ASYMPTOTIC PROFILE FOR THE SUB-EXTREMALS OF THE SHARP SOBOLEV INEQUALITY ON THE SPHERE

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0 - Introduction and statements of the results

Let (M, g) be a smooth compact Riemannian n-manifold, $n \geq 3$, without boundary. We denote by $H_1^2(M)$ the standard Sobolev space, that is the completion of $C^\infty(M)$ for the norm

$$\|u\|_{H_1^2} = \|\nabla u\|_2 + \|u\|_2$$

where $\|.\|_p$, as in the sequel, is the L^p-norm. It follows from the Sobolev embedding theorem that $H_1^2(M) \subset L^{2^*}(M)$, where $2^* = \frac{2n}{n-2}$ is the critical exponent. This leads to the existence of two constants A and B such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^*}^2 \leq A\|\nabla u\|_2^2 + B\|u\|_2^2 \qquad (I)$$

As usual, we define the best first constant α_2 in (I) by

$$\alpha_2 = \inf \{A \text{ for which there exists } B \text{ such that } (I) \text{ is valid with } A \text{ and } B\}$$

where, by (I) is valid, we mean that (I) holds for any $u \in H_1^2(M)$. It is now well-known, see for instance [17] for an exposition in book form, that $\alpha_2 = K_n$, where K_n is the best constant in the Euclidean Sobolev inequality. Its value has been found independently by Aubin [2] and Talenti [27] :

$$K_n = \frac{4}{n(n-2)} \omega_n^{-\frac{2}{n}}$$

where ω_n denotes the volume of the unit sphere in $\mathbb{R}^{n+1}$.

Since the work of Hebey and Vaugon [19], [20], we know that α_2 is attained in (I). In other words, there exists a constant B such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^*}^2 \leq K_n \|\nabla u\|_2^2 + B\|u\|_2^2 \qquad (I_{opt})$$

This inequality is optimal with respect to the first constant. One can then lower B to its minimum in (I_{opt}), and thus define

$$B_0(g) = \inf \{B \text{ s.t. } (I_{opt}) \text{ is valid with } B\}$$

Clearly, for any $u \in H_1^2(M)$,

$$\|u\|_{2^*}^2 \leq K_n \|\nabla u\|_2^2 + B_0(g)\|u\|_2^2 \qquad (I_{g,OPT})$$

and this inequality is optimal with respect to the first and second constants. Lower and upper-bounds for $B_0(g)$ may be found in [17]. Following usual terminology, we say that $u_0 \in H_1^2(M)$ is an extremal function for $(I_{g,OPT})$ if $u_0 \not\equiv 0$ and

$$\|u_0\|_{2^*}^2 = K_n \|\nabla u_0\|_2^2 + B_0(g)\|u_0\|_2^2$$

Results concerning the existence of extremal functions for $(I_{g,OPT})$ on general compact manifolds are in Djadli-Druet [8]. In particular, it is shown there that $(I_{g,OPT})$ possesses extremal functions if the scalar curvature of g is either nonpositive or constant.

In this paper, we concentrate our attention on the case of the conformal class of the standard unit sphere. We let (S^n, h) be the unit n-sphere of $\mathbb{R}^{n+1}$ with its standard metric h, and we let

$$[h] = \left\{ g = \varphi^{\frac{4}{n-2}} h, \varphi \in C^\infty(M), \varphi > 0 \right\}$$

be the conformal class of h. Given $g \in [h]$ some conformal metric to h, the existence of extremal functions for $(I_{g,OPT})$ has been studied by Hebey [16]. His result, that we recall below, should be regarded as the starting point of our paper. As in all the sequel, S_g denotes the scalar curvature of g.

Theorem 0.1 ([16]) - *Let (S^n, h) be the unit n-sphere. If $n \geq 4$, then for any $g \in [h]$,*

$$B_0(g) = \frac{n-2}{4(n-1)} K_n \max_{S^n} S_g \quad ,$$

and there exist extremal functions for $(I_{g,OPT})$ if and only if, up to a positive constant scale factor, g and h are isometric. If $n = 3$, then for any $g \in [h]$,

$$B_0(g) \leq \frac{1}{8} K_3 \max_{S^3} S_g \quad ,$$

but there now exists $g \in [h]$ for which this inequality is strict. In case of equality, there exist extremal functions for $(I^2_{g,OPT})$ if and only if, up to a positive constant scale factor, g and h are isometric.

Note that, see [17] for an exposition in book form, the extremal functions for $(I_{h,OPT})$ are explicitly known. More precisely, if u is an extremal function for $(I_{h,OPT})$, and for instance $\int_{S^n} u^{2^*}(x)\, dv_h(x) = 1$, then

$$u(x) = \omega_n^{-\frac{1}{2^*}} \left(\beta^2 - 1\right)^{\frac{n-2}{4}} (\beta - (x_0, x))^{1-\frac{n}{2}}$$

where $\beta > 1$ is some real number, $x_0 \in S^n$, and (x_0, x) denotes the scalar product in $\mathbb{R}^{n+1}$.

In what follows we assume that $n \geq 4$ and we let $g = \varphi^{\frac{4}{n-2}} h$ be some metric conformal to h. Given $\alpha < B_0(g)$, we set

$$\lambda_\alpha = \inf_{u \in H_1^2(S^n), u \not\equiv 0} \frac{\|\nabla u\|_2^2 + \alpha K_n^{-1} \|u\|_2^2}{\|u\|_{2^*}^2}$$

It follows from the definition of $B_0(g)$ that $\lambda_\alpha < K_n^{-1}$, while, according to Theorem 0.1,

$$B_0(g) = c_n K_n \max_{S^n} S_g$$

where $c_n = \frac{n-2}{4(n-1)}$. By standard variational technics, the strict inequality $\lambda_\alpha < K_n^{-1}$ leads to the existence of $z_\alpha \in C^\infty(S^n)$, $z_\alpha > 0$, such that

$$\begin{cases} \Delta_g z_\alpha + \alpha K_n^{-1} z_\alpha = \lambda_\alpha z_\alpha^{2^*-1} \\ \int_{S^n} z_\alpha^{2^*} dv_g = 1 \end{cases}$$

where $\Delta_g = -div_g \nabla$. We refer to the z_α's as sub-extremals for the sharp Sobolev inequality $(I_{g,OPT})$. If, up to a positive constant scale factor, g and h are isometric, then, by a result of Gidas and Spruck [10] and Bidaut-Véron and Véron [5], z_α is constant, and hence explicitly known.

Theorem 0.2 ([10], [5]) - *Let (S^n, h) be the unit n-sphere, let $g = \varphi^{\frac{4}{n-2}} h$ be some conformal metric to h such that, up to a positive constant scale factor, g and h are isometric, and let (z_α) be as above. Then $z_\alpha = \left(\alpha K_n^{-1} \lambda_\alpha^{-1}\right)^{\frac{n-2}{4}}$.*

Given $g = \varphi^{\frac{4}{n-2}} h$ as above, and $n \geq 4$, we assume now that, up to any positive constant scale factor, g and h are not isometric. Then, according to Theorem 0.1, $(I_{g,OPT})$ does not possess extremal functions, and one gets from standard elliptic theory that $z_\alpha \rightharpoonup 0$ weakly in $H_1^2(S^n)$ as $\alpha \to B_0(g)$. It thus follows that there exists $x_0 \in S^n$ such that $z_\alpha \to 0$ in $C^2_{loc}(S^n \backslash \{x_0\})$ and $z_\alpha^{2^*} \to \delta_{x_0}$ in the sense of distributions. We study here the asymptotic profile of the z_α's as $\alpha \to B_0(g)$, and answer a question that was asked to us by Hebey. Such studies were initiated by Atkinson-Peletier [1] and Brézis-Peletier [6] in the Euclidean context when considering the equation

$$\Delta u = n(n-2) f(x) u^{2^*-1} \quad \text{in } B\ ,\ u \equiv 0 \text{ on } \partial B$$

where B is the unit ball of $\mathbb{R}^n$, and u and f are radially symmetrical. With arguments from ODE's theory, assuming that $f \equiv 1$, Atkinson and Peletier [1] got that

$$\lim_{\varepsilon \to 0} \varepsilon u_\varepsilon(0)^2 = \frac{4\Gamma(n)}{(n-2)\Gamma(\frac{n}{2})^2}\ ,$$

and that, for all $x \in B \backslash \{0\}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = \frac{\sqrt{n-2}\,\Gamma(\frac{n}{2})}{2\sqrt{\Gamma(n)}} \left(\frac{1}{|x|^{n-2}} - 1 \right)$$

Brézis and Peletier [6] returned to this problem, but with arguments from PDE's theory, and they conjectured that a similar behaviour should occur in the non radial case. This was proved to be true independently by Han [12] and Rey [25]. When f is nonconstant, the problem has been considered by Hebey [15], [18], and also Robert [26] with the addition of a linear term $a(x)u$ in the equation. Similar studies have also been developed on the whole of $\mathbb{R}^n$. See for instance Pan-Wang [24]. Note that a key idea to get information on blow-up rate and location is to use the Pohozaev identity, respectively the Kazdan-Warner identity. This idea goes back to Brézis-Peletier [6] and Schoen [27]. See also Han [12] and Hebey [13].

For $P \in S^n$, and $t \in [1;\infty)$, we let $\Phi_{P,t} : S^n \to S^n$ be the conformal diffeomorphism defined by

$$\Phi_{P,t}(x) = \pi_P^{-1}\left(t\pi_P(x)\right)$$

where π_P is the stereographic projection of north pole P. We then let $|det \ \ d\Phi_{P,t}|$ be defined by

$$\Phi_{P,t}^* h = |det \ \ d\Phi_{P,t}|^{\frac{2}{n}} h$$

We also denote by $G(x_0, x)$ the Green's function at x_0 of $\Delta_g + B_0(g)K_n^{-1}$. More precisely, $G(x_0, x)$ is the only function such that

$$\Delta_g G(x_0, x) + B_0(g)K_n^{-1}G(x_0, x) = \delta_{x_0}$$

in the sense of distributions. See [3] for existence and basic properties of such functions. We set

$$MaxS_g = \left\{ x \in S^n, S_g(x) = \max_{y \in S^n} S_g(y) \right\}$$

Our first result is the following :

Theorem 0.3 - *Let (S^n, h) be the unit n-sphere, $n \geq 4$, let $g = \varphi^{\frac{4}{n-2}} h$ be some conformal metric to h with the property that, up to any positive constant scale factor, g and h are not isometric, and let (z_α) be as above. There exist $x_0 \in MaxS_g$, a sequence $(x_\alpha) \in S^n$, with the property that $x_\alpha \to x_0$ as $\alpha \to B_0(g)$, and a sequence $(t_\alpha) \in \mathbb{R}$, with the property that $t_\alpha \to +\infty$ as $\alpha \to B_0(g)$, such that*

$$t_\alpha^{\frac{n}{2}-1} z_\alpha(x) \quad \to \quad \varphi(x_0)\omega_n^{-\frac{1}{2^*}} 2^{n-2}\omega_{n-1}(n-2)G(x_0, x) \qquad in \ \ C^2_{loc}\left(S^n \backslash \{x_0\}\right)$$

and

$$|det\ \ d\Phi_{x_\alpha,t_\alpha}|^{\frac{1}{2^*}} z_\alpha \circ \Phi_{x_\alpha,t_\alpha} \quad \to \quad \omega_n^{-\frac{1}{2^*}} \varphi(x_0)^{-1} \qquad in \quad C^2_{loc}\left(S^n \backslash \{-x_0\}\right)$$

as $\alpha \to B_0(g)$. *Moreover,* $(|det\ \ d\Phi_{x_\alpha,t_\alpha}|^{\frac{1}{2^*}} z_\alpha \circ \Phi_{x_\alpha,t_\alpha})$ *is uniformly bounded in* $L^\infty(S^n)$.

The first part of this theorem provides us with a rather standard description of asymptotic profiles. The second part is more specific to the sphere. Our next result, Theorem 0.4 below, gives informations on the sequences (t_α) and (x_α) involved in Theorem 0.3. We let here C_n be the dimensional constant defined by

$$C_4 = \frac{1}{24}\omega_4^{-\frac{1}{2}}\ ,\ C_5 = \frac{3}{10\pi}\omega_5^{-\frac{3}{5}}\ ,$$
$$C_6 = \frac{2}{15}\omega_6^{-\frac{1}{3}}\ ,\ C_n = \frac{4}{n(n-1)(n-6)}\omega_n^{-\frac{2}{n}}$$

if $n \geq 7$. We then have the following :

Theorem 0.4 - *Let* (S^n, h) *be the unit n-sphere,* $n \geq 4$, *and* $g = \varphi^{\frac{4}{n-2}} h$ *be some conformal metric to h with the property that, up to any positive constant scale factor, g and h are not isometric. We assume that for any* $x \in MaxS_g$, $\nabla^2 S_g(x)$ *is nondegenerate. For* t_α *as in Theorem 0.3, one then has the following :*

(1) If $n = 4$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) \ln t_\alpha$$
$$= C_4 \varphi(x_0)^{-1} \int_{S^4} (S_g(x_0) - S_g(x))\, \varphi(x)^{-1} \left(\frac{1-(x_0,x)}{2}\right)^{-1} G(x_0,x) dv_g$$

(2) If $n = 5$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha)\, t_\alpha$$
$$= C_5 \varphi(x_0)^{-\frac{1}{3}} \int_{S^5} (S_g(x_0) - S_g(x))\, \varphi(x)^{-1} \left(\frac{1-(x_0,x)}{2}\right)^{-\frac{3}{2}} G(x_0,x) dv_g$$

(3) If $n = 6$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) \frac{t_\alpha^2}{\ln t_\alpha} = C_6 \varphi(x_0) \Delta_g S_g(x_0)$$

(4) If $n \geq 7$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha)\, t_\alpha^2 = C_n \varphi(x_0)^{2^*-2} \Delta_g S_g(x_0)$$

Moreover, at least when $n \geq 7$, and for x_α and x_0 as in Theorem 0.3, one may take $x_\alpha = x_0$ for any α.

