z_1 \leq 0\})} = O(1)$$

when $k \to +\infty$, where $\rho_\infty = \lim_{k\to+\infty} \frac{\tilde{x}_{k,1}}{|\tilde{y}_k - \tilde{x}_k|}$. Since $\tilde{G}_k$ vanishes on $\{z_1 = 0\}$, it then follows that there exists $C > 0$ such that $|\tilde{G}_k(z)| \leq C|z_1|$ for all $z \in [B_{R/2}(0) \setminus \overline{B}_{2\delta}(\rho_\infty, 0, ..., 0)] \cap \{z_1 \leq 0\}$. Coming back to the definition of $\tilde{G}_k$ and noting that $d(y_k, \partial\Omega) = (1 + o(1))|\tilde{y}_{k,1}|$ when $k \to +\infty$, we get a contradiction with (144) as in Case 2.2 of Step 9.2.

In all the cases, we have contradicted (144). This proves the claim and ends Step 9.3.

The proof of Theorem 9.1 is complete. We prove Theorem 9.2.

**Step 9.4:** We let $H(x) = -\partial_\nu G_x(0)$ for any $x \in \overline{\Omega} \setminus \{0\}$. It follows from (143) that there exists $C = C(\Omega, K, \lambda) > 0$ such that

$$0 \leq H(x) \leq \frac{Cd(x, \partial\Omega)}{|x|^n} \leq \frac{C}{|x|^{n-1}} \tag{145}$$

for all $x \in \Omega$. Since $\Delta G_x + aG_x = 0$ in $\Omega \setminus \{x\}$, using the symetry (G4) of $G$ and (145), we get that $H \in C^2(\overline{\Omega} \setminus \{0\})$ and that $\Delta H + aH = 0$ in $\Omega$ and $H(x) = 0$ for all $x \in \partial\Omega \setminus \{0\}$. Derivating (G3), we get that

$$\partial_\nu \varphi(0) = -\int_\Omega H(x)(\Delta\varphi + a\varphi)(x)\, dx \tag{146}$$

for all $\varphi \in C^2(\overline{\Omega})$ such that $\varphi \equiv 0$ on $\partial\Omega$.

**Step 9.5:** Assume that there exists a sequence $(a_k)_{k>0} \in C^{0,\theta}(\overline{\Omega})$ such that (135) and (134) hold, that there exists a sequence $(r_k)_{k>0} \in \mathbb{R}$ such that $r_k > 0$, $\lim_{k\to+\infty} r_k = 0$ and

$$\lim_{k\to+\infty}\sup_{|x|=r_k}\frac{H_k(x)|x|^n}{d(x,\partial\Omega)}=0,$$

where $H_k$ comes from the Green's function of $\Delta + a_k$. We claim that in this situation, we have that

$$\lim_{k\to+\infty}\sup_{\frac{1}{2}r_k\le|x|\le 3r_k}\left(\frac{H_k(x)|x|^n}{d(x,\partial\Omega)}+|x|^n|\nabla H_k(x)|\right)=0. \tag{147}$$

Indeed, we let $\varphi : U \to V$ as in (13) where $U, V$ are open neighborhoods of 0. We let

$$\tilde{H}_k(x) = r_k^{n-1}H_k(\varphi(r_k x))$$

for all $x \in \frac{U}{r_k}\cap\{x_1 \le 0\}$. It follows from (145) that for any $R > \delta > 0$, there exists $C(R,\delta) > 0$ such that $|\tilde{H}_k(x)| \le C(R,\delta)$ for all $x \in [B_R(0)\setminus B_\delta(0)]\cap\{x_1\le 0\}$. In addition $\tilde{H}_k$ vanishes when $x_1 = 0$. Moreover, we have that

$$\Delta_{g_k}\tilde{H}_k + r_k^2 a_k(\varphi(r_k x))\tilde{H}_k = 0,$$

where $(g_k)_{ij} = (\partial\varphi,\partial_j\varphi)(r_k x)$ for $i,j \in \{1,...,n\}$. It then follows from standard elliptic theory that there exists $\tilde{H} \in C^2(\overline{\mathbb{R}^n_-}\setminus\{0\})$ such that $\Delta\tilde{H} = 0$ in $\mathbb{R}^n_-\setminus\{0\}$ and

