ON THE LOCAL NIRENBERG PROBLEM FOR THE Q-CURVATURES

PH. DELANOË, F. ROBERT

Abstract. The local image of each conformal Q-curvature operator on the sphere admits no scalar constraint although identities of Kazdan–Warner type hold for its graph.

1. Introduction

Let us call admissible any couple of positive integers (m,n) such that $n>1$, and $n\geq 2m$ in case n is even. Given such a couple (m,n), we will work on the standard n-sphere $(\mathbb{S}^n, g_0)$ with pointwise conformal metrics[1] $g_u = e^{2u}g_0$ and discuss the structure near $u=0$ of the image of the conformal $2m$-th order Q-curvature increment operator $u\mapsto \mathbf{Q}_{m,n}[u]=Q_{m,n}(g_u)-Q_{m,n}(g_0)$ (see section 2), thus considering a local Nirenberg-type problem (Nirenberg's one was for $m=1$, *cf. e.g.* [19, 14, 15] or [1, p.122]). At the infinitesimal level, the situation looks as follows (dropping henceforth the subscript (m,n)):

Lemma 1. *Let $L=d\mathbf{Q}[0]$ stand for the linearization at $u=0$ of the conformal Q-curvature increment operator and Λ_1, for the $(n+1)$-space of first spherical harmonics on $(\mathbb{S}^n,g_0)$. Then L is self-adjoint and Ker $L=\Lambda_1$.*

Besides, the graph $\Gamma(\mathbf{Q}) := \{(u,\mathbf{Q}[u]), u\in C^\infty(\mathbb{S}^n)\}$ of $\mathbf{Q}$ in $C^\infty(\mathbb{S}^n)\times C^\infty(\mathbb{S}^n)$ admits scalar constraints which are the analogue for $\mathbf{Q}$ of the so-called Kazdan–Warner identities for the conformal scalar curvature (*i.e.* when $m=1$) [14, 15, 5]. Here, a scalar constraint means a real-valued submersion defined near $\Gamma(\mathbf{Q})$ in $C^\infty(\mathbb{S}^n)\times C^\infty(\mathbb{S}^n)$ and vanishing on $\Gamma(\mathbf{Q})$. Specifically, we have:

Theorem 1. *For each $(u,q)\in C^\infty(\mathbb{S}^n)\times C^\infty(\mathbb{S}^n)$ and each conformal Killing vector field X on $(\mathbb{S}^n,g_0)$:*

$$(u,q)\in\Gamma(\mathbf{Q}) \implies \int_{\mathbb{S}^n}(X\cdot q)\ d\mu_u = 0$$

where $d\mu_u=e^{nu}d\mu_0$ stands for the Lebesgue measure of the metric g_u. In particular, there is no solution $u\in C^\infty(\mathbb{S}^n)$ to the equation:

$$Q(g_u) = z + \text{constant}$$

with $z\in\Lambda_1$.

Key words and phrases. conformal Q-curvature, Nirenberg problem, Paneitz–Branson operators, local image, constraints, Kazdan–Warner identities.

The first author is supported by the CNRS.

[1] all objects will be taken smooth

Due to the naturality of Q (*cf.* Remark 2) and the self-adjointness of $d\mathbf{Q}[u]$ in $L^2(M_n, d\mu_u)$ (*cf.* Remarks 3 and 4), this theorem holds as a particular case of a general result (Theorem 3 below).

Can one do better than Theorem 1, drop the u variable occuring in the constraints and find constraints bearing on the sole image of the operator $\mathbf{Q}$? Since L is self-adjoint in $L^2(\mathbb{S}^n, g_0)$ [12], Lemma 1 shows that the map $u \mapsto \mathbf{Q}[u]$ misses infinitesimally at $u = 0$ a vector space of dimension $(n+1)$. How does this translate at the local level? Calling now a real valued map K, a scalar constraint for the local image of $\mathbf{Q}$ near 0, if K is a submersion defined near 0 in $C^\infty(\mathbb{S}^n)$ such that $K \circ \mathbf{Q} = 0$ near 0 in $C^\infty(\mathbb{S}^n)$, a spherical symmetry argument (as in [8, Corollary 5]) shows that if the local image of $\mathbf{Q}$ admits a scalar constraint near 0, it must admit $(n + 1)$ independent such ones, that is the maximal expectable number. In this context, our main result is quite in contrast with Theorem 1, namely:

