Mountain pass critical points for Paneitz-Branson operators

by

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Abstract

Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 5$, we study fourth order equations involving Paneitz-Branson type operators and the critical Sobolev exponent.

1 Introduction and statement of the results

In 1983, Paneitz [14] introduced a conformally fourth order operator defined on 4-dimensional Riemannian manifolds. Branson [3] generalized the definition to n-dimensional Riemannian manifolds. We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and denote by Ric_g and S_g the Ricci and scalar curvature of g. For $u \in C^\infty(M)$, the Paneitz-Branson operator is given by

$$P_g^n u = \Delta_g^2 u - div_g \left[(a_n S_g g + b_n Ric_g)^\# du \right] + \frac{n-4}{2} Q_g^n u,$$

where $\Delta_g u = -div_g(\nabla u)$ is the Laplace-Beltrami operator,

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \ , \ b_n = -\frac{4}{n-2} \ ,$$

the symbol # stands for the musical isomorphism (index are raised with the metric), and

$$Q_g^n = \frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|_g^2.$$

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The Paneitz-Branson operator is conformally invariant in the following sense: if $\tilde{g} = \varphi^{4/(n-4)} g$ is a metric conformal to g, then for all $u \in C^\infty(M)$,

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u).$$

Taking $u \equiv 1$, we then find that

$$P_g^n \varphi = \frac{n-4}{2} Q_{\tilde{g}}^n \varphi^{\frac{n+4}{n-4}}.$$

In particular, the Paneitz-Branson operator possesses conformal properties that are very similar to the ones satisfied by the conformal laplacian. We are then naturally led to study extensions to this operator of some classical problems.

The geometric Paneitz-Branson operator falls into two types of operators, depending on the manifold we consider. Given $A \in \Lambda^\infty_{(2,0)}(M)$ a smooth symmetric $(2,0)$-tensor field, and $a \in C^\infty(M)$, we refer to a Paneitz-Branson type operator with general coefficients as an operator of the form

$$P_g u = \Delta_g^2 u - div_g \left[A^\# du \right] + au. \tag{1}$$

Given $\alpha, a \in \mathbb{R}$, we refer to a Paneitz-Branson type operator with constant coefficients as an operator of the form

$$P_g u = \Delta_g^2 u + \alpha \Delta_g u + au. \tag{2}$$

With such a terminology, introduced by Hebey, it is easily seen that the Paneitz-Branson type operator with constant coefficients given by (2) is the Paneitz-Branson type operator with general coefficients (1) when $A = \alpha g$, and $\alpha, a \in \mathbb{R}$. Moreover, whatever (M, g) is, the geometric Paneitz-Branson operator P_g^n is of the type (1), and when (M, g) is Einstein, the geometric Paneitz-Branson operator P_g^n is of the type (2). We indeed do find that

$$P_g^n u = \Delta_g^2 u + \frac{n^2 - 2n - 4}{2n(n-1)} S_g \Delta_g u + \frac{(n-4)(n^2-4)}{16n(n-1)^2} S_g^2 u \tag{3}$$

when (M, g) is Einstein. In particular, when $(M, g) = (S^n, h)$ is the unit n-sphere,

$$P_h^n u = \Delta_g^2 u + c_n \Delta_g u + d_n u \tag{4}$$

where $c_n = \frac{n^2-2n-4}{2}$ and $d_n = \frac{(n-4)n(n^2-4)}{16}$. In what follows we refer to a Paneitz-Branson type operator as an operator given either by (1), or (2).

We let $H_2^2(M)$ be the standard Sobolev space consisting of functions in $L^2(M)$ whose derivatives up to the order 2 are in $L^2(M)$, and let $2^\sharp$ be the critical exponent given by $2^\sharp = \frac{2n}{n-4}$. The Sobolev embedding theorem asserts that $H_2^2(M)$ is continuously embedded in $L^q(M)$ for $1 < q \le 2^\sharp$, with the property that this embedding is compact when $q < 2^\sharp$. We now define $K_0 > 0$ to be the sharp constant in the Euclidean Sobolev inequality $\|u\|_{2^\sharp}^2 \le K \|\Delta u\|_2^2$. We know from the work of [12], [13] and [9], that

$$\frac{1}{K_0} = \frac{n(n^2-4)(n-4)\omega_n^{\frac{4}{n}}}{16},$$

where for $k \in \mathbb{N}^\star$, ω_k denotes the volume of the unit $k-$sphere $(\mathbb{S}^k, h)$. Moreover, the extremals for the sharp Euclidean Sobolev inequality are precisely the functions

$$u(x) = \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-4}{2}} \tag{5}$$

where $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

Given (M, g) a smooth compact Riemannian manifold of dimension $n \ge 5$, f, h two continuous functions on M, and $q \in (1, 2^\sharp - 1)$, the goal in this paper is to study equations like

$$P_g u = f|u|^{2^\sharp - 2} u + h|u|^{q-1} u \tag{6}$$

where P_g is a Paneitz-Branson type operator, namely either with general coefficients as in (1), or with constant coefficients as in (2). Solutions of (6) can be seen as critical points of the functional

$$E(u) = \frac{1}{2} \int_M (P_g u) u \, dv_g - \frac{1}{2^\sharp} \int_M f|u|^{2^\sharp} \, dv_g - \frac{1}{q+1} \int_M h|u|^{q+1} \, dv_g. \tag{7}$$

Because of the failure (in general) of the maximum principle, getting positive solutions to (6) is still an open problem when P_g is with general coefficients. When P_g is with constant coefficients, there are particular cases (see below) where a maximum principle is available and the positivity of the solutions can be obtained. This includes the geometric Paneitz-Branson operator P_g^n when (M, g) is Einstein of positive scalar curvature. Equation (6) when $h \equiv 0$, with a special emphasis on the case of the unit sphere, was studied by Djadli-Hebey-Ledoux [6]. An equivalent problem when the fourth order Paneitz-Branson type operator is replaced by a second order Laplacian type operator was studied by Brézis-Nirenberg [4] in the Euclidean case, and then by Djadli [5] in the Riemannian context.

We assume in what follows that P_g is coercive in the sense that there exists $c > 0$ such that for all $u \in H_2^2(M)$,

$$\int_M (P_g u)u\, dv_g \geq c \int_M u^2\, dv_g$$

Necessary and sufficient conditions for P_g to be coercive are in Hebey-Robert [10] when P_g is with constant coefficients. These necessary and sufficient conditions imply sufficient conditions for P_g to be coercive when P_g is with general coefficients.

Our first result is the following. The main tool there is the Mountain-Pass Lemma of Ambrosetti and Rabinowitz [1].

Theorem 1 *Let (M, g) be a compact Riemannian n-manifold, $n \geq 5$, f, h be two functions in $C^\eta(M)$, $0 < \eta < 1$, $q \in (1, 2^\sharp - 1)$, and P_g be a Paneitz-Branson type operator. We assume that P_g is coercive, that f is positive and that there exists $v_0 \in H_2^2(M)$ such that*

$$\sup_{t \geq 0} E(tv_0) \quad < \quad \frac{2}{nK_0^{\frac{n}{4}} \left(\sup_M f\right)^{\frac{n-4}{4}}}. \tag{8}$$

where E is as in (7). Then the equation

$$P_g u = f|u|^{2^\sharp - 2}u + h|u|^{q-1}u$$

possesses a nontrivial solution $u \in C^{4,\eta}(M)$. Moreover, the solution can be assumed to be positive if P_g has constant coefficents, h is nonnegative, $\alpha, a > 0$, and $a \leq \alpha^2/4$, where α and a are as in (2).

With such a theorem we are left with finding conditions on A, a, f, h such that (8) is satisfied. For this purpose, we compute the left-hand-side of (8) for some suitable function $v_0 \in H_2^2(M)$, essentially given by (5). We denote by $Maxf$ the set consisting of the points in M where f is maximum. Our first application of Theorem 1 is the following:

Theorem 2 *Let (M, g) be a compact Riemannian n-manifold, $n \geq 6$, f, h be two smooth functions on M, $q \in (\frac{n}{n-4}, \frac{n+4}{n-4})$, and P_g be a Paneitz-Branson type operator. We assume that P_g is coercive, that f is positive and that there exists $x_0 \in Maxf$ such that $h(x_0) > 0$. Then the equation*

$$P_g u = f|u|^{2^\sharp - 2}u + h|u|^{q-1}u$$

possesses a nontrivial solution $u \in C^{4,\eta}(M)$, $0 < \eta < 1$. Moreover, the solution can be assumed to be positive if P_g has constant coefficents, h is nonnegative, $\alpha, a > 0$, and $a \leq \alpha^2/4$, where α and a are as in (2).

