THE HARDY–SCHRÖDINGER OPERATOR ON THE POINCARÉ BALL: COMPACTNESS AND MULTIPLICITY

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Abstract. Let Ω be a compact smooth domain containing zero in the Poincaré ball model of the Hyperbolic space $\mathbb{B}^n$ ($n \geq 3$) and let $-\Delta_{\mathbb{B}^n}$ be the Laplace-Beltrami operator on $\mathbb{B}^n$, associated with the metric $g_{\mathbb{B}^n} = \frac{4}{(1-|x|^2)^2} g_{\text{Eucl}}$. We consider issues of non-existence, existence, and multiplicity of variational solutions for the borderline Dirichlet problem,

$$\begin{cases} -\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u &= V_{2^\star(s)} |u|^{2^\star(s)-2} u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $0 \leq \gamma \leq \frac{(n-2)^2}{4}$, $0 < s < 2$, $2^\star(s) := \frac{2(n-s)}{n-2}$ is the corresponding critical Sobolev exponent, V_2 (resp., $V_{2^\star(s)}$) is a Hardy-type potential (resp., Hardy-Sobolev weight) that is invariant under hyperbolic scaling and which behaves like $\frac{1}{r^2}$ (resp., $\frac{1}{r^s}$) at the origin. The bulk of this paper is a sharp blow-up analysis that we perform on approximate solutions of (1) with bounded but arbitrary high energies. When these approximate solutions are positive, our analysis leads to improvements of results in [6] regarding positive ground state solutions for (9), as we show that they exist whenever $n \geq 4$, $0 \leq \gamma \leq \frac{(n-2)^2}{4} - 1$ and $\lambda > 0$. The latter result also holds true for $n \geq 3$ and $\gamma > \frac{(n-2)^2}{4} - 1$ provided the domain has a positive "hyperbolic mass". On the other hand, the same analysis yields that if $\gamma > \frac{(n-2)^2}{4} - 1$ and the mass is non vanishing, then there is a surprising stability of regimes where no variational positive solution exists. As for higher energy solutions to (1), we show that there are infinitely many of them provided $n \geq 5$, $0 \leq \gamma < \frac{(n-2)^2}{4} - 4$ and $\lambda > \frac{n-2}{n-4}\left(\frac{n(n-4)}{4} - \gamma\right)$.

Contents

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1. Introduction

Consider the Poincaré ball model of the Hyperbolic space $\mathbb{B}^n$, $n \geq 3$, which is the Euclidean unit ball $B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the metric $g_{\mathbb{B}^n} = \left(\frac{2}{1-|x|^2}\right)^2 g_{\text{Eucl}}$. Let

$$f(r) := \frac{(1-r^2)^{n-2}}{r^{n-1}} \text{ and } G(r) := \int_r^1 f(t)dt, \tag{2}$$

where $r = \sqrt{\sum_{i=1}^n x_i^2}$ denotes the Euclidean distance from a point x to the origin.
The function $\frac{1}{n\omega_{n-1}}G(r)$ is then a fundamental solution of the Hyperbolic Laplacian $\Delta_{\mathbb{B}^n} u = div_{\mathbb{B}^n}(\nabla_{\mathbb{B}^n} u)$. Moreover, if we consider the hyperbolic scaling of a given function $u : \mathbb{B}^n \to \mathbb{R}$, defined for $\lambda > 0$ by

$$u_\lambda(r) = \lambda^{-\frac{1}{2}} u\left(G^{-1}(\lambda G(r))\right),$$

then for any radially symmetric $u \in H_1^2(\mathbb{B}^n)$, the completion of $C_c^\infty(\mathbb{B}^n)$ for $\|\cdot\|_2$, and $p \geq 1$, one has the following invariance property:

$$\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u_\lambda|^2 \, dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 dv_{g_{\mathbb{B}^n}} \quad \text{and} \quad \int_{\mathbb{B}^n} V_p |u_\lambda|^p dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} V_p |u|^p dv_{g_{\mathbb{B}^n}}, \tag{3}$$

where

$$V_p(r) := \frac{f(r)^2(1-r^2)^2}{4(n-2)^2 G(r)^{\frac{p+2}{2}}}. \tag{4}$$

The weights V_p have the following asymptotic behaviors: for $n \geq 3$ and $p > 1$,

$$\begin{aligned} V_p(r) &= \frac{c_0(n,p)}{r^{n(1-p/2^*)}}(1+o(1)) && \text{as } r \to 0 \\ V_p(r) &= \frac{c_1(n,p)}{(1-r)^{(n-1)(p-2)/2}}(1+o(1)) && \text{as } r \to 1. \end{aligned} \tag{5}$$

In particular for $n \geq 3$, the weight $V_2(r) = \frac{1}{4(n-2)^2}\left(\frac{f(r)(1-r^2)}{G(r)}\right)^2 \sim_{r\to 0} \frac{1}{4r^2}$, while at $r = 1$ it has a finite positive value. In other words, V_2 is qualitatively similar to the Euclidean Hardy potential, which led Sandeep–Tintarev to establish the following Hyperbolic Hardy inequality on $\mathbb{B}^n$ (Theorem 3.4 of [25]):

$$\frac{(n-2)^2}{4} \int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H_1^2(\mathbb{B}^n). \tag{6}$$

They also show the following Hyperbolic Sobolev inequality: for some constant $C > 0$.

$$C\left(\int_{\mathbb{B}^n} V_{2^\star} |u|^{2^\star} \, dv_{g_{\mathbb{B}^n}}\right)^{2/2^\star} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H_1^2(\mathbb{B}^n). \tag{7}$$

By interpolating between these two inequalities, one then easily obtains for $0 \leq s \leq 2$, the following Hyperbolic Hardy-Sobolev inequality [6]:
If $\gamma < \dfrac{(n-2)^2}{4}$, then there exists a constant $C > 0$ such that for any $u \in H_1^2(\mathbb{B}^n)$,

$$C\left(\int_{\mathbb{B}^n} V_{2^*(s)} |u|^{2^*(s)} \, dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} - \gamma \int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}}, \tag{8}$$

where $2^\star(s) := \frac{2(n-s)}{(n-2)}$. Note that $V_{2^\star(s)}$ behaves like $\frac{1}{r^s}$ at the origin, making (8) the exact analogue of the Euclidean Hardy-Sobolev inequality:

$$C\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)} \leq \int_\Omega |\nabla u|^2\,dx - \gamma\int_\Omega \frac{u^2}{|x|^2}\,dx \quad \text{for any } u \in C_c^\infty(\Omega).$$

Let $\Omega_{\mathbb{B}^n}$ be a compact smooth subdomain of $\mathbb{B}^n$, $n \geq 3$, such that $0 \in \Omega_{\mathbb{B}^n}$, but $\overline{\Omega}_{\mathbb{B}^n}$ does not touch the boundary of $\mathbb{B}^n$. We write $\Omega_{\mathbb{B}^n} \Subset \mathbb{B}^n$. In this paper, we are interested in the question of existence and multiplicity of solutions to the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u &= V_{2^\star(s)}|u|^{2^\star(s)-2}u \quad \text{in } \Omega_{\mathbb{B}^n} \\ u &= 0 \quad \text{on } \partial\Omega_{\mathbb{B}^n}. \end{cases} \tag{9}$$

It is clear that (9) is the Euler-Lagrange equation for the following energy functional on $H^2_{1,0}(\Omega_{\mathbb{B}^n})$:

$$\mathcal{J}_{\gamma,s,\lambda}(u) := \frac{\int_{\Omega_{\mathbb{B}^n}} \left(|\nabla_{\mathbb{B}^n} u|^2 - \gamma V_2 u^2 - \lambda u^2\right) dv_{g_{\mathbb{B}^n}}}{\left(\int_{\Omega_{\mathbb{B}^n}} |u|^{2^\star(s)} V_{2^\star(s)} dv_{g_{\mathbb{B}^n}}\right)^{2/2^\star(s)}},$$

where $H^2_{1,0}(\Omega_{\mathbb{B}^n})$ is the completion of $C_c^\infty(\Omega_{\mathbb{B}^n})$ with respect to the norm given by $\|u\| = \sqrt{\int_{\Omega_{\mathbb{B}^n}} |\nabla_{\mathbb{B}^n} u|^2 dv_{g_{\mathbb{B}^n}}}$.

The hyperbolic mass $m_{\gamma,\lambda}(\Omega_{\mathbb{B}^n})$ of a domain $\Omega_{\mathbb{B}^n}$, associated to the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ was defined in [6] and is recalled below.

The existence of a positive ground state solution for (9) has already been addressed in [6]. While our main objective in this paper is to study higher energy solutions, our methods improve in a couple of ways their results about ground state solutions, in particular in the critical dimensions 3 and 4. We get the following.

Theorem 1.1. *Let $\Omega_{\mathbb{B}^n} \Subset \mathbb{B}^n$ be a smooth compact domain containing 0 and let $0 < s < 2$. Assume that $0 \leq \gamma < \frac{(n-2)^2}{4}$ Let $\lambda \in \mathbb{R}$ be such that the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive. Then*

(1) *For $n \geq 4$ and $\gamma \leq \frac{(n-2)^2}{4} - 1$, equation (9) has a positive ground state solution whenever $\lambda > 0$.*
(2) *For $n \geq 3$ and $\gamma > \frac{(n-2)^2}{4} - 1$, equation (9) has a positive ground state solution whenever the mass $m_{\gamma,\lambda}(\Omega_{\mathbb{B}^n}) > 0$.*

We note that statement (1) of the above theorem was established in [6] in the case where $n \geq 5$, $0 \leq \gamma \leq \frac{(n-2)^2}{4} - 1$ and $\lambda > \dfrac{n-2}{n-4}\left(\dfrac{n(n-4)}{4} - \gamma\right)$. As for statement (2), it was proved in [6] under additional assumptions on λ, at least in dimensions 3 and 4.

The above theorem is actually a consequence of the following more general compactness result for positive approximate solutions.

Theorem 1.2. *Let $\Omega_{\mathbb{B}^n} \Subset \mathbb{B}^n$ be a smooth compact domain containing 0 and let $0 < s < 2$ and $\lambda \in \mathbb{R}$ such that the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive. Let $(p_\epsilon)_{\epsilon>0}$ be such that $0 \leq p_\epsilon < 2^\star(s) - 2$ and $\lim_{\epsilon\to 0} p_\epsilon = 0$ and consider a sequence of functions $(u_\epsilon)_{\epsilon>0}$ that is uniformly bounded in $H^2_{1,0}(\Omega_{\mathbb{B}^n})$ such that for each $\epsilon > 0$, u_ϵ is a solution to the equation:*

$$\begin{cases} -\Delta_{\mathbb{B}^n} u_\epsilon - \gamma V_2 u_\epsilon - \lambda u_\epsilon &= V_{2^\star(s)} u_\epsilon^{2^\star(s)-1-p_\epsilon} \quad \text{in } \Omega_{\mathbb{B}^n}, \\ u_\epsilon &> 0 \quad \text{in } \Omega_{\mathbb{B}^n}, \\ u_\epsilon &= 0 \quad \text{on } \partial\Omega_{\mathbb{B}^n}. \end{cases}$$

Assume that $0 \leq \gamma \leq \frac{(n-2)^2}{4}$. *Assuming one of the following conditions*

- $n \geq 4$, $0 \leq \gamma \leq \frac{(n-2)^2}{4} - 1$ *and* $\lambda > 0$;
- $\gamma > \frac{(n-2)^2}{4} - 1$ *and* $m_{\gamma,\lambda}(\Omega)_{\mathbb{B}^n} > 0$.

Then the sequence $(u_\epsilon)_{\epsilon>0}$ *is pre-compact in the space* $H^2_{1,0}(\Omega_{\mathbb{B}^n})$.

We now recall the notion of hyperbolic mass of a domain Ω associated with a coercive operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$. This notion was introduced in the Euclidean case in [16] and was extended to the hyperbolic setting in [6].

Theorem 1.3 (The hyperbolic interior mass). *Let* $\Omega_{\mathbb{B}^n} \Subset \mathbb{B}^n$ $(n \geq 3)$ *be a smooth compact domain containing* 0. *Let* $\gamma < \dfrac{(n-2)^2}{4}$ *and let* $\lambda \in \mathbb{R}$ *be such that the operator* $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ *is coercive on* $\Omega_{\mathbb{B}^n}$. *Then, there exists* $H \in C^\infty(\overline{\Omega_{\mathbb{B}^n}} \setminus \{0\})$ *such that*

$$\begin{cases} -\Delta_{\mathbb{B}^n} H - \gamma V_2 H - \lambda H = 0 & \text{in } \Omega_{\mathbb{B}^n} \setminus \{0\} \\ H > 0 & \text{in } \Omega_{\mathbb{B}^n} \setminus \{0\} \\ H = 0 & \text{on } \partial\Omega_{\mathbb{B}^n}. \end{cases} \tag{10}$$

These solutions are unique up to a positive multiplicative constant, and there exists $c_+ > 0$ *such that*

$$H(x) \simeq_{x \to 0} \frac{c_+}{|x|^{\alpha_+(\gamma)}}.$$

Moreover, when $\gamma > \max\left\{\dfrac{n(n-4)}{4}, 0\right\}$, *then there exists* $c_- \in \mathbb{R}$ *such that*

$$H(x) = \frac{c_+}{|x|^{\alpha_+(\gamma)}} + \frac{c_-}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{1}{|x|^{\alpha_-(\gamma)}}\right) \text{ as } x \to 0$$

where $\alpha_+(\gamma), \alpha_-(\gamma)$ *stand for*

$$\alpha_\pm(\gamma) := \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}. \tag{11}$$

We define the interior mass as $m_{\gamma,\lambda}(\Omega_{\mathbb{B}^n}) := \dfrac{c_+}{c_-}$, *which is independent of the choice of* H.

This existence theorem will follow directly from Theorem 1.7 below.

In this paper we are concerned with showing the multiplicity of higher energy solutions. Here is our main result.

Theorem 1.4. *Let* $\Omega_{\mathbb{B}^n} \Subset \mathbb{B}^n$, $n \geq 5$, *be a smooth compact domain containing* 0 *and let* $0 < s < 2$. *Let* $\lambda \in \mathbb{R}$ *be such that the operator* $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ *is coercive.*

If $0 \leq \gamma < \dfrac{(n-2)^2}{4} - 4$, *then equation* (9) *has an infinite number of solutions corresponding to higher energy critical levels for* $\mathcal{J}_{\gamma,s}$ *whenever* $\lambda \geq \dfrac{n-2}{n-4}\left(\dfrac{n(n-4)}{4} - \gamma\right)$.

Note that -unlike the case of ground state solutions- we do not have a multiplicity result neither in *low dimensions*, nor when $\frac{(n-2)^2}{4} - 4 \leq \gamma \leq \frac{(n-2)^2}{4} - 1$, even under the assumption of positivity of the *mass* of the compact subdomain. Note also the condition $\lambda \geq \dfrac{n-2}{n-4}\left(\dfrac{n(n-4)}{4} - \gamma\right)$, which is more restrictive than $\lambda > 0$.