$$\lim_{k\to+\infty}\tilde{H}_k = \tilde{H}$$

in $C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0\})$. As easily checked, we have that

$$\begin{aligned}&\lim_{k\to+\infty}\sup_{\frac{1}{2}r_k\le|x|\le 3r_k}\left(\frac{H_k(x)|x|^n}{d(x,\partial\Omega)}+|x|^n|\nabla H_k(x)|\right)\\ &= \sup_{\frac{1}{2}\le|x|\le 3}\left(\frac{\tilde{H}(x)|x|^n}{|x_1|}+|x|^n|\nabla\tilde{H}(x)|\right)\end{aligned} \tag{148}$$

and

$$0=\lim_{k\to+\infty}\sup_{|x|=r_k}\frac{H_k(x)|x|^n}{d(x,\partial\Omega)}=\sup_{|x|=1}\left(\frac{\tilde{H}(x)|x|^n}{|x_1|}\right). \tag{149}$$

Assume that $\tilde{H}\not\equiv 0$. Then, since $\tilde{H}\ge 0$ vanishes on $\partial\mathbb{R}^n_-$, we have that $\tilde{H} > 0$ in $\mathbb{R}^n_-$ and $\partial_1\tilde{H} < 0$ on $\partial\mathbb{R}^n_-\setminus\{0\}$. It then follows that the RHS of (149) is positive. A contradiction, since the LHS is 0. Then $\tilde{H}\equiv 0$, and (147) follows from (148). This ends Step 9.5.

**Step 9.6:** We claim that there exists $\epsilon(\Omega,K,\lambda) > 0$ such that

$$\liminf_{r\to 0}\sup_{|x|=r}\frac{H(x)|x|^n}{d(x,\partial\Omega)}\ge\epsilon(\Omega,K,\lambda). \tag{150}$$

Indeed, we argue by contradiction and assume that there exists a sequence $(a_k)_{k>0}\in C^{0,\theta}(\overline{\Omega})$ such that (135) and (134) hold, that there exists a sequence $(r_k)_{k>0}\in\mathbb{R}$ such that $r_k > 0$, $\lim_{k\to+\infty} r_k = 0$ and

$$\lim_{k\to+\infty}\sup_{|x|=r_k}\frac{H_k(x)|x|^n}{d(x,\partial\Omega)}=0,$$

where $H_k$ comes from the Green's function of $\Delta+a_k$. It then follows from Step 9.5. that

$$\lim_{k\to+\infty} m_k = 0. \tag{151}$$

where

$$m_k := \sup_{\frac{1}{2}r_k\leq|x|\leq 3r_k}\left(\frac{H_k(x)|x|^n}{d(x,\partial\Omega)}+|x|^n|\nabla H_k(x)|\right).$$

We let $\tilde{\eta}\in C^\infty(\mathbb{R}^n)$ such $\tilde{\eta}\equiv 0$ in $B_1(0)$ and $\tilde{\eta}\equiv 1$ in $\mathbb{R}^n\setminus B_2(0)$. We let $\eta_k(x)=\tilde{\eta}(x/r_k)$ for all $x\in\mathbb{R}^n$ and all $k>0$. We let $\varphi_k\in C^2(\overline{\Omega})$ such that

$$\Delta\varphi_k + a_k\varphi_k = 1 \text{ in } \Omega \text{ and } \varphi_k\equiv 0 \text{ on } \partial\Omega.$$