Theorem 2. *The local image of* $\mathbf{Q}$ *near 0 admits no scalar constraint.*

Finally, the picture about the local image of the Q-curvature increment operator on $(\mathbb{S}^n, g_0)$ may be completed with a remark:

Remark 1. The local Nirenberg problem for $\mathbf{Q}$ near 0 is governed by the nonlinear Fredholm formula (9) (*cf. infra*). In particular, as in [8, Corollary 5], a local result of Moser type [19] holds. Specifically, if $f \in C^\infty(\mathbb{S}^n)$ is close enough to zero and invariant under a nontrivial group of isometries of $(\mathbb{S}^n, g_0)$ acting without fixed points[2], then $\mathcal{D}(f) = 0$ in (9), hence f lies in the local image of $\mathbf{Q}$.

The outline of the paper is as follows. We first present (section 2) an independent account on general Kazdan–Warner type identities, implying Theorem 1. Then we focus on Theorem 2: we recall basic facts for the Q-curvature operators on spheres (section 3), then sketch the proof of Theorem 2 (section 4) relying on [8], reducing it to Lemma 1 and another key-lemma; we then carry out the proofs of the lemmas (sections 4 and 5), defering to Appendice A some eigenvalues calculations.

2. General identities of Kazdan–Warner type

The following statement is essentially due to Jean–Pierre Bourguignon [4]:

Theorem 3. *Let M_n be a compact n-manifold and $g \mapsto D(g) \in C^\infty(M)$ be a scalar natural*[3] *differential operator defined on the open cone of Riemannian metrics on M_n. Given a conformal class $\mathbf{c}$ and a Riemannian metric $g_0 \in \mathbf{c}$, sticking to the notation $g_u = e^{2u} g_0$ for $u \in C^\infty(M)$, consider the operator $u \mapsto \mathbf{D}[u] := D(g_u)$ and its linearization $L_u = d\mathbf{D}[u]$ at u. Assume that, for each $u \in C^\infty(M)$, the linear differential operator L_u is formally self-adjoint in $L^2(M, d\mu_u)$, where $d\mu_u = e^{nu} d\mu_0$ stands for the Lebesgue measure of g_u. Then, for any conformal Killing vector field X on $(M_n, \mathbf{c})$ and any $u \in C^\infty(M)$, the following identity holds:*

$$\int_M X \cdot \mathbf{D}[u] \, d\mu_u = 0 \ .$$

[2] which is more general than a free action

[3] in the sense of [21], see (5) below

In particular, if $(M_n, \mathbf{c})$ is equal to $\mathbb{S}^n$ equipped with its standard conformal class, there is no solution $u \in C^\infty(\mathbb{S}^n)$ to the equation:

$$\mathbf{D}[u] = z + \text{constant}$$

with $z \in \Lambda_1$ (a first spherical harmonic).

Proof. We rely on Bourguignon's functional integral invariants approach and follow the proof of [4, Proposition 3] (using freely notations from [4, p.101]), presenting its functional geometric framework with some care. We consider the affine Fréchet manifold Γ whose generic point is the volume form (possibly of odd type in case M is not orientable [9]) of a Riemannian metric $g \in \mathbf{c}$; we denote by ω_g the volume form of a metric g (recall the tensor ω_g is natural [21, Definition 2.1]). The metric $g_0 \in \mathbf{c}$ yields a global chart of Γ defined by:

$$\omega_g \in \Gamma \to u := \frac{1}{n} \log \left(\frac{d\omega_g}{d\omega_{g_0}} \right) \in C^\infty(M_n)$$

(viewing volume-forms like measures and using the Radon–Nikodym derivative) in other words, such that $\omega_g = e^{nu}\omega_{g_0}$; changes of such charts are indeed affine (and pure translations). It will be easier, though, to avoid the use of charts on Γ, except for proving that a 1-form is closed (*cf. infra*). The tangent bundle to Γ is trivial, equal to $T\Gamma = \Gamma \times \Omega^n(M_n)$ (setting $\Omega^k(A)$ for the k-forms on a manifold A), and there is a canonical Riemannian metric on Γ (of Fischer type [10]) given at $\omega_g \in \Gamma$ by:

$$\forall (v, w) \in T_{\omega_g}\Gamma,\ <v, w>:= \int_M \frac{dv}{d\omega_g} \frac{dw}{d\omega_g}\, \omega_g\ .$$

