The Bochner formula for Riemannian flows

Fida El Chami[*], Georges Habib[†]

Abstract

On a Riemannian manifold (M, g) endowed with a Riemannian flow, we study in this paper the curvature term in the Bochner-Weitzenböck formula of the basic Laplacian. We prove that this term splits into two parts; a first part that depends on the curvature operator of the manifold M and a second part that can be expressed in terms of the O'Neill tensor of the flow. After getting a lower bound for this curvature term depending on a bound of each of these two parts, we establish an eigenvalue estimate for the basic Laplacian. We then discuss the limiting case of this latter estimate and prove that, when equality occurs, the manifold M is isometric to a local product.

Key words: Riemannian flow, basic Laplacian, eigenvalue, O'Neill tensor, Bochner formula.

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1 Introduction

Given a Riemannian manifold (M^n, g) of dimension n, the Hodge Laplacian $\Delta = d\delta + \delta d$ (δ being the L^2-adjoint of d) is related to the curvature operator on M through the Bochner-Weitzenböck formula. Namely, the formula is

$$\Delta = \nabla^*\nabla + \mathcal{B}^{[p]},$$

where $\mathcal{B}^{[p]}$, usually called the *Bochner operator*, is a symmetric endomorphism on the bundle of p-forms $\Lambda^p(M)$ given by $\mathcal{B}^{[p]} = \sum_{i,j=1}^n e_j \wedge (e_i \lrcorner R^M(e_j, e_i))$. Here R^M is the curvature operator associated to the Levi-Civita connection ∇^M on M considered as $R^M(X,Y) = \nabla^M_{[X,Y]} - [\nabla^M_X, \nabla^M_Y]$ and $\{e_i\}_{i=1,\cdots,n}$ denotes a local orthonormal frame of TM. In all the paper, we shall identify vector fields with their corresponding 1-forms through the usual musical isomorphism.

It is clear that the Bochner-Weitzenböck formula is a useful tool to estimate the eigenvalues of the Laplacian on a compact manifold, since any lower bound of the Bochner operator provides a lower bound for the eigenvalues. For example, when $p = 1$, A. Lichnerowicz [13] proved that if $\mathcal{B}^{[1]}$ (that is, the Ricci tensor of the manifold) is greater than some positive number k, the first positive eigenvalue is greater than $\frac{n}{n-1}k$. This inequality was later characterized by M. Obata in [19] who shows that equality occurs if and only if the manifold is isometric to a round sphere.

Another estimate of the Bochner operator was obtained by Gallot and Meyer in [7] when $p = 1, \cdots, n-1$. Indeed, they showed that if the curvature operator of M, considered as a symmetric 2-tensor, has a lower bound k then $\mathcal{B}^{[p]}$ is always greater than $p(n-p)k$. This inequality has led to the following rigidity result [7, Prop. 2.9]: when the lower bound k is positive, all the cohomology groups $H^p(M)$ vanish which means that the manifold M is cohomologically isometric to a round

[*]Lebanese University, Faculty of Sciences II, Department of Mathematics, P.O. Box 90656 Fanar-Matn, Lebanon, E-mail: `fchami@ul.edu.lb`

[†]Corresponding author: Lebanese University, Faculty of Sciences II, Department of Mathematics, P.O. Box 90656 Fanar-Matn, Lebanon and Université de Lorraine, CNRS, IECL, F-54506 Nancy, France, E-mail: `ghabib@ul.edu.lb`

sphere. Moreover, based on the same inequality, they proved the following estimates for the first eigenvalues $\lambda'_{1,p}$ and $\lambda''_{1,p}$ of the Laplacian restricted respectively to closed and co-closed p-forms. Namely,

$$\lambda'_{1,p} \geq kp(n-p+1) \quad \text{and} \quad \lambda''_{1,p} \geq k(p+1)(n-p). \tag{1}$$

Besides the round sphere of curvature k, the authors provided examples of hypersurfaces in the complex projective space where the equality in (1) is attained [7, Prop. 8.1].

In [23], A. Savo used a new technique to bound the Bochner operator for submanifolds. In fact, on a given Riemannian manifold M of dimension n and a submanifold Σ, he expressed the curvature operator of Σ in terms of the one on M and of the second fundamental form of the immersion through the Gauss formula. Indeed, he showed that the term $\mathcal{B}^{[p]}$, acting on p-forms of Σ, can be split into two parts: the restriction part $\mathcal{B}^{[p]}_{\text{res}}$ that mainly depends on the ambient manifold M and the exterior part $\mathcal{B}^{[p]}_{\text{ext}}$ that is determined by the Weingarten tensor S [23, Thm. 1]. The proof is based on the expression of the Bochner operator $\mathcal{B}^{[p]}$ in terms of the curvature of the manifold Σ through the Clifford Lie bracket used in [21, Lemma 4.7] (see also [23, Thm. 17]). In particular he proved that, for a hypersurface Σ, the following inequality

$$\mathcal{B}^{[p]} \geq p(n-p)(\gamma_M + \beta_p(\Sigma)) \tag{2}$$

holds, where γ_M is a lower bound of the curvature operator of M and $\beta_p(\Sigma)$ is the lowest eigenvalue of the operator $T^{[p]} := (\operatorname{tr} S)S^{[p]} - S^{[p]} \circ S^{[p]}$. The operator $S^{[p]}$ is some canonical extension of S to differential p-forms on Σ. Then, he deduced from Inequality (2) several rigidity results; among them on the vanishing of de Rham cohomology groups of Σ, on the existence of compact hypersurfaces of the round sphere having nonnegative sectional curvature, etc. Also with the help of the Bochner-Weitzenböck formula, he found a new sharp estimate for the eigenvalues of the Laplacian on Σ. Note that this last eigenvalue estimate has been later generalized to all codimensions in [4].

In this paper, we consider Riemannian foliations (see Section 2 for the definition) which are the global geometric aspects of Riemannian submersions. As in Savo's work, we aim to express the Bochner operator in the transverse Bochner-Weitzenböck formula [11] in terms of the geometric data of the foliations. With the help of the O'Neill formulas [20] we prove that, for Riemannian flows (Riemannian foliations of 1-dimensional leaves), the Bochner operator splits into a restriction part and an exterior part (see Equation (8)). The former part depends on the geometry of the ambient manifold while the latter involves the O'Neill tensor. We deduce in Corollary 5.3 a lower bound for this operator leading to vanishing results on the basic cohomology groups (see Corollary 5.4). Also, we establish a sharp estimate for the first positive eigenvalue $\lambda_{1,p}$ of the basic Laplacian restricted to p-forms ($1 \leq p \leq [\frac{q}{2}]$ where q is the codimension of the flow) on minimal flows (that is, the leaves are minimal submanifolds). Namely, we show

Theorem 1.1 *Let (M, g) be a compact Riemannian manifold endowed with a minimal Riemannian flow of codimension q. Let p be any integer number such that $1 \leq p \leq m$ with $m = [\frac{q}{2}]$. Then the first positive eigenvalue of the basic Laplacian acting on basic p-forms satisfies*

$$\lambda_{1,p} \geq p(q-p+1)(\gamma_M + \beta^1_M),$$

where γ_M is a lower bound of the curvature operator on M restricted to the normal bundle and β^1_M is the lowest eigenvalue of the square of the O'Neill tensor. If equality is attained, then M is isometric to the quotient of $\mathbb{R} \times \Sigma$ by some fixed-point-free cocompact discrete subgroup $\Gamma \subset \mathbb{R} \times \mathrm{SO}_{q+1}$, where Σ is a compact simply connected manifold of positive curvature.

The paper is organized as follows. In Section 2, we review the definitions of foliations and the basic Laplacian. We also state an eigenvalue estimate for the basic Laplacian that involves a lower bound of the Bochner operator (see Proposition 2.1). In Section 3, we adapt the technique used in [21, Lemma 4.7] for writing the Bochner operator in terms of Clifford Lie bracket to the set-up of foliations. As a consequence, we state a rigidity result for the basic cohomology groups (see Proposition 3.4). In Section 4, we made some technical computations on Riemannian flows to prove the main results in Section 5. Several examples are also provided. The last section is devoted to some general results on flows that we use in our study.

2 Preliminaries

In this section, we recall the basic facts on Riemannian foliations and some well-known results that can be found in [26].

Let $(M^n, g, \mathcal{F})$ be a Riemannian manifold of dimension n endowed with a Riemannian foliation $\mathcal{F}$ of codimension q. We assume, throughout this paper, the metric g to be a bundle-like. That means, $\mathcal{F}$ is given by an integrable subbundle L of TM and the metric g satisfies the holonomy-invariance condition $\mathcal{L}_X g|_Q = 0$ on the normal vector bundle $Q = TM/L$, for all $X \in \Gamma(L)$. Here $\mathcal{L}$ denotes the Lie derivative. In this case, the tangent bundle of M decomposes orthogonally into L and Q and the bundle Q is identified with $L^\perp$. We equip the bundle Q with the transverse Levi-Civita connection ∇ [26, p. 48] which is the unique metric and free torsion connection with respect to the induced metric on Q. The curvature of this connection vanishes along the leaves and therefore data on Q are defined along orthogonal directions.

A basic form ω is a differential form on M that does uniquely depend on the transverse variables, in other words, ω satisfying $X \lrcorner \omega = 0$ and $X \lrcorner d\omega = 0$, for all $X \in \Gamma(L)$. Roughly speaking, basic forms can be seen as differential forms on the base manifold of the local submersions that define the foliation. The set of basic forms, denoted by $\Omega(M, \mathcal{F})$, is preserved by the exterior derivative d and is used to define the basic Laplacian $\Delta_b = d_b\delta_b + \delta_b d_b$. Here, d_b is the restriction of d to $\Omega(M, \mathcal{F})$ and δ_b is its L^2-adjoint. Locally, the exterior differential and its adjoint are given by the formulas $d_b = \sum_{i=1}^q e_i \wedge \nabla_{e_i}$ and $\delta_b = -\sum_{i=1}^q e_i \lrcorner \nabla_{e_i} + \kappa_b \lrcorner$, where $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of $\Gamma(Q)$ and κ_b denotes the basic component of the mean curvature field κ of the foliation [14]. The basic Laplacian yields the basic Hodge theory that can be used to compute the basic cohomology groups

$$H_b^p(\mathcal{F}) = \frac{\ker\{d : \Omega^p(M, \mathcal{F}) \to \Omega^{p+1}(M, \mathcal{F})\}}{\mathrm{image}\{d : \Omega^{p-1}(M, \mathcal{F}) \to \Omega^p(M, \mathcal{F})\}},$$

for $0 \leq p \leq q$.