For A as in (1), we let $tr_g(A)$ be the trace of A given in local coordinates by $tr_g(A) = A_{ij}g^{ij}$. For x in M we also let F be the function given by

$$\begin{aligned} F(x) &= 8(n-1)tr_g(A)(x) - 4(n^2-2n-4)S_g(x) \\ &\quad +(n+2)(n-4)(n-6)\frac{\Delta_g f}{f}(x) \end{aligned} \tag{9}$$

The limit case of Theorem 1 where $h(x_0) = 0$ is treated in the following theorem:

Theorem 3 *Let (M, g) be a compact Riemannian n-manifold, $n \geq 6$, f, h be two smooth functions on M, $q \in (\frac{n}{n-4}, \frac{n+4}{n-4})$, and P_g be a Paneitz-Branson type operator. We assume that P_g is coercive, that f is positive, and that for some $x_0 \in Maxf$, $h(x_0) = 0$ and $F(x_0) < 0$, where F is as in (9). Then the equation*

$$P_g u = f|u|^{2^\sharp-2}u + h|u|^{q-1}u$$

possesses a nontrivial solution $u \in C^{4,\eta}(M)$, $0 < \eta < 1$. The same conclusion holds if $n \geq 8$ and for some $x_0 \in Maxf$, $h(x_0) = 0$, $F(x_0) = 0$, and $\Delta_g h(x_0) < 0$. Moreover, in both cases, the solution can be assumed to be positive if P_g has constant coefficents, h is nonnegative, $\alpha, a > 0$, and $a \leq \alpha^2/4$, where α and a are as in (2).

For A as in (1), and $x \in M$, we let G be the function given by

$$G(x) = F(x) - \frac{8n(n-1)(n+2)(n-6)}{\sqrt{n(n-4)(n^2-4)}}\frac{h}{\sqrt{f}}(x) \tag{10}$$

Theorems 2 and 3 deal with the case $q \in (\frac{n}{n-4}, \frac{n+4}{n-4})$. When $q = \frac{n}{n-4}$, we get that the following theorem holds:

Theorem 4 *Let (M, g) be a compact Riemannian n-manifold, $n \geq 6$, f, h be two smooth functions on M, $q = \frac{n}{n-4}$, and P_g be a Paneitz-Branson type operator. We assume that P_g is coercive, that f is positive, and that for some $x_0 \in Maxf$, $G(x_0) < 0$, where G is as in (10). Then the equation*

$$P_g u = f|u|^{2^\sharp-2}u + h|u|^{q-1}u$$

possesses a nontrivial solution $u \in C^{4,\eta}(M)$, $0 < \eta < 1$. Moreover, the solution can be assumed to be positive if P_g has constant coefficents, h is nonnegative, $\alpha, a > 0$, and $a \leq \alpha^2/4$, where α and a are as in (2).

With Theorems 2, 3, 4 we are left with the case where $q \in (1, \frac{n}{n-4})$. This is the subject of the following theorem:

Theorem 5 *Let (M,g) be a compact Riemannian n-manifold, $n \geq 8$, f, h be two smooth functions in M, $q < \frac{n}{n-4}$, and P_g be a Paneitz-Branson type operator. We assume that P_g is coercive, that f is positive, and that for some $x_0 \in Maxf$, either $F(x_0) < 0$, or $F(x_0) = 0$ and $h(x_0) > 0$, where F is as in (9). Then the equation*

$$P_g u = f|u|^{2^\sharp - 2}u + h|u|^{q-1}u$$

possesses a nontrivial solution $u \in C^{4,\eta}(M)$, $0 < \eta < 1$. Moreover, the solution can be assumed to be positive if P_g has constant coefficents, h is nonnegative, $\alpha, a > 0$, and $a \leq \alpha^2/4$, where α and a are as in (2).

Our last theorem deals with the geometric case and the geometric Paneitz-Branson operator P_g^n. In such a case, $h \equiv 0$ and $P_g = P_g^n$. Then,

$$A = a_n S_g g + b_n Ric_g$$

and it is easily seen that $8(n-1)tr_g(A) - 4(n^2-2n-4)S_g \equiv 0$. In particular, Theorems 2-5 do not apply to such a case since if $x_0 \in Maxf$, $\Delta_g f(x_0) \geq 0$. Independently, when (M,g) is Einstein, then P_g^n is with constant coefficients α and a where, thanks to (3),

$$\alpha = \frac{n^2-2n-4}{2n(n-1)}S_g \text{ and } a = \frac{(n-4)(n^2-4)}{16n(n-1)^2}S_g^2$$

In particular, $a + S_g^2/(n^2(n-1)^2) = \alpha^2/4$ so that $a \leq \alpha^2/4$. If in addition S_g is positive, P_g^n is coercive (see [10]) and, as above, we can get the positivity of the solutions of the equation we consider. For $x \in M$ we let

$$\begin{aligned} H(x) \quad = \quad & \frac{4(n^2-4n-4)}{3(n+2)}|Weyl_g|_g^2(x) + (n-6)(n-8)\frac{\Delta_g^2 f}{f}(x) \\ & +2(n-6)(n-8)\frac{(\nabla^2 f, Ric_g)_g}{f}(x) \end{aligned} \tag{11}$$

where $(.,.)_g$ stands for the pointwise scalar product with respect to g, and $Weyl_g$ stands for the Weyl curvature tensor of g. In local coordinates,

$$(\nabla^2 f)_{ij} = \partial_{ij}^2 f - \Gamma_{ij}^k \partial_k f$$

where the Γ_{ij}^k's are the Christoffel symbols of the Levi-Civita connexion, and $(\nabla^2 f, Ric_g)_g = R^{ij}(\nabla^2 f)_{ij}$ where an index is raised with the metric. Our last theorem is as follows:

Theorem 6 (The geometric case) *Let (M, g) be a compact Riemannian n-manifold, $n \geq 8$, f be a smooth positive function on M, and P_g^n be the geometric Paneitz-Branson operator. We assume that P_g^n is coercive, and that there exists $x_0 \in Maxf$ such that $\Delta_g f(x_0) = 0$ and $H(x_0) > 0$, where H is given by (11). Then the equation*

$$P_g^n u = f|u|^{2^\sharp - 2}u$$

possesses a nontrivial solution $u \in C^{4,\eta}(M)$, $0 < \eta < 1$. When (M, g) is Einstein with positive scalar curvature, this solution can be assumed to be smooth and positive. Then there exists $\tilde{g}$ conformal to g such that $\frac{n-4}{2} Q_{\tilde{g}}^n = f$.

The paper is divided as follows. In section 2, we apply the Mountain-Pass Lemma to the functional E and study the associated Palais-Smale sequences. We deal with the regularity of solutions to the type of fourth-order equations we consider in section 3. Section 4 to 6 are devoted to fairly general test-function computations. These computations have their analogue in [2] when dealing with the conformal Laplacian. We prove Theorems 2-6 in section 7.

2 Mountain-Pass lemma and Palais-Smale sequences

As already mentioned, the main tool in this section is the Mountain-Pass lemma of Ambrosetti-Rabinowitz [1]. We use the following statement of the lemma:

Proposition 1 *Let $F \in C^1(V, \mathbb{R})$ where $(V, \|.\|)$ is a Banach space. We assume that:*
(i) $F(0) = 0$,
(ii) $\exists \lambda, R > 0$ such that $F(u) \geq \lambda$ for all $u \in V$ such that $\|u\| = R$,
(iii) $\exists v_0 \in V$ such that $\limsup_{t \to +\infty} F(tv_0) < 0$.
We let $t_0 > 0$ large be such that $\|t_0 v_0\| > R$ and $F(t_0 v_0) < 0$, and $\beta = \inf_{\gamma \in \Gamma} \sup F(\gamma(t))$, where $\Gamma = \{\gamma : [0, 1] \to V \text{ s.t. } \gamma(0) = 0, \gamma(1) = t_0 v_0\}$. Then there exists a sequence (u_n) in V such that

$$F(u_n) \to \beta \quad , \quad F'(u_n) \to 0 \text{ strongly in } V'.$$

Moreover, we have that $\beta \leq \sup_{t \geq 0} F(tv_0)$.

We say that a sequence (u_n) in $H_2^2(M)$ is a Palais-Smale (P-S) sequence for E if there exists $\beta \in \mathbb{R}$ such that $E(u_n) \to \beta$ and $E'(u_n) \to 0$ strongly in $H_2^2(M)'$. Let $\beta \in \mathbb{R}$. We say that E satisfies the (P-S) condition at the level β if for any (u_n) a (P-S) sequence for E in $H_2^2(M)$ such that $E(u_n) \to \beta$, there exists a subsequence (u_n)

of (u_n) such that (u_n) converges strongly in $H_2^2(M)$. As easily checked, this limit is then a critical point for E. The lack of compactness for Palais-Smale sequence in the case where $h \equiv 0$ was described in Hebey-Robert [10]. We prove here the following result:

Proposition 2 *Let (M,g) be a compact Riemannian n-manifold, $n \geq 5$, f,h be two functions in $C^\eta(M)$, $0<\eta<1$, $q \in (1, 2^\sharp - 1)$, and P_g be a Paneitz-Branson type operator. We assume that P_g is coercive, and that f is positive. For any*

$$\beta < \frac{2}{nK_0^{\frac{n}{4}}(\max f)^{\frac{n-4}{4}}},$$

the functional E satisfies the (P-S) condition at the level β.