From Riesz theorem, a tangent covector $a \in T^*_{\omega_g}\Gamma$ may thus be identified with a tangent vector $a^\sharp \in \Omega^n(M_n)$ or else with the function $\dfrac{da^\sharp}{d\omega_g} =: \rho_g(a) \in C^\infty(M_n)$ such that:

$$\forall \varpi \in T_{\omega_g}\Gamma,\ a(\varpi) = \int_M \rho_g(a)\varpi\ . \tag{1}$$

We also consider the Lie group G of conformal maps on $(M_n, \mathbf{c})$, acting on the manifold Γ by:

$$(\varphi, \omega_g) \in G \times \Gamma \to \varphi^*\omega_g \in \Gamma$$

(indeed, we have $\varphi^*\omega_g = \omega_{\varphi^* g}$ by naturality and $\varphi \in G \Rightarrow \varphi^* g \in \mathbf{c}$). For each conformal Killing field X on $(M_n, \mathbf{c})$, the flow of X as a map $t \in \mathbb{R} \to \varphi_t \in G$ yields a vector field $\bar{X}$ on Γ defined by:

$$\omega_g \mapsto \bar{X}(\omega_g) := \frac{d}{dt}\left(\varphi_t^*\omega_g\right)_{t=0} \equiv L_X\omega_g$$

(L_X standing here for the Lie derivative on M_n). In this context, regardless of any Banach completion, one may define the (global) flow $t \in \mathbb{R} \to \bar{\varphi}_t \in \mathrm{Diff}(\Gamma)$ of $\bar{X}$ on the Fréchet manifold Γ by setting:

$$\forall \omega_g \in \Gamma,\ \bar{\varphi}_t(\omega_g) := \varphi_t^*\omega_g\ ;$$

indeed, the latter satisfies (see *e.g.* [16, p.33]):

$$\frac{d}{dt}\left(\varphi_t^*\omega_g\right) = \varphi_t^*(L_X\omega_g) \equiv L_X(\varphi_t^*\omega_g) = \bar{X}\left[\bar{\varphi}_t(\omega_g)\right]\ .$$

With the flow $(\bar{\varphi}_t)_{t\in\mathbb{R}}$ at hand, we can define the Lie derivative $L_{\bar{X}}$ of forms on Γ as usual, by $L_{\bar{X}} a := \frac{d}{dt}\left({\bar{\varphi}_t}^* a\right)_{t=0}$. Finally, one can checks Cartan's formula for $\bar{X}$, namely (setting $i_{\bar{X}}$ for the interior product with $\bar{X}$):

$$L_{\bar{X}} = i_{\bar{X}} d + d i_{\bar{X}} \tag{2}$$

by verifying it for a generic function f on Γ and for its exterior derivative df (with d defined as in [17]).

Following [4], and using our global chart $\omega_g \mapsto u$ (*cf. supra*), we apply (2) to the 1-form σ on Γ defined at ω_g by the function $\rho_g(\sigma) := \mathbf{D}[u]$ (see (1)). Arguing as in [4, p.102], one readily verifies in the chart u (and using constant local vector fields on Γ) that the 1-form σ is closed due to the self-adjointness of the linearized operator L_u in $L^2(M_n, d\mu_u)$; furthermore (dropping the chart u), one derives at once the G-invariance of σ from the naturality of $g \mapsto D(g)$. We thus have $d\sigma = 0$ and $L_{\bar{X}}\sigma = 0$, hence $d(i_{\bar{X}}\sigma) = 0$ by (2). So the function $i_{\bar{X}}\sigma$ is constant on Γ, in other words $\int_M \mathbf{D}[u]\ L_X \omega_u$ is independent of u, or else, integrating by parts, so is $\int_M X \cdot \mathbf{D}[u]\, d\mu_u$ (where $X\cdot$ stands for X acting as a derivation on real-valued functions on M_n).

To complete the proof of the first part of Theorem 3, let us show that the integrand of the latter expression at $u = 0$, namely $X \cdot D(g_0)$, vanishes for a suitable choice of the metric g_0 in the conformal class $\mathbf{c}$. To do so, we recall the Ferrand–Obata theorem [18, 20] according to which, either the conformal group G is compact, or if not then $(M_n, \mathbf{c})$ is equal to $\mathbb{S}^n$ equipped with its standard conformal class. In the former case, averaging on G, we may pick $g_0 \in \mathbf{c}$ invariant under the action of G: with g_0 such, so is $D(g_0)$ by naturality, hence indeed $X \cdot D(g_0) \equiv 0$. In the latter case, as observed below (section 5.1) $D(g_0)$ is constant on $\mathbb{S}^n$ hence the desired result follows again.

