# Eigenvalue estimate for the basic Laplacian on manifolds with foliated boundary, part II

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**Abstract**

On a compact Riemannian manifold $N$ whose boundary is endowed with a Riemannian flow, we gave in [4] a sharp lower bound for the first non-zero eigenvalue of the basic Laplacian acting on basic 1-forms. In this paper, we extend this result to the set of basic $p$-forms when $p > 1$. We then characterize the limiting case by showing that the manifold $N$ is isometric to $\Gamma\backslash\mathbb{R} \times B'$ for some group $\Gamma$ where $B'$ denotes the unit closed ball. As a consequence, we describe the Riemannian product $\mathbb{S}^1 \times \mathbb{S}^n$ as the boundary of a manifold.

**Key words**: Riemannian flow, manifolds with boundary, basic Laplacian, eigenvalue, second fundamental form, O'Neill tensor, basic Killing forms, rigidity results.

**Mathematics Subject Classification**: 53C12, 53C24, 58J50, 58J32.

## 1 Introduction

On a compact Riemannian manifold $(N^{n+2}, g)$ whose boundary $M$ carries a minimal Riemannian flow given by a unit vector field $\xi$ (see Section 2 for the definition of a Riemannian flow), we derived in [4] a sharp lower bound for the first non-zero eigenvalue $\lambda'_{1,p}$ of the basic Laplacian acting on (closed) basic $p$-forms on $M$ when $p = 1$. This lower bound involves the $q$-curvatures $\sigma_q$ $(1 \leq q \leq n+1)$ of $M$ [4, Thm. 1.1] which are defined as the sum of the first $q$ principal curvatures assumed to be arranged in an increasing way. Here the word minimal means that the mean curvature of the integral curves of $\xi$ is zero. The main tool to prove the estimate is the use of a suitable extension of a basic closed $p$-form to the whole manifold $N$ (see Lemma 3.1) and the so-called *Reilly formula* established in [11]. As a consequence of the limiting case, we got several rigidity theorems that characterize the flow as a local product and the boundary as an $\eta$-umbilical submanifold (see [4, Sec. 5]). These results can be seen as a foliated version of the work of S. Raulot and A. Savo in [11].

In this paper, we generalize the results stated in [4] to any basic $p$-form when $p > 1$. Some of the techniques used in this paper are similar to those of $p = 1$ (Lemmas 3.1 and 3.2) but some new results and techniques are needed to extend the estimate and study the equality case (Lemmas 3.3, 4.2, 4.4 and 4.5). First, we prove

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**Theorem 1.1** *Let $(N^{n+2}, g)$ be a compact Riemannian manifold with positive curvature operator. Assume that the boundary $M$ is endowed with a minimal Riemannian flow given by a unit vector field $\xi$ and that $\sigma_{n+1-p}(M) > 0$ for some $2 \leq p \leq \frac{n}{2}$. Then,*

$$\lambda'_{1,p} + 4[\frac{n}{2}] \sup_M |\Omega|^2 \geq \Big(\sigma_{p+1}(M) - \sup_M g(S(\xi),\xi)\Big)\sigma_{n+2-p}(M), \tag{1}$$

*where $\Omega$ is the 2-form associated to the O'Neill tensor of the flow and $S$ is the second fundamental form of $M$.*

Inequality (1) differs from the one in [4, Thm. 1.1] by the additional term $4[\frac{n}{2}] \sup\limits_M |\Omega|^2$ that results from the computation of the norm of the interior product of the O'Neill tensor with any basic $p$-form (see Lemma 3.3). As an example of manifolds that satisfy the assumptions of Theorem 1.1, one can consider the unit closed ball $B'$ in $\mathbb{R}^{n+2}$ where the boundary $\mathbb{S}^{n+1}$ is endowed with the Riemannian flow given by the Hopf fibration.

When equality is realized in (1), it turns out that $\Omega$ vanishes which yields to the following characterization:

**Theorem 1.2** *Under the same assumptions as in Theorem 1.1 with $n \geq 1$ even, if $\sigma_1(M) \geq 0$ and the equality case is realized, the manifold $N$ is then isometric to the quotient ${}_{\Gamma}\backslash^{\mathbb{R} \times B'}$ for some fixed-point-free cocompact discrete subgroup $\Gamma \subset \mathbb{R} \times \mathrm{SO}_{n+1}$, where $B'$ is the unit closed ball in $\mathbb{R}^{n+1}$.*

As a consequence, we describe manifolds whose boundaries are isometric to the Riemannian product $\mathbb{S}^1 \times \mathbb{S}^n$ and get the following rigidity result:

**Corollary 1.3** *Let $N$ be an $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary $M$ is $\mathbb{S}^1 \times \mathbb{S}^n$ and $(n+1-p)\sup_M g(S(\xi),\xi) + 4c^2[\frac{n}{2}]$ does not change sign. If the inequality $\sigma_{p+1}(M) \geq p$ holds, the manifold $N$ is isometric to $\mathbb{S}^1 \times B'$.*

## 2 Riemannian flows and manifolds with boundary

Throughout this section, we recall the main ingredients of Riemannian flows defined on a manifold and the basic facts on manifolds with boundary (see [2, 17, 11] and [4] for manifolds with foliated boundary). In the sequel, we will use the musical isomorphism between the tangent space of a Riemannian manifold and its dual. In particular, for any vector field $X$ and a differential form $\varphi$, we write $X \wedge \varphi$ for the form $X^* \wedge \varphi$.

Let $(M^{n+1}, g)$ be a Riemannian manifold and $\xi$ be a smooth unit vector field on $M$ defining the structure of a Riemannian flow on $M$. That is, the vector field $\xi$ foliates the manifold $M$ by its integral curves (called the leaves) in a way that those curves are locally equidistant [17]. In other words, when one restricts the metric $g$ to the bundle $Q = \xi^\perp$ it is then constant along the leaves. That means the relation $\mathcal{L}_\xi g|_{\xi^\perp} = 0$ holds. Equivalently to this last relation, the endomorphism $h := \nabla^M \xi$ (known as the O'Neill tensor [10]) defines a skew-symmetric tensor field on the bundle $Q$. Moreover, the normal bundle carries a covariant derivative $\nabla$ (called transversal Levi-Civita connection) compatible with the induced metric $g$ on $Q$ [17] that can be related with the Levi-Civita connection on $M$ through the Gauss-type formulas: For all sections $Z, W$ in $\Gamma(Q)$, we have

$$\begin{cases} \nabla^M_Z W = \nabla_Z W - \Omega(Z,W)\xi, \\ \\ \nabla^M_\xi Z = \nabla_\xi Z + h(Z) - \kappa(Z)\xi, \end{cases}$$

where $\kappa := \nabla^M_\xi \xi$ is the mean curvature of the leaves and $\Omega(Z,W) := g(h(Z),W)$ is the differential 2-form associated to $h$. In all the paper, we will consider minimal Riemannian flows, that is $\kappa = 0$. Recall now that a basic form is a differential form $\varphi$ on $M$ such that $\xi \lrcorner \varphi = 0$ and $\xi \lrcorner d^M \varphi = 0$. We will denote by $\Omega_b(M)$ the set of all such forms. Clearly, basic forms are preserved by the exterior derivative $d^M$ and therefore we can set $d_b := d^M|_{\Omega_b(M)}$. For a compact manifold $M$, we consider the $L^2$-adjoint of $d_b$, denoted by $\delta_b$, and define the basic Laplacian as $\Delta_b = d_b\delta_b + \delta_b d_b$. From the spectral theory of transversally elliptic operators, the basic Laplacian has a discrete spectrum [5, 6].