Proof: From the coercivity of P_g, there exists $c>0$ such that

$$c\|u\|^2_{H_2^2(M)} \leq \int_M (\Delta_g u)^2 dv_g + \int_M A^\#(du,du)dv_g + \int_M au^2 dv_g \tag{12}$$

We take any sequence $\{u_n\}_{n\in N} \subseteq H_2^2(M)$ such that $E(u_n) \to \beta$ for some $\beta < \frac{2}{n}K_0^{-\frac{n}{4}}(\max f)^{-\frac{n-4}{4}}$ and $E'(u_n) \to 0$. We prove that this sequence is relatively compact in $H_2^2(M)$. A first claim is that (u_n) is bounded in $H_2^2(M)$. Standard computations lead to

$$\begin{aligned} O(1) + o(\|u_n\|) &= 2E(u_n) - \langle E'(u_n), u_n \rangle \\ &= \frac{4}{n}\int_M f|u_n|^{2^\sharp} dv_g + \frac{q-1}{q+1}\int_M h|u_n|^{q+1} dv_g. \end{aligned}$$

With (12), it comes that

$$\begin{aligned} c\|u_n\|^2_{H_2^2(M)} &\leq 2E(u_n) + \frac{2}{2^\sharp}\int_M f|u_n|^{2^\sharp} dv_g + \frac{2}{q+1}\int_M h|u_n|^{q+1} dv_g \\ &= O(1) + o(\|u_n\|) \end{aligned}$$

As easily checked, for all $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that $t^{q+1} \leq \epsilon t^{2^\sharp} + K_\epsilon$ for all $t \geq 0$. As a consequence,

$$\left| \int_M h|u|^{q+1} dv_g \right| \leq K_\varepsilon \|h\|_\infty Vol_g(M) + \epsilon \frac{\|h\|_\infty}{\min_M f}\int_M f|u|^{2^\sharp} dv_g$$

where $Vol_g(M)$ is the volume of M with respect to g. Then $\|u_n\|_{H_2^2(M)}$ is bounded, and this proves the claim. In particular, up to the extraction of a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in $H_2^2(M)$. With the compactness of the embedding $H_2^2(M) \hookrightarrow L^p(M)$ for all $1 \leq p < 2^\sharp$ we can also assume that $u_n \to u$ for all $1 \leq p < 2^\sharp$. By standard variational arguments, we infer that u is a distributional solution in $H_2^2(M)$ of our equation. For all $\varphi \in H_2^2(M)$, we get that

$$\int_M \Delta_g u \Delta_g \varphi dv_g + \int_M A^\#(du, d\varphi) dv_g + \int_M au\varphi dv_g$$
$$= \int_M f|u|^{2^\sharp-2} u\varphi \, dv_g + \int_M h|u|^{q-1} u\varphi \, dv_g$$

Taking $\varphi = u$ yields the following expression for $E(u)$:

$$\begin{aligned} E(u) &= \frac{q-1}{2(q+1)} \left[\int_M (\Delta_g u)^2 dv_g + \int_M A^\#(du, du) dv_g + \int_M au^2 dv_g \right] \\ &+ \left(\frac{1}{q+1} - \frac{1}{2^\sharp} \right) \int_M f|u|^{2^\sharp} dv_g \geq 0 \end{aligned}$$

We compare the energy of u_n and u. Taking into account the weak convergence of u_n to u, we obtain

$$\begin{aligned} E(u_n) - E(u) &= \frac{1}{2} \int_M (\Delta_g(u_n - u))^2 \, dv_g \\ &- \frac{1}{2^\sharp} \int_M f \left(|u_n|^{2^\sharp} - |u|^{2^\sharp} \right) dv_g + o(1). \end{aligned} \tag{13}$$

By standard integration theory

$$\int_M f \left(|u_n|^{2^\sharp} - |u_n - u|^{2^\sharp} \right) dv_g = \int_M f|u|^{2^\sharp} dv_g + o(1). \tag{14}$$

Testing $E'(u_n)$ on $u_n - u \rightharpoonup 0$ in $H_2^2(M)$ and using (14), we get

$$\begin{aligned} o(1) &= \langle u_n - u, E'(u_n) \rangle \\ &= \langle u_n - u, E'(u_n) - E'(u) \rangle \\ &= \int_M (\Delta_g(u_n - u))^2 \, dv_g - \int_M f|u_n - u|^{2^\sharp} dv_g + o(1) \end{aligned} \tag{15}$$

From (13) and (15), we get

$$\begin{aligned} &\frac{1}{2} \int_M (\Delta_g(u_n - u))^2 \, dv_g - \frac{1}{2^\sharp} \int_M f|u_n - u|^{2^\sharp} dv_g \\ &= \frac{2}{n} \int_M (\Delta_g(u_n - u))^2 \, dv_g + o(1) \\ &= E(u_n) - E(u) + o(1) \leq E(u_n) + o(1) \to \beta \end{aligned} \tag{16}$$

with the coercivity of P_g. As stated in [6], for all $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that for all $u \in H_2^2(M)$,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}} \leq (1+\epsilon)K_0 \int_M \left[(\Delta_g u)^2 + |\nabla u|_g^2\right] dv_g + B_\epsilon \int_M u^2 dv_g$$

Testing on $u_n - u$, we obtain that

$$\int_M f|u_n - u|^{2^\sharp} dv_g \leq (\max_M f) K_0^{\frac{2^\sharp}{2}} (1+\epsilon)^{\frac{2^\sharp}{2}} \left[\int_M (\Delta_g(u_n - u))^2 \, dv_g\right]^{\frac{2^\sharp}{2}} + o(1).$$

At last, from (15), for $\epsilon > 0$ small enough

$$\begin{aligned} o(1) &= \int_M (\Delta_g(u_n - u))^2 \, dv_g - \int_M f|u_n - u|^{2^\sharp} dv_g \\ &\geq \left[\int_M (\Delta_g(u_n - u))^2 \, dv_g\right] \\ &\quad \times \left\{1 - (\max_M f) K_0^{\frac{2^\sharp}{2}} (1+\epsilon)^{\frac{2^\sharp}{2}} \left[\int_M (\Delta_g(u_n - u))^2 \, dv_g\right]^{\frac{2^\sharp - 2}{2}}\right\}. \end{aligned}$$

With (16), it comes that

$$\int_M (\Delta_g(u_n - u))^2 \, dv_g \leq \frac{n}{2}\beta + o(1).$$

Using that $\beta < \frac{2}{nK_0^{\frac{n}{4}} (\max_M f)^{\frac{n-4}{4}}}$, it comes that there exists $C > 0$ such that

$$o(1) \geq C \int_M (\Delta_g(u_n - u))^2 \, dv_g + o(1).$$

Hence $u_n \to u$ in $H_2^2(M)$. This ends the proof of the proposition. □

Up to the regularity of the solution, that we prove in the following section, it is clear that the first part of Theorem 1 follows from Propositions 1 and 2. Concerning the second part, when P_g has constant coefficients, we can proceed as follows. We apply the mountain pass lemma to the functional

$$E_+(u) = \frac{1}{2}\int_M (P_g u) u dv_g - \frac{1}{2^\sharp}\int_M f u_+^{2^\sharp} dv_g - \frac{1}{q+1}\int_M h u_+^{q+1} dv_g,$$

where $u_+ = \max(0, u)$. Critical points of E_+ are weak solutions of

$$\Delta_g^2 u + \alpha \Delta_g u + au = \left(f u_+^{2^\sharp - 2} + h u_+^{q-1} \right) u$$

Similar arguments to the ones we used to prove the first part of Theorem 1 give that E_+ has a critical point u. It is then easily seen, mimicking what we do in Proposition 3 below, that $u \in C^{4,\eta}(M)$, $\eta \in (0,1)$. We let

$$\beta_1 = \frac{\alpha + \sqrt{\alpha^2 - 4a}}{2} \ , \ \beta_2 = \frac{\alpha - \sqrt{\alpha^2 - 4a}}{2}.$$

Then, $\beta_1, \beta_2 > 0$ and

$$(\Delta_g + \beta_1) \circ (\Delta_g + \beta_2) u = \Delta_g^2 u + \alpha \Delta_g u + au \geq 0.$$

Applying the maximum principle twice, it comes that $u > 0$. Hence u is a C^4-positive solution of

$$\Delta_g^2 u + \alpha \Delta_g u + au = f u^{2^\sharp - 1} + h u^{q+1}.$$

Standard regularity results then give that u is smooth, and the second part of Theorem 1 is proved.

3 Regularity results

We are here concerned with the regularity of critical points for E. We claim that the following regularity result holds:

Proposition 3 *Let (M, g) be a compact Riemannian n-manifold, $n \geq 5$, f, h be two functions in $C^\eta(M)$, $0 < \eta < 1$, $q \in (1, 2^\sharp - 1)$, and P_g be a Paneitz-Branson type operator. If $u \in H_2^2(M)$ is a weak solution of*

$$P_g u = f |u|^{2^\sharp - 2} u + h |u|^{q-1} u \tag{17}$$

then $u \in C^{4,\eta}(M)$ and u is a strong solution of the equation. Moreover, if f and h are smooth, and u is positive, then u is also smooth.