Finally, the last assertion of the theorem[4] follows from the first one, by taking for the vector field X the gradient of z with respect to the standard metric of $\mathbb{S}^n$, which is conformal Killing as well-known. □

3. Back to Q-curvatures on spheres: basic facts recalled

3.1. **The special case $n = 2m$.** Here we will consider the Q-curvature increment operator given by $\mathbf{Q}[u] = Q(g_u) - Q_0$, with

$$Q(g_u) = e^{-2mu}(Q_0 + P_0[u]) \tag{3}$$

where, on $(\mathbb{S}^n, g_0)$, $Q_0 = Q(g_0)$ is equal to $Q_0 = (2m-1)!$ and (see [6, 2]):

$$P_0 = \prod_{k=1}^{m} \left[\Delta_0 + (m-k)(m+k-1)\right], \tag{4}$$

setting henceforth Δ_0 (resp. ∇_0) for the positive laplacian (resp. the gradient) operator of g_0 (P_0 is the so-called Paneitz–Branson operator of the metric g_0).

Remark 2. One can define [7] a Paneitz–Branson operator P_0 for *any* metric g_0 (given by a formula more general than (4) of course), and a Q-curvature $Q(g_0)$ transforming like (3) under the conformal change of metrics $g_u = e^{2u} g_0$. Importantly

[4] morally consistent with Proposition 1 (below) and Fredholm theorem if L_0 is elliptic

then, the map $g \mapsto Q(g) \in C^\infty(\mathbb{S}^n)$ is natural, meaning (see *e.g.* [21, Definition 2.1]) that for any diffeomeorphism ψ we have:

$$\psi^* Q(g) = Q(\psi^* g). \tag{5}$$

Remark 3. From (3) and the formal self-adjointness of P_0 in $L^2(\mathbb{S}^n, d\mu_0)$ [12, p.91], one readily verifies that, for each $u \in C^\infty(\mathbb{S}^n)$, the linear differential operator $d\mathbf{Q}[u]$ is formally self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$.

3.2. **The case** $n \neq 2m$**.** The expression of the Paneitz–Branson operator on $(\mathbb{S}^n, g_0)$ becomes [13, Proposition 2.2]:

$$P_0 = \prod_{k=1}^{m} \left[\Delta_0 + \left(\frac{n}{2} - k\right)\left(\frac{n}{2} + k - 1\right)\right], \tag{6}$$

while the corresponding one for the metric $g_u = e^{2u} g_0$ is given by:

$$P_u(.) = e^{-\left(\frac{n}{2}+m\right)u} P_0 \left[e^{\left(\frac{n}{2}-m\right)u} .\right], \tag{7}$$

with the Q-curvature of g_u given accordingly by $\left(\frac{n}{2} - m\right) Q(g_u) = P_u(1)$. The analogue of Remark 2 still holds (now see [11, 12]). We will consider the (renormalized) Q-curvature increment operator: $\mathbf{Q}[u] = \left(\frac{n}{2} - m\right)[Q(g_u) - Q_0]$, now with:

$$\left(\frac{n}{2} - m\right) Q_0 = \left(\frac{n}{2} - m\right) Q(g_0) = P_0(1) = \prod_{k=0}^{2m-1} \left(k + \frac{n}{2} - m\right). \tag{8}$$

Remark 4. Finally, we note again that the linearized operator $d\mathbf{Q}[u]$ is formally self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$. Indeed, a straightforward calculation yields

$$d\mathbf{Q}[u](v) = \left(\frac{n}{2} - m\right) P_u(v) - \left(\frac{n}{2} + m\right) P_u(1)\ v\ ,$$

and the Paneitz–Branson operator P_u is known to be self-adjoint in $L^2(\mathbb{S}^n, d\mu_u)$ [12, p.91].

For later use, and in all the cases for (m, n), we will set p_0 for the degree m polynomial such that $P_0 = p_0(\Delta_0)$.

