ON A p–LAPLACE EQUATION WITH MULTIPLE CRITICAL NONLINEARITIES

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Abstract. Using the Mountain–Pass Theorem of Ambrosetti and Rabinowitz we prove that $-\Delta_p u - \mu|x|^{-p}u^{p-1} = |x|^{-s}u^{p^\star(s)-1} + u^{p^\star-1}$ admits a positive weak solution in $\mathbb{R}^n$ of class $D_1^p(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, whenever $\mu < \mu_1$, and $\mu_1 = [(n-p)/p]^p$. The technique is based on the existence of extremals of some Hardy–Sobolev type embeddings of independent interest. We also show that if $u \in D_1^p(\mathbb{R}^n)$ is a weak solution in $\mathbb{R}^n$ of $-\Delta_p u - \mu|x|^{-p}|u|^{p-2}u = |x|^{-s}|u|^{p^\star(s)-2}u + |u|^{q-2}u$, then $u \equiv 0$ when either $1 < q < p^\star$, or $q > p^\star$ and u is also of class $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$.

Abstract. Résumé: En utilisant le lemme du col d'Ambrosetti et Rabinowitz, nous prouvons que l'équation $-\Delta_p u - \mu|x|^{-p}u^{p-1} = |x|^{-s}u^{p^\star(s)-1} + u^{p^\star-1}$ admet une solution faible positive dans $D_1^p(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ dès que $\mu < \mu_1$, avec $\mu_1 = [(n-p)/p]^p$. La technique utilisée repose sur l'existence d'extrémales pour certains plongements de Hardy–Sobolev. Nous montrons parallèlement que si $u \in D_1^p(\mathbb{R}^n)$ est une solution faible dans $\mathbb{R}^n$ de $-\Delta_p u - \mu|x|^{-p}|u|^{p-2}u = |x|^{-s}|u|^{p^\star(s)-2}u + |u|^{q-2}u$, alors $u \equiv 0$ lorsque $1 < q < p^\star$, ou bien lorsque $q > p^\star$ et u est de classe $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$.

Contents

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1. Introduction

In this paper, we are interested in weak solutions $u \in D_1^p(\mathbb{R}^n)$, $u \geq 0$ a.e, of the *double* critical equation of Emden-Fowler type

$$-\Delta_p u - \mu \frac{u^{p-1}}{|x|^p} = u^{p^\star -1} + \frac{u^{p^\star(s)-1}}{|x|^s} \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p–Laplace operator, $n \geq 2$ is an integer, μ is a real parameter, $p \in (1, n)$ and $p^\star := np/(n-p)$, while $s \in (0, p)$ and $p^\star(s) := p(n-s)/(n-p)$. The space $D_1^p(\mathbb{R}^n)$ is defined as the completion of $C_c^\infty(\mathbb{R}^n)$, the set of smooth compactly supported function on $\mathbb{R}^n$, for the norm

$$u \mapsto \|\nabla u\|_p,$$

where here and in the sequel, $\|\cdot\|_q$ denotes the L^q–norm on the Lebesgue space $L^q(\mathbb{R}^n)$.

Throughout the paper, we say that $u \in D_1^p(\mathbb{R}^n)$ is a *weak solution* of $-\Delta_p u = f$, where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\, dx = \int_{\mathbb{R}^n} f\varphi\, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Existence and non–existence, as well as qualitative properties, of non–trivial non–negative solutions for elliptic equations with singular potentials were recently studied by several authors, but, essentially, only with a solely critical exponent. We refer, e.g., in bounded domains and for $p = 2$ to [4, 12, 13, 18, 19], and for general $p > 1$ to [5, 7, 14, 16]; while in $\mathbb{R}^n$ and for $p = 2$ to [6, 10, 20, 32], and for general $p > 1$ to [1, 11, 24], and the references therein. The large literature on p–Laplacian equations in the entire $\mathbb{R}^n$ differs somehow for the nonlinear structure, objectives and methods from those presented in this paper.

Indeed, the combination of the two critical exponents induces more subtilizes and difficulties. When just one critical exponent is involved, there are solutions to the corresponding equations (see for instance [24]): in general, these solutions are radially symmetrical with respect to a point of the domain (0 in general) and are explicit. In our context, very few is known: yet, we refer to an interessant approach by Kang and Li [17].

A natural strategy is to construct the solutions of (1) as critical points of a suitable functional via the mountain–pass lemma of Ambrosetti and Rabinowitz. Due to the invariance of (1) by the conformal one parameter transformation group

$$\left\{ \begin{array}{lll} T_r: & D_1^p(\mathbb{R}^n) & \to \quad D_1^p(\mathbb{R}^n) \\ & u & \mapsto \quad \left[x \mapsto r^{(n-p)/p} u(rx)\right] \end{array} \right\}, \; r > 0, \tag{2}$$

it is well–known that the mountain–pass lemma does not yield critical points, but only Palais–Smale sequences. The main issue of the paper is to understand the behavior of these Palais–Smale sequences. Indeed, the principal difficulty here is that there is an asymptotic competition between the energy carried by the two critical nonlinearities. If one dominates the other, then there is vanishing of the weakest one and one recovers solutions to an equation with only one critical nonlinearity: in this situation, we do not get solutions of equation (1). Therefore, the crucial point here is to avoid the domination of one term on the other.

Sections 2, 3 and 4 of the paper are devoted to the proof of the following main existence result:

Theorem 1. *For any $\mu \in (-\infty, \mu_1)$, $\mu_1 := [(n-p)/p]^p$, and $s \in (0,p)$, there exists a positive weak solution of* (1)*. More precisely, there exists $u \in D_1^p(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ such that $u > 0$ in $\mathbb{R}^n \setminus \{0\}$ and u solves* (1) *weakly in $\mathbb{R}^n$.*

Theorem 1 is proved via the choice of a suitable energy level for the mountain–pass lemma: with this choice, a careful analysis of concentration allows us to show that there is a balance between the energies of the two nonlinearities mentioned above, and therefore none can dominate the other. There we make a full use of the conformal invariance of (1) under the transformation (2); this guarantees the convergence to a solution to (1). As an offshoot of this analysis, we prove that the blow–up energy is quantized for both nonlinearities.

The choice of the energy level involves the best constants in the Hardy–Sobolev inequalities (see (5) and (6) of Section 2). We are then led to considering the possible extremals for them. As far as we know, the result in its full generality, that we need, does not appear in the literature: therefore, for the sake of completeness, we prove the existence of extremals when $s > 0$ in the Appendix given in Section 6. Concerning the case $s = 0$, there is no extremal in general when $\mu < 0$ and the analysis relies on the radial case and is made in the Appendix given in Section 7. For details concerning the extremals in the case $s = 0$ we remind to both Sections 6 and 7.

It is to be noticed that the exponents $p^\star$ and $p^\star(s)$ are exactly the ones that make the equation invariant under the transformation group (2). One can therefore naturally wonder what happens for different exponents: in Section 5, we present a non–existence theorem, when $q \neq p^\star$, cf. Theorem 3 and Claims 5.4–5.5 (we also refer to [24] for other nonexistence results in the same spirit). In particular, in general, there is no solution to the corresponding equation (except the null one) when one takes exponents different from $p^\star$ and $p^\star(s)$ in (1).

The paper is organized as follows: in Sections 2, 3 and 4 we prove Theorem 1 when $\mu \geq 0$. In Section 5 we deal with the non–existence result in the spirit of Pohozaev. In Section 6, we prove the existence of extremals for some Hardy–Sobolev type embeddings, see Theorem 4. While Section 7 deals with the situation in which $\mu < 0$.

2. Preliminaries and construction of the appropriate Palais–Smale sequence

Clearly equation (1) is related to some specific functional embeddings and inequalities. The standard Hardy inequality asserts that $D_1^p(\mathbb{R}^n)$ is embedded in the weighted space $L^p(\mathbb{R}^n, |x|^{-p})$ and that this embedding is continuous: more precisely,

$$\mu_1 \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p}\, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dx, \qquad \mu_1 := \left(\frac{n-p}{p}\right)^p, \tag{3}$$

for all $u \in D_1^p(\mathbb{R}^n)$. Moreover, the constant μ_1 is optimal. If $\mu < \mu_1$, it follows from the Hardy inequality (3) that

$$\|u\| := \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx - \mu \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p}\, dx\right)^{1/p}$$

is well–defined on $D_1^p(\mathbb{R}^n)$. Note that, $\|\cdot\|$ is *comparable* to the norm $\|\nabla\cdot\|_p$ since the following inequalities hold

$$\left(1-\frac{\mu_+}{\mu_1}\right)\|\nabla u\|_p^p \le \|u\|^p \le \left(1+\frac{\mu_-}{\mu_1}\right)\|\nabla u\|_p^p \tag{4}$$

for any $u\in D_1^p(\mathbb{R}^n)$, where $\mu_+=\max\{\mu,\,0\}$ and $\mu_-=\max\{-\mu,\,0\}$.

It follows from Sobolev's embedding theorem that $D_1^p(\mathbb{R}^n)$ is continuously embedded in $L^{p^\star}(\mathbb{R}^n)$ where $p^\star := np/(n-p)$. Therefore, there exists $C>0$ such that $\|u\|_{p^\star}\le C\|u\|$. Taking C as small as possible, we define the optimal constant $K(n,p,\mu,0)>0$ associated to this embedding as

$$\frac{1}{K(n,p,\mu,0)} := \inf_{u\in D_1^p(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}|\nabla u|^p\,dx-\mu\int_{\mathbb{R}^n}|u|^p|x|^{-p}\,dx}{\left(\int_{\mathbb{R}^n}|u|^{p^\star}\,dx\right)^{p/p^\star}}, \tag{5}$$

that is $C^p = K(n,p,\mu,0)$. Combining the Hardy inequality and the Sobolev inequality, we obtain the Hardy–Sobolev inequality. Indeed, let $s\in(0,p)$ be a real number: then $D_1^p(\mathbb{R}^n)$ is continuously embedded in the weighted space $L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})$, where $p^\star(s):=p(n-s)/(n-p)$. Here again, taking the smallest constant associated to this embedding, we let

$$\frac{1}{K(n,p,\mu,s)} := \inf_{u\in D_1^p(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}|\nabla u|^p\,dx-\mu\int_{\mathbb{R}^n}|u|^p|x|^{-p}\,dx}{\left(\int_{\mathbb{R}^n}|u|^{p^\star(s)}|x|^{-s}\,dx\right)^{p/p^\star(s)}}. \tag{6}$$

Let the functional Φ defined on $D_1^p(\mathbb{R}^n)$ as follows:

$$\Phi(u) := \frac{1}{p}\|u\|^p - \frac{1}{p^\star}\int_{\mathbb{R}^n}(u_+)^{p^\star}\,dx - \frac{1}{p^\star(s)}\int_{\mathbb{R}^n}\frac{(u_+)^{p^\star(s)}}{|x|^s}\,dx$$

for $u\in D_1^p(\mathbb{R}^n)$. Here and in the sequel, $u_+=\max\{u,0\}$. It follows from the Hardy, Sobolev and Hardy–Sobolev embeddings that Φ is well–defined and that $\Phi\in C^1(D_1^p(\mathbb{R}^n))$. Note that a positive weak solution to (1) is a nontrivial critical point of Φ; and we actually show, in the proof of Claim 4.3, that a nonnegative nontrivial weak limit of a Palais–Smale sequence of Φ is a positive solution of (1) by the Tolksdorf regularity theory [33] and the Vazquez strong maximum principle [34].