In the study of the basic Poincaré duality which fails to hold for the basic cohomology groups, the authors in [11] introduce a new cohomology group $\widetilde{H}_b(\mathcal{F})$ that uses the *twisted exterior derivative* $\widetilde{d}_b := d_b - \frac{1}{2}\kappa_b\wedge$, [14]. They prove that the twisted derivative $\widetilde{d}_b$ and its L^2-adjoint $\widetilde{\delta}_b := \delta_b - \frac{1}{2}\kappa_b\lrcorner$ share the same formulas with the basic Hodge star operator $\bar{\star}$ as on ordinary manifolds. Also, the corresponding *twisted Laplacian* $\widetilde{\Delta}_b := \widetilde{d}_b\widetilde{\delta}_b + \widetilde{\delta}_b\widetilde{d}_b$ commutes with $\bar{\star}$ and, therefore, the Poincaré duality holds for those twisted cohomology groups. They state a Bochner-Weitzenböck formula for $\widetilde{\Delta}_b$ which allows to get several rigidity results on the twisted cohomology groups as well as on the usual basic cohomology [11, Thm. 6.16]. Namely, on basic p-forms, the formula is [11, Prop. 6.7]

$$\widetilde{\Delta}_b = \nabla^*\nabla + \mathcal{B}^{[p]} + \frac{1}{4}|\kappa_b|^2, \tag{3}$$

where $\nabla^*\nabla := -\sum_{i=1}^q \nabla_{e_i,e_i} + \nabla_{\kappa_b}$ and $\mathcal{B}^{[p]} = \sum_{i,j=1}^q e_j \wedge (e_i \lrcorner R(e_j, e_i))$ with $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ is the transversal curvature operator and $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of Q. Here the basic component of the mean curvature κ_b is assumed to be a harmonic 1-form, i.e. $d_b\kappa_b = \delta_b\kappa_b = 0$.

As in [23, Prop. 3], we can state the following result

Proposition 2.1 *Let $(M, g, \mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g. Let p be an integer number such that $1 \leq p \leq q$.*

1) *If $\mathcal{B}^{[p]} \geq 0$ for some integer p and κ_b is a basic-harmonic one form, then any basic harmonic p-form is transversally parallel. If the strict inequality $\mathcal{B}^{[p]} > 0$ holds, then $H_b^p(\mathcal{F}) = 0$.*

2) *If the foliation is minimal and $\mathcal{B}^{[p]} \geq p(q-p)\Lambda$ for some $\Lambda > 0$, then the first positive eigenvalue $\lambda_{1,p}$ of the basic Laplacian satisfies*
$$\lambda_{1,p} \geq p(q-p+1)\Lambda, \tag{4}$$
where p is chosen such that $1 \leq p \leq \frac{q}{2}$.

Proof. The proof of the point 1) is a direct consequence of the Bochner-Weitzenböck formula. Indeed, take any basic harmonic p-form ω, that is $d_b\omega = \delta_b\omega = 0$, one can easily see that $|\widetilde{d_b}\omega|^2 + |\widetilde{\delta_b}\omega|^2 = \frac{1}{4}|\kappa_b|^2|\omega|^2$. Hence, applying Equation (3) to ω and taking the scalar product with the same form, we get after integrating over M
$$\frac{1}{4}\int_M |\kappa|^2|\omega|^2 dv_g = \int_M |\nabla\omega|^2 dv_g + \int_M \langle \mathcal{B}^{[p]}\omega, \omega\rangle dv_g + \frac{1}{4}\int_M |\kappa|^2|\omega|^2 dv_g \geq \frac{1}{4}\int_M |\kappa|^2|\omega|^2 dv_g,$$
which yields the first statement. Now, if $\mathcal{B}^{[p]} > 0$ then it is clear that any basic harmonic p-form vanishes. By [5] and [17, Thm 6.2], one can always change the bundle-like metric into another bundle-like metric with the same transverse metric so that the basic component of the mean curvature κ_b is a basic harmonic 1-form with respect to the new metric. Therefore, we can work with such a metric keeping the same condition on $\mathcal{B}^{[p]}$. Hence the assumption on the mean curvature can be dropped off and we deduce the statement as the basic cohomology is independent of the choice of the bundle-like metric.

The proof of the point 2) follows the same steps as in [23, Prop. 3]. Indeed, by the pointwise inequality $|\nabla\omega|^2 \geq \frac{|d_b\omega|^2}{p+1} + \frac{|\delta_T\omega|^2}{q-p+1}$ that is valid for any basic p-form ω where $\delta_T = \delta_b - \kappa_b \lrcorner$ [7], [16], we get on minimal foliations that
$$\int_M |\nabla\omega|^2 dv_g \geq \int_M \left(\frac{|d_b\omega|^2}{p+1} + \frac{|\delta_b\omega|^2}{q-p+1}\right) dv_g \geq \frac{\lambda_{1,p}}{q-p+1}\int_M |\omega|^2 dv_g.$$
Here, we use the fact that $p+1 \leq q-p+1$, as $p \leq \frac{q}{2}$. Applying Equality (3) to ω and taking the scalar product with ω itself yields the result after integration. $\square$

Remark.

1. The estimate (4) is not valid for nonminimal Riemannian foliations when considering the eigenvalues of the twisted Laplacian. Indeed by a straightforward computation we have, for any p-form ω where $p \leq \frac{q}{2}$, that
$$|\nabla\omega|^2 \geq \frac{|d_b\omega|^2}{p+1} + \frac{|\delta_T\omega|^2}{q-p+1} \geq \frac{1}{q-p+1}\left(|\widetilde{d_b}\omega|^2 + |\widetilde{\delta_b}\omega|^2 - \frac{1}{4}|\kappa_b|^2|\omega|^2 + \langle(\kappa_b\lrcorner d_b - \kappa_b \wedge \delta_T)\omega, \omega\rangle\right).$$

2. When the equality case in (4) is attained, the associated eigenform is a *basic conformal Killing form* [24, 16] which is either closed or of degree $p = \frac{q}{2}$ (that is, q should be even). Recall here that a basic conformal Killing form ω is a basic form that satisfies, for all $X \in \Gamma(Q)$, the equation
$$\nabla_X\omega = \frac{1}{p+1}X \lrcorner d_b\omega - \frac{1}{q-p+1}X \wedge \delta_T\omega,$$
where we recall $\delta_T = \delta_b - \kappa_b\lrcorner$.

3 Clifford multiplication on basic forms

In this section, we will review the approach of [21, Sect. 4] to write the curvature term in the Bochner-Weitzenböck formula in terms of the Clifford Lie bracket. We also refer to [23] for more details.

Let $(M, g, \mathcal{F})$ be a Riemannian manifold endowed with a Riemannian foliation $\mathcal{F}$ and let Q be the normal bundle of codimension q. For $X \in \Gamma(Q)$ and $\omega \in \Lambda^p Q$, the Clifford multplication of X with ω is defined as

$$X \cdot \omega = X \wedge \omega - X \lrcorner \omega \quad \text{and} \quad \omega \cdot X = (-1)^p (X \wedge \omega + X \lrcorner \omega). \tag{5}$$

A direct consequence of the definition is that the following relation

$$X \cdot Y + Y \cdot X = -2g(X, Y)$$

holds, for any two sections X and Y on Q. Given any two forms ω and θ in $\Lambda^p Q$, we can extend the definition (5) to the Clifford multplication between ω and θ as follows: write locally $\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$ in any orthonormal frame $\{e_1, \cdots, e_q\}$ of Q and define

$$\omega \cdot \theta = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p} e_{i_1} \cdot \cdots \cdot e_{i_p} \cdot \theta.$$

The Clifford multiplication "$\cdot$" is associative by construction. The Lie bracket between differential forms is now defined as $[\omega, \theta] = \omega \cdot \theta - \theta \cdot \omega$. In particular, for a 2-form Ψ and a p-form ω, the Lie bracket between Ψ and ω can be expressed explicitly as

Lemma 3.1 *Let Ψ be a 2-form and let ω be a p-form. We have*

$$[\Psi, \omega] = 2 \sum_{i=1}^{q} (e_i \lrcorner \Psi) \wedge (e_i \lrcorner \omega),$$

where $\{e_1, \cdots, e_q\}$ is an orthonormal frame of Q. In particular, the degree of $[\Psi, \omega]$ is the same as the one of the form ω.

Proof. The proof relies mainly on the use of Equations (5) and the fact that $X \cdot \Phi = (-1)^p \Phi \cdot X - 2X \lrcorner \Phi$ that is valid for any vector X and a p-form Φ. Indeed if we write $\Psi = \sum_{i<j} \Psi_{ij} e_i \wedge e_j$, we compute

$$\begin{aligned}
\Psi \cdot \omega &= \sum_{i<j} \Psi_{ij} e_i \cdot e_j \cdot \omega = \sum_{i<j} \Psi_{ij} e_i \cdot ((-1)^p \omega \cdot e_j - 2 e_j \lrcorner \omega) \\
&= \sum_{i<j} \Psi_{ij} \left(\omega \cdot e_i \cdot e_j - 2(-1)^p (e_i \lrcorner \omega) \cdot e_j - 2(-1)^{p-1} (e_j \lrcorner \omega) \cdot e_i + 4 e_i \lrcorner (e_j \lrcorner \omega) \right) \\
&= \omega \cdot \Psi - 2(-1)^p \sum_{i<j} \Psi_{ij} (e_i \lrcorner \omega) \cdot e_j - 2(-1)^{p-1} \sum_{i<j} \Psi_{ij} (e_j \lrcorner \omega) \cdot e_i + 2 \sum_{i,j} \Psi_{ij} e_i \lrcorner (e_j \lrcorner \omega) \\
&= \omega \cdot \Psi - 2(-1)^p \sum_{i,j} \Psi_{ij} (e_i \lrcorner \omega) \cdot e_j + 2 \sum_{i,j} \Psi_{ij} e_i \lrcorner (e_j \lrcorner \omega) \\
&= \omega \cdot \Psi + 2 \sum_{i,j} \Psi_{ij} (e_j \wedge (e_i \lrcorner \omega) + e_j \lrcorner (e_i \lrcorner \omega)) + 2 \sum_{i,j} \Psi_{ij} e_i \lrcorner (e_j \lrcorner \omega).
\end{aligned}$$

Finally, we deduce that $[\Psi, \omega] = 2 \sum_{i,j} \Psi_{ij} e_j \wedge (e_i \lrcorner \omega)$ which finishes the proof of the lemma. $\square$

Now, we state another useful property of the Lie bracket that will be used later in this paper.