4. Proof of Theorem 2

The case $m = 1$ was settled in [8] with a proof robust enough to be followed again. For completeness, let us recall how it goes (see [8] for details).
If $\mathcal{P}_1$ stands for the orthogonal projection of $L^2(\mathbb{S}^n, g_0)$ onto Λ_1, Lemma 1 and the self-adjointsess of L imply [8, Theorem 7] that the modified operator

$$u \mapsto \mathbf{Q}[u] + \mathcal{P}_1 u$$

is a local diffeomorphism of a neighborhood of 0 in $C^\infty(\mathbb{S}^n)$ onto another one: set $\mathcal{S}$ for its inverse and $\mathcal{D} = \mathcal{P}_1 \circ \mathcal{S}$ (defect map). Then $u = \mathcal{S}f$ satisfies the local non-linear Fredholm-like equation:

$$\mathbf{Q}[u] = f - \mathcal{D}(f). \tag{9}$$

Moreover [8, Theorem 2] if a local constraint exists for $\mathbf{Q}$ at 0, then $\mathcal{D} \circ \mathbf{Q} = 0$ (recalling the above symmetry fact). Fixing $z \in \Lambda_1$, we will prove Theorem 2 by showing that $\mathcal{D} \circ \mathbf{Q}[tz] \neq 0$ for small $t \in \mathbb{R}$; here is how.

On the one hand, setting

$$u_t = \mathcal{S} \circ \mathbf{Q}[tz] := tu_1 + t^2 u_2 + t^3 u_3 + O(t^4),$$

Lemma 1 yields $u_1 = 0$ and the following expansion holds (as a general fact, easily verified):

$$\mathbf{Q}[u_t] + \mathcal{P}_1 u_t = t^2 (L + \mathcal{P}_1) u_2 + t^3 (L + \mathcal{P}_1) u_3 + O(t^4). \tag{10}$$

On the other hand, let us consider the expansion of $\mathbf{Q}[tz]$:

$$\mathbf{Q}[tz] = t^2 c_2[z] + t^3 c_3[z] + O(t^4)\ , \tag{11}$$

and focus on its third order coefficient $c_3[z]$, for which we will prove:

Lemma 2. *Let (m, n) be admissible, then*

$$\int_{\mathbb{S}^n} z\, c_3[z]\, d\mu_0 \neq 0\ .$$

Granted Lemma 2, we are done: indeed, the equality

$$\mathbf{Q}[u_t] + \mathcal{P}_1 u_t = \mathbf{Q}[tz]\ ,$$

combined with (10)-(11), yields

$$(L + \mathcal{P}_1) u_3 = c_3[z],$$

which, integrated against z, implies:

$$\int_{\mathbb{S}^n} z \mathcal{P}_1 u_3\, d\mu_0 \neq 0$$

(recalling L is self-adjoint and $z \in \mathrm{Ker}\ L$ by Lemma 1). Therefore $\mathcal{P}_1 u_3 \neq 0$, hence also $\mathcal{D} \circ \mathbf{Q}[tz] \neq 0$.
We have thus reduced the proof of Theorem 2 to those of Lemmas 1 and 2, which we now present.

5. Proof of Lemma 1

5.1. **Proof of the inclusion $\Lambda_1 \subset$ Ker L.** We need neither ellipticity nor conformal covariance for this inclusion to hold; the naturality (5) suffices. Let us provide a general result implying at once the one we need, namely:

Proposition 1. *Let $g \mapsto D(g)$ be any scalar natural differential operator on $\mathbb{S}^n$, defined on the open cone of Riemannian metrics, valued in $C^\infty(\mathbb{S}^n)$. For each $u \in C^\infty(\mathbb{S}^n)$, set $\mathbf{D}[u] = D(g_u) - D(g_0)$ and $L = d\mathbf{D}[0]$, where $g_u = e^{2u} g_0$. Then $\Lambda_1 \subset$ Ker L.*

Proof. Let us first observe that $D(g_0)$ must be constant. Indeed, for each isometry ψ of $(\mathbb{S}^n, g_0)$, the naturality of D implies $\psi^* D(g_0) \equiv D(g_0)$; so the result follows because the group of such isometries acts transitively on $\mathbb{S}^n$. Morally, since g_0 has constant curvature, this result is also expectable from the theory of Riemannian invariants (see [21] and references therein), here though, without any regularity (or polynomiality) assumption.
Given an arbitrary nonzero $z \in \Lambda_1$, let $S = S(z) \in \mathbb{S}^n$ stand for its corresponding "south pole" (where $z(S) = -M$ is minimum) and, for each small real t, let ψ_t denote the conformal diffeomorphism of $\mathbb{S}^n$ fixing S and composed elsewhere of:

Ster_S, the stereographic projection with pole S, the dilation $X \in \mathbb{R}^n \mapsto e^{Mt}X \in \mathbb{R}^n$, and the inverse of Ster_S. As t varies, the family ψ_t satisfies :

$$\psi_0 = I, \qquad \frac{d}{dt}(\psi_t)_{t=0} = -\nabla_0 z$$

and if we set $e^{2u_t}g_0 = \psi_t^* g_0$ we get:

$$\frac{d}{dt}(u_t)_{t=0} \equiv z.$$

Recalling $D(g_0)$ is constant, the naturality of D implies

$$\mathbf{D}[u_t] = \psi_t^* D(g_0) - D(g_0) = 0;$$

in particular, differentiating this equation at $t = 0$ yields $Lz = 0$ hence we may conclude: $\Lambda_1 \subset \mathrm{Ker}\ L$. $\square$

5.2. **Proof of the reversed inclusion Ker L $\subset \Lambda_1$.** To prove Ker L $\subset \Lambda_1$, let us argue by contradiction and assume the existence of a nonzero $v \in \Lambda_1^\perp \cap \mathrm{Ker\ L}$. If $\mathcal{B}$ is an othonormal basis of eigenfunctions of Δ_0 in $L^2(\mathbb{S}^n, d\mu_0)$, there exists an integer $i \neq 1$ and a function $\varphi_i \in \Lambda_i \cap \mathcal{B}$ (where Λ_i henceforth denotes the space of i-th spherical harmonics) such that

$$\int_{\mathbb{S}^n} \varphi_i v \, d\mu_0 \neq 0$$

(actually $i \neq 0$, due to $\int_{\mathbb{S}^n} v \, d\mu_0 = 0$, obtained just by averaging $Lv = 0$ on $\mathbb{S}^n$). By the self-adjointness of L, we may write:

$$0 = \int_{\mathbb{S}^n} \varphi_i L v \, d\mu_0 = \int_{\mathbb{S}^n} v L\varphi_i \, d\mu_0,$$

infer (see below):

$$0 = [p_0(\lambda_i) - p_0(\lambda_1)] \int_{\mathbb{S}^n} \varphi_i v \, d\mu_0,$$

and get the desired contradiction, because $p_0(\lambda_i) \neq p_0(\lambda_1)$ for $i \neq 1$ (cf. Appendix A). Here, we used the following auxiliary facts, obtained by differentiating (3) or (7) at $u = 0$ in the direction of $w \in C^\infty(\mathbb{S}^n)$:

$$\begin{array}{lll} n = 2m & \Rightarrow & Lw = P_0(w) - n!w \\ n \neq 2m & \Rightarrow & Lw = \left(\frac{n}{2} - m\right) P_0(w) - \left(\frac{n}{2} + m\right) p_0(\lambda_0) w. \end{array}$$

From $\Lambda_1 \subset \mathrm{Ker}\ L$, we get, taking $w = z \in \Lambda_1$:

$$\begin{array}{lll} n = 2m & \Rightarrow & p_0(\lambda_1) - n! = 0 \\ n \neq 2m & \Rightarrow & \left(\frac{n}{2} - m\right) p_0(\lambda_1) - \left(\frac{n}{2} + m\right) p_0(\lambda_0) = 0. \end{array} \tag{12}$$

Moreover, taking $w = \varphi_i \in \Lambda_i$, we then have:

$$\begin{array}{lll} n = 2m & \Rightarrow & L\varphi_i = [p_0(\lambda_i) - p_0(\lambda_1)]\, \varphi_i \\ n \neq 2m & \Rightarrow & L\varphi_i = \left(\frac{n}{2} - m\right) [p_0(\lambda_i) - p_0(\lambda_1)]\, \varphi_i. \end{array}$$

6. Proof of Lemma 2

6.1. **Case $m = 2n$.** For fixed $z \in \Lambda_1$ and for $t \in \mathbb{R}$ close to 0, let us compute the third order expansion of $\mathbf{Q}[tz]$. By Lemma 1 it vanishes up to first order. Noting the identity

$$\forall v \in \Lambda_1, \frac{\mathbf{Q}[v]}{Q_0} \equiv e^{-nv}(1+nv) - 1 \ ,$$

we find at once:

$$\frac{\mathbf{Q}[tz]}{Q_0} = -2m^2t^2z^2 + \frac{8}{3}m^3t^3z^3 + O(t^4) \ ,$$

in particular (with the notation of section 1)

$$c_3[z] = \frac{8}{3}m^3 Q_0 z^3$$

and Lemma 2 holds trivially.