In this section, we prove the following:

Proposition 1. *Assume*

$$\mu\in[0,\mu_1) \ \textit{and}\ s\in[0,p). \tag{7}$$

Then there exists $(u_k)_{k\in\mathbb{N}}\in D_1^p(\mathbb{R}^n)$ *such that*

$$\lim_{k\to\infty}\Phi'(u_k)=0 \quad \textit{strongly in}\ (D_1^p(\mathbb{R}^n))' \quad \textit{and} \quad \lim_{k\to\infty}\Phi(u_k)=c,$$

where

$$0<c<c_\star := \min\left\{\frac{1}{n}K(n,p,\mu,0)^{-n/p},\, \frac{p-s}{p(n-s)}K(n,p,\mu,s)^{-(n-s)/(p-s)}\right\}. \tag{8}$$

Note that $1/p-1/p^\star=1/n$, $p^\star/(p^\star-p)=n/p$, $1/p-1/p^\star(s)=(p-s)/p(n-s)$ and $p^\star(s)/(p^\star(s)-p)=(n-s)/(p-s)$. The proof of Proposition 1 uses the following version of the Mountain–Pass lemma:

Theorem 2 (Ambrosetti and Rabinowitz, [2]). *Let (V, N) be a Banach space and let $F \in C^1(V)$. We assume that*

(i) $F(0) = 0$,

(ii) There exist $\lambda, R > 0$ such that $F(u) \geq \lambda$ for all $u \in V$, with $N(u) = R$,

(iii) There exists $v_0 \in V$ such that $\limsup_{t\to\infty} F(tv_0) < 0$.

Let $t_0 > 0$ be such that $N(t_0 v_0) > R$ and $F(t_0 v_0) < 0$ and let

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} F(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C^0([0, 1], V) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_0 v_0\}.$$

Then there exists a Palais–Smale sequence at level c, that is there exists a sequence $(u_k)_{k\in\mathbb{N}} \in V$ such that

$$\lim_{k\to\infty} F(u_k) = c \quad \text{and} \quad \lim_{k\to\infty} F'(u_k) = 0 \quad \text{strongly in } V'.$$

Claim 2.1. *The functional Φ verifies the hypotheses of the Mountain–Pass lemma at any $u \in D_1^p(\mathbb{R}^n)$, with $u_+ \not\equiv 0$.*

Proof of Claim 2.1*:* Clearly $\Phi \in C^1(D_1^p(\mathbb{R}^n))$ and $\Phi(0) = 0$. Using the definition of the best constants in (5), (6), we get that

$$\begin{aligned}\Phi(u) &\geq \frac{1}{p}\|u\|^p - \frac{K(n,p,\mu,0)^{p^\star/p}}{p^\star}\|u\|^{p^\star} - \frac{K(n,p,\mu,s)^{p^\star(s)/p}}{p^\star(s)}\|u\|^{p^\star(s)}\\ &= \left(\frac{1}{p} - \frac{K(n,p,\mu,0)^{p^\star/p}}{p^\star}\|u\|^{p^\star-p} - \frac{K(n,p,\mu,s)^{p^\star(s)/p}}{p^\star(s)}\|u\|^{p^\star(s)-p}\right)\|u\|^p.\end{aligned}$$

Then, since (4) holds and since $p < p^\star(s) \leq p^\star$ being $s \in [0, p)$, there exists $R > 0$ such that $\Phi(u) \geq \lambda$ for all $u \in D_1^p(\mathbb{R}^n)$ such that $\|\nabla u\|_p = R$: point (ii) of Theorem 2 is satisfied. Moreover, given any $u \in D_1^p(\mathbb{R}^n)$, with $u_+ \not\equiv 0$, we have that

$$\lim_{t\to\infty} \Phi(tu) = -\infty.$$

We then, let $t_u > 0$ be such that $\Phi(tu) < 0$ for $t \geq t_u$ and $\|\nabla(t_u u)\|_p > R$. Consider

$$\Gamma_u := \{\gamma \in C^0([0, 1], D_1^p(\mathbb{R}^n)) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_u u\}$$

and

$$c_u := \inf_{\gamma\in\Gamma_u} \sup_{t\in[0,1]} \Phi(\gamma(t)).$$

Then the hypotheses of Theorem 2 are satisfied. This ends the proof of Claim 2.1. □

It follows from Theorem 2 that there exists $(u_k)_{k\in\mathbb{N}} \in D_1^p(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \Phi(u_k) = c_u \quad \text{and} \quad \lim_{k\to\infty} \Phi'(u_k) = 0 \quad \text{strongly in } (D_1^p(\mathbb{R}^n))'.$$

Moreover, from the definition of c_u it is also clear that $c_u \geq \lambda$, and so

$$c_u > 0$$

for all $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$.

Claim 2.2. *Assume* (7)*. Then there exists $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$ such that $u \geq 0$ and*

$$c_u < \frac{1}{n} K(n,p,\mu,0)^{-n/p}. \tag{9}$$

Proof of Claim 2.2*:* By (7), let $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$ be a non-negative extremal for $1/K(n,p,\mu,0)$ in (5) (see Theorem 4 in Section 6). Since $u = u_+$, by the definition of t_u and the fact that $c_u > 0$, we have

$$c_u \leq \sup_{t \geq 0} \Phi(tu) \leq \sup_{t \geq 0} f(t),$$

where

$$f(t) := \frac{t^p}{p}\|u\|^p - \frac{t^{p^\star}}{p^\star}\int_{\mathbb{R}^n} |u|^{p^\star}\,dx$$

for all $t \geq 0$. Straightforward computations yield

$$c_u \leq \left(\frac{1}{p} - \frac{1}{p^\star}\right)\left(\frac{\|u\|^p}{\left(\int_{\mathbb{R}^n} |u|^{p^\star}\,dx\right)^{p/p^\star}}\right)^{p^\star/(p^\star - p)} = \frac{1}{n}K(n,p,\mu,0)^{-n/p},$$

since u is a non-negative extremal for (5). Hence, if equality would hold in (9), then $0 < c_u = \sup_{t \geq 0} \Phi(tu) = \sup_{t \geq 0} f(t)$. Letting $t_1, t_2 > 0$ be points where the two suprema are attained respectively, we get that

$$f(t_1) - \frac{t_1^{p^\star(s)}}{p^\star(s)}\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s}\,dx = f(t_2),$$

that is $f(t_2) < f(t_1)$, being $u_+ \not\equiv 0$ and $t_1 > 0$. This gives the required contradiction and the claim is proved when (7) holds. $\square$

Claim 2.3. *Assume* (7)*. There exists* $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$ *such that* $u \geq 0$ *and*

$$0 < c_u < c_\star,$$

where $c_\star$ *is defined in* (8)*.*

Proof of Claim 2.3*:* In case

$$\frac{1}{n}K(n,p,\mu,0)^{-n/p} \leq \frac{p-s}{p(n-s)}K(n,p,\mu,s)^{-(n-s)/(p-s)},$$

we take $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$ as in Claim 2.2 to get the result. Otherwise we take $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$ a non–negative extremal for (6) (which exists by Theorem 4 of Section 6) and proceed as in the first part of the proof of Claim 2.2, with f replaced by

$$\tilde{f}(t) := \frac{t^p}{p}\|u\|^p - \frac{t^{p^\star(s)}}{p^\star(s)}\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s}\,dx,$$

which gives now the contradiction

$$\tilde{f}(t_1) - \frac{t_1^{p^\star}}{p^\star}\int_{\mathbb{R}^n} |u|^{p^\star}\,dx = \tilde{f}(t_2).$$

This proves Claim 2.3. $\square$

Proposition 1 is a consequence of Claims 2.1 and 2.3 for a suitable u in $D_1^p(\mathbb{R}^n)$.

3. The structure of Palais–Smale sequence going to zero weakly

From now on, we assume that $s \in (0,p)$. We prove the following proposition:

Proposition 2. *Let $(u_k)_{k\in\mathbb{N}} \in D_1^p(\mathbb{R}^n)$ be a Palais–Smale sequence at level $c \in (0, c_\star)$ as in Proposition 1, with $s \neq 0$ in (7). If $u_k \rightharpoonup 0$ weakly in $D_1^p(\mathbb{R}^n)$ as $k \to \infty$, then there exists $\epsilon_0 = \epsilon_0(n,p,\mu,s,c) > 0$ such that*

$$\text{either} \quad \lim_{k\to\infty} \int_{B_\delta(0)} (u_k)_+^{p^\star}\, dx = 0 \quad \text{or} \quad \limsup_{k\to\infty} \int_{B_\delta(0)} (u_k)_+^{p^\star}\, dx \geq \epsilon_0$$

for all $\delta > 0$.

The proof of Proposition 2 goes through four claims.

Claim 3.1. *Let $(u_k)_{k\in\mathbb{N}} \in D_1^p(\mathbb{R}^n)$ be a Palais–Smale sequence as in Proposition 2. If $u_k \rightharpoonup 0$ weakly in $D_1^p(\mathbb{R}^n)$ as $k \to \infty$, then for all $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$, up to a subsequence, we have that*

$$\lim_{k\to\infty} \int_\omega \frac{|u_k|^p}{|x|^p}\, dx = \lim_{k\to\infty} \int_\omega \frac{|u_k|^{p^\star(s)}}{|x|^s}\, dx = 0, \tag{10}$$

$$\lim_{k\to\infty} \int_\omega |u_k|^{p^\star}\, dx = \lim_{k\to\infty} \int_\omega |\nabla u_k|^p\, dx = 0. \tag{11}$$

Proof of Claim 3.1*:* Fix $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$. Clearly the embedding $D_1^p(\mathbb{R}^n) \hookrightarrow L^q(\omega)$ is compact for $1 \leq q < p^\star$ and $|x| + |x|^{-1}$ is bounded on ω. Hence (10) follows at once, being $p < p^\star$ and $p^\star(s) < p^\star$ since $s \in (0,p)$ by assumption.