It follows from standard elliptic theory that $\lim_{k\to+\infty}\varphi_k=\varphi\not\equiv 0$ in $C^2(\overline{\Omega})$. It then follows from Hopf's maximum principle that

$$\partial_\nu\varphi(0)<0. \tag{152}$$

Integrating by parts and using that $\Delta H_k + a_k H_k = 0$, we obtain that

$$\begin{aligned}
\int_\Omega H_k(x)(\Delta\varphi_k + a_k\varphi_k)(x)\,dx &= \int_\Omega (\eta_k H_k)(x)(\Delta\varphi_k + a_k\varphi_k)(x)\,dx + o(1)\\
&= \int_\Omega (\Delta(\eta_k H_k) + a\eta_k H_k)\varphi_k\,dx + o(1)\\
&= \int_\Omega ((\Delta\eta_k)H_k - 2\nabla\eta_k\nabla H_k)\varphi_k\,dx + o(1)\\
&= \int_{\Omega\cap B_{2r_k}(0)\setminus B_{r_k}(0)} ((\Delta\eta_k)H_k - 2\nabla\eta_k\nabla H_k)\varphi_k\,dx\\
&\quad + o(1)
\end{aligned}$$

where $\lim_{k\to+\infty}o(1)=0$. Since $\varphi_k(0)=0$ and $\lim_{k\to+\infty}\varphi_k=\varphi$ in $C^1(\overline{\Omega})$, using the definition of $m_k$ we get that

$$\int_\Omega H_k(x)(\Delta\varphi_k + a_k\varphi_k)(x)\,dx = O\left(r_k^n(m_k r_k^{-2} r_k^{1-n} r_k)\right) + o(1) = O(m_k) + o(1).$$

With (151), letting $k\to+\infty$, and using (146) we get that

$$\partial_\nu\varphi(0) = \partial_\nu\varphi_k(0) + o(1) = -\int_\Omega H_k(x)(\Delta\varphi_k + a_k\varphi_k)(x)\,dx + o(1) = 0.$$

A contradiction with (152), and the claim is proved.

**Step 9.7:** We claim that there exists $\epsilon(\Omega,K,\lambda)>0$ such that

$$\liminf_{r\to 0}\inf_{|x|=r}\frac{H(x)|x|^n}{d(x,\partial\Omega)}\geq\epsilon(\Omega,K,\lambda). \tag{153}$$

Indeed, we argue by contradiction and assume that there exists a sequence $(a_k)_{k>0} \in C^{0,\theta}(\overline{\Omega})$ such that (135) and (134) hold, that there exists a sequence $(r_k)_{k>0} \in \mathbb{R}$ such that $r_k > 0$, $\lim_{k\to+\infty} r_k = 0$ and

$$\lim_{k\to+\infty} \inf_{|x|=r_k} \frac{H_k(x)|x|^n}{d(x,\partial\Omega)} = 0,$$

where $H_k$ comes from the Green's function of $\Delta + a_k$. Mimicking the proof of Step 9.5, we obtain that $\tilde{H}_k(x) := r_k^{n-1} H_k(\varphi(r_k x))$ converges to $\tilde{H}$ in $C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0\})$. We get that

$$\inf_{|x|=1} \frac{\tilde{H}(x)|x|^n}{|x_1|} = \lim_{k\to+\infty} \inf_{|x|=r_k} \frac{H_k(x)|x|^n}{d(x,\partial\Omega)} = 0.$$

Since $\tilde{H} \geq 0$ is harmonic and vanishes on $\partial\mathbb{R}^n_- \setminus \{0\}$, it follows from Hopf's maximum principle that $\tilde{H} \equiv 0$. We then get that

$$\lim_{k\to+\infty} \sup_{|x|=r_k} \frac{H_k(x)|x|^n}{d(x,\partial\Omega)} = \sup_{|x|=1} \frac{\tilde{H}(x)|x|^n}{|x_1|} = 0.$$

A contradiction with Step 9.6. This proves the claim.

**Step 9.8:** We claim that there exists $C = C(\Omega, K, \lambda) > 0$ such that

$$\frac{d(x,\partial\Omega)}{C|x|^n} \leq H(x) \leq \frac{Cd(x,\partial\Omega)}{|x|^n}$$

for all $x \in \Omega \setminus \{0\}$. Indeed, this claim is a consequence of (145), Step 9.7 and standard elliptic theory. This proves point (G11).