Assume now that $(N^{n+2}, g)$ is a Riemannian manifold of dimension $n+2$ with boundary $M$. Recall that the shape operator (or the Weingarten tensor) is defined for all $X \in \Gamma(TM)$ as $S(X) = -\nabla^N_X \nu$ where $\nabla^N$ is the Levi-Civita connection of $N$ and $\nu$ is the inward unit nomal vector field along $M$. The Gauss-Codazzi equation is given for any $X, Y \in \Gamma(TM)$, by

$$(\nabla^M_X S)(Y) - (\nabla^M_Y S)(X) = R^N(Y,X)\nu, \tag{2}$$

where $R^N$ denotes the curvature tensor operator on $N$. At any point $x \in M$, we let $\eta_1(x), \cdots, \eta_{n+1}(x)$ be the principal curvatures of $M$ and arrange them so that $\eta_1(x) \leq \eta_2(x) \leq \cdots \leq \eta_{n+1}(x)$. For any $p \in \{1, \cdots, n+1\}$, we define the lowest $p$-curvatures $\sigma_p(x)$ by $\sigma_p(x) = \eta_1(x) + \cdots + \eta_p(x)$. It is a clear fact that the inequality $\frac{\sigma_p(x)}{p} \leq \frac{\sigma_q(x)}{q}$ holds for $p \leq q$, and the equality is achieved if and only if either $\eta_1(x) = \eta_2(x) = \cdots = \eta_q(x)$ or $p = q$. As mentioned in [11], the Weingarten tensor admits a canonical extension to any $p$-form $\varphi$ on $M$ by the following:

$$S^{[p]}(\varphi)(X_1, \cdots, X_p) = \sum_{i=1}^{p} \varphi(X_1, \cdots, S(X_i), \cdots, X_p),$$

for all vector fields $X_1, \cdots, X_p$ on $M$. The eigenvalues of $S^{[p]}$ are exactly the $p$-curvatures (that is, $\sum_{k=1}^{p} \eta_{i_k}$, with $i_1 < \cdots < i_p$) and that means the following inequality

$$\langle S^{[p]}(\varphi), \varphi \rangle \geq \sigma_p(M)|\varphi|^2 \tag{3}$$

holds, where $\sigma_p(M)$ is the infimum over $M$ of the lowest $p$-curvatures $\sigma_p(x)$. We point out that this extension can be done for any symmetric tensor field on $TM$ by the same definition. In particular, we will use it later for the tensor $\nabla^M_X S$ for any $X \in \Gamma(TM)$.

Now, we recall the Reilly formula established in [11, Thm. 3]. Let $J : M \to N$ be the inclusion map of $M$ into $N$ and let $J^*$ be the pull-back of a form on $N$ into $M$, that is $J^*$ is the restriction of differential forms on $N$ to the boundary $M$. For any $p$-form $\alpha$ on $N$, the formula is the following

$$\int_N (|d^N\alpha|^2 + |\delta^N\alpha|^2)v_g = \int_N |\nabla^N\alpha|^2 v_g + \int_N \langle W^N_p(\alpha), \alpha\rangle v_g + 2\int_M \langle \nu\lrcorner\alpha, \delta^M(J^*\alpha)\rangle v_g + \int_M \mathcal{B}(\alpha,\alpha)v_g$$

where $v_g$ is the volume element of $g$ on $N$ (also on $M$) and $W^N_p$ is the curvature term that appears in the Bochner-Weitzenböck formula for the Laplacian on $N$. The last term is defined by

$$\begin{aligned}\mathcal{B}(\alpha,\alpha) &= \langle S^{[p]}(J^*\alpha), J^*\alpha\rangle + \langle S^{[n+2-p]}(J^*(*_N\alpha)), J^*(*_N\alpha)\rangle \\ &= \langle S^{[p]}(J^*\alpha), J^*\alpha\rangle + (n+1)H|\nu\lrcorner\alpha|^2 - \langle S^{[p-1]}(\nu\lrcorner\alpha), \nu\lrcorner\alpha\rangle.\end{aligned}$$

We mention here that $J^*(*_N\alpha)$ is equal (up to a sign) to $*_M(\nu\lrcorner\alpha)$ and the relation $|J^*\alpha|^2 + |\nu\lrcorner\alpha|^2 = |\alpha|^2$ is true at any point of the boundary. We point out that the term $W^N_p \geq 0$ when the curvature operator of $N$ is non-negative.

Finally, the following boundary value problem will be of interest in our study. In fact, given any $p$-form $\varphi$ on $M$, the solution of the system

$$\begin{cases} \Delta^N\hat{\varphi} = 0 & \text{on } N, \\ \\ J^*\hat{\varphi} = \varphi,\ J^*(\delta^N\hat{\varphi}) = 0 & \text{on } M \end{cases} \tag{4}$$

is unique on $N$ by Lemma 3.5.6 in [14]. Moreover, the $p$-form $\hat{\varphi}$ is co-closed on $N$ and $d^N\hat{\varphi}$ belongs to the de Rham cohomology group $H^{p+1}(N)$ (see [1, Lemma 3.1] for more details).

# 3 Eigenvalue estimate for the basic Laplacian on manifolds with foliated boundary

In this section, we are going to prove Theorem 1.1. For this purpose, we need to state the following lemmas already proved in [4].

**Lemma 3.1** [4] *Let $(N^{n+2}, g)$ be a compact Riemannian manifold with boundary $M$ with $W^N_{p+1} \geq 0$ for some $1 \leq p \leq n$. Assume that $M$ carries a Riemannian flow defined by a unit vector field $\xi$ such that $\sigma_{n+1-p}(M) > 0$. Given a non-zero basic closed p-form $\varphi$, the corresponding solution $\hat{\varphi}$ of the boundary value problem* (4) *is then closed and co-closed on $N$.*

**Lemma 3.2** [4] *Let $(N^{n+2}, g)$ be a Riemannian manifold with boundary $M$. Assume that $M$ carries a Riemannian flow given by a unit vector field $\xi$. For any basic p-form $\varphi$ on $M$ where $1 \leq p \leq n$, we have*

$$\langle S^{[p]}(\varphi), \varphi\rangle \geq (\sigma_{p+1}(M) - g(S(\xi), \xi))|\varphi|^2.$$

This last estimate is optimal when $N$ is isometric to $\mathbb{S}^1 \times B'$, where $B'$ is the unit closed ball in $\mathbb{R}^{n+1}$ while Inequality (3) is not sharp in this example. Indeed, consider the flow by $\mathbb{S}^1$ on the boundary $M = \mathbb{S}^1 \times \mathbb{S}^n$ (i.e. the trivial fibration over $\mathbb{S}^n$) and let $\varphi$ be any non-zero $p$-form on $\mathbb{S}^n$. It is not difficult to check that $S^{[p]}(\varphi) = p\varphi$ and that $\sigma_q = q - 1$ for any $1 \leq q \leq n + 1$.