Lemma 3.2 *Let Ψ be a 2-form and let ω be a p-form. Then we have*

$$[\Psi, X \wedge \omega] = X \cdot [\Psi, \omega] + 2(X \lrcorner \Psi) \cdot \omega + [\Psi, X \lrcorner \omega],$$

for any $X \in \Gamma(Q)$.

Proof. Using the definition of the Lie bracket and again the formula $X \cdot \Phi = (-1)^p \Phi \cdot X - 2X \lrcorner \Phi$ as before, we write

$$\begin{aligned}
[\Psi, X \wedge \omega] &= \Psi \cdot (X \wedge \omega) - (X \wedge \omega) \cdot \Psi \\
&= \Psi \cdot (X \cdot \omega + X \lrcorner \omega) - (X \cdot \omega + X \lrcorner \omega) \cdot \Psi \\
&= X \cdot \Psi \cdot \omega + 2(X \lrcorner \Psi) \cdot \omega + \Psi \cdot (X \lrcorner \omega) - X \cdot \omega \cdot \Psi - (X \lrcorner \omega) \cdot \Psi \\
&= X \cdot [\Psi, \omega] + 2(X \lrcorner \Psi) \cdot \omega + [\Psi, X \lrcorner \omega].
\end{aligned}$$

The proof of the lemma is then finished. $\square$

Next, we recall the definition of the basic Dirac operator restricted to basic forms [8]. Given any orthonormal frame $\{e_i\}_{i=1,\cdots,q}$ of $\Gamma(Q)$, the basic Dirac operator is given by

$$D_b = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i} - \frac{1}{2}\kappa_b \cdot,$$

where κ_b is as usual the basic component of the mean curvature. From its definition, one can easily check that this operator is transversally elliptic and self-adjoint, if the manifold M is compact. Also from the first equation in (5), it can be split as $D_b = \widetilde{d}_b + \widetilde{\delta}_b$ where we recall that $\widetilde{d}_b = d_b - \frac{1}{2}\kappa_b \wedge$ and $\widetilde{\delta}_b = \delta_b - \frac{1}{2}\kappa_b \lrcorner$. Hence by squaring both sides and using the fact that $\widetilde{d}_b^2 = \widetilde{\delta}_b^2 = 0$ we get that $D_b^2 = \widetilde{\Delta}_b$. Following the same steps as in [21, Thm. 50], we have the corresponding Bochner-Weitzenböck formula for the square of the basic Dirac operator.

Proposition 3.3 *Let $(M, g, \mathcal{F})$ be a Riemannian manifold endowed with a Riemannian foliation of codimension q. Assume that the basic component of the mean curvature is closed and co-closed. Then, we have*

$$D_b^2 = \nabla^*\nabla - \frac{1}{2}\sum_{i,j=1}^{q} e_i \cdot e_j \cdot R(e_i, e_j) + \frac{1}{4}|\kappa_b|^2,$$

and

$$D_b^2 = \nabla^*\nabla + \frac{1}{2}\sum_{i,j=1}^{q} R(e_i, e_j) \cdot e_i \cdot e_j + \frac{1}{4}|\kappa_b|^2.$$

Proof. We begin to prove the first identity. At any point $x \in M$, we consider a local orthonormal frame $\{e_i\}_{i=1,\cdots,q}$ of $\Gamma(Q)$ such that $\nabla e_i|_x = 0$. Then, we write at x

$$\begin{aligned}
D_b^2 &= \left(\sum_{i=1}^{q} e_i \cdot \nabla_{e_i} - \frac{1}{2}\kappa_b \cdot\right)\left(\sum_{j=1}^{q} e_j \cdot \nabla_{e_j} - \frac{1}{2}\kappa_b \cdot\right) \\
&= \sum_{i,j=1}^{q} e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j} - \frac{1}{2}\sum_{i=1}^{q} e_i \cdot \nabla_{e_i}\kappa_b \cdot - \frac{1}{2}\sum_{i=1}^{q} e_i \cdot \kappa_b \cdot \nabla_{e_i} - \frac{1}{2}\sum_{j=1}^{q} \kappa_b \cdot e_j \cdot \nabla_{e_j} - \frac{1}{4}|\kappa_b|^2 \\
&= \sum_{i,j=1}^{q} e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j} - \frac{1}{2}(d_b\kappa_b + \delta_b\kappa_b - |\kappa_b|^2) + \nabla_{\kappa_b} - \frac{1}{4}|\kappa_b|^2 \\
&= -\sum_{i=1}^{q} \nabla_{e_i}\nabla_{e_i} + \sum_{i \neq j} e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j} + \nabla_{\kappa_b} + \frac{1}{4}|\kappa_b|^2 \\
&= \nabla^*\nabla - \frac{1}{2}\sum_{i,j=1}^{q} e_i \cdot e_j \cdot R(e_i, e_j) + \frac{1}{4}|\kappa_b|^2.
\end{aligned}$$

This shows the desired identity by using that $\nabla^*\nabla = -\sum_{i=1}^q \nabla_{e_i}\nabla_{e_i} + \nabla_{\kappa_b}$. The second identity can be done in the same way as the first one. Indeed we introduce, for any basic p-form ω, the operator

$$\hat{D}_b\omega = \sum_{i=1}^q \nabla_{e_i}\omega\cdot e_i - \frac{1}{2}\omega\cdot\kappa_b = (-1)^p(\widetilde{d}_b - \widetilde{\delta}_b)\omega$$

and we observe that $\hat{D}_b^2 = \widetilde{\Delta}_b = D_b^2$. This ends the proof. □

Now adding the two equations in Proposition 3.3 and dividing by 2, we deduce after comparing with Equation (3) that

$$\mathcal{B}^{[p]}\omega = \frac{1}{4}[R(e_i,e_j)\omega, e_i\cdot e_j].$$

Moreover following the same lines of the proof of [23, Thm. 17] one can show that, for any basic p-forms ω and φ, the pointwise scalar product

$$\langle\mathcal{B}^{[p]}\omega,\varphi\rangle = \frac{1}{4}\sum_{r,s=1}^{\binom{q}{2}}\langle R\psi_r,\psi_s\rangle\langle[\hat{\psi}_r,\omega],[\hat{\psi}_s,\varphi]\rangle \tag{6}$$

holds. Here $\{\psi_r\}_{r=1,\dots,\binom{q}{2}}$ is any orthonormal frame of $\wedge^2 Q$ and $\{\hat{\psi}_r\}_{r=1,\dots,\binom{q}{2}}$ is its dual basis. The curvature $R:\Lambda^2 Q\to\Lambda^2 Q$ is viewed as a symmetric operator by $\langle R(X\wedge Y), Z\wedge W\rangle = g(R(X,Y)Z,W)$ for all $X,Y,Z,W\in\Gamma(Q)$.

A direct consequence of (6) is that the Bochner operator is nonnegative when the transversal curvature operator is assumed to be nonnegative. Therefore applying Proposition 2.1, we deduce the following result as in [21, Thm. 51] (see also [18, Cor. D] for a different proof)

Proposition 3.4 *Let $(M,g,\mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation of codimension q.*

1. *If the transversal curvature operator is nonnegative and κ_b is basic-harmonic, then any basic harmonic form is transversally parallel.*
2. *If the transversal curvature operator is positive, then $H_b^p(\mathcal{F}) = 0$ for all $p\in\{1,\cdots,q-1\}$.*

4 Curvature operator for Riemannian flows

In this section we will consider a Riemannian flow, that is a Riemannian foliation of 1-dimensional leaves given by a unit vector field. As mentioned in the introduction, we will prove throughout this section that the curvature operator of the normal bundle splits into two parts. The first part, that we call restriction part, depends on the curvature operator of the underlying manifold. The second part, that we call exterior part, is expressed in terms of the O'Neill tensor of the flow.

Let (M,g,ξ) be a Riemannian manifold endowed with a Riemannian flow given by a unit vector field ξ. Recall the condition on the metric is that $\mathcal{L}_\xi g|_{\xi^\perp} = 0$ which means in this situation that the tensor $h := \nabla^M\xi$, called the O'Neill tensor, is a skew-symmetric endomorphism on $\Gamma(Q)$. From the relation $g(h(X),Y) = -\frac{1}{2}g([X,Y],\xi)$ that is valid for any $X,Y\in\Gamma(Q)$, one can characterize the integrability of the normal bundle of a Riemannian flow by the vanishing of the O'Neill tensor [20]. Moreover, when both the O'Neill tensor and the mean curvature $\kappa := \nabla^M_\xi\xi$ vanish, the manifold M is in this case isometric to a local product.

By a straightforward computation, one can show that the endomorphism h is a basic tensor, that is $\nabla_\xi h = 0$, when the mean curvature κ is a basic 1-form [12, Lemma 2.4]. This fact fails to hold for higher Riemannian foliations and that's why we shall restrict our study for Riemannian flows. Based on this fact, the curvature R^M restricted to sections of the form $\xi \wedge X$ for $X \in \Gamma(Q)$ can be expressed as follows.