**Step 9.9:** We claim that there exists $C(\Omega, K, \lambda) > 0$ such that

$$|x|^n |\nabla H(x)| \leq C(\Omega, K, \lambda) \tag{154}$$

for all $x \in \Omega \setminus \{0\}$. We proceed by contradiction and assume that that there exists a sequence $(a_k)_{k>0} \in C^{0,\theta}(\overline{\Omega})$ such that (135) and (134) hold, that there exists a sequence $(x_k)_{k>0} \in \Omega$ such that

$$\lim_{k\to+\infty} |x_k|^n |\nabla H_k(x_k)| = +\infty, \tag{155}$$

where $H_k$ comes from the Green's function of $\Delta + a_k$.

*Case 1:* $\lim_{k\to+\infty} x_k \neq 0$. In this case, since $\Delta H_k + a_k H_k = 0$, it follows from (145) and standard elliptic theory that $|\nabla H_k(x_k)| = O(1)$ when $k \to +\infty$.

*Case 2:* $\lim_{k\to+\infty} x_k = 0$. We consider $\varphi : U \to V$ as in (13) with $y_0 = 0$ and $U, V$ are open neighborhoods of 0. We let $x_k = \varphi(\tilde{x}_k)$. We let

$$\tilde{H}_k(x) = |\tilde{x}_k|^{n-1} H_k(\varphi(|\tilde{x}_k| x))$$

for all $x \in \frac{U}{|\tilde{x}_k|} \cap \{x_1 \leq 0\}$. As in Step 9.5, we get that there exists $C > 0$ such that

$$\|\tilde{H}_k\|_{C^1(\{x_1\leq 0\}\cap B_2(0)\setminus B_{1/2}(0))} \leq C.$$

Estimating the gradient at $\tilde{x}_k/|\tilde{x}_k|$, we get that

$$|x_k|^n |\nabla H_k(x_k)| = O(1)$$

when $k \to +\infty$.

In both cases, we have contradicted (155). This proves (154).

**Step 9.10:** We claim that there exists $\delta(\Omega,K,\lambda), C(\Omega,K,\lambda)>0$ such that

$$|x|^n|\nabla H(x)|\geq C(\Omega,K,\lambda) \tag{156}$$

for all $x\in\Omega\setminus\{0\}$ such that $|x|\leq\delta(\Omega,K,\lambda)$. We proceed by contradiction and assume that that there exists a sequence $(a_k)_{k>0}\in C^{0,\theta}(\overline{\Omega})$ such that (135) and (134) hold, that there exists a sequence $(x_k)_{k>0}\in\Omega$ such that $\lim_{k\to+\infty}x_k=0$ and

$$\lim_{k\to+\infty}|x_k|^n|\nabla H_k(x_k)|=0, \tag{157}$$

where $H_k$ comes from the Green's function of $\Delta+a_k$. We let $x_k=\varphi(\tilde{x}_k)$ and $y_k=\varphi(\tilde{y}_k)$. We let

$$\tilde{H}_k(x)=|\tilde{x}_k|^{n-1}H_k(\varphi(|\tilde{x}_k|x))$$

for all $x\in\frac{U}{|\tilde{x}_k|}\cap\{x_1\leq 0\}$. Mimicking the proof of Steps 9.5 and 9.9, we get that there exists $\tilde{H}\in C^2(\overline{\mathbb{R}^n_-}\setminus\{0\})$ such that

$$\lim_{k\to+\infty}\tilde{H}_k=\tilde{H} \tag{158}$$

in $C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{0\})$. In particular, we have that $\tilde{H}$ is harmonic. It follows from Step 9.8 and (158) that there exists $C>0$ such that

$$\frac{|x_1|}{C|x|^n}\leq\tilde{H}(x)\leq\frac{C|x_1|}{|x|^n}$$

for all $x\in\mathbb{R}^n_-\setminus\{0\}$. It then follows from the rigidity Property 9.1 below that $\nabla\tilde{H}(x)\neq 0$ for all $x\in\overline{\mathbb{R}^n_-}\setminus\{0\}$. It follows from (157) and (158) that there exists $\hat{x}\in\overline{\mathbb{R}^n_-}\setminus\{0\}$ such that $\nabla\tilde{H}(\hat{x})=0$. A contradiction. This proves (156).