6.2. **Case $m \neq 2n$.** In this case, calculations are drastically simplified by picking the nonlinear argument of P_0 in $P_u(1)$, namely $w := \exp[(\frac{n}{2} - m)u]$ (see (7)), as new parameter for the local image of the conformal curvature-increment operator. Since w is close to 1, we further set $w = 1 + v$, so the conformal factor becomes:

$$e^{2u} = (1+v)^{\frac{4}{n-2m}}$$

and the renormalized Q-curvature increment operator reads accordingly:

$$\mathbf{Q}[u] \equiv \tilde{Q}[v] := (1+v)^{1-2^\star} P_0(1+v) - \left(\frac{n}{2} - m\right) Q_0 \tag{13}$$

where $2^\star$ stands in our context for $\frac{2n}{n-2m}$ (admittedly a loose notation, customary for critical Sobolev exponents). Of course, Lemma 1 still holds for the operator $\tilde{Q}$ (with $\tilde{L} := d\tilde{Q}[0] \equiv \frac{2^\star}{n} L$) and proving Theorem 2 (section 4) for $\tilde{Q}$ is equivalent to proving it for $\mathbf{Q}$. Altogether, we may thus focus on the proof of Lemma 2 for $\tilde{Q}$ instead of $\mathbf{Q}$[5].

Picking z and t as above, plugging $v = tz$ in (13), and using (from (12)):

$$P_0(z) = p_0(\lambda_1)z \equiv (2^\star - 1)\left(\frac{n}{2} - m\right) Q_0 z \ ,$$

we readily calculate the expansion:

$$\frac{1}{\left(\frac{n}{2} - m\right) Q_0} \tilde{Q}[tz] = -\frac{1}{2}(2^\star - 2)(2^\star - 1)\ t^2z^2 + \frac{1}{3}(2^\star - 2)(2^\star - 1)2^\star\ t^3z^3 + O(t^4)$$

thus find for its third order coefficient:

$$\frac{1}{\left(\frac{n}{2} - m\right) Q_0} \tilde{c}_3[z] = \frac{1}{3}(2^\star - 2)(2^\star - 1)2^\star\ z^3 \ .$$

So Lemma 2 obviously holds.

[5] exercise (for the frustrated reader): prove Lemma 2 directly for $\mathbf{Q}$ (it takes a few pages)

Appendix A. Eigenvalues calculations

As well known (see e.g. [3]), for each $i \in \mathbb{N}$, the i-th eigenvalue of Δ_0 on $\mathbb{S}^n$ is equal to $\lambda_i = i(i+n-1)$. Recalling (6), we have to calculate

$$p_0(\lambda_i) = \prod_{k=1}^{m} \left[\lambda_i + \left(\frac{n}{2} - k\right)\left(\frac{n}{2} + k - 1\right)\right].$$

Setting provisionally

$$r = \frac{n-1}{2}\ ,\ s_k = k - \frac{1}{2},$$

so that:

$$\frac{n}{2} - k = r - s_k\ ,\ \frac{n}{2} + k - 1 = r + s_k\ ,\ \lambda_i = i^2 + 2ir,$$

we can rewrite:

$$\begin{aligned} p_0(\lambda_i) &= \prod_{k=1}^{m} \left[(i+r)^2 - s_k^2\right] \\ &= \prod_{k=1}^{m} \left(\frac{1}{2} + i + r - k\right)\left(\frac{1}{2} + i + r + k - 1\right) \\ &\equiv \prod_{k=0}^{2m-1} \left(\frac{1}{2} + i + r - m + k\right), \end{aligned}$$

getting (back to m, n and k only)

$$p_0(\lambda_i) = \prod_{k=0}^{2m-1} \left(i + \frac{n}{2} - m + k\right).$$

In particular, we have:

$$P_0(1) \equiv p_0(\lambda_0) = \left(\frac{n}{2} - m\right) \prod_{k=1}^{2m-1} \left(\frac{n}{2} - m + k\right)$$

as asserted in (8) (and consistently there with the value of Q_0 in case $n = 2m$). An easy induction argument yields:

$$\forall i \in \mathbb{N},\ p_0(\lambda_{i+1}) = \frac{\left(\frac{n}{2} + m + i\right)}{\left(\frac{n}{2} - m + i\right)}\ p_0(\lambda_i)$$