Next, we need to get an upper bound for the norm of the interior product of the 2-form $\Omega$ with any basic $p$-form. Indeed,

**Lemma 3.3** *Let $(N^{n+2}, g)$ be a Riemannian manifold with boundary $M$. Assume that $M$ carries a Riemannian flow given by a unit vector field $\xi$. For any basic p-form $\varphi$ with $p \geq 2$, we have*

$$|\Omega\lrcorner\varphi| \leq [\frac{n}{2}]^{\frac{1}{2}}|\Omega||\varphi|. \tag{5}$$

*For $n$ even, the equality is realized if and only if $\Omega = 0$.*

**Proof.** As $\Omega$ is a skew-symmetric 2-tensor field on $Q$, we can always find a local orthonormal frame $\{e_i\}$ of $\Gamma(Q)$ such that $\Omega = \sum_{j=1}^{[\frac{n}{2}]} \lambda_j e_{2j-1} \wedge e_{2j}$. Therefore, we compute

$$|\Omega\lrcorner\varphi| = |\sum_{j=1}^{[\frac{n}{2}]} \lambda_j(e_{2j-1} \wedge e_{2j})\lrcorner\varphi| \leq \sum_{j=1}^{[\frac{n}{2}]} |\lambda_j||e_{2j-1}\lrcorner(e_{2j}\lrcorner\varphi)| \leq \sum_{j=1}^{[\frac{n}{2}]} |\lambda_j||\varphi| \leq [\frac{n}{2}]^{\frac{1}{2}}|\Omega||\varphi|.$$

Here we used the fact that $|v\lrcorner\varphi| \leq |v||\varphi|$ and the Cauchy-Schwarz inequality (in the last estimate). Assume now that the equality is realized, then either all the $\lambda_j's$ are of the same absolute value and there exists a $j$ such that $\lambda_j = 0$ (in this case, all the $\lambda_j$'s are 0) or for all $j$, $e_{2j} \wedge \varphi = 0$ and $e_{2j-1} \wedge \varphi = 0$. But for $n$ even, the last statement just means that $X \wedge \varphi = 0$ for all $X \in \Gamma(Q)$ and thus $\varphi = 0$. This leads to a contradiction; hence $\lambda_j = 0$ for all $j$ which yields to $\Omega = 0$. $\square$

Now, we have all the ingredients to prove Theorem 1.1:

**Proof of Theorem 1.1.** For any basic closed $p$-eigenform $\varphi$ on $M$ corresponding to the eigenvalue $\lambda'_{1,p}$ of the basic Laplacian, we associate its extension $\hat{\varphi}$ to $N$ from Lemma 3.1. Applying now the

Reilly formula to $\hat{\varphi}$ gives, under the curvature assumption and with the use of Lemma 3.2, the following

$$0 \geq 2\int_M \langle \nu\lrcorner\hat{\varphi}, \delta^M\varphi\rangle v_g + \sigma_{p+1}(M)\int_M |\varphi|^2 v_g - \int_M g(S(\xi),\xi)|\varphi|^2 v_g + \sigma_{n+2-p}(M)\int_M |\nu\lrcorner\hat{\varphi}|^2 v_g.$$

With the help of the pointwise inequality $|\nu\lrcorner\hat{\varphi}+\frac{1}{\sigma_{n+2-p}(M)}\delta^M\varphi|^2 \geq 0$, the above one can be reduced to

$$\int_M |\delta^M\varphi|^2 v_g \geq \sigma_{p+1}(M)\sigma_{n+2-p}(M)\int_M |\varphi|^2 v_g - \sigma_{n+2-p}(M)\sup_M(g(S(\xi),\xi))\int_M |\varphi|^2 v_g. \tag{6}$$

Now from the relation $\delta_b = \delta^M - 2\Omega\lrcorner(\xi\wedge)$ on basic forms [12, Prop.2.4] and the estimate in Lemma 3.3, we compute

$$\begin{aligned}
|\delta^M\varphi|^2 &= |\delta_b\varphi|^2 + 4|\Omega\lrcorner(\xi\wedge\varphi)|^2 + 4\langle\delta_b\varphi, \Omega\lrcorner(\xi\wedge\varphi)\rangle \\
&= |\delta_b\varphi|^2 + 4|\xi\wedge(\Omega\lrcorner\varphi)|^2 + \underbrace{4\langle\delta_b\varphi, \xi\wedge(\Omega\lrcorner\varphi)\rangle}_{=0 \text{ since } \delta_b\varphi \text{ is basic}} \\
&= |\delta_b\varphi|^2 + 4|\Omega\lrcorner\varphi|^2 \\
&\overset{(5)}{\leq} |\delta_b\varphi|^2 + 4[\frac{n}{2}]|\Omega|^2|\varphi|^2.
\end{aligned}$$

Therefore after integrating over the manifold $M$, we get the desired result. □

**Remark 3.4** *The assumptions in Theorem 1.1 on the curvature can be weakened. The positivity of the curvature operator can be replaced by the positivity of $W_p^N$ and $W_{p+1}^N$.*

# 4 The equality case

This section is devoted to prove Theorem 1.2. In other words, we are going to study the limiting case of Inequality (1). We will show that, under some conditions, the second fundamental form vanishes along $\xi$ and is equal to $\eta\,\mathrm{Id}$ in the direction of $Q$ for some constant $\eta$, i.e. the boundary is $\eta$-umbilical. We will also prove that the O'Neill tensor defining the flow vanishes which is equivalent to the integrability of the normal bundle. Consequently, this allows to classify all manifolds on which Inequality (1) is optimal.

It is clear to see that when the equality is realized, the estimate in Lemma 3.3 is optimal which means that $h = 0$. On the other hand, the eigenform $\hat{\varphi}$ is parallel on $N$ and $\sigma_{p+1}, \sigma_{n+2-p}$ and $g(S(\xi),\xi)$ are constant on $M$. Moreover, we have the identity

$$\delta^M\varphi = -\sigma_{n+2-p}\nu\lrcorner\hat{\varphi}. \tag{7}$$

In particular, using the relations in [11, Eq. (23) and Lemma 18 (ii)], we get for all $X \in \Gamma(TM)$,

$$\begin{cases}
\nabla^M_X\varphi = S(X)\wedge(\nu\lrcorner\hat{\varphi}) \\
\nabla^M_X(\nu\lrcorner\hat{\varphi}) = -S(X)\lrcorner\varphi \\
\delta^M\varphi = S^{[p-1]}(\nu\lrcorner\hat{\varphi}) - \sigma_{n+1}\nu\lrcorner\hat{\varphi} \\
d^M(\nu\lrcorner\hat{\varphi}) = -S^{[p]}(\varphi).
\end{cases} \tag{8}$$

By replacing Equation (7) (recall that $\sigma_{n+2-p}$ is constant) in the last two equations in (8) and by using the fact that $d^M(\delta^M\varphi) = d_b(\delta_b\varphi) = \lambda'_{1,p}\varphi = \sigma_{n+2-p}(\sigma_{p+1} - g(S(\xi),\xi))\varphi$, we deduce that

$$\begin{cases} S^{[p-1]}(\nu\lrcorner\hat{\varphi}) = (\sigma_{n+1} - \sigma_{n+2-p})\nu\lrcorner\hat{\varphi} \\ \\ S^{[p]}(\varphi) = (\sigma_{p+1} - g(S(\xi),\xi))\varphi. \end{cases} \tag{9}$$