Clearly Theorem 9.2 is a consequence of Steps 9.4 to 9.10.

**Step 9.11:** Our last step is the proof of the following rigidity result:

**Proposition 9.1.** *Let $h\in C^2(\overline{\mathbb{R}^n_-}\setminus\{0\})$. We assume that $h$ is nonnegative in a neighborhood of 0, harmonic and vanishes on $\partial\mathbb{R}^n_-\setminus\{0\}$. We assume that there exists $C>0$ such that $|h(x)|\leq C|x|^{1-n}$ for all $x\in\mathbb{R}^n_-\setminus\{0\}$. Then there exists $\alpha\geq 0$ such that*

$$h(x)=\alpha\frac{|x_1|}{|x|^n}$$

*for all $x\in\mathbb{R}^n_-\setminus\{0\}$.*

*Proof.* Up to rescaling, we assume that $h\geq 0$ in $B_2(0)\setminus\{0\}$. We let

$$\alpha:=\max\left\{\lambda\geq 0/\,h(x)\geq\lambda\frac{|x_1|}{|x|^n}\text{ for all }x\in\mathbb{R}^n_-\cap\overline{B}_1(0)\right\}.$$

We let $\tilde{h}(x)=h(x)-\alpha\frac{|x_1|}{|x|^n}$ for all $x\in\mathbb{R}^n_-$. The new function $\tilde{h}$ satisfies the hypothesis of Proposition 9.1. In addition, it follows from the definition of $\alpha$ and Hopf's maximum principle that

$$\liminf_{r\to 0}\inf_{|x|=r}\frac{\tilde{h}(x)|x|^n}{-x_1}=0.$$

Mimicking what was done in Steps 9.5 and 9.7, we get that

$$\liminf_{r\to 0}\sup_{|x|=r,\, x\in\mathbb{R}^n_-}\frac{\tilde{h}(x)|x|^n}{-x_1}=0.$$

We let

$$\hat{h}(x_1,\tilde{x}):=\left\{\begin{array}{ll}\tilde{h}(x_1,\tilde{x}) & \text{if } x_1\leq 0 \text{ and } (x_1,\tilde{x})\neq 0\\ -\tilde{h}(-x_1,\tilde{x}) & \text{if } x_1>0.\end{array}\right.$$

As easily checked, we have that $\hat{h}\in C^2(\mathbb{R}^n\setminus\{0\})$ and $\Delta\hat{h}=0$ in $\mathbb{R}^n\setminus\{0\}$. With the definition of $\hat{h}$, we immediately get that

$$\liminf_{r\to 0}\sup_{|x|=r}\frac{|\hat{h}(x)|\cdot|x|^n}{|x_1|}=0.$$

We let $(r_k)_{k>0}$ such that $\lim_{k\to+\infty}r_k=0$ and

$$\liminf_{k\to+\infty}\sup_{|x|=r_k}\frac{|\hat{h}(x)|\cdot|x|^n}{|x_1|}=0.$$

We let $\tilde{\eta}\in C^\infty(\mathbb{R}^n)$ such that $\tilde{\eta}\equiv 0$ in $B_1(0)$ and $\tilde{\eta}\equiv 1$ in $\mathbb{R}^n\setminus B_2(0)$. We let $\eta_k(x):=\tilde{\eta}(x/r_k)$. Mimicking what was done in Step 9.6, we let $\varphi\in C^\infty_c(\mathbb{R}^n)$ and get that

$$\begin{aligned}\int_{\mathbb{R}^n}\hat{h}\Delta\varphi\,dx &= \int_{\mathbb{R}^n}\eta_k\hat{h}\Delta\varphi\,dx+o(1)\\ &= \int_{\mathbb{R}^n}\Delta(\eta_k\hat{h})\cdot(\varphi-\varphi(0))\,dx+\varphi(0)\int_{\mathbb{R}^n}\Delta(\eta_k\hat{h})\,dx+o(1)\\ &= o(1)+\varphi(0)\int_{\mathbb{R}^n}\Delta(\eta_k\hat{h})\,dx\end{aligned}$$