By a well-known result of Obata [23], if $g \in [h]$ and S_g is constant, then, up to a positive constant scale factor, g and h are isometric. Given $g \in [h]$, the limits involved in points (1) and (2) of Theorem 0.4 are then nonnegative, and null if and only if, up to a positive constant scale factor, g and h are isometric. Under the assumption that $\nabla^2 S_g(x)$ is definite negative for any $x \in MaxS_g$, the limits involved in (3) and (4) are also positive.

In our last result we restrict ourselves to a particular case where we can drop the assumption of nondegeneracy we made on S_g in Theorem 0.4, and where we get a complete description of the asymptotic profile of the sub-extremals, hence of the sequences (x_α) and (t_α) involved in Theorem 0.3. More precisely, we assume now that φ is radially symmetrical with respect to some point $x_0 \in S^n$ and that S_g, which is therefore also radially symmetrical with respect to x_0, achieves its maximum at x_0. Under such assumptions, one easily checks that we can choose the sub-extremals z_α to be radially symmetrical with respect to x_0, and to blow-up at x_0 (see section 4 below for more details on such an assertion). We let $p \in I\!N^*$ be such that

$$\Delta_g^i S_g(x_0) = 0 \text{ for any } 1 \leq i < p \ , \quad \text{and } \ \Delta_g^p S_g(x_0) \neq 0$$

with the convention that $p = +\infty$ if $\Delta_g^i S_g(x_0) = 0$ for any $i \in I\!N^*$, where

$$\Delta_g^i = \Delta_g \circ \ldots \circ \Delta_g \qquad (i \text{ times})$$

We let also

$$D_1(n,p) = \frac{(p+1)4^p}{n(n-1)(2p)!}\omega_n^{-\frac{2}{n}} \left(\Pi_{k=0\ldots p-1} \frac{2k+1}{n-6-2k} \right)$$

$$D_2(n) = \frac{2^{n-5}(n-2)\omega_n^{-\frac{2}{n}}}{(n-4)!n(n-1)} \left(\prod_{k=0}^{\frac{n-6}{2}} \frac{2k+1}{n+2k} \right) \left(\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-2}} dr \right)^{-1}$$

$$D_3(n) = \frac{(n-2)^2}{8n(n-1)}\omega_n^{-\frac{2}{n}}\omega_{n-1} \left(\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-2}} dr \right)^{-1}$$

Our last result is then as follows :

Theorem 0.5 - *Let (S^n, h) be the unit n-sphere, $n \geq 4$, and let $g = \varphi^{\frac{4}{n-2}} h$ be some conformal metric to h with the property that, up to any positive constant scale factor, g and h are not isometric. We assume that φ is radially symmetrical with respect to some $x_0 \in S^n$, that S_g achieves its maximum at x_0, and we choose the sub-extremals z_α to be radially symmetrical with respect to x_0, and to blow-up at x_0. Then Theorem 0.3 holds for z_α with $x_\alpha = x_0$ for every α. Moreover, t_α verifies :*

(1) If $n = 4$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) \ln t_\alpha = \frac{1}{24} \omega_4^{-\frac{1}{2}} \varphi(x_0)^{-1} \times \int_{S^4} (S_g(x_0) - S_g(x)) \varphi(x)^{-1} \left(\frac{1 - (x_0, x)}{2} \right)^{-1} G(x_0, x) dv_g$$

(2) If $n = 5$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) t_\alpha = \frac{3}{10\pi} \omega_5^{-\frac{3}{5}} \varphi(x_0)^{-\frac{1}{3}} \times \int_{S^5} (S_g(x_0) - S_g(x)) \varphi(x)^{-1} \left(\frac{1 - (x_0, x)}{2} \right)^{-\frac{3}{2}} G(x_0, x) dv_g$$

(3) If $n \geq 6$, *and*

(3a) $2p < n - 4$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) t_\alpha^{2p} = -D_1(n, p) \varphi(x_0)^{p(2^* - 2)} (-\Delta_g)^p S_g(x_0)$$

(3b) $2p = n - 4$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) \frac{t_\alpha^{n-4}}{\ln t_\alpha} = -D_2(n) \varphi(x_0)^{\frac{2(n-4)}{n-2}} (-\Delta_g)^{\frac{n}{2} - 2} S_g(x_0)$$

(3c) $2p > n - 4$,

$$\lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) t_\alpha^{n-4} = D_3(n) \varphi(x_0)^{\frac{n-6}{n-2}} \times \int_{S^n} (S_g(x_0) - S_g(x)) \varphi(x)^{-1} \left(\frac{1 - (x_0, x)}{2} \right)^{1 - \frac{n}{2}} G(x_0, x) dv_g$$

where p, $D_1(n, p)$, $D_2(n)$, and $D_3(n)$ are as above.

As in Theorem 0.4, the limits in Theorem 0.5 are always positive (since S_g is nonconstant by Obata's theorem [23]). Clearly, in the radial case, Theorem 0.5 provides us with a complete description of the asymptotic profile of the sub-extremals.

1 - Proof of Theorem 0.3

As a starting point, we list some useful formulae regarding the $\Phi_{P,t}$'s introduced above. Let $P \in S^n$ and $t \geq 1$. As easily seen, see for instance [17], p.130-131, one has that

$$|det\ \ d\Phi_{P,t}|^{\frac{2}{n}}(x) = 4t^2\left[\left(1+t^2\right)+\left(t^2-1\right)(P,x)\right]^{-2} \tag{1.1}$$

where (P,x) denotes the scalar product in $I\!R^{n+1}$, as in the rest of this paper. As $t \to +\infty$, $\Phi_{P,t}(x) \to P$ for all $x \neq -P$. This is easily seen on the following: for any $x \in S^n$,

$$(\Phi_{P,t}(x), P) = 1 - 2\frac{1-(x,P)}{t^2\left(1+(x,P)\right)+1-(x,P)} \tag{1.2}$$

Given $Q \in S^n$, we let π_Q be the stereographic projection of north pole Q. Easy computations lead to the following : for any $x \in I\!R^n$,

$$\left(P, \pi_{-P}^{-1}(x)\right) = \frac{1-|x|^2}{1+|x|^2} \tag{1.3}$$

and

$$|det\ \ d\Phi_{P,t}|^{\frac{2}{n}}\left(\pi_{-P}^{-1}(x)\right) = t^2\left(1+|x|^2\right)^2\left(t^2+|x|^2\right)^{-2} \tag{1.4}$$

At last, for any $f \in C^0\left(S^n\right)$ and any $x \in I\!R^n$, we have

$$f\ \circ\ \Phi_{P,t}\ \circ\ \pi_{-P}^{-1}(x) = f\ \circ\ \pi_{-P}^{-1}\left(\frac{x}{t}\right) \tag{1.5}$$

Now, we go on with the proof of Theorem 0.3.

1.1 - The concentration phenomenon

We let z_α be as in the introduction, $\alpha < B_0(g)$. As already mentioned, $z_\alpha \rightharpoonup 0$ weakly in $H_1^2(S^n)$. Another easy claim is that

$$\lim_{\alpha \to B_0(g)} \lambda_\alpha = K_n^{-1}$$

We list in this subsection results on the concentration phenomenon that the z_α's develop. These results have already been proved in [8] (see also Druet [9]). We therefore omit giving too many details.

As a starting claim, there exists a unique point x_0 in S^n such that, after passing to a subsequence,

$$\lim_{\alpha\to B_0(g)}\int_{B(x_0,\delta)} z_\alpha^{2^*}\,dv_g = 1 \quad \text{for any } \delta>0$$

and

$$\lim_{\alpha\to B_0(g)} z_\alpha = 0 \qquad \text{in } C^2_{loc}\left(S^n\backslash\left\{x_0\right\}\right) \tag{1.6}$$

The L^{2^*}-mass of (z_α) therefore concentrates around x_0. We set $u_\alpha = z_\alpha\varphi$. As easily checked, u_α is such that

$$\begin{aligned} &L_h u_\alpha + \left(\alpha K_n^{-1} - c_n S_g\right)\varphi^{2^*-2}u_\alpha = \lambda_\alpha u_\alpha^{2^*-1} \\ &\int_{S^n} u_\alpha^{2^*}\,dv_h = 1 \end{aligned} \tag{E_α}$$

where $L_h = \Delta_h + \frac{n(n-2)}{4}$ is the conformal Laplacian for the metric h. We let also $x_\alpha\in S^n$ be a point where u_α is maximum. Clearly, $u_\alpha(x_\alpha)\to+\infty$ and $x_\alpha\to x_0$ as $\alpha\to B_0(g)$. We let

$$t_\alpha = \omega_n^{\frac{1}{n}} u_\alpha(x_\alpha)^{\frac{2}{n-2}} \tag{1.7}$$

and set

$$v_\alpha = u_\alpha\circ\Phi_\alpha |det\ \ d\Phi_\alpha|^{\frac{1}{2^*}}$$

where $\Phi_\alpha = \Phi_{x_\alpha,t_\alpha}$. As easily checked,

$$\begin{aligned} &L_h v_\alpha + \left(\alpha K_n^{-1} - c_n S_g\circ\Phi_\alpha\right)\left(\varphi\circ\Phi_\alpha\right)^{2^*-2}|det\ \ d\Phi_\alpha|^{\frac{2}{n}} v_\alpha = \lambda_\alpha v_\alpha^{2^*-1} \\ &\int_{S^n} v_\alpha^{2^*}\,dv_h = 1 \end{aligned} \tag{F_α}$$

For π_{-x_α} the stereographic projection of north pole $-x_\alpha$, we set

$$\tilde S_\alpha = S_g\circ\pi^{-1}_{-x_\alpha}\ ,\quad \tilde\varphi_\alpha = \varphi\circ\pi^{-1}_{-x_\alpha}\ ,\quad \tilde v_\alpha = v_\alpha\circ\pi^{-1}_{-x_\alpha}$$

Since

$$\left(\pi^{-1}_{-x_\alpha}\right)^* h = \psi^{\frac{4}{n-2}}\xi \quad\text{and}\quad \psi(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n}{2}-1}$$

where ξ is the Euclidean metric, we get with (1.4) and (1.5) that

$$\begin{aligned} \Delta_\xi\left(\tilde v_\alpha\psi\right) = \lambda_\alpha\left(\tilde v_\alpha\psi\right)^{2^*-1} \\ + 4\left(c_n\tilde S_\alpha\left(\frac{x}{t_\alpha}\right) - \alpha K_n^{-1}\right)\tilde\varphi_\alpha\left(\frac{x}{t_\alpha}\right)^{2^*-2} t_\alpha^{-2}\left(1+\frac{|x|^2}{t_\alpha^2}\right)^{-2}\tilde v_\alpha\psi \end{aligned}$$

Moreover, one easily checks with (1.7) and (1.4) that

$$\|\tilde{v}_\alpha \psi\|_\infty = \tilde{v}_\alpha(0)\psi(0) = 2^{\frac{n}{2}-1}\omega_n^{-\frac{1}{2^*}}$$

Independently, it is clear that $(\tilde{v}_\alpha \psi)$ is bounded in $H_1^2(I\!R^n)$ and in $L^\infty(I\!R^n)$. By standard elliptic theory (see [11], theorem 8.17), $(\tilde{v}_\alpha \psi)$ is then uniformly continuous. It hence follows from Ascoli's theorem that $\tilde{v}_\alpha \psi \to \tilde{v}$ in $C^0_{loc}(I\!R^n)$ as $\alpha \to B_0(g)$, where

$$\|\tilde{v}\|_\infty = \tilde{v}(0) = 2^{\frac{n}{2}-1}\omega_n^{-\frac{1}{2^*}}$$

and $\tilde{v}$ is such that

$$\Delta_\xi \tilde{v} = K_n^{-1}\tilde{v}^{2^*-1}$$

By a well known result of Caffarelli, Gidas and Spruck [7],

$$\tilde{v} = \omega_n^{-\frac{1}{2^*}}\psi(x)$$

Thus, up to standard elliptic theory,

$$\lim_{\alpha \to B_0(g)} \tilde{v}_\alpha = \omega_n^{-\frac{1}{2^*}} \quad \text{in } C^2_{loc}(I\!R^n) \tag{1.8}$$

This convergence result gives informations on the speed of concentration of the L^{2^*}-norm of (z_α). Indeed, for any $R > 0$,

$$\begin{aligned}\int_{B\left(x_\alpha, Rt_\alpha^{-1}\right)} z_\alpha^{2^*}\,dv_g &= \int_{B\left(x_\alpha, Rt_\alpha^{-1}\right)} u_\alpha^{2^*}\,dv_h \\ &= \int_{B(x_\alpha, \pi-\varepsilon'(R))} v_\alpha^{2^*}\,dv_h \qquad \text{(see (1.3))} \\ &\to 1 - \varepsilon(R) \quad \text{as } \alpha \to B_0(g) \quad \text{(by (1.8))}\end{aligned}$$

where $\varepsilon(R)$, $\varepsilon'(R)$ go to 0 as $R \to +\infty$. In other words,

$$\lim_{R \to +\infty} \lim_{\alpha \to B_0(g)} \int_{B\left(x_\alpha, Rt_\alpha^{-1}\right)} z_\alpha^{2^*}\,dv_g = 1 \tag{1.9}$$

As it was shown in [8], see also [9], this integral estimate leads to pointwise estimates: there exists some positive constant C such that

$$d_g(x_\alpha, x)^{\frac{n}{2}-1} z_\alpha(x) \le C \quad \text{for any } \alpha \text{ , and any } x \in S^n \tag{1.10}$$

and one also has that

$$\lim_{R\to+\infty}\lim_{\alpha\to B_0(g)}\sup_{x\in S^n\backslash B\left(x_\alpha,Rt_\alpha^{-1}\right)} d_g(x_\alpha,x)^{\frac{n}{2}-1}z_\alpha(x)=0 \tag{1.11}$$

We refer the reader to [8] for details on these assertions. Moreover, it was also proved there that in such a situation, if there are no extremal functions for $(I_{g,OPT})$ (which is the case here), then

$$B_0(g)\le c_nK_nS_g(x_0)$$

where x_0 is the point of concentration of (z_α). Thus $x_0\in MaxS_g$. Together with (1.8), this proves the second part of Theorem 0.3 :

$$\lim_{\alpha\to B_0(g)} z_\alpha\circ\Phi_\alpha|det\ \ d\Phi_\alpha|^{\frac{1}{2^*}}=\varphi(x_0)^{-1}\omega_n^{-\frac{1}{2^*}}\qquad\text{in }\ C^2_{loc}\left(S^n\backslash\left\{-x_0\right\}\right)$$

We provide in the next subsection a stronger pointwise estimate than (1.10) and (1.11). This will allow us to conclude the proof of Theorem 0.3.