Proof: Let $u \in H_2^2(M)$ be a weak solution of (17). From the work of [17] and [6], u satisfies

$$\begin{aligned} (\Delta_g + 1)^2 u &= div_g \left(A^\# du \right) + (1 - a) u + 2 \Delta_g u + f |u|^{2^\sharp - 2} u + h |u|^{q-1} u \\ &= b + q_\epsilon u + f_\epsilon \end{aligned} \tag{18}$$

where $b = div_g\left(A^{\#}du\right) + (1-a)u + 2\Delta_g u \in L^2(M)$, $q_\epsilon \in L^{\frac{n}{4}}(M)$ satisfies $\|q_\varepsilon\|_{\frac{n}{4}} \le \epsilon$, and $f_\epsilon \in L^\infty(M)$. We now follow [6]. For $s > 1$, we can define the operator

$$H_\epsilon : v \in L^s(M) \to (\Delta_g + 1)^{-2}(q_\epsilon v) \in L^s(M)$$

with

$$\begin{aligned} \|H_\epsilon v\|_{L^s} &= O(\|(\Delta_g+1)^{-2}(q_\epsilon v)\|_{H_4^{\frac{ns}{n+4s}}}) = O(\|q_\epsilon v\|_{L^{\frac{ns}{n+4s}}}) \\ &= O(\|q_\epsilon\|_{L^{\frac{n}{4}}}\|v\|_{L^s}) \le C\varepsilon\|v\|_{L^s}. \end{aligned}$$

It follows from the Sobolev theorem and classical regularity results that for any $f \in L^p(M)$ with $p > 1$, there exists a unique function $u \in H_2^p(M)$ such that $(\Delta_g+1)u = f$ with $\|u\|_{H_2^p} \le C\|f\|_{L^p}$. Hence, for $\varepsilon > 0$ small enough,

$$\|H_\epsilon\|_{L^s \to L^s} \le C\varepsilon < \frac{1}{2}.$$

We rewrite (18) in the form

$$(Id - H_\epsilon)u = (\Delta_g + 1)^{-2}(b + f_\epsilon)$$

where for $s > 1$, $Id - H_\epsilon : L^s \to L^s$ is an invertible operator. We have $b + f_\epsilon \in L^2(M)$ and then $(\Delta_g+1)^{-2}(b+f_\epsilon) \in H_4^2(M)$. By the Sobolev theorem, we obtain that, if $n \le 8$, $u \in L^p(M)$ for all $p > 1$ and, if $n > 8$, $u \in L^{\frac{2n}{n-8}}(M)$. Since for $n > 8$ there holds $\frac{2n(n-4)}{(n+4)(n-8)} > 2$, we get that

$$(\Delta_g + 1)^2 u \in L^2(M)$$

We now use a bootstrap argument. We construct a non-decreasing sequence $s_k \in \mathbb{R} \cup \{+\infty\}$ such that $u \in H_4^{s_k}(M)$ for all $k \in \mathbb{N}$. We define s_k by induction. We let $s_0 = 2$. For all $k \ge 0$ such that $u \in H_4^{s_k}(M)$, the Sobolev theorem asserts that

$$div_g\left(A^{\#}du\right) + (1-a)u + 2\Delta_g u \in L^{\frac{ns_k}{n-2s_k}}(M),$$

with the convention that $\frac{ns_k}{n-2s_k} = +\infty$ if $s_k \ge \frac{n}{2}$, and

$$f|u|^{2^\sharp-2}u + h|u|^{q-1}u \in L^{\frac{(ns_k)(n-4)}{(n-4s_k)(n+4)}}(M),$$

with the convention that $\frac{ns_k}{n-4s_k} = +\infty$ if $s_k \ge \frac{n}{4}$. Then $(\Delta_g+1)^2 u \in L^{s_{k+1}}(M)$, where

$$s_{k+1} = \min\left\{\frac{ns_k}{n-2s_k}, \frac{(ns_k)(n-4)}{(n-4s_k)(n+4)}\right\} \ge s_k.$$

By standard elliptic arguments, $u \in H_4^{s_{k+1}}(M)$. The sequence (s_k) is then well-defined. We assume now that (s_k) is bounded. Then it goes to a limit $L \geq 2$ such that

$$L = \min\left\{\frac{nL}{n-2L}, \frac{nL(n-4)}{(n+4)(n-4L)}\right\}$$

if $L < \frac{n}{4}$. A contradiction. If $L \geq \frac{n}{4}$, the same kind of arguments lead also to a contradiction. Hence $s_k \to +\infty$, and $u \in H_4^s(M)$ for all $s > 1$. From the Sobolev theorem, it comes that $u \in C^{3,\nu}(M)$ for all $0 < \nu < 1$. Plugging this result in (18), it comes that $u \in C^{4,\eta}(M)$. This proves the first part of the proposition. Now if a, f, h, α are smooth and $u > 0$, we note that $fu^{2^\sharp-1} + hu^q \in C^4(M)$ and standard bootstrap arguments show that $u \in C^\infty(M)$. This ends the proof of the proposition. □

For the sake of completeness, we mention that the same method leads to the following bounds:

Proposition 4 *Assume that* α *and* a *are smooth. Let* $u \in H_2^s(M)$ *and* $\Phi \in L^s(M)$, $s > 1$, *such that* $P_g u = \Phi$ *in the weak sense. Then* $u \in H_4^s(M)$ *and there exists* $C(s) > 0$ *depending only on* (M, g), s *and* a, α *such that*

$$\|u\|_{H_4^s(M)} \leq C(s)\left(\|\Phi\|_s + \|u\|_{L^s(M)}\right).$$

Moreover, if $\Phi \in H_k^s(M)$ *with* $k \in \mathbb{N}$, *then* $u \in H_{k+4}^s(M)$ *and there exists* $C(s,k) > 0$ *depending only on* (M, g), s, k, a *and* α *such that*

$$\|u\|_{H_{k+4}^s(M)} \leq C(s)\left(\|\Phi\|_{H_k^s(M)} + \|u\|_{L^s(M)}\right).$$

We are now left with finding conditions for (8) to be true. This is the purpose of the following sections.

4 First order estimates for Paneitz-Branson type operators

We let $\delta \in (0, \frac{i_g(M)}{2})$, where $i_g(M)$ is the injectivity radius, and $x_0 \in M$. We let also $\eta \in C^\infty(M)$ be such that $\eta(x) = 1$ for all $x \in B_g(x_0, \delta)$ and $\eta(x) = 0$ for all $x \in M - B_g(x_0, 2\delta)$. For $\varepsilon > 0$, we define the function $u_\epsilon \in C^\infty(M)$ by

$$u_\epsilon(x) = \frac{\eta(x)}{(\varepsilon^2 + d_g(x, x_0)^2)^{\frac{n-4}{2}}}.$$

Given P_g a Paneitz-Branson type operator, $q \in (1, 2^\sharp - 1)$, and f, h smooth functions on M, the aim of this section is to compute expansions of

$$\int_M P_g u_\epsilon u_\epsilon \, dv_g \ , \ \int_M f u_\epsilon^{2^\sharp} \, dv_g \ , \ \int_M h u_\epsilon^{q+1} \, dv_g .$$

We compute the different terms separately. We start with the leading term $\int_M (\Delta_g u_\epsilon)^2 \, dv_g$. The function u_ϵ is radially symmetrical. Computing in the exponential chart, it comes that

$$\begin{aligned} \Delta_g u_\epsilon &= -\frac{1}{r^{n-1}\sqrt{|g|}} \partial_r \left(r^{n-1} \sqrt{|g|} \partial_r u_\epsilon \right) \\ &= \Delta_\xi u_\epsilon - \partial_r \left(\ln \sqrt{|g|} \right) \partial_r u_\epsilon , \end{aligned}$$

where $r = d_g(x, x_0)$, and $|g|$ is the determinant of the components of g in the chart. We let

$$\theta_\varepsilon = \frac{1}{\varepsilon^{n-8}} \ \text{ if } n \geq 9 \ , \ \theta_\varepsilon = |\ln \varepsilon| \ \text{ if } n = 8 \ , \ \theta_\varepsilon = 1 \ \text{ if } n = 6, 7$$

We first assume that $n \geq 7$. Then,

$$\begin{aligned} \int_M (\Delta_g u_\epsilon)^2 \, dv_g &= \int_{B_\xi(0,\delta)} (\Delta_\xi u_\epsilon)^2 \, dv_g \\ &\quad -2 \int_{B_\xi(0,\delta)} \Delta_\xi u_\epsilon \partial_r u_\epsilon \partial_r \left(\ln \sqrt{|g|} \right) \, dv_g + O(\theta_\varepsilon). \end{aligned}$$

We write now, thanks to the Cartan expansion of the metric, that

$$\sqrt{|g|}(x) = 1 - \frac{1}{6} R_{ij} x^i x^j - \frac{1}{12} \nabla_k R_{ij} x^i x^j x^k + O(|x|^4), \tag{19}$$

where the R_{ij}'s are the components of the Ricci tensor in the exponential chart. With (19), it comes that

$$\begin{aligned} \int_M (\Delta_g u_\epsilon)^2 \, dv_g &= \int_{B_\xi(0,\delta)} (\Delta_\xi u_\epsilon)^2 \, dx - \frac{1}{6} R_{ij} \int_{B_\xi(0,\delta)} x^i x^j (\Delta_\xi u_\epsilon)^2 \, dx \\ &\quad -2 \int_{B_\xi(0,\delta)} \Delta_\xi u_\epsilon \partial_r u_\epsilon \partial_r \left(\ln \sqrt{|g|} \right) \, dx + O(\theta_\varepsilon). \end{aligned}$$