We let $R>3$, and choose $k_0$ such that $0<r_k<1$ for $k>k_0$. We then get that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\Delta(\eta_k\hat{h})\,dx\right| &= \left|\int_{B_R(0)}\Delta(\eta_k\hat{h})\,dx\right|=\left|\int_{\partial B_R(0)}\partial_\nu(\eta_k\hat{h})\,d\sigma\right|\\ &= \left|\int_{\partial B_R(0)}\partial_\nu\hat{h}\,d\sigma\right|\leq CR^{n-1}R^{-n}\leq\frac{C}{R}\end{aligned}$$

Letting $R\to+\infty$, we get that $\int_{\mathbb{R}^n}\Delta(\eta_k\hat{h})\,dx=0$. We finally get that

$$\int_{\mathbb{R}^n}\hat{h}\Delta\varphi\,dx=0$$

for all $\varphi\in C^\infty_c(\mathbb{R}^n)$. As a consequence, $\Delta\hat{h}=0$ in $\mathcal{D}'(\mathbb{R}^n)$, and $\hat{h}\in C^2(\mathbb{R}^n)$. Since there exists $C>0$ such that $|\hat{h}(x)|\leq C|x|^{1-n}$, we then get that $\hat{h}$ is uniformly bounded on $\mathbb{R}^n$. Since $\Delta\hat{h}=0$, we get that $\hat{h}\equiv 0$. In particular,

$$h(x) = \alpha \frac{|x_1|}{|x|^n}$$

for all $x \in \mathbb{R}^n_- \setminus \{0\}$. □

## 10. Appendix C: Symmetry of the positive solutions to the limit equation

This section is devoted to the proof of a symmetry property for the positive solutions to the limit equations involved in Proposition 3.1.

**Proposition 10.1.** *Let $n \geq 3$ and $s \in (0,2)$. We let $u \in C^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ such that*

$$\begin{cases} \Delta u = \frac{u^{2^\star-1}}{|x|^s} & \text{in } \mathbb{R}^n_- \\ u > 0 & \text{in } \mathbb{R}^n_- \\ u = 0 & \text{on } \partial\mathbb{R}^n_-, \end{cases} \tag{159}$$

*where $2^\star = \frac{2(n-s)}{n-2}$. We assume that there exists $C > 0$ such that $u(x) \leq C(1+|x|)^{1-n}$ for all $x \in \mathbb{R}^n_-$. Then we have that $u \circ \sigma = u$ for all isometry of $\mathbb{R}^n$ such that $\sigma(\mathbb{R}^n_-) = \mathbb{R}^n_-$. In particular, there exists $v \in C^2(\mathbb{R}^\star_- \times \mathbb{R}) \cap C^1(\mathbb{R}_- \times \mathbb{R})$ such that for all $x_1 < 0$ and all $x' \in \mathbb{R}^{n-1}$, we have that $u(x_1, x') = v(x_1, |x'|)$.*

We prove the Proposition in the sequel. We let $u \in C^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ that verifies the system (159) and such that there exists $C > 0$ such that

$$u(x) \leq \frac{C}{(1+|x|)^{n-1}} \tag{160}$$

for all $x \in \mathbb{R}^n_-$. We $\vec{e}_1$ be the first vector of the canonical basis of $\mathbb{R}^n$. We let the open ball

$$D := B_{1/2}\left(-\frac{1}{2}\vec{e}_1\right).$$

We define

$$v(x) := |x|^{2-n} u\left(\vec{e}_1 + \frac{x}{|x|^2}\right) \tag{161}$$

for all $x \in \overline{D} \setminus \{0\}$. We prolongate $v$ by 0 at 0. Clearly, this is well-defined.

**Step 10.1:** We claim that

$$v \in C^2(D) \cap C^1(\overline{D}) \text{ and } \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial D \tag{162}$$

where $\partial/\partial\nu$ denotes the outward normal derivative.