Now, in order to prove that the manifold $N$ is isometric to the quotient $_{\Gamma}\backslash^{\mathbb{R}\times B'}$, we need first to establish a series of lemmas:

**Lemma 4.1** *If the equality is realized in* (1)*, then* $S(\xi) = 0$.

**Proof.** Using Equation (7), we deduce that the form $\nu\lrcorner\hat{\varphi}$ is basic (recall here that $\delta^M\varphi = \delta_b\varphi$). Hence by applying the first equation in (9) to the vector fields $\xi$ and $X_1,\cdots,X_{p-2}\in\Gamma(Q)$, we find that $S(\xi)\lrcorner(\nu\lrcorner\hat{\varphi}) = 0$. On the other hand, since the O'Neill tensor vanishes, then $\nabla^M_\xi\varphi = \nabla_\xi\varphi$ which is equal to zero, because the form $\varphi$ is basic. Here, we recall that $\nabla$ is the extension of the transversal Levi-Civita connection to basic forms. Finally, by taking $X=\xi$ in the first equation of (8), we find that $S(\xi)\wedge(\nu\lrcorner\hat{\varphi}) = 0$. Mainly, that means $S(\xi) = 0$. We mention here that $\nu\lrcorner\hat{\varphi}$ cannot vanish, since this would imply that $\nabla^M_X\varphi = 0$ for all $X\in\Gamma(TM)$ which would give $\lambda'_{1,p} = 0$ (recall that $\lambda'_{1,p}$ is the first non-zero eigenvalue). $\square$

In the sequel, we aim to prove that the principal curvatures of $S$ are constant and are all equal to a number $\eta$, along transversal principal directions. The proof of this statement is a technical computation and will be splitted into several lemmas (see Lemmas 4.2, 4.4 and 4.5). In the sequel, $\{f_i\}_{i=1,\cdots,n+1}$ will denote an orthonormal frame of $\Gamma(TM)$.

**Lemma 4.2** *If the equality is realized in* (1)*, then*

$$\begin{aligned} \sum_{i=1}^{n+1}\langle(\nabla^M_{f_i}S)^{[p]}\varphi, f_i\wedge(\nu\lrcorner\hat{\varphi})\rangle &= ((\sigma_{p+1} - \sigma_{n+1} + \sigma_{n+2-p})\sigma_{n+2-p} - |S|^2)|\nu\lrcorner\hat{\varphi}|^2 \\ &\quad + \sum_{i=1}^{n+1}\langle f_i\lrcorner(\nu\lrcorner\hat{\varphi}), S^2(f_i)\lrcorner(\nu\lrcorner\hat{\varphi})\rangle. \end{aligned} \tag{10}$$

**Proof.** By differentiating the second equation in (9) along any vector field $X\in\Gamma(TM)$, we get after using (18)

$$S^{[p]}(SX\wedge(\nu\lrcorner\hat{\varphi})) + (\nabla^M_X S)^{[p]}\varphi = \sigma_{p+1}SX\wedge(\nu\lrcorner\hat{\varphi}).$$

Here we also used the first equation in (8). Setting $X = f_i$ and taking the scalar product of the last equality with $f_i\wedge(\nu\lrcorner\hat{\varphi})$, we obtain after tracing and using (19) that,

$$\begin{aligned} \sum_{i=1}^{n+1}\langle(\nabla^M_{f_i}S)^{[p]}\varphi, f_i\wedge(\nu\lrcorner\hat{\varphi})\rangle &= \sigma_{p+1}\sum_{i=1}^{n+1}\langle S(f_i)\wedge(\nu\lrcorner\hat{\varphi}), f_i\wedge(\nu\lrcorner\hat{\varphi})\rangle \\ &\quad - \sum_{i=1}^{n+1}\langle S^2(f_i)\wedge(\nu\lrcorner\hat{\varphi}), f_i\wedge(\nu\lrcorner\hat{\varphi})\rangle \\ &\quad - \sum_{i=1}^{n+1}\langle S(f_i)\wedge S^{[p-1]}(\nu\lrcorner\hat{\varphi}), f_i\wedge(\nu\lrcorner\hat{\varphi})\rangle. \end{aligned}$$

Then with the help of (9), the last equality reduces to

$$\sum_{i=1}^{n+1}\langle(\nabla^M_{f_i}S)^{[p]}\varphi, f_i\wedge(\nu\lrcorner\hat\varphi)\rangle = (\sigma_{p+1}-\sigma_{n+1}+\sigma_{n+2-p})\sum_{i=1}^{n+1}\langle S(f_i)\wedge(\nu\lrcorner\hat\varphi), f_i\wedge(\nu\lrcorner\hat\varphi)\rangle - \sum_{i=1}^{n+1}\langle S^2(f_i)\wedge(\nu\lrcorner\hat\varphi), f_i\wedge(\nu\lrcorner\hat\varphi)\rangle. \quad (11)$$

In order to finish the proof, it is sufficient to calculate the two sums in the r.h.s. of (11). In fact, the first sum is equal to

$$\sum_{i=1}^{n+1}\langle S(f_i)\wedge(\nu\lrcorner\hat\varphi), f_i\wedge(\nu\lrcorner\hat\varphi)\rangle = \sigma_{n+1}|\nu\lrcorner\hat\varphi|^2 - \sum_{i=1}^{n+1}\langle f_i\lrcorner(\nu\lrcorner\hat\varphi), S(f_i)\lrcorner(\nu\lrcorner\hat\varphi)\rangle \overset{(20),(9)}{=} \sigma_{n+2-p}|\nu\lrcorner\hat\varphi|^2,$$

while the second one is

$$\sum_{i=1}^{n+1}\langle S^2(f_i)\wedge(\nu\lrcorner\hat\varphi), f_i\wedge(\nu\lrcorner\hat\varphi)\rangle = |S|^2|\nu\lrcorner\hat\varphi|^2 - \sum_{i=1}^{n+1}\langle S^2(f_i)\lrcorner(\nu\lrcorner\hat\varphi), f_i\lrcorner(\nu\lrcorner\hat\varphi)\rangle.$$

The substitution into (11) gives the desired result. $\square$

In the following lemma, we will expand the curvature operator of the manifold $N$. Indeed,

**Lemma 4.3** *We have the splitting*

$$\langle W_p^N(\nu\wedge(\nu\lrcorner\hat\varphi)),\varphi\rangle = I+J+K, \quad (12)$$

*where $I, J$ and $K$ are given by*

$$I = \sum_{i=1}^{n+1}\langle R^N(\nu,f_i)\nu\lrcorner\hat\varphi, f_i\lrcorner\varphi\rangle$$

$$J = (-1)^{\frac{p(p-1)}{2}}\sum_{i=1}^{n+1}R^N(\nu,f_i,f_i,(\nu\lrcorner\hat\varphi)\lrcorner\varphi)$$

$$K = -\sum_{i,j=1}^{n+1}\langle R^N(f_i,f_j)\nu\wedge(f_i\lrcorner(\nu\lrcorner\hat\varphi)), f_j\lrcorner\varphi\rangle.$$

*Moreover, we have that $I=K$. When the equality is realized in (1), we get*

$$I = -(-1)^{\frac{p(p-1)}{2}}((\nu\lrcorner\hat\varphi)\lrcorner\varphi)(\sigma_{n+1}) + (\sigma_{n+1}-\sigma_{n+2-p})\sigma_{p+1}|\varphi|^2 - \sigma_{p+1}^2|\varphi|^2 + \sum_{i=1}^{n+1}\langle f_i\lrcorner\varphi, S^2(f_i)\lrcorner\varphi\rangle. \quad (13)$$