It is easily seen that,

$$\int_{B_\xi(0,\delta)} (\Delta_\xi u_\epsilon)^2 \, dx = \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + O(1)$$

and that

$$R_{ij}\int_{B_\xi(0,\delta)} x^i x^j \left(\Delta_\xi u_\epsilon\right)^2\, dx = \frac{(n-4)^2\omega_{n-1}S_g(x_0)}{n\varepsilon^{n-6}}\int_0^{\frac{\delta}{\varepsilon}} \frac{s^{n+1}(n+2s^2)^2\, ds}{(1+s^2)^n}.$$

In the same order of ideas, thanks to (19), we get that

$$\int_{B_\xi(0,\delta)} \Delta_\xi u_\epsilon \partial_r u_\epsilon \partial_r \left(\ln\sqrt{|g|}\right)\, dx$$
$$= \frac{(n-4)^2\omega_{n-1}S_g(x_0)}{3n\varepsilon^{n-6}}\int_0^{\frac{\delta}{\varepsilon}} \frac{(n+2s^2)s^{n+1}\, ds}{(1+s^2)^{n-1}} + O(\theta_\varepsilon).$$

Then, when $n \geq 7$,

$$\int_M \left(\Delta_g u_\epsilon\right)^2\, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n\varepsilon^{n-4}}$$
$$-\frac{n(n^2+4n-20)(n-4)\omega_n}{6(n-6)2^n}S_g(x_0)\frac{1}{\varepsilon^{n-6}} + O(\theta_\varepsilon).$$

Similarly, when $n = 6$, we find that

$$\int_M \left(\Delta_g u_\epsilon\right)^2\, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n\varepsilon^{n-4}}$$
$$-\frac{2(n-4)^2\omega_{n-1}}{n}S_g(x_0)|\ln\varepsilon| + O(1).$$

We let $A \in \Lambda^\infty_{(2,0)}(M)$ be a smooth symmetric $(2,0)$-tensor field, and we let $a \in C^\infty(M)$. Then, with similar computations to the ones we just developed, we get that

$$\int_M a u_\epsilon^2\, dv_g = O(\theta_\varepsilon)$$

when $n \geq 6$, and that

$$\int_M A^\#(du_\epsilon, du_\epsilon)\, dv_g = \frac{4(n-1)(n-4)\omega_n}{2^n(n-6)}\frac{Tr_g A(x_0)}{\varepsilon^{n-6}} + O(\theta_\varepsilon) \text{ if } n \geq 7,$$
$$\int_M A^\#(du_\epsilon, du_\epsilon)\, dv_g = \frac{(n-4)^2\omega_{n-1}}{n}Tr_g A(x_0)|\ln\varepsilon| + O(1) \text{ if } n = 6.$$

Hence,

$$\int_M P_g u_\epsilon u_\epsilon \, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}}$$
$$+\frac{(n-4)\omega_n}{(n-6)2^n}\left(4(n-1)Tr_g A(x_0) - \frac{n(n^2+4n-20)}{6} S_g(x_0)\right)\frac{1}{\varepsilon^{n-6}}$$
$$+O(\theta_\varepsilon),$$

when $n \geq 7$, and

$$\int_M P_g u_\epsilon u_\epsilon \, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}}$$
$$+\frac{(n-4)^2 \omega_{n-1}}{n}\left(Tr_g A(x_0) - 2S_g(x_0)\right) |\ln \varepsilon| + O(1)$$

when $n = 6$. We now compute $\int_M f u_\epsilon^{2^\sharp} \, dv_g$. Clearly

$$\begin{aligned}\int_M f u_\epsilon^{2^\sharp} \, dv_g &= \int_{B_g(x_0,\delta)} \frac{f(x)}{\left(\varepsilon^2 + d_g(x,x_0)^2\right)^n} \, dv_g + O(1) \\ &= \int_{B(0,\delta)} \frac{\tilde{f}(x)\sqrt{|g|}(x)}{\left(\varepsilon^2 + |x|^2\right)^n} \, dx + O(1),\end{aligned}$$

where $\tilde{f} = f \circ exp_{x_0} g$. Thanks to (19), it follows that for $n \geq 5$,

$$\int_M f u_\epsilon^{2^\sharp} \, dv_g = \frac{f(x_0)\omega_n}{2^n \varepsilon^n}$$
$$-\frac{\omega_n}{6(n-2)2^n}\left(S_g(x_0) f(x_0) + 3\Delta_g f(x_0)\right)\frac{1}{\varepsilon^{n-2}} + O\left(\frac{1}{\varepsilon^{n-4}}\right).$$

At last we compute an expansion of $\int_M h u_\epsilon^{q+1} \, dv_g$. It easily comes that

$$\int_M h u_\epsilon^{q+1} \, dv_g = \frac{\omega_{n-1} h(x_0)}{\varepsilon^{(q+1)(n-4)-n}} \int_0^{+\infty} \frac{s^{n-1} \, ds}{(1+s^2)^{(q+1)\frac{n-4}{2}}} + o\left(\frac{1}{\varepsilon^{(q+1)(n-4)-n}}\right)$$

if $q+1 > \frac{n}{n-4}$, that

$$\int_M h u_\epsilon^{q+1} \, dv_g = \omega_{n-1} h(x_0) |\ln \varepsilon| + o\left(|\ln \varepsilon|\right)$$

if $q+1=\frac{n}{n-4}$, and that

$$\int_M h u_\epsilon^{q+1}\, dv_g = O(1)$$

if $q+1<\frac{n}{n-4}$. Moreover, when $h(x_0)=0$, then we can write that

$$\int_M h u_\varepsilon^{q+1}\, dv_g = -\frac{\Lambda\omega_{n-1}}{2n}\Delta_g h(x_0)\varepsilon^{2+n-(n-4)(q+1)} + o\left(\varepsilon^{2+n-(n-4)(q+1)}\right)$$

where $\Lambda = \int_0^{+\infty}\frac{s^{n+1}\, ds}{(1+s^2)^{\frac{n-4}{2}(q+1)}}$ and $q+1>\frac{n+2}{n-4}$.

5 Second order estimates for the geometric Paneitz-Branson operator

Let $x_0\in M$. Up to changing conformally the metric, see [11], we may assume that

$$\begin{aligned} & Ric_g(x_0)=0,\ S_g(x_0)=0,\ \nabla S_g(x_0)=0,\\ & \Delta_g S_g(x_0)=\frac{1}{6}|Weyl_g(x_0)|_g^2, \text{ and}\\ & dv_g = dv_\xi(1+O(r^N)) \end{aligned} \tag{20}$$

where N is arbitrarily large. We let $0<\delta<\frac{i_g(M)}{2}$ and $\eta\in C^\infty(M)$ be a radially symmetrical function such that $\eta\equiv 1$ in $B_g(x_0,\delta)$ and $\eta\equiv 0$ in $M-B_g(x_0,2\delta)$, where $B_g(x,r)$ denotes the geodesic ball of center $x\in M$ and radius $r>0$. We let also $u_\epsilon\in C^\infty(M)$ be the function given by

$$u_\epsilon(x)=\frac{\eta(x)}{(\varepsilon^2+d_g(x,x_0)^2)^{\frac{n-4}{2}}}.$$

Our aim in this section is to estimate

$$\int_M P_g u_\epsilon u_\epsilon\, dv_g \quad \text{and} \quad \int_M f u_\epsilon^{2^\sharp}\, dv_g$$

We compute the different terms separately. We start with

$$I_1=\int_M (\Delta_g u_\epsilon)^2\, dv_g$$

We have that

$$\int_M (\Delta_g u_\epsilon)^2\, dv_g = \int_{B_g(x_0,\delta)} (\Delta_g u_\epsilon)^2\, dv_g + O(1).$$

Since u_ϵ is radially symmetrical on $B_g(x_0,\delta)$, we have that

$$\Delta_g u_\epsilon = -\frac{1}{r^{n-1}\sqrt{|g|}}\partial_r\left(r^{n-1}\sqrt{|g|}\partial_r\left(\frac{1}{\varepsilon^2+|x|^2}\right)^{\frac{n-4}{2}}\right),$$

where $\sqrt{|g|} = \sqrt{\det(g_{ij})}$ and the g_{ij}'s are the components of g in the exponential chart at x_0. We have $\sqrt{|g|} = 1 + O(r^N)$. Then, with N large enough,

$$(\Delta_g u_\epsilon)^2 = \left(\Delta_\xi \frac{1}{(\varepsilon^2+r^2)^{\frac{n-4}{2}}}\right)^2 + O(1),$$

where $r = d_g(x,x_0) < \delta$, and

$$\begin{aligned}\int_M (\Delta_g u_\epsilon)^2\, dv_g &= \int_{B_\xi(0,\delta)}\left(\Delta_\xi \frac{1}{(\varepsilon^2+r^2)^{\frac{n-4}{2}}}\right)^2 dv_\xi + O(1)\\ &= \frac{1}{\varepsilon^{n-4}}\int_{\mathbb{R}^n}(\Delta_\xi u_0)^2\, dv_\xi + O(1)\end{aligned}$$

Considering that u_0 is an extremal function for the sharp Euclidean Sobolev inequality, we obtain that

$$\int_M (\Delta_g u_\epsilon)^2\, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n\varepsilon^{n-4}} + O(1).$$