In order to prove Theorem 1.4, we shall proceed as in [6] and use a conformal transformation

$$g_{\mathbb{B}^n} = \varphi^{\frac{4}{n-2}} \text{Eucl} \quad \text{where} \quad \varphi = \left(\frac{2}{1-r^2}\right)^{\frac{n-2}{2}},$$

to reduce equation (9) to a Dirichlet boundary value problem on Euclidean space. We shall denote by Ω the subdomain $\Omega_{\mathbb{B}^n}$ considered as a subset of $\mathbb{R}^n$. The following sharpens a few estimates obtained in [6]. These will be needed when dealing with the critical (low dimensional) cases.

Lemma 1. *$u \in H^2_{1,0}(\Omega_{\mathbb{B}^n})$ satisfies (9) if and only if $v := \varphi u \in H^2_{1,0}(\Omega)$ (the completion of $C_c^\infty(\Omega)$ for $\|\cdot\|_2$) satisfies*

$$\begin{cases} -\Delta v - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right) v &= b(x)\frac{v^{2^\star(s)-1}}{|x|^s} \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $x \mapsto b(x)$ is a positive function in $C^{0,1}(\overline{\Omega})$ with $b(0) = \frac{(n-2)^{\frac{2-s}{n-2}}}{2^{2-s}}$ satisfying the following estimates:

- *For $n = 3$, we have that*
$$(x, \nabla b(x)) = (2^\star(s) + 2)b(0)|x| + O(|x|^2) \text{ as } x \to 0 \tag{13}$$
- *For $n = 4$, we have $b \in C^1(\overline{\Omega})$, $\nabla b(0) = 0$ and*
$$(x, \nabla b(x)) = 4(2^\star(s) + 2)b(0)|x|^2 \ln\frac{1}{|x|} + O(|x|^2) \text{ as } x \to 0 \tag{14}$$
- *For $n \geq 5$, we have $b \in C^2(\overline{\Omega})$, $\nabla b(0) = 0$ and*
$$\frac{\Delta b(0)}{b(0)} = \frac{4n(2n-2-s)}{n-4}. \tag{15}$$

We also have that $h_{\gamma,\lambda} \in C^1\left(\overline{\Omega} \setminus \{0\}\right)$ in such a way that there exist $c_3, c_4 \in \mathbb{R}$ such that

$$h_{\gamma,\lambda}(x) = h_{\gamma,\lambda}(r) = \begin{cases} \frac{4\gamma}{r} + c_3 + O(r) & \text{for } n = 3, \\ 8\gamma \ln\frac{1}{r} + c_4 + O(r) & \text{for } n = 4, \\ 4\lambda + \frac{4(n-2)}{n-4}\left(\gamma - \frac{n(n-4)}{4}\right) + O(r) & \text{for } n \geq 5. \end{cases} \tag{16}$$

In addition, when $n = 4$ and $\gamma = 0$, we have that $c_4 = 4(\lambda - 2)$.

Moreover, the hyperbolic operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive if and only if the corresponding Euclidean operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right)$ is coercive.

We note that equation (12) have been studied extensively in the case where $h_{\gamma,\lambda} \equiv \lambda$ and $b \equiv 1$, that is

$$\begin{cases} -\Delta u - \gamma\frac{u}{|x|^2} - \lambda u &= \frac{|u|^{2^\star(s)-2}u}{|x|^s} \quad \text{in } \Omega \setminus \{0\}, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{17}$$

and more so in the non-singular case (i.e., when $\gamma = s = 0$), with major contributions to the existence of ground state solutions by Brezis-Nirenberg [2] when $n \geq 4$ and $\lambda > 0$, and by Druet [9, 10] and Druet-Laurain [12] when $n = 3$. The multiplicity of solutions was established in that case by Devillanova-Solimini [8] in dimension $n \geq 7$ and $0 < \lambda < \lambda_1(-\Delta)$.

The existence of ground state solutions when $s = 0$, $0 < \gamma \leq \frac{(n-2)^2}{4} - 1$, $0 < \lambda < \lambda_1(-\Delta - \frac{\gamma}{|x|^2})$, and $0 \in \Omega$ was established by Janelli [22], while the multiplicity of solutions for when $0 \leq \gamma < \frac{(n-2)^2}{4} - 4$ and $n \geq 7$ was proved by Cao-Yan [4]. We also note that if Ω is the unit ball, then Catrina-Lavine [5] showed that (17) admits no radial positive solutions [5] for $\gamma > \frac{(N-2)^2}{4} - 1$, while Esposito et al. show in

[13] that for λ small, problem (17) has no radial sign-changing solutions in B whenever $\gamma \geq \frac{(N-2)^2}{4} - 4$. There are also many results in the case where $\lambda > \lambda_1(-\Delta)$. They are summarized in [13].

The cases where both γ and s are not zero, including the limiting cases were also studied by Ghoussoub-Robert [19] and many other authors [3, 23, 26, 28].

The case when $0 \in \partial\Omega$ is more involved and was studied extensively in Ghoussoub-Robert [18] and by the above authors [15]. This paper is mostly concerned with the influence of the potential $h_{\gamma,\lambda}$ on existence and multiplicity issues related to the equation (12).

Existence and Multiplicity of solutions. Our multiplicity result will therefore follow from the following more general Euclidean statement. We set

$$C^1(\overline{\Omega}, |x|^{-\theta}) := \left\{ \begin{array}{l} h \in C^1(\overline{\Omega} \setminus \{0\}); \quad \exists c \in \mathbb{R} \text{ such that} \\ \\ \lim_{x\to 0} |x|^\theta h(x) = c \;\&\; \lim_{x\to 0} |x|^\theta (x, \nabla h(x)) = -c\theta. \end{array} \right. \tag{18}$$

The main multiplicity result in this paper is the following theorem

Theorem 1.5. *Consider a bounded, smooth domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) such that $0 \in \Omega$ and assume that $0 < s < 2$ and $0 \leq \theta < 2$. Let $h \in C^1(\overline{\Omega}, |x|^{-\theta})$ be such that $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive in Ω. Let $b \in C^2(\overline{\Omega})$ be such that $b \geq 0$, $b(0) > 0$ and $\nabla b(0) = 0$, while if $\theta = 0$, we shall assume in addition that $\nabla^2 b(0) \geq 0$ in the sense of bilinear forms.*

If $0 \leq \gamma < \dfrac{(n-2)^2}{4} - (2-\theta)^2$ and $\lim_{x\to 0} |x|^\theta h(x) = K_h > 0$, then the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u - \gamma \dfrac{u}{|x|^2} - h(x) u &=& b(x) \dfrac{|u|^{2^\star(s)-2} u}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{19}$$

has an infinite number of possibly sign-changing solutions in $H^2_{1,0}(\Omega)$. Moreover, these solutions belong to $C^2(\overline{\Omega} \setminus \{0\})$ while around 0 they behave like

$$\lim_{x\to 0} |x|^{\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - \gamma}} u(x) = K \in \mathbb{R}. \tag{20}$$

Note that when $h_{\gamma,\lambda} \equiv \lambda$ and $b \equiv 1$, the above yields an infinite number of solutions for equation (17), provided $0 < s < 2$, $n \geq 5$, $0 \leq \gamma < \dfrac{(n-2)^2}{4} - 4$ and $0 < \lambda < \lambda_1(-\Delta - \frac{\gamma}{|x|^2})$. Note that the fact that $s > 0$ allows for an improvement on the dimension $n \geq 7$ established for $s = 0$ by Devillanova-Solimini [8] when $\gamma = 0$ and Cao-Yan [4] when $\gamma > 0$.

The multiplicity result will follow from standard min-max arguments once we prove the required compactness, which relies on blow-up analysis techniques. The proof consists of analyzing the asymptotic behaviour of a family of solutions to the related subcritical equations –potentially developing a singularity at zero– as we approach the critical exponent.

Compactness of approximate solutions. For $0 \leq \theta < 2$, let $(h_\epsilon)_{\epsilon>0}$ be functions in $C^1(\overline{\Omega} \setminus \{0\})$, and $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$ such that

$$\lim_{\epsilon\to 0} \sup_{x\in\Omega} \left(|x|^\theta |h_\epsilon(x) - h_0(x)| + |x|^{\theta+1} |\nabla (h_\epsilon - h_0)(x)| \right) = 0. \tag{21}$$

Theorem 1.6. *Consider a bounded, smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $0 \leq \theta < 2$. Let $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$ be such that $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$ is coercive in Ω and consider $(h_\epsilon)_{\epsilon>0}$ such that (21) hold. Let $b \in C^2(\overline{\Omega})$ be such that $b \geq 0$, $b(0) > 0$ and $\nabla b(0) = 0$, while if $\theta = 0$, we shall assume in addition that $\nabla^2 b(0) \geq 0$ in the sense of bilinear forms.*

Let $(p_\epsilon)_{\epsilon>0}$ such that $0 \leq p_\epsilon < 2^\star(s) - 2$ and $\lim_{\epsilon\to 0} p_\epsilon = 0$ and consider a sequence of functions $(u_\epsilon)_{\epsilon>0}$ that is uniformly bounded in $H^2_{1,0}(\Omega)$ and such that for each $\epsilon > 0$, u_ϵ is a solution to the equation:

$$\begin{cases} -\Delta u_\epsilon - \gamma \frac{u_\epsilon}{|x|^2} - h_\epsilon(x) u_\epsilon = b(x) \frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon} u_\epsilon}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

If $\gamma < \frac{(n-2)^2}{4} - (2-\theta)^2$ and $\lim_{|x|\to 0} |x|^\theta h_0(x) = K_{h_0} > 0$, then the sequence $(u_\epsilon)_{\epsilon>0}$ is pre-compact in the space $H^2_{1,0}(\Omega)$.

The interior mass and compactness for approximate positive solutions at any energy level. Note that the function $x \mapsto |x|^{-\alpha}$ is a solution of

$$\left(-\Delta - \frac{\gamma}{|x|^2}\right) u = 0 \qquad \text{on } \mathbb{R}^n \setminus \{0\}, \tag{23}$$

if and only if $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$, where $\alpha_-(\gamma), \alpha_+(\gamma)$ are as in (11). Actually, one can show that any non-negative solution $u \in C^2(\mathbb{R}^n \setminus \{0\})$ of (23) is of the form

$$u(x) = C_- |x|^{-\alpha_-(\gamma)} + C_+ |x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{24}$$

where $C_-, C_+ \geq 0$, see [19].

For showing compactness in low-dimensions, the arguments are not any more local, but global. We are naturally led to introducing a notion of mass:

Theorem 1.7 (The Euclidean interior mass, see [19])**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \Omega$ is an interior point. Let $\gamma < \frac{(n-2)^2}{4}$ and let $h \in C^1(\overline{\Omega}, |x|^{-\theta})$ be such that the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive in Ω. Then there exists $H \in C^\infty(\overline{\Omega} \setminus \{0\})$ such that*

$$\begin{cases} -\Delta H - \frac{\gamma}{|x|^2} H - h(x) H = 0 & \text{in } \Omega \setminus \{0\} \\ H > 0 & \text{in } \Omega \setminus \{0\} \\ H = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

These solutions are unique up to a positive multiplicative constant, and there exists $c_+ > 0$ such that

$$H(x) \simeq_{x\to 0} \frac{c_+}{|x|^{\alpha_+(\gamma)}}.$$

Moreover, when $\gamma > \frac{(n-2)^2}{4} - \left(1 - \frac{\theta}{2}\right)^2$, there exists $c_- \in \mathbb{R}$ such that

$$H(x) = \frac{c_+}{|x|^{\alpha_+(\gamma)}} + \frac{c_-}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{1}{|x|^{\alpha_-(\gamma)}}\right) \text{ as } x \to 0. \tag{26}$$

We define the interior mass as $m_{\gamma,h}(\Omega) := \frac{c_+}{c_-}$, which is independent of the choice of H.

We then establish the following compactness result for positive solutions.

Theorem 1.8. *Consider a bounded, smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $0 \leq \theta < 2$. Let $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$ be such that $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$ is coercive in Ω and consider $(h_\epsilon)_{\epsilon>0}$ such that (21) hold. Let $b \in C^2(\overline{\Omega})$ be such that $b \geq 0$, $b(0) > 0$ and $\nabla b(0) = 0$, while if $\theta = 0$, we shall assume in addition that $\Delta b(0) \geq 0$.*

Let $(p_\epsilon)_{\epsilon>0}$ be such that $0 \leq p_\epsilon < 2^\star(s)-2$ and $\lim_{\epsilon\to 0} p_\epsilon = 0$ and consider a sequence of functions $(u_\epsilon)_{\epsilon>0}$ that is uniformly bounded in $H^2_{1,0}(\Omega)$ such that for each $\epsilon>0$, u_ϵ is a solution to the equation:

$$\left\{\begin{array}{rcll} -\Delta u_\epsilon - \gamma\frac{u_\epsilon}{|x|^2} - h_\epsilon(x)u_\epsilon & = & b(x)\frac{u_\epsilon^{2^\star(s)-1-p_\epsilon}}{|x|^s} & \text{in } \Omega\setminus\{0\},\\ u_\epsilon & > & 0 & \text{in } \Omega,\\ u_\epsilon & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{27}$$

Assuming one of the following conditions

- $0 \leq \gamma \leq \frac{(n-2)^2}{4} - (1-\frac{\theta}{2})^2$ *and* $\lim_{|x|\to 0}|x|^\theta h_0(x) = K_{h_0} > 0$;
- $\gamma > \frac{(n-2)^2}{4} - (1-\frac{\theta}{2})^2$ *and* $m_{\gamma,h_0}(\Omega) > 0$.

Then the sequence $(u_\epsilon)_{\epsilon>0}$ is pre-compact in the space $H^2_{1,0}(\Omega)$.

Non-existence: Stability of the Pohozaev obstruction. To address issues of non-existence of solutions, we shall prove the following surprising stability of regimes where variational positive solutions do not exist.

Theorem 1.9. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n\geq 3)$ such that $0\in\Omega$ is an interior point. Assume that $0<s<2$, $0\leq\theta<2$ and $\gamma<(n-2)^2/4$. Let $h_0\in C^1(\overline{\Omega},|x|^{-\theta})$ be such that $-\Delta-\gamma|x|^{-2}-h_0$ is coercive and let $b\in C^2(\overline{\Omega})$ be such that $b\geq 0$, $b(0)>0$ and $\nabla b(0)=0$.*

Assume that $\gamma > \frac{(n-2)^2}{4} - \left(1-\frac{\theta}{2}\right)^2$, the mass $m_{\gamma,h_0}(\Omega)$ is non-zero, and that there is no positive variational solution to the boundary value problem:

$$\left\{\begin{array}{rcll} -\Delta u - \gamma\frac{u}{|x|^2} - h_0(x)u & = & b(x)\frac{u^{2^\star(s)-1-p_\epsilon}}{|x|^s} & \text{in } \Omega\setminus\{0\},\\ u & > & 0 & \text{in } \Omega,\\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{28}$$

Then, for all $\Lambda>0$, there exists $\epsilon := \epsilon(\Lambda,h_0)>0$ such that for any $h\in C^1(\overline{\Omega},|x|^{-\theta})$ satisfying

$$\sup_{x\in\Omega}(|x|^\theta|h(x)-h_0(x)| + |x|^{\theta+1}|\nabla(h-h_0)(x)|) < \epsilon, \tag{29}$$

there is no positive solution to (19) such that $\|\nabla u\|_2 \leq \Lambda$.