1.2 - A fundamental estimate

We prove here the following estimate : there exists $C>0$ such that for any $\alpha<B_0(g)$, and any $x\in S^n$,

$$d_g(x,x_\alpha)^{n-2}\,z_\alpha(x_\alpha)\,z_\alpha(x)\le C \tag{1.12}$$

Similar estimates are in Han [12], Hebey-Vaugon [19], Li [22], Schoen-Zhang [28]. We divide the proof of (1.12) into two steps. We first claim that for any $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$t_\alpha^{\frac{n}{2}-1-\varepsilon}d_g(x_\alpha,x)^{n-2-\varepsilon}z_\alpha(x)\le C(\varepsilon) \tag{1.13}$$

As a remark, note that (1.13) is true on any ball $B(x_\alpha,Rt_\alpha^{-1})$ with $R>0$ (see (1.8)). We therefore just have to prove (1.13) on $S^n\backslash B(x_\alpha,Rt_\alpha^{-1})$ for some $R>0$ (to be chosen later). We let L_α be the operator given by:

$$L_\alpha u=\Delta_g u+\left(\alpha K_n^{-1}-\lambda_\alpha z_\alpha^{2^*-2}\right)u$$

Since $L_\alpha z_\alpha=0$ and $z_\alpha>0$, the maximum principle holds for L_α (see [4]). Set

$$\theta_\alpha(x)=\frac{C_\alpha}{\varphi(x)}\left(1-(x_\alpha,x)\right)^{\varepsilon+1-\frac{n}{2}}$$

where $g = \varphi^{\frac{4}{n-2}} h$. We claim that for R sufficiently large,

$$L_\alpha \theta_\alpha \geq 0$$

and that choosing suitably C_α,

$$\theta_\alpha \geq z_\alpha \quad \text{on } \partial B(x_\alpha, R t_\alpha^{-1})$$

We first prove that $L_\alpha \theta_\alpha \geq 0$ in $S^n \backslash B(x_\alpha, R t_\alpha^{-1})$. We have

$$L_g \theta_\alpha = \varphi^{1-2^*} L_h(\theta_\alpha \varphi)$$

and

$$\Delta_h(\theta_\alpha \varphi) = C_\alpha \Delta_h \left((1 - (x_\alpha, x))^{\varepsilon + 1 - \frac{n}{2}} \right)$$

Hence,

$$\frac{1}{C_\alpha} \Delta_h(\theta_\alpha \varphi) = \\ - \frac{1}{\left(1 - (x_\alpha, x)^2\right)^{\frac{n-1}{2}}} \partial_r \left(\left(1 - (x_\alpha, x)^2\right)^{\frac{n-1}{2}} \partial_r \left(1 - (x_\alpha, x)\right)^{\varepsilon + 1 - \frac{n}{2}} \right)$$

Easy computations then lead to

$$\begin{aligned} &(1 - (x_\alpha, x)) \frac{L_\alpha \theta_\alpha}{\theta_\alpha} \\ &= (1 - (x_\alpha, x)) \left(\alpha K_n^{-1} - c_n S_g \right) - \lambda_\alpha \left(1 - (x_\alpha, x)\right) z_\alpha^{2^* - 2} \\ &\quad + \varepsilon \varphi^{2 - 2^*} \left(n - 1 - \varepsilon - (1 + \varepsilon)(x_\alpha, x)\right) \\ &\geq (1 - (x_\alpha, x)) K_n^{-1} \left(\alpha - B_0(g)\right) - C \lambda_\alpha d_h(x_\alpha, x)^2 z_\alpha^{\frac{4}{n-2}} \\ &\quad + (n - 2 - 2\varepsilon) \varepsilon \varphi^{2 - 2^*} \end{aligned}$$

By (1.11),

$$d_h(x_\alpha, x)^2 z_\alpha^{\frac{4}{n-2}} \leq \varepsilon(R)$$

when $d_h(x_\alpha, x) \geq R t_\alpha^{-1}$, and $\varepsilon(R) \to 0$ as $R \to \infty$. Therefore,

$$(1 - (x_\alpha, x)) \frac{L_\alpha \theta_\alpha}{\theta_\alpha} \geq \\ (1 - (x_\alpha, x)) K_n^{-1} \left(\alpha - B_0(g)\right) - C \varepsilon(R) + (n - 2 - 2\varepsilon) \varepsilon \varphi^{2 - 2^*}$$

and the RHS of this inequality is positive as $\alpha \to B_0(g)$ if we choose R sufficiently large. This proves that $L_\alpha \theta_\alpha \geq 0$ for all $x \in S^n \backslash B(x_\alpha, Rt_\alpha^{-1})$. Let $x \in S^n$ be such that $d_h(x_\alpha, x) = Rt_\alpha^{-1}$. We easily get that

$$\theta_\alpha(x) \geq CC_\alpha t_\alpha^{(n-2-2\varepsilon)}$$

and since $z_\alpha \leq Ct_\alpha^{\frac{n}{2}-1}$, it follows that

$$z_\alpha \leq \theta_\alpha \quad \text{on } \partial B(x_\alpha, Rt_\alpha^{-1})$$

if we take $C_\alpha = Ct_\alpha^{1+2\varepsilon-\frac{n}{2}}$ for some $C > 0$ independent of α. According to the maximum principle, this leads to

$$z_\alpha \leq \theta_\alpha \quad \text{on } S^n \backslash B(x_\alpha, Rt_\alpha^{-1})$$

and hence, (1.13) is proved.

We now prove (1.12). We follow here [26] and we refer the reader to this reference for more details. Let G_α be the Green's function for the operator $\Delta_g + \alpha K_n^{-1}$, the only function $G_\alpha : S^n \times S^n \backslash \{(x,x)/x \in S^n\} \to I\!R$ which is such that:

$$\Delta_{g,y} G_\alpha(x,y) + \alpha K_n^{-1} G_\alpha(x,y) = \delta_x$$

By standard elliptic theory and standard properties of the Green's function, there exists some $C > 0$, independent of α, such that for any $x \neq y$ and any $\alpha \to B_0(g)$,

$$|G_\alpha(x,y)| \leq C d_g(x,y)^{2-n}$$

We first prove (1.12) on any compact subset K of $S^n \backslash \{-x_0\}$. Let (y_α) be a sequence of points in such a K. Up to a subsequence, we may assume that $y_\alpha \to y_0$ as $\alpha \to B_0(g)$. Of course, $y_0 \neq -x_0$. Since

$$\Delta_g z_\alpha + \alpha K_n^{-1} z_\alpha = \lambda_\alpha z_\alpha^{2^*-1}$$

we write that

$$\begin{aligned} z_\alpha(y_\alpha) &= \lambda_\alpha \int_{S^n} G_\alpha(y_\alpha, x) z_\alpha^{2^*-1} dv_g \\ &\leq C \int_{S^n} G_\alpha(y_\alpha, x) u_\alpha^{2^*-1} dv_h \end{aligned}$$

for some $C > 0$ independent of α. Through the stereographic projection π_{-x_α} of north pole $-x_\alpha$, this gives

$$z_\alpha(y_\alpha) \le C \int_{I\!R^n} G_\alpha(y_\alpha, \pi_{-x_\alpha}^{-1}(x)) \left(u_\alpha \circ \pi_{-x_\alpha}^{-1}\right)^{2^*-1} \left(1+|x|^2\right)^{-n} dx$$

Since $y_0 \ne -x_0$, and by standard properties of the Green's function, one easily checks that

$$\begin{aligned} &G_\alpha(y_\alpha, \pi_{-x_\alpha}^{-1}(x)) \le C(R)|\tilde{y}_\alpha - x|^{2-n} && \text{for any } x \in B(0,R) \\ &G_\alpha(y_\alpha, \pi_{-x_\alpha}^{-1}(x)) \le C(R) && \text{for any } x \in I\!R^n \backslash B(0,R) \end{aligned}$$

where $\tilde{y}_\alpha = \pi_{-x_\alpha}(y_\alpha)$, R is some positive real large enough and $C(R)$ depends only on R. Independently, by (1.7) and (1.13), one has that for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\left(u_\alpha \circ \pi_{-x_\alpha}^{-1}\right)(x) \le C(\varepsilon) t_\alpha^{\frac{n}{2}-1} \left(\frac{1+t_\alpha^2|x|^2}{1+|x|^2}\right)^{\frac{\varepsilon+2-n}{2}}$$

for any $x \in I\!R^n$. The above inequality then becomes for R large enough

$$\begin{aligned} &z_\alpha(y_\alpha) \le C(\varepsilon, R) t_\alpha^{\frac{n}{2}+1} \\ &\times \int_{B(0,R)} |\tilde{y}_\alpha - x|^{2-n} \left(1+t_\alpha^2|x|^2\right)^{\frac{(n+2)}{2(n-2)}\varepsilon - 1 - \frac{n}{2}} \left(1+|x|^2\right)^{\frac{2-n}{2} - \frac{(n+2)}{2(n-2)}\varepsilon} dx \\ &\quad + C(\varepsilon, R) \int_{I\!R^n \backslash B(0,R)} \left(1+t_\alpha^2|x|^2\right)^{\frac{(n+2)}{2(n-2)}\varepsilon - 1 - \frac{n}{2}} \left(1+|x|^2\right)^{\frac{2-n}{2} - \frac{(n+2)}{2(n-2)}\varepsilon} dx \end{aligned}$$

where $C(\varepsilon, R)$ depends only on R and ε. We take $\varepsilon = \frac{n-2}{n+2}$. Setting $y = t_\alpha x$, we get

$$\begin{aligned} z_\alpha(y_\alpha) \le\ & C t_\alpha^{\frac{n}{2}-1} \int_{I\!R^n} |t_\alpha \tilde{y}_\alpha - y|^{2-n} \left(1+|y|^2\right)^{-\frac{n+1}{2}} \left(1+t_\alpha^{-2}|y|^2\right)^{\frac{1-n}{2}} dy \\ & + o\left(t_\alpha^{1-\frac{n}{2}}\right) \end{aligned}$$

The proof splits now into the study of two cases. In the first one, we assume that, after passing to a subsequence, $\lim_{\alpha \to B_0(g)} t_\alpha \tilde{y}_\alpha = \tilde{y}$. Then, by Lebesgue's theorem, we get that for ε small enough,

$$z_\alpha(y_\alpha) \le C t_\alpha^{\frac{n}{2}-1} \int_{I\!R^n} |\tilde{y} - y|^{2-n} \left(1+|y|^2\right)^{-\frac{n+1}{2}} dy$$

Since $d_g(x_\alpha, y_\alpha) \leq C|\tilde{y}_\alpha| \leq Ct_\alpha^{-1}$ and $z_\alpha(x_\alpha) \leq Ct_\alpha^{\frac{n}{2}-1}$, it follows that

$$d_g(x_\alpha, y_\alpha)^{n-2} z_\alpha(x_\alpha) z_\alpha(y_\alpha) \leq C$$

In the second case, we assume that $t_\alpha|\tilde{y}_\alpha| \to +\infty$ as $\alpha \to B_0(g)$. Then

$$\begin{aligned}
z_\alpha(y_\alpha) &\leq Ct_\alpha^{1-\frac{n}{2}}|\tilde{y}_\alpha|^{2-n} \int_{I\!R^n \setminus B\left(t_\alpha \tilde{y}_\alpha, \frac{t_\alpha|\tilde{y}_\alpha|}{2}\right)} \left(1+|y|^2\right)^{-\frac{n+1}{2}} dy + o\left(t_\alpha^{1-\frac{n}{2}}\right) \\
&\quad + Ct_\alpha^{-2-\frac{n}{2}}|\tilde{y}_\alpha|^{-1-n} \int_{B\left(t_\alpha \tilde{y}_\alpha, \frac{t_\alpha|\tilde{y}_\alpha|}{2}\right)} |y - t_\alpha \tilde{y}_\alpha|^{2-n} dy \\
&\leq Ct_\alpha^{1-\frac{n}{2}}|\tilde{y}_\alpha|^{2-n}
\end{aligned}$$

This concludes the proof of (2.12) on any compact subset of $S^n \backslash \{-x_0\}$. Since $t_\alpha^{\frac{n}{2}-1} z_\alpha$ verifies

$$\Delta_g \left(t_\alpha^{\frac{n}{2}-1} z_\alpha\right) + \alpha K_n^{-1} \left(t_\alpha^{\frac{n}{2}-1} z_\alpha\right) = \lambda_\alpha t_\alpha^{-2} \left(t_\alpha^{\frac{n}{2}-1} z_\alpha\right)^{2^*-1}$$

standard Harnack's inequalities conclude the proof of (2.12).