**Proof.** Recall that the curvature $W_p^N$ is given by the expression $W_p^N := \sum_{i,j} e_j^*\wedge\left(e_i\lrcorner R^N(e_i,e_j)\right)$ where $\{e_i\}_{i=1,\cdots,n+2}$ is any orthonormal frame of $TN$. At a point $x\in M$, we choose the orthonormal frame $\{f_i,\nu\}_{i=1,\cdots,n+1}$ of $T_xN$ to compute

$$\langle W_p^N(\nu\wedge(\nu\lrcorner\hat\varphi)),\varphi\rangle = \sum_{i=1}^{n+1}\langle\nu\lrcorner\left(R^N(\nu,f_i)(\nu\wedge(\nu\lrcorner\hat\varphi))\right), f_i\lrcorner\varphi\rangle + \sum_{i,j=1}^{n+1}\langle f_i\lrcorner\left(R^N(f_i,f_j)(\nu\wedge(\nu\lrcorner\hat\varphi))\right), f_j\lrcorner\varphi\rangle$$

$$= \sum_{i=1}^{n+1}\langle R^N(\nu,f_i)\nu\lrcorner\hat\varphi, f_i\lrcorner\varphi\rangle + \sum_{i,j=1}^{n+1}\langle R^N(\nu,f_i)f_i,f_j\rangle\langle\nu\lrcorner\hat\varphi, f_j\lrcorner\varphi\rangle - \sum_{i,j=1}^{n+1}\langle R^N(f_i,f_j)\nu\wedge(f_i\lrcorner(\nu\lrcorner\hat\varphi)), f_j\lrcorner\varphi\rangle.$$

$$= \sum_{i=1}^{n+1} \langle R^N(\nu, f_i)\nu \lrcorner \hat{\varphi}, f_i \lrcorner \varphi\rangle + (-1)^{p-1} \sum_{i,j=1}^{n+1} \langle R^N(\nu, f_i) f_i, f_j\rangle \langle (\nu \lrcorner \hat{\varphi}) \wedge f_j, \varphi\rangle$$

$$- \sum_{i,j=1}^{n+1} \langle R^N(f_i, f_j)\nu \wedge (f_i \lrcorner (\nu \lrcorner \hat{\varphi})), f_j \lrcorner \varphi\rangle.$$

Then using (21), we deduce (12). To prove that $I = K$, we first remark that the term $-K$ is equal to the following:

$$\begin{aligned}
&\sum_{\substack{i_1 < \cdots < i_{p-1} \\ k=1\cdots,p-1 \\ i,j=1,\cdots,n+1}} (-1)^{k+1} (\nu \lrcorner \hat{\varphi})_{i_1,\cdots,i_{p-1}} \delta_{ii_k} \langle R^N(f_i, f_j)\nu \wedge f_{i_1} \wedge \cdots \hat{f}_{i_k} \wedge \cdots \wedge f_{i_{p-1}}, f_j \lrcorner \varphi\rangle \\
&= \sum_{\substack{i_1 < \cdots < i_{p-1} \\ k=1\cdots,p-1 \\ j=1,\cdots,n+1}} (\nu \lrcorner \hat{\varphi})_{i_1,\cdots,i_{p-1}} \langle f_{i_1} \wedge \cdots \wedge R^N(f_{i_k}, f_j)\nu \wedge \cdots \wedge f_{i_{p-1}}, f_j \lrcorner \varphi\rangle \\
&\overset{(2)}{=} \sum_{\substack{i_1 < \cdots < i_{p-1} \\ k=1\cdots,p-1 \\ j=1,\cdots,n+1}} (\nu \lrcorner \hat{\varphi})_{i_1,\cdots,i_{p-1}} \langle f_{i_1} \wedge \cdots \wedge (\nabla^M_{f_j} S)(f_{i_k}) \wedge \cdots \wedge f_{i_{p-1}}, f_j \lrcorner \varphi\rangle \\
&= \sum_{j=1}^{n+1} \langle \varphi, f_j \wedge (\nabla^M_{f_j} S)^{[p-1]}(\nu \lrcorner \hat{\varphi})\rangle.
\end{aligned} \tag{14}$$

Hence, using the second equation in (14), $I - K$ is equal to

$$\begin{aligned}
&\sum_{\substack{i_1 < \cdots < i_{p-1} \\ k=1\cdots,p-1 \\ i=1,\cdots,n+1}} (\nu \lrcorner \hat{\varphi})_{i_1,\cdots,i_{p-1}} \{\langle f_{i_1} \wedge \cdots \wedge R^N(\nu, f_i) f_{i_k} \wedge \cdots \wedge f_{i_{p-1}}, f_i \lrcorner \varphi\rangle \\
&\qquad + \langle f_{i_1} \wedge \cdots \wedge R^N(f_{i_k}, f_i)\nu \wedge \cdots \wedge f_{i_{p-1}}, f_i \lrcorner \varphi\rangle\} \\
&= \sum_{\substack{i_1 < \cdots < i_{p-1} \\ k=1,\cdots,p-1 \\ i,l=1,\cdots,n+1}} (\nu \lrcorner \hat{\varphi})_{i_1,\cdots,i_{p-1}} \{R^N(f_{i_k}, f_l, \nu, f_i) \langle f_{i_1} \wedge \cdots \wedge f_l \wedge \cdots \wedge f_{i_{p-1}}, f_i \lrcorner \varphi\rangle \\
&\qquad + R^N(f_{i_k}, f_i, \nu, f_l) \langle f_{i_1} \wedge \cdots \wedge f_l \wedge \cdots \wedge f_{i_{p-1}}, f_i \lrcorner \varphi\rangle\},
\end{aligned}$$

which is zero when we interchange the role of the indices $i$ and $l$ in the first summation. When the equality is realized in (1), and since $I = K = -\sum_{i=1}^{n+1} \langle f_i \lrcorner \varphi, (\nabla^M_{f_i} S)^{[p-1]}(\nu \lrcorner \varphi)\rangle$, we get

$$\begin{aligned}
I &\overset{(18)}{=} -\sum_{i=1}^{n+1} \langle f_i \lrcorner \varphi, \nabla^M_{f_i}(S^{[p-1]}(\nu \lrcorner \hat{\varphi}))\rangle + \sum_{i=1}^{n+1} \langle f_i \lrcorner \varphi, S^{[p-1]}(\nabla^M_{f_i}(\nu \lrcorner \hat{\varphi}))\rangle \\
&\overset{(8),(9)}{=} -\sum_{i=1}^{n+1} f_i(\sigma_{n+1}) \langle f_i \lrcorner \varphi, \nu \lrcorner \hat{\varphi}\rangle + (\sigma_{n+1} - \sigma_{n+2-p}) \sum_{i=1}^{n+1} \langle f_i \lrcorner \varphi, S(f_i) \lrcorner \varphi\rangle \\
&\quad -\sum_{i=1}^{n+1} \langle f_i \lrcorner \varphi, S^{[p-1]}(S(f_i) \lrcorner \varphi)\rangle,
\end{aligned}$$

which gives the result, using Equations (21),(20),(23) and (9). $\square$

In the following, we will compute the l.h.s. of Equation (10) in terms of the curvature operator $W_p^N$. Indeed, we have