Lemma 4.1 *On a Riemannian manifold (M^n, g, ξ) endowed with a Riemannian flow with basic mean curvature κ, we have that*

$$R^M(\xi, X)\xi = -h^2(X) + g(\kappa, h(X))\xi + \nabla^M_X \kappa - g(\kappa, X)\kappa,$$

for any $X \in \Gamma(Q)$. In particular, for a minimal Riemannian flow, the matrix of R^M in the orthonormal frame $\{\xi \wedge e_i\}_{i=1,\cdots,n-1}$ is the same as $-h^2$.

Proof. Let X be any foliated vector field, that is $\nabla_\xi X = 0$. The curvature R^M applied to ξ and X is equal to

$$\begin{aligned} R^M(\xi, X)\xi &= -\nabla^M_\xi \nabla^M_X \xi + \nabla^M_X \kappa + \nabla^M_{[\xi,X]}\xi \\ &= -\nabla^M_\xi h(X) + \nabla^M_X \kappa - g(\kappa, X)\kappa. \end{aligned}$$

The last equality comes from the fact that $[\xi, X] = g([\xi, X], \xi)\xi = -g(\kappa, X)\xi$, as X is foliated. Now using the O'Neill formula for Riemannian flows [10, Eq. 4.4]

$$\nabla^M_\xi Y = \nabla_\xi Y + h(Y) - g(\kappa, Y)\xi,$$

that is valid for all $Y \in \Gamma(Q)$ and the fact that the tensor h is a basic tensor as mentioned before, the curvature reduces to

$$R^M(\xi, X)\xi = -h^2(X) + g(\kappa, h(X))\xi + \nabla^M_X \kappa - g(\kappa, X)\kappa.$$

This finishes the proof of the lemma. $\square$

At any point $x \in M$, we denote by $\gamma_0^M(x)$ and $\gamma_1^M(x)$ the smallest and largest eigenvalues of the symmetric tensor $R^M : \Lambda^2(Q) \to \Lambda^2(Q)$ defined by $g(R^M(X \wedge Y), Z \wedge W) := R^M_{XYZW}$ for $X, Y, Z, W \in \Gamma(Q)$. Again using the O'Neill formulas in [20], this curvature term is related to the one on the normal bundle Q by the following relation: for all sections X, Y, Z, W of Q, we have

$$R^M_{XYZW} = R_{XYZW} - 2g(h(X), Y)g(h(Z), W) + g(h(Y), Z)g(h(X), W) + g(h(Z), X)g(h(Y), W). \tag{7}$$

Therefore, according to Equation (7), the curvature of Q splits into R_{ext} and R_{res}, where we set

$$g(R_{\text{ext}}(X \wedge Y), Z \wedge W) = 2g(h(X), Y)g(h(Z), W) - g(h(Y), Z)g(h(X), W) - g(h(Z), X)g(h(Y), W)$$

and

$$g(R_{\text{res}}(X \wedge Y), Z \wedge W) = R^M_{XYZW}.$$

Hence, Equation (6) can be written in the following way

$$\mathcal{B}^{[p]} = \mathcal{B}^{[p]}_{\text{ext}} + \mathcal{B}^{[p]}_{\text{res}}, \tag{8}$$

where

$$\langle \mathcal{B}^{[p]}_{\text{ext}} \omega, \varphi \rangle = \frac{1}{4} \sum_{r,s=1}^{\binom{q}{2}} \langle R_{\text{ext}} \psi_r, \psi_s \rangle \langle [\hat{\psi}_r, \omega], [\hat{\psi}_s, \varphi] \rangle \tag{9}$$

and

$$\langle \mathcal{B}^{[p]}_{\text{res}} \omega, \varphi \rangle = \frac{1}{4} \sum_{r,s=1}^{\binom{q}{2}} \langle R_{\text{res}} \psi_r, \psi_s \rangle \langle [\hat{\psi}_r, \omega], [\hat{\psi}_s, \varphi] \rangle.$$

In order to find a lower bound of the operator $\mathcal{B}^{[p]}$, we need to bound both terms $\mathcal{B}^{[p]}_{\text{ext}}$ and $\mathcal{B}^{[p]}_{\text{res}}$. For this, we first choose an orthonormal basis of eigenvectors of R_{res} and use the formula

$$\frac{1}{4}\sum_{r=1}^{\binom{q}{2}} |[\hat{\psi}_r, \omega]|^2 = p(q-p)|\omega|^2 \tag{10}$$

which follows from [23, Lem. 18], to get the pointwise estimate

$$p(q-p)\gamma_0^M(x) \leq \mathcal{B}^{[p]}_{\text{res}} \leq p(q-p)\gamma_1^M(x). \tag{11}$$

Second, for a lower bound of the term $\langle \mathcal{B}^{[p]}_{\text{ext}}\omega, \omega\rangle$, we will compute the eigenvalues of R_{ext} in terms of the eigenvalues of the tensor h.

Computation of the eigenvalues of the tensor R_{ext} : Let us treat the case where q is even, say $q = 2m$. Since the tensor h is skew-symmetric and a basic form, we can always find a local basic orthonormal frame $\{e_i\}_{i=1,\dots,q}$ of Q such that the matrix of h can be written in this basis as

$$\begin{pmatrix} \begin{pmatrix} 0 & -b_1 \\ b_1 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} 0 & -b_2 \\ b_2 & 0 \end{pmatrix} & \ddots & \vdots \\ \vdots & & \ddots & \\ & \ddots & & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} 0 & -b_m \\ b_m & 0 \end{pmatrix} \end{pmatrix}$$

where $b_1, \cdots, b_m$ are smooth basic functions on M chosen in a way such that $|b_1| \leq |b_2| \leq \cdots \leq |b_m|$. That is, $h(e_{2i-1}) = b_i e_{2i}$ and $h(e_{2i}) = -b_i e_{2i-1}$ for all $i = 1, \cdots, m$. Depending on the different choices of indices, we will now compute R_{ext}. For all $i, j, k, l \in \{1, \dots, q\}$, we have

$$\begin{cases} g(R_{\text{ext}}(e_{2i-1}\wedge e_{2i}), e_{2i-1}\wedge e_{2i}) &= 3b_i^2 \\ g(R_{\text{ext}}(e_{2i-1}\wedge e_{2i}), e_{2k-1}\wedge e_{2k}) &= 2b_ib_k \text{ for } k \neq i \\ g(R_{\text{ext}}(e_{2i-1}\wedge e_{2j-1}), e_{2k}\wedge e_{2l}) &= -b_ib_j\delta_{jk}\delta_{il} + b_ib_j\delta_{ik}\delta_{jl} \\ g(R_{\text{ext}}(e_{2i-1}\wedge e_{2j}), e_{2k-1}\wedge e_{2l}) &= 2b_ib_k\delta_{ij}\delta_{kl} + b_ib_j\delta_{jk}\delta_{il}. \end{cases}$$

The other terms are all equal to zero. Therefore, in the basis $\{e_i \wedge e_j\}_{1\leq i<j\leq 2m}$, arranged as follows

$$\{e_{2i-1}\wedge e_{2i}\}_{1\leq i\leq m}, \quad \{e_{2i-1}\wedge e_{2j-1}, e_{2i}\wedge e_{2j}\}_{1\leq i<j\leq m}, \quad \{e_{2i-1}\wedge e_{2j}, e_{2i}\wedge e_{2j-1}\}_{1\leq i<j\leq m}$$

the tensor R_{ext} is a block diagonal matrix having diagonal blocks matrices $D, D_{i,j}, -D_{i,j}$, for $1 \leq i < j \leq m$ where:

- D is the matrix representation of the restriction of R_{ext} to the subspace generated by $\{e_{2i-1}\wedge e_{2i}\}_{1\leq i\leq m}$ and is given by

$$D = \begin{pmatrix} 3b_1^2 & 2b_1b_2 & \dots & 2b_1b_m \\ 2b_1b_2 & 3b_2^2 & \dots & 2b_2b_m \\ \vdots & & \ddots & \\ 2b_1b_m & 2b_2b_m & \dots & 3b_m^2 \end{pmatrix}.$$

- $D_{i,j}$ is the matrix representation of the restriction of R_{ext} to the subspace generated by $\{e_{2i-1} \wedge e_{2j-1}, e_{2i} \wedge e_{2j}\}$ which is given by

$$\begin{pmatrix} 0 & b_i b_j \\ b_i b_j & 0 \end{pmatrix}.$$

- The last block $-D_{i,j}$ is the matrix representation of the restriction of R_{ext} to the subspace generated by $\{e_{2i-1} \wedge e_{2j}, e_{2i} \wedge e_{2j-1}\}$.

One can easily check that the eigenvalues of the matrices $D_{i,j}$ are $\pm b_i b_j$ with unit eigenvectors $\theta_{ij}^{\pm} = \frac{1}{\sqrt{2}}(e_{2i-1} \wedge e_{2j-1} \pm e_{2i} \wedge e_{2j})$. Also the eigenvalues of the matrices $-D_{i,j}$ are $\pm b_i b_j$ with unit eigenvectors given by $\rho_{ij}^{\mp} = \frac{1}{\sqrt{2}}(e_{2i-1} \wedge e_{2j} \mp e_{2i} \wedge e_{2j-1})$. The eigenvalues of the matrix D are not easy to compute but we know that they are all nonnegative since $\langle DX, X\rangle = \sum_{i=1}^m b_i^2 X_i^2 + 2(\sum_{i=1}^m b_i X_i)^2 \geq 0$ for any vector X.

In conclusion, the eigenvalues $\{\lambda_r\}_{r=1,\cdots,\binom{q}{2}}$ of the tensor R_{ext} consist of three families (q is even):

- **Type I :** The eigenvalues are $\pm b_i b_j$ $(i < j)$ with unit eigenvectors $\theta_{ij}^{\pm} = \frac{-1}{\sqrt{2}}(e_{2i-1} \wedge e_{2j-1} \pm e_{2i} \wedge e_{2j})$.
- **Type II :** The eigenvalues are $\pm b_i b_j$ $(i < j)$ with unit eigenvectors given by $\rho_{ij}^{\mp} = \frac{1}{\sqrt{2}}(e_{2i-1} \wedge e_{2j} \mp e_{2i} \wedge e_{2j-1})$.
- **Type III :** The eigenvalues are those of the matrix D which are all nonnegative and the eigenvectors are in the subspace generated by $\{e_{2i-1} \wedge e_{2i}\}_{i=1\cdots,m}$.

The case where q is odd can be treated in a similar way as the even case but an additional direction e_0 is involved corresponding to the eigenvalue 0 of h. Since $g(R_{\text{ext}}(e_0 \wedge X), Y \wedge Z) = 0$ for every $X, Y, Z \in \Gamma(Q)$, we deduce that the eigenvalues of R_{ext} consist of families of type **I, II, III** (the same as defined above) and **IV**, where in the last family 0 is an eigenvalue and the corresponding eigenvector is in the subspace generated by $\{e_0 \wedge e_i\}_{i=1,\cdots,2m}$.

5 Main results

In this section we will show that, on a given Riemannian flow (M, g, ξ), the Bochner operator $\mathcal{B}^{[p]}$ on basic p-forms is bounded from below by the lowest eigenvalue of the curvature operator of M and by the O'Neill tensor (see Corollary 5.3). In particular, this will allow us to estimate the first eigenvalue of the basic Laplacian by some lower bound that we will completely characterize its limiting case (see Theorem 1.1).

Theorem 5.1 *Let (M, g, ξ) be a Riemannian manifold endowed with a Riemannian flow of codimension q given by a unit vector field ξ. For any integer p such that $1 \leq p \leq q-1$ and a basic p-form ω, we have*

$$\langle \mathcal{B}_{\text{ext}}^{[p]} \omega, \omega\rangle \geq -p(q-p) b_m^2 |\omega|^2,$$

with $m = [\frac{q}{2}]$. If the equality is attained for some $p \in \{1, \cdots, m\}$ where $m > 1$, then $|b_1| = \cdots = |b_m|$. If $m = 1$ and the equality is attained, then $b_1 = 0$.

Proof: The proof of the inequality in Theorem 5.1 is based on the use of Equation (9) and the computation of the eigenvalues of the tensor R_{ext} in Section 4. For this, we denote by $\tilde{\lambda}_r$ $(1 \leq r \leq m)$

the eigenvalues of the matrix D and by $\{\tilde{\theta}_r\}$ an orthonormal family of eigenvectors associated with the eigenvalues $\tilde{\lambda}_r$. Now, we compute

$$\begin{aligned}
\langle \mathcal{B}^{[p]}_{\text{ext}}\omega,\omega\rangle &= \frac{1}{4}\sum_{1\leq i<j\leq m} b_ib_j(|[\theta^+_{ij},\omega]|^2+|[\rho^-_{ij},\omega]|^2)-\frac{1}{4}\sum_{1\leq i<j\leq m} b_ib_j(|[\theta^-_{ij},\omega]|^2+|[\rho^+_{ij},\omega]|^2)\\
&\quad +\frac{1}{4}\sum_{r=1}^{m}\tilde{\lambda}_r|[\tilde{\theta}_r,\omega]|^2\\
&\geq \frac{1}{4}\sum_{1\leq i<j\leq m} b_ib_j(|[\theta^+_{ij},\omega]|^2+|[\rho^-_{ij},\omega]|^2)-\frac{1}{4}\sum_{1\leq i<j\leq m} b_ib_j(|[\theta^-_{ij},\omega]|^2+|[\rho^+_{ij},\omega]|^2)\\
&\geq -\frac{1}{4}b_m^2\left(\sum_{1\leq i<j\leq m}\left(|[\theta^+_{ij},\omega]|^2+|[\rho^-_{ij},\omega]|^2+|[\theta^-_{ij},\omega]|^2+|[\rho^+_{ij},\omega]|^2\right)\right)\\
&\overset{(10)}{\geq} -p(q-p)b_m^2|\omega|^2. \qquad (12)
\end{aligned}$$

In the inequalities above, we use the fact that the eigenvalues $\tilde{\lambda}_r$ are all nonnegative and that $|b_1|\leq|b_2|\leq\cdots\leq|b_m|$. Therefore, we get the first part of the theorem.

To discuss the equality case of (12), we first treat the even codimension case $q=2m>2$. Two possibilities may occur: either for all (i,j) one of the Lie bracket of the coefficient b_ib_j does not vanish and in this case we get $|b_1|=\cdots=|b_m|$ or there exist i and j with $i<j$ and such that all the coefficients of b_ib_j vanish, that is

$$[\theta^{\pm}_{ij},\omega]=[\rho^{\pm}_{ij},\omega]=0. \qquad (13)$$

Let us now prove that the second alternative gives $|b_1|=\cdots=|b_m|$ as well. When Equalities (13) hold for some i and j, we have an explicit expression for the form ω that we describe it in the following lemma:

Lemma 5.2 *Assume that there exist i,j with $i<j$ such that Equalities (13) hold. Then, there exist basic forms ω_1 and ω_2 such that*

$$\omega=e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j}\wedge\omega_1+\omega_2,$$

with

$$\begin{cases} e_{2i-1}\lrcorner\,\omega_1=e_{2i}\lrcorner\,\omega_1=0\\ e_{2j-1}\lrcorner\,\omega_1=e_{2j}\lrcorner\,\omega_1=0,\end{cases}$$

The same system holds for ω_2.

Proof. By adding (and substracting) the brackets $[\theta^+_{ij},\omega]$ and $[\theta^-_{ij},\omega]$ together as well as the bracket $[\rho^+_{ij},\omega]$ with $[\rho^-_{ij},\omega]$, we deduce that the following equations

$$[e_{2i-1}\wedge e_{2j-1},\omega]=[e_{2i}\wedge e_{2j},\omega]=[e_{2i-1}\wedge e_{2j},\omega]=[e_{2i}\wedge e_{2j-1},\omega]=0$$

hold. Now, using Lemma 3.1 for each of the above brackets, these equations reduce to the following system

$$\begin{cases}
e_{2j-1}\wedge(e_{2i-1}\lrcorner\omega)=e_{2i-1}\wedge(e_{2j-1}\lrcorner\omega)\\[2ex]
e_{2j}\wedge(e_{2i}\lrcorner\omega)=e_{2i}\wedge(e_{2j}\lrcorner\omega)\\[2ex]
e_{2j}\wedge(e_{2i-1}\lrcorner\omega)=e_{2i-1}\wedge(e_{2j}\lrcorner\omega)\\[2ex]
e_{2j-1}\wedge(e_{2i}\lrcorner\omega)=e_{2i}\wedge(e_{2j-1}\lrcorner\omega).
\end{cases}$$

In order to solve this system, we take the interior product of the first equation with e_{2i-1} (resp. with e_{2j-1}) to get that

$$e_{2i-1}\lrcorner\omega = e_{2j-1}\wedge\beta_0 \quad \text{and} \quad e_{2j-1}\lrcorner\omega = e_{2i-1}\wedge\beta_1,$$

where β_0 (resp. β_1) is a form that does not contain neither e_{2i-1} nor e_{2j-1}. The same can be done for the third equation with respect to e_{2i-1} and e_{2j} to obtain

$$e_{2i-1}\lrcorner\omega = e_{2j}\wedge\beta_3 \quad \text{and} \quad e_{2j}\lrcorner\omega = e_{2i-1}\wedge\beta_4,$$

for some β_3, β_4. Comparing the above equations and using the fact that the general solution of any equation of type $X\wedge\alpha = Y\wedge\beta$ where X and Y are orthogonal and $X\lrcorner\alpha = Y\lrcorner\beta = 0$ is given by $\alpha = Y\wedge(X\lrcorner\beta)$, we conclude that β_0 should be of the form $e_{2j}\wedge\beta_5$ for some form β_5. The same technique can be used for the second and fourth equations in the system. This allows us to finish the proof of the lemma by using the fact that the general solution of any equation of the form $X\lrcorner\omega = \alpha$ is $\omega = X\wedge\alpha+\beta$ where $X\lrcorner\beta = 0$. $\square$

We now proceed with the proof of Theorem 5.1. According to Lemmas 5.2, 3.2 and to Equality (9), we set $\Phi := e_{2i}\wedge e_{2j-1}\wedge e_{2j}\wedge\omega_1$ and compute

$$\begin{aligned}
\langle\mathcal{B}^{[p]}_{\text{ext}}\omega,\omega\rangle &= \langle\mathcal{B}^{[p]}_{\text{ext}}(e_{2i-1}\wedge\Phi), e_{2i-1}\wedge\Phi\rangle + 2\langle\mathcal{B}^{[p]}_{\text{ext}}(e_{2i-1}\wedge\Phi),\omega_2\rangle + \langle\mathcal{B}^{[p]}_{\text{ext}}\omega_2,\omega_2\rangle\\
&= \frac{1}{4}\sum_{r=1}^{\binom{q}{2}}\lambda_r|[\hat\theta_r, e_{2i-1}\wedge\Phi]|^2 + \frac{1}{2}\sum_{r=1}^{\binom{q}{2}}\lambda_r\langle[\hat\theta_r, e_{2i-1}\wedge\Phi],[\hat\theta_r,\omega_2]\rangle + \langle\mathcal{B}^{[p]}_{\text{ext}}\omega_2,\omega_2\rangle\\
&= \frac{1}{4}\sum_{r=1}^{\binom{q}{2}}\lambda_r|[\hat\theta_r,\Phi]|^2 + \sum_{r=1}^{\binom{q}{2}}\lambda_r|e_{2i-1}\lrcorner\hat\theta_r|^2|\Phi|^2 + \sum_{r=1}^{\binom{q}{2}}\lambda_r\langle e_{2i-1}\cdot[\hat\theta_r,\Phi],(e_{2i-1}\lrcorner\hat\theta_r)\cdot\Phi\rangle\\
&\quad + \frac{1}{2}\sum_{r=1}^{\binom{q}{2}}\lambda_r\langle e_{2i-1}\cdot[\hat\theta_r,\Phi],[\hat\theta_r,\omega_2]\rangle + \sum_{r=1}^{\binom{q}{2}}\lambda_r\langle(e_{2i-1}\lrcorner\hat\theta_r)\cdot\Phi,[\hat\theta_r,\omega_2]\rangle + \langle\mathcal{B}^{[p]}_{\text{ext}}\omega_2,\omega_2\rangle\\
&= \langle\mathcal{B}^{[p-1]}_{\text{ext}}\Phi,\Phi\rangle + \sum_{r=1}^{\binom{q}{2}}\lambda_r|e_{2i-1}\lrcorner\hat\theta_r|^2|\Phi|^2 - \sum_{r=1}^{\binom{q}{2}}\lambda_r\langle[\hat\theta_r,\Phi], e_{2i-1}\cdot(e_{2i-1}\lrcorner\hat\theta_r)\cdot\Phi\rangle\\
&\quad + \frac{1}{2}\sum_{r=1}^{\binom{q}{2}}\lambda_r\langle e_{2i-1}\wedge[\hat\theta_r,\Phi],[\hat\theta_r,\omega_2]\rangle + \sum_{r=1}^{\binom{q}{2}}\lambda_r\langle(e_{2i-1}\lrcorner\hat\theta_r)\wedge\Phi,[\hat\theta_r,\omega_2]\rangle + \langle\mathcal{B}^{[p]}_{\text{ext}}\omega_2,\omega_2\rangle.
\end{aligned} \tag{14}$$