1.3 - Proof of Theorem 0.3 (continued)

As one easily checks, (1.12), together with (1.1), (1.2) and (1.7), gives that

$$\begin{aligned}
&z_\alpha\left(\Phi_\alpha(x)\right)|det \ d\Phi_\alpha|(x)^{\frac{1}{2^*}} \\
&= (2t_\alpha)^{\frac{n}{2}-1} z_\alpha\left(\Phi_\alpha(x)\right)\left[(1+t_\alpha^2) + (t_\alpha^2 - 1)(x, x_\alpha)\right]^{1-\frac{n}{2}} \\
&= \left(\frac{t_\alpha^2+1}{2t_\alpha}\right)^{\frac{n}{2}-1} z_\alpha\left(\Phi_\alpha(x)\right)\left[1 - \frac{t_\alpha^2-1}{t_\alpha^2+1}\left(x_\alpha, \Phi_\alpha(x)\right)\right]^{\frac{n}{2}-1} \\
&\leq Ct_\alpha^{\frac{n}{2}-1} z_\alpha\left(\Phi_\alpha(x)\right)\left[1 - (x_\alpha, \Phi_\alpha(x))\right]^{\frac{n}{2}-1} + Ct_\alpha^{1-\frac{n}{2}} z_\alpha\left(\Phi_\alpha(x)\right) \\
&\leq C
\end{aligned}$$

This proves the last part of Theorem 0.3. Moreover, it easily follows from standard elliptic theory that

$$\lim_{\alpha \to B_0(g)} t_\alpha^{\frac{n}{2}-1} z_\alpha = \lambda G(x_0, x) \quad \text{in } C_{loc}^2\left(S^n \backslash \{x_0\}\right)$$

To compute λ, we just pick a point $x \neq x_0 \in S^n$ and write by the Green's formula that

$$\begin{aligned}
& t_\alpha^{\frac{n}{2}-1} z_\alpha(x) \\
& = \lambda_\alpha t_\alpha^{\frac{n}{2}-1} \int_{S^n} G_\alpha(x,y) z_\alpha(y)^{2^*-1} dv_g(y) \\
& = \lambda_\alpha t_\alpha^{\frac{n}{2}-1} \int_{S^n} G_\alpha(x,y) u_\alpha(y)^{2^*-1} \varphi(y) dv_h(y) \\
& = \lambda_\alpha t_\alpha^{\frac{n}{2}-1} \int_{S^n} G_\alpha(x,\Phi_\alpha(y)) v_\alpha(y)^{2^*-1} \varphi \circ \Phi_\alpha(y) |det \;\; d\Phi_\alpha|^{\frac{1}{2^*}} dv_h(y) \\
& = 2^{\frac{n}{2}-1} \lambda_\alpha \frac{t_\alpha^{n-2}}{(t_\alpha^2+1)^{\frac{n}{2}-1}} \\
& \quad \times \int_{S^n} G_\alpha(x,\Phi_\alpha(y)) v_\alpha(y)^{2^*-1} \varphi \circ \Phi_\alpha(y) \left[1 + \frac{t_\alpha^2-1}{t_\alpha^2+1}(x,x_\alpha) \right]^{1-\frac{n}{2}} dv_h(y)
\end{aligned}$$

Since (v_α) is bounded (see section 1.1) and converges pointwise to $\omega_n^{-\frac{1}{2^*}}$, and $\lambda_\alpha \to K_n^{-1}$, we obtain by Lebesgue's theorem that

$$\begin{aligned}
& \lim_{\alpha \to B_0(g)} t_\alpha^{\frac{n}{2}-1} z_\alpha(x) \\
& = 2^{\frac{n}{2}-1} K_n^{-1} G(x_0,x) \varphi(x_0) \omega_n^{-1+\frac{1}{2^*}} \int_{S^n} [1+(x,x_0)]^{1-\frac{n}{2}} dv_h \\
& = 2^n K_n^{-1} G(x_0,x) \varphi(x_0) \omega_n^{-1+\frac{1}{2^*}} \int_{\mathbb{R}^n} \left(1+|y|^2\right)^{-1-\frac{n}{2}} dy \\
& = \varphi(x_0) \omega_n^{-\frac{1}{2^*}} 2^{n-2} \omega_{n-1} (n-2) G(x_0,x)
\end{aligned}$$

This ends the proof of the theorem.

2 - Proof of Theorem 0.4

We begin by setting up some notations we use in the sequel. For (x_α) and (t_α) given by Theorem 0.3, we let

$$\Phi_\alpha = \Phi_{x_\alpha, t_\alpha}$$

Then, for π_{-x_α} the stereographic projection of north pole $-x_\alpha$, we let

$$\tilde{S}_\alpha = S_g \circ \pi_{-x_\alpha}^{-1}, \quad \tilde{\varphi}_\alpha = \varphi \circ \pi_{-x_\alpha}^{-1}, \quad \tilde{v}_\alpha = v_\alpha \circ \pi_{-x_\alpha}^{-1}$$

and, for π_{-x_0} the stereographic projection of north pole $-x_0$, we let

$$\tilde{S}_g = S_g \circ \pi_{-x_0}^{-1}, \quad \tilde{\varphi} = \varphi \circ \pi_{-x_0}^{-1}, \quad \tilde{G}(x_0, .) = G\left(x_0, \pi_{-x_0}^{-1}(.)\right)$$

A main tool in the proof of Theorem 0.4 is the Kazdan-Warner identity [21]: if f and u in $C^\infty(S^n)$ verify

$$L_h u = f u^{2^*-1}$$

then for any first eigenfunction ψ of Δ_h associated to the first nonzero eigenvalue $\lambda_1 = n$ of Δ_h,

$$\int_{S^n} (\nabla f, \nabla \psi)_h u^{2^*} dv_h = 0$$

We apply this identity to equation (F_α) with $\psi_\alpha(x) = (x_\alpha, x)$. Integrating by parts, this leads to

$$\int_{S^n} \left(\nabla\left[\left(c_n S_g \circ \Phi_\alpha - \alpha K_n^{-1}\right)(\varphi \circ \Phi_\alpha)^{2^*-2}\right], \nabla(x_\alpha, x)\right)_h |det\ \ d\Phi_\alpha|^{\frac{2}{n}} v_\alpha^2 dv_h$$
$$+ \int_{S^n} \left(\nabla\left[|det\ \ d\Phi_\alpha|^{\frac{2}{n}}\right], \nabla(x_\alpha, x)\right)_h \left(c_n S_g \circ \Phi_\alpha - \alpha K_n^{-1}\right)(\varphi \circ \Phi_\alpha)^{2^*-2} v_\alpha^2 dv_h$$
$$= 2\int_{S^n} \left(c_n S_g \circ \Phi_\alpha - \alpha K_n^{-1}\right)(\varphi \circ \Phi_\alpha)^{2^*-2} (x_\alpha, x)|det\ \ d\Phi_\alpha|^{\frac{2}{n}} v_\alpha^2 dv_h \qquad (2.1)$$

With the help of relations (1.3), (1.4), (1.5), and thanks to the fact that $\left(\pi_{-x_\alpha}^{-1}\right)^* h = 4\left(1 + |x|^2\right)^{-2} \xi$, (2.1) may be written as

$$I_\alpha + II_\alpha = III_\alpha \qquad (2.2)$$

where

$$I_\alpha = \frac{1}{2t_\alpha} \int_{I\!R^n} \frac{\left(\nabla\left[\left(c_n \tilde{S}_\alpha - \alpha K_n^{-1}\right)\tilde{\varphi}_\alpha^{2^*-2}\right]\left(\frac{x}{t_\alpha}\right), x\right)\tilde{v}_\alpha^2}{\left(t_\alpha^2 + |x|^2\right)^2 \left(1 + |x|^2\right)^{n-2}} dx$$

$$II_\alpha = t_\alpha^2 \int_{I\!R^n} \frac{\left(c_n \tilde{S}_\alpha\left(\frac{x}{t_\alpha}\right) - \alpha K_n^{-1}\right)\tilde{\varphi}_\alpha\left(\frac{x}{t_\alpha}\right)^{2^*-2} \tilde{v}_\alpha^2}{\left(t_\alpha^2 + |x|^2\right)^3 \left(1 + |x|^2\right)^{n-2}} dx$$

$$III_\alpha = \int_{I\!R^n} \frac{\left(c_n \tilde{S}_\alpha\left(\frac{x}{t_\alpha}\right) - \alpha K_n^{-1}\right)\tilde{\varphi}_\alpha\left(\frac{x}{t_\alpha}\right)^{2^*-2} |x|^2 \tilde{v}_\alpha^2}{\left(t_\alpha^2 + |x|^2\right)^3 \left(1 + |x|^2\right)^{n-2}} dx$$

We now derive useful relations for $\tilde v_\alpha$. Let $x_0 \in MaxS_g$ and $G(x_0,x)$ be the Green's function of $\Delta_g + B_0(g)K_n^{-1}$ at x_0. We set

$$H(x_0,x) = \frac{1}{2^{n-2}\omega_{n-1}(n-2)\varphi(x)\varphi(x_0)}\left(\frac{1-(x_0,x)}{2}\right)^{1-\frac{n}{2}}$$

so that

$$L_g H(x_0,x) = \delta_{x_0}$$

where the conformal Laplacian L_g is given by $L_g = \Delta_g + c_n S_g$. We write then

$$G(x_0,x) = \frac{1}{2^{n-2}\omega_{n-1}(n-2)\varphi(x)\varphi(x_0)}\left[\left(\frac{1-(x_0,x)}{2}\right)^{1-\frac{n}{2}} + \sigma(x)\right] \quad (2.3)$$

Hence, σ satisfies in the sense of distributions

$$L_h\sigma = \left(c_nS_g - B_0(g)K_n^{-1}\right)\varphi^{2^*-2}\left[\left(\frac{1-(x_0,x)}{2}\right)^{1-\frac{n}{2}} + \sigma(x)\right] \quad (2.4)$$

By standard elliptic theory, since

$$|c_nS_g - B_0(g)K_n^{-1}| = c_n|S_g(x) - S_g(x_0)| \le Cd_h(x_0,x)^2$$

we find that $\sigma \in C^1(S^n)$ for $n=4$ and $\sigma \in C^0(S^n)$ for $n=5$. We set $\tilde\sigma = \sigma\circ\pi^{-1}_{-x_0}$, and come back to the study of the $\tilde v_\alpha$'s. We already know by Theorem 0.3 that $\tilde v_\alpha$ is bounded uniformly in α. We claim now that

$$\tilde v_\alpha(t_\alpha x) \to \omega_n^{-\frac{1}{2^*}}\left[1+|x|^{n-2}\left(1+|x|^2\right)^{1-\frac{n}{2}}\tilde\sigma(x)\right] \quad \text{in } C^0_{loc}(I\!R^n\setminus\{0\}) \quad (2.5)$$

This follows from Theorem 0.3. Indeed, using in particular (1.4) and (1.5),

$$\begin{aligned}
\tilde v_\alpha(t_\alpha x) &= v_\alpha\circ\Phi_\alpha^{-1}\circ\pi^{-1}_{-x_\alpha}(x)\\
&= u_\alpha\circ\pi^{-1}_{-x_\alpha}(x)|det\ \ d\Phi_\alpha|\left(\pi_{-x_\alpha}(t_\alpha x)\right)^{\frac{1}{2^*}}\\
&= t_\alpha^{\frac{n}{2}-1}u_\alpha\circ\pi^{-1}_{-x_\alpha}(x)\left(t_\alpha^{-2}+|x|^2\right)^{\frac{n}{2}-1}\left(1+|x|^2\right)^{1-\frac{n}{2}}\\
&\to \omega_n^{-\frac{1}{2^*}}\left[1+|x|^{n-2}\left(1+|x|^2\right)^{1-\frac{n}{2}}\tilde\sigma(x)\right] \quad \text{in } C^0_{loc}(I\!R^n\setminus\{0\})
\end{aligned}$$

by Theorem 0.3. We derive now an estimate of $\tilde v_\alpha(t_\alpha x)$ near the origin. More precisely, we claim that

$$\lim_{R\to+\infty}\lim_{\alpha\to B_0(g)}\sup_{x\in B(0,R^{-1})}|\tilde v_\alpha(t_\alpha x) - \omega_n^{-\frac{1}{2^*}}| = 0 \quad (2.6)$$

Let $y_\alpha \in B\left(0, R^{-1}\right)$ be such that

$$|\tilde{v}_\alpha(t_\alpha y_\alpha) - \omega_n^{-\frac{1}{2^*}}| = \sup_{x \in B(0,R^{-1})} |\tilde{v}_\alpha(t_\alpha x) - \omega_n^{-\frac{1}{2^*}}|$$

We want to prove that

$$\lim_{\alpha \to B_0(g)} |\tilde{v}_\alpha(t_\alpha y_\alpha) - \omega_n^{-\frac{1}{2^*}}| \le \varepsilon(R)$$

where $\varepsilon(R) \to 0$ as $R \to +\infty$. The proof here splits into three cases. In the first one we assume that $|y_\alpha| \not\to 0$ as $\alpha \to B_0(g)$. Up to the extraction of a subsequence, $y_\alpha \to y$, $y \neq 0$, so that by (2.5), as $\alpha \to B_0(g)$,

$$\begin{aligned}\tilde{v}_\alpha(t_\alpha y_\alpha) &\to \omega_n^{-\frac{1}{2^*}} \left[1 + |y|^{n-2}\left(1+|y|^2\right)^{\frac{n}{2}-1} \tilde{\sigma}(y)\right] \\ &= \omega_n^{-\frac{1}{2^*}} + \varepsilon(R)\end{aligned}$$

according to standard properties of the Green's function. In the second case we assume that $t_\alpha y_\alpha \to y$ after passing to a subsequence. By (1.8), we have

$$\tilde{v}_\alpha(t_\alpha y_\alpha) \to \omega_n^{-\frac{1}{2^*}}$$

as $\alpha \to B_0(g)$. In the third case we assume that $|y_\alpha| \to 0$ and $t_\alpha |y_\alpha| \to +\infty$ as $\alpha \to B_0(g)$. The Green's formula on $\mathbb{R}^n$ gives that

$$\begin{aligned}\tilde{v}_\alpha(t_\alpha y_\alpha) = {} & \frac{2^{1-\frac{n}{2}}}{\omega_{n-1}(n-2)} \left(1 + |t_\alpha y_\alpha|^2\right)^{\frac{n}{2}-1} \\ & \times \int_{\mathbb{R}^n} |x - t_\alpha y_\alpha|^{2-n} \Delta_\xi \left(2^{\frac{n}{2}-1}\left(1+|x|^2\right)^{1-\frac{n}{2}} \tilde{v}_\alpha(x)\right) dx\end{aligned}$$

where Δ_ξ is the Laplacian with respect to the Euclidean metric. Recall that

$$\left(\pi_{-x_\alpha}^{-1}\right)^* h = 4\left(1+|x|^2\right)^{-2} \xi$$

Equation (F_α) together with (1.4) and (1.5) then gives that

$$\begin{aligned}&\left(\frac{1+|x|^2}{2}\right)^{\frac{n}{2}+1} \Delta_\xi \left(2^{\frac{n}{2}-1}\left(1+|x|^2\right)^{1-\frac{n}{2}} \tilde{v}_\alpha(x)\right) = \lambda_\alpha \tilde{v}_\alpha(x)^{2^*-1} \\ &\quad - \left(\alpha K_n^{-1} - c_n \tilde{S}_\alpha\left(\frac{x}{t_\alpha}\right)\right) \tilde{\varphi}_\alpha\left(\frac{x}{t_\alpha}\right)^{2^*-2} t_\alpha^2 \left(1+|x|^2\right)^2 \left(t_\alpha^2 + |x|^2\right)^{-2} \tilde{v}_\alpha(x)\end{aligned}$$

Coming back to the above Green's formula, we obtain

$$\begin{aligned}\tilde{v}_\alpha(t_\alpha y_\alpha) &= \frac{4}{\omega_{n-1}(n-2)}\lambda_\alpha \left(1+|t_\alpha y_\alpha|^2\right)^{\frac{n}{2}-1} A_\alpha \\ &- \frac{4t_\alpha^2}{\omega_{n-1}(n-2)} \left(1+|t_\alpha y_\alpha|^2\right)^{\frac{n}{2}-1} B_\alpha\end{aligned}$$

where

$$\begin{aligned}A_\alpha &= \int_{I\!R^n} |x - t_\alpha y_\alpha|^{2-n} \left(1+|x|^2\right)^{-1-\frac{n}{2}} \tilde{v}_\alpha(x)^{2^*-1} dx \\ B_\alpha &= \int_{I\!R^n} |x - t_\alpha y_\alpha|^{2-n} C_\alpha\left(\frac{x}{t_\alpha}\right)\left(t_\alpha^2 + |x|^2\right)^{-2}\left(1+|x|^2\right)^{1-\frac{n}{2}} \tilde{v}_\alpha(x) dx\end{aligned}$$

and

$$C_\alpha(x) = \left(\alpha K_n^{-1} - c_n \tilde{S}_\alpha(x)\right)\tilde{\varphi}_\alpha(x)^{2^*-2}$$

Regarding the first term in the RHS of the above relation, we have that

$$\begin{aligned}|t_\alpha y_\alpha|^{n-2} A_\alpha &= \int_{I\!R^n} \left|\frac{x}{t_\alpha |y_\alpha|} - \frac{y_\alpha}{|y_\alpha|}\right|^{2-n}\left(1+|x|^2\right)^{-1-\frac{n}{2}} \tilde{v}_\alpha(x)^{2^*-1} dx \\ &\to \omega_n^{-1+\frac{1}{2^*}} \int_{I\!R^n}\left(1+|x|^2\right)^{-1-\frac{n}{2}} dx = \frac{\omega_{n-1}}{n}\omega_n^{-1+\frac{1}{2^*}}\end{aligned}$$

by Lebesgue's theorem. Regarding the second term, we get by Theorem 0.3, and performing the change of variables $x = t_\alpha|y_\alpha|y$, that

$$\begin{aligned}&t_\alpha^n |y_\alpha|^{n-2} B_\alpha \\ &\le C t_\alpha^2 \int_{I\!R^n} \left|\frac{x}{t_\alpha|y_\alpha|} - \frac{y_\alpha}{|y_\alpha|}\right|^{2-n}\left(t_\alpha^2 + |x|^2\right)^{-2}\left(1+|x|^2\right)^{1-\frac{n}{2}} dx \\ &\le C|y_\alpha|^2 \int_{I\!R^n} \left|y - \frac{y_\alpha}{|y_\alpha|}\right|^{2-n}\left(t_\alpha^{-2}|y_\alpha|^{-2} + |y|^2\right)^{1-\frac{n}{2}}\left(1+|y_\alpha|^2|y|^2\right)^{-2} dy \\ &\le C|y_\alpha|^2\end{aligned}$$

Since $|y_\alpha| \to 0$ and $\lambda_\alpha \to K_n^{-1}$, we get that

$$\tilde{v}_\alpha(t_\alpha y_\alpha) \to \omega_n^{-\frac{1}{2^*}}$$

as $\alpha \to B_0(g)$. Thus, (2.6) is proved.

Going on with the proof of Theorem 0.4, we divide it into two parts. On the one hand, we deal with dimensions $n \ge 6$. On the other hand, we deal with dimensions $n = 4, 5$.

2.1 - The case $n \geq 6$

We need here one more ingredient to estimate the speed of convergence of x_α to x_0 as $\alpha \to B_0(g)$. By the standard Sobolev inequality on (S^n, h),

$$1 = \left(\int_{S^n} v_\alpha^{2^*} dv_h \right)^{\frac{2}{2^*}} \leq K_n \left[\int_{S^n} |\nabla v_\alpha|_h^2 dv_h + \frac{n(n-2)}{4} \int_{S^n} v_\alpha^2 dv_h \right]$$

Together with equation (F_α), this gives

$$K_n^{-1} - \lambda_\alpha \leq \int_{S^n} \left(c_n S_g \circ \Phi_\alpha - \alpha K_n^{-1} \right) (\varphi \circ \Phi_\alpha)^{2^*-2} |det \;\; d\Phi_\alpha|^{\frac{2}{n}} v_\alpha^2 dv_h \quad (2.7)$$

Independently, by the definition of λ_α, we have that for any $\alpha < B_0(g)$

$$\begin{aligned} &\int_{S^n} |\nabla \left(\varphi^{-1} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-\frac{1}{2^*}} \right)|_g^2 dv_g \\ &+ \alpha K_n^{-1} \int_{S^n} \varphi^{-2} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-\frac{2}{2^*}} dv_g \\ &\quad \geq \lambda_\alpha \left(\int_{S^n} \varphi^{-2^*} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-1} dv_g \right)^{\frac{2}{2^*}} \end{aligned}$$

where t_α is given by Theorem 0.3. By conformal change of the metric, this inequality becomes

$$\begin{aligned} &\int_{S^n} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-\frac{1}{2^*}} L_h \left(|det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-\frac{1}{2^*}} \right) dv_h \\ &\geq \int_{S^n} \left(c_n S_g - \alpha K_n^{-1} \right) \varphi^{2^*-2} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-\frac{2}{2^*}} dv_h \\ &\quad + \lambda_\alpha \left(\int_{S^n} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-1} dv_h \right)^{\frac{2}{2^*}} \end{aligned}$$

As easily checked,

$$\begin{aligned} L_h &\left(|det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-\frac{1}{2^*}} \right) = \\ &\frac{n(n-2)}{4} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-1+\frac{1}{2^*}} \end{aligned}$$

and

$$\int_{S^n} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi^{-1}_{x_0,t_\alpha}(x) \right)^{-1} dv_h = \omega_n$$

Therefore,

$$\omega_n^{-\frac{2}{2^*}} \int_{S^n} \left(c_n S_g - \alpha K_n^{-1}\right) \varphi^{2^*-2} |det \;\; d\Phi_{x_0,t_\alpha}| \left(\Phi_{x_0,t_\alpha}^{-1}(x)\right)^{-\frac{2}{2^*}} dv_h$$
$$\leq K_n^{-1} - \lambda_\alpha$$

and

$$\omega_n^{-\frac{2}{2^*}} \int_{S^n} \left(c_n S_g \circ \Phi_{x_0,t_\alpha} - \alpha K_n^{-1}\right) \left(\varphi \circ \Phi_{x_0,t_\alpha}\right)^{2^*-2} |det \;\; d\Phi_{x_0,t_\alpha}|^{\frac{2}{n}} dv_h$$
$$\leq K_n^{-1} - \lambda_\alpha$$

Combining this inequality with (2.7), we obtain

$$\omega_n^{-\frac{2}{2^*}} \int_{S^n} \left(c_n S_g \circ \Phi_{x_0,t_\alpha} - \alpha K_n^{-1}\right) \left(\varphi \circ \Phi_{x_0,t_\alpha}\right)^{2^*-2} |det \;\; d\Phi_{x_0,t_\alpha}|^{\frac{2}{n}} dv_h$$
$$\leq \int_{S^n} \left(c_n S_g \circ \Phi_\alpha - \alpha K_n^{-1}\right) \left(\varphi \circ \Phi_\alpha\right)^{2^*-2} |det \;\; d\Phi_\alpha|^{\frac{2}{n}} v_\alpha^2 dv_h \qquad (2.8)$$

We now prove the theorem when $n \geq 7$, and leave details to the reader when $n = 6$ (see the end of this subsection). First, we compute the LHS term in (2.8). Through the stereographic projection π_{-x_0}, we get using (1.4) and (1.5), and performing the change of variables $x = t_\alpha y$, that

$$\begin{aligned}
&\text{LHS of (2.8)} \\
&= 2^n \omega_n^{-\frac{2}{2^*}} t_\alpha^2 \int_{I\!R^n} \frac{\left(c_n \tilde{S}_g\left(\frac{x}{t_\alpha}\right) - \alpha K_n^{-1}\right) \tilde{\varphi}\left(\frac{x}{t_\alpha}\right)^{2^*-2}}{\left(t_\alpha^2 + |x|^2\right)^2 \left(1+|x|^2\right)^{n-2}} dx \\
&= 2^n \omega_n^{-\frac{2}{2^*}} t_\alpha^{2-n} \left(B_0(g) - \alpha\right) K_n^{-1} \int_{I\!R^n} \frac{\tilde{\varphi}\left(y\right)^{2^*-2}}{\left(1+|y|^2\right)^2 \left(t_\alpha^{-2} + |y|^2\right)^{n-2}} dy \\
&\quad + 2^n \omega_n^{-\frac{2}{2^*}} c_n t_\alpha^{2-n} \int_{I\!R^n} \frac{\left(\tilde{S}_g\left(y\right) - \tilde{S}_g(0)\right) \tilde{\varphi}\left(y\right)^{2^*-2}}{\left(1+|y|^2\right)^2 \left(t_\alpha^{-2} + |y|^2\right)^{n-2}} dy
\end{aligned}$$

On the one hand,

$$\begin{aligned}
&\int_{I\!R^n} \tilde{\varphi}\left(y\right)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} dy \\
&\qquad = \tilde{\varphi}(0)^{2^*-2} t_\alpha^{n-4} \int_{I\!R^n} \left(1+|y|^2\right)^{2-n} dy + o\left(t_\alpha^{n-4}\right)
\end{aligned}$$

On the other hand, writing that

$$\tilde{S}_g(y) - \tilde{S}_g(0) = \frac{1}{2}\partial_{ij}\tilde{S}_g(0)y^i y^j + O\left(|y|^3\right)$$

we get for the second term that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\tilde{S}_g(y) - \tilde{S}_g(0)\right)\tilde{\varphi}(y)^{2^*-2}\left(t_\alpha^{-2}+|y|^2\right)^{2-n}\left(1+|y|^2\right)^{-2}dy\\
&\quad= \frac{1}{2}\partial_{ij}\tilde{S}_g(0)\int_{\mathbb{R}^n} y^i y^j\tilde{\varphi}(y)^{2^*-2}\left(t_\alpha^{-2}+|y|^2\right)^{2-n}\left(1+|y|^2\right)^{-2}dy\\
&\quad\quad+O\left(\int_{\mathbb{R}^n}|y|^3\left(t_\alpha^{-2}+|y|^2\right)^{2-n}\left(1+|y|^2\right)^{-2}dy\right)\\
&\quad= -\frac{1}{2n}\Delta_\xi\tilde{S}_g(0)\tilde{\varphi}(0)^{2^*-2}t_\alpha^{n-6}\int_{\mathbb{R}^n}|y|^2\left(1+|y|^2\right)^{2-n}dy + o\left(t_\alpha^{n-6}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\text{LHS of (2.8)}\\
&= 2^n K_n^{-1}\omega_n^{-\frac{2}{2^*}}\varphi(x_0)^{2^*-2}\left(\int_{\mathbb{R}^n}\left(1+|y|^2\right)^{2-n}dy\right)\left(B_0(g)-\alpha\right)t_\alpha^{-2}\\
&\quad-\frac{2^{n-1}}{n}c_n\omega_n^{-\frac{2}{2^*}}\varphi(x_0)^{2^*-2}\Delta_\xi\tilde{S}_g(0)\left(\int_{\mathbb{R}^n}|y|^2\left(1+|y|^2\right)^{2-n}dy\right)t_\alpha^{-4}\\
&\quad+o\left(\left(B_0(g)-\alpha\right)t_\alpha^{-2}\right)+o\left(t_\alpha^{-4}\right)
\end{aligned}$$

We now deal with the RHS term in (2.8). Through the stereographic projection π_{-x_α}, we get using (1.4) and (1.5), and performing the change of variables $x = t_\alpha y$, that

$$\begin{aligned}
&\text{RHS of (2.8)}\\
&= \int_{S^n}\left(c_n S_g\circ\Phi_\alpha - \alpha K_n^{-1}\right)\left(\varphi\circ\Phi_\alpha\right)^{2^*-2}|det\ \ d\Phi_\alpha|^{\frac{2}{n}}v_\alpha^2 dv_h\\
&= 2^n t_\alpha^{2-n}\left(c_n K_n S_g(x_\alpha)-\alpha\right)K_n^{-1}\int_{\mathbb{R}^n}\frac{\tilde{\varphi}_\alpha(y)^{2^*-2}\tilde{v}_\alpha(t_\alpha y)^2}{\left(1+|y|^2\right)^2\left(t_\alpha^{-2}+|y|^2\right)^{n-2}}dy\\
&\quad+2^n c_n t_\alpha^{2-n}\int_{\mathbb{R}^n}\frac{\left(\tilde{S}_\alpha(y)-\tilde{S}_\alpha(0)\right)\tilde{\varphi}_\alpha(y)^{2^*-2}\tilde{v}_\alpha(t_\alpha y)^2}{\left(1+|y|^2\right)^2\left(t_\alpha^{-2}+|y|^2\right)^{n-2}}dy
\end{aligned}$$

For any $R > 0$,

$$\int_{\mathbb{R}^n}\tilde{\varphi}_\alpha(y)^{2^*-2}\left(t_\alpha^{-2}+|y|^2\right)^{2-n}\left(1+|y|^2\right)^{-2}\tilde{v}_\alpha(t_\alpha y)^2 dy$$

$$= \omega_n^{-\frac{2}{2^*}} \int_{B(0,R^{-1})} \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} dy$$
$$+ \int_{B(0,R^{-1})} \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \left(\tilde{v}_\alpha(t_\alpha y)^2 - \omega_n^{-\frac{2}{2^*}}\right) dy$$
$$+ \int_{\mathbb{R}^n \setminus B(0,R^{-1})} \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \tilde{v}_\alpha(t_\alpha y)^2 dy$$