Concerning the two equalities in (11), let $\eta \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ such that $0 \leq \eta \leq 1$ and $\eta_{|\omega} \equiv 1$. Since $\eta^p u_k \in D_1^p(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, we get that

$$\langle \Phi'(u_k), \eta^p u_k \rangle = o(\|\eta^p u_k\|) = o(\|u_k\|) = o(1) \tag{12}$$

as $k \to \infty$, being $(\|u_k\|)_{k\in\mathbb{N}}$ bounded by the weak convergence of $(u_k)_{k\in\mathbb{N}}$ in $D_1^p(\mathbb{R}^n)$ and (4). Since $\lim_{k\to\infty} \|u_k\|_{L^p(\mathrm{Supp}|\nabla\eta|)} = 0$ by (10) and $(\|\nabla u_k\|_p)_{k\in\mathbb{N}}$ is bounded, we have as $k \to \infty$

$$\int_{\mathbb{R}^n} |\nabla u_k|^{p-1} |\nabla(\eta^p)| \cdot |u_k|\, dx \leq \|\nabla u_k\|_p^{p-1} \|u_k\|_{L^p(\mathrm{Supp}|\nabla\eta|)} = o(1),$$

and so by (12)

$$\begin{aligned}
o(1) &= \langle \Phi'(u_k), \eta^p u_k \rangle \\
&= \int_{\mathbb{R}^n} |\nabla u_k|^{p-2} (\nabla u_k, \nabla(\eta^p u_k))\, dx - \int_{\mathbb{R}^n} \eta^p (u_k)_+^{p^\star}\, dx + o(1) \\
&= \int_{\mathbb{R}^n} |\eta \nabla u_k|^p\, dx - \int_{\mathbb{R}^n} \eta^p (u_k)_+^{p^\star}\, dx \\
&\qquad + O\left(\int_{\mathbb{R}^n} |\nabla u_k|^{p-1} |\nabla(\eta^p)| \cdot |u_k|\, dx \right) + o(1) \\
&= \int_{\mathbb{R}^n} |\nabla(\eta u_k)|^p dx - \int_{\mathbb{R}^n} \eta^p (u_k)_+^{p^\star}\, dx + o(1) \\
&\geq \|\eta u_k\|^p - \int_{\mathbb{R}^n} \eta^p (u_k)_+^{p^\star}\, dx + o(1),
\end{aligned} \tag{13}$$

since

$$\int_{\mathbb{R}^n} |\nabla(\eta u_k)|^p dx = \int_{\mathbb{R}^n} |\eta \nabla u_k|^p dx + o(1). \tag{14}$$

We prove (14). Indeed, by the elementary inequality $|\,|X+Y|^p-|X|^p| \le C_p(|X|^{p-1}+|Y|^{p-1})|Y|$ for all $X,\ Y \in \mathbb{R}^n$, we have $|\,|\nabla(\eta u_k)|^p - |\eta\nabla u_k|^p| \le C_p(|\eta\nabla u_k|^{p-1} + |u_k\nabla\eta|^{p-1})|u_k\nabla\eta|$, and by Hölder's inequality

$$\int_{\mathbb{R}^n} |\eta\nabla u_k|^{p-1}|u_k\nabla\eta|\,dx \le \|\nabla u_k\|_p^{p-1}\|u_k\|_{L^p(\mathrm{Supp}|\nabla\eta|)} = o(1)$$

by (10), as well as $\int_{\mathbb{R}^n} |u_k\nabla\eta|^p dx \le \|u_k\|^p_{L^p(\mathrm{Supp}|\nabla\eta|)} = o(1)$. This proves (14).

Formula (13) above shows that

$$\|\eta u_k\|^p \le \int_{\mathbb{R}^n} (u_k)_+^{p^\star-p}|\eta u_k|^p\,dx + o(1)$$

as $k\to\infty$. By Hölder's inequality and (5), we then have

$$\begin{aligned}\|\eta u_k\|^p &\le \left(\int_{\mathbb{R}^n} (u_k)_+^{p^\star}\,dx\right)^{(p^\star-p)/p^\star}\left(\int_{\mathbb{R}^n} |\eta u_k|^{p^\star}\,dx\right)^{p/p^\star} + o(1)\\ &\le \left(\int_{\mathbb{R}^n} (u_k)_+^{p^\star}\,dx\right)^{(p^\star-p)/p^\star} K(n,p,\mu,0)\,\|\eta u_k\|^p + o(1),\end{aligned}$$

which gives

$$\left(1-\left(\int_{\mathbb{R}^n} (u_k)_+^{p^\star}\,dx\right)^{(p^\star-p)/p^\star} K(n,p,\mu,0)\right)\|\eta u_k\|^p \le o(1). \tag{15}$$

Independently, $\Phi(u_k) - \frac{1}{p}\langle\Phi'(u_k),u_k\rangle = c + o(\|u_k\|) = c+o(1)$ as $k\to\infty$ since $(\|u_k\|)_{k\in\mathbb{N}}$ in bounded, which yields

$$\left(\frac{1}{p}-\frac{1}{p^\star}\right)\int_{\mathbb{R}^n}(u_k)_+^{p^\star}\,dx + \left(\frac{1}{p}-\frac{1}{p^\star(s)}\right)\int_{\mathbb{R}^n}\frac{(u_k)_+^{p^\star(s)}}{|x|^s}\,dx = c+o(1) \tag{16}$$

as $k\to\infty$. Therefore,

$$\int_{\mathbb{R}^n}(u_k)_+^{p^\star}\,dx \le c\,n + o(1) \tag{17}$$

as $k\to\infty$. Plugging (17) into (15) we get that

$$\left(1-(c\,n)^{p/n}\,K(n,p,\mu,0)\right)\|\eta u_k\|^p \le o(1)$$

as $k\to\infty$. The upper bound (8) on c yields

$$\lim_{k\to\infty}\|\eta u_k\|^p = 0,$$

and in turn by (5)

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}|\eta u_k|^{p^\star}\,dx = 0.$$

Since $\eta_{|\omega}\equiv 1$, these two latest inequalities and (4) yield (11). This proves Claim 3.1. $\square$

For $\delta>0$, we define

$$\begin{aligned}\alpha &:= \limsup_{k\to\infty}\int_{B_\delta(0)}(u_k)_+^{p^\star}\,dx; \qquad \beta := \limsup_{k\to\infty}\int_{B_\delta(0)}\frac{(u_k)_+^{p^\star(s)}}{|x|^s}\,dx\,;\\ \gamma &:= \limsup_{k\to\infty}\int_{B_\delta(0)}\left(|\nabla u_k|^p - \mu\frac{|u_k|^p}{|x|^p}\right)dx.\end{aligned} \tag{18}$$

It follows from Claim 3.1 that these three quantities are well–defined and independent of the choice of $\delta > 0$.

Claim 3.2. *Let $(u_k)_{k\in\mathbb{N}} \in D_1^p(\mathbb{R}^n)$ be a Palais–Smale sequence as in Proposition* 2*, and let α, β and γ be defined as in* (18)*. If $u_k \rightharpoonup 0$ weakly in $D_1^p(\mathbb{R}^n)$ as $k \to \infty$, then*

$$\alpha^{p/p^\star} \leq K(n,p,\mu,0)\gamma \quad \text{and} \quad \beta^{p/p^\star(s)} \leq K(n,p,\mu,s)\gamma. \tag{19}$$

Proof of Claim 3.2*:* Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta_{|B_\delta(0)} \equiv 1$, with $\delta > 0$. Inequality (5) and Claim 3.1 yield

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |(\eta u_k)_+|^{p^\star}\, dx\right)^{p/p^\star} &\leq K(n,p,\mu,0)\, \|\eta u_k\|^p \\ \left(\int_{B_\delta(0)} (u_k)_+^{p^\star}\, dx\right)^{p/p^\star} &\leq K(n,p,\mu,0) \int_{B_\delta(0)} \left(|\nabla u_k|^p - \mu\frac{|u_k|^p}{|x|^p}\right) dx + o(1) \end{aligned}$$

as $k \to \infty$. Letting $k \to \infty$, we get that $\alpha^{p/p^\star} \leq K(n,p,\mu,0)\gamma$. Similarly, we obtain the second inequality of (19). This proves Claim 3.2. □

Claim 3.3. *Let $(u_k)_{k\in\mathbb{N}} \in D_1^p(\mathbb{R}^n)$ be a Palais–Smale sequence as in Proposition* 2*, and let α, β and γ be defined as in* (18)*. If $u_k \rightharpoonup 0$ weakly in $D_1^p(\mathbb{R}^n)$ as $k \to \infty$, then $\gamma \leq \alpha + \beta$.*

Proof of Claim 3.3*:* Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta_{|B_\delta(0)} \equiv 1$. Since $\eta u_k \in D_1^p(\mathbb{R}^n)$ and since $\lim_{k\to\infty}\langle \Phi'(u_k), \eta u_k\rangle = 0$, using Claim 3.1 and the definitions of α, β and γ in (18), we get that $\gamma \leq \alpha + \beta$. This proves Claim 3.3. □

Proof of Proposition 2: Let $(u_k)_{k\in\mathbb{N}}$ be as in Proposition 1, with $s \neq 0$. Claims 3.2 and 3.3 yield

$$\begin{aligned} \alpha^{p/p^\star} &\leq K(n,p,\mu,0)\alpha + K(n,p,\mu,0)\beta, \\ \alpha^{p/p^\star}\left(1 - K(n,p,\mu,0)\alpha^{(p^\star-p)/p^\star}\right) &\leq K(n,p,\mu,0)\beta. \end{aligned} \tag{20}$$

Moreover, by (17), we obtain

$$\alpha \leq c\,n. \tag{21}$$

Plugging (21) into (20), we have

$$\left(1 - (c\,n)^{p/n}\, K(n,p,\mu,0)\right)\alpha^{p/p^\star} \leq K(n,p,\mu,0)\beta.$$

By the upper bound (8) on c there exists δ_1, depending on n, p, μ and c, such that $\alpha^{p/p^\star} \leq \delta_1\beta$. Similarly, there exists δ_2, depending on n, p, μ, s and c, such that $\beta^{p/p^\star(s)} \leq \delta_2\alpha$. In particular, it follows from these two latest inequalities that there exists $\epsilon_0 = \epsilon_0(n,p,\mu,s,c) > 0$ such that

$$\text{either} \quad \alpha = \beta = 0 \quad \text{or} \quad \{\alpha \geq \epsilon_0 \ \text{and} \ \beta \geq \epsilon_0\}. \tag{22}$$

By the definitions of α and β given in (18), this proves Proposition 2. □

4. Proof of Theorem 1 in the case $\mu \geq 0$

The final argument goes through the three following claims.

Claim 4.1. *Let $(u_k)_{k\in\mathbb{N}}$ be as in Proposition 2. Then*

$$\limsup_{k\to\infty}\int_{\mathbb{R}^n}(u_k)_+^{p^\star}\,dx>0.$$

Proof of Claim 4.1: We argue by contradiction and assume that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}(u_k)_+^{p^\star}\,dx=0. \tag{23}$$

Estimating $\langle\Phi'(u_k),u_k\rangle$ and using inequality (6) and (23), we get as $k\to\infty$

$$\|u_k\|^p=\|(u_k)_+\|^{p^\star(s)}_{L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})}+o(1),$$

$$\|(u_k)_+\|^p_{L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})}\leq K(n,p,\mu,s)\|(u_k)_+\|^{p^\star(s)}_{L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})}+o(1),$$

$$\|(u_k)_+\|^p_{L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})}\Big(1-K(n,p,\mu,s)\|(u_k)_+\|^{p^\star(s)-p}_{L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})}\Big)\leq o(1). \tag{24}$$

As in (16) and (18), we have that

$$\int_{\mathbb{R}^n}\frac{(u_k)_+^{p^\star(s)}}{|x|^s}\,dx=\frac{c\,p(n-s)}{p-s}+o(1)$$

as $k\to\infty$. Plugging this inequality in (24) and using the upper bound (8) on c, we get that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{(u_k)_+^{p^\star(s)}}{|x|^s}\,dx=0.$$

A contradiction with (16) and (23) since $c>0$. This proves Claim 4.1. □

Claim 4.2. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence as in Proposition 2. Then there exists $\epsilon_1\in(0,\epsilon_0/2]$, with ϵ_0 given in (22), such that for all $\epsilon\in(0,\epsilon_1)$, there exists a sequence $(r_k)_{k\in\mathbb{N}}$ of $\mathbb{R}_{>0}$ such that the sequence $(\tilde{u}_k)_{k\in\mathbb{N}}$ of $D_1^p(\mathbb{R}^n)$, defined by*

$$\tilde{u}_k(x):=r_k^{(n-p)/p}u_k(r_kx)\quad\text{ for } x\in\mathbb{R}^n,$$

is again a Palais–Smale sequence of type given in Proposition 2 and verifies

$$\int_{B_1(0)}(\tilde{u}_k)_+^{p^\star}\,dx=\epsilon \tag{25}$$

for all $k\in\mathbb{N}$.

Proof of Claim 4.2: Let $\lambda:=\limsup_{k\to\infty}\int_{\mathbb{R}^n}(u_k)_+^{p^\star(s)}dx$. It follows from Claim 4.1 that $\lambda>0$. Let $\epsilon_1:=\min\{\epsilon_0/2,\lambda\}$, with $\epsilon_0>0$ given in (22), see also Proposition 1, and fix $\epsilon\in(0,\epsilon_1)$. Up to a subsequence, still denoted by $(u_k)_{k\in\mathbb{N}}$, for any $k\in\mathbb{N}$ there exists $r_k>0$ such that

$$\int_{B_{r_k}(0)}(u_k)_+^{p^\star}\,dx=\epsilon.$$

Due to scaling invariance, it is then straightforward to check that $(\tilde{u}_k)_{k\in\mathbb{N}}$ satisfies (25) and the properties of Proposition 2. This proves Claim 4.2. □

Claim 4.3 (Proof of Theorem 1 when $\mu\geq 0$)**.** *Let $\tilde{u}_\infty\in D_1^p(\mathbb{R}^n)$ be the weak limit of $(\tilde{u}_k)_{k\in\mathbb{N}}$ as $k\to\infty$ (after a subsequence). Then $\tilde{u}_\infty\in C^1(\mathbb{R}^n\setminus\{0\})$, $\tilde{u}_\infty>0$ in $\mathbb{R}^n\setminus\{0\}$ and $\tilde{u}_\infty$ is a weak solution of (1).*

Proof of Claim 4.3*:* We first assert that $(\tilde{u}_k)_k$ is bounded in $D_1^p(\mathbb{R}^n)$. Indeed, since $p < p^\star < p^\star(s)$ and $(\tilde{u}_k)_k$ is a Palais–Smale sequence, there exist two positive constants c_1 and c_2 such that

$$\begin{aligned} c_1 + c_2\|\tilde{u}_k\| &\geq \Phi(\tilde{u}_k) - \frac{1}{p^\star(s)}\langle \Phi'(\tilde{u}_k), \tilde{u}_k\rangle \\ &= \left(\frac{1}{p} - \frac{1}{p^\star(s)}\right)\|\tilde{u}_k\|^p + \left(\frac{1}{p^\star} - \frac{1}{p^\star(s)}\right)\|(\tilde{u}_k)_+\|_{p^\star}^{p^\star} \\ &\geq \left(\frac{1}{p} - \frac{1}{p^\star(s)}\right)\|\tilde{u}_k\|^p \end{aligned}$$

and the assertion follows at once by (4), being $p > 1$. Let $\tilde{u}_\infty \in D_1^p(\mathbb{R}^n)$ be the weak limit of $(\tilde{u}_k)_{k\in\mathbb{N}}$ as $k \to \infty$, up to a subsequence. In case $\tilde{u}_\infty \equiv 0$, Proposition 2 yields that either we have that $\lim_{k\to\infty}\int_{B_1(0)}(\tilde{u}_k)_+^{p^\star}\,dx = 0$ or we have that $\limsup_{k\to\infty}\int_{B_1(0)}(\tilde{u}_k)_+^{p^\star}\,dx \geq \varepsilon_0$. Since $0 < \varepsilon < \varepsilon_0/2$, this is a contradiction with (25). Then $\tilde{u}_\infty \not\equiv 0$. It follows from Evans [9] and Demengel–Hebey [7] (Lemmae 2 and 3) (see also Saintier [27] Step 1.2 on p.303) that $\tilde{u}_\infty$ is a nontrivial weak solution of

$$-\Delta_p \tilde{u}_\infty - \mu\frac{|\tilde{u}_\infty|^{p-2}\tilde{u}_\infty}{|x|^p} = (\tilde{u}_\infty)_+^{p^\star-1} + \frac{(\tilde{u}_\infty)_+^{p^\star(s)-1}}{|x|^s} \quad \text{in } \mathbb{R}^n. \tag{26}$$

We write (26) as $-\Delta_p\tilde{u}_\infty = f(x,\tilde{u}_\infty)$, with an obvious choice of f. Indeed, for all $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$, there exists $C(\omega) > 0$ such that $|f(x,u)| \leq C(\omega)(1 + |u|^{p^\star-1})$ for all $x \in \omega$ and $u \in \mathbb{R}$: it then follows from Theorem 2.1 of Pucci–Servadei [25] (see also Druet [8, Lemmas 2.1 and 2.2], Guedda–Veron [15, Proposition 1.1]) that $\tilde{u}_\infty \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Hence it follows from Tolksdorf [33, Theorem 1] that $\tilde{u}_\infty \in C^1(\mathbb{R}^n \setminus \{0\})$.
Multiplying (26) by $(\tilde{u}_\infty)_-$ and integrating, we get that $\|(\tilde{u}_\infty)_-\| = 0$, and therefore $(\tilde{u}_\infty)_- \equiv 0$ thanks to (4). It then follows that $\tilde{u}_\infty \in C^1(\mathbb{R}^n \setminus \{0\})$ is a non–negative nontrivial weak solution to (26): thus $\tilde{u}_\infty > 0$ by the strong maximum principle of Vàzquez [34]. Therefore, $\tilde{u}_\infty \in D_1^p(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ is a *positive weak solution* of (1). This proves Claim 4.3 and therefore Theorem 1. ☐

Remark: Consider the functional

$$\tilde{\Phi}(u) := \frac{1}{p}\|u\|^p - \frac{1}{p^\star}\int_{\mathbb{R}^n}|u|^{p^\star}\,dx - \frac{1}{p^\star(s)}\int_{\mathbb{R}^n}\frac{|u|^{p^\star(s)}}{|x|^s}\,dx$$

for $u \in D_1^p(\mathbb{R}^n)$. Then the analysis above can be carried out for the functional $\tilde{\Phi}$, with only minor modifications. The main difference here is that the weak limit $\tilde{u}_\infty$ is not necessarily positive.

5. A non–existence result

In this section we require only that $\mu < \mu_1$ and prove the following result:

Theorem 3. *Let $1 < p < n$. If $u \in D_1^p(\mathbb{R}^n)$ is a weak solution to*

$$-\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^\star(s)-2}u}{|x|^s} + |u|^{q-2}u \quad \text{in } \mathbb{R}^n, \tag{27}$$

when $s \in (0,p)$ and $1 < q < p^\star$, then $u \equiv 0$.

Remark 1: note that, since $1 < q < p^\star$, we get that $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and the definition of the weak solution makes sense.

Remark 2: when $q > p^\star$, the same conclusion holds if $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ (see Claims 5.4 and 5.5).

The proof of Theorem 3 uses a Pohozaev–type identity. It proceeds in five claims:

Claim 5.1. *Let $\eta,\ u \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}(\nabla u, \nabla(x, \nabla(\eta u)))\, dx + \frac{n-p}{p}\int_{\mathbb{R}^n} \eta|\nabla u|^p\, dx = B(u,\eta), \tag{28}$$

where

$$B(u,\eta) = \int_{\mathbb{R}^n}\Bigg(u|\nabla u|^{p-2}(\nabla u, \nabla\eta) + \nabla^2\eta(x,\nabla u)|\nabla u|^{p-2}u \\ +|\nabla u|^{p-2}(\nabla u,\nabla\eta)(x,\nabla u) + \frac{1}{p'}(x,\nabla\eta)|\nabla u|^p\Bigg)dx,$$

and $p' = p/(p-1)$.

Proof of Claim 5.1*:* A similar identity was proved by Guedda–Veron [15] on bounded domains of $\mathbb{R}^n$. Expanding $\nabla(x, \nabla(\eta u))$, we obtain that

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}(\nabla u, \nabla(x,\nabla(\eta u)))\, dx = \int_{\mathbb{R}^n} \eta|\nabla u|^p\, dx + \int_{\mathbb{R}^n}\eta|\nabla u|^{p-2}x^i\partial_{ij}u\partial_j u\, dx \\ + \int_{\mathbb{R}^n}\Big(u|\nabla u|^{p-2}(\nabla u,\nabla\eta) + \nabla^2\eta(x,\nabla u)|\nabla u|^{p-2}u + |\nabla u|^{p-2}(\nabla u,\nabla\eta)(x,\nabla u) \\ +(x,\nabla\eta)|\nabla u|^p\Big)dx, \tag{29}$$

with Einstein's summation convention being used. Independently, we have that

$$\int_{\mathbb{R}^n}\eta|\nabla u|^{p-2}x^i\partial_{ij}u\partial_j u\, dx = \int_{\mathbb{R}^n}\eta x^i\partial_i\left(\frac{|\nabla u|^p}{p}\right)dx \\ = -\int_{\mathbb{R}^n}\frac{\partial_i(\eta x^i)}{p}|\nabla u|^p dx. \tag{30}$$

Plugging together (29) and (30), we get (28) and Claim 5.1 is proved. □

Claim 5.2. *If $u \in D_1^p(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})\cap H^1_{2,\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ and $\eta \in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, then identity* (28) *holds.*

Proof of Claim 5.2*:* By a density argument, we get that there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n\setminus\{0\})$ such that $\lim_{k\to\infty}\varphi_k = u$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\}) \cap H^1_{2,\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. We then apply Claim 5.1 to η, φ_k and let $k \to \infty$. Claim 5.2 is now proved. □

Claim 5.3. *Let $f \in C^0((\mathbb{R}^n\setminus\{0\})\times\mathbb{R})$ and let $u \in D_1^p(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})\cap H^1_{2,\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ be a weak solution of*

$$-\Delta_p u = f(x,u) \quad \text{in } \mathbb{R}^n. \tag{31}$$

Define $F(x,u) := \int_0^u f(x,v)\,dv$ *and assume that* $F \in C^1((\mathbb{R}^n\setminus\{0\})\times\mathbb{R})$. *Moreover, along the solution* u, *assume that* $uf(\cdot,u)$, $F(\cdot,u)$ *and* $x^i(\partial_i F)(\cdot,u) \in L^1(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n}\left[\frac{n-p}{p}uf(x,u) - nF(x,u) - x^i(\partial_i F)(x,u)\right]dx = 0. \tag{32}$$

Proof of Claim 5.3*:* Fix $\eta \in C_c^\infty(\mathbb{R}^n\setminus\{0\})$. Using the notations of the proof of Claim 5.2 and (31), we get that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla u|^{p-2}(\nabla u, \nabla(x,\nabla(\eta u)))\,dx\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^n}|\nabla u|^{p-2}(\nabla u, \nabla(x,\nabla(\eta\varphi_k)))\,dx\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^n} f(x,u)(x,\nabla(\eta\varphi_k))\,dx = \int_{\mathbb{R}^n} f(x,u)(x,\nabla(\eta u))\,dx\\
&= \int_{\mathbb{R}^n} uf(x,u)(x,\nabla\eta)\,dx + \int_{\mathbb{R}^n}\eta x^i[\partial_i(F(x,u)) - (\partial_i F)(x,u)]\,dx\\
&= \int_{\mathbb{R}^n} uf(x,u)(x,\nabla\eta)\,dx - \int_{\mathbb{R}^n}\partial_i(\eta x^i)F(x,u)\,dx - \int_{\mathbb{R}^n}\eta x^i(\partial_i F)(x,u)\,dx.
\end{aligned}\tag{33}$$

Independently, using (31), we have that

$$\begin{aligned}
\int_{\mathbb{R}^n}|\nabla u|^{p-2}(\nabla u,\nabla(\eta u))\,dx &= \lim_{k\to\infty}\int_{\mathbb{R}^n}|\nabla u|^{p-2}(\nabla u,\nabla(\eta\varphi_k))\,dx\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^n} f(x,u)\eta\varphi_k\,dx = \int_{\mathbb{R}^n} f(x,u)\eta u\,dx
\end{aligned}$$

and therefore

$$\int_{\mathbb{R}^n}\eta|\nabla u|^p\,dx = \int_{\mathbb{R}^n}\eta uf(x,u)\,dx - \int_{\mathbb{R}^n}u|\nabla u|^{p-2}(\nabla u,\nabla\eta)\,dx. \tag{34}$$

Plugging (33) and (34) into (28), we get by Hölder's inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\eta\left[\frac{n-p}{p}uf(x,u) - nF(x,u) - x^i(\partial_i F)(x,u)\right]dx\right|\\
&\qquad\le \|\nabla u\|_{L^p(\mathrm{Supp}|\nabla\eta|)}^{p-1}\|u\|_{p^\star}\left(\frac{n}{p}\|\nabla\eta\|_n + \|\,|x|\cdot|\nabla^2\eta|\,\|_n\right)\\
&\qquad+ \|\,|x|\cdot|\nabla\eta|\,\|_\infty\int_{\mathrm{Supp}|\nabla\eta|}|uf(x,u) - F(x,u)|\,dx\\
&\qquad+\left(1+\frac{1}{p'}\right)\|\,|x|\cdot|\nabla\eta|\,\|_\infty\|\nabla u\|_{L^p(\mathrm{Supp}|\nabla\eta|)}.
\end{aligned}\tag{35}$$

We are left with choosing an appropriate cut–off function η. Let $h \in C^\infty(\mathbb{R})$ be such that $h_{|\{t\le 1\}} \equiv 0$, $h_{|\{t\ge 2\}} \equiv 1$ and $0 \le h \le 1$. Given $\epsilon > 0$ small, define η_ϵ as follows: $\eta_\epsilon(x) = h(|x|/\epsilon)$ if $|x| \le 3\epsilon$, $\eta_\epsilon(x) = h(1/\epsilon|x|)$ if $|x| \ge (2\epsilon)^{-1}$ and $\eta_\epsilon(x) = 1$ elsewhere. Clearly $\eta_\epsilon \in C_c^\infty(\mathbb{R}^n\setminus\{0\})$. Taking $\eta = \eta_\epsilon$ in (35) and letting $\epsilon \to 0$, we get (32) and Claim 5.3 is proved. □

Claim 5.4. *If* $u \in D_1^p(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})\cap H^1_{2,\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ *is a weak solution to* (27) *when* $q > 1$ *and* $q \neq p^\star$, *then* $u \equiv 0$.

Proof of Claim 5.4*:* In order to use Claim 5.3, we need to prove that $u \in L^q(\mathbb{R}^n)$. Indeed, testing (27) on $\eta_\epsilon u$, where $\eta_\epsilon \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is as above (this is a valid test–function, see the proof of (34)), we get that

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}(\nabla u, \nabla(\eta_\epsilon u))\, dx - \mu \int_{\mathbb{R}^n} \frac{\eta_\epsilon |u|^p}{|x|^p}\, dx = \int_{\mathbb{R}^n} \frac{\eta_\epsilon |u|^{p^\star(s)}}{|x|^s}\, dx + \int_{\mathbb{R}^n} \eta_\epsilon |u|^q\, dx.$$

The Hardy inequality (3), the Hardy–Sobolev inequality (6) and Hölder's inequality yield the existence of $C > 0$, independent of ϵ, such that $\int_{\mathbb{R}^n} \eta_\epsilon |u|^q\, dx \le C$ for all $\epsilon > 0$. Letting $\epsilon \to 0$, we get that $u \in L^q(\mathbb{R}^n)$. Then we can use Claim 5.3 and, applying (32), we have that

$$\left(\frac{1}{p^\star} - \frac{1}{q}\right) \int_{\mathbb{R}^n} |u|^q\, dx = 0.$$

The fact that $q \neq p^\star$ implies that $u \equiv 0$, and Claim 5.4 is proved. □

Claim 5.5. *Let $u \in D_1^p(\mathbb{R}^n)$ be a weak solution of* (27)*, with $q > 1$. Moreover, assume in addition that $u \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ in case $q > p^\star$. Then*

$$u \in D_1^p(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}) \cap H^1_{2,\,\text{loc}}(\mathbb{R}^n \setminus \{0\}).$$

Proof of Claim 5.5*:* The argument relies essentially on the works of Tolksdorf [33], Druet [8] and Guedda–Veron [15]. We write (27) as $-\Delta_p u = f(x, u)$, with an obvious choice of f. Indeed, when $1 < q \le p^\star$, we get that for all $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$, there exists $C(\omega) > 0$ such that $-\Delta_p u = f(x, u)$, with $|f(x, u)| \le C(\omega)(1 + |u|^{p^\star - 1})$ for all $x \in \omega$ and $u \in \mathbb{R}$: it then follows from Druet [8, Lemmas 2.1 and 2.2], Guedda–Veron [15, Proposition 1.1] that $u \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$.

When also $u \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, then u satisfies $-\Delta_p u = f(x, u)$ weakly in $\mathbb{R}^n$, with $f(\cdot, u) \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Hence it follows from Tolksdorf [33, Theorem 1 and Proposition 1] that $u \in C^1(\mathbb{R}^n \setminus \{0\}) \cap H^{\inf\{2,p\}}_{2,\,\text{loc}}(\mathbb{R}^n \setminus \{0\})$. This proves Claim 5.5. □

Proof of Theorem 3: The proof follows from the combination of Claims 5.4 and 5.5.

6. Appendix 1: Extremals for Sobolev–type inequalities

In this section we allow μ to be possible negative.

Theorem 4. *Let $p \in (1, n)$, $\mu < \mu_1$ and $s \in [0, p)$. If $s = 0$, we assume that $\mu \ge 0$. Then the infimum $1/K(n, p, \mu, s)$ in* (6) *is achieved. More precisely, if $(u_k)_{k\in\mathbb{N}}$ is a minimizing sequence for $1/K(n, p, \mu, s)$ in $D_1^p(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |u_k|^{p^\star(s)}|x|^{-s} dx = 1$, then there exists a sequence $(r_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $(r_k^{(n-p)/p} u_k(r_k \cdot))_{k\in\mathbb{N}}$ is relatively compact in $D_1^p(\mathbb{R}^n)$ and converges to a minimizer for $1/K(n, p, \mu, s)$ up to a subsequence. Moreover, the infimum is achieved by a non–negative extremal.*

Finally, if $\mu \in [0, \mu_1)$ and if $s \in (0, p)$ when $\mu = 0$, then any non–negative minimizer of (6) *in $D_1^p(\mathbb{R}^n) \setminus \{0\}$ is positive, radially symmetric, radially decreasing with respect to* 0 *and approaches zero as $|x| \to \infty$.*

Remark 1: The assumption that $\mu \ge 0$ in case $s = 0$ is not technical. Indeed, as shown in Claim 7.1, it is not difficult to prove that $K(n, p, \mu, 0) = K(n, p, 0, 0)$ when $\mu < 0$: then, since there are extremals for $K(n, p, 0, 0)$, there is no extremal for $K(n, p, \mu, 0)$. We refer to Lions [22] for further considerations on this phenomenon.

Remark 2: When $p = 2$, the statement of Theorem 4 is essentially contained in Catrina-Wang [6]. In particular, the assumption that $\mu \geq 0$ in the last assertion of the theorem is not technical: indeed, it follows from Catrina-Wang [6] that when $p = 2$, for any $\mu < 0$, there exists $s_\mu > 0$ such that for all $s \in (0, s_\mu)$, then no minimizer of (6) is radially symmetrical.

The proof of Theorem 4 relies essentially on Lions's proof of the existence of extremals for the classical Sobolev inequalities [22]. We mainly follow the proof given in the book of Struwe [30]. Note that when $s = \mu = 0$, the extremals exist (see Rodemich [26], Aubin [3], Talenti [31], see also Lions [22]).

Let $(\tilde{u}_k)_{k\in\mathbb{N}} \subset D_1^p(\mathbb{R}^n) \setminus \{0\}$ be a minimizing sequence for $1/K(n,p,\mu,s)$ in (6). Up to multiplying by a positive constant, we assume that

$$\int_{\mathbb{R}^n} \frac{|\tilde{u}_k|^{p^\star(s)}}{|x|^s}\, dx = 1 \qquad \text{and} \qquad \lim_{k\to\infty} \int_{\mathbb{R}^n} \left(|\nabla \tilde{u}_k|^p - \mu \frac{|\tilde{u}_k|^p}{|x|^p} \right) dx = \frac{1}{K(n,p,\mu,s)}.$$

Since $\int_{\mathbb{R}^n} |\tilde{u}_k|^{p^\star(s)} |x|^{-s} dx = 1$ for all $k \in \mathbb{N}$, there exists $r_k > 0$ such that

$$\int_{B_{r_k}(0)} \frac{|\tilde{u}_k|^{p^\star(s)}}{|x|^s}\, dx = \frac{1}{2}$$

for all $k \in \mathbb{N}$. We define the rescaled sequence

$$u_k(x) := r_k^{(n-p)/p} \tilde{u}_k(r_k x)$$

for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Clearly $u_k \in D_1^p(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and $(u_k)_{k\in\mathbb{N}}$ is a minimizing sequence for $1/K(n,p,\mu,s)$, that is

$$\int_{\mathbb{R}^n} \frac{|u_k|^{p^\star(s)}}{|x|^s}\, dx = 1 \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{R}^n} \left(|\nabla u_k|^p - \mu \frac{|u_k|^p}{|x|^p} \right) dx = \frac{1}{K(n,p,\mu,s)}. \tag{36}$$