Here, we recall that $\{\lambda_r\}$ are the eigenvalues of the tensor R_{ext} and $\{\hat\theta_r\}$ are the corresponding dual eigenvectors found previously. In the last equality, we use the fact that the degrees of the differential forms $[\hat\theta_r,\omega_2]$ and $[\hat\theta_r,\Phi]$ are p and $p-1$ respectively, according to Lemma 3.1. In the following, we will compute each sum separately with respect to the family of eigenvalues of type (I), (II) and (III) that we already got in Section 4. For this, we denote by $\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3,\mathbf{S}_4$ the respective sums in Equality (14).

Type I : We will prove that $\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3$ and $\mathbf{S}_4$ all vanish with respect to an orthonormal basis of type I. In fact, as we have that

$$e_s\lrcorner\theta^{\pm}_{kl} = \frac{-1}{\sqrt{2}}(\delta_{s2k-1}e_{2l-1} - \delta_{s2l-1}e_{2k-1} \pm \delta_{s2k}e_{2l} \mp \delta_{s2l}e_{2k}), \tag{15}$$

we first deduce that $|e_{2i-1}\lrcorner\theta^{\pm}_{kl}|^2 = \frac{1}{2}$ if $i=k$ or $i=l$ and thus $\mathbf{S}_1$ is zero (the sum of all the eigenvalues). Next, from Lemma 3.1, we have that

$$[\theta^{\pm}_{kl},\Theta] = \frac{-2}{\sqrt{2}}\left(e_{2l-1}\wedge(e_{2k-1}\lrcorner\Theta) - e_{2k-1}\wedge(e_{2l-1}\lrcorner\Theta) \pm e_{2l}\wedge(e_{2k}\lrcorner\Theta) \mp e_{2k}\wedge(e_{2l}\lrcorner\Theta)\right), \tag{16}$$

for any form Θ. Therefore, we get that

$$(e_{2i-1}\lrcorner\theta_{kl}^{\pm})\lrcorner[\theta_{kl}^{\pm},\omega_2] = \begin{cases} \pm e_{2i}\wedge(e_{2k-1}\lrcorner e_{2k}\lrcorner\omega_2) & \text{for} \quad i=l \\ \\ \pm e_{2i}\wedge(e_{2l-1}\lrcorner e_{2l}\lrcorner\omega_2) & \text{for} \quad i=k \end{cases}$$

(up to a factor $\frac{-1}{\sqrt{2}}$) which, by taking the scalar product with Φ, gives that $\mathbf{S}_4 = 0$. Here we used the fact that ω_2 does not contain any factor in e_i and e_j. For the sum $\mathbf{S}_3$, we first compute

$$e_{2i-1}\lrcorner[\theta_{kl}^{\pm},\omega_2] = \frac{-2}{\sqrt{2}}\left(\delta_{il}e_{2k-1}\lrcorner\omega_2 - \delta_{ik}e_{2l-1}\lrcorner\omega_2\right).$$

Hence, the term (up to the factor $\frac{-2}{\sqrt{2}}$)

$$\langle[\theta_{kl}^{\pm},\Phi], e_{2i-1}\lrcorner[\theta_{kl}^{\pm},\omega_2]\rangle = \begin{cases} \langle[\theta_{ki}^{\pm},\Phi], e_{2k-1}\lrcorner\omega_2\rangle & \text{for} \quad i=l \\ \\ -\langle[\theta_{il}^{\pm},\Phi], e_{2l-1}\lrcorner\omega_2\rangle & \text{for} \quad i=k \end{cases}$$

also vanishes by Equation (16) (replace Θ by Φ and l or k by i). Hence $\mathbf{S}_3 = 0$. Now, we are left with the sum $\mathbf{S}_2$ that we will prove that it vanishes as well. Indeed, we write

$$\begin{aligned} \mathbf{S}_2 &= \sum_{k<l} b_k b_l \langle[\theta_{kl}^{+},\Phi], e_{2i-1}\cdot(e_{2i-1}\lrcorner\theta_{kl}^{+})\cdot\Phi\rangle - \sum_{k<l} b_k b_l \langle[\theta_{kl}^{-},\Phi], e_{2i-1}\cdot(e_{2i-1}\lrcorner\theta_{kl}^{-})\cdot\Phi\rangle \\ &\overset{(15)}{=} \sum_{i<l} b_i b_l \langle[\theta_{il}^{+}-\theta_{il}^{-},\Phi], e_{2i-1}\cdot e_{2l-1}\cdot\Phi\rangle - \sum_{k<i} b_k b_i \langle[\theta_{ki}^{+}-\theta_{ki}^{-},\Phi], e_{2i-1}\cdot e_{2k-1}\cdot\Phi\rangle. \end{aligned}$$

Now from the expression of the vector fields θ_{kl}^{+} and θ_{kl}^{-} and using again Lemma 3.2, we have that

$$\begin{aligned} \langle[\theta_{il}^{+}-\theta_{il}^{-},\Phi], e_{2i-1}\cdot e_{2l-1}\cdot\Phi\rangle &= \frac{-2}{\sqrt{2}}\langle[e_{2i}\wedge e_{2l},\Phi], e_{2i-1}\cdot e_{2l-1}\cdot\Phi\rangle \\ &= \frac{-4}{\sqrt{2}}\langle e_{2l}\wedge(e_{2i}\lrcorner\Phi) - e_{2i}\wedge(e_{2l}\lrcorner\Phi), e_{2i-1}\cdot e_{2l-1}\cdot\Phi\rangle \\ &= \frac{-4}{\sqrt{2}}\langle e_{2l}\wedge(e_{2i}\lrcorner\Phi), e_{2l-1}\lrcorner(e_{2i-1}\wedge\Phi)\rangle \\ &= \frac{-4}{\sqrt{2}}\langle e_{2l-1}\wedge e_{2l}\wedge(e_{2i}\lrcorner\Phi), e_{2i-1}\wedge\Phi\rangle = 0, \end{aligned}$$

which means that the first sum vanishes. By interchanging the roles of i and l, we also deduce that the second sum $\mathbf{S}_2$ is zero.

Type II : The computation can be done in the same way as for **Type I** and shows that all of the sums vanish.

Type III : Recall that in this case, the eigenvectors of R_{ext} are in the subspace generated by $\{e_{2k-1}\wedge e_{2k}\}_{k=1\cdots,m}$. Hence any eigenvector $\tilde{\theta}_r$ $(1\leq r\leq m)$ can be written as $\tilde{\theta}_r = \sum_{k=1}^{m}\alpha_r^k e_{2k-1}\wedge e_{2k}$ for some functions α_r^k. Thus, we have

$$e_{2i-1}\lrcorner\tilde{\theta}_r = \alpha_r^i e_{2i}. \tag{17}$$

The first sum $\mathbf{S}_1$ is then equal to $\sum_{r=1}^{m}\tilde{\lambda}_r(\alpha_r^i)^2|\Phi|^2$, where $\tilde{\lambda}_r$ are the eigenvalues of the matrix D defined before. Next, we will show that $\mathbf{S}_3$ and $\mathbf{S}_4$ are equal to zero. Indeed, using (17), we can easily see that $(e_{2i-1}\lrcorner\tilde{\theta}_r)\wedge\Phi = 0$ which gives that $\mathbf{S}_4 = 0$. Now using Lemma 3.2, we have

$$[\tilde{\theta}_r,\Theta] = \sum_{k=1}^{m}\alpha_r^k[e_{2k-1}\wedge e_{2k},\Theta] = 2\sum_{k=1}^{m}\alpha_r^k(e_{2k}\wedge(e_{2k-1}\lrcorner\Theta) - e_{2k-1}\wedge(e_{2k}\lrcorner\Theta)),$$

for any form Θ. This gives that $e_{2i-1}\lrcorner[\tilde{\theta}_r,\omega_2]=0$ and thus $\mathbf{S}_3=0$. Here, we used the fact that ω_2 does not contain any factor in e_i. The term $\mathbf{S}_2$ is now equal to

$$\begin{aligned}
\mathbf{S}_2 &= \sum_{r=1}^m \lambda_r\alpha_r^i\langle[\tilde{\theta}_r,\Phi],e_{2i-1}\cdot e_{2i}\cdot\Phi\rangle\\
&= 2\sum_{k,r=1}^m \lambda_r\alpha_r^i\alpha_r^k\langle(e_{2k}\wedge(e_{2k-1}\lrcorner\Phi)-e_{2k-1}\wedge(e_{2k}\lrcorner\Phi),e_{2i-1}\wedge(e_{2i}\lrcorner\Phi)\rangle\\
&= -2\sum_{k,r=1}^m \lambda_r\alpha_r^i\alpha_r^k\langle e_{2k-1}\wedge(e_{2k}\lrcorner\Phi),e_{2i-1}\wedge(e_{2i}\lrcorner\Phi)\rangle\\
&= -2\sum_{k,r=1}^m \lambda_r\alpha_r^i\alpha_r^k\delta_{ik}|e_{2i}\lrcorner\Phi|^2=-2\sum_{r=1}^m\lambda_r(\alpha_r^i)^2|\Phi|^2.
\end{aligned}$$