**Lemma 4.4** *If the equality is realized in* (1), *we have*

$$\sum_{i=1}^{n+1}\langle(\nabla^M_{f_i}S)^{[p]}\varphi, f_i\wedge(\nu\lrcorner\hat{\varphi})\rangle = -\langle W_p^N(\nu\wedge(\nu\lrcorner\hat{\varphi})),\varphi\rangle+\sum_{i=1}^{n+1}\langle f_i\lrcorner\varphi, S^2(f_i)\lrcorner\varphi\rangle + (\sigma_{n+1}-\sigma_{n+2-p}-\sigma_{p+1})\sigma_{p+1}|\varphi|^2. \tag{15}$$

**Proof.** Using the symmetry property of the tensor $\nabla^M S$ and Equation (19), the l.h.s. of Equation (15) is equal to

$$\begin{aligned}&\sum_{i=1}^{n+1}\langle\varphi,(\nabla^M_{f_i}S)(f_i)\wedge(\nu\lrcorner\hat{\varphi})\rangle+\langle\varphi, f_i\wedge(\nabla^M_{f_i}S)^{[p-1]}(\nu\lrcorner\hat{\varphi})\rangle\\ &\overset{(21)}{=}(-1)^{\frac{p(p-1)}{2}}\sum_{i=1}^{n+1}\langle(\nu\lrcorner\hat{\varphi})\lrcorner\varphi,(\nabla^M_{f_i}S)(f_i)\rangle+\sum_{i=1}^{n+1}\langle\varphi, f_i\wedge(\nabla^M_{f_i}S)^{[p-1]}(\nu\lrcorner\hat{\varphi})\rangle.\end{aligned}$$

Therefore, from Equation (2) and again from the symmetry of $\nabla^M S$, the above expression reduces to

$$\begin{aligned}&(-1)^{\frac{p(p-1)}{2}}\sum_{i=1}^{n+1}\langle(\nabla^M_{(\nu\lrcorner\hat{\varphi})\lrcorner\varphi}S)(f_i),f_i\rangle+(-1)^{\frac{p(p-1)}{2}}R^N((\nu\lrcorner\hat{\varphi})\lrcorner\varphi,f_i,\nu,f_i)+\sum_{i=1}^{n+1}\langle\varphi, f_i\wedge(\nabla^M_{f_i}S)^{[p-1]}(\nu\lrcorner\hat{\varphi})\rangle\\ &=(-1)^{\frac{p(p-1)}{2}}((\nu\lrcorner\hat{\varphi})\lrcorner\varphi)(\sigma_{n+1})-(-1)^{\frac{p(p-1)}{2}}\sum_{i=1}^{n+1}R^N(\nu,f_i,f_i,(\nu\lrcorner\hat{\varphi})\lrcorner\varphi)+\sum_{i=1}^{n+1}\langle\varphi, f_i\wedge(\nabla^M_{f_i}S)^{[p-1]}(\nu\lrcorner\hat{\varphi})\rangle.\\ &\overset{(12),(14)}{=}(-1)^{\frac{p(p-1)}{2}}((\nu\lrcorner\hat{\varphi})\lrcorner\varphi)(\sigma_{n+1})-\langle W_p^N(\nu\wedge(\nu\lrcorner\hat{\varphi})),\varphi\rangle+I+K-K.\end{aligned}\tag{16}$$

Substituting Equation (13) into Equation (16), we finally get the result. □

In the next lemma, we will compare the sign of the l.h.s. of Equation (10) which is given by (15) to the sign of the r.h.s. Under a curvature assumption, we will find that they are of opposite signs. In particular, this will mean that all principal curvatures along transversal directions are equal. More precisely, we have

**Lemma 4.5** *If the equality in* (1) *is realized and if moreover* $\sigma_1(M)\geq 0$*, then* $S(X)=\eta X$ *for all* $X$ *orthogonal to* $\xi$*.*

**Proof.** We will show that the l.h.s. of Equation (10) is nonnegative while the r.h.s. is nonpositive which means that both terms vanish. We first begin to check the l.h.s. which is given by Equation (15) in Lemma 4.4. Indeed, the eigenform $\hat{\varphi}$ is parallel thus the term $\langle W_p^N\hat{\varphi},\hat{\varphi}\rangle$ vanishes. Therefore by writing $\hat{\varphi}=\varphi+\nu\wedge(\nu\lrcorner\hat{\varphi})$ at any point of the boundary and using the fact that $W_p^N$ is nonnegative, we deduce that the term $\langle W_p^N(\nu\wedge(\nu\lrcorner\hat{\varphi})),\varphi\rangle$ is nonpositive. On the other hand, the tensor $S$ has 0 as an eigenvalue (recall that $S(\xi)=0$) and that $\sigma_1(M)\geq 0$, therefore all the $\eta_i$'s are greater than 0 for $i=2,\cdots,n+1$. Hence

$$\sum_{i=1}^{n+1}\langle f_i\lrcorner\varphi, S^2(f_i)\lrcorner\varphi\rangle\geq\eta_2\sum_{i=1}^{n+1}\langle f_i\lrcorner\varphi, S(f_i)\lrcorner\varphi\rangle\overset{(20),(9)}{=}\eta_2\sigma_{p+1}|\varphi|^2.$$

Thus, the l.h.s. of Equation (10) is bounded from below by

$$(\sigma_{n+1}-\sigma_{n+2-p}-\sigma_{p+1}+\eta_2)\sigma_{p+1}|\varphi|^2=((\eta_{n+3-p}-\eta_3)+\cdots+(\eta_{n+1}-\eta_{p+1}))\sigma_{p+1}|\varphi|^2\geq 0,$$

since the sequence $\eta_i$ is increasing. We can easily see that all inequalities are sharp when all the $\eta_i's$ $(i=2,\cdots,n+1)$ are equal. Concerning the r.h.s. of Equation (10), recall that it is given by

$$((\sigma_{p+1}-\sigma_{n+1}+\sigma_{n+2-p})\sigma_{n+2-p}-|S|^2)|\nu\lrcorner\hat{\varphi}|^2+\sum_{i=1}^{n+1}\langle f_i\lrcorner(\nu\lrcorner\hat{\varphi}),S^2(f_i)\lrcorner(\nu\lrcorner\hat{\varphi})\rangle. \tag{17}$$

In the sequel, we will take the vectors $\{f_i\}_{i=1,\cdots,n+1}$ as the principal directions associated with the principal curvatures $\eta_i$ of the tensor $S$. We first have

$$\begin{aligned}\sum_{i=1}^{n+1}\langle f_i\lrcorner(\nu\lrcorner\hat{\varphi}),S^2(f_i)\lrcorner(\nu\lrcorner\hat{\varphi})\rangle &= \sum_{i=1}^{n+1}\eta_i\langle f_i\lrcorner(\nu\lrcorner\hat{\varphi}),S(f_i)\lrcorner(\nu\lrcorner\hat{\varphi})\rangle\\ &\leq \eta_{n+1}\sum_{i=1}^{n+1}\langle f_i\lrcorner(\nu\lrcorner\hat{\varphi}),S(f_i)\lrcorner(\nu\lrcorner\hat{\varphi})\rangle\\ &\overset{(20),(9)}{=} \eta_{n+1}(\sigma_{n+1}-\sigma_{n+2-p})|\nu\lrcorner\hat{\varphi}|^2.\end{aligned}$$