Moreover, we have that

$$\int_{B_1(0)} \frac{|u_k|^{p^\star(s)}}{|x|^s}\, dx = \frac{1}{2} \tag{37}$$

for all $k \in \mathbb{N}$. In addition, $\|u_k\|^p = K(n,p,\mu,s)^{-1} + o(1)$ as $k \to \infty$, and then, using (4), the $(\|\nabla u_k\|_p)_{k\in\mathbb{N}}$ is bounded. Therefore, without loss of generality, we assume that there exists $u \in D_1^p(\mathbb{R}^n)$ such that

$$\begin{aligned} &u_k \rightharpoonup u \quad \text{weakly in } D_1^p(\mathbb{R}^n) \text{ as } k \to \infty, \\ &\lim_{k\to\infty} u_k(x) = u(x) \quad \text{for a.a. } x \in \mathbb{R}^n. \end{aligned}$$

We define the measures

$$\nu_k := \frac{|u_k|^{p^\star(s)}}{|x|^s}\, dx \quad \text{and} \quad \lambda_k := \left(|\nabla u_k|^p - \mu \frac{|u_k|^p}{|x|^p} \right) dx. \tag{38}$$

Hence (36) simply reduces to

$$\int_{\mathbb{R}^n} d\nu_k = 1 \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{R}^n} d\lambda_k = \frac{1}{K(n,p,\mu,s)}. \tag{39}$$

Clearly, $\nu_k \geq 0$ by (36). Moreover, in the sense of measures, we get that $|\lambda_k| \leq (|\nabla u_k|^p + |\mu||u_k|^p|x|^{-p})\, dx$ is a bounded measure with respect to $k \in \mathbb{N}$. Up to a subsequence, there exist two measures ν and λ such that

$$\nu_k \rightharpoonup \nu \quad \text{and} \quad \lambda_k \rightharpoonup \lambda \quad \text{weakly in the sense of measures as } k \to \infty.$$

We now apply Lions's first concentration–compactness Lemma [22] to the sequence of measures $(\nu_k)_{k\in\mathbb{N}}$. Indeed, up to a subsequence, three situations can occur (cf. [30, Lemma 1, page 39]):

(a) (*Compactness*) There exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that for any $\epsilon>0$ there exists $R_\epsilon>0$ for which

$$\int_{B_{R_\varepsilon}(x_k)} d\nu_k \geq 1-\epsilon \qquad \text{for all } k\in\mathbb{N} \text{ large.}$$

(b) (*Vanishing*) For all $R>0$ there holds

$$\lim_{k\to\infty}\left(\sup_{x\in\mathbb{R}^n}\int_{B_R(x)} d\nu_k\right)=0.$$

(c) (*Dichotomy*) There exists $\alpha\in(0,1)$ such that for any $\epsilon>0$ there exists $R_\epsilon>0$ and a sequence $(x_k^\epsilon)_{k\in\mathbb{N}}\in\mathbb{R}^n$, with the following property: given $R'>R_\epsilon$, there are non–negative measures ν_k^1 and ν_k^2 such that

$$\begin{gathered}
0\leq \nu_k^1+\nu_k^2\leq \nu_k, \quad \mathrm{Supp}(\nu_k^1)\subset B_{R_\epsilon}(x_k^\epsilon), \quad \mathrm{Supp}(\nu_k^2)\subset \mathbb{R}^n\setminus B_{R'}(x_k^\epsilon),\\
\nu_k^1=\nu_k\big|_{B_{R_\epsilon}(x_k^\epsilon)}, \qquad \nu_k^2=\nu_k\big|_{\mathbb{R}^n\setminus B_{R'}(x_k^\epsilon)},\\
\limsup_{k\to\infty}\left(\left|\alpha-\int_{\mathbb{R}^n} d\nu_k^1\right|+\left|(1-\alpha)-\int_{\mathbb{R}^n} d\nu_k^2\right|\right)\leq\epsilon.
\end{gathered}$$

Claim 6.1. *Compactness* (*point* (a)) *holds. In particular, we have that* $\int_{\mathbb{R}^n} d\nu=1$.

Proof. It follows from (37) that *Vanishing*, point (b), does not hold. We argue by contradiction and assume that *Dichotomy* holds, that is there exists $\alpha\in(0,1)$ such that (c) above holds. Taking $\epsilon=(k+1)^{-1}$, we can assume that, up to a subsequence, there exist sequences $(R_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_{>0}$, $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ and two sequences of non–negative measures, $(\nu_k^1)_{k\in\mathbb{N}}$ and $(\nu_k^2)_{k\in\mathbb{N}}$, such that

$$\begin{aligned}
&0\leq \nu_k^1+\nu_k^2\leq\nu_k, \qquad \lim_{k\to\infty}R_k=\infty,\\
&\mathrm{Supp}(\nu_k^1)\subset B_{R_k}(x_k), \qquad \mathrm{Supp}(\nu_k^2)\subset\mathbb{R}^n\setminus B_{2R_k}(x_k),\\
&\nu_k^1=\nu_k\big|_{B_{R_k}(x_k)}, \qquad \nu_k^2=\nu_k\big|_{\mathbb{R}^n\setminus B_{2R_k}(x_k)},\\
&\lim_{k\to\infty}\int_{\mathbb{R}^n} d\nu_k^1=\alpha \quad\text{and}\quad \lim_{k\to\infty}\int_{\mathbb{R}^n} d\nu_k^2=1-\alpha.
\end{aligned}\tag{40}$$

In particular, by $(39)_1$ and (40), we have

$$\lim_{k\to\infty}\int_{D_k} d\nu_k=0, \qquad D_k:=B_{2R_k}(x_k)\setminus B_{R_k}(x_k).\tag{41}$$

Step 6.1.1: We claim that

$$\lim_{k\to\infty}\int_{D_k}\frac{|u_k|^p}{|x|^p}\,dx=0.\tag{42}$$

Indeed, by Hölder's inequality, we get that

$$\begin{aligned}\int_{D_k}\frac{|u_k|^p}{|x|^p}\,dx &= \int_{D_k}\frac{1}{|x|^{p-ps/p^\star(s)}}\left(\frac{|u_k|}{|x|^{s/p^\star(s)}}\right)^p dx\\ &\le \left(\int_{D_k}\left(\frac{1}{|x|^{p-ps/p^\star(s)}}\right)^{(n-s)/(p-s)} dx\right)^{1-\frac{p}{p^\star(s)}}\left(\int_{D_k}\frac{|u_k|^{p^\star(s)}}{|x|^s}\,dx\right)^{p/p^\star(s)}\\ &\le C\left(\int_{D_k} d\nu_k\right)^{p/p^\star(s)}.\end{aligned}$$

Therefore, (41) yields (42), and the claim is proved.

Step 6.1.2: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$, $\varphi_{|B_1(0)} \equiv 1$ and $\varphi_{|B_2(0)^c} \equiv 0$. We define $\varphi_k(x) := \varphi(R_k^{-1}(x - x_k))$ for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{N}$. By (40), (41), (6) and the fact that $p < p^\star(s)$, we get that

$$\begin{aligned}1 &= \left(\int_{\mathbb{R}^n}\varphi_k^{p^\star(s)} d\nu_k^1 + \int_{\mathbb{R}^n}(1-\varphi_k)^{p^\star(s)} d\nu_k^2\right)^{p/p^\star(s)} + o(1)\\ &\le \left(\int_{\mathbb{R}^n}\varphi_k^{p^\star(s)} d\nu_k^1\right)^{p/p^\star(s)} + \left(\int_{\mathbb{R}^n}(1-\varphi_k)^{p^\star(s)} d\nu_k^2\right)^{p/p^\star(s)} + o(1)\\ &\le \left(\int_{\mathbb{R}^n}\varphi_k^{p^\star(s)} d\nu_k\right)^{p/p^\star(s)} + \left(\int_{\mathbb{R}^n}(1-\varphi_k)^{p^\star(s)} d\nu_k\right)^{p/p^\star(s)} + o(1)\\ &\le K(n,p,\mu,s)\int_{\mathbb{R}^n}\left(|\nabla(\varphi_k u_k)|^p - \mu\frac{|\varphi_k u_k|^p}{|x|^p}\right) dx\\ &\quad + K(n,p,\mu,s)\int_{\mathbb{R}^n}\left(|\nabla((1-\varphi_k)u_k)|^p - \mu\frac{|(1-\varphi_k)u_k|^p}{|x|^p}\right) dx + o(1).\end{aligned}\tag{43}$$

Step 6.1.3: As shown in (14), we shall prove that

$$\int_{\mathbb{R}^n}|\nabla(\varphi_k u_k)|^p\,dx = \int_{\mathbb{R}^n}|\varphi_k|^p|\nabla u_k|^p\,dx + o(1)\tag{44}$$

as $k \to \infty$. Indeed,

$$||\nabla(\varphi_k u_k)|^p - |\varphi_k|^p|\nabla u_k|^p| \le C_p(|\varphi_k\nabla u_k|^{p-1}|u_k\nabla\varphi_k| + |u_k\nabla\varphi_k|^p)$$

for all $k \in \mathbb{N}$, which, integrated over $\mathbb{R}^n$, gives

$$\left|\int_{\mathbb{R}^n}|\nabla(\varphi_k u_k)|^p dx - \int_{\mathbb{R}^n}|\varphi_k|^p|\nabla u_k|^p dx\right| \le C_p\int_{\mathbb{R}^n}(|\varphi_k\nabla u_k|^{p-1}|u_k\nabla\varphi_k| + |u_k\nabla\varphi_k|^p)dx.$$

By Hölder's inequality, and since $\mathrm{Supp}\,(\nabla\varphi_k) \subset D_k$, we get that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}|\nabla(\varphi_k u_k)|^p dx - \int_{\mathbb{R}^n}|\varphi_k|^p|\nabla u_k|^p dx\right|\\ &\quad\le C_p\|\nabla u_k\|_p^{p-1}\left(\int_{\mathbb{R}^n}|u_k\nabla\varphi_k|^p\,dx\right)^{1/p} + C_p\int_{\mathbb{R}^n}|u_k\nabla\varphi_k|^p\,dx\\ &\quad\le C\left[\left(\int_{D_k}\frac{|u_k|^p}{|x|^p}\,dx\right)^{1/p} + \int_{D_k}\frac{|u_k|^p}{|x|^p}\,dx\right]\end{aligned}$$

for all $k \in \mathbb{N}$. Therefore, (42) yields (44). This ends Step 6.1.3.