The above result is surprising for the following reason: Assuming Ω is starshaped with respect to 0, then the classical Pohozaev obstruction (see Section 13) yields that (19) has no positive variational solution whenever $h_0\in C^1(\overline{\Omega},|x|^{-\theta})$ satisfies

$$h_0(x) + \frac{1}{2}(x,\nabla h_0(x)) \leq 0 \text{ for all } x\in\Omega. \tag{30}$$

We then get the following corollaries.

Corollary 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n\geq 3)$ that is starshaped around 0. Assume $0<s<2$, and*

$$\frac{(n-2)^2}{4} - \left(1-\frac{\theta}{2}\right)^2 < \gamma < \frac{(n-2)^2}{4}.$$

If $h_0\in C^1(\overline{\Omega},|x|^{-\theta})$ is a potential satisfying (30), then for all $\Lambda>0$, there exists $\epsilon(\Lambda,h_0)>0$ such that for all $h\in C^1(\overline{\Omega},|x|^{-\theta})$ satisfying (29), there is no positive solution to (28) with $b\equiv 1$, such that $\|\nabla u\|_2\leq\Lambda$.

Corollary 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ that is starshaped around 0. Assume $0 < s < 2$, and*
$$\frac{(n-2)^2}{4} - 1 < \gamma < \frac{(n-2)^2}{4}.$$
Then, for all $\Lambda > 0$, there exists $\epsilon(\Lambda) > 0$ such that for all $\lambda \in [0, \epsilon(\Lambda))$, there is no positive solution to equation (17) with $\|\nabla u\|_2 \leq \Lambda$.

Remark 1. *It is worth comparing these results to what happens in the nonsingular case, i.e., when $\gamma = s = \theta = 0$. Note that when $\gamma = \theta = 0$, the condition $\gamma > \frac{(n-2)^2}{4} - \left(1 - \frac{\theta}{2}\right)^2$ reads $n = 3$. In contrast to the singular case, a celebrated result of Brezis-Nirenberg [2] shows that, for $\gamma = s = \theta = 0$, a variational solution to (17) always exists whenever $n \geq 4$ and $0 < \lambda < \lambda_1(\Omega)$, with the geometry of the domain playing no role whatsoever. On the other hand, Druet-Laurain [12] showed that the geometry plays a role in dimension $n = 3$, still for $\gamma = s = \theta = 0$, by proving that when Ω is star-shaped, then there is no solution to (17) for all small values of $\lambda > 0$ (with no apriori bound on $\|\nabla u\|_2$). Another point of view is that for $n = 3$, the nonexistence of solutions persists under small perturbations, but it does not for $n \geq 4$, i.e., only for $n = 3$ that the Pohozaev obstruction is stable in the nonsingular case.*

When $0 \in \partial\Omega$, the situation is different. Indeed, the authors showed in [15] that there is no low-dimensional phenomenon and that in the singular case, the Pohozaev obstruction is stable in all dimensions.

Here are some other extensions related to this phenomenon.

- Our stability result still holds under an additional smooth perturbation of the domain Ω, just as was done by Druet-Hebey-Laurain [11] when $n = 3$, $\gamma = s = 0$.
- On the other hand, the stability result of Druet-Laurain [12] is not conditional on an apriori bound $\|\nabla u\|_2$, while ours is. We expect however to be able to get rid of the apriori bound in the singular case, as long as $0 \in \Omega$ and $s > 0$. This is a project in progress.

2. Setting the Blow-up

Throughout this paper, Ω will denote a bounded, smooth domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$. We will always assume that $\gamma < \frac{(n-2)^2}{4}$, $s \in (0, 2)$. For $\epsilon > 0$, we let $p_\epsilon \in [0, 2^\star(s) - 2)$ be such that
$$\lim_{\epsilon \to 0} p_\epsilon = 0. \tag{31}$$
We also consider $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$, $0 \leq \theta < 2$ such that there exists $K_{h_0} \in \mathbb{R}$ with
$$\lim_{x \to 0} |x|^\theta h_0(x) = K_{h_0} \text{ and } \lim_{x \to 0} |x|^\theta (x, \nabla h_0(x)) = -\theta K_{h_0}, \tag{32}$$
and $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$ is coercive in Ω. We also let $(h_\epsilon)_{\epsilon > 0}$ be such that $h_\epsilon \in C^1(\overline{\Omega} \setminus \{0\})$ for all $\epsilon > 0$ and
$$\lim_{\epsilon \to 0} \sup_{x \in \Omega} \left(|x|^\theta |h_\epsilon(x) - h_0(x)| + |x|^{\theta+1} |\nabla (h_\epsilon - h_0)(x)| \right) = 0. \tag{33}$$
Concerning b, we assume that
$$b(x) \geq 0 \text{ with } b \in C^2(\Omega) \text{ and } b(0) > 0, \nabla b(0) = 0. \tag{34}$$
The exponents $\alpha_\pm(\gamma)$ will denote
$$\alpha_\pm(\gamma) := \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}. \tag{35}$$

We consider a sequence of functions $(u_\epsilon)_{\epsilon>0}$ in $H^2_{1,0}(\Omega)$ such that for all $\epsilon>0$ the function u_ϵ is a solution to the Dirichlet boundary value problem:

$$\left\{\begin{array}{rcll} -\Delta u_\epsilon - \gamma\frac{u_\epsilon}{|x|^2} - h_\epsilon(x)u_\epsilon & = & b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}u_\epsilon}{|x|^s} & \text{in } H^2_{1,0}(\Omega), \\ u_\epsilon & = & 0 & \text{on } \partial\Omega. \end{array}\right. \qquad (E_\epsilon)$$

where (p_ϵ), $h_\epsilon(x)$ and b is such that (31), (33) and (34) holds.

By the regularity Theorem 13.1, $u_\epsilon \in C^2(\overline{\Omega}\setminus\{0\})$ and there exists $K_\epsilon \in \mathbb{R}$ such that $\lim\limits_{x\to 0}|x|^{\alpha_-(\gamma)}u_\epsilon(x) = K_\epsilon$. In addition, we assume that the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $H^2_{1,0}(\Omega)$ and we let $\Lambda>0$ be such that

$$\int_\Omega \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}dx \le \Lambda \qquad \text{for all } \epsilon>0. \tag{36}$$

It then follows from the weak compactness of the unit ball of $H^2_{1,0}(\Omega)$ that there exists $u_0 \in H^2_{1,0}(\Omega)$ such that

$$u_\epsilon \rightharpoonup u_0 \qquad \text{weakly in } H^2_{1,0}(\Omega) \text{ as } \epsilon \to 0. \tag{37}$$

Then u_0 is a weak solution to the Dirichlet boundary value problem

$$\left\{\begin{array}{rcll} -\Delta u_0 - \gamma\frac{u_0}{|x|^2} - h_0(x)u_0 & = & b(x)\frac{|u_0|^{2^\star(s)-2}u_0}{|x|^s} & \text{in } \Omega\setminus\{0\}, \\ u_0 & = & 0 & \text{on } \partial\Omega. \end{array}\right. \qquad (E_0)$$

Again from the regularity Theorem 13.1 it follows that $u_0 \in C^{2,\theta}(\overline{\Omega}\setminus\{0\})$ and $\lim\limits_{x\to 0}|x|^{\alpha_-(\gamma)}u_0(x) = K_0 \in \mathbb{R}$. Fix $\tau\in\mathbb{R}$ such that

$$\alpha_-(\gamma) < \tau < \frac{n-2}{2}. \tag{38}$$

The following proposition shows that the sequence $(u_\epsilon)_\epsilon$ is pre-compact in $H^2_{1,0}(\Omega)$ if $x\mapsto |x|^\tau u_\epsilon$ is uniformly bounded in $L^\infty(\Omega)$.

Proposition 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\ge 3$, such that $0\in\Omega$ and assume that $0<s<2$, $\gamma<\frac{(n-2)^2}{4}$. We let (u_ϵ), (p_ϵ), $h(x)$ and b be such that (E_ϵ), (31), (33), (34) and (36) holds. Suppose there exists τ as in (38) and $C>0$ such that $|x|^\tau|u_\epsilon(x)|\le C$ for all $x\in\Omega\setminus\{0\}$ and for all $\epsilon>0$. Then up to a subsequence, $\lim\limits_{\epsilon\to 0}u_\epsilon = u_0$ in $H^2_{1,0}(\Omega)$ where u_0 is as in (37).*

Proof: We have assumed that $|x|^\tau|u_\epsilon(x)|\le C$ for all $x\in\Omega$ and for all $\epsilon>0$. So the sequence (u_ϵ) is uniformly bounded in $L^\infty(\Omega')$ for any $\Omega'\subset\subset\overline{\Omega}\setminus\{0\}$. Then by standard elliptic estimates and from (37) it follows that $u_\epsilon\to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$.
Now since $|x|^\tau|u_\epsilon(x)|\le C$ for all $x\in\Omega$ and for all $\epsilon>0$ and since $\tau<\frac{n-2}{2}$, we have

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{B_\delta(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}dx = 0 \quad\text{and}\quad \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{B_\delta(0)}\frac{|u_\epsilon|^2}{|x|^2}dx = 0. \tag{39}$$

Therefore

$$\lim_{\epsilon\to 0}\int_\Omega b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}dx = \int_\Omega b(x)\frac{|u_0|^{2^\star(s)}}{|x|^s}dx \quad\text{and}\quad \lim_{\epsilon\to 0}\int_\Omega\frac{|u_\epsilon|^2}{|x|^2}dx = \int_\Omega\frac{|u_0|^2}{|x|^2}dx.$$

From (E_ϵ) and (37) we then obtain

$$\lim_{\epsilon\to 0}\int_\Omega\left(|\nabla u_\epsilon|^2-\gamma\frac{u_\epsilon^2}{|x|^2}-h(x)u_\epsilon^2\right)dx=\int_\Omega\left(|\nabla u_0|^2-\gamma\frac{u_0^2}{|x|^2}-h(x)u_0^2\right)dx$$

$$\text{and so then }\lim_{\epsilon\to 0}\int_\Omega|\nabla u_\epsilon|^2=\lim_{\epsilon\to 0}\int_\Omega|\nabla u_0|^2.$$

And hence $\lim_{\epsilon\to 0}u_\epsilon=u_0$ in $H^2_{1,0}(\Omega)$. □

From now on we shall assume that

$$\lim_{\epsilon\to 0}\||x|^\tau u_\epsilon\|_{L^\infty(\Omega)}=+\infty\quad\text{where}\quad\alpha_-(\gamma)<\tau<\frac{n-2}{2},\tag{40}$$

and work towards obtaining a contradiction. We shall say that blow-up occurs whenever (40) holds.

3. Some Scaling Lemmas

We start with two scaling lemmas which we shall use many times in our analysis.

Lemma 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, such that $0\in\Omega$ and assume that $0<s<2$ and $\gamma<\frac{(n-2)^2}{4}$. Let (u_ϵ), (p_ϵ), $h(x)$ and b be such that (E_ϵ), (31), (33), (34) and (36) holds. Let $(y_\epsilon)_\epsilon\in\Omega\setminus\{0\}$ and let*

$$\nu_\epsilon^{-\frac{n-2}{2}}:=|u_\epsilon(y_\epsilon)|,\quad \ell_\epsilon:=\nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}}\quad\textit{and}\quad\kappa_\epsilon:=|y_\epsilon|^{s/2}\,\ell_\epsilon^{\frac{2-s}{2}}\qquad\textit{for }\epsilon>0$$

Suppose $\lim_{\epsilon\to 0}y_\epsilon=0$ and $\lim_{\epsilon\to 0}\nu_\epsilon=0$. Assume that for any $R>0$ there exists $C(R)>0$ such that for all $\epsilon>0$

$$|u_\epsilon(x)|\leq C(R)\,\frac{|y_\epsilon|^\tau}{|x|^\tau}|u_\epsilon(y_\epsilon)|\qquad\textit{for all }x\in B_{R\kappa_\epsilon}(y_\epsilon)\setminus\{0\}.\tag{41}$$

Then

$$|y_\epsilon|=O(\ell_\epsilon)\qquad\textit{as }\epsilon\to 0.\tag{42}$$

Proof of Lemma 2: We proceed by contradiction and assume that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon}=+\infty.\tag{43}$$

Then it follows from the definition of κ_ϵ that

$$\lim_{\epsilon\to 0}\kappa_\epsilon=0,\ \lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{\ell_\epsilon}=+\infty\text{ and }\lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{|y_\epsilon|}=0.\tag{44}$$

Fix a $\rho>0$. We define for all $\epsilon>0$

$$v_\epsilon(x):=\nu_\epsilon^{\frac{n-2}{2}}u_\epsilon(y_\epsilon+\kappa_\epsilon x)\qquad\text{for }x\in B_{2\rho}(0)$$

Note that this is well defined since $\lim_{\epsilon\to 0}|y_\epsilon|=0\in\Omega$ and $\lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{|y_\epsilon|}=0$. It follows from (41) that there exists $C(\rho)>0$ such that all $\epsilon>0$

$$|v_\epsilon(x)|\leq C(\rho)\frac{1}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^\tau}\qquad\forall x\in B_{2\rho}(0)\tag{45}$$

using (44) we then get as $\epsilon\to 0$

$$|v_\epsilon(x)|\leq C(\rho)\,(1+o(1))\qquad\forall x\in B_{2\rho}(0).$$

From equation (E_ϵ) we obtain that v_ϵ satisfies the equation

$$-\Delta v_\epsilon - \frac{\kappa_\epsilon^2}{|y_\epsilon|^2}\frac{\gamma}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^2}\, v_\epsilon - \kappa_\epsilon^2\ h_\epsilon(y_\epsilon+\kappa_\epsilon x)\ v_\epsilon = b(y_\epsilon+\kappa_\epsilon x)\frac{|v_\epsilon|^{2^\star(s)-2-p_\epsilon}v_\epsilon}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}$$

weakly in $B_{2\rho}(0)$ for all $\epsilon>0$. With the help of (44), (33) and standard elliptic theory it then follows that there exists $v\in C^1(B_{2\rho}(0))$ such that

$$\lim_{\epsilon\to 0} v_\epsilon = v \qquad \text{in } C^1(B_\rho(0)).$$

In particular,

$$v(0)=\lim_{\epsilon\to 0} v_\epsilon(0)=1 \tag{46}$$

and therefore $v\not\equiv 0$.

On the other hand, change of variables and the definition of κ_ϵ yields

$$\begin{aligned}\int_{B_{\rho\kappa_\epsilon}(y_\epsilon)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx =& \frac{|u_\epsilon(y_\epsilon)|^{2^\star(s)-p_\epsilon}\kappa_\epsilon^n}{|y_\epsilon|^s}\int_{B_\rho(0)}\frac{|v_\epsilon|^{2^\star(s)-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}\,dx\\ &= \ell_\epsilon^{-\left(1+\frac{2(2-s)}{2^\star(s)-2-p_\epsilon}\right)}\left(\frac{|y_\epsilon|}{\ell_\epsilon}\right)^{s\left(\frac{n-2}{2}\right)}\int_{B_\rho(0)}\frac{|v_\epsilon|^{2^\star(s)-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}\,dx\\ &\geq \left(\frac{|y_\epsilon|}{\ell_\epsilon}\right)^{s\left(\frac{n-2}{2}\right)}\int_{B_\rho(0)}\frac{|v_\epsilon|^{2^\star(s)-p_\epsilon}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{\kappa_\epsilon}{|y_\epsilon|}x\right|^s}\,dx.\end{aligned}$$

Using the equation (E_ϵ), (36), (43), (44) and passing to the limit $\epsilon\to 0$ we get that $\int_{B_\rho(0)} |v|^{2^\star(s)}\,dx = 0$, and so then $v\equiv 0$ in $B_\rho(0)$, a contradiction with (46). Thus (43) cannot hold. This proves that $y_\epsilon = O(\ell_\epsilon)$ when $\epsilon\to 0$, which proves the lemma. □

Here and in the sequel, we let $D^{1,2}(\mathbb{R}^n)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u\mapsto \|\nabla u\|_2$.

Lemma 3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, such that $0\in\Omega$ and assume that $0<s<2$ and $\gamma<\frac{(n-2)^2}{4}$. Let (u_ϵ), (p_ϵ), $h(x)$ and b be such that (E_ϵ), (31), (33), (34) and (36) holds. Let $(y_\epsilon)_\epsilon\in\Omega\setminus\{0\}$ and let*

$$\nu_\epsilon^{-\frac{n-2}{2}} := |u_\epsilon(y_\epsilon)| \quad \text{and} \quad \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}} \qquad \text{for } \epsilon>0$$

Suppose $\nu_\epsilon\to 0$ and $|y_\epsilon| = O(\ell_\epsilon)$ as $\epsilon\to 0$.

For $\epsilon>0$ we rescale and define

$$w_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} u_\epsilon(\ell_\epsilon x) \qquad \text{for } x\in \ell_\epsilon^{-1}\Omega\setminus\{0\}.$$

Assume that for any $R>\delta>0$ there exists $C(R,\delta)>0$ such that for all $\epsilon>0$

$$|w_\epsilon(x)| \leq C(R,\delta) \qquad \text{for all } x\in B_R(0)\setminus\overline{B_\delta(0)}. \tag{47}$$

Then there exists $w\in D^{1,2}(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})$ such that

$$\begin{aligned} w_\epsilon &\rightharpoonup w \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \epsilon\to 0\\ w_\epsilon &\to w \qquad \text{in } C^1_{loc}(\mathbb{R}^n\setminus\{0\}) \quad \text{as } \epsilon\to 0\end{aligned}$$

And w satisfies weakly the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0)\frac{|w|^{2^\star(s)-2}w}{|x|^s} \ \textit{in } \mathbb{R}^n \setminus \{0\}.$$

Moreover if $w \not\equiv 0$, then

$$\int_{\mathbb{R}^n} \frac{|w|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

where

$$\mu_{\gamma,s,0}(\mathbb{R}^n) := \inf_{u \in D^{1,2}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n}\left(|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2\right)\,dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)} \tag{48}$$

and there exists $t \in (0,1]$ such that $\lim_{\epsilon\to 0} \nu_\epsilon^{p_\epsilon} = t$.

Proof of Lemma 3: The proof proceeds in four steps.
Step 3.1: Let $\eta \in C_c^\infty(\mathbb{R}^n)$. One has that $\eta w_\epsilon \in H_0^1(\mathbb{R}^n)$ for $\epsilon > 0$ sufficiently small. We claim that there exists $w_\eta \in D^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$\begin{cases} \eta w_\epsilon \rightharpoonup w_\eta & \text{weakly in } D^{1,2}(\mathbb{R}^n) \ \text{ as } \epsilon \to 0, \\ \eta w_\epsilon \to w_\eta(x) & a.e \ \text{ in } \mathbb{R}^n \ \text{ as } \epsilon \to 0. \end{cases}$$

We prove the claim. Let $x \in \mathbb{R}^n$, then

$$\nabla\left(\eta w_\epsilon\right)(x) = w_\epsilon(x)\nabla\eta(x) + \nu_\epsilon^{\frac{n-2}{2}}\ell_\epsilon\ \eta(x)\nabla u_\epsilon(\ell_\epsilon x).$$

Now for any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$

$$(a+b)^2 \leq C(\theta)a^2 + (1+\theta)b^2$$

With this inequality we then obtain

$$\int_{\mathbb{R}^n} |\nabla(\eta w_\epsilon)|^2\ dx \leq C(\theta)\int_{\mathbb{R}^n} |\nabla\eta|^2 w_\epsilon^2\ dx + (1+\theta)\nu_\epsilon^{n-2}\ell_\epsilon^2 \int_{\mathbb{R}^n} \eta^2 \left|\nabla u_\epsilon(\ell_\epsilon x)\right|^2\ dx$$

With Hölder inequality and a change of variables this becomes

$$\int_{\mathbb{R}^n} |\nabla(\eta w_\epsilon)|^2\,dx \leq C(\theta)\,\|\nabla\eta\|_{L^n}^2\left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2}\left(\int_\Omega |u_\epsilon|^{2^\star}\ dx\right)^{\frac{n-2}{n}}$$
$$+ (1+\theta)\left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2}\int_\Omega \left(\eta\left(\frac{x}{\ell_\epsilon}\right)\right)^2 |\nabla u_\epsilon|^2\ dx. \tag{49}$$

Since $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$, so for $\epsilon > 0$ small enough

$$\|\eta w_\epsilon\|_{D^{1,2}(\mathbb{R}^n)} \leq C_\eta$$

Where C_η is a constant depending on the function η. The claim then follows from the reflexivity of $D^{1,2}(\mathbb{R}^n)$.

Step 3.2: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \eta_1 \leq 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for} \quad x \in B_0(1) \\ 0 & \text{for} \quad x \in \mathbb{R}^n \backslash B_0(2) \end{cases} \tag{50}$$

For any $R>0$ we let $\eta_R=\eta_1(x/R)$. Then with a diagonal argument we can assume that upto a subsequence for any $R>0$ there exists $w_R \in D^{1,2}(\mathbb{R}^n)$ such that

$$\left\{\begin{array}{ll} \eta_R w_\epsilon \rightharpoonup w_R & \text{weakly in } D^{1,2}(\mathbb{R}^n) \text{ as } \epsilon \to 0 \\ \eta_R w_\epsilon(x) \to w_R(x) & a.e\ x \ \text{ in } \mathbb{R}^n \ \text{ as } \epsilon \to 0 \end{array}\right.$$

Since $\|\nabla \eta_R\|_n^2 = \|\nabla \eta_1\|_n^2$ for all $R>0$, letting $\epsilon \to 0$ in (49) we obtain that

$$\int_{\mathbb{R}^n_-} |\nabla w_R|^2\, dx \leq C \qquad \text{for all } R>0$$

where C is a constant independent of R. So there exists $w \in D^{1,2}(\mathbb{R}^n)$ such that

$$\left\{\begin{array}{ll} w_R \rightharpoonup w & \text{weakly in } D^{1,2}(\mathbb{R}^n) \text{ as } R \to +\infty \\ w_R(x) \to w(x) & a.e\ x \ \text{ in } \mathbb{R}^n \ \text{ as } R \to +\infty \end{array}\right.$$

Step 3.3: We claim that $w \in C^1(\mathbb{R}^n \setminus \{0\})$ and it satisfies weakly the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0) \frac{|w|^{2^\star(s)-2} w}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

We prove the claim. From (E_ϵ) it follows that for any $\epsilon>0$ and $R>0$, $\eta_R w_\epsilon$ satisfies weakly the equation

$$-\Delta(\eta_R w_\epsilon) - \frac{\gamma}{|x|^2}(\eta_R w_\epsilon) - \ell_\epsilon^2\, h_\epsilon(\ell_\epsilon x)(\eta_R w_\epsilon) = b(\ell_\epsilon x) \frac{|(\eta_R w_\epsilon)|^{2^\star(s)-2-p_\epsilon}(\eta_R w_\epsilon)}{|x|^s}. \tag{51}$$

From (47) and (33), using the standard elliptic estimates it follows that $w_R \in C^1(B_R(0)\setminus\{0\})$ and that up to a subsequence

$$\lim_{\epsilon \to 0} \eta_R w_\epsilon = w_R \qquad \text{in } C^1_{loc}(B_R(0)\setminus\{0\}).$$

Letting $\epsilon \to 0$ in eqn (51) gives that w_R satisfies weakly the equation

$$-\Delta w_R - \frac{\gamma}{|x|^2} w_R = b(0) \frac{|w_R|^{2^\star(s)-2-p_\epsilon} w_R}{|x|^s}.$$

Again we have that $|w_R(x)| \leq C(R,\delta)$ for all $x \in \overline{B_{R/2}(0)} \setminus \overline{B_{2\delta}(0)}$ and then again from standard elliptic estimates it follows that $w \in C^1(\mathbb{R}^n \setminus \{0\})$ and $\lim_{R\to+\infty} \tilde{w}_R = \tilde{w}$ in $C^1_{loc}(\mathbb{R}^n\setminus\{0\})$, up to a subsequence. Letting $R \to +\infty$ we obtain that w satisfies weakly the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0) \frac{|w|^{2^\star(s)-2} w}{|x|^s}.$$

This proves our claim.

Step 3.4: Coming back to equation (49) we have for $R>0$

$$\int_{\mathbb{R}^n} |\nabla(\eta_R w_\epsilon)|^2\ dx \leq C(\theta) \left(\int_{B_0(2R)\setminus B_0(R)} (\eta_{2R} w_\epsilon)^{2^*}\ dx \right)^{\frac{n-2}{n}} + (1+\theta)\left(\frac{\nu_\epsilon}{\ell_\epsilon}\right)^{n-2} \int_\Omega |\nabla u_\epsilon|^2\ dx. \tag{52}$$

Since the sequence $(u_\epsilon)_\epsilon$ is bounded in $H^2_{1,0}(\Omega)$, letting $\epsilon \to 0$ and then $R \to +\infty$ we obtain for some constant C

$$\int_{\mathbb{R}^n} |\nabla w|^2 \, dx \leq C \left(\lim_{\epsilon \to 0} \left(\frac{\nu_\epsilon}{\ell_\epsilon} \right) \right)^{n-2}. \tag{53}$$

Now if $w \not\equiv 0$ weakly satisfies the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0) \frac{|w|^{2^\star(s)-2} w}{|x|^s}$$

using the definition 48 of $\mu_{\gamma,s,0}(\mathbb{R}^n)$ it then follows that

$$\int_{\mathbb{R}^n} \frac{|w|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)} \right)^{\frac{2^\star(s)}{2^\star(s)-2}}.$$

Hence $\lim_{\epsilon \to 0} \left(\frac{\nu_\epsilon}{\ell_\epsilon} \right) > 0$ which implies that

$$t := \lim_{\epsilon \to 0} \nu_\epsilon^{p_\epsilon} > 0. \tag{54}$$

Since $\lim_{\epsilon \to 0} \nu_\epsilon = 0$, therefore we have that $0 < t \leq 1$. This completes the lemma. □

4. Construction and Exhaustion of the Blow-up scales

In this section we prove the following proposition:

Proposition 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (p_ϵ), (h_ϵ) and b be such that (E_ϵ), (31), (33), (34) and (36) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon \to 0} \| |x|^\tau u_\epsilon \|_{L^\infty(\Omega)} = +\infty \quad \text{where} \quad \alpha_-(\gamma) < \tau < \frac{n-2}{2}.$$

Then, there exists $N \in \mathbb{N}^\star$ families of scales $(\mu_{i,\epsilon})_{\epsilon > 0}$ such that we have:

- **(A1)** $\lim_{\epsilon \to 0} u_\epsilon = u_0$ *in* $C^2_{loc}(\overline{\Omega} \setminus \{0\})$ *where u_0 is as in*(37).
- **(A2)** $0 < \mu_{1,\epsilon} < \ldots < \mu_{N,\epsilon}$, *for all $\epsilon > 0$.*
- **(A3)**

$$\lim_{\epsilon \to 0} \mu_{N,\epsilon} = 0 \text{ and } \lim_{\epsilon \to 0} \frac{\mu_{i+1,\epsilon}}{\mu_{i,\epsilon}} = +\infty \text{ for all } 1 \leq i \leq N-1.$$

- **(A4)** *For any $1 \leq i \leq N$ and for $\epsilon > 0$ we rescale and define*

$$\tilde{u}_{i,\epsilon}(x) := \mu_{i,\epsilon}^{\frac{n-2}{2}} u_\epsilon(k_{i,\epsilon} x) \qquad \text{for } x \in k_{i,\epsilon}^{-1} \Omega \setminus \{0\}$$

where $k_{i,\epsilon} = \mu_{i,\epsilon}^{1 - \frac{p_\epsilon}{2^\star(s)-2}}$.
Then there exists $\tilde{u}_i \in D^{1,2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, $\tilde{u}_i \not\equiv 0$ such that $\tilde{u}_i$ weakly solves the equation

$$-\Delta \tilde{u}_i - \frac{\gamma}{|x|^2} \tilde{u}_i = b(0) \frac{|\tilde{u}_i|^{2^\star(s)-2} \tilde{u}_i}{|x|^s} \tag{55}$$

and

$$\tilde{u}_{i,\epsilon} \longrightarrow \tilde{u}_i \qquad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \qquad \text{as } \epsilon \to 0,$$
$$\tilde{u}_{i,\epsilon} \rightharpoonup \tilde{u}_i \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \qquad \text{as } \epsilon \to 0.$$

(A5) *There exists $C>0$ such that*

$$|x|^{\frac{n-2}{2}} |u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq C$$

for all $\epsilon > 0$ and all $x \in \Omega \setminus \{0\}$.

(A6) $\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\Omega\setminus B_{Rk_{N,\epsilon}}(0)} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0.$

(A7) $\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{B_{\delta k_{1,\epsilon}}(0)\setminus\{0\}} |x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_1\left(\frac{x}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0.$

(A8) *For any $\delta > 0$ and any $1 \leq i \leq N-1$, we have*

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{\delta k_{i+1,\epsilon} \geq |x| \geq R k_{i,\epsilon}} |x|^{\frac{n-2}{2}} \left| u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}} \tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0.$$

(A9) *For any $i \in \{1,...,N\}$ there exists $t_i \in (0,1]$ such that $\lim_{\epsilon\to 0} \mu_{i,\epsilon}^{p_\epsilon} = t_i$.*

The proof of this proposition proceeds in five steps.
Since $s > 0$, the subcriticality $2^\star(s) < 2^\star := 2^\star(0)$ of equations (E_ϵ) in $\Omega\setminus\{0\}$ along with (37) yields that $u_\epsilon \to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. So the only blow-up point is the origin.

Step 4.1: The construction of the $\mu_{i,\epsilon}$'s proceeds by induction. This step is the initiation.

By the regularity Theorem 13.1 and the definition of τ it follows that for any $\epsilon > 0$ there exists $x_{1,\epsilon} \in \Omega\setminus\{0\}$ such that

$$\sup_{x\in\Omega\setminus\{0\}} |x|^\tau |u_\epsilon(x)| = |x_{1,\epsilon}|^\tau |u_\epsilon(x_{1,\epsilon})|. \tag{56}$$

We define $\mu_{1,\epsilon}$ and $k_{1,\epsilon} > 0$ as follows

$$\mu_{1,\epsilon}^{-\frac{n-2}{2}} := |u_\epsilon(x_{1,\epsilon})| \quad \text{and} \quad k_{1,\epsilon} := \mu_{1,\epsilon}^{1-\frac{p_\epsilon}{2^\star-2}}. \tag{57}$$

Since blow-up occurs, that is (40) holds, we have

$$\lim_{\epsilon\to 0} \mu_{1,\epsilon} = 0$$

It follows that u_ϵ satisfies the hypothesis (41) of Lemma 2 with $y_\epsilon = x_{1,\epsilon}$, $\nu_\epsilon = \mu_{1,\epsilon}$. Therefore

$$|x_{1,\epsilon}| = O\left(k_{1,\epsilon}\right) \quad \text{as } \epsilon \to 0.$$

Infact, we claim that there exists $c_1 > 0$ such that

$$\lim_{\epsilon\to 0} \frac{|x_{1,\epsilon}|}{k_{1,\epsilon}} = c_1. \tag{58}$$

We argue by contradiction and we assume that $|x_{1,\epsilon}| = o(k_{1,\epsilon})$ as $\epsilon \to 0$. We define for $\epsilon > 0$

$$\tilde{v}_\epsilon(x) := \mu_{1,\epsilon}^{\frac{n-2}{2}} u_\epsilon(|x_{1,\epsilon}|x) \qquad \text{for } x \in |x_{1,\epsilon}|^{-1}\Omega\setminus\{0\}$$

Using (E_ϵ) we obtain that $\tilde{v}_\epsilon$ weakly satisfies the equation in $|x_{1,\epsilon}|^{-1}\Omega\setminus\{0\}$

$$-\Delta\tilde{v}_\epsilon - \frac{\gamma}{|x|^2}\tilde{v}_\epsilon - |x_{1,\epsilon}|^2 h_\epsilon(|x_{1,\epsilon}|x)\ \tilde{v}_\epsilon = b(|x_{1,\epsilon}|x)\left(\frac{|x_{1,\epsilon}|}{k_{1,\epsilon}}\right)^{2-s} \frac{|\tilde{v}_\epsilon|^{2^\star(s)-2}\tilde{v}_\epsilon}{|x|^s}.$$

The definition (56) yields $|x|^\tau|\tilde{v}_\epsilon(x)| \leq 1$ for all $x \in |x_{1,\epsilon}|^{-1}\Omega \setminus \{0\}$. Standard elliptic theory then yield the existence of $\tilde{v} \in C^2(\mathbb{R}^n \setminus \{0\})$ such that $\tilde{v}_\epsilon \to \tilde{v}$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ where

$$-\Delta\tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} = 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

In addition, we have that $\left|\tilde{v}_\epsilon\left(|x_{1,\epsilon}|^{-1}x_{1,\epsilon}\right)\right| = 1$ and so $\tilde{v} \not\equiv 0$. Also since $|x|^\tau|\tilde{v}(x)| \leq 1$ in $\mathbb{R}^n \setminus \{0\}$, we have the bound

$$|\tilde{v}(x)| < 2|x|^{-\alpha_+(\gamma)} + 2|x|^{-\alpha_-(\gamma)} \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

The classification of positive solutions of $-\Delta v - \frac{\gamma}{|x|^2}v = 0$ in $\mathbb{R}^n \setminus \{0\}$ (see (197)) yields the existence of $A, B \in \mathbb{R}$ such that $\tilde{v}(x) = A|x|^{-\alpha_+(\gamma)} + B|x|^{-\alpha_-(\gamma)}$ in $\mathbb{R}^n \setminus \{0\}$. Then the pointwise control $|x|^\tau|\tilde{v}(x)| \leq 1$ in $\mathbb{R}^n \setminus \{0\}$ yields $A = B = 0$, contradicting $\tilde{v} \not\equiv 0$. This proves the claim (58).

We rescale and define

$$\tilde{u}_{1,\epsilon}(x) := \mu_{1,\epsilon}^{\frac{n-2}{2}} u_\epsilon(k_{1,\epsilon}x) \qquad \text{for } x \in k_{1,\epsilon}^{-1}\Omega \setminus \{0\}$$

It follows from (56) and (58) that $\tilde{u}_{1,\epsilon}$ satisfies the hypothesis (47) of Lemma 3 with $y_\epsilon = x_{1,\epsilon}$, $\nu_\epsilon = \mu_{1,\epsilon}$. Then using lemma (3) we get that there exists $\tilde{u}_1 \in D^{1.2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ weakly satisfying the equation:

$$-\Delta\tilde{u}_1 - \frac{\gamma}{|x|^2}\tilde{u}_1 = b(0)\frac{|\tilde{u}_1|^{2^\star(s)-2}\tilde{u}_1}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

and

$$\tilde{u}_{1,\epsilon} \rightharpoonup \tilde{u}_1 \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \epsilon \to 0$$
$$\tilde{u}_{1,\epsilon} \to \tilde{u}_1 \qquad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \qquad \text{as } \epsilon \to 0$$

It follows from the definition that $\left|\tilde{u}_{1,\epsilon}\left(\frac{x_{1,\epsilon}}{k_{1,\epsilon}}\right)\right| = 1$. From (58) we therefore have that $\tilde{u}_1 \not\equiv 0$. And hence again from Lemma 3 we get that

$$\int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_1|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t_1 \in (0,1]$ such that $\lim\limits_{\epsilon\to 0}\mu_{1,\epsilon}^{p_\epsilon} = t_1$.

Next, since $|x|^{\alpha_-(\gamma)}\tilde{u}_1 \in C^0(\mathbb{R}^n)$, we have

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \sup_{B_{\delta k_{1,\epsilon}}(0)\setminus\{0\}} |x|^{\frac{n-2}{2}}\left|\mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0$$

and then using the definitions (56), (57) it follows that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \sup_{B_{\delta k_{1,\epsilon}}(0)\setminus\{0\}} |x|^{\frac{n-2}{2}}\left|u_\epsilon(x) - \mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0.$$

□

Step 4.2: There exists $C > 0$ such that for all $\epsilon > 0$ and all $x \in \Omega \setminus \{0\}$,

$$|x|^{\frac{n-2}{2}}|u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq C. \tag{59}$$

Proof of Step 4.2: We argue by contradiction and let $(y_\epsilon)_{\epsilon>0} \in \Omega \setminus \{0\}$ be such that

$$\sup_{x\in\Omega\setminus\{0\}} |x|^{\frac{n-2}{2}}|u_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = |y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \to +\infty \;\text{ as } \epsilon \to 0. \tag{60}$$

By the regularity Theorem 13.1 it follows that the sequence $(y_\epsilon)_{\epsilon>0}$ is well-defined and moreover $\lim_{\epsilon\to 0} y_\epsilon = 0$, since $u_\epsilon \to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. For $\epsilon>0$ we let

$$\nu_\epsilon := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}},\ \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}} \text{ and } \kappa_\epsilon := |y_\epsilon|^{s/2}\,\ell_\epsilon^{\frac{2-s}{2}}.$$

Then it follows from (60) that

$$\lim_{\epsilon\to 0}\nu_\epsilon = 0,\ \lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon} = +\infty \text{ and } \lim_{\epsilon\to 0}\frac{\kappa_\epsilon}{|y_\epsilon|} = 0. \tag{61}$$

Let $R>0$ and let $x\in B_R(0)$ be such that $y_\epsilon+\kappa_\epsilon x\in\Omega\setminus\{0\}$. It follows from the definition (60) of y_ϵ that for all $\epsilon>0$

$$|y_\epsilon+\kappa_\epsilon x|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon+\kappa_\epsilon x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq |y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}$$

and then, for all $\epsilon>0$

$$\left(\frac{|u_\epsilon(y_\epsilon+\kappa_\epsilon x)|}{|u_\epsilon(y_\epsilon)|}\right)^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{1}{1-\frac{\kappa_\epsilon}{|y_\epsilon|}R}\right)^{\frac{n-2}{2}}$$

for all $x\in B_R(0)$ such that $y_\epsilon+\kappa_\epsilon x\in\Omega\setminus\{0\}$. Using (61), we get that there exists $C(R)>0$ such that the hypothesis (41) of Lemma 2 is satisfied and therefore one has $|y_\epsilon| = O(\ell_\epsilon)$ when $\epsilon\to 0$, contradiction to (61). This proves (59). □

Let $\mathcal{I}\in\mathbb{N}^\star$. We consider the following assertions:

- **(B1)** $0<\mu_{1,\epsilon}<\ldots<\mu_{\mathcal{I},\epsilon}$.
- **(B2)** $\lim_{\epsilon\to 0}\mu_{\epsilon,\mathcal{I}} = 0$ and $\lim_{\epsilon\to 0}\frac{\mu_{\epsilon,i+1}}{\mu_{i,\epsilon}} = +\infty$ for all $1\leq i\leq\mathcal{I}-1$
- **(B3)** For all $1\leq i\leq\mathcal{I}$ there exists $\tilde{u}_i\in D^{1,2}(\mathbb{R}^n)\cap C^2(\mathbb{R}^n\setminus\{0\})$ such that $\tilde{u}_i$ weakly solves the equation

$$-\Delta\tilde{u}_i - \frac{\gamma}{|x|^2}\tilde{u}_i = b(0)\frac{|\tilde{u}_i|^{2^\star(s)-2}\tilde{u}_i}{|x|^s} \qquad \text{in } \mathbb{R}^n\setminus\{0\}$$

  with

$$\int_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}},$$

  and

$$\begin{aligned}\tilde{u}_{i,\epsilon} &\longrightarrow \tilde{u}_i \qquad \text{in } C^1_{loc}(\mathbb{R}^n\setminus\{0\}) \qquad \text{as } \epsilon\to 0,\\ \tilde{u}_{i,\epsilon} &\rightharpoonup \tilde{u}_i \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \qquad \text{as } \epsilon\to 0.\end{aligned}$$

  where for $\epsilon>0$

$$\tilde{u}_{i,\epsilon}(x) := \mu_{i,\epsilon}^{\frac{n-2}{2}}u_\epsilon(k_{i,\epsilon}x) \qquad \text{for } x\in k_{i,\epsilon}^{-1}\Omega\setminus\{0\}$$

  with $k_{i,\epsilon} = \mu_{i,\epsilon}^{1-\frac{p_\epsilon}{2^\star(s)-2}}$.
- **(B4)** For all $1\leq i\leq\mathcal{I}$, there exists $t_i\in(0,1]$ such that $\lim_{\epsilon\to 0}\mu_{i,\epsilon}^{p_\epsilon} = t_i$.

We say that $\mathcal{H}_\mathcal{I}$ holds if there exists $\mathcal{I}$ sequences $(\mu_{i,\epsilon})_{\epsilon>0}$, $i=1,\ldots,\mathcal{I}$ such that points (B1), (B2) (B3) and (B4) holds. Note that it follows from Step 4.1 that $\mathcal{H}_1$ holds. Next we show the following holds:

Step 4.3 Let $I\geq 1$. We assume that $\mathcal{H}_\mathcal{I}$ holds. Then either $\mathcal{H}_{\mathcal{I}+1}$ holds or

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{\Omega\setminus B_0(Rk_{\mathcal{I},\epsilon})}|x|^{\frac{n-2}{2}}|u_\epsilon(x)-u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0.$$

Proof of Step 4.3: Suppose

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{\Omega\setminus B_0(Rk_{\mathcal{I},\epsilon})}|x|^{\frac{n-2}{2}}|u_\epsilon(x)-u_0(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}\neq 0.$$

Then there exists a sequence of points $(y_\epsilon)_{\epsilon>0}\in\Omega\setminus\{0\}$ such that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{k_{\mathcal{I},\epsilon}}=+\infty \text{ and } \lim_{\epsilon\to 0}|y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)-u_0(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=a>0. \tag{62}$$

Since $u_\epsilon\to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$ it follows that $\lim_{\epsilon\to 0}y_\epsilon=0$. Then by the regularity Theorem 13.1 and since $\alpha_-(\gamma)<\frac{n-2}{2}$, we get

$$\lim_{\epsilon\to 0}|y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}=a>0 \tag{63}$$

for some positive constant a. In particular, $\lim_{\epsilon\to 0}|u_\epsilon(y_\epsilon)|=+\infty$. Let

$$\mu_{\mathcal{I}+1,\epsilon}:=|u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } k_{\mathcal{I}+1,\epsilon}:=\mu_{\mathcal{I}+1,\epsilon}^{1-\frac{p_\epsilon}{2^\star(s)-2}}.$$

As a consequence we have

$$\lim_{\epsilon\to 0}\mu_{\mathcal{I}+1,\epsilon}=0 \qquad\text{and}\qquad \lim_{\epsilon\to 0}\frac{|y_\epsilon|}{k_{\mathcal{I}+1,\epsilon}}=a>0. \tag{64}$$

We rescale and define

$$\tilde{u}_{\mathcal{I}+1,\epsilon}(x):=\mu_{\mathcal{I}+1,\epsilon}^{\frac{n-2}{2}}u_\epsilon(k_{\mathcal{I}+1,\epsilon}\,x) \qquad \text{for } x\in k_{\mathcal{I}+1,\epsilon}^{-1}\Omega\setminus\{0\}$$

It follows from (59) that for all $\epsilon>0$

$$|x|^{\frac{n-2}{2}}|\tilde{u}_{\mathcal{I}+1,\epsilon}(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}}\leq C \qquad \text{for } x\in k_{\mathcal{I}+1,\epsilon}^{-1}\Omega\setminus\{0\}.$$

so hypothesis (47) of Lemma 3 is satisfied. Then using Lemma 3 we get that there exists $\tilde{u}_{\mathcal{I}+1}\in D^{1,2}(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})$ weakly satisfying the equation:

$$-\Delta\tilde{u}_{\mathcal{I}+1}-\frac{\gamma}{|x|^2}\tilde{u}_{\mathcal{I}+1}=b(0)\frac{|\tilde{u}_{\mathcal{I}+1}|^{2^\star(s)-2}\tilde{u}_{\mathcal{I}+1}}{|x|^s} \text{ in } \mathbb{R}^n\setminus\{0\}.$$

and

$$\begin{aligned}
&\tilde{u}_{\mathcal{I}+1,\epsilon}\rightharpoonup\tilde{u}_{\mathcal{I}+1} && \text{weakly in } D^{1,2}(\mathbb{R}^n) && \text{as } \epsilon\to 0\\
&\tilde{u}_{\mathcal{I}+1,\epsilon}\to\tilde{u}_{\mathcal{I}+1} && \text{in } C^1_{loc}(\mathbb{R}^n\setminus\{0\}) && \text{as } \epsilon\to 0
\end{aligned}$$

We denote $\tilde{y}_\epsilon:=\dfrac{y_\epsilon}{k_{\mathcal{I}+1,\epsilon}}$. From (64) it follows that that $\lim_{\epsilon\to 0}|\tilde{y}_\epsilon|:=|\tilde{y}_0|=a\neq 0$. Therefore $|\tilde{u}_{\mathcal{I}+1}(\tilde{y}_0)|=\lim_{\epsilon\to 0}|\tilde{u}_{\mathcal{I}+1,\epsilon}(\tilde{y}_\epsilon)|=1$, and hence $\tilde{u}_{\mathcal{I}+1}\not\equiv 0$. And hence again from Lemma 3 we get

$$\int_{\mathbb{R}^n}\frac{|\tilde{u}_{\mathcal{I}+1}|^{2^\star(s)}}{|x|^s}\geq\left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t_{\mathcal{I}+1}\in(0,1]$ such that $\lim_{\epsilon\to 0}\mu_{\mathcal{I}+1,\epsilon}^{p_\epsilon}=t_{\mathcal{I}+1}$. Moreover, it follows from (62) and (64) that

$$\lim_{\epsilon\to 0}\frac{\mu_{\mathcal{I}+1,\epsilon}}{\mu_{\mathcal{I},\epsilon}}=+\infty \text{ and } \lim_{\epsilon\to 0}\mu_{\mathcal{I}+1,\epsilon}=0.$$

Hence the families $(\mu_{i,\epsilon})_{\epsilon>0}$, $1\leq i\leq\mathcal{I}+1$ satisfy $\mathcal{H}_{\mathcal{I}+1}$. □

The next step is equivalent to step 4.3 at intermediate scales.

Step 4.4 Let $I \geq 1$. We assume that $\mathcal{H}_{\mathcal{I}}$ holds. Then for any $1 \leq i \leq \mathcal{I}-1$ and for any $\delta > 0$, either $\mathcal{H}_{\mathcal{I}+1}$ holds or

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{B_{\delta k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} |x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = 0.$$

Proof of Step 4.4: We assume that there exists an $i \leq \mathcal{I}-1$ and $\delta > 0$ such that

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{B_{\delta k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} |x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} > 0.$$

It then follows that there exists a sequence $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon|}{k_{i,\epsilon}} = +\infty, \qquad |y_\epsilon| \leq \delta k_{i+1,\epsilon} \text{ for all } \epsilon > 0 \tag{65}$$

$$|y_\epsilon|^{\frac{n-2}{2}}\left|u_\epsilon(y_\epsilon)-\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{y_\epsilon}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0. \tag{66}$$

for some positive constant a. Note that $a < +\infty$ since

$$|x|^{\frac{n-2}{2}}\left|u_\epsilon(x)-\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}}$$

is uniformly bounded for all $x \in B_{\delta k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$.

We let $\tilde{y}_\epsilon^* \in \mathbb{R}^n$ be such that $y_\epsilon = k_{i+1,\epsilon}\, \tilde{y}_\epsilon^*$. It follows from (65) that $|\tilde{y}_\epsilon^*| \leq \delta$ for all $\epsilon > 0$. . We rewrite (66) as

$$\lim_{\epsilon\to 0}|\tilde{y}_\epsilon^*|^{\frac{n-2}{2}}\left|\tilde{u}_{i+1,\epsilon}(\tilde{y}_\epsilon^*)-\tilde{u}_{i+1}(\tilde{y}_\epsilon^*)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0.$$

Then from point (B3) of $\mathcal{H}_{\mathcal{I}}$ it follows that $\tilde{y}_\epsilon^* \to 0$ as $\epsilon \to 0$. And since $|x|^{\alpha_-(\gamma)}\tilde{u}_{i+1} \in C^0(\mathbb{R}^n)$, we get as $\epsilon \to 0$

$$|y_\epsilon|^{\frac{n-2}{2}}\left|\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{y_\epsilon}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = O\left(\frac{|y_\epsilon|}{k_{i+1,\epsilon}}\right)^{\frac{n-2}{2}-\alpha_-(\gamma)} = o(1)$$

Then (66) becomes

$$\lim_{\epsilon\to 0}|y_\epsilon|^{\frac{n-2}{2}}|u_\epsilon(y_\epsilon)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} = a > 0. \tag{67}$$

In particular, $\lim_{\epsilon\to 0}|u_\epsilon(y_\epsilon)| = +\infty$. We let

$$\nu_\epsilon := |u_\epsilon(y_\epsilon)|^{-\frac{2}{n-2}} \text{ and } \ell_\epsilon := \nu_\epsilon^{1-\frac{p_\epsilon}{2^\star(s)-2}}.$$

Then we have

$$\lim_{\epsilon\to 0}\nu_\epsilon = 0 \qquad \text{and} \qquad \lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\ell_\epsilon} = a > 0. \tag{68}$$

We rescale and define

$$\tilde{u}_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}}u_\epsilon(\ell_\epsilon\, x) \qquad \text{for } x \in \ell_\epsilon^{-1}\Omega \setminus \{0\}$$

It follows from (59) that for all $\epsilon > 0$

$$|x|^{\frac{n-2}{2}}|\tilde{u}_\epsilon(x)|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq C \qquad \text{for } x \in \ell_\epsilon^{-1}\Omega \setminus \{0\}.$$

so hypothesis (47) of Lemma 3 is satisfied. Then using lemma (3) we get that there exists $\tilde{u} \in D^{1,2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ weakly satisfying the equation:

$$-\Delta \tilde{u} - \frac{\gamma}{|x|^2}\tilde{u} = b(0)\frac{|\tilde{u}|^{2^\star(s)-2}\tilde{u}}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

and

$$\tilde{u}_\epsilon \rightharpoonup \tilde{u} \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \epsilon \to 0$$
$$\tilde{u}_\epsilon \to \tilde{u} \qquad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \quad \text{as } \epsilon \to 0$$

We denote $\tilde{y}_\epsilon := \dfrac{y_\epsilon}{\ell_\epsilon}$. From (67) it follows that that $\lim\limits_{\epsilon\to 0} |\tilde{y}_\epsilon| := |\tilde{y}_0| = a \neq 0$. Therefore $|\tilde{u}(\tilde{y}_0)| = \lim_{\epsilon\to 0} |\tilde{u}_\epsilon(\tilde{y}_\epsilon)| = 1$, and hence $\tilde{u} \not\equiv 0$. And hence again from Lemma 3 we get

$$\int_{\mathbb{R}^n} \frac{|\tilde{u}|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t \in (0,1]$ such that $\lim\limits_{\epsilon\to 0} \nu_\epsilon^{p_\epsilon} = t$. Moreover it follows from(67), (65) and since $\lim\limits_{\epsilon\to 0} \frac{|y_\epsilon|}{k_{i+1,\epsilon}} = 0$, that

$$\lim_{\epsilon\to 0} \frac{\nu_\epsilon}{\mu_{i,\epsilon}} = +\infty \quad \text{and} \quad \lim_{\epsilon\to 0} \frac{\mu_{i+1,\epsilon}}{\nu_\epsilon} = +\infty.$$

Hence the families $(\mu_{1,\epsilon})$,..., $(\mu_{i,\epsilon})$, (ν_ϵ), $(\mu_{i+1,\epsilon})$,..., $(\mu_{\mathcal{I},\epsilon})$ satisfy $\mathcal{H}_{\mathcal{I}+1}$. □

The last step tells us that family $\{\mathcal{H}_{\mathcal{I}}\}$ is finite.

Step 4.5: Let $N_0 = \max\{\mathcal{I} : \mathcal{H}_{\mathcal{I}} \text{ holds }\}$. Then $N_0 < +\infty$ and the conclusion of Proposition 2 holds with $N = N_0$.

Proof of Step 4.5: Indeed, assume that $\mathcal{H}_{\mathcal{I}}$ holds. Since $\mu_{i,\epsilon} = o(\mu_{i+1,\epsilon})$ for all $1 \leq i \leq N-1$, we get with a change of variable and the definition of $\tilde{u}_{i,\epsilon}$ that for any $R > \delta > 0$

$$\begin{aligned}\int_\Omega \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} dx &\geq \sum_{i=1}^{\mathcal{I}} \int_{B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\delta k_{i,\epsilon}}(0)} \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} dx \\ &\geq \sum_{i=1}^{\mathcal{I}} \int_{B_R(0)\setminus \overline{B}_\delta(0)} \frac{|\tilde{u}_{i,\epsilon}|^{2^\star(s)-p_\epsilon}}{|x|^s} dx.\end{aligned}$$

$$\text{Then from (36) we have } \Lambda \geq \sum_{i=1}^{\mathcal{I}} \int_{B_R(0)\setminus \overline{B}_\delta(0)} \frac{|\tilde{u}_{i,\epsilon}|^{2^\star(s)-p_\epsilon}}{|x|^s} dx. \tag{69}$$

Passing to the limit $\epsilon \to 0$ and then $\delta \to 0$, $R \to +\infty$ we obtain using point (B3) of $\mathcal{H}_{\mathcal{I}}$, that

$$\Lambda \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}} \mathcal{I}.$$

It then follows that $N_0 < +\infty$. □

We let families $(\mu_{1,\epsilon})_{\epsilon>0}$,..., $(\mu_{N_0,\epsilon})_{\epsilon>0}$ such that $\mathcal{H}_{N_0}$ holds. We argue by contradiction and assume that the conclusion of Proposition 2 does not hold with $N = N_0$. Assertions (A1), (A2), (A3), (A4), (A5), (A7) and (A9) holds. Assume that (A6) or (A8) does not hold. It then follows from Steps (4.3), (4.4) and (4.5) that $\mathcal{H}_{N+1}$ holds. A contradiction with the choice of $N = N_0$ and the proposition is proved. □

5. Strong Pointwise Estimates

The objective of this section is the proof of the following strong pointwise control.

Proposition 3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (p_ϵ), (h_ϵ) and b be such that (E_ϵ), (31), (33), (34) and (36) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0} \||x|^\tau u_\epsilon\|_{L^\infty(\Omega)} = +\infty \quad \text{where} \quad \alpha_-(\gamma) < \tau < \frac{n-2}{2}.$$

Consider the $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ from Proposition 2. Then there exists $C > 0$ such that for all $\epsilon > 0$

$$|u_\epsilon(x)| \leq C \left(\sum_{i=1}^{N} \frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}|x|^{\alpha_-(\gamma)} + |x|^{\alpha_+(\gamma)}} + \frac{\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}} \right) \tag{70}$$

for all $x \in \Omega \setminus \{0\}$.

The proof of this estimate proceeds in seven steps.

Step 5.1: We claim that for any $\sigma > 0$ small and any $R > 0$, there exists $C(\sigma, R) > 0$ such that for all $\epsilon > 0$ sufficiently small

$$|u_\epsilon(x)| \leq C(\sigma, R) \left(\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}-\sigma}}{|x|^{\alpha_+(\gamma)-\sigma}} + \frac{\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)+\sigma}} \right) \text{ for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0). \tag{71}$$

Proof of Step 5.1: We fix γ' such that $\gamma < \gamma' < \frac{(n-2)^2}{4}$. Since the operator $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$ is coercive, taking γ' close to γ it follows that the operator $-\Delta - \frac{\gamma'}{|x|^2} - h_0(x)$ is also coercive in Ω. From Theorem 25 it get that there exists $H \in C^\infty(\overline{\Omega} \setminus \{0\})$ such that

$$\begin{cases} -\Delta H - \frac{\gamma'}{|x|^2}H - h_0(x)H = 0 & \text{in } \Omega \setminus \{0\} \\ H > 0 & \text{in } \Omega \setminus \{0\} \\ H = 0 & \text{on } \partial\Omega. \end{cases} \tag{72}$$

And we have the following bound on H: there exists $\delta_1, C_1 > 0$ such that

$$\frac{1}{C_1}\frac{1}{|x|^{\beta_+(\gamma')}} \leq H(x) \leq C_1 \frac{1}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in B_{2\delta_1}(0). \tag{73}$$

We let $\lambda_1^{\gamma'} > 0$ be the first eigenvalue of the coercive operator $-\Delta - \frac{\gamma'}{|x|^2} - h$ on Ω and we let $\varphi \in H^2_{1,0}(\Omega)$ be the unique eigenfunction such that

$$\begin{cases} -\Delta\varphi - \frac{\gamma'}{|x|^2}\varphi - h_0(x)\varphi & = \lambda_1^{\gamma'}\varphi \quad \text{in } \Omega \\ \varphi & > 0 \quad \text{in } \Omega \setminus \{0\} \\ \varphi & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{74}$$

It follows from the regularity result, Theorem 13.1 that there exists $C_2, \delta_2 > 0$ such that

$$\frac{1}{C_2}\frac{1}{|x|^{\beta_-(\gamma')}} \leq \varphi(x) \leq C_2 \frac{1}{|x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \Omega \cap B_{2\delta_2}(0). \tag{75}$$

We define the operator

$$\mathcal{L}_\epsilon := -\Delta - \left(\frac{\gamma}{|x|^2} + h_\epsilon \right) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}.$$

Step 5.1.1: We claim that given any $\gamma < \gamma' < \frac{(n-2)^2}{4}$ there exist $\delta_0 > 0$ and $R_0 > 0$ such that for any $0 < \delta < \delta_0$ and $R > R_0$, we have for $\epsilon > 0$ sufficiently small

$$\begin{aligned}\mathcal{L}_\epsilon H(x) > 0, \text{ and } \mathcal{L}_\epsilon \varphi(x) > 0 \qquad &\text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \not\equiv 0.\\ \mathcal{L}_\epsilon H(x) > 0 \qquad &\text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \equiv 0.\end{aligned} \tag{76}$$

As one checks for all $\epsilon > 0$ and $x \neq 0$

$$\frac{\mathcal{L}_\epsilon H(x)}{H(x)} = \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}$$

and

$$\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} = \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s} + \lambda_1^{\gamma'}.$$

For $\epsilon > 0$ sufficiently small $\|h_0 - h_\epsilon\|_\infty \leq \frac{\gamma'-\gamma}{4(1+\sup\limits_\Omega |x|^2)}$ and we choose $0 < \delta_0 < \min\{1, \delta_1, \delta_2\}$ such that

$$||b||_{L^\infty(\Omega)} \delta_0^{(2^\star(s)-2)\left(\frac{n-2}{2} - \alpha_-(\gamma)\right)} |||x|^{\alpha_-(\gamma)} u_0||_{L^\infty(\Omega)}^{2^\star(s)-2} \leq \frac{\gamma' - \gamma}{2^{2^\star(s)+3}} \tag{77}$$

It follows from point (A6) of Proposition 2 that, there exists $R_0 > 0$ such that for any $R > R_0$, we have for all $\epsilon > 0$ sufficiently small

$$|b(x)|^{\frac{1}{2^\star(s)-2}} |x|^{\frac{n-2}{2}} |u_\epsilon(x) - u_0(x)|^{1 - \frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma' - \gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}} \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0)$$

With this choice of δ_0 and R_0 we get that for any $0 < \delta < \delta_0$ and $R > R_0$, we have for $\epsilon > 0$ small enough

$$\begin{aligned}|b(x)|\ |x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}|b(x)||u_\epsilon(x) - u_0(x)|^{2^\star(s)-2-p_\epsilon}\\ &+ 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}|b(x)||u_0(x)|^{2^\star(s)-2-p_\epsilon}\\ &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \not\equiv 0\\ \text{and } |b(x)|\ |x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \equiv 0.\end{aligned}$$

Hence we obtain that for $\epsilon > 0$ small enough

$$\begin{aligned}\frac{\mathcal{L}_\epsilon H(x)}{H(x)} &= \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}\\ &\geq \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - \frac{\gamma'-\gamma}{4|x|^2}\\ &> 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \not\equiv 0\\ \text{and } \quad \frac{\mathcal{L}_\epsilon H(x)}{H(x)} &> 0 \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0), \text{ if } u_0 \equiv 0.\end{aligned} \tag{78}$$

And

$$\begin{aligned}\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} &\geq \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}.\\ &\geq \frac{\gamma' - \gamma}{|x|^2} + (h_0 - h_\epsilon) - \frac{\gamma'-\gamma}{4|x|^2}\\ &> 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0)).\end{aligned} \tag{79}$$

□

Step 5.1.2: It follows from point (A4) of Proposition 2 that there exists $C_1'(R)>0$ such that for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq C_1'(R)\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}}\qquad \text{for all } x\in\partial B_{Rk_{N,\epsilon}}(0) \tag{80}$$

By estimate (73) on H, we then have for some constant $C_1(R)>0$

$$|u_\epsilon(x)|\leq C_1(R)\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}H(x)\qquad \text{for all } x\in\partial B_{Rk_{N,\epsilon}}(0). \tag{81}$$

It follows from point (A1) of Proposition 2 and the regularity result (13.1), that there exists $C_2'(\delta)>0$ such that for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq C_2'(\delta)\frac{\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}}\qquad \text{for all } x\in\partial B_\delta(0),\ \text{if } u_0\not\equiv 0. \tag{82}$$

And then by the estimate (75) on φ we then have for some constant $C_2(\delta)>0$

$$|u_\epsilon(x)|\leq C_2(\delta)\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}\ \varphi(x)\qquad \text{for all } x\in\partial B_\delta(0) \text{ if } u_0\not\equiv 0.. \tag{83}$$

We now let for $\epsilon>0$,

$$\Psi_\epsilon(x):=C_1(R)\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}H(x)+C_2(\delta)\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}\varphi(x)\qquad \text{for } x\in\Omega\setminus\{0\}.$$

Then (83) and (81) implies that for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq\Psi_\epsilon(x)\qquad \text{for all } x\in\partial(B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)),\ \text{if } u_0\not\equiv 0. \tag{84}$$

and if $u_0\equiv 0$ then

$$|u_\epsilon(x)|\leq\Psi_\epsilon(x)\qquad \text{for all } x\in\partial(\Omega\setminus\overline{B}_{Rk_{N,\epsilon}}(0)). \tag{85}$$

Therefore when $u_0\not\equiv 0$ it follows from (76)) and (84) that for all $\epsilon>0$ sufficiently small

$$\begin{cases}\mathcal{L}_\epsilon\Psi_\epsilon\geq 0=\mathcal{L}_\epsilon u_\epsilon & \text{in } B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\\ \Psi_\epsilon\geq u_\epsilon & \text{on } \partial(B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0))\\ \mathcal{L}_\epsilon\Psi_\epsilon\geq 0=-\mathcal{L}_\epsilon u_\epsilon & \text{in } B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\\ \Psi_\epsilon\geq -u_\epsilon & \text{on } \partial(B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)).\end{cases}$$

and from (76) and (85), in case $u_0\equiv 0$, we have for $\epsilon>0$ sufficiently small

$$\begin{cases}\mathcal{L}_\epsilon\Psi_\epsilon\geq 0=\mathcal{L}_\epsilon u_\epsilon & \text{in } \Omega\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\\ \Psi_\epsilon\geq u_\epsilon & \text{on } \partial(\Omega\setminus\overline{B}_{Rk_{N,\epsilon}}(0))\\ \mathcal{L}_\epsilon\Psi_\epsilon\geq 0=-\mathcal{L}_\epsilon u_\epsilon & \text{in } \Omega\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\\ \Psi_\epsilon\geq -u_\epsilon & \text{on } \partial(\Omega\setminus\overline{B}_{Rk_{N,\epsilon}}(0)).\end{cases}$$

Since $\Psi_\epsilon>0$ and $\mathcal{L}_\epsilon\Psi_\epsilon>0$, it follows from the comparison principle of Berestycki-Nirenberg-Varadhan [1] that the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle on $B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)$. Therefore

$$\begin{aligned}|u_\epsilon(x)|\leq\Psi_\epsilon(x)&\qquad \text{for all } x\in B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0),\\ \text{and}\ \ |u_\epsilon(x)|\leq\Psi_\epsilon(x)&\qquad \text{for all } x\in\Omega\setminus\overline{B}_{Rk_{N,\epsilon}}(0)\ \ \text{if } u_0\equiv 0.\end{aligned}$$

Therefore when $u_0\not\equiv 0$, we have for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq C_1(R)\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}H(x)+C_2(\delta)\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}\varphi(x)$$

for all $x\in B_\delta(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)$, for R large and δ small.

Then when $u_0 \not\equiv 0$, using the estimates (73) and (75) we have or all $\epsilon > 0$ small

$$|u_\epsilon(x)| \ \leq C_1(R)\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} + C_2(\delta)\frac{\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma')}}$$

for all $x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\epsilon}}(0)$, for R large and δ small.

And if $u_0 \equiv 0$, then all $\epsilon > 0$ small and $R > 0$ large

$$|u_\epsilon(x)| \leq C_1(R)\frac{\mu_{N,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} \quad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\epsilon}}(0).$$

Taking γ' close to γ, along with points (A1) and (A4) of Proposition 2 it then follows that estimate (71) holds on $\Omega \setminus \overline{B}_{Rk_{\epsilon,N}}(0)$ for all $R > 0$. □

Step 5.2: Let $1 \leq i \leq N-1$. We claim that for any $\sigma > 0$ small and any $R, \rho > 0$, there exists $C(\sigma, R, \rho) > 0$ such that all $\epsilon > 0$.

$$|u_\epsilon(x)| \leq C(\sigma, R, \rho)\left(\frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}-\sigma}}{|x|^{\alpha_+(\gamma)-\sigma}} + \frac{1}{\mu_{i+1,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}-\sigma}|x|^{\alpha_-(\gamma)+\sigma}}\right) \tag{86}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$.

Proof of Step 5.2: We let $i \in \{1, ..., N-1\}$. We emulate the proof of Step 5.1. Fix γ' such that $\gamma < \gamma' < \frac{(n-2)^2}{4}$. Consider the functions H and φ defined in Step 5.1 satisfying (72) and (72) respectively. We define the operator

$$\mathcal{L}_\epsilon := -\Delta - \left(\frac{\gamma}{|x|^2} + h_\epsilon\right) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}.$$

Step 5.2.1: We claim that given any $\gamma < \gamma' < \frac{(n-2)^2}{4}$ there exist $\rho_0 > 0$ and $R_0 > 0$ such that for any $0 < \rho < \rho_0$ and $R > R_0$, we have for $\epsilon > 0$ sufficiently small

$$\mathcal{L}_\epsilon H(x) > 0, \text{ and } \mathcal{L}_\epsilon \varphi(x) > 0 \qquad \text{for all } x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0) \tag{87}$$

As one checks for all $\epsilon > 0$ and $x \neq 0$

$$\begin{aligned}\frac{\mathcal{L}_\epsilon H(x)}{H(x)} &= \frac{\gamma'-\gamma}{|x|^2} + (h_0 - h_\epsilon) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s},\\ \frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} &\geq \frac{\gamma'-\gamma}{|x|^2} + (h_0 - h_\epsilon) - b(x)\frac{|u_\epsilon|^{2^\star(s)-2-p_\epsilon}}{|x|^s}.\end{aligned}$$

We choose $0 < \rho_0 < 1$ such that

$$\begin{aligned}&\rho_0^2 \sup_\Omega |h_0 - h_\epsilon| \leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } \epsilon > 0 \text{ small and}\\ &||b||_{L^\infty}\rho_0^{(2^\star(s)-2)\left(\frac{n-2}{2}-\alpha_-(\gamma)\right)}|||x|^{\alpha_-(\gamma)}\tilde{u}_{i+1}||^{2^\star(s)-2}_{L^\infty(B_2(0)\cap\mathbb{R}^n)} \leq \frac{\gamma'-\gamma}{2^{2^\star(s)+3}}\end{aligned} \tag{88}$$

It follows from point (A8) of Proposition 2 that there exists $R_0 > 0$ such that for any $R > R_0$ and any $0 < \rho < \rho_0$, we have for all $\epsilon > 0$ sufficiently small

$$|b(x)|^{\frac{1}{2^\star(s)-2}}|x|^{\frac{n-2}{2}}\left|u_\epsilon(x) - \mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma'-\gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$.

With this choice of ρ_0 and R_0 we get that for any $0<\rho<\rho_0$ and $R>R_0$, we have for $\epsilon>0$ small enough

$$\begin{aligned}|b(x)||x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}|b(x)|\left|u_\epsilon(x)-\mu_{i+1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{2^\star(s)-2-p_\epsilon}\\ &\quad+2^{2^\star(s)-1-p_\epsilon}\left(\frac{|x|}{k_{i+1,\epsilon}}\right)^{2-s}|b(x)|\left|\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\epsilon}}\right)\right|^{2^\star(s)-2-p_\epsilon}\\ &\leq\frac{\gamma'-\gamma}{4}\qquad\text{for all } x\in B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0).\end{aligned}\tag{89}$$

Hence as in Step 5.1 we have that for $\epsilon>0$ small enough

$$\frac{\mathcal{L}_\epsilon H(x)}{H(x)}>0\ \text{ and }\ \frac{\mathcal{L}_\epsilon\varphi(x)}{\varphi(x)}>0\ \text{ for all } x\in B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0).$$

□

Step 5.2.2: let $i\in\{1,...,N-1\}$. It follows from point (A4) of Proposition 2 that there exists $C_1'(R)>0$ such that for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq C_1'(R)\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}}\qquad\text{for all } x\in\partial B_{Rk_{i,\epsilon}}(0)\tag{90}$$

And then by the estimate (73) on H we have for some constant $C_1(R)>0$

$$|u_\epsilon(x)|\leq C_1(R)\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}H(x)\qquad\text{for all } x\in\partial B_{Rk_{i,\epsilon}}(0).\tag{91}$$

From point (A4) of Proposition 2 it follows that there exists $C_2'(\rho)>0$ such that for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq C_2'(\rho)\frac{1}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|^{\beta_-(\gamma')}}\qquad\text{for all } x\in\partial B_{\rho k_{i+1,\epsilon}}(0).\tag{92}$$

Then by the estimate (75) on φ we have for some constant $C_2(\delta)>0$

$$|u_\epsilon(x)|\leq C_2(\rho)\frac{\varphi(x)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}\qquad\text{for all } x\in\partial B_{\rho k_{i+1,\epsilon}}(0).\tag{93}$$

We let for all $\epsilon>0$

$$\tilde{\Psi}_\epsilon(x):=C_1(R)\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}H(x)+C_2(\rho)\frac{\varphi(x)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}\qquad\text{for } x\in\Omega\setminus\{0\}$$

Then (91) and (93) implies that for all $\epsilon>0$ small

$$|u_\epsilon(x)|\leq\tilde{\Psi}_\epsilon(x)\qquad\text{for all } x\in\partial\left(B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)\right).\tag{94}$$

Therefore it follows from (87) and (94) that $\epsilon>0$ sufficiently small

$$\begin{cases}\mathcal{L}_\epsilon\tilde{\Psi}_\epsilon\geq 0=\mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)\\ \tilde{\Psi}_\epsilon\geq u_\epsilon & \text{on } \partial\left(B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)\right)\\ \mathcal{L}_\epsilon\tilde{\Psi}_\epsilon\geq 0=-\mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)\\ \tilde{\Psi}_\epsilon\geq -u_\epsilon & \text{on } \partial\left(B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)\right).\end{cases}$$

Since $\tilde{\Psi}_\epsilon>0$ and $\mathcal{L}_\epsilon\tilde{\Psi}_\epsilon>0$, it follows from the comparison principle of Berestycki-Nirenberg-Varadhan [1] that the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle on $B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)$. Therefore

$$|u_\epsilon(x)|\leq\tilde{\Psi}_\epsilon(x)\qquad\text{for all } x\in B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)).$$

So for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_1(R)\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\rho)\frac{\varphi(x)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$, for R large and ρ small. Then using the estimates (73) and (75) we have or all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_1(R)\frac{\mu_{i,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} + \frac{C_2(\rho)}{\mu_{i+1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}|x|^{\beta_-(\gamma')}} \quad \text{for all } x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0).$$

for R large and ρ small.

Taking γ' close to γ, along with point (A4) of Proposition 2 it then follows that estimate (86) holds on $B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$ for all $R, \rho > 0$. □

Step 5.3: We claim that for any $\sigma > 0$ small and any $\rho > 0$, there exists $C(\sigma, \rho) > 0$ such that all $\epsilon > 0$.

$$|u_\epsilon(x)| \leq C(\sigma,\rho)\frac{1}{\mu_{1,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}-\sigma}|x|^{\alpha_-(\gamma)+\sigma}} \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\}. \tag{95}$$

Proof of Step 5.3: Fix γ' such that $\gamma < \gamma' < \frac{(n-2)^2}{4}$. Consider the function φ defined in Step 5.1 satisfying (72).

Step 5.3.1: We claim that given any $\gamma < \gamma' < \frac{(n-2)^2}{4}$ there exist $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$ we have for $\epsilon > 0$ sufficiently small

$$\mathcal{L}_\epsilon \varphi(x) > 0 \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\} \tag{96}$$

We choose $0 < \rho_0 < 1$ such that

$$\begin{aligned} &\rho_0^2 \sup_\Omega |h_0 - h_\epsilon| \leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } \epsilon > 0 \text{ small and} \\ &||b||_{L^\infty}\rho_0^{(2^\star(s)-2)\left(\frac{n-2}{2}-\alpha_-(\gamma)\right)}|||x|^{\alpha_-(\gamma)}\tilde{u}_1||_{L^\infty(B_2(0)\cap\mathbb{R}^n)}^{2^\star(s)-2} \leq \frac{\gamma'-\gamma}{2^{2^\star(s)+3}} \end{aligned} \tag{97}$$

It follows from point (A7) of Proposition 2 that for any $0 < \rho < \rho_0$, we have for all $\epsilon > 0$ sufficiently small

$$|b(x)|^{\frac{1}{2^\star(s)-2}}|x|^{\frac{n-2}{2}}\left|u_\epsilon(x) - \mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\epsilon}}\right)\right|^{1-\frac{p_\epsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma'-\gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}}$$

for all $x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\}$.

With this choice of ρ_0 we get that for any $0 < \rho < \rho_0$ we have for $\epsilon > 0$ small enough

$$\begin{aligned} |b(x)||x|^{2-s}|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon} &\leq 2^{2^\star(s)-1-p_\epsilon}|x|^{2-s}|b(x)|\left|u_\epsilon(x) - \mu_{1,\epsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\epsilon}}\right)\right|^{2^\star(s)-2-p_\epsilon} \\ &\quad + 2^{2^\star(s)-1-p_\epsilon}\left(\frac{|x|}{k_{1,\epsilon}}\right)^{2-s}|b(x)|\left|\tilde{u}_1\left(\frac{x}{k_{1,\epsilon}}\right)\right|^{2^\star(s)-2-p_\epsilon} \\ &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\}. \end{aligned} \tag{98}$$

Hence similarly as in Step 5.1, we obtain that for $\epsilon > 0$ small enough

$$\frac{\mathcal{L}_\epsilon \varphi(x)}{\varphi(x)} > 0 > 0 \qquad \text{for all } x \in x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\}. \tag{99}$$

□

Step 5.3.2: It follows from point (A4) of Proposition 2 that there exists $C_2'(\rho) > 0$ such that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_2'(\rho) \frac{1}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \partial B_{\rho k_{1,\epsilon}}(0) \tag{100}$$

and then by the estimate (75) on φ we have for some constant $C_2(\delta) > 0$

$$|u_\epsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for all } x \in \partial B_{\rho k_{1,\epsilon}}(0). \tag{101}$$

We let for all $\epsilon > 0$

$$\Psi_\epsilon^0(x) := C_2(\rho) \frac{\varphi(x)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for } x \in \Omega \setminus \{0\}.$$

Then (101) implies that for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq \Psi_\epsilon^0(x) \qquad \text{for all } x \in \partial B_{\rho k_{1,\epsilon}}(0). \tag{102}$$

Therefore it follows from (96) and (102) that $\epsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\epsilon \Psi_\epsilon^0 \geq 0 = \mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{1,\epsilon}}(0) \setminus \{0\} \\ \Psi_\epsilon^0 \geq u_\epsilon & \text{on } \partial B_{\rho k_{1,\epsilon}}(0) \setminus \{0\} \\ \mathcal{L}_\epsilon \Psi_\epsilon^0 \geq 0 = -\mathcal{L}_\epsilon u_\epsilon & \text{in } B_{\rho k_{1,\epsilon}}(0) \\ \Psi_\epsilon^0 \geq -u_\epsilon & \text{on } \partial B_{\rho k_{1,\epsilon}}(0). \end{cases}$$

Since the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle on $B_{\rho k_{1,\epsilon}}(0)$. Therefore

$$|u_\epsilon(x)| \leq \Psi_\epsilon^0(x) \qquad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0).$$

And so for all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \quad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\}.$$

for ρ small. Using the estimate (75) we have or all $\epsilon > 0$ small

$$|u_\epsilon(x)| \leq \frac{C_2(\rho)}{\mu_{1,\epsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \quad \text{for all } x \in B_{\rho k_{1,\epsilon}}(0) \setminus \{0\}.$$

for ρ small. It then follows from point (A4) of Proposition 2 that estimate (95) holds on $x \in B_{\rho k_{1,\epsilon}}(0)$ for all $\rho > 0$. □

Step 5.4: Combining the previous three steps, it follows from (71), (86), (95) and Proposition 2 that for any $\sigma > 0$ small, there exists $C(\sigma) > 0$ such that for all $\epsilon > 0$ we have

$$|u_\epsilon(x)| \leq C(\sigma) \left(\sum_{i=1}^N \frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2} - \sigma}}{\mu_{i,\epsilon}^{(\alpha_+(\gamma)-\alpha_-(\gamma))-2\sigma} |x|^{\alpha_-(\gamma)+\sigma} + |x|^{\alpha_+(\gamma)-\sigma}} + \frac{\| |x|^{\alpha_-(\gamma)} u_0 \|_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)+\sigma}} \right) \tag{103}$$

for all $x \in \Omega \setminus \{0\}$.

Next we improve the above estimate and show that one can take $\sigma = 0$ in (103).

For ϵ small, we let G_ϵ be the Green's function for the coercive operator $-\Delta - \frac{\gamma}{|x|^2} - h_\epsilon$ on Ω with Dirichlet boundary condition. Green's representation formula, the pointwise bounds on the Green's function (205) [see Ghoussoub-Robert [19]] yields for any $z \in \Omega$

$$u_\epsilon(z) = \int\limits_\Omega G_\epsilon(z,x) b(x) \frac{|u_\epsilon(x)|^{2^\star(s)-2-p_\epsilon}\, u_\epsilon(x)}{|x|^s}\, dx$$

$$|u_\epsilon(z)| \leq \int\limits_\Omega G_\epsilon(z,x)|b(x)| \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx$$

$$\leq C \int\limits_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}} \right)^{\alpha_-(\gamma)} \frac{1}{|x-z|^{n-2}} \frac{|u_\epsilon(x)|^{2^\star(s)-1-p_\epsilon}}{|x|^s}\, dx. \tag{104}$$

Using (103) we then obtain with $0 < \sigma < \frac{2^\star(s)-2}{2^\star(s)-1} \left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2} \right)$ that

$$|u_\epsilon(z)| \leq$$

$$C \sum_{i=1}^N \int\limits_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}} \right)^{\alpha_-(\gamma)} \frac{1}{|x-z|^{n-2}|x|^s} \left(\frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}-\sigma}}{\mu_{i,\epsilon}^{(\alpha_+(\gamma)-\alpha_-(\gamma))-2\sigma}|x|^{\alpha_-(\gamma)+\sigma} + |x|^{\alpha_+(\gamma)-\sigma}} \right)^{2^\star(s)-1-p_\epsilon} dx$$

$$+ C \||x|^{\alpha_-(\gamma)} u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\epsilon} \int\limits_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}} \right)^{\alpha_-(\gamma)} \frac{1}{|x-z|^{n-2}|x|^s} \frac{1}{|x|^{(\alpha_-(\gamma)+\sigma)(2^\star(s)-1-p_\epsilon)}}\, dx. \tag{105}$$

The first term in the above integral was computed for each bubble in Ghoussoub-Robert in [19] when $p_\epsilon = 0$. The proof goes exactly the same with $p_\epsilon > 0$. The last last term is straightforward to estimate. We then get that there exists a constant $C > 0$ such that for any sequence of points (z_ϵ) in $\Omega \setminus \{0\}$ we have

$$|u_\epsilon(z_\epsilon)| \leq\ C \left(\sum_{i=1}^N \frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}|z_\epsilon|^{\alpha_-(\gamma)} + |z_\epsilon|^{\alpha_+(\gamma)}} + \frac{\||x|^{\alpha_-(\gamma)} u_0\|_{L^\infty(\Omega)}}{|z_\epsilon|^{\alpha_-(\gamma)}} \right). \tag{106}$$

This completes the proof of Proposition 3. □

6. Sharp Blow-up rates and proof of Compactness

When the expression makes sense, we define

$$\mathcal{C}_{n,s} := \frac{\left(\frac{2-\theta}{2}\right) \frac{1}{t_N^{\frac{n-\theta}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|^\theta}\, dx}{\frac{1}{2^\star(s)} \left(\frac{n-s}{2^\star(s)}\right) \left(\sum\limits_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx \right)} \tag{107}$$

The proof of compactness rely on the following two key propositions.

Proposition 4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $\gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs, that is*

$$\lim_{\epsilon\to 0} \||x|^\tau u_\epsilon\|_{L^\infty(\Omega)} = +\infty \quad \textit{for some} \quad \alpha_-(\gamma) < \tau < \frac{n-2}{2}. \tag{108}$$

Consider the $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Suppose that

$$\textit{either } \{\, \alpha_+(\gamma) - \alpha_-(\gamma) > 4 - 2\theta \,\} \textit{ or } \{\alpha_+(\gamma) - \alpha_-(\gamma) > 2 - \theta \textit{ and } u_0 \equiv 0\}. \tag{109}$$

Then, we have the following blow-up rate:

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)}\left(K_{h_0} + \frac{\mathbf{1}_{\theta=0}}{2^\star(s)}\sum_{i,j=1}^{n}\partial_{ij}b(0)\frac{\int_{\mathbb{R}^n} X^i X^j \frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}\tilde{u}_N^2\,dx}\right). \tag{110}$$

Proposition 5 (*The positive case*)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, such that $0\in\Omega$ and assume that $0<s<2$, $0\leq\gamma<\frac{(n-2)^2}{4}$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be as in Proposition 4. Assume that blow-up occurs as in (108). Consider $\mu_{1,\epsilon},...,\mu_{N,\epsilon}$ and $t_1,...,t_N$ from Proposition 2. Suppose in addition that*

$$u_\epsilon > 0 \qquad \textit{for all } \epsilon > 0.$$

Then, we have the following blow-up rates:

(1) *When $\alpha_+(\gamma)-\alpha_-(\gamma) > 4-2\theta$, we have*

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)}\left(K_{h_0} + \mathbf{1}_{\theta=0}\frac{(n-2)(n(n-4)-4\gamma)}{4n(2n-2-s)}\frac{\Delta b(0)}{b(0)}\right).$$

(2) *When $\alpha_+(\gamma)-\alpha_-(\gamma) = 4-2\theta$, we have*

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)}\left(K_{h_0} + \mathbf{1}_{\theta=0}\frac{(n-2)(n(n-4)-4\gamma)}{4n(2n-2-s)}\frac{\Delta b(0)}{b(0)}\right) \quad \textit{if } u_0\equiv 0.$$

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)}\left(K_{h_0} + \tilde{K} + \mathbf{1}_{\theta=0}\frac{(n-2)(n(n-4)-4\gamma)}{4n(2n-2-s)}\frac{\Delta b(0)}{b(0)}\right) \quad \textit{if } u_0 > 0,$$

for some $\tilde{K}>0$.

(3) *When $\alpha_+(\gamma)-\alpha_-(\gamma) < 4-2\theta$, we have $u_0\equiv 0$ and*

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)}\left(K_{h_0} + \mathbf{1}_{\theta=0}\frac{(n-2)(n(n-4)-4\gamma)}{4n(2n-2-s)}\frac{\Delta b(0)}{b(0)}\right) \quad \textit{if } \alpha_+(\gamma)-\alpha_-(\gamma) > 2-\theta.$$

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}\ln\frac{1}{\mu_{N,\epsilon}}} = -\mathcal{C}'_{n,s}\frac{K_{h_0}}{b(0)} \qquad \textit{if } \alpha_+(\gamma)-\alpha_-(\gamma) = 2-\theta, \tag{111}$$

where

$$\mathcal{C}'_{n,s} := \frac{\left(\frac{2-\theta}{2}\right)\mathcal{K}^2\omega_{n-1}}{\frac{1}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^{N}\frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx\right)},$$

with $\mathcal{K}$ as defined in (143).

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}} = -\chi\cdot m_{\gamma,h_0}(\Omega) \qquad \textit{if } \alpha_+(\gamma)-\alpha_-(\gamma) < 2-\theta, \tag{112}$$

where $\chi>0$ is a constant and $m_{\gamma,h_0}(\Omega)$ is the mass of Ω associated with the operator $-\Delta-\frac{\gamma}{|x|^2}-h_0(x)$, defined in Theorem 1.7.

Proof of Theorems 1.6 and 1.8: We argue by contradiction and assume that the family is not pre-compact. Then, up to a subsequence, it blows up. We then apply Propositions 4 and 5 to get the blow-up rate (that is nonegative). However, the hypothesis of Theorems 1.5, 1.6 and 1.8 yield exactly negative blow-up rates. This is a contradiction, and therefore the family is pre-compact. This proves the three compactness theorems. □

We now establish Propositions 4 and 5. The proof is divided in 14 steps in Sections 7 to 8. These steps are numbered Steps P1, P2, etc.

7. Estimates on the localized Pohozaev identity

We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs as in (108). Note that

$$\gamma < \frac{(n-2)^2}{4} - \left(1 - \frac{\theta}{2}\right)^2 \iff \alpha_+(\gamma) - \alpha_-(\gamma) > 2 - \theta,$$

In the following, we will use the following consequence of (A9) of Proposition 2: for all $i = 1, ..., N$, there exists $c_i > 1$ such that

$$c_i^{-1}\mu_{i,\epsilon} \leq k_{i,\epsilon} \leq c_i \mu_{i,\epsilon}. \tag{113}$$

Step P1 (Pohozaev identity)**.** *For $\delta_0 > 0$ small, we define*

$$\rho_\epsilon := \begin{cases} \delta_0 & \text{if } u_0 \equiv 0, \\ r_\epsilon := \sqrt{\mu_{N,\epsilon}} & \text{if } u_0 \not\equiv 0 \end{cases} \tag{114}$$

and

$$\begin{aligned} F_\epsilon(x) := (x,\nu)\left(\frac{|\nabla u_\epsilon|^2}{2} - \frac{\gamma}{2}\frac{u_\epsilon^2}{|x|^2} - \frac{h_\epsilon(x)}{2}u_\epsilon^2 - \frac{b(x)}{2^\star(s)-p_\epsilon}\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right) \\ - \left(x^i\partial_i u_\epsilon + \frac{n-2}{2}u_\epsilon\right)\partial_\nu u_\epsilon. \end{aligned} \tag{115}$$

We then have

$$\begin{aligned} &\int_{B_{\rho_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2}\right)u_\epsilon^2\, dx \\ &+ \frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p_\epsilon}\right)\int_{B_{\rho_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx \\ &+ \frac{1}{2^\star(s)-p_\epsilon}\int_{B_{\rho_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} (x,\nabla b(x))\,\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx \\ &= -\int_{\partial B_{\rho_\epsilon}(0)} F_\epsilon(x)\, d\sigma + \int_{\partial B_{k_{1,\epsilon}^2}(0)} F_\epsilon(x)\, d\sigma. \end{aligned} \tag{116}$$

Proof of Step P1: We apply the Pohozaev identity (194) with $y_0 = 0$ and $U_\epsilon = B_{\rho_\epsilon}(0) \setminus B_{k_{1,\epsilon}^2}(0)$. This ends the proof of Step P1. □

We will estimate each of the terms in the above integral identities and calculate the limit as $\epsilon \to 0$.

Estimates of the $L^{2^\star(s)}$ term in the localized Pohozaev identity.

Step P2. *Assuming that b satisfies (34), we claim as $\epsilon \to 0$*

$$\int_{B_{\rho_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx = \sum_{i=1}^{N} \frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx + o(1), \tag{117}$$

where (ρ_ϵ) is as in (114) and the $\tilde{u}_i$'s are as defined in Proposition 2 (A4).

Proof of Step P2: For any $R, \rho > 0$ we decompose the above integral as

$$\begin{aligned}\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx &= \int_{B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx\\ &+ \sum_{i=1}^{N}\int_{B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\rho k_{i,\epsilon}}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx\\ &+ \sum_{i=1}^{N-1}\int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx\\ &+ \int_{B_{Rk_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx.\end{aligned}$$

We will evaluate each of the above terms and calculate the limit $\lim_{R\to+\infty}\lim_{\rho\to 0}\lim_{\epsilon\to 0}$.

From the estimate (70), we get as $\epsilon \to 0$

$$
\begin{aligned}
&\int_{B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} \left| b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right| \, dx \\
\leq\, & C \int_{B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} \left[\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_+(\gamma)(2^\star(s)-p_\epsilon)+s}} + \frac{1}{|x|^{\alpha_-(\gamma)(2^\star(s)-p_\epsilon)+s}} \right] dx \\
\leq\, & C \int_{B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} \frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_+(\gamma)(2^\star(s)-p_\epsilon)+s}} