BUBBLING PHENOMENA FOR FOURTH-ORDER FOUR-DIMENSIONAL PDES WITH EXPONENTIAL GROWTH

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ABSTRACT. We are concerned in this short paper with the bubbling phenomenon for nonlinear fourth-order four-dimensional PDE's. The operators in the equations are perturbations of the bi-Laplacian. The nonlinearity is of exponential growth. Such equations arise naturally in statistical physics and geometry. As a consequence of our theorem we get a priori bounds for solutions of our equations.

We are concerned in this paper with understanding the bubbling phenomenon for fourth-order four-dimensional PDE's of exponential growth. Such equations arise naturally in statistical physics and in geometry (see [7] and [9]). In what follows, we let (M, g) be a smooth compact Riemannian 4-manifold without boundary. We let also $(b_\varepsilon)_{\varepsilon>0}$ and $(f_\varepsilon)_{\varepsilon>0}$ be sequences of smooth functions on M, and $(A_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth $(2,0)$-symetric tensor fields. We assume that (b_ε), (f_ε) and (A_ε) converge as $\varepsilon \to 0$ in the C^k-topologies, k positive integer, to limiting objects of the same nature, b_0, f_0 and A_0. Then we consider sequences $(u_\varepsilon)_{\varepsilon>0}$ of solutions of

$$\Delta_g^2 u_\varepsilon + R_\varepsilon\left(x, du_\varepsilon\right) = f_\varepsilon(x) e^{u_\varepsilon} \tag{1}$$

where $\Delta_g = -div_g\left(\nabla\, .\right)$ is the Laplace-Beltrami operator and

$$R_\varepsilon\left(x, du\right) = -div_g\left(A_\varepsilon du\right) + b_\varepsilon\ . \tag{2}$$

Following standard terminology, we say that the u_ε's blow up if $u_\varepsilon\left(x_\varepsilon\right) \to +\infty$ as $\varepsilon \to 0$ for a sequence (x_ε) of points in M. We let

$$L_0 = \Delta_g^2 u - div_g\left(A_0 du\right) \tag{3}$$

be the limit operator in (1). At last, we let G be the Green function of L_0. The Green function is unique up to a constant when the kernel of L_0 consists only of constants. We write G as

$$G(x,y) = \frac{1}{8\pi^2} \ln \frac{1}{d_g\left(x,y\right)} + \beta\left(x,y\right)$$

for $(x,y) \in M \times M \setminus D$, with $D = \{(x,x),\, x \in M\}$ is the diagonal in $M \times M$, where $\beta \in C^1\left(M \times M\right)$. We let φ be the function given by

$$\varphi(x) = \int_M G\left(x,y\right) b_0(y)\, dv_g(y)\ .$$

For u a function on M we let

$$\bar{u} = \frac{1}{Vol_g\left(M\right)} \int_M u\, dv_g$$

Date: September 2004.
1991 *Mathematics Subject Classification.* 58E30, 58J05, 35J35.
Key words and phrases. Concentration estimates, fourth-order equations, compactness.

be the mean value of u, where $Vol_g(M)$ is the volume of M with respect to g. Our theorem states as follows :

Theorem 1. *Let (M,g) be a smooth compact Riemannian manifold of dimension 4 without boundary. Let (u_ε) be a blowing-up sequence of solutions of (1). Assume that the kernel of L_0 consists only of constants and that f_0 is a positive function on M. Then*

$$\int_M b_0\, dv_g = 64\pi^2 N$$

for some $N \in \mathbb{N}^\star$. Moreover there exists a finite subset $S \subset M$, consisting of N points x_i's, $i = 1, \ldots, N$, such that

$$u_\varepsilon - \bar{u}_\varepsilon \to 64\pi^2 \sum_{i=1}^{N} G\left(x_i, \,.\right) - \varphi$$

in $C^4_{loc}(M\backslash S)$. At last, we have that

$$64\pi^2 \nabla_y \beta\left(x_i, x_i\right) + 64\pi^2 \sum_{j\neq i} \nabla_x G\left(x_i, x_j\right) - \nabla\varphi(x_i) = -\frac{\nabla f_0\left(x_i\right)}{f_0\left(x_i\right)}$$

for all $i = 1, \ldots, N$.

The proof of Theorem 1 comes with strong pointwise estimates on the u_ε's and the observation that concentration points are isolated (we refer to section 1 for details). This should be compared with the more intricate situation of Yamabe type equations for which concentration points are not necessarily isolated (see [3, 4, 5, 6]). Independently, as is easily checked, a priori C^4-bounds on sequences of solutions follow from the above theorem when $\int_M b_0\, dv_g \notin 64\pi^2\mathbb{N}$. This includes compactness of the geometric Paneitz equation with arbitrary prescribed Q-curvature (we refer to the nice surveys [1] and [2] for material on the Q-curvature). Such a priori C^4-bounds should be regarded as a first step towards a Morse theory for the equations we consider in this paper. We refer to [11] where this question was handled in the case of the Yamabe equation.

1. Proof of Theorem 1

Let us assume that we have a sequence (u_ε) of smooth solutions of

$$L_\varepsilon u_\varepsilon + b_\varepsilon(x) = f_\varepsilon(x) e^{u_\varepsilon}\ . \tag{4}$$

where $L_\varepsilon = \Delta_g^2 - div_g\left(A_\varepsilon d\,.\right)$. Since we assumed that $Ker\, L_0 = \{constants\}$, it is clear that $Ker\, L_\varepsilon = \{constants\}$ for all $\varepsilon > 0$ small enough. Thus, if the sequence (u_ε) is bounded from above, it follows from standard elliptic theory that (u_ε) is uniformly bounded in $C^4(M)$ except if $\int_M b_0\, dv_g = 0$. This clarifies the remarks after the theorem. From now on, we assume that the u_ε's blow-up, i.e. that

$$\max_M u_\varepsilon \to +\infty \ \text{ as } \varepsilon \to 0\ . \tag{5}$$

Before starting the proof of Theorem 1, we note that, integrating equation (4),

$$\int_M f_\varepsilon e^{u_\varepsilon}\, dv_g = \int_M b_\varepsilon\, dv_g = \int_M b_0\, dv_g + o(1)\ . \tag{6}$$

We divide the proof into several steps. The first step goes as follows :

Step 1 - *Assume that (5) holds. Then there exist $N \in \mathbb{N}^\star$ and N sequences $(x_{i,\varepsilon})$ of converging points in M such that, after passing to a subsequence, the following assertions hold :*

a) $\frac{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty$ *as $\varepsilon \to 0$ for all $i, j = 1, \ldots, N$, $i \neq j$ where*

$$f_\varepsilon\left(x_{i,\varepsilon}\right) \mu_{i,\varepsilon}^4 e^{u_\varepsilon(x_{i,\varepsilon})} = 1\ .$$

b) We have that

$$v_{i,\varepsilon}(x) = u_\varepsilon\big(\exp_{x_{i,\varepsilon}}(\mu_{i,\varepsilon}x)\big) - u_\varepsilon(x_{i,\varepsilon}) \to V_0(x) = -4\ln\left(1+\frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$ *as* $\varepsilon \to 0$ *for all* $i = 1, \dots, N$.

c) For all $i = 1, \dots, N$, *we have that*

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon}\,dv_g = 64\pi^2\ .$$

d) At last, there exists $C > 0$ *such that*

$$\big(\inf_{i=1,\dots,N} d_g(x_{i,\varepsilon},x)^4\big)e^{u_\varepsilon(x)} \le C$$

for all $\varepsilon > 0$ *and all* $x \in M$.

Proof of Step 1 - We briefly sketch the proof below and we refer to [10] for the details. We let $x_\varepsilon \in M$ be such that $u_\varepsilon(x_\varepsilon) = \max_M u_\varepsilon$. By (5), $u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. We let $\mu_\varepsilon > 0$ be defined by

$$f_\varepsilon(x_\varepsilon)\,\mu_\varepsilon^4 e^{u_\varepsilon(x_\varepsilon)} = 1 \tag{7}$$

so that $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. We let for $x \in B_0\left(\delta\mu_\varepsilon^{-1}\right)$, the Euclidean ball of center 0 and radius $\delta\mu_\varepsilon^{-1}$, $\delta > 0$ small fixed,

$$\begin{aligned}
&v_\varepsilon(x) = u_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right) - u_\varepsilon(x_\varepsilon)\ ,\\
&g_\varepsilon(x) = \left(\exp^\star_{x_\varepsilon} g\right)(\mu_\varepsilon x)\ ,\ \tilde{A}_\varepsilon(x) = \left(\exp^\star_{x_\varepsilon} A_\varepsilon\right)(\mu_\varepsilon x)\ ,\\
&\tilde{b}_\varepsilon(x) = b_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right) \text{ and } \tilde{f}_\varepsilon(x) = f_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right)\ .
\end{aligned} \tag{8}$$

We then have that

$$\Delta^2_{g_\varepsilon} v_\varepsilon - \mu_\varepsilon^2 div_{g_\varepsilon}\left(\tilde{A}_\varepsilon dv_\varepsilon\right) + \mu_\varepsilon^4\tilde{b}_\varepsilon = \frac{\tilde{f}_\varepsilon}{f_\varepsilon(x_\varepsilon)}e^{v_\varepsilon} \tag{9}$$

in $B_0\left(\delta\mu_\varepsilon^{-1}\right)$. We write with the Green representation formula that

$$u_\varepsilon(x) - \bar{u}_\varepsilon = \int_M G_\varepsilon(x,y)\,L_\varepsilon u_\varepsilon(y)\,dv_g(y)$$

for all $x \in M$ where G_ε is the Green function of L_ε. Using equation (4) and differentiating the above with respect to x, we obtain for $k = 1, 2, 3$ that

$$\begin{aligned}
\left|\nabla^k u_\varepsilon\right|_g(x) &\le \int_M \left|\nabla^k_x G_\varepsilon(x,y)\right|_g \left|f_\varepsilon(y)e^{u_\varepsilon(y)} - b_\varepsilon(y)\right| dv_g(y)\\
&\le \int_M \left|\nabla^k_x G_\varepsilon(x,y)\right|_g f_\varepsilon(y)e^{u_\varepsilon(y)}\,dv_g(y) + O(1)
\end{aligned}$$

since $b_\varepsilon \to b_0$ in $C^0(M)$ as $\varepsilon \to 0$. Let $y_\varepsilon \in B_{x_\varepsilon}(R\mu_\varepsilon)$, $R > 0$ fixed. We write that

$$\begin{aligned}
&\int_M \left|\nabla^k_x G(y_\varepsilon,y)\right|_g e^{u_\varepsilon(y)}\,dv_g(y)\\
&= O\left(\mu_\varepsilon^{-k}\int_{M\setminus B_{y_\varepsilon}(\mu_\varepsilon)} e^{u_\varepsilon}\,dv_g\right) + O\left(e^{u_\varepsilon(x_\varepsilon)}\int_{B_{y_\varepsilon}(\mu_\varepsilon)} d_g(y_\varepsilon,y)^{-k}\,dv_g(y)\right)\\
&= O\left(\mu_\varepsilon^{-k}\right)
\end{aligned}$$

thanks to the fact that $u_\varepsilon \le u_\varepsilon(x_\varepsilon)$, to (7) and to standard estimates on the Green function (which are uniform in ε). Together with the definition (8) of v_ε, this gives that (v_ε) is uniformly bounded in $C^3(K)$ for all compact subset K of $\mathbb{R}^4$. Standard elliptic theory gives then thanks to equation (9) that

$$\lim_{\varepsilon\to 0} v_\varepsilon = V_0 \text{ in } C^4_{loc}\left(\mathbb{R}^4\right) \tag{10}$$

where V_0 is a solution of

$$\Delta_\xi^2 V_0 = e^{V_0} \tag{11}$$

in $\mathbb{R}^4$ satisfying $V_0(x) \le V_0(0) = 0$ for all $x \in \mathbb{R}^4$. Moreover, since

$$\lim_{\varepsilon \to 0} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon e^{u_\varepsilon} \, dv_g = \int_{B_0(R)} e^{V_0} \, dx \ ,$$

equation (6) implies that $e^{V_0} \in L^1\left(\mathbb{R}^4\right)$. From the classification of the solutions of equation (11) by Lin [8], we get that either

$$V_0(x) = -4 \ln \left(1 + \frac{|x|^2}{8\sqrt{6}}\right) \tag{12}$$

or there exists $a > 0$ such that

$$\Delta_\xi V_0 \ge a \tag{13}$$

in $\mathbb{R}^4$. Let us prove that we are in the first situation. For that purpose, we write with the Green representation formula and equation (4) that

$$\begin{aligned}
&\int_{B_0(R)} \left|\Delta_{g_\varepsilon} v_\varepsilon\right|_{g_\varepsilon} \, dv_{g_\varepsilon} = \mu_\varepsilon^{-2} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} \left|\Delta_g u_\varepsilon\right|_g \, dv_g \\
&\le C\mu_\varepsilon^{-2} \int_{x \in B_{x_\varepsilon}(R\mu_\varepsilon)} \int_{y \in M} \left|\Delta_{g,x} G_\varepsilon(x,y)\right|_g \left(e^{u_\varepsilon(y)} + 1\right) \, dv_g(y) \, dv_g(x) \\
&\le C\mu_\varepsilon^{-2} \int_{y \in M} \left(e^{u_\varepsilon(y)} + 1\right) \left(\int_{x \in B_{x_\varepsilon}(R\mu_\varepsilon)} d_g(x,y)^{-2} \, dv_g(x)\right) dv_g(y) \\
&\le CR^2
\end{aligned}$$

thanks to standard estimates on the Green function and to (6) where $C > 0$ denotes some constant independent of R and $\varepsilon > 0$. Letting $\varepsilon \to 0$, we get that

$$\int_{B_0(R)} \left|\Delta_\xi V_0\right|_\xi \, dx \le CR^2$$

for all $R > 0$. This clearly eliminates the possibility (13). Then (12) must hold. It is then easily checked that

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon e^{u_\varepsilon} \, dv_g = \int_{\mathbb{R}^4} e^{V_0} \, dx = 64\pi^2 \ . \tag{14}$$

For $k \ge 1$, we say that $\mathcal{H}_k$ holds if there exist $\left(x_{i,\varepsilon}\right)_{i=1,\dots,k}$ k converging sequences of points in M and $\left(\mu_{i,\varepsilon}\right)_{i=1,\dots,k}$ k sequences of positive real numbers going to 0 as $\varepsilon \to 0$ such that $f_\varepsilon\left(x_{i,\varepsilon}\right) \mu_{i,\varepsilon}^4 e^{u_\varepsilon\left(x_{i,\varepsilon}\right)} = 1$ and such that, after passing to a subsequence, the following assertions hold :

(A_k^1) $\frac{d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right)}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ for all $i, j = 1, \dots, N$, $i \ne j$.

(A_k^2) We have that

$$v_{i,\varepsilon}(x) = u_\varepsilon\big(\exp_{x_{i,\varepsilon}}\left(\mu_{i,\varepsilon} x\right)\big) - u_\varepsilon\left(x_{i,\varepsilon}\right) \to V_0(x) = -4 \ln \left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}\left(\mathbb{R}^4\right)$ as $\varepsilon \to 0$ for all $i = 1, \dots, N$.

(A_k^3) For all $i = 1, \dots, N$, we have that

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} \, dv_g = 64\pi^2 \ .$$

Clearly, with what we said above, $\mathcal{H}_1$ holds. We let now $k \ge 1$ and assume that $\mathcal{H}_k$ holds. We also assume that

$$\sup_M R_{k,\varepsilon}(x)^4 e^{u_\varepsilon(x)} \to +\infty \ \text{ as } \varepsilon \to 0 \tag{15}$$

where

$$R_{k,\varepsilon}(x) = \min_{i=1,\dots,k} d_g\left(x_{i,\varepsilon}, x\right)\ .$$

We prove in the following that, in this situation, $\mathcal{H}_{k+1}$ holds. For that purpose, we let $x_{k+1,\varepsilon} \in M$ be such that

$$R_{k,\varepsilon}\left(x_{k+1,\varepsilon}\right)^4 e^{u_\varepsilon(x_{k+1,\varepsilon})} = \sup_M R_{k,\varepsilon}(x)^4 e^{u_\varepsilon(x)} \tag{16}$$

and we set

$$\mu_{k+1,\varepsilon} = \left(\frac{1}{f_\varepsilon\left(x_{k+1,\varepsilon}\right) e^{u_\varepsilon(x_{k+1,\varepsilon})}}\right)^{\frac{1}{4}}\ .$$

Since M is compact, (15) implies that $\mu_{k+1,\varepsilon} \to 0$ as $\varepsilon \to 0$ and that

$$\frac{d_g\left(x_{i,\varepsilon}, x_{k+1,\varepsilon}\right)}{\mu_{k+1,\varepsilon}} \to +\infty \ \text{ as } \varepsilon \to 0 \tag{17}$$

for all $i = 1, \dots, k$. Thanks to (A_k^2), it is also easily checked that $\frac{d_g(x_{i,\varepsilon}, x_{k+1,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ for all $i = 1, \dots, k$ so that (A_{k+1}^1) holds. It follows from (16) and (17) that

$$\lim_{\varepsilon\to 0} \sup_{z\in B_{x_{k+1,\varepsilon}}(R\mu_{k+1,\varepsilon})} \left(u_\varepsilon(z) - u_\varepsilon\left(x_{k+1,\varepsilon}\right)\right) = 0\ .$$

Mimicking what we did above thanks to the Green representation formula, one proves then that, after passing to a subsequence,

$$u_\varepsilon\left(\exp_{x_{k+1,\varepsilon}}\left(\mu_{k+1,\varepsilon}x\right)\right) - u_\varepsilon\left(x_{k+1,\varepsilon}\right) \to V_0(x)$$

in $C^4_{loc}\left(\mathbb{R}^4\right)$ as $\varepsilon \to 0$. And, as a consequence,

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_{k+1,\varepsilon}}(R\mu_{k+1,\varepsilon})} f_\varepsilon e^{u_\varepsilon}\, dv_g = 64\pi^2\ .$$

Recollecting the informations above, one gets that $\mathcal{H}_{k+1}$ holds. Since (A_k^1) and (A_k^3) of $\mathcal{H}_k$ imply that

$$\int_M f_\varepsilon e^{u_\varepsilon}\, dv_g \geq 64\pi^2 k + o(1)\ ,$$

we easily get thanks to (6) that there exists a maximal k, $1 \leq k \leq \frac{1}{64\pi^2}\int_M b_0\, dv_g$, such that $\mathcal{H}_k$ holds. Arriving to this maximal k, we get that (15) can not hold. Writing $k = N$, we have finished the proof of Step 1. $\diamondsuit$

STEP 2 - *For $k = 1$, 2, 3, there exists $C_k > 0$ such that*

$$R_\varepsilon(x)^k \left|\nabla^k u_\varepsilon\right|_g(x) \leq C_k$$

for all $x \in M$ and all $\varepsilon > 0$. Here,

$$R_\varepsilon(x) = \inf_{i=1,\dots,N} d_g\left(x_{i,\varepsilon}, x\right)$$

where the $x_{i,\varepsilon}$'s are as in Step 1.