(consistently when $i = 0$ with (12)), which implies: $\forall i \in \mathbb{N}, |p_0(\lambda_{i+1})| > |p_0(\lambda_i)|$, hence in particular $p_0(\lambda_i) \neq p_0(\lambda_1)$ for $i > 1$ as required in the proof of Lemma 1. Moreover, it readily implies the final formula:

$$\forall i \geq 1,\ p_0(\lambda_i) = \frac{\left(\frac{n}{2} + m\right) \ldots \left(\frac{n}{2} + m + i - 1\right)}{\left(\frac{n}{2} - m\right) \ldots \left(\frac{n}{2} - m + i - 1\right)}\ p_0(\lambda_0)\ .$$

Acknowledgement: It is a pleasure to thank Colin Guillarmou for providing us with the explicit formula of the conformally covariant powers of the laplacian on the sphere [13, Proposition 2.2].

References

[1] Th. Aubin, *Nonlinear analysis on manifolds. Monge–Ampère equations*, Grundlehren der math. Wissensch. **252** (Springer, New-York, 1982)

[2] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality, *Annals of Math* **138** (1993) 213-242

[3] M. Berger, P. Gauduchon & E. Mazet, Le spectre d'une variété riemannienne, *Lecture Notes in Math.* **194** Springer–Verlag (1971)

[4] J.-P. Bourguignon, Invariants intégraux fonctionnels pour des équations aux dérivées partielles d'origine géométrique, *Lecture Notes in Math.* **1209** Springer–Verlag (1986) 100-108

[5] J-P. Bourguignon & J-P. Ezin, Scalar curvature functions in a conformal class of metrics and conformal transformations, *Trans. Amer. Math. Soc.* **301** (1987) 723-736

[6] T. Branson, Group representations arising from Lorentz conformal geometry, *J. Funct. Anal.* **74** (1987) 199-293

[7] T. Branson, Sharp inequalities, the functional determinant, and the complementary series, *Trans. Amer. Math. Soc.* **347** (1995) 3671-3742

[8] Ph. Delanoë, Local solvability of elliptic, and curvature, equations on compact manifolds, *J. reine angew. Math.* **558** (2003) 23-45

[9] G. De Rham, *Variétés différentiables*, Publ. Univ. Nancago III, Hermann, Paris 1960

[10] Th. Friedrich, Die Fischer-Information und symplektische Strukturen, *Math. Nachr.* **153** (1991) 273-296

[11] C. Robin Graham, R. Jenne, L.J. Mason & G.A.J. Sparling, Conformally invariant powers of the laplacian, I: existence, *J. London Math. Soc.* **46**:2 (1992) 557-565

[12] C. Robin Graham & M. Zworski, Scattering matrix in conformal geometry, *Invent. math.* **152** (2003) 89-118

[13] C. Guillarmou & F. Naud, Wave 0-trace and length spectrum on convex co-compact hyperbolic manifolds, *Comm. Anal. Geom.*, to appear.

[14] J.L. Kazdan & F.W. Warner, Curvature functions on compact 2-manifolds, *Annals of Math.* **99** (1974) 14-47

[15] J.L. Kazdan & F.W. Warner, Scalar curvature and conformal deformation of Riemannian structure, *J. Diff. Geom.* **10** (1975) 113-134

[16] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Interscience, vol. **I** (1963)

[17] S. Lang, *Introduction to differentiable manifolds*, John Wiley & Sons, Inc., New–York 1962

[18] J. Lelong–Ferrand, Transformations conformes et quasi-conformes des variétés riemanniennes: application à la démonstration d'une conjecture de A. Lichnerowicz, *C. R. Acad. Sci. Paris, Sér. A* **269** (1969) 583-586

[19] J. Moser, On a nonlinear problem in differential geometry, in: *Dynamical systems* (edit. M. Peixoto), Academic Press (1973), 273-279

[20] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, *J. Diff. Geom.* **6** (1971) 247-258

[21] P. Stredder, Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal groups, *J. Diff. Geom.* **10** (1975) 647-660.

Authors common address:

Université de Nice–Sophia Antipolis
Laboratoire J.–A. Dieudonné, Parc Valrose
F-06108 Nice CEDEX 2

first author's e-mail: delphi@math.unice.fr
second author's e-mail: frobert@math.unice.fr