Step 6.1.4: Similarly to (44) it results

$$\int_{\mathbb{R}^n} |\nabla\{(1-\varphi_k)u_k\}|^p\,dx = \int_{\mathbb{R}^n} |1-\varphi_k|^p|\nabla u_k|^p\,dx + o(1) \tag{45}$$

as $k\to\infty$. Plugging (44) and (45) into (43), we obtain

$$\begin{aligned} 1 &\le \left(\int_{\mathbb{R}^n} \varphi_k^{p^\star(s)}d\nu_k^1\right)^{p/p^\star(s)} + \left(\int_{\mathbb{R}^n} (1-\varphi_k)^{p^\star(s)}d\nu_k^2\right)^{p/p^\star(s)} + o(1) \\ &\le K(n,p,\mu,s)\int_{\mathbb{R}^n} [\varphi_k^p + (1-\varphi_k)^p]\,d\lambda_k + o(1) \\ &= 1 + K(n,p,\mu,s)\int_{\mathbb{R}^n} [\varphi_k^p + (1-\varphi_k)^p - 1]\,d\lambda_k + o(1) \end{aligned} \tag{46}$$

by $(39)_2$. We now deal with the second term of the right hand side above. Since $\mathrm{Supp}\,(1-\varphi_k^p-(1-\varphi_k)^p)\subset D_k$ and $0\le\varphi_k^p+(1-\varphi_k)^p\le 1$, we get that

$$\begin{aligned} \int_{\mathbb{R}^n} [\varphi_k^p + (1-\varphi_k)^p - 1]\,d\lambda_k &= -\int_{\mathbb{R}^n} [1-\varphi_k^p-(1-\varphi_k)^p]\,|\nabla u_k|^p dx \\ &\quad - \mu\int_{D_k}[\varphi_k^p + (1-\varphi_k)^p - 1]\,\frac{|u_k|^p}{|x|^p}dx \\ &\le 2|\mu|\int_{D_k}\frac{|u_k|^p}{|x|^p}\,dx. \end{aligned} \tag{47}$$

Letting $k\to\infty$ in (46) and using (47), (42) and (40), we get that $1=\alpha^{p/p^\star(s)}+(1-\alpha)^{p/p^\star(s)}$. This is impossible when $\alpha\in(0,1)$, since $p<p^\star(s)$ being $s\in[0,p)$. This contradiction proves Claim 6.1. □

Claim 6.2. *There exist $J\subset\mathbb{N}$ at most countable, a subset $I\subset J$ and a family $\{x_i\}_{i\in J}$ in $\mathbb{R}^n$ such that*

$$\nu = \frac{|u|^{p^\star(s)}}{|x|^s}\,dx + \sum_{i\in I}\nu^i\delta_{x_i}, \tag{48}$$

where $\nu^i=\nu(\{x^i\})>0$ for all $i\in I$. In particular, $\{x_i\,/\,i\in I\}\subset\{0\}$ when $s>0$. Moreover, there exists a bounded non–negative measure $\lambda_0\ge 0$ with no atoms (that is $\lambda_0(\{x\})=0$ for all $x\in\mathbb{R}^n$) such that

$$\lambda = \lambda_0 + \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx + \sum_{i\in J}\lambda^i\delta_{x_i}, \tag{49}$$

where $\lambda^i=\lambda(\{x_i\})>0$ for all $i\in J$. In addition,

$$(\nu^i)^{p/p^\star(s)} \le K(n,p,\mu,s)\lambda^i \quad \text{for all } i\in I. \tag{50}$$

Proof. This proof is essentially an adaptation of Lions's second concentration–compactness Lemma [22]. When $s=0$, (48) is a consequence of Lions's result. When $s>0$, since $(u_k)_{k\in\mathbb{N}}$ goes to u strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for $q<p^\star$, we get that $\nu=|u|^{p^\star(s)}|x|^{-s}\,dx+\nu(\{0\})\delta_0$. This proves (48) in the case $s\ge 0$.

We are left with proving (49). As above, we get that there exists $L\ge 0$ such that

$$\frac{|u_k|^p}{|x|^p}\,dx \rightharpoonup \frac{|u|^p}{|x|^p}\,dx + L\,\delta_0 \tag{51}$$

in the sense of measures as $k \to \infty$. Up to extraction, we let λ' be the weak limit of $(|\nabla u_k|^p\,dx)$ as $k \to \infty$ in the sense of measures. Since $u_k \rightharpoonup u$ weakly in $D_1^p(\mathbb{R}^n)$ as $k \to \infty$, we get that $\lambda' \geq |\nabla u|^p\,dx$. Therefore, we decompose λ' as follows:

$$\lambda' = \lambda_0 + |\nabla u|^p\,dx + \sum_{j\in K} \lambda'(\{z_j\})\delta_{z_j}, \tag{52}$$

where $\lambda_0 \geq 0$ and the z_j's, $j \in K$ countable, are the atoms of λ'. Combining (51) and (52), we have that

$$\lambda = \lambda_0 + \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - L\mu\,\delta_0 + \sum_{j\in K}\lambda'(\{z_j\})\delta_{z_j}. \tag{53}$$

We claim that

$$[\nu(\{x\})]^{p/p^\star(s)} \leq K(n,p,\mu,s)\lambda(\{x\}) \quad \text{ for all } x \in \mathbb{R}^n. \tag{54}$$

Indeed, take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi_{|B_1(0)} \equiv 1$ and $\varphi_{|\mathbb{R}^n\setminus B_2(0)} \equiv 0$. Given $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, we define $\varphi_\epsilon(x) = \varphi(\epsilon^{-1}(x - x_0))$ for $x \in \mathbb{R}^n$. It follows from the Sobolev inequality (6) that

$$\left(\int_{\mathbb{R}^n}\frac{|\varphi_\epsilon u_k|^{p^\star(s)}}{|x|^s}\,dx\right)^{p/p^\star(s)} \leq K(n,p,\mu,s)\int_{\mathbb{R}^n}\left(|\nabla(\varphi_\epsilon u_k)|^p - \mu\frac{|\varphi_\epsilon u_k|^p}{|x|^p}\right)dx.$$

As in the proof of (44), we have

$$\left(\int_{\mathbb{R}^n}|\varphi_\epsilon|^{p^\star(s)}\,d\nu_k\right)^{p/p^\star(s)} \leq K(n,p,\mu,s)\int_{\mathbb{R}^n}|\varphi_\epsilon|^p\,d\lambda_k + C\theta_k + C(\theta_k)^{1/p} \tag{55}$$

for all $k \in \mathbb{N}$ and all $\epsilon > 0$, where

$$\theta_k := \int_{B_{2\epsilon}(x_0)\setminus B_\epsilon(x_0)}\frac{|u_k|^p}{|x|^p}\,dx.$$

Letting $k \to \infty$ and then $\epsilon \to 0$, we get that

$$[\nu(\{x_0\})]^{p/p^\star(s)} \leq K(n,p,\mu,s)\lambda(\{x_0\})$$

and the claim is proved.

Combining (53) with (54) and considering separately the cases $0 \in \{x_i /\, i \in J\}$ or not, we get (49). This proves Claim 6.2. $\square$

Claim 6.3. *We assert that*

$$\textit{either } \left\{\nu = \frac{|u|^{p^\star(s)}}{|x|^s}\,dx \textit{ and } \lambda = \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx\right\}$$

$$\textit{or there exists } x_0 \in \mathbb{R}^n \textit{ such that } \left\{\nu = \delta_{x_0} \textit{ and } \lambda = \frac{\delta_{x_0}}{K(n,p,\mu,s)}\right\}.$$

Proof. Integrating (48) and (49), using (6), (50) and the fact that $\int_{\mathbb{R}^n} d\nu = 1$ (see Claim 6.1) and inequality (5), we have

$$
\begin{aligned}
1 = \left(\int_{\mathbb{R}^n} d\nu\right)^{p/p^\star(s)} &= \left(\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s}\, dx + \sum_{i\in I} \nu^i\right)^{p/p^\star(s)} \\
&\le \left(\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s} dx\right)^{p/p^\star(s)} + \sum_{i\in I} (\nu^i)^{p/p^\star(s)} \\
&\le K(n,p,\mu,s)\left(\int_{\mathbb{R}^n}\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right) dx + \sum_{i\in I}\lambda^i\right) \\
&\le K(n,p,\mu,s)\int_{\mathbb{R}^n} d\lambda.
\end{aligned} \tag{56}
$$

We are then left with estimating $\int_{\mathbb{R}^n} d\lambda$ from above. Let $\psi \in C^\infty(\mathbb{R}^n)$ such that $0 \le \psi \le 1$, $\psi_{|B_1(0)} \equiv 0$ and $\psi_{|\mathbb{R}^n\setminus B_2(0)} \equiv 1$. Given $R > 0$, we let $\psi_R(x) = \psi(R^{-1}x)$ for $x \in \mathbb{R}^n$. In particular, $1 - \psi_R^p \in C_c^0(\mathbb{R}^n)$. Hence, since $\mu < \mu_1$, by $(39)_2$ and (3) we find that

$$
\begin{aligned}
\int_{\mathbb{R}^n} (1-\psi_R^p)\, d\lambda_k &= \int_{\mathbb{R}^n} d\lambda_k - \int_{\mathbb{R}^n}\left(\psi_R^p|\nabla u_k|^p - \mu\frac{|\psi_R u_k|^p}{|x|^p}\right) dx \\
&= \int_{\mathbb{R}^n} d\lambda_k - \int_{\mathbb{R}^n}\left(|\nabla(\psi_R u_k)|^p - \mu\frac{|\psi_R u_k|^p}{|x|^p}\right) dx \\
&\qquad + \int_{\mathbb{R}^n} \left(|\nabla(\psi_R u_k)|^p - \psi_R^p|\nabla u_k|^p\right) dx \\
&\le \frac{1}{K(n,p,\mu,s)} + \int_{\mathbb{R}^n} \left(|\nabla(\psi_R u_k)|^p - \psi_R^p|\nabla u_k|^p\right) dx + o(1).
\end{aligned} \tag{57}
$$

Mimicking what was worked out in (44), we obtain

$$
\left|\int_{\mathbb{R}^n} \left(|\nabla(\psi_R u_k)|^p - \psi_R^p|\nabla u_k|^p\right) dx\right| \le C\theta_k(R) + C\theta_k(R)^p,
$$

where

$$
\theta_k(R) := \int_{B_{2R}(0)\setminus B_R(0)} \frac{|u_k|^p}{|x|^p}\, dx.
$$

Therefore, letting $k \to \infty$ in (57), and then $R \to \infty$, we get that

$$
\int_{\mathbb{R}^n} d\lambda \le \frac{1}{K(n,p,\mu,s)}.
$$

Plugging this latest inequality in (56), we get that $\int_{\mathbb{R}^n} d\lambda = K(n,p,\mu,s)^{-1}$. Therefore, there is equality in (56). By convexity, this means that one and only one term in (48) is nonzero and that there is equality in all the inequalities used to prove (56). The conclusion of the claim then follows. This proves Claim 6.3. ☐

Claim 6.4. *We assert that* $\nu = |u|^{p^\star(s)}|x|^{-s}\, dx$ *and* $\lambda = (|\nabla u|^p - \mu|u|^p|x|^{-p})\, dx$.

Proof. We argue by contradiction. If Claim 6.4 does not hold, it follows from Claim 6.3, that there exists $x_0 \in \mathbb{R}^n$ such that $\nu = \delta_{x_0}$ and $\lambda = \delta_{x_0}/K(n,p,\mu,s)$: in particular, $u \equiv 0$. If $x_0 = 0$, then $\int_{B_{1/2}(0)} d\nu = 1$, which contradicts the initial hypotheses (37) and proves Claim 6.4 when $x_0 = 0$. We are then left with proving that $x_0=0$. We argue by contradiction and assume that $x_0\neq 0$. We distinguish two cases:

Case 1: $s>0$. Then, since $u\equiv 0$, we get that $\lim_{k\to\infty}u_k=0$ in $L^{p^\star(s)}_{\rm loc}(\mathbb{R}^n)$, and then $\lim_{k\to\infty}\int_{B_\delta(x_0)}|u_k|^{p^\star(s)}|x|^{-s}\,dx=0$ for $\delta>0$ small enough: a contradiction with the fact that $\nu=\delta_{x_0}$. This ends Case 1.