We now compute

$$I_2 = \int_M Q_g^n u_\epsilon^2\, dv_g$$

We write that $Q_g^n(x) = Q_g^n(x_0) + O(d_g(x,x_0))$. Then,

$$\begin{aligned}\int_M Q_g^n u_\epsilon^2\, dv_g &= Q_g^n(x_0)\int_{B_\xi(0,\delta)}\frac{dx}{(\varepsilon^2+|x|^2)^{n-4}}\\ &\quad +O\left(\int_{B_\xi(0,\delta)}\frac{|x|dx}{(\varepsilon^2+|x|^2)^{n-4}}\right)\\ &= \frac{Q_g^n(x_0)\omega_{n-1}}{\varepsilon^{n-8}}\int_0^{\frac{\delta}{\varepsilon}}\frac{s^{n-1}\, ds}{(1+s^2)^{n-4}}\\ &\quad +O\left(\frac{\varepsilon}{\varepsilon^{n-8}}\int_0^{\frac{\delta}{\varepsilon}}\frac{s^n\, ds}{(1+s^2)^{n-4}}\right)\end{aligned}$$

Here, we have used a polar change of coordinates and the change of variable $r = \varepsilon s$. Since

$$Q_g^n(x_0) = \frac{1}{2(n-1)}\Delta_g S_g(x_0) = \frac{1}{12(n-1)}|Weyl_g(x_0)|_g^2,$$

it follows that

$$\begin{aligned} I_2 &= \frac{(n-3)\omega_n}{2^{n-2}3(n-6)(n-8)}|Weyl_g(x_0)|_g^2\frac{1}{\varepsilon^{n-8}} + o\left(\frac{1}{\varepsilon^{n-8}}\right) \text{ if } n \geq 9\,, \\ &= \frac{\omega_{n-1}}{12(n-1)}|Weyl_g(x_0)|_g^2|\ln\varepsilon| + o\left(\ln\varepsilon\right) \text{ if } n = 8\,, \text{ and} \\ &= O(1) \text{ if } 5 \leq n \leq 7\,. \end{aligned}$$

Going on with these estimates, we compute

$$I_3 = \int_M S_g|\nabla u_\epsilon|_g^2\, dv_g$$

We have that

$$\int_M S_g|\nabla u_\epsilon|_g^2\, dv_g = \int_{B_g(x_0,\delta)} S_g|\nabla u_\epsilon|_g^2\, dv_g + O(1),$$

Moreover, u_ϵ is radially symmetrical and

$$|\nabla u_\epsilon|_g^2(x) = (n-4)^2\frac{r^2}{(\varepsilon^2+r^2)^{n-2}},$$

where $r = d_g(x,x_0)$. Since $S_g(x_0) = 0$, we obtain that

$$\begin{aligned} \int_M S_g|\nabla u_\epsilon|_g^2\, dv_g &= \frac{1}{2}\partial_{ij}S_g(x_0)\int_{B_\xi(0,\delta)}(n-4)^2\frac{x^i x^j r^2\, dx}{(\varepsilon^2+r^2)^{n-2}} \\ &\quad + O\left(\int_{B_\xi(0,\delta)}\frac{|x|^5\, dx}{(\varepsilon^2+|x|^2)^{n-2}}\right). \end{aligned}$$

A polar change of coordinates and the change of variable $r = \varepsilon s$, gives that

$$\begin{aligned} \int_{B_\xi(0,\delta)}\frac{x^i x^j r^2\, dx}{(\varepsilon^2+r^2)^{n-2}} &= \int_{\mathbb{S}^{n-1}} x^i x^j\, d\sigma \int_0^\delta \frac{r^{n+3}\, dr}{(\varepsilon^2+r^2)^{n-2}} \\ &= \frac{\delta_{ij}\omega_{n-1}}{n}\frac{1}{\varepsilon^{n-8}}\int_0^{\frac{\delta}{\varepsilon}}\frac{s^{n+3}\, ds}{(1+s^2)^{n-2}}. \end{aligned}$$

where $d\sigma$ denotes the surface element of the standard unit sphere $\mathbb{S}^{n-1}$. Noting that in geodesic coordinates, $\Delta_g S_g(x_0) = -\partial_{ii} S_g(x_0)$, (20) gives that

$$\begin{aligned} & I_3 \\ & = -\frac{\omega_n (n+2)(n-1)(n-4)}{2^n 3 (n-6)(n-8)} |Weyl_g(x_0)|_g^2 \frac{1}{\varepsilon^{n-8}} + o\left(\frac{1}{\varepsilon^{n-8}}\right) \text{ if } n \geq 9 \, , \\ & = -\frac{(n-4)^2 \omega_{n-1}}{12n} |Weyl_g(x_0)|_g^2 |\ln \varepsilon| + o\left(|\ln \varepsilon|\right) \text{ if } n = 8 \, , \text{ and} \\ & = O(1) \text{ if } 5 \leq n \leq 7 \, . \end{aligned}$$

At last, we compute

$$I_4 = \int_M Ric_g^{\#}(du_\epsilon, du_\epsilon)\, dv_g$$

We have that

$$\begin{aligned} I_4 &= \int_{B_g(x_0,\delta)} Ric_g^{\#}(du_\epsilon, du_\epsilon)\, dv_g + O(1) \\ &= \int_{B_\xi(0,\delta)} R^{ij} \partial_i u_\epsilon \partial_j u_\epsilon \, dx + O(1) \\ &= (n-4)^2 \int_{B_\xi(0,\delta)} \frac{\psi(x)}{(\varepsilon^2 + r^2)^{n-2}}\, dx + O(1). \end{aligned}$$

where $\psi(x) = R^{ij}(x) x_i x_j$. We write that

$$\psi(x) = \frac{1}{2} D^2\psi_0(x^2) + \frac{1}{3!} D^3\psi_0(x^3) + \frac{1}{4!} D^4\psi_0(x^4) + O(|x|^5).$$

For parity reasons, it follows that

$$\begin{aligned} I_4 &= \frac{(n-4)^2}{2} \int_{\mathbb{S}^{n-1}} D^2\psi_0(x^2)\, d\sigma \int_0^\delta \frac{r^{n+1}\, dr}{(\varepsilon^2 + r^2)^{n-2}} \\ & + \frac{(n-4)^2}{4!} \int_{\mathbb{S}^{n-1}} D^4\psi_0(x^4)\, d\sigma \int_0^\delta \frac{r^{n+3}\, dr}{(\varepsilon^2 + r^2)^{n-2}} \\ & + O\left(\int_0^\delta \frac{r^{n+4}\, dr}{(\varepsilon^2 + r^2)^{n-2}}\right) \end{aligned}$$

We have here, see [7], that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{S}^{n-1}} D^2\psi_0(x^2)\, d\sigma = -\frac{\omega_{n-1}}{2n} \Delta_\xi \psi(0) \, , \text{ and} \\ & \frac{1}{4!} \int_{\mathbb{S}^{n-1}} D^4\psi_0(x^4)\, d\sigma = \frac{\omega_{n-1}}{8n(n+2)} \Delta_\xi^2 \psi(0) \end{aligned}$$

Noting that we use a normal chart at x_0 and that $Ric_g(x_0) = 0$, we get that

$$\begin{aligned} &\Delta_\xi \psi(0) = 0\ , \text{ and} \\ &\Delta_\xi^2 \psi(0) = 4(\partial_{ii} R_{jj} + 2\partial_{ij} R_{ij}). \end{aligned}$$

The Bianchi identity and $Ric_g(x_0) = 0$ lead to

$$\sum_{i,j} 2\partial_{ij} R_{ij}(x_0) = \sum_{i,j} \partial_{ii} R_{jj}(x_0) = -\Delta_g S_g(x_0).$$

Then, with (20) and the change of variable $r = \varepsilon s$, it comes that

$$\begin{aligned} I_4 &= -\frac{(n-4)^2 \omega_{n-1}}{6n(n+2)} \int_0^{\frac{\delta}{\varepsilon}} \frac{s^{n+3}\, ds}{(1+s^2)^{n-2}} \frac{|Weyl_g(x_0)|_g^2}{\varepsilon^{n-8}} \\ &\quad + O\left(\frac{\varepsilon}{\varepsilon^{n-8}} \int_0^{\frac{\delta}{\varepsilon}} \frac{s^{n+4}\, ds}{(1+s^2)^{n-2}} \right). \end{aligned}$$

Consequently,

$$\begin{aligned} I_4 &= -\frac{3\omega_n (n-1)(n-4)}{2^{n-1}(n-6)(n-8)} |Weyl_g(x_0)|_g^2 \frac{1}{\varepsilon^{n-8}} + o\left(\frac{1}{\varepsilon^{n-8}} \right) \ \text{ if } n \geq 9\ , \\ &= -\frac{(n-4)^2 \omega_{n-1}}{6n(n+2)} |Weyl_g(x_0)|_g^2 |\ln \varepsilon| + o\left(|\ln \varepsilon|\right) \ \text{ if } n = 8\ , \text{ and} \\ &= O(1) \ \text{ if } 5 \leq n \leq 7\ . \end{aligned}$$