*Proof.* It follows from the assumptions on $u$ that $v \in C^2(D) \cap C^1(\overline{D} \setminus \{0\})$. Moreover, $v(x) > 0$ for all $x \in D$ and $v(x) = 0$ for all $x \in \partial D \setminus \{0\}$. It follows from the estimate (160) that there exists $C > 0$ such that

$$v(x) \leq C|x| \tag{163}$$

for all $x \in \overline{D} \setminus \{0\}$. Since $v(0) = 0$, we have that $v \in C^0(\overline{D})$. The function $v$ verifies the equation

$$\Delta v = \frac{v^{2^\star-1}}{|x + |x|^2\vec{e}_1|^s} = \frac{v^{2^\star-1}}{|x|^s \, |x + \vec{e}_1|^s} \tag{164}$$

in $D$. Since $-\vec{e}_1 \in \partial D \setminus \{0\}$ and $v \in C^1(\overline{D} \setminus \{0\}) \cap C^0(\overline{D})$, there exists $C > 0$ such that

$$v(x) \leq C|x + \vec{e}_1| \tag{165}$$

for all $x \in \overline{D}$. It then follows from (163), (164), (165) and standard elliptic theory that $v \in C^1(\overline{D})$. Since $v > 0$ in $D$, it follows from Hopf's Lemma that $\frac{\partial v}{\partial \nu} < 0$ on $\partial D$. □

We prove the symmetry of $u$ by proving a symmetry property of $v$, which is defined on a ball. Our proof uses the moving plane method. We take largely inspiration in [24] and [8]. Classically, for any $\mu \geq 0$ and any $x = (x', x_n) \in \mathbb{R}^n$ ($x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$), we let

$$x_\mu = (x', 2\mu - x_n) \text{ and } D_\mu = \{x \in D / \, x_\mu \in D\}.$$

It follows from Hopf's Lemma (see (162)) that there exists $\epsilon_0 > 0$ such that for any $\mu \in (\frac{1}{2} - \epsilon_0, \frac{1}{2})$, we have that $D_\mu \neq \emptyset$ and $v(x) \geq v(x_\mu)$ for all $x \in D_\mu$ such that $x_n \leq \mu$. We let $\mu \geq 0$. We say that $(P_\mu)$ holds if $D_\mu \neq \emptyset$ and

$$v(x) \geq v(x_\mu)$$

for all $x \in D_\mu$ such that $x_n \leq \mu$. We let

$$\lambda := \min \left\{ \mu \geq 0 / \, (P_\nu) \text{ holds for all } \nu \in \left(\mu, \frac{1}{2}\right) \right\}. \tag{166}$$

**Step 10.2:** We claim that $\lambda = 0$.

*Proof.* We proceed by contradiction and assume that $\lambda > 0$. We then get that $D_\lambda \neq \emptyset$ and that $(P_\lambda)$ holds. We let

$$w(x) := v(x) - v(x_\lambda)$$

for all $x \in D_\lambda \cap \{x_n < \lambda\}$. Since $(P_\lambda)$ holds, we have that $w(x) \geq 0$ for all $x \in D_\lambda \cap \{x_n < \lambda\}$. With the equation (164) of $v$ and $(P_\lambda)$, we get that

$$\begin{aligned} \Delta w &= \frac{v(x)^{2^\star - 1}}{|x + |x|^2 \vec{e}_1|^s} - \frac{v(x_\lambda)^{2^\star - 1}}{|x_\lambda + |x_\lambda|^2 \vec{e}_1|^s} \\ &\geq v(x_\lambda)^{2^\star - 1} \left( \frac{1}{|x + |x|^2 \vec{e}_1|^s} - \frac{1}{|x_\lambda + |x_\lambda|^2 \vec{e}_1|^s} \right) \end{aligned}$$

for all $x \in D_\lambda \cap \{x_n < \lambda\}$. With straightforward computations, we have that