Together with (2.6), and letting R go to $+\infty$, this gives

$$\int_{\mathbb{R}^n} \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \tilde{v}_\alpha(t_\alpha y)^2 dy$$
$$= \tilde{\varphi}(0)^{2^*-2} \omega_n^{-\frac{2}{2^*}} t_\alpha^{n-4} \int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} dy + o\left(t_\alpha^{n-4}\right)$$

Independently, since $\tilde{v}_\alpha$ is bounded (see Theorem 0.3), we get that

$$\int_{\mathbb{R}^n} \left(\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0)\right) \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \tilde{v}_\alpha(t_\alpha y)^2 dy$$
$$= \partial_i \tilde{S}_\alpha(0) \int_{\mathbb{R}^n} y^i \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \tilde{v}_\alpha(t_\alpha y)^2 dy$$
$$+ \frac{1}{2} \partial_{ij} \tilde{S}_\alpha(0) \int_{\mathbb{R}^n} y^i y^j \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \tilde{v}_\alpha(t_\alpha y)^2 dy$$
$$+ O\left(\int_{\mathbb{R}^n} |y|^3 \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} dy\right)$$

As above, using (2.6), one easily gets that

$$\int_{\mathbb{R}^n} \left(\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0)\right) \tilde{\varphi}_\alpha(y)^{2^*-2} \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-2} \tilde{v}_\alpha(t_\alpha y)^2 dy$$
$$= -\frac{1}{2n} \omega_n^{-\frac{2}{2^*}} \tilde{\varphi}(0)^{2^*-2} \Delta_\xi \tilde{S}_g(0) \left(\int_{\mathbb{R}^n} |y|^2 \left(1+|y|^2\right)^{2-n} dy\right) t_\alpha^{n-6}$$
$$+ o\left(|\nabla S_g(x_\alpha)| t_\alpha^{n-5}\right) + o\left(t_\alpha^{n-6}\right)$$

It then follows that

$$\text{RHS of (2.8)}$$
$$= 2^n \varphi(x_0)^{2^*-2} K_n^{-1} \omega_n^{-\frac{2}{2^*}} \left(\int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} dy\right) \left(c_n K_n S_g(x_\alpha) - \alpha\right) t_\alpha^{-2}$$
$$- \frac{2^{n-1}}{n} c_n \omega_n^{-\frac{2}{2^*}} \varphi(x_0)^{2^*-2} \Delta_\xi \tilde{S}_g(0) \left(\int_{\mathbb{R}^n} |y|^2 \left(1+|y|^2\right)^{2-n} dy\right) t_\alpha^{-4}$$
$$+ o\left(\left(c_n K_n S_g(x_\alpha) - \alpha\right) t_\alpha^{-2}\right) + o\left(t_\alpha^{-4}\right) + o\left(|\nabla S_g(x_\alpha)| t_\alpha^{-3}\right)$$

Writing that LHS of (2.8) $\leq$ RHS of (2.8), this is just inequality (2.8), we get that

$$S_g(x_0) - S_g(x_\alpha) \leq o\left(t_\alpha^{-1}\right) |\nabla S_g(x_\alpha)| + o\left(B_0(g) - \alpha\right) + o\left(t_\alpha^{-2}\right)$$

By the assumption made in Theorem 0.4, $\nabla^2 S_g(x_0) < 0$, so that

$$\begin{aligned} d_h(x_0, x_\alpha)^2 &\leq O\left(S_g(x_0) - S_g(x_\alpha)\right) \\ &\leq o\left(t_\alpha^{-1}\right) d_h(x_0, x_\alpha) + o\left(B_0(g) - \alpha\right) + o\left(t_\alpha^{-2}\right) \end{aligned}$$

and

$$d_h(x_0, x_\alpha)^2 \leq o\left(B_0(g) - \alpha\right) + o\left(t_\alpha^{-2}\right)$$

Summarizing, one gets with (2.8) that

$$S_g(x_0) - S_g(x_\alpha) \leq o\left(B_0(g) - \alpha\right) + o\left(t_\alpha^{-2}\right) \tag{2.9}$$

$$|\nabla S_g(x_\alpha)| \leq o\left(\left(B_0(g) - \alpha\right)^{\frac{1}{2}}\right) + o\left(t_\alpha^{-1}\right) \tag{2.10}$$

We now compute an expansion of the different terms in (2.2). Set $x = t_\alpha y$. Since $\tilde{v}_\alpha$ is bounded, we get that

$$\begin{aligned} III_\alpha = t_\alpha^{-n} \left(c_n \tilde{S}_\alpha(0) - \alpha K_n^{-1}\right) \int_{I\!R^n} \tilde{\varphi}_\alpha(y)^{2^*-2} \tilde{v}_\alpha(t_\alpha y)^2 |y|^2 \frac{\left(1 + |y|^2\right)^{-3}}{\left(t_\alpha^{-2} + |y|^2\right)^{n-2}} dy \\ + O\left(t_\alpha^{-n}\right) \int_{I\!R^n} |y|^3 \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1 + |y|^2\right)^{-3} dy \end{aligned}$$

which gives, since $c_n \tilde{S}_\alpha(0) - \alpha K_n^{-1} \to 0$ as $\alpha \to B_0(g)$,

$$III_\alpha = o\left(t_\alpha^{-6}\right)$$

Similarly,

$$\begin{aligned} II_\alpha = K_n^{-1} \left(c_n K_n S_g(x_\alpha) - \alpha\right) t_\alpha^{-n} \int_{I\!R^n} \frac{\tilde{\varphi}_\alpha(y)^{2^*-2} \tilde{v}_\alpha(t_\alpha y)^2}{\left(1 + |y|^2\right)^3 \left(t_\alpha^{-2} + |y|^2\right)^{n-2}} dy \\ + c_n t_\alpha^{-n} \int_{I\!R^n} \frac{\left(\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0)\right) \tilde{\varphi}_\alpha(y)^{2^*-2} \tilde{v}_\alpha(t_\alpha y)^2}{\left(1 + |y|^2\right)^3 \left(t_\alpha^{-2} + |y|^2\right)^{n-2}} dy \end{aligned}$$

With (2.6), one sees with the same kind of arguments than those used above that

$$\int_{\mathbb{R}^n} \tilde{\varphi}_\alpha(y)^{2^*-2} \tilde{v}_\alpha(t_\alpha y)^2 \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-3} dy$$

$$= \varphi(x_0)^{2^*-2} \omega_n^{-\frac{2}{2^*}} \left(\int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} dy\right) t_\alpha^{n-4} + o\left(t_\alpha^{n-4}\right)$$

On the other hand, we write with (2.10)

$$\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0) = o\left((B_0(g)-\alpha)^{\frac{1}{2}}\right)|y| + o\left(t_\alpha^{-1}\right)|y| + \frac{1}{2}\partial_{ij}\tilde{S}_\alpha(0) y^i y^j + O\left(|y|^3\right)$$

This gives, after some computations similar to the ones we developed above, that

$$\int_{\mathbb{R}^n} \left(\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0)\right) \tilde{\varphi}_\alpha(y)^{2^*-2} \tilde{v}_\alpha(t_\alpha y)^2 \left(t_\alpha^{-2} + |y|^2\right)^{2-n} \left(1+|y|^2\right)^{-3} dy$$

$$= -\frac{1}{2n}\varphi(x_0)^{2^*-2}\omega_n^{-\frac{2}{2^*}} \Delta_\xi \tilde{S}_g(0) \left(\int_{\mathbb{R}^n} |y|^2 \left(1+|y|^2\right)^{2-n} dy\right) t_\alpha^{n-6}$$

$$+ o\left(t_\alpha^{n-6}\right) + o\left((B_0(g)-\alpha)^{\frac{1}{2}} t_\alpha^{n-5}\right)$$

Noting that

$$o\left((B_0(g)-\alpha)^{\frac{1}{2}} t_\alpha^{n-5}\right) = o\left((B_0(g)-\alpha) t_\alpha^{n-4}\right) + o\left(t_\alpha^{n-6}\right)$$

we therefore get with (2.9) that

$$II_\alpha = \varphi(x_0)^{2^*-2} K_n^{-1} \omega_n^{-\frac{2}{2^*}} \left(\int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} dy\right) (B_0(g)-\alpha) t_\alpha^{-4}$$

$$- \frac{c_n}{2n}\varphi(x_0)^{2^*-2}\omega_n^{-\frac{2}{2^*}} \Delta_\xi \tilde{S}_g(0) \left(\int_{\mathbb{R}^n} |y|^2 \left(1+|y|^2\right)^{2-n} dy\right) t_\alpha^{-6}$$

$$+ o\left(t_\alpha^{-6}\right) + o\left((B_0(g)-\alpha) t_\alpha^{-4}\right)$$

At last, we deal with I_α. We have here

$$I_\alpha = \frac{t_\alpha^{-n}}{2} \int_{\mathbb{R}^n} \frac{\left(\nabla\left[\left(c_n \tilde{S}_\alpha(y) - \alpha K_n^{-1}\right)\tilde{\varphi}_\alpha(y)^{2^*-2}\right], y\right) \tilde{v}_\alpha(t_\alpha y)^2}{\left(t_\alpha^{-2} + |y|^2\right)^{n-2} \left(1+|y|^2\right)^2} dy$$

From (2.10), we deduce that

$$\left(\nabla\left[\left(c_n \tilde{S}_\alpha(y) - \alpha K_n^{-1}\right)\tilde{\varphi}_\alpha(y)^{2^*-2}\right], y\right)$$

$$= c_n \tilde{\varphi}_\alpha(0)^{2^*-2} \partial_{ij}\tilde{S}_\alpha(0) y^i y^j + o\left((B_0(g)-\alpha)^{\frac{1}{2}}\right)|y| + o\left(t_\alpha^{-1}\right)|y|$$

$$+ o\left((B_0(g)-\alpha)^{\frac{1}{2}}\right)|y|^2 + o\left(t_\alpha^{-1}\right)|y|^2 + O\left(|y|^3\right)$$

This gives :

$$I_\alpha = -\frac{c_n}{2n}\varphi(x_0)^{2^*-2}\omega_n^{-\frac{2}{2^*}}\Delta_\xi \tilde{S}_g(0)\left(\int_{I\!R^n} |y|^2\left(1+|y|^2\right)^{2-n} dy\right) t_\alpha^{-6}$$
$$+ o\left(t_\alpha^{-6}\right) + o\left(\left(B_0(g)-\alpha\right) t_\alpha^{-4}\right)$$

By letting α go to $B_0(g)$ in (2.2), we obtain

$$\lim_{\alpha\to B_0(g)} \left(B_0(g)-\alpha\right) t_\alpha^2 = \frac{c_n K_n}{n}\Delta_\xi \tilde{S}_g(0)\frac{\int_{I\!R^n} |y|^2\left(1+|y|^2\right)^{2-n} dy}{\int_{I\!R^n}\left(1+|y|^2\right)^{2-n} dy}$$

Easy computations lead then to

$$\Delta_\xi \tilde{S}_g(0) = 4\varphi(x_0)^{2^*-2}\Delta_g S_g(x_0)$$

and

$$\frac{c_n K_n}{n}\frac{\int_{I\!R^n} |y|^2\left(1+|y|^2\right)^{2-n} dy}{\int_{I\!R^n}\left(1+|y|^2\right)^{2-n} dy} = \frac{\omega_n^{-\frac{2}{n}}}{n(n-1)(n-6)}$$

Hence,

$$\lim_{\alpha\to B_0(g)} \left(B_0(g)-\alpha\right) t_\alpha^2 = \frac{4\omega_n^{-\frac{2}{n}}}{n(n-1)(n-6)}\varphi(x_0)^{2^*-2}\Delta_g S_g(x_0)$$

and the first part of Theorem 0.4 is proved when $n \geq 7$. To prove the second part, just note that by (2.9),

$$d_h(x_0, x_\alpha) = o\left(t_\alpha^{-1}\right)$$

As easily seen, this allows us to replace x_α by x_0 in the proof of Theorem 0.3.

In order to end this subsection, we list the different results we obtain in dimension $n = 6$ when computing the LHS and RHS terms of (2.8), and the different terms of (2.2). The details are left to the reader. Only few changes are needed with respect to the case $n \geq 7$. We obtain

$$\text{LHS of (2.8)}$$
$$= 2^6 K_6^{-1}\omega_6^{-\frac{2}{3}}\varphi(x_0)\left(\int_{I\!R^6}\left(1+|y|^2\right)^{-4} dy\right)\left(B_0(g)-\alpha\right) t_\alpha^{-2}$$
$$- \frac{2^4}{3}c_6\omega_6^{-\frac{2}{3}}\omega_5\varphi(x_0)\Delta_\xi \tilde{S}_g(0) t_\alpha^{-4}\ln t_\alpha$$
$$+ o\left(t_\alpha^{-4}\ln t_\alpha\right) + o\left(\left(B_0(g)-\alpha\right) t_\alpha^{-2}\right)$$

and

RHS of (2.8)

$$= 2^6 K_6^{-1}\omega_6^{-\frac{2}{3}}\varphi(x_0)\left(\int_{I\!R^6}\left(1+|y|^2\right)^{-4}dy\right)\left(c_6K_6S_g(x_\alpha)-\alpha\right)t_\alpha^{-2}$$
$$+o\left(\left(c_6K_6S_g(x_\alpha)-\alpha\right)t_\alpha^{-2}\right)-\frac{2^4}{3}c_6\omega_6^{-\frac{2}{3}}\omega_5\varphi(x_0)\Delta_\xi\tilde{S}_g(0)t_\alpha^{-4}\ln t_\alpha$$
$$+o\left(t_\alpha^{-4}\ln t_\alpha\right)+o\left(|\nabla S_g(x_\alpha)|t_\alpha^{-3}\right)$$

Together with (2.8), this gives

$$S_g(x_0)-S_g(x_\alpha)\le o\left(B_0(g)-\alpha\right)+o\left(t_\alpha^{-2}\ln t_\alpha\right)$$
$$|\nabla S_g(x_\alpha)|\le o\left(\left(B_0(g)-\alpha\right)^{\frac{1}{2}}\right)+o\left(t_\alpha^{-1}\left(\ln t_\alpha\right)^{\frac{1}{2}}\right)$$

As for (2.2), using these estimates, we get

$$I_\alpha=-\frac{c_6}{12}\varphi(x_0)\omega_6^{-\frac{2}{3}}\omega_5\Delta_\xi\tilde{S}_g(0)t_\alpha^{-6}\ln t_\alpha+o\left(t_\alpha^{-6}\ln t_\alpha\right)+o\left(\left(B_0(g)-\alpha\right)t_\alpha^{-4}\right)\quad,$$

$$II_\alpha=\varphi(x_0)K_6^{-1}\omega_6^{-\frac{2}{3}}\left(\int_{I\!R^6}\left(1+|y|^2\right)^{-4}dy\right)\left(B_0(g)-\alpha\right)t_\alpha^{-4}$$
$$-\frac{c_6}{12}\varphi(x_0)\omega_6^{-\frac{2}{3}}\omega_5\Delta_\xi\tilde{S}_g(0)t_\alpha^{-6}\ln t_\alpha$$
$$+o\left(t_\alpha^{-6}\ln t_\alpha\right)+o\left(\left(B_0(g)-\alpha\right)t_\alpha^{-4}\right)$$

and

$$III_\alpha=o\left(t_\alpha^{-6}\ln t_\alpha\right)\quad,$$

Letting α go to $B_0(g)$ in (2.2), we therefore get that

$$\lim_{\alpha\to B_0(g)}\left(B_0(g)-\alpha\right)t_\alpha^2\left(\ln t_\alpha\right)^{-1}=\frac{2}{15}\omega_6^{-\frac{1}{3}}\varphi(x_0)\Delta_gS_g(x_0)$$

This proves Theorem 0.4 when $n=6$.