Hence, (17) can be bounded from above by $A|\nu\lrcorner\hat{\varphi}|^2$, where $A$ is given by

$$A=(\sigma_p-\sigma_{n+1}+\sigma_{n+2-p})\sigma_{n+2-p}+\eta_{p+1}\sigma_{n+2-p}-\eta_2^2-\cdots-\eta_n^2+\eta_{n+1}(\eta_{n+3-p}+\cdots+\eta_n).$$

Let us prove that $A$ is nonpositive, which implies in particular that (17) is nonpositive. We write

$$\begin{aligned}A &= (\sigma_p-\sigma_{n+1}+\sigma_{n+2-p})\sigma_{p+1}+(\sigma_p-\sigma_{n+1}+\sigma_{n+2-p})(\eta_{p+2}+\cdots+\eta_{n+2-p})\\ &\quad+\eta_{p+1}\sigma_p+\eta_{p+1}(\eta_{p+1}+\cdots+\eta_{n+2-p})-\eta_2^2-\cdots-\eta_n^2+\eta_{n+1}(\eta_{n+3-p}+\cdots+\eta_n)\\ &= \sigma_p\sigma_{p+1}-(\sigma_{n+1}-\sigma_{n+2-p})\sigma_{p+1}+(\sigma_p-\sigma_{n+1}+\sigma_{n+2-p})(\eta_{p+2}+\cdots+\eta_{n+2-p})\\ &\quad+\eta_{p+1}\sigma_p+B-\eta_2^2-\cdots-\eta_p^2-\eta_{n+3-p}^2-\cdots-\eta_n^2+\eta_{n+1}(\eta_{n+3-p}+\cdots+\eta_n)\end{aligned}$$

where $B$ is given by

$$B=\eta_{p+2}(\eta_{p+1}-\eta_{p+2})+\eta_{p+3}(\eta_{p+1}-\eta_{p+3})+\cdots+\eta_{n+2-p}(\eta_{p+1}-\eta_{n+2-p}).$$

Clearly $B$ is nonpositive, since $1\leq p\leq\frac{n}{2}$. On the other hand, using that $S(\xi)=0$ we have

$$S^{[p]}(\xi\wedge(\nu\lrcorner\hat{\varphi}))\overset{(19)}{=}\xi\wedge S^{[p-1]}(\nu\lrcorner\hat{\varphi})\overset{(9)}{=}(\sigma_{n+1}-\sigma_{n+2-p})(\xi\wedge(\nu\lrcorner\hat{\varphi})),$$

then $\sigma_p+\sigma_{n+2-p}\leq\sigma_{n+1}$. Therefore

$$\begin{aligned}A &\leq \sigma_p^2+2\eta_{p+1}\sigma_p-(\sigma_{n+1}-\sigma_{n+2-p})\sigma_{p+1}-\eta_2^2-\cdots-\eta_p^2\\ &\quad-\eta_{n+3-p}^2-\cdots-\eta_n^2+\eta_{n+1}(\eta_{n+3-p}+\cdots+\eta_n)\\ &= 2\sum_{2\leq i<j\leq p}\eta_i\eta_j+2\eta_{p+1}\sigma_p-\eta_{n+3-p}(\eta_{n+3-p}+\sigma_{p+1})-\cdots-\eta_n(\eta_n+\sigma_{p+1})\\ &\quad-\eta_{n+1}(\sigma_{p+1}-\eta_{n+3-p}-\cdots-\eta_n).\end{aligned}$$

It is clear that $\eta_{n+3-p}+\cdots+\eta_n\leq\sigma_{p+1}$, as $\sigma_{p+1}-(\eta_{n+3-p}+\cdots+\eta_n)$ is an eigenvalue of $S^{[2]}$ (just apply $S^{[2]}$ to the eigenform $(f_{n+3-p}\wedge\cdots\wedge f_n)\lrcorner\varphi$ by using the formula (22)). This fact combined with $\eta_i\geq\eta_{p+2}$ for $i=n+3-p,\cdots,n+1$, gives that

$$A\leq 2\sum_{2\leq i<j\leq p}\eta_i\eta_j+2\eta_{p+1}\sigma_p-\eta_{p+2}(p-1)\sigma_{p+1}\leq 0.$$

This last inequality is true because the number of positive terms is equal to the number of negative terms which is $p(p-1)$. $\square$

Now we are able to prove Theorem 1.2.

**Proof of Theorem 1.2.** The proof follows exactly the same lines as in [4, Thm. 1.2] (see also [3] for more details) that we briefly explain the main idea. We first show that the vector field $\xi$ defining the flow can be extended to a unique parallel vector field $\hat{\xi}$ on $N$ which is orthogonal to $\nu$ [4]. The proof mainly relies on the use of the Reilly formula applied to the solution $\hat{\xi}$ of the boundary value problem (4). Second, we consider a connected integral submanifold $N_1$ of the bundle $(\mathbb{R}\hat{\xi})^\perp$, where the orthogonal complement is taken in $N$. The manifold $N_1$ is complete with totally umbilical boundary and the Ricci tensor of $\partial N_1$ is bounded from below by the constant $(n-1)\eta^2$. That is, the manifold $\partial N_1$ is compact as a consequence of Myers's theorem and from the main result in [9, Thm. 1.1], we deduce that $N_1$ is compact.

On the other hand, we have, from (8), $\varphi = -\frac{1}{p\eta} d^M(\nu\lrcorner\hat{\varphi})$ which means that it is $d^M$-exact and thus $d^{\partial N_1}$-exact, since $\partial N_1$ is totally geodesic in $M$ and both $\varphi$ and $\nu\lrcorner\hat{\varphi}$ are basic. Moreover, the basic form $\varphi$ is an eigenform of the Laplacian on $\partial N_1$, that is $\Delta^{\partial N_1}\varphi = \Delta_b\varphi = \lambda'_{1,p}\varphi$. Therefore, if we denote by $\lambda_{1,p}^{\partial N_1}$ the first eigenvalue of $\Delta^{\partial N_1}$ restricted to exact $p$-forms on $N_1$ and by $\tilde{\sigma}_p$ the $p$-curvatures of $\partial N_1$ into the compact manifold $N_1$, we get from the main estimate in [11, Thm. 5] that

$$p(n+1-p)\eta^2 = \tilde{\sigma}_p\tilde{\sigma}_{n+1-p} \leq \lambda_{1,p}^{\partial N_1} \leq \lambda'_{1,p} = \sigma_{p+1}\sigma_{n+2-p} = p(n+1-p)\eta^2.$$

Hence the equality is attained in the estimate of S. Raulot and A. Savo [11, Thm. 5] and therefore $N_1$ is isometric to the Euclidean closed ball $B'$. Finally, by the de Rham theorem, the manifold $\widetilde{N}$ is isometric to $\mathbb{R}\times B'$ and $N$ is the quotient of the Riemannian product $\mathbb{R}\times B'$ by its fundamental group. Since $\pi_1(N)$ embeds into $\pi_1(M)$, the manifold $N$ is then isometric to $\Gamma\backslash\mathbb{R}\times B'$, for some fixed-point-free cocompact discrete subgroup $\Gamma\subset\mathbb{R}\times \mathrm{SO}_{n+1}$. $\square$

## 5 Rigidity results on manifolds with foliated boundary

Our objective, in this section, is to derive rigidity results on manifolds with foliated boundary. These results generalize the ones in [4, Sect. 5]. For this end, we recall that a basic special Killing $p$-form $\omega$ is a basic co-closed (with respect to the basic codifferential $\delta_b$) form satisfying for all $X\in\Gamma(Q)$ the relations [14, 16, 8]

$$\nabla_X\omega = \frac{1}{p+1}X\lrcorner d_b\omega \qquad \text{and} \qquad \nabla_X d_b\omega = -c(p+1)X\wedge\omega,$$

where $\nabla$ is the transversal Levi-Civita connection and $c$ is a non-negative constant. In general, one can prove that a basic special Killing $p$-form is a co-closed eigenform of the basic Laplacian corresponding to the eigenvalue $c(p+1)(n-p)$ where $n$ is the rank of $Q$.