We now replace all the computations done above in Equation (14) to deduce that

$$\begin{aligned}
-p(q-p)b_m^2|\omega|^2=\langle\mathcal{B}_{\mathrm{ext}}^{[p]}\omega,\omega\rangle &= \langle\mathcal{B}_{\mathrm{ext}}^{[p-1]}\Phi,\Phi\rangle+3\sum_{r=1}^m\lambda_r(\alpha_r^i)^2|\Phi|^2+\langle\mathcal{B}_{\mathrm{ext}}^{[p]}\omega_2,\omega_2\rangle\\
&\overset{(12)}{\geq} -(p-1)(q-p+1)b_m^2|\Phi|^2-p(q-p)b_m^2|\omega_2|^2.
\end{aligned}$$

Here, we use the fact that all the eigenvalues λ_r are nonnegative. As $|\omega|^2=|\Phi|^2+|\omega_2|^2$, the last inequality implies that either $b_m=0$ or that $\Phi=0$. Recall here that the integer p is chosen such that $1\leq p\leq m$. The fact that the b_i's are chosen in a way that $|b_1|\leq\cdots\leq|b_m|$, then $b_m=0$ implies that $|b_1|=\cdots=|b_m|=0$, which is the statement of Theorem 5.1. We are now left with the case when $\Phi=0$, which means by Lemma 5.2 that $\omega=\omega_2$ with $e_{2i-1}\lrcorner\omega=e_{2i}\lrcorner\omega=e_{2j-1}\lrcorner\omega=e_{2j}\lrcorner\omega=0$. But recall that i and j are chosen in a way so that all the Lie bracket coefficients of b_ib_j in Equation (12) are equal to zero. Therefore the same choice holds for $i=1$ and $1\leq j\leq m$, since otherwise we would get $|b_1|=\cdots=|b_m|$. Hence by varying j, we arrive at $X\lrcorner\omega=0$ for any X, which leads to $\omega=0$; that is a contradiction. This finishes the proof for $m>1$.

Now, we discuss the equality when q is odd, say $q=2m+1$. In this case, we have $[e_0\wedge e_l,\omega]=0$ for all $l=1,\cdots,2m$. Recall here that e_0 is the eigenvector of h that corresponds to the eigenvalue 0. As in the even case, either for all (i,j) one of the Lie bracket of the coefficient of b_ib_j in (12) does not vanish and we get $|b_1|=\cdots=|b_m|$ or there exist i and j with $i<j$ and such that all the coefficients vanish. In the second alternative, Equations (13) still hold and we get the same description as in Lemma 5.2. That means, we write $\omega=e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j}\wedge\omega_1+\omega_2$. From the one hand, we take $l=2i-1$ in the equation $[e_0\wedge e_l,\omega]=0$ and make the interior product of this last identity with e_{2i-1} to get after using Lemma 3.1 that

$$e_0\lrcorner\omega_2=0\quad\text{and}\quad e_0\wedge\omega_1=0. \tag{18}$$

From the other hand, we take $l\notin\{2i-1,2i,2j-1,2j\}$ and make the interior product of the same equation with $e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j}$ to find that

$$e_l\wedge(e_0\lrcorner\omega_1)=0\quad\text{and}\quad e_0\wedge(e_l\lrcorner\omega_2)=0. \tag{19}$$

Now, the interior product of the first equation in (18) with e_l combined with the second equation in (19) allows us to deduce that $e_l\lrcorner\omega_2=0$ for any $l=1,\cdots,q$. Therefore, we deduce that $\omega_2=0$ and hence $\omega=e_{2i-1}\wedge\Phi$. The rest of the proof can be done in the same way as the even case. We notice that the family IV of eigenvalues does not contribute to Equation (14), since in this case all the eigenvalues are equal to zero.

We are now left with the case when $m = 1$. As from the first line of Equation (12) the term $\langle \mathcal{B}^{[1]}_{\mathrm{ext}}\omega, \omega\rangle$ is nonnegative, we then deduce that the equality in Theorem 5.1 is attained if $b_1 = 0$. This ends the proof. $\square$

In the following, we will give some direct consequences of Theorem 5.1. By adding the estimate in Theorem 5.1 to the l.h.s. of Inequality (11), we get the following

Corollary 5.3 *Let (M, g, ξ) be a Riemannian manifold endowed with a Riemannian flow of codimension q given by a unit vector field ξ. For any integer p such that $1 \le p \le q-1$ and any basic p-form ω, we have*

$$\langle \mathcal{B}^{[p]}\omega, \omega\rangle \ge p(q-p)(\gamma_0^M - b_m^2)|\omega|^2, \qquad (20)$$

where γ_0^M is the lowest eigenvalue of the curvature operator of M restricted to Q and $m = [\frac{q}{2}]$. If $m > 1$ and the equality is attained for some $p \in \{1, \cdots, m\}$, then $|b_1| = \cdots = |b_m|$. If $m = 1$ and the equality is attained, then $b_1 = 0$.

Examples.

1. One can easily check that for the Hopf fibration $\mathbb{S}^{2m+1} \to \mathbb{C}\mathrm{P}^m$ for $m > 1$, the Kähler form Ω on $\mathbb{C}\mathrm{P}^m$, which is a parallel basic 2-form, satisfies the equality of the estimate in the above theorem. Here $\gamma_0^M = b_m^2 = 1$. For $m = 1$, the inequality is strict for any basic form ω.

2. On the Riemannian product $\mathbb{S}^1 \times \mathbb{S}^{2m+1}$ for $m > 1$, we consider the flow defined by the unit vector field $\xi := \frac{1}{\sqrt{2}}(\xi_1 + \xi_2)$ where ξ_1 is the unit parallel vector field on $\mathbb{S}^1$ and ξ_2 is the unit Killing vector field that defines the Hopf fibration. The Kähler form on $\mathbb{C}\mathrm{P}^m$ is transversally parallel and satisfies the equality of the estimate since $\gamma_0^M = b_m^2 = \frac{1}{2}$.

3. Another example of the equality case for nonminimal Riemannian flows. Consider in the 3-dimensional case the Carrière example [3]. Let B be any matrix in $\mathrm{SL}(2, \mathbb{Z})$ with two eigenvalues $\lambda > 1$ and $\frac{1}{\lambda}$. We define the hyperbolic torus $\mathbb{T}^3_B$ as the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m, t) to $(Bm, t+1)$. We chose a bundle-like metric so that the vectors $e_1 := \lambda^{-t}V_1, e_2 := \lambda^t V_2, e_3 := \partial t$ form an orthonormal frame. Here V_1 and V_2 are respectively the eigenvectors associated to λ and $\frac{1}{\lambda}$. An easy computation shows that the Christoffel symbols $\Gamma^k_{ij} = g(\nabla_{e_i} e_j, e_k)$ are given by [10, p. 68]

$$\Gamma^3_{11} = \Gamma^2_{23} = -\Gamma^1_{13} = -\Gamma^3_{22} = -\ln(\lambda).$$

   The others are zero. The flow defined by the vector field e_1 is Riemannian with vanishing O'Neill tensor and the mean curvature is equal to $\kappa = -\ln(\lambda)e_3$. Also, one can easily check that $\gamma_0^M = -(\ln(\lambda))^2$ and the transversal Ricci tensor (which corresponds to $\mathcal{B}^{[1]}$) is equal to $-(\ln(\lambda))^2\mathrm{Id}$. Therefore, for any basic 1-form, the equality is satisfied.

4. Consider the Riemannian fibration $\mathbb{S}^1 \times \mathbb{S}^{2m+1} \to \mathbb{S}^1 \times \mathbb{C}\mathrm{P}^m$ and let Ω be again the Kähler form on $\mathbb{C}\mathrm{P}^m$. Here, we have the strict inequality for Ω since $|b_1| = \cdots = |b_m| = 1$ and $\gamma_0^M = 0$.

When the term in the lower bound of Corollary 5.3 is positive, we get the following rigidity result:

Corollary 5.4 *Let (M, g) be a compact Riemannian manifold endowed with a Riemannian flow of codimension q given by a unit vector field ξ. If $\gamma_0^M \ge b_m^2$ and κ is basic-harmonic, then every harmonic basic p-form is transversally parallel. If the strict inequality holds, then $H^s_b(\mathcal{F}) = \{0\}$ for any $s \in \{1, \cdots, q-1\}$.*

The proof of this corollary uses the first statement of Proposition 2.1. Another direct consequence of Corollary 5.3 that characterizes minimal Riemannian flow on round spheres is the following (see [9])

Corollary 5.5 *Let $\mathbb{S}^n$ be the round sphere of constant sectional curvature* 1 *and assume that it is endowed with a minimal Riemannian flow. Then, the flow defines a Sasakian structure on $\mathbb{S}^n$.*

Proof: As the curvature on the sphere $\mathbb{S}^n$ is given for all vector fields X, Y, Z by $R^M(X,Y)Z = g(X,Z)Y - g(Y,Z)X$, we find from Lemma 4.1 that $h^2(X) = -X$ for all $X \in \Gamma(Q)$, that is $|b_1| = \cdots = |b_m| = 1$. Using the first part of Proposition 6.2 in the Appendix, we get that the basic 2-form $\Omega := -\frac{1}{2}d\xi$ is a basic-harmonic 2-form. Now, Corollary 5.4 allows us to deduce that it is transversally parallel. This ends the proof. $\square$

Proof of Theorem 1.1: Using the second statement in Proposition 2.1 and Corollary 5.3, we easily get the estimate for the first eigenvalue of the basic Laplacian. It now remains to treat the equality case. Assume we have equality, then the inequality in Corollary 5.3 is also attained and therefore $|b_1| = \cdots = |b_m| = \mathrm{cst}$ for $m > 1$ and $b_1 = 0$ for $m = 1$. In the following, we will prove that this constant should be also zero. Assume it were not, then we would deduce that $\gamma_M > \mathrm{cst} > 0$ since $\lambda_{1,p} = p(q-p+1)(\gamma_M - \mathrm{cst}) > 0$. Therefore, from Corollary 5.4, we would get that $H_b^2(\mathcal{F}) = 0$ on one hand. On the other hand, using Lemma 4.1, the Ricci curvature on M would be equal to

$$\mathrm{Ric}^M(\xi,\xi) = \sum_{i=1}^q R^M(\xi, e_i, \xi, e_i) = -\sum_{i=1}^q g(h^2 e_i, e_i) = |h|^2 = 2m\mathrm{cst} > 0,$$

and

$$\mathrm{Ric}^M(X,X) = \sum_{i=1}^q R^M(X, e_i, X, e_i) + R^M(X,\xi,X,\xi) \geq \gamma_M \sum_{i=1}^q |X \wedge e_i|^2 + |hX|^2 > \mathrm{cst}'|X|^2 > 0,$$

for all $X \in \Gamma(Q)$ which would mean that $H^1(M) = 0$. From Proposition 6.1 in the Appendix, this would give a contradiction. Thus, we deduce that $|b_1| = \cdots = |b_m| = 0$ which means that the normal bundle is integrable. In this case, the universal cover of M is isometric to the Riemannian product of $\mathbb{R} \times \Sigma$ where Σ is a simply connected compact manifold with positive curvature. This ends the proof. $\square$

Remarks.