2.2 - The case $n=4,5$

We prove now Theorem 0.4 when $n=4,5$. As a starting point, we need a substitute for (2.8) in order to get informations on the speed of convergence of x_α to x_0. We note here that

$$\Delta_\xi\left(\tilde{v}_\alpha\psi\right)=$$
$$\left[\lambda_\alpha+\left(c_n\tilde{S}_\alpha\left(\frac{x}{t_\alpha}\right)-\alpha K_n^{-1}\right)t_\alpha^2\frac{\left(1+|x|^2\right)^2}{\left(t_\alpha^2+|x|^2\right)^2}\tilde{\varphi}_\alpha\left(\frac{x}{t_\alpha}\right)^{2^*-2}\tilde{v}_\alpha^{2-2^*}\right]\left(\tilde{v}_\alpha\psi\right)^{2^*-1}$$

where $\psi = \left(\frac{1+|x|^2}{2}\right)^{1-\frac{n}{2}}$. Integrations by parts then give that

$$\int_{I\!R^n} \frac{\tilde{v}_\alpha(t_\alpha x)^2}{\left(t_\alpha^{-2} + |x|^2\right)^{n-2} \left(1 + |x|^2\right)^2} \nabla \left[\left(c_n \tilde{S}_\alpha(x) - \alpha K_n^{-1} \right) \tilde{\varphi}_\alpha(x)^{2^*-2} \right] dx$$

$$= 4 \int_{I\!R^n} \frac{\left(c_n \tilde{S}_\alpha(x) - \alpha K_n^{-1} \right) \tilde{\varphi}_\alpha(x)^{2^*-2} \tilde{v}_\alpha(t_\alpha x)^2}{\left(t_\alpha^{-2} + |x|^2\right)^{n-2} \left(1 + |x|^2\right)^3} x \, dx \tag{2.11}$$

We assume now that $n = 4$, and refer to the end of this subsection for the case where $n = 5$. Together with Lebesgue's theorem, with the fact that $(\tilde{v}_\alpha)$ is bounded and with (2.5), (2.11) gives that

$$c_4 \omega_4^{-\frac{1}{2}} \int_{I\!R^4} \nabla \left[\left(\tilde{S}_g(x) - \tilde{S}_g(0) \right) \tilde{\varphi}(x)^2 \right] \frac{\left[1 + |x|^2 \left(1 + |x|^2\right)^{-1} \tilde{\sigma}(x) \right]}{|x|^4 \left(1 + |x|^2\right)^2} dx$$

$$+ \nabla \left[\left(c_4 \tilde{S}_\alpha(x) - \alpha K_4^{-1} \right) \tilde{\varphi}_\alpha(x)^2 \right](0) \int_{I\!R^4} \frac{\tilde{v}_\alpha(t_\alpha x)^2}{\left(1 + |x|^2\right)^2 \left(t_\alpha^{-2} + |x|^2\right)^2} dx + o(1)$$

$$= 4 c_4 \omega_4^{-\frac{1}{2}} \int_{I\!R^4} \left(\tilde{S}_g(x) - \tilde{S}_g(0) \right) \tilde{\varphi}(x)^2 \frac{\left[1 + |x|^2 \left(1 + |x|^2\right)^{-1} \tilde{\sigma}(x) \right]}{|x|^4 \left(1 + |x|^2\right)^3} x dx$$

Together with (2.6), we get that

$$\int_{I\!R^4} \tilde{v}_\alpha(t_\alpha x)^2 \left(t_\alpha^{-2} + |x|^2\right)^{-2} \left(1 + |x|^2\right)^{-2} dx = \omega_4^{-\frac{1}{2}} \omega_3 \ln t_\alpha + o\left(\ln t_\alpha\right)$$

Therefore,

$$|\nabla \left[\left(c_4 \tilde{S}_\alpha(x) - \alpha K_4^{-1} \right) \tilde{\varphi}_\alpha(x)^2 \right](0)| = O\left(\left(\ln t_\alpha\right)^{-1} \right) \tag{2.12}$$

We now compute the different terms in (2.2). Setting $x = t_\alpha y$, and by Lebesgue's theorem, we get that

$$III_\alpha = t_\alpha^{-4} \int_{I\!R^4} \left(c_4 \tilde{S}_\alpha(y) - \alpha K_4^{-1} \right) \tilde{\varphi}_\alpha(y)^2 \frac{|y|^2 \tilde{v}_\alpha(t_\alpha y)^2}{\left(t_\alpha^{-2} + |y|^2\right)^2 \left(1 + |y|^2\right)^3} dy$$

$$= c_4 \omega_4^{-\frac{1}{2}} t_\alpha^{-4} \int_{I\!R^4} \left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \frac{\left[1 + |y|^2 \left(1 + |y|^2\right)^{-1} \tilde{\sigma}(y) \right]^2}{|y|^2 \left(1 + |y|^2\right)^3} dy$$

$$+ o\left(t_\alpha^{-4}\right)$$

Performing the same change of variables in II_α, we get that

$$II_\alpha = t_\alpha^{-4} K_4^{-1} \left(c_4 S_g(x_\alpha) K_4 - \alpha\right) \int_{I\!R^4} \frac{\tilde{\varphi}_\alpha(y)^2 \tilde{v}_\alpha(t_\alpha y)^2}{\left(1+|y|^2\right)^3 \left(t_\alpha^{-2}+|y|^2\right)^2} dy$$
$$+ c_4 t_\alpha^{-4} \int_{I\!R^4} \frac{\left(\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0)\right) \tilde{\varphi}_\alpha(y)^2 \tilde{v}_\alpha(t_\alpha y)^2}{\left(1+|y|^2\right)^3 \left(t_\alpha^{-2}+|y|^2\right)^2} dy$$

Since

$$|\tilde{S}_\alpha(y) - \tilde{S}_\alpha(0)| \le C|y|$$

the integral in the second term of the RHS of this relation converges. For the first term in the RHS of this relation, (2.6), together with the same trick than we used in subsection 2.1, gives that

$$\int_{I\!R^4} \tilde{\varphi}_\alpha(y)^2 \tilde{v}_\alpha(t_\alpha y)^2 \left(t_\alpha^{-2}+|y|^2\right)^{-2} \left(1+|y|^2\right)^{-3} dy$$
$$= \varphi(x_0)^2 \omega_4^{-\frac{1}{2}} \omega_3 \ln t_\alpha + o\left(\ln t_\alpha\right)$$

Therefore,

$$II_\alpha = K_4^{-1} \varphi(x_0)^2 \omega_4^{-\frac{1}{2}} \omega_3 \left(c_4 S_g(x_\alpha) K_4 - \alpha\right) t_\alpha^{-4} \ln t_\alpha$$
$$+ o\left(\left(c_4 S_g(x_\alpha) K_4 - \alpha\right) t_\alpha^{-4} \ln t_\alpha\right) + o\left(t_\alpha^{-4}\right)$$
$$+ c_4 \omega_4^{-\frac{1}{2}} t_\alpha^{-4} \int_{I\!R^4} \left(\tilde{S}_g(y) - \tilde{S}_g(0)\right) \tilde{\varphi}(y)^2 \frac{\left[1+|y|^2\left(1+|y|^2\right)^{-1} \tilde{\sigma}(y)\right]^2}{|y|^4 \left(1+|y|^2\right)^3} dy$$

Similar computations give that

$$I_\alpha = \frac{c_4}{2} \omega_4^{-\frac{1}{2}} t_\alpha^{-4} \int_{I\!R^4} \left(\nabla \left[\left(\tilde{S}_g(y) - \tilde{S}_g(0)\right) \tilde{\varphi}(y)^2\right], y\right)$$
$$\times \frac{\left[1+|y|^2\left(1+|y|^2\right)^{-1} \tilde{\sigma}(y)\right]^2}{\left(1+|y|^2\right)^2 |y|^4} dy + o\left(t_\alpha^{-4}\right)$$

Coming back to (2.2), and passing to the limit $\alpha \to B_0(g)$ in this relation, we obtain

$$\lim_{\alpha \to B_0(g)} \left(c_4 S_g(x_\alpha) K_4 - \alpha\right) \ln t_\alpha$$
$$= \frac{\varphi(x_0)^{-2}}{24\omega_3} \omega_4^{-\frac{1}{2}}$$

$$\times \left[2 \int_{I\!R^4} \left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \frac{\left(|y|^2 - 1\right) \left[|y|^{-2} \left(1 + |y|^2\right) + \tilde{\sigma}(y) \right]^2}{\left(1 + |y|^2\right)^5} dy \right.$$

$$\left. - \int_{I\!R^4} \left(\nabla \left[\left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \right], y \right) \frac{\left[|y|^{-2} \left(1 + |y|^2\right) + \tilde{\sigma}(y) \right]^2}{\left(1 + |y|^2\right)^4} dy \right]$$

We use (2.12) to conclude. By (2.12),

$$|\nabla \tilde{S}_\alpha(0)| = O\left((\ln t_\alpha)^{-1} \right)$$

so that, together with the assumption $\nabla^2 S_g(x_0) < 0$ we made in Theorem 0.4,

$$d_h(x_0, x_\alpha) = O\left((\ln t_\alpha)^{-1} \right)$$

and

$$S_g(x_0) - S_g(x_\alpha) = o\left((\ln t_\alpha)^{-1} \right)$$

Hence,

$$\begin{aligned} & \lim_{\alpha \to B_0(g)} (B_0(g) - \alpha) \ln t_\alpha \\ = \quad & \frac{\varphi(x_0)^{-2}}{4\omega_3} \omega_4^{-\frac{1}{2}} \qquad (2.13) \\ \times & \left[2c_4 \int_{I\!R^4} \left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \frac{\left(|y|^2 - 1\right) \left[|y|^{-2} \left(1 + |y|^2\right) + \tilde{\sigma}(y) \right]^2}{\left(1 + |y|^2\right)^5} dy \right. \\ & \left. - c_4 \int_{I\!R^4} \left(\nabla \left[\left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \right], y \right) \frac{\left[|y|^{-2} \left(1 + |y|^2\right) + \tilde{\sigma}(y) \right]^2}{\left(1 + |y|^2\right)^4} dy \right] \end{aligned}$$

In order to simplify this expression, we let

$$\psi(y) = 2 \left(1 + |y|^2\right)^{-1}$$

so that $\left(\pi_{-x_0}^{-1}\right)^* h = \psi^2 \xi$. By (2.4), we have :

$$\Delta_\xi \left(\psi \tilde{\sigma} \right) = c_4 \left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}^2 \psi^2 \left[2|y|^{-2} + \tilde{\sigma} \psi \right]$$

Regarding the first integral in the RHS term of (2.13), we write that

$$\begin{aligned} & 2c_4 \int_{I\!R^4} \left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \frac{\left(|y|^2 - 1\right)}{\left(1 + |y|^2\right)^5} \left[|y|^{-2} \left(1 + |y|^2\right) + \tilde{\sigma}(y) \right]^2 dy \\ & \quad = 2^{-4} \int_{I\!R^4} \Delta_\xi \left(\psi \tilde{\sigma} \right)(y) \left[2|y|^{-2} + \tilde{\sigma}(y) \psi(y) \right] \psi(y) \left(|y|^2 - 1\right) dy \end{aligned}$$

As for the second one, integrations by parts give that

$$\begin{aligned}
&- c_4 \int_{I\!R^4} \left(\nabla \left[\left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^2 \right], y \right) \frac{\left[|y|^{-2} \left(1+|y|^2\right) + \tilde{\sigma}(y) \right]^2}{\left(1+|y|^2\right)^4} dy \\
&= 2^{-4} \times 4 \int_{I\!R^4} \Delta_\xi \left(\psi \tilde{\sigma} \right)(y) \left[2|y|^{-2} + \tilde{\sigma}(y)\psi(y) \right] dy \\
&\quad + 2^{-3} \int_{I\!R^4} \Delta_\xi \left(\psi \tilde{\sigma} \right)(y) \psi(y)^{-1} \left(\nabla \left(\left[2|y|^{-2} + \tilde{\sigma}(y)\psi(y) \right] \psi(y) \right), y \right) dy
\end{aligned}$$

Some more integrations by parts then lead to

$$\text{RHS of (2.13)} = - \frac{\varphi(x_0)^{-2}}{16\omega_3} \omega_4^{-\frac{1}{2}} \int_{I\!R^4} \Delta_\xi \left(\psi \tilde{\sigma} \right)(y) |y|^{-2} dy$$

Since

$$\Delta_\xi \left(|y|^{-2} \right) = 2\omega_3 \delta_0$$

it follows that

$$\lim_{\alpha \to B_0(g)} \left(B_0(g) - \alpha \right) \ln t_\alpha = - \frac{1}{4} \omega_4^{-\frac{1}{2}} \varphi(x_0)^{-2} \sigma(x_0)$$

Note here that $\sigma(x_0)$ makes sense, since for $n = 4$, $\sigma \in C^0(S^4)$. By the Green's formula on (S^n, h), together with (2.4) and (2.3),

$$\begin{aligned}
\sigma(x_0) &= \frac{1}{8\omega_3} \int_{S^4} L_h \sigma(x) \left(\frac{1 - (x_0, x)}{2} \right)^{-1} dv_h \\
&= \frac{1}{6} \varphi(x_0) \int_{S^4} \left(S_g(x) - S_g(x_0) \right) \varphi(x)^{-1} \left(\frac{1 - (x_0, x)}{2} \right)^{-1} G(x_0, x) dv_g
\end{aligned}$$

This proves Theorem 0.4 when $n = 4$.