In particluar, thanks to the previous estimates, we get that

$$\begin{aligned} &\int_M P_g u_\epsilon u_\epsilon \, dv_g \\ &= \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} \\ &\quad -\frac{(n-4)(n^2-4n-4)\omega_n}{2^{n+1} 3(n-6)(n-8)} \frac{|Weyl_g(x_0)|_g^2}{\varepsilon^{n-8}} + o\left(\frac{1}{\varepsilon^{n-8}} \right) \ \text{ if } n \geq 9\ , \\ &= \frac{15\omega_8}{2\varepsilon^4} - \frac{\omega_7}{30} |Weyl_g(x_0)|_g^2 |\ln \varepsilon| + o\left(|\ln \varepsilon|\right) \ \text{ if } n = 8\ , \text{ and} \\ &= \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + O(1) \ \text{ if } 5 \leq n \leq 7\ . \end{aligned} \tag{21}$$

Similarly we now compute

$$I_5 = \int_M f u_\epsilon^{2^\sharp} \, dv_g$$

Since $dv_g = dv_\xi(1+O(r^N))$ with N large enough, we can write that

$$I_5 = \int_{B_\xi(0,\delta)} \frac{f\circ exp_{x_0}}{(\varepsilon^2+|x|^2)^n}\,dx + O(1).$$

With the same techniques as before, we easily find that, for $n\geq 5$,

$$\begin{aligned} I_5 &= \omega_{n-1}\int_0^{+\infty}\frac{s^{n-1}\,ds}{(1+s^2)^n}\frac{f(x_0)}{\varepsilon^n} - \frac{\omega_{n-1}\int_0^{+\infty}\frac{s^{n+1}\,ds}{(1+s^2)^n}}{2n}\frac{\Delta_\xi f(x_0)}{\varepsilon^{n-2}} + \\ &\quad + \frac{\omega_{n-1}\int_0^{+\infty}\frac{s^{n+3}\,ds}{(1+s^2)^n}}{8n(n+2)}\frac{\Delta_\xi^2 f(x_0)}{\varepsilon^{n-4}} + o\left(\frac{1}{\varepsilon^{n-4}}\right). \end{aligned}$$

Since we are in a normal coordinate chart, and since $Ric_g(x_0)=0$ and $\nabla S_g(x_0)=0$, we obtain that $\Delta_g f(x_0) = \Delta_\xi f(x_0)$ and $\Delta_g^2 f(x_0) = \Delta_\xi^2 f(x_0)$. As a consequence,

$$\begin{aligned} \int_M f u_\epsilon^{2^\sharp}\,dv_g &= \frac{\omega_n f(x_0)}{2^n\varepsilon^n} - \frac{\omega_n}{2^{n+1}(n-2)}\frac{\Delta_g f(x_0)}{\varepsilon^{n-2}} \\ &\quad + \frac{\omega_n}{2^{n+3}(n-2)(n-4)}\frac{\Delta_g^2 f(x_0)}{\varepsilon^{n-4}} + o\left(\frac{1}{\varepsilon^{n-4}}\right) \end{aligned} \tag{22}$$

when $n\geq 5$.

6 General estimates for Paneitz-Branson type operators

We let $x_0\in M$ and $N\in\mathbb{N}^\star$. Then, see [11], there exists $\tilde g = \varphi^{\frac{4}{n-4}}g$, $\varphi>0$ is a smooth function on M, such that

$$\begin{aligned} &Ric_{\tilde g}(x_0) = 0\ ,\ \nabla S_{\tilde g}(x_0) = 0, \\ &\Delta_{\tilde g} S_{\tilde g}(x_0) = \frac{1}{6}|Weyl_g(x_0)|_g^2,\ \text{ and} \\ &dv_{\tilde g} = dv_\xi(1+O(r^N)) \end{aligned}$$

We let $\delta\in(0,\frac{i_{\tilde g}(M)}{2})$, where $i_{\tilde g}(M)$ is the injectivity radius of $\tilde g$, and $\eta\in C^\infty(M)$ be such that $\eta(x)=1$ for all $x\in B_{\tilde g}(x_0,\delta)$ and $\eta(x)=0$ for all $x\in M-B_{\tilde g}(x_0,2\delta)$. For $\varepsilon>0$, we define the function $\tilde u_\varepsilon\in C^\infty(M)$ by

$$\tilde u_\varepsilon(x) = \frac{\eta(x)\varphi(x)}{(\varepsilon^2+d_{\tilde g}(x,x_0)^2)^{\frac{n-4}{2}}}.$$

We let also P_g be a Paneitz-Branson type operator, and f be a smooth function on M. For the sake of completeness, we quote in this section results concerning the expansions of

$$I = \int_M P_g \tilde{u}_\varepsilon \tilde{u}_\varepsilon \, dv_g \text{ and } J = \int_M f \tilde{u}_\varepsilon^{2^\sharp} \, dv_g .$$

Such expansions are not required to prove our theorems. Nevertheless, they can be useful in another context. Details on these expansions can be found in Esposito-Robert [8]. Writing that

$$P_g = \Delta_g^2 u - div_g \left[A^{\#} du \right] + au,$$

we set

$$\begin{aligned} \tilde{A} &= A - a_n S_g g - b_n Ric_g \text{ , and} \\ \tilde{a} &= a - \frac{n-4}{2} Q_g^n . \end{aligned}$$

We define Φ

$$\begin{aligned} \Phi = &- \frac{n^2 - 4n - 4}{96(n-1)(n-3)} |Weyl_g|_g^2 \\ &+ \frac{g^{ij} g^{kl} (\nabla^2 \tilde{A})_{ijkl} + 2 g^{ik} g^{jl} (\nabla^2 \tilde{A})_{ijkl}}{8(n-3)} - \frac{(n-4)(Ric_g, \tilde{A})_g}{4(n-2)(n-3)} \\ &- \frac{n S_g tr_g(\tilde{A})}{8(n-1)(n-2)(n-3)} + \frac{1}{n-4} \left(a - \frac{n-4}{2} Q_g^n \right) , \end{aligned}$$

where $(\cdot,\cdot)_g$ is the scalar product with respect to g. We let also

$$\Theta = \Delta_g^2 f + \frac{2 S_g}{n-1} \Delta_g f + 2 \left(\nabla^2 f, Ric_g \right)_g + \frac{n-2}{2(n-1)} (\nabla f, \nabla S_g)_g$$

We then find that the following holds. Concerning I, we find that

$$\begin{aligned} &\int_M P_g \tilde{u}_\varepsilon \tilde{u}_\varepsilon \, dv_g \\ &= \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + \frac{\omega_n (n-1)(n-4) tr_g(\tilde{A})(x_0)}{2^{n-2}(n-6)\varepsilon^{n-6}} \\ &\quad + \frac{\omega_n (n-1)(n-3)(n-4)\Phi(x_0)}{2^{n-4}(n-6)(n-8)\varepsilon^{n-8}} + o\left(\frac{1}{\varepsilon^{n-8}} \right) \text{ if } n \geq 9 \text{ , and} \\ &= \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + \frac{\omega_n (n-1)(n-4) tr_g(\tilde{A})(x_0)}{2^{n-2}(n-6)\varepsilon^{n-6}} \\ &\quad + (n-4)\omega_{n-1} \Phi(x_0) |\ln \varepsilon| + o(\ln \varepsilon) \text{ if } n = 8. \end{aligned}$$

Similarly,

$$\int_M P_g \tilde{u}_\varepsilon \tilde{u}_\varepsilon \, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + \frac{4\omega_n (n-1)(n-4) tr_g(\tilde{A})(x_0)}{2^n (n-6) \varepsilon^{n-6}} + o\left(\frac{1}{\varepsilon^{n-6}}\right) \text{ if } n = 7 \text{ , and}$$

$$= \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + \frac{(n-4)^2 \omega_{n-1} tr_g(\tilde{A})(x_0)}{n} |\ln \varepsilon| + o(|\ln \varepsilon|) \text{ if } n = 6.$$

When $n = 5$, we just find that

$$\int_M P_g \tilde{u}_\varepsilon \tilde{u}_\varepsilon \, dv_g = \frac{n(n-4)(n^2-4)\omega_n}{2^n \varepsilon^{n-4}} + O(1).$$

Concerning J, we find that for $n \geq 5$,

$$\int_M f \tilde{u}_\varepsilon^{2^\sharp} \, dv_g = \frac{\omega_n f(x_0)}{2^n \varepsilon^n} - \frac{\omega_n}{2^{n+1}(n-2)} \frac{\Delta_g f(x_0)}{\varepsilon^{n-2}} + \frac{\omega_n \Theta(x_0)}{2^{n+3}(n-2)(n-4)\varepsilon^{n-4}} + o\left(\frac{1}{\varepsilon^{n-4}}\right).$$

7 Proof of the theorems 2-6

We prove Theorems 2-6 using Theorem 1 and the expansions we got in sections 4 and 5. We let $x_0 \in M$ and consider the paths γ_ε's given by

$$\gamma_\varepsilon(t) = t \frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}.$$

Thanks to Theorem 1, it suffices to prove Theorems 2-6 to show that there exists $\varepsilon > 0$ such that

$$\sup_{t \geq 0} E(\gamma_\varepsilon(t)) < \frac{2}{n K_0^{\frac{n}{4}} (\max_M f)^{\frac{n-4}{4}}} \tag{23}$$

where, in its general form,

$$\begin{aligned} E(\gamma_\varepsilon(t)) &= \frac{t^2}{2\|u_\epsilon\|_{2^\sharp}^2}\int_M P_g u_\epsilon u_\epsilon\, dv_g - \frac{t^{2^\sharp}}{2^\sharp\|u_\epsilon\|_{2^\sharp}^{2^\sharp}}\int_M f u_\epsilon^{2^\sharp}\, dv_g \\ &\quad -\frac{t^{q+1}}{(q+1)\|u_\epsilon\|_{2^\sharp}^{q+1}}\int_M h u_\epsilon^{q+1}\, dv_g \\ &= \frac{t^2}{2}A_\varepsilon - \frac{t^{2^\sharp}}{2^\sharp}B_\varepsilon - \frac{t^{q+1}}{q+1}C_\varepsilon \end{aligned}$$