1. In the equality case of the estimate in Theorem 1.1, the O'Neill tensor vanishes. Therefore, the basic Laplacian on M restricts to the usual Laplacian on the manifold Σ and thus the first eigenvalue on Σ satisfies the equality case in the Gallot-Meyer estimate [7, Thm. 6.13]. In view of the remark after Theorem 2.1 and if p is chosen such that $p < \frac{q}{2}$, we deduce that $d\omega = 0$ where ω is an eigenform associated with the first eigenvalue. If $p = 2$ and $q > 4$, the form $\alpha = \delta\omega$ is a coclosed 1-form which is still an eigenform of the Laplacian (the form α does not vanish since this would imply that ω vanishes). Hence, by a result of S. Tachibana [25, Thm. 3.3] the manifold Σ is either isometric to a Sasakian manifold or to a round sphere with constant curvature.

2. By the result in [2], the manifold Σ is a spherical space form. In case Σ is isometric to a round sphere, the group $\Gamma = \pi_1(M)$ preserves the orthogonal splitting $T_{(t,x)}\widetilde{M} = \mathbb{R} \oplus T_x\mathbb{S}^q$ (the vertical distribution $\mathbb{R}$ is the kernel of the Ricci tensor), as it is acting by isometries on the universal cover $\widetilde{M}$. Therefore, the fundamental group is embedded in the product $\mathrm{Isom}_+(\mathbb{R}) \times \mathrm{Isom}_+(\mathbb{S}^q)$ where Isom_+ is the group of isometries that preserve the orientation of the corresponding manifold. For q even, we deduce that $\Gamma \simeq \mathbb{Z}$ and that it acts as $(t,x) \to$

$(t+a, A(x))$ for some $(a, A) \in \mathbb{R}^* \times \mathrm{SO}(q+1)$. For q odd, the group Γ is not necessarily isomorphic to $\mathbb{Z}$, since one might consider the group $\Gamma = \mathbb{Z} \times \Gamma_2$ where Γ_2 is a finite subgroup of $\mathrm{SO}(q+1)$ consisting of rotations in orthogonal 2-planes in $\mathbb{R}^{q+1}$.

6 Appendix

The following results are partially contained in [15, Rem. 2.14], [1, Prop. 1.8] and [6] but we include them here for completeness. Let us denote by $b_s(M) = \dim H^s(M)$ (resp. $b_s(\mathcal{F}) = \dim H^s_b(\mathcal{F})$) the Betti numbers (resp. basic Betti numbers).

Proposition 6.1 *Let (M, g, ξ) be a compact Riemannian manifold endowed with a Riemannian flow of codimension q with basic mean curvature κ. Assume that the first cohomology group satisfies $H^1(M) = \{0\}$. Then we have that $b_2(\mathcal{F}) = 1 + b_2(M)$.*

Proof. We use the long exact sequence of cohomologies stated in [22, Thm. 3.2]

$$0 \to H^1_b(\mathcal{F}) \to H^1(M) \xrightarrow{j} H^q_b(\mathcal{F}) \xrightarrow{i_1} H^2_b(\mathcal{F}) \xrightarrow{i_2} H^2(M) \to H^{q-1}_b(\mathcal{F}),$$

where $i_1 = \wedge[\Omega]$ and i_2 is the inclusion map. Since $H^1(M) = 0$, we have that $H^q_b(\mathcal{F}) \simeq \mathbb{R}$ and $H^{q-1}_b(\mathcal{F}) \simeq H^1_b(\mathcal{F}) = \{0\}$ (see [26]). From the fact that the map i_1 is injective, i_2 is surjective and $\mathrm{Im}\, i_1 = \mathrm{Ker}\, i_2$, we find that $\mathrm{Ker}\, i_2 \simeq \mathbb{R}$ and $\mathrm{Im}\, i_2 = H^2(M)$. Therefore, we deduce the statement of the proposition. □

Proposition 6.2 *Let (M, g, ξ) be a compact Riemannian manifold endowed with a minimal Riemannian flow of codimension q. Assume that $\mathrm{Ric}^M(\xi) = \lambda\xi$ with $\lambda > 0$. Then the Euler class $[d\xi]$ is a non-zero cohomology class in $H^2_b(\mathcal{F})$. Moreover, we have that $b_1(\mathcal{F}) = b_1(M)$ and $1 \le b_2(\mathcal{F}) \le 1 + b_2(M)$.*

Proof. Take an orthonormal frame $\{e_i\}_{i=1,\cdots,q}$ in $\Gamma(Q)$ and consider $Y = Z = e_i$ in the formula $g(R^M(X,Y)\xi, Z) = g(-(\nabla_X h)Y + (\nabla_Y h)X, Z)$. After tracing over i, we get that $\mathrm{Ric}^M(\xi, X) = (\delta_b h)(X)$ for all $X \in \Gamma(Q)$. The assumption $\mathrm{Ric}^M(\xi) = \lambda\xi$ gives that the basic 2-form $\Omega := -\frac{1}{2}d\xi = g(h\cdot, \cdot)$ is co-closed. As Ω is also a closed form, it then becomes a basic-harmonic form. But the choice of $\lambda = |h|^2$ to be positive implies that the form Ω does not vanish. This shows the first part. To prove the second part, we use again the Gysin sequence as in the previous proposition and the fact that $H^q_b(\mathcal{F}) \simeq \mathbb{R}$ (recall the flow is minimal) to get that i_1 is injective and thus $b_2(\mathcal{F}) = 1 + \dim \mathrm{Im}\, i_2$. Also, we get that $j = 0$ and therefore $H^1_b(\mathcal{F}) \simeq H^1(M)$. This finishes the proof. □

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Declarations
Conflict of interest No potential conflict of interest was reported by the authors.

References

[1] C. Boyer, K. Galicki and M. Nakayame, *On positive Sasakian Geometry*, Geom. Dedicata **101** (2003), 93–102.

[2] C. Böhm and B. Wilking, *Manifolds with positive curvature operators are space forms*, Annals of Math. **167** (2008), 1079–1097.

[3] Y. Carrière, *Flots riemanniens*, Astérique **116** (1984), 31–52.

[4] Q. Cui and L. Sun, *A sharp lower bounds of eigenvalues for differential forms and homology sphere theorems*, arxiv:1704.00668v1.

[5] D. Domínguez, *A tenseness theorem for Riemannian foliations*, C. R. Acad. Sci. Sér. I **320** (1995), 1331–1335.

[6] A. El Kacimi, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Mathematica **79** (1990), 57–106.

[7] S. Gallot and D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d'une variété riemanienne*, J. Math. Pures. Appl. **54** (1975), 259–284.

[8] J. F. Glazebrook and F.W. Kamber, *Transversal Dirac families in Riemannian foliations*, Comm. Math. Phys. **140** (1991), 217–240.

[9] D. Gromoll and K. Grove, *One dimensional metric foliations in constant curvature space*, Diff. Geom. Comp. Anal. H.E. Rauch memorial volume, Springer, Berlin (1985), 165–167.

[10] G. Habib, *Energy-Momentum tensor on foliations*, J. Geom. Phys. **57** (2007), 2234-2248.

[11] G. Habib and K. Richardson, *Modified differentials and basic cohomology for Riemannian foliations*, J. Geom. Anal. **23** (2013), 1314–1342.

[12] G. Habib and K. Richardson, *Riemannian flows and adiabatic limits*, Int. J. Math. **28** (2018), 185011.

[13] A. Lichnerowicz, *Géométrie des groupes de transformations*, Travaux et Recherches Mathématiques, III, Dunod, Paris, 1958.

[14] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. **118** (1996), 1249–1275.

[15] P. Jammes, *Effondrement, spectre et propriétés diophantiennes des flots riemanniens*, Ann. Inst. Fourier. **60** (2010), 257–290.

[16] S. D. Jung and K. Richardson, *Transversal conformal Killing forms and a Gallot-Meyer theorem for foliations*, Math. Z. **270** (2012), 337–350.

[17] A. Mason, *An application of stochastic flows of Riemannian foliations*, Houst. J. Math. **26** (2000), 481–515.

[18] M. Min-Oo, E. Ruh and Ph. Tondeur, *Vanishing theorems for the basic cohomology of Riemannian foliations*, J. Reine Angew. Math. **415** (1991), 167–174.

[19] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340.

[20] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 459–469.

[21] P. Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer, New York (1998).

[22] J. I. Royo Prieto, *The Gysin sequence for Riemannian flows*, Contem. Math. **288** (2001), 415–419.

[23] A. Savo, *The Bochner formula for isometric immersions*, Pacific J. Math. **272** No. 2 (2014), 395–422.

[24] U. Semmelmann, *On conformal Killing tensor in a Riemannian space*, Math. Z. **245** (2003), 503–527.

[25] S. Tachibana, *On Killing tensors in Riemannian manifolds with positive curvature operator*, Tohoku. Math. J. **28** (1976), 177–184.

[26] Ph. Tondeur, *Geometry of Foliations*, Birkhäuser, Boston, 1997.