$$|x_\lambda|^2 - |x|^2 = 4\lambda(\lambda - x_n)$$
$$|x_\lambda + |x_\lambda|^2 \vec{e}_1|^2 - |x + |x|^2 \vec{e}_1|^2 = (|x_\lambda|^2 - |x|^2)\left(1 + |x_\lambda|^2 + |x|^2 + 2x_1\right)$$

for all $x \in \mathbb{R}^n$. It follows that $\Delta w(x) > 0$ for all $x \in D_\lambda \cap \{x_n < \lambda\}$. Note that we have used that $\lambda > 0$. It then follows from Hopf's Lemma and the strong comparison principle that

$$w > 0 \text{ in } D_\lambda \cap \{x_n < \lambda\} \text{ and } \frac{\partial w}{\partial \nu} < 0 \text{ on } D_\lambda \cap \{x_n = \lambda\}. \tag{167}$$

By definition, there exists a sequence $(\lambda_i)_{i \in \mathbb{N}} \in \mathbb{R}$ and a sequence $(x^i)_{i \in \mathbb{N}} \in D$ such that $\lambda_i < \lambda$, $x^i \in D_{\lambda_i}$, $(x^i)_n < \lambda_i$, $\lim_{i \to +\infty} \lambda_i = \lambda$ and

$$v(x^i) < v((x^i)_{\lambda_i}) \tag{168}$$

for all $i \in \mathbb{N}$. Up to extraction a subsequence, we assume that there exists $x \in \overline{(D_\lambda \cap \{x_n < \lambda\})}$ such that $\lim_{i \to +\infty} x^i = x$ with $x_n \leq \lambda$. Passing to the limit

$i \to +\infty$ in (168), we get that $v(x) \le v(x_\lambda)$. It follows from this last inequality and (167) that $v(x) - v(x_\lambda) = w(x) = 0$, and then $x \in \partial(D_\lambda \cap \{x_n < \lambda\})$.

*Case 1:* We assume that $x \in \partial D$. Then $v(x_\lambda) = 0$ and $x_\lambda \in \partial D$. Since $D$ is a ball and $\lambda > 0$, we get that $x = x_\lambda \in \partial D$. Since $v$ is $C^1$, we get that there exists $\tau_i \in ((x^i)_n, 2\lambda_i - (x^i)_n)$ such that

$$v(x^i) - v((x^i)_{\lambda_i}) = \partial_n v((x')^i, \tau_i) \times 2((x^i)_n - \lambda_i)$$

Letting $i \to +\infty$, using that $(x^i)_n < \lambda_i$ and (168), we get that $\partial_n v(x) \ge 0$. On the other hand, we have that

$$\partial_n v(x) = \frac{\partial v}{\partial \nu}(x) \cdot (\nu(x)|\vec{e}_n) = \frac{\lambda}{|x + \vec{e}_1/2|} \frac{\partial v}{\partial \nu}(x) < 0.$$

A contradiction.

*Case 2:* We assume that $x \in D$. Since $v(x_\lambda) = v(x)$, we then get that $x_\lambda \in D$. Since $x \in \partial(D_\lambda \cap \{x_n < \lambda\})$, we then get that $x \in D \cap \{x_n = \lambda\}$. With the same argument as in the preceding step, we get that $\partial_n v(x) \ge 0$. On the other hand, since $x_n = \lambda$, we get with (167) that $\partial_n v(x) = \frac{\partial_n w(x)}{2} < 0$. A contradiction.

In all the cases, we have obtained a contradiction. This proves that $\lambda = 0$. □

**Step 10.3:** Here goes the final argument. Since $\lambda = 0$, it follows from the definition (166) of $\lambda$ that $v(x', x_n) \ge v(x', -x_n)$ for all $x \in D$ such that $x_n \le 0$. With the same technique, we get the reverse inequality, and then, we get that

$$v(x', x_n) = v(x', -x_n)$$

for all $x = (x', x_n) \in D$. In other words, $v$ is symmetric with respect to the hyperplane $\{x_n = 0\}$. The same analysis holds for any hyperplane containing $\vec{e_1}$. Coming back to the initial function $u$, this proves the Theorem.

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