Case 2: $s=0$. Let $\delta>0$ and $\varphi\in C^\infty_c(\mathbb{R}^n)$ such that $0\leq\varphi\leq\varphi(x_0)=1$ and $\varphi_{\mathbb{R}^n\setminus B_\delta(x_0)}\equiv 0$. Since $\lim_{k\to\infty}u_k=0$ in $L^p_{\rm loc}(\mathbb{R}^n)$, it follows from the Hardy inequality (3) and computations similar to the ones leading to (45) that there exists $C>0$ such that

$$
\begin{aligned}
\int_{\mathbb{R}^n}\frac{|(1-\varphi)u_k|^p}{|x|^p}\,dx &\leq C\int_{\mathbb{R}^n}\left(|\nabla\{(1-\varphi)u_k\}|^p-\mu\frac{|(1-\varphi)u_k|^p}{|x|^p}\right)dx\\
&=C\int_{\mathbb{R}^n}(1-\varphi)^p d\lambda_k+o(1)\\
&=C\left(\frac{1}{K(n,p,\mu,s)}-\int_{\mathbb{R}^n}[1-(1-\varphi)^p]d\lambda_k\right)+o(1)\\
&=\frac{C}{K(n,p,\mu,s)}\left\{1-[1-(1-\varphi(x_0))^p]\right\}+o(1)=o(1)
\end{aligned}
$$

as $k\to\infty$, since clearly (45) holds when φ replaces φ_k, and $\lambda_k\rightharpoonup\lambda=\delta_{x_0}/K(n,p,\mu,s)$. In particular, for all $\delta>0$, we get that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n\setminus B_\delta(x_0)}\frac{|u_k|^p}{|x|^p}\,dx=0.$$

Moreover, since $x_0\neq 0$ and $u_k\to 0$ strongly in $L^p_{\rm loc}(\mathbb{R}^n)$, we have

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{|u_k|^p}{|x|^p}\,dx=0,$$

which implies by (39), since $s=0$, that

$$\frac{\int_{\mathbb{R}^n}|\nabla u_k|^p\,dx}{\left(\int_{\mathbb{R}^n}|u_k|^{p^\star}\,dx\right)^{p/p^\star}}=\frac{1}{K(n,p,\mu,0)}+o(1)$$

as $k\to\infty$. It then follows from (5) that

$$\frac{1}{K(n,p,0,0)}\leq\frac{1}{K(n,p,\mu,0)}. \tag{58}$$

Let $u\in D^p_1(\mathbb{R}^n)\setminus\{0\}$ be an extremal for $K(n,p,0,0)$ (this exists, see Rodemich [26], Talenti [31], Aubin [3] and also Lions [22]). Estimating the functional of $K(n,p,\mu,0)$ at u and using that $\mu>0$, we get that

$$\frac{1}{K(n,p,0,0)}>\frac{1}{K(n,p,\mu,0)}.$$

A contradiction with inequality (58). This rules out the case $x_0\neq 0$, and Case 2 is finished. This also ends the proof of Claim 6.4. $\square$

Proof of Theorem 4. Since $\nu=|u|^{p^\star(s)}|x|^{-s}dx$ and $\lambda=(|\nabla u|^p-\mu\,|u|^p|x|^{-p})dx$, we get that $\lim_{k\to\infty}u_k=u$ in $L^{p^\star(s)}(\mathbb{R}^n,|x|^{-s})\cap L^p(\mathbb{R}^n,|x|^{-p})$. Consequently, we get that $\|\nabla u_k\|_p\to\|\nabla u\|_p$ as $k\to\infty$ and by Clarkson's uniform convexity, we find that $\lim_{k\to\infty}u_k=u$ in $D^p_1(\mathbb{R}^n)$. Hence u is an extremal for (6). In addition, $|u|$

is in $D_1^p(\mathbb{R}^n)$ and $|\nabla|u|| = |\nabla u|$ a.e on $\mathbb{R}^n$: therefore, $|u|$ is also an extremal, and then there exist non–negative extremals. The first part of Theorem 4 is proved.

Assume now that $\mu \in [0, \mu_1)$ and $s \in (0,p)$ when $\mu = 0$. Let $u \geq 0$ be a minimizer of (6) in $D_1^p(\mathbb{R}^n) \setminus \{0\}$, which exists from the first part of Theorem 4 already proved. Following Talenti [31], see also [21, Section 3.2], we define the Schwarz symmetrization of u by

$$u_*(x) := \inf\{t \geq 0 \, : \, \mathrm{meas}(U^t) < \omega_n |x|^n\},$$

where U^t are the level sets of $u = |u|$, that is, $U^t = \{x \in \mathbb{R}^n \, : \, |u(x)| > t\}$, and ω_n denotes the measure of the standard unit ball of $\mathbb{R}^n$. In particular, $(|x|^{-\alpha})_* = |x|^{-\alpha}$ for all $\alpha > 0$, see [21, 3.3–(ii)]. By the well known Polya–Szego inequality (see [31] and [23])

$$\int_{\mathbb{R}^n} |\nabla u_*|^p dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dx,$$

and $u_* \in D_1^p(\mathbb{R}^n)$, being $\int_{\mathbb{R}^n} |u_*|^{p^\star} dx = \int_{\mathbb{R}^n} |u|^{p^\star} dx$. Furthermore, by Theorem 3.4. of [21]

$$\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s} dx \leq \int_{\mathbb{R}^n} \frac{|u_*|^{p^\star(s)}}{|x|^s} dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} \frac{|u_*|^p}{|x|^p} dx.$$

Combining the above inequalities and the fact that $\mu \geq 0$, we get that also u_* is a minimizer and achieves the infimum of (6). Hence the equality sign holds in all the inequalities above. In particular,

$$\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s} dx = \int_{\mathbb{R}^n} \frac{|u_*|^{p^\star(s)}}{|x|^s} dx \quad \text{and} \quad \mu \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx = \mu \int_{\mathbb{R}^n} \frac{|u_*|^p}{|x|^p} dx.$$

From Theorem 3.4 of [21], in the case of equality, it then follows that $u = |u| = u_*$ if either $\mu \neq 0$ or if $s \neq 0$. In particular, u is positive, radially symmetric and decreasing with respect to 0. Hence u must approach a limit as $|x| \to \infty$, which must be zero, being $u \in L^{p^\star}(\mathbb{R}^n)$. □

7. Appendix 2: The case $\mu < 0$

As mentioned above, when $s = 0$ and $\mu < 0$, there is no extremal for (6). More precisely, we have the following:

Claim 7.1: *Condition $\mu \leq 0$ entails that*

$$K(n, p, \mu, 0) = K(n, p, 0, 0).$$

In particular, there are no extremals when $\mu < 0$.

Proof of Claim 7.1: Since $\mu \leq 0$, we have that

$$K(n, p, \mu, 0)^{-1} \geq K(n, p, 0, 0)^{-1}. \tag{59}$$

Let $u \in D_1^p(\mathbb{R}^n) \setminus \{0\}$ be an extremal for $K(n, p, 0, 0)^{-1}$. Fix $\alpha \in \mathbb{R}$ and let e_1 be a nontrivial vector of $\mathbb{R}^n$. We define

$$u_\alpha(x) := u(x - \alpha e_1) \tag{60}$$

for all $x \in \mathbb{R}^n$. With a change of variables, we have

$$\frac{\int_{\mathbb{R}^n} |\nabla u_\alpha|^p dx - \mu \int_{\mathbb{R}^n} |u_\alpha|^p |x|^{-p} dx}{\left(\int_{\mathbb{R}^n} |u_\alpha|^{p^\star}\right)^{p/p^\star}} = \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx - \mu \int_{\mathbb{R}^n} |u|^p |x + \alpha e_1|^{-p} dx}{\left(\int_{\mathbb{R}^n} |u|^{p^\star}\right)^{p/p^\star}},$$

so that

$$\lim_{\alpha\to\infty}\frac{\int_{\mathbb{R}^n}|\nabla u_\alpha|^p dx-\mu\int_{\mathbb{R}^n}|u_\alpha|^p|x|^{-p}dx}{\left(\int_{\mathbb{R}^n}|u_\alpha|^{p^\star}\right)^{p/p^\star}}=\frac{\int_{\mathbb{R}^n}|\nabla u|^p dx}{\left(\int_{\mathbb{R}^n}|u|^{p^\star}\right)^{p/p^\star}}=\frac{1}{K(n,p,0,0)}.$$

Therefore, $K(n,p,\mu,0)^{-1}\leq K(n,p,0,0)^{-1}$. Combining this with (59), we obtain that $K(n,p,\mu,0)^{-1}=K(n,p,0,0)^{-1}$. This proves Claim 7.1. $\square$

Taking u an extremal for $K(n,p,0,0)^{-1}$ and u_α as in (60), we get after some computations that

$$\max_{t\geq 0}\Phi(tu_\alpha)<\frac{1}{n}K(n,p,\mu,0)^{-n/p}$$

for α large when $0<s<\min\{p,(n-p)/(p-1)\}$. *This permits to extend the proof given in Sections* 2 *and* 3 *to the case* $\mu<0$ *and* $0<s<\min\{p,(n-p)/(p-1)\}$.

We present here an alternative approach that allows to recover the full range $\mu<\mu_1$. Define

$$D^p_{1,r}(\mathbb{R}^n):=\{u\in D^p_1(\mathbb{R}^n)/\ u \text{ is radially symmetrical}\}$$

and for all $p\in(1,n)$, $s\in(0,p)$ and $\mu<\mu_1$, we let

$$\frac{1}{K_r(n,p,\mu,s)}:=\inf_{u\in D^p_{1,r}(\mathbb{R}^n)\setminus\{0\}}\frac{\int_{\mathbb{R}^n}|\nabla u|^p\,dx-\mu\int_{\mathbb{R}^n}|u|^p|x|^{-p}\,dx}{\left(\int_{\mathbb{R}^n}|u|^{p^\star(s)}|x|^{-s}\,dx\right)^{p/p^\star(s)}}. \tag{61}$$

Arguing as in Section 6, we have

Proposition 3. *For all* $p\in(1,n)$, $s\in(0,p)$ *and* $\mu<\mu_1$, *there are nonnegative extremals for* $K_r(n,p,\mu,s)^{-1}$.

In particular, a consequence of Theorem 4 and the remarks following this theorem is that

$$K(n,p,\mu,s)=K_r(n,p,\mu,s),$$
$$\text{when } \mu\in[0,\mu_1) \text{ and } s\in[0,p),\ \text{with } \mu+s>0;$$

while

$$K(n,p,\mu,s)>K_r(n,p,\mu,s) \text{ when } \mu<0 \text{ and } s\in(0,s_\mu).$$

Since we have the existence of extremals in the radial case, one can carry out the proofs of Sections 2 and 3 by restricting to radial functions and by replacing $K(n,p,\mu,s)$ in the definition (8) of c_* by $K_r(n,p,\mu,s)$. This proves Theorem 1 in the case $\mu<0$.

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