In the following, we will consider a compact manifold $N$ whose boundary carries a basic special Killing $p$-form. Due to the equality case of our main estimate, we will be able to characterize the boundary as the product $\mathbb{S}^1\times\mathbb{S}^n$. We first prove the following result:

**Corollary 5.1** *Let $N$ be an $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary $M$ carries a minimal Riemannian flow such that $(n+1-p)\sup_M g(S(\xi),\xi) + 4c^2[\frac{n}{2}] \leq 0$ and also admits a basic special Killing $(n-p)$-form for some $2\leq p\leq\frac{n}{2}$. If the inequality $\sigma_{p+1}(M)\geq p$ holds, the manifold $N$ is isometric to $\Gamma\backslash\mathbb{R}\times B'$.*

**Proof.** Let $\varphi$ be a basic special Killing $(n-p)$-form on $M$. Then $*_b\varphi$ is a basic closed $p$-eigenform for the basic Laplacian, that is $\Delta_b(*_b\varphi) = p(n+1-p)(*_b\varphi)$. Here, we used the minimality of the flow to say that the basic Hodge operator "$*_b$" commutes with the basic Laplacian. Hence

$\lambda'_{1,p} \leq p(n+1-p)$ [17]. To get the upper bound, we will use the estimate in Theorem 1.1. First, we have $\sigma_{n+1-p} \geq \sigma_{p+1} > 0$ as $2 \leq p \leq \frac{n}{2}$. On the other hand, by considering the functions $\theta_i = \sigma_{i+1} - \eta_1$ for all $i = 1, \cdots, n$, and using the estimate

$$\sigma_{n+2-p} \geq \frac{n+1-p}{p}\theta_p + \eta_1 \geq (n+1-p) - \eta_1\frac{(n+1-2p)}{p} \geq n+1-p,$$

we finish the proof because $\eta_1 \leq g(S(\xi), \xi) \leq 0$. $\square$

Using this last result, we can prove Corollary 1.3 as in [4, Cor. 1.3]. Also a similar result holds as in [4, Cor. 5.2]

**Corollary 5.2** *Let $N$ be an $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that $M = \mathbb{S}^1 \times \mathbb{S}^n$ with $n \geq 3$, the sectional curvature $K^N$ of $N$ vanishes on $M$, the mean curvature $H > 0$ and $(n+1-p)\sup_M g(S(\xi), \xi) + 4c^2[\frac{n}{2}] \leq 0$. Then, the manifold $N$ is isometric to $\mathbb{S}^1 \times B'$.*

Finally, the analogue result holds as in [4, Cor. 5.3]

**Corollary 5.3** *Let $N$ be an $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary $M$ carries a minimal Riemannian flow such that $(n+1-p)\sup_M g(S(\xi), \xi) + 4c^2[\frac{n}{2}] \geq 0$ and also admits a basic special Killing $(n-p)$-form for some $2 \leq p \leq \frac{n}{2}$. If the inequality $\sigma_{p+1}(M) \geq p + \sup\limits_M g(S(\xi), \xi) + \frac{4}{n+1-p}c^2[\frac{n}{2}]$ holds, the manifold $N$ is isometric to $\Gamma\backslash\mathbb{R} \times B'$.*

**Proof.** We follow the same proof as in Corollary 5.1. We just remark that

$$\begin{aligned}
\sigma_{n+2-p} &\geq \frac{n+1-p}{p}(p + \sup_M g(S(\xi), \xi) + \frac{4c^2[\frac{n}{2}]}{n+1-p}) + \eta_1(\frac{2p-n-1}{p}) \\
&\geq (n+1-p) + \sup_M g(S(\xi), \xi) + \frac{4c^2[\frac{n}{2}]}{p} \geq n+1-p,
\end{aligned}$$

which finishes the proof of the corollary. $\square$

# 6 Appendix

In this Appendix, we will state some technical formulas that we use in our computations. We omit the proofs of these formulas and leave them to the reader. For any $X \in \Gamma(TM)$, we have

$$(\nabla^M_X S)^{[p]} = \nabla^M_X S^{[p]} - S^{[p]}(\nabla^M_X). \tag{18}$$

Also, for any $p$-form $\varphi$ and $X \in \Gamma(TM)$,

$$\begin{cases} S^{[p+1]}(X \wedge \varphi) = S(X) \wedge \varphi + X \wedge S^{[p]}(\varphi) \\ \\ (\nabla^M_X S)^{[p+1]}(X \wedge \varphi) = (\nabla^M_X S)(X) \wedge \varphi + X \wedge (\nabla^M_X S)^{[p]}(\varphi). \end{cases} \tag{19}$$

For any orthonormal frame $\{f_i\}_{i=1,\cdots,n+1}$ of $TM$, we have

$$\langle S^{[p]}\varphi, \varphi\rangle = \sum_{i=1}^{n+1}\langle S(f_i)\lrcorner\varphi, f_i\lrcorner\varphi\rangle. \tag{20}$$

Next, we define the interior product of an $s$-form with a $p$-form $\varphi$, as follows:

$$((X_1 \wedge \cdots \wedge X_s) \lrcorner \varphi)(Y_1, \cdots, Y_{p-s}) = \varphi(X_s, \cdots, X_1, Y_1, \cdots, Y_{p-s}).$$

Therefore, for any $(p-s)$-form $\psi$, one has

$$\langle (X_1 \wedge \cdots \wedge X_s) \lrcorner \varphi, \psi \rangle = (-1)^{\frac{s(s-1)}{2}} \langle \varphi, X_1 \wedge \cdots \wedge X_s \wedge \psi \rangle. \tag{21}$$

Finally,

$$S^{[p-s]}((X_1 \wedge \cdots \wedge X_s) \lrcorner \varphi) = (X_1 \wedge \cdots \wedge X_s) \lrcorner S^{[p]}(\varphi) - (S^{[s]}(X_1 \wedge \cdots \wedge X_s)) \lrcorner \varphi. \tag{22}$$

In particular, this gives for $s = 1$

$$S^{[p-1]}(X \lrcorner \varphi) = X \lrcorner S^{[p]}(\varphi) - S(X) \lrcorner \varphi. \tag{23}$$

**Acknowledgment:** The first two named authors were supported by a fund from the Lebanese University. The second named author would like to thank the Alexander von Humboldt Foundation for its support.

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