PROOF OF STEP 2 - We use again the Green representation for u_ε that we differentiate. We let $x_\varepsilon \in M$ be such that $x_\varepsilon \neq x_{i,\varepsilon}$ for all $i = 1, \dots, N$. Note that, for $x_\varepsilon = x_{i,\varepsilon}$, the estimates of the proposition are obvious. We write thanks to standard estimates on the Green function that

$$\left|\nabla^k u_\varepsilon\right|_g(x_\varepsilon) = O\left(\int_M \frac{1}{d_g\left(x_\varepsilon, y\right)^k} e^{u_\varepsilon(y)}\, dv_g(y)\right) + O(1)\ .$$

For $i = 1, \dots, N$, we let

$$\Omega_{i,\varepsilon} = \left\{y \in M,\ R_\varepsilon(y) = d_g\left(x_{i,\varepsilon}, y\right)\right\}$$

and we write that

$$\begin{aligned}
&\int_{\Omega_{i,\varepsilon}} \frac{1}{d_g\left(x_\varepsilon, y\right)^k} e^{u_\varepsilon(y)}\, dv_g(y)\\
&= O\left(\frac{1}{d_g\left(x_\varepsilon, x_{i,\varepsilon}\right)^k} \int_{\Omega_{i,\varepsilon}\cap B_{x_{i,\varepsilon}}\left(\frac{d_g\left(x_\varepsilon, x_{i,\varepsilon}\right)}{2}\right)} e^{u_\varepsilon}\, dv_g\right)\\
&\quad + O\left(\int_{\Omega_{i,\varepsilon}\setminus B_{x_{i,\varepsilon}}\left(\frac{d_g\left(x_{i,\varepsilon}, x_\varepsilon\right)}{2}\right)} \frac{1}{d_g\left(x_\varepsilon, y\right)^k}\frac{1}{d_g\left(y, x_{i,\varepsilon}\right)^4}\, dv_g(y)\right)\\
&= O\left(\frac{1}{d_g\left(x_\varepsilon, x_{i,\varepsilon}\right)^k}\right)
\end{aligned}$$

thanks to assertion d) of Step 1, to (6) and to some straightforward computations. Step 2 clearly follows. ◊

STEP 3 - *For any $1 \le \nu < 2$, there exists $\delta_\nu > 0$ and $C_\nu > 0$ such that*

$$\mu_{i,\varepsilon}^{4(1-\nu)} d_g\left(x_{i,\varepsilon}, x\right)^{4\nu} e^{u_\varepsilon(x)} \le C_\nu$$

for all $i = 1, \dots, N$, all $\varepsilon > 0$ and all $x \in B_{x_{i,\varepsilon}}\left(\delta_\nu\right)$ where $x_{i,\varepsilon}$ and $\mu_{i,\varepsilon}$ are as in Step 1. In particular, we have that

$$d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right) \ge \delta_0$$

for all i, $j \in \{1, \dots, N\}$, $i \ne j$, where $\delta_0 > 0$ is independent of ε and i, j. At last, this implies that $\bar{u}_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

PROOF OF STEP 3 - Fix $1 \le \nu < 2$. We set for $i = 1, \dots, N$

$$R_{i,\varepsilon} = \min_{j \ne i} d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right) \tag{18}$$

and we take some $i \in \{1, \dots, N\}$ such that there exists $\theta > 0$ such that

$$R_{i,\varepsilon} \le \theta R_{j,\varepsilon} \tag{19}$$

for all $j \in \{1, \dots, N\}$. We set

$$\varphi_{i,\varepsilon}(r) = r^{4\nu} \exp\left(\left(Vol_g\left(\partial B_{x_{i,\varepsilon}}(r)\right)\right)^{-1} \int_{\partial B_{x_{i,\varepsilon}}(r)} u_\varepsilon\, d\sigma_g\right) \tag{20}$$

for $0 \le r < inj_g(M)$. A simple consequence of assertion b) of Step 1 is that

$$\varphi_{i,\varepsilon}'\left(R\mu_{i,\varepsilon}\right) < 0 \tag{21}$$

for $\varepsilon > 0$ small and all $R \ge R_\nu$ where $R_\nu^2 = \frac{16\sqrt{6}\nu}{2-\nu}$. We define $r_{i,\varepsilon}$ by

$$r_{i,\varepsilon} = \inf\left\{R_\nu \mu_{i,\varepsilon} \le r \le \frac{R_{i,\varepsilon}}{2} \text{ s.t. } \varphi_{i,\varepsilon}'(r) < 0 \text{ in } \left[R_\nu \mu_{i,\varepsilon}, r\right)\right\}. \tag{22}$$

Note that, by (21), we have that

$$\frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 . \tag{23}$$

Let us assume that

$$r_{i,\varepsilon} \to 0 \text{ as } \varepsilon \to 0 . \tag{24}$$

We set for $x \in B_0\left(\delta r_{i,\varepsilon}^{-1}\right)$, $\delta > 0$ small fixed,

$$v_{i,\varepsilon}(x) = u_\varepsilon\left(\exp_{x_{i,\varepsilon}}\left(r_{i,\varepsilon} x\right)\right) - C_{i,\varepsilon} \tag{25}$$

where

$$C_{i,\varepsilon} = \left(Vol_g\left(\partial B_{x_{i,\varepsilon}}(r_{i,\varepsilon})\right)\right)^{-1} \int_{\partial B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} u_\varepsilon\, d\sigma_g . \tag{26}$$

We also set, for $j \in \mathcal{S}_i = \{j \neq i \text{ s.t. } d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) = O(r_{i,\varepsilon})\}$,

$$\tilde{x}_{j,\varepsilon} = r_{i,\varepsilon}^{-1} \exp_{x_{i,\varepsilon}}^{-1}(x_{j,\varepsilon}) \quad \text{and} \quad \tilde{x}_j = \lim_{\varepsilon \to 0} \tilde{x}_{j,\varepsilon} , \tag{27}$$

after passing to a subsequence, if necessary. Note that, thanks to (18), to (22) and to the choice of i we made (see (19)), we have that $|\tilde{x}_j| \geq 2$ for all $j \in \mathcal{S}_i$ and that $|\tilde{x}_j - \tilde{x}_k| \geq \frac{2}{\theta}$ for all j, $k \in \mathcal{S}_i$, $j \neq k$. By equation (4), we have that

$$\Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} - r_{i,\varepsilon}^2 div_{g_{i,\varepsilon}}(A_{i,\varepsilon} \nabla v_{i,\varepsilon}) + r_{i,\varepsilon}^4 b_{i,\varepsilon} = f_{i,\varepsilon} \varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} e^{v_{i,\varepsilon}} \tag{28}$$

in $B_0\left(\delta r_{i,\varepsilon}^{-1}\right)$ where

$$\begin{aligned} g_{i,\varepsilon}(x) &= \left(\exp_{x_{i,\varepsilon}}^{\star} g\right)(r_{i,\varepsilon} x) \ , \quad A_{i,\varepsilon}(x) = \left(\exp_{x_{i,\varepsilon}}^{\star} A_\varepsilon\right)(r_{i,\varepsilon} x) \ , \\ b_{i,\varepsilon}(x) &= b_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon} x)\right) \ \text{and} \ f_{i,\varepsilon}(x) = f_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon} x)\right) \ . \end{aligned} \tag{29}$$

Thanks to Step 2, we know that $(v_{i,\varepsilon})$ is uniformly bounded in $C^3(K)$ for all compact subsets K of $\mathbb{R}^4 \backslash \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i}$. Thanks to the definition (22) of $r_{i,\varepsilon}$ and to (23), we have that

$$\varphi_{i,\varepsilon}(r_{i,\varepsilon}) \leq \varphi_{i,\varepsilon}(R\mu_{i,\varepsilon})$$

for all $R > R_\nu$. Thanks to assertion b) of Step 1 and to (23), it is now rather easily checked that

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \varphi_{i,\varepsilon}(R\mu_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} = 0$$

since $1 \leq \nu < 2$. Thus standard elliptic theory leads thanks to (28) and (29) that, after passing to a subsequence,

$$v_{i,\varepsilon} \to H_i \ \text{ in } C_{loc}^4\left(\mathbb{R}^4 \backslash \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i}\right) \ \text{ as } \varepsilon \to 0 \tag{30}$$

where H_i satisfies

$$\Delta_\xi^2 H_i = 0 \ \text{ in } \mathbb{R}^4 \backslash \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i} \ . \tag{31}$$

Moreover, thanks to Step 2, we have that, for $l = 1, 2, 3$,

$$R(x)^l |\nabla^l H_i(x)|_\xi \leq C_l \ \text{ in } \mathbb{R}^4 \backslash \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i} \tag{32}$$

where

$$R(x) = \min\{|x|; |x - \tilde{x}_j|\}_{j \in \mathcal{S}_i} \ .$$

Equation (32) easily permits to prove that

$$H_i(x) = \alpha \ln \frac{1}{|x|} + \sum_{j \in \mathcal{S}_i} \alpha_j \ln \frac{1}{|x - \tilde{x}_j|} + \beta \tag{33}$$

where α, β and the α_j's are real numbers. Integrating equation (28) over $B_0(1)$ and passing to the limit as $\varepsilon \to 0$ thanks to (29), (30) and (33), we obtain that

$$\lim_{\varepsilon \to 0} \varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(1)} f_{i,\varepsilon} e^{v_{i,\varepsilon}} \, dv_{g_{i,\varepsilon}} = -\int_{\partial B_0(1)} \partial_\nu \Delta_\xi H_i \, d\sigma_\xi = 8\alpha\pi^2 \ .$$

With a change of variable, we get that

$$\varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(1)} f_{i,\varepsilon} e^{v_{i,\varepsilon}} \, dv_{g_{i,\varepsilon}} = \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} \, dv_g$$

so that

$$\lim_{\varepsilon \to 0} \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} \, dv_g = 8\alpha\pi^2 \ . \tag{34}$$

Step 2 with $k = 1$ together with the definitions of $R_{i,\varepsilon}$ and $r_{i,\varepsilon}$ gives the existence of some $C > 0$ such that for any $0 \leq r \leq 3/2$,

$$\left| u_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon} x)\right) - u_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon} y)\right) \right| \leq C$$

for all $x, y \in \mathbb{R}^4$ such that $|x| = |y| = r$. With point b) of Step 1, (22) and (23), we then get that for any $\eta > 0$, there exists $R_\eta > 0$ such that for any $R > R_\eta$, we have that

$$d_g(x, x_{i,\varepsilon})^{4\nu} e^{u_\varepsilon(x)} \leq \eta \mu_{i,\varepsilon}^{4(\nu-1)} \tag{35}$$

for all $x \in B_{x_{i,\varepsilon}}(r_{i,\varepsilon}) \setminus B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})$. With point b) of Step 1 and (35), we get that

$$\lim_{\varepsilon \to 0} \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} \, dv_g = 64\pi^2 \ .$$

With (34), we obtain that $\alpha = 8$. Integrating on $B_{\tilde{x}_j}(\delta)$ for $\delta > 0$ small instead of $B_0(1)$, one proves in the same way that $\alpha_j \geq 8$ for all $j \in \mathcal{S}_i$. We let

$$\bar{H}_i(r) = \frac{1}{2\pi^2 r^3} \int_{\partial B_0(r)} H_i(x) \, d\sigma \ .$$

A simple computation gives that

$$\frac{d}{dr}\left(r^{4\nu} e^{\bar{H}_i(r)}\right) = 4\left(\nu - 2 - \left(\sum_{j \in \mathcal{S}_i} \frac{\alpha_j}{8|\tilde{x}_j|^2}\right) r^2\right) r^{4\nu-1} e^{\bar{H}_i(r)}$$

for $r \in \left(0, \frac{3}{2}\right)$. Since $\nu < 2$, we get in particular that

$$\frac{d}{dr}\left(r^{4\nu} e^{\bar{H}_i(r)}\right)(1) < 0 \ .$$

This clearly proves that

$$r_{i,\varepsilon} = \frac{R_{i,\varepsilon}}{2} \tag{36}$$

for all i such that (19) holds. Thanks to (24), this in turn implies that $R_{i,\varepsilon} \to 0$ and that $\mathcal{S}_j \neq \emptyset$. Note that, for the moment, we have proved, with the help of Step 2 (see (35)), that the estimate of Step 3 holds if for any $i \in \{1, ..., N\}$, we have that $R_{i,\varepsilon} \not\to 0$ as $\varepsilon \to 0$. Indeed, if this is the case, there exists some $\delta > 0$ such that $R_{j,\varepsilon} \geq \delta$ for all $j \in \{1, \ldots, N\}$ and one can easily repeat the above arguments with any of the j's in $\{1, \ldots, N\}$. Thus, in order to end the proof of the step, it remains to prove that $R_{i,\varepsilon} \not\to 0$ as $\varepsilon \to 0$ for all $i \in \{1, ..., N\}$. We let $i_0 \in \{1, \ldots, N\}$ be such that, up to a subsequence,

$$R_{i_0,\varepsilon} = \min_{i=1,\ldots,N} R_{i,\varepsilon} \ .$$

We assume by contradiction that

$$\lim_{\varepsilon \to 0} R_{i_0,\varepsilon} = 0 \ .$$

Clearly (19) holds for $i = i_0$, and (36) holds. It then follows from the definition of $\mathcal{S}_{i_0}$ that for any $i \in \mathcal{S}_{i_0}$, there exists $C(i) > 0$ such that

$$R_{i,\varepsilon} \leq C(i) R_{j,\varepsilon}$$

for all $j \in \{1, ..., N\}$. It follows that (19) holds for all $i \in \mathcal{S}_{i_0}$, and that the preceding analysis can be carried out. We pick up $i \in \mathcal{S}_{i_0}$ such that

$$d_g(x_{i,\varepsilon}, x_{i_0,\varepsilon}) \geq d_g(x_{j,\varepsilon}, x_{i_0,\varepsilon})$$

for all $j \in \mathcal{S}_{i_0}$ and all $\varepsilon > 0$. With (27), we get that $|\tilde{x}_{i_0}| \geq |\tilde{x}_j - \tilde{x}_{i_0}|$ for all $j \in \mathcal{S}_{i_0}$. Since $\mathcal{S}_i = (\mathcal{S}_{i_0} \setminus \{i\}) \cup \{i_0\}$, we have that

$$|\tilde{x}_{i_0}| \geq |\tilde{x}_j - \tilde{x}_{i_0}|$$

for all $j \in \mathcal{S}_i$. A consequence of this inequality is that

$$(\tilde{x}_{i_0}, \tilde{x}_j) > 0 \tag{37}$$

for all $j \in \mathcal{S}_i$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product. This amounts to assuming that all the $\tilde{x}_j$'s, $j \in \mathcal{S}_i$ lie in the same half-space which boundary contains 0. Let $0 < \delta < 1$. We write thanks to equation (28) that

$$\begin{aligned}\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon} dv_{g_{i,\varepsilon}} - r^2_{i,\varepsilon} \int_{B_0(\delta)} \nabla v_{i,\varepsilon} div_{g_{i,\varepsilon}} \left(A_{i,\varepsilon} \nabla v_{i,\varepsilon}\right) dv_{g_{i,\varepsilon}} \\ = \varphi_{i,\varepsilon}\left(r_{i,\varepsilon}\right) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(\delta)} f_{i,\varepsilon} \nabla e^{v_{i,\varepsilon}}\, dv_{g_{i,\varepsilon}} - r^4_{i,\varepsilon} \int_{B_0(\delta)} b_{i,\varepsilon} \nabla v_{i,\varepsilon}\, dv_{g_{i,\varepsilon}} \ .\end{aligned}$$

Integrating by parts, using the estimates of Step 2, (6) and (30), one can easily estimate the different terms involved in this equation to arrive to

$$\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon}\, dv_{g_{i,\varepsilon}} \to 0 \text{ as } \varepsilon \to 0 \ . \tag{38}$$

Using the Cartan expansion of the metric in the exponential chart and the estimates on the derivatives of $v_{i,\varepsilon}$, some integrations by parts then lead with (30) to

$$\begin{aligned}\left(\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon}\, dv_{g_{i,\varepsilon}}\right)_k \quad \to \quad & -\int_{\partial B_0(\delta)} \partial_k H_i \left(\nabla \Delta_\xi H_i, \nu\right)_\xi d\sigma_\xi \\ & + \int_{\partial B_0(\delta)} \partial_{lk} H_i \nu^l \Delta_\xi H_i\, d\sigma_\xi \\ & + \frac{1}{2} \int_{\partial B_0(\delta)} \left(\Delta_\xi H_i\right)^2 \nu_k\, d\sigma_\xi\end{aligned}$$

as $\varepsilon \to 0$. We let

$$H_i(x) = 8 \ln \frac{1}{|x|} + G_i(x) \ .$$

Simple computations then give that

$$\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon}\, dv_\xi \to 64\pi^2 \nabla G_i(0)$$

as $\varepsilon \to 0$. Coming back to (38), we obtain that $\nabla G_i(0) = 0$, a contradiction with the choice of i we made in (37). This ends the proof of Step 3. Note that the fact that $\bar{u}_\varepsilon \to -\infty$ is a direct consequence of the estimate we just proved and of Step 2. ♢

We are now in position to conclude the proof of Theorem 1. Using the estimates of Step 3, it is easily checked that

$$\int_M f_\varepsilon e^{u_\varepsilon}\, dv_g \to 64\pi^2 N \ \text{ as } \varepsilon \to 0 \ ,$$

which gives the first assertion of the theorem thanks to (6). Since we already proved that $\bar{u}_\varepsilon \to -\infty$ as $\varepsilon \to 0$, it remains to prove the convergence of $u_\varepsilon - \bar{u}_\varepsilon$ outside the concentration points and to prove the last property of the theorem concerning the location of concentration points. We let $\mathcal{S} = \{x_i\}_{i=1,\dots,N}$ where $x_i = \lim_{\varepsilon\to 0} x_{i,\varepsilon}$. We let $x_0 \in M \setminus \mathcal{S}$ and we write with the Green representation formula that

$$u_\varepsilon\left(x_0\right) - \bar{u}_\varepsilon = \int_M G_\varepsilon\left(x_0, y\right) \left(f_\varepsilon(y) e^{u_\varepsilon(y)} - b_\varepsilon(y)\right) dv_g(y)$$

where G_ε is the Green function of L_ε. It is then easy to compute an asymptotic expansion of the different terms involved to get that

$$u_\varepsilon\left(x_0\right) - \bar{u}_\varepsilon \to 64\pi^2 \sum_{i=1}^N G\left(x_0, x_i\right) - \int_M G\left(x_0, y\right) b_0(y)\, dv_g(y) \tag{39}$$

as $\varepsilon \to 0$ where G is the Green function of the limit operator L_0. The convergence result in the theorem easily follows. The last part of the theorem is a consequence of a Pohozaev-type identity. More precisely, we write in the exponential chart around $x_i \in \mathcal{S}$ and for $\delta > 0$ small enough that

$$\int_{B_{x_i}(\delta)} (L_\varepsilon u_\varepsilon + b_\varepsilon) \nabla u_\varepsilon \, dv_g = \int_{B_{x_i}(\delta)} f_\varepsilon e^{u_\varepsilon} \nabla u_\varepsilon \, dv_g$$

thanks to equation (4). Integration by parts together with dominated convergence theorem then lead to

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{B_{x_i}(\delta)} f_\varepsilon e^{u_\varepsilon} \nabla u_\varepsilon \, dv_g = -64\pi^2 \frac{\nabla f_0(x_i)}{f_0(x_i)}$$

thanks to Steps 1 to 3 and to (39). On the other hand, after integration by parts, using (39), rather long but easy computations lead to

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{B_{x_i}(\delta)} (L_\varepsilon u_\varepsilon + b_\varepsilon) \nabla u_\varepsilon \, dv_g = 64\pi^2 \nabla G_i(x_i)$$

where

$$G_i(x) = 64\pi^2 \beta(x_i, x) + 64\pi^2 \sum_{j \neq i}^{N} G(x, x_j) - \int_M G(x, y) \, b_0(y) \, dv_g(y)$$

with β is the regular part of G. The last assertion of the theorem follows.

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