Thanks to the results of section 4, $\|\tilde u_\varepsilon\|_{2^\sharp} \sim \left(\frac{\omega_n}{2^n}\right)^{\frac{n-4}{2n}} \varepsilon^{-\frac{n-4}{2}}$, and

$$A_\varepsilon \to \frac{1}{K_0}\ ,\ \ B_\varepsilon \to f(x_0)\ ,\ \ C_\varepsilon \to 0$$

Then, it is easily checked that

$$\sup_{t\geq 0} E(\gamma_\varepsilon(t)) = \frac{2}{n}\cdot \frac{A_\varepsilon^{\frac{n}{4}}}{B_\varepsilon^{\frac{n-4}{4}}} - \frac{T_0^{q+1}}{q+1}C_\varepsilon + o(C_\varepsilon) \tag{24}$$

where $T_0 = (K_0 f(x_0))^{-\frac{n-4}{8}}$. We let

$$K = \frac{2^{\frac{n(q+1)}{2^\sharp}} \int_0^{+\infty} \frac{s^{n-1}\, ds}{(1+s^2)^{\frac{(n-4)(q+1)}{2}}}}{(q+1)(K_0 f(x_0))^{\frac{(n-4)(q+1)}{8}} \omega_n^{\frac{q+1}{2^\sharp}}}$$

Using the estimates we got in section 4, and (24), the following expansions hold. We assume first that $q > \frac{n}{n-4}$. Then we get that

$$\begin{aligned} \sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) &= \frac{2}{nK_0^{\frac{n}{4}} f(x_0)^{\frac{n-4}{4}}} \\ &\quad - K\omega_{n-1} h(x_0)\varepsilon^{\frac{n-4}{2}(2^\sharp-1-q)} + o\left(\varepsilon^{\frac{n-4}{2}(2^\sharp-1-q)}\right) \end{aligned}$$

when $n \geq 6$, and if $h(x_0) = 0$, we get that

$$\begin{aligned} \sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) &= \frac{2}{nK_0^{\frac{n}{4}} f(x_0)^{\frac{n-4}{4}}} \\ &\quad + \frac{2^n(n-4)\omega_{n-1}(Tr_g A - 2S_g)(x_0)}{2n^2(n^2-4)\omega_n K_0^{\frac{n}{4}} f(x_0)^{\frac{n-4}{4}}}\varepsilon^2|\ln\varepsilon| \\ &\quad + o\left(\varepsilon^2|\ln\varepsilon|\right) \end{aligned}$$

when $n=6$, and when $n\geq 7$,

$$\sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) = \frac{2}{nK_0^{\frac{n}{4}}f(x_0)^{\frac{n-4}{4}}} + \frac{F(x_0)}{4n(n^2-4)(n-6)K_0^{\frac{n}{4}}f(x_0)^{\frac{n-4}{4}}}\varepsilon^2 + o(\varepsilon^2)$$

where F is as in the introduction. Theorem 2 and the first part of Theorem 3 easily follow from these expansions and Theorem 1, since, under the assumptions we made in these theorems, (23) holds true. Moreover, still when $q>\frac{n}{n-4}$, we find with the estimates of section 4 that if $h(x_0)=F(x_0)=0$, and $n\geq 8$, then

$$\sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) = \frac{2}{nK_0^{\frac{n}{4}}f(x_0)^{\frac{n-4}{4}}} + \frac{K\omega_{n-1}}{2\left((q+1)(n-4)-(n+2)\right)}\left(\Delta_g h(x_0)\right)\varepsilon^{n+2-(q+1)\frac{n-4}{2}} + o\left(\varepsilon^{n+2-(q+1)\frac{n-4}{2}}\right).$$

Thanks to Theorem 1, this implies the second part of Theorem 3. We assume now that $q=\frac{n}{n-4}$. Then, when $n=6$, we get that

$$\sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) = \frac{2}{nK_0^{n/4}f(x_0)^{\frac{n-4}{4}}}\left(1+\frac{2^n(n-4)\omega_{n-1}(Tr_gA-2S_g)(x_0)}{4n(n^2-4)\omega_n}\varepsilon^2|\ln\varepsilon|\right) + o(\varepsilon^2|\ln\varepsilon|)$$

and when $n\geq 7$, we get that

$$\sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) = \frac{2}{nK_0^{\frac{n}{4}}f(x_0)^{\frac{n-4}{4}}} + \frac{G(x_0)}{4n(n^2-4)(n-6)K_0^{\frac{n}{4}}f(x_0)^{\frac{n-4}{4}}}\varepsilon^2 + o(\varepsilon^2)$$

where G is as in the introduction. Theorem 4 easily follows from these expansions and Theorem 1, since, under the assumptions we made in this theorem, (23) holds true. At last we assume that $q < \frac{n}{n-4}$. Then, when $n \geq 8$, we get that

$$\begin{aligned}\sup_{t\geq 0} E\left(t\frac{u_\epsilon}{\|u_\epsilon\|_{2^\sharp}}\right) &= \frac{2}{nK_0^{\frac{n}{4}} f(x_0)^{\frac{n-4}{4}}} \\ &+ \frac{F(x_0)}{4n(n^2-4)(n-6)K_0^{\frac{n}{4}} f(x_0)^{\frac{n-4}{4}}}\varepsilon^2 \\ &- K\omega_{n-1}h(x_0)\varepsilon^{\frac{n-4}{2}(2^\sharp-1-q)} \\ &+ o\left(\varepsilon^{\frac{n-4}{2}(2^\sharp-1-q)}\right).\end{aligned}$$

and Theorems 5 easily follows from this expansion and Theorem 1, since, under the assumptions we made in this theorem, (23) holds true. We are now left with the proof of Theorem 6. We use here the estimates we got in section 5. We let $\tilde{g}$ be a conformal metric to g which satisfies (20), and denote by $\tilde{u}_\varepsilon$ the functions we introduced in section 5 which we consider now with respect to $\tilde{g}$. Assuming that $\Delta_g f(x_0) = 0$, we also have that $\Delta_{\tilde{g}} f(x_0) = 0$. Then, thanks to the estimates of section 5, we get that when $n \geq 9$,

$$\sup_{t\geq 0} E\left(t\frac{\tilde{u}_\varepsilon}{\|\tilde{u}_\varepsilon\|_{2^\sharp}}\right) = \frac{2}{nK_0^{\frac{n}{4}} f(x_0)^{\frac{n-4}{4}}}\left(1 - \frac{\tilde{H}(x_0)}{C(n)}\varepsilon^4 + o(\varepsilon^4)\right)$$

where E is with respect to $\tilde{g}$, $C(n) = 32(n-2)(n-6)(n-8)$, and

$$\tilde{H}(x_0) = \frac{4(n^2-4n-4)}{3(n+2)}|Weyl_{\tilde{g}}(x_0)|^2_{\tilde{g}} + (n-6)(n-8)\frac{\Delta^2_{\tilde{g}} f}{f}(x_0).$$

Writing that $\tilde{g} = \varphi^{4/(n-2)}g$, see [11], we do have that $\varphi(x_0) = 1$, $\nabla\varphi(x_0) = 0$, and

$$\nabla^2\varphi(x_0) = \frac{n-4}{2(n-2)}\left(Ric_g - \frac{S_g}{2(n-1)}g\right)(x_0)$$

Then, since $x_0 \in Maxf$ and $\Delta_g f(x_0) = 0$, we get that

$$\Delta^2_{\tilde{g}} f(x_0) = \Delta^2_g f(x_0) + 2\left(\nabla^2 f, Ric_g\right)_g(x_0)$$

Hence, thanks to the conformal invariance of the Weyl tensor, $\tilde{H}(x_0) > 0$ if and only if $H(x_0) > 0$, where H is as in the introduction. Similarly, when $n = 8$,

$$\sup_{t\geq 0} E\left(t\frac{\tilde{u}_\varepsilon}{\|\tilde{u}_\varepsilon\|_{2^\sharp}}\right) = \frac{1}{4K_0^2 f(x_0)}\left(1 - \frac{2\omega_7|Weyl_{\tilde{g}}(x_0)|^2_{\tilde{g}}}{225\omega_8}\varepsilon^4|\ln\varepsilon| + o(\varepsilon^4|\ln\varepsilon|)\right)$$

Thanks to Theorem 1, and the conformal invariance of the geometric Paneitz-Branson operator, Theorem 6 follows from the above estimates. Under the assumptions we made in this theorem, we indeed do have that (23) with respect to $\tilde{g}$ holds true. Hence, our equation with respect to $\tilde{g}$ has a solution u. Writing that $\tilde{g} = \varphi^{4/(n-4)}g$, the conformal invariance then gives that $u\varphi$ is a solution of our equation with respect to g. This proves Theorem 6.

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