When $n = 5$, similar computations lead to

$$\begin{aligned}
I_\alpha = \frac{1}{2} c_5 \omega_5^{-\frac{2}{2^*}} t_\alpha^{-5} &\int_{I\!R^5} \left(\nabla \left[\left(\tilde{S}_g(y) - \tilde{S}_g(0) \right) \tilde{\varphi}(y)^{2^*-2} \right], y \right) \\
&\times \frac{\left(|y|^{-3} \left(1+|y|^2\right)^{\frac{3}{2}} + \tilde{\sigma}(y) \right)^2}{\left(1+|y|^2\right)^5} dy \\
+ o\left(t_\alpha^{-5}\right) &+ o\left(\left(B_0(g) - \alpha\right) t_\alpha^{-4} \right) \quad ,
\end{aligned}$$

$$II_\alpha = K_5^{-1}\omega_5^{-\frac{2}{2^*}}\varphi(x_0)^{2^*-2}\left(\int_{I\!R^5}\left(1+|y|^2\right)^{-3}dy\right)(B_0(g)-\alpha)\,t_\alpha^{-4}$$
$$+\,o\left((B_0(g)-\alpha)\,t_\alpha^{-4}\right)+o\left(t_\alpha^{-5}\right)$$
$$+\,c_5\omega_5^{-\frac{2}{2^*}}t_\alpha^{-5}\int_{I\!R^5}\left(\tilde S_g(y)-\tilde S_g(0)\right)\tilde\varphi(y)^{2^*-2}\frac{\left[|y|^{-3}\left(1+|y|^2\right)^{\frac{3}{2}}+\tilde\sigma(y)\right]^2}{\left(1+|y|^2\right)^6}dy$$

and

$$III_\alpha = c_5\omega_5^{-\frac{2}{2^*}}t_\alpha^{-5}$$
$$\times\int_{I\!R^5}\left(\tilde S_g(y)-\tilde S_g(0)\right)\tilde\varphi(y)^{2^*-2}|y|^2\frac{\left[|y|^{-3}\left(1+|y|^2\right)^{\frac{3}{2}}+\tilde\sigma(y)\right]^2}{\left(1+|y|^2\right)^6}dy$$
$$+\,o\left(t_\alpha^{-5}\right)$$

Coming back to (2.2), and letting α go to $B_0(g)$, we get that

$$\lim_{\alpha\to B_0(g)}(B_0(g)-\alpha)\,t_\alpha = \frac{3}{10\pi}\omega_5^{-\frac{3}{5}}\varphi(x_0)^{-\frac{1}{3}}$$
$$\times\int_{S^5}\left(S_g(x_0)-S_g(x)\right)\varphi(x)^{-1}\left(\frac{1-(x_0,x)}{2}\right)^{-\frac{3}{2}}G(x_0,x)dv_g$$

This ends the proof of Theorem 0.4.

3 - The radial case : proof of Theorem 0.5

Let $x_0\in S^n$, and $g=\varphi^{\frac{4}{n-2}}h$, where φ is radially symmetrical with respect to x_0. Clearly, S_g is also radially symmetrical with respect to x_0. We assume in what follows that for any $\lambda>0$, g and λh are not isometric. We then define

$$B_0(g)_r = \inf\left\{B>0 \text{ s.t. } (I_{opt}) \text{ is valid with } B \text{ for any } u\in C_r^\infty(S^n)\right\}$$

where $C_r^\infty(S^n)$ is the set of functions in $C^\infty(S^n)$ which are radially symmetrical with respect to x_0. We have $B_0(g)_r\le B_0(g)=c_nK_n\max_{S^n}S_g$, while, using test functions as in Hebey [16],

$$B_0(g)_r\ge c_nK_n\max_{\{x_0,-x_0\}}S_g$$

Assume now that

$$\max_{\{x_0,-x_0\}}S_g=\max_{S^n}S_g=S_g(x_0)$$

Then

$$B_0(g)_r = B_0(g) = c_n K_n \max_{S^n} S_g$$

As easily seen, it follows that we can choose the sub-extremals (z_α) to be radially symmetrical with respect to x_0. If x_1 is the concentration point of (z_α), we know (see for instance Hebey [14]) that $x_1 \in \{x_0, -x_0\}$. Moreover, see section 1, x_1 has to be a point where S_g is maximum. Without loss of generality, we may then assume that $x_1 = x_0$.

We let now x_α be as in section 1, and claim that

$$d_h(x_0, x_\alpha) = o\left(t_\alpha^{-1}\right) \tag{3.1}$$

To prove this claim, let π_{-x_0} be the stereographic projection of north pole $-x_0$. Since $u_\alpha = z_\alpha \varphi$ is radially symmetrical with respect to x_0, we get that for any $k \in I\!N$ and any $i = 1, \ldots, n$

$$\int_{S^n_+} \pi^i_{-x_0}(x) u^k_\alpha dv_h = 0$$

where $S^n_+ = B\left(x_0, \frac{\pi}{2}\right)$. Hence,

$$\int_{S^n_+} \pi^i_{-x_0}(\Phi_\alpha(x)) v^k_\alpha |det \;\; d\Phi_\alpha|^{1-\frac{k}{2^*}} dv_h = 0$$

We let

$$\tilde{v}_\alpha = v_\alpha \circ \pi^{-1}_{-x_\alpha}$$

so that, by (1.4) and (1.5),

$$\int_{\pi_{-x_\alpha}(S^n_+)} \pi^i_{-x_0}(\pi^{-1}_{-x_\alpha}\left(\frac{x}{t_\alpha}\right)) \tilde{v}_\alpha(x)^k \frac{\left(1 + t_\alpha^{-2}|x|^2\right)^{-n(1-\frac{k}{2^*})}}{(1+|x|^2)^{\frac{kn}{2^*}}} dx = 0$$

Performing the change of variables $x = t_\alpha y$, we obtain

$$\int_{\pi_{-x_\alpha}(S^n_+)} \pi^i_{-x_0}(\pi^{-1}_{-x_\alpha}(y)) \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2} + |y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy = 0$$

We have that for any $y \in \pi_{-x_\alpha}\left(S^n_+\right)$,

$$\pi^i_{-x_0}\left(\pi^{-1}_{-x_\alpha}(y)\right) = \pi^i_{-x_0}(x_\alpha) + C(\alpha)_{ij} y^j + O\left(|y|^2\right)$$

where the $C(\alpha)_{ij}$'s are bounded. This leads to

$$\pi^i_{-x_0}(x_\alpha)\int_{\pi_{-x_\alpha}(S^n_+)} \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2}+|y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy$$

$$= -C(\alpha)_{ij}\int_{\pi_{-x_\alpha}(S^n_+)} y^j \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2}+|y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy$$

$$+ O\left(\int_{\pi_{-x_\alpha}(S^n_+)} |y|^2 \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2}+|y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy\right)$$

When $2^* > k > \frac{n+2}{n-2}$,

$$\int_{\pi_{-x_\alpha}(S^n_+)} \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2}+|y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy$$

$$= C t_\alpha^{k(n-2)-n} + o\left(t_\alpha^{k(n-2)-n}\right)$$

$$\int_{\pi_{-x_\alpha}(S^n_+)} y^j \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2}+|y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy$$

$$= o\left(t_\alpha^{k(n-2)-n-1}\right)$$

$$\int_{\pi_{-x_\alpha}(S^n_+)} |y|^2 \tilde{v}_\alpha(t_\alpha y)^k \left(t_\alpha^{-2}+|y|^2\right)^{-\frac{kn}{2^*}} \left(1+|y|^2\right)^{-n(1-\frac{k}{2^*})} dy$$

$$= O\left(t_\alpha^{k(n-2)-n-2}\right)$$

so that

$$\pi^i_{-x_0}(x_\alpha) = o\left(t_\alpha^{-1}\right)$$

This proves (3.1). As in subsection 2.1, we may then take $x_\alpha = x_0$ in Theorem 0.3.

We now compute t_α. For length reasons, we give details in the case $n \geq 6$, $2p < n-4$, and leave the proof of Theorem 0.5 to the reader for the other cases. The necessary material will be found in subsections 2.1 and 2.2, and in Robert [26]. Performing once more the change of variables $x = t_\alpha y$, and since one may take $x_\alpha = x_0$, the different terms in (2.2) are :

$$I_\alpha = \frac{1}{2t_\alpha^n}\int_{I\!R^n} \frac{\left(\nabla\left[\left(c_n\tilde{S}_g - \alpha K_n^{-1}\right)\tilde{\varphi}^{2^*-2}\right](y), y\right)\tilde{v}_\alpha(t_\alpha y)^2}{\left(t_\alpha^{-2}+|y|^2\right)^{n-2}\left(1+|y|^2\right)^2} dy$$

$$II_\alpha = t_\alpha^{-n} \int_{\mathbb{R}^n} \frac{\left(c_n \tilde{S}_g(y) - \alpha K_n^{-1}\right) \tilde{\varphi}(y)^{2^*-2} \tilde{v}_\alpha(t_\alpha y)^2}{\left(t_\alpha^{-2} + |y|^2\right)^{n-2} \left(1+|y|^2\right)^3} dy$$

$$III_\alpha = t_\alpha^{-n} \int_{\mathbb{R}^n} \frac{\left(c_n \tilde{S}_g(y) - \alpha K_n^{-1}\right) \tilde{\varphi}(y)^{2^*-2} |y|^2 \tilde{v}_\alpha(t_\alpha y)^2}{\left(t_\alpha^{-2} + |y|^2\right)^{n-2} \left(1+|y|^2\right)^3} dy$$

We first deal with I_α. We write that

$$\left(\nabla \left[\left(c_n \tilde{S}_g - \alpha K_n^{-1}\right) \tilde{\varphi}^{2^*-2}\right] \left(\frac{x}{t_\alpha}\right), x\right) = c_n \varphi(x_0)^{2^*-2} \frac{\tilde{S}_g^{(2p)}(0)}{(2p-1)!} r^{2p} + o\left(r^{2p}\right) + o\left(B_0(g) - \alpha\right)$$

so that, with (2.6) and using the assumption $2p < n-4$,

$$I_\alpha = \frac{c_n \omega_n^{-\frac{2}{2^*}}}{2} \varphi(x_0)^{2^*-2} \frac{\tilde{S}_g^{(2p)}(0)}{(2p-1)!} t_\alpha^{-2p-4} \int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} |y|^{2p} dy + o\left(t_\alpha^{-2p-4}\right) + o\left(\left(B_0(g) - \alpha\right) t_\alpha^{-4}\right)$$

For II_α and III_α, we write that

$$\left(c_n \tilde{S}_g(y) - \alpha K_n^{-1}\right) \tilde{\varphi}(y)^{2^*-2} = K_n^{-1} \left(B_0(g) - \alpha\right) \varphi(x_0)^{2^*-2} + c_n \varphi(x_0)^{2^*-2} \frac{\tilde{S}_g^{(2p)}(0)}{(2p)!} r^{2p} + o\left(B_0(g) - \alpha\right) + o\left(r^{2p}\right)$$

This leads to

$$II_\alpha = K_n^{-1} \omega_n^{-\frac{2}{2^*}} \left(B_0(g) - \alpha\right) \varphi(x_0)^{2^*-2} t_\alpha^{-4} \int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} dy + c_n \omega_n^{-\frac{2}{2^*}} \varphi(x_0)^{2^*-2} \frac{\tilde{S}_g^{(2p)}(0)}{(2p)!} t_\alpha^{-4-2p} \int_{\mathbb{R}^n} |y|^{2p} \left(1+|y|^2\right)^{2-n} dy + o\left(\left(B_0(g) - \alpha\right) t_\alpha^{-4}\right) + o\left(t_\alpha^{-4-2p}\right)$$

and

$$III_\alpha = o\left(\left(B_0(g) - \alpha\right) t_\alpha^{-4}\right) + o\left(t_\alpha^{-4-2p}\right)$$

Coming back to (2.2), and passing to the limit $\alpha \to B_0(g)$, we get that

$$\lim_{\alpha \to B_0(g)} \left(B_0(g) - \alpha\right) t_\alpha^{2p} = \frac{c_n K_n (p+1)}{(2p)!} \frac{\int_{\mathbb{R}^n} |y|^{2p} \left(1+|y|^2\right)^{2-n} dy}{\int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} dy} \left(-\tilde{S}_g^{(2p)}(0)\right) \tag{3.2}$$

We now compute $\tilde{S}_g^{(2p)}(0)$. For that purpose, let $f \in C^\infty(I\!R^n)$ be radially symmetrical with respect to x_0 and such that

$$f^{(k)}(0) = 0 \text{ for all } 0 \le k < 2p \quad \text{and } f^{(2p)}(0) \neq 0$$

As one easily checks,

$$(-\Delta_\xi)^k f(0) = 0 \qquad \text{for any } 0 \le k < p$$
$$(-\Delta_\xi)^p f(0) = \frac{n(n+2).......(n-2+2p)}{3 \times 5 \times .... \times (2p-1)} f^{(2p)}(0)$$

Moreover, thanks to the formula that relates the scalar curvatures of two conformal metrics,

$$(-\Delta_\xi)^p \tilde{S}_g(0) = 4^p \varphi(x_0)^{p(2^*-2)} (-\Delta_g)^p S_g(x_0)$$

Hence,

$$\tilde{S}_g^{(2p)}(0) = 2^{2p} \frac{3 \times 5 \times .... \times (2p-1)}{n(n+2).......(n-2+2p)} \varphi(x_0)^{p(2^*-2)} (-\Delta_g)^p S_g(x_0)$$

Together with (3.2), this proves the theorem when $n \ge 6$ and $2p < n-4$. As already mentioned, the proof for the other cases goes in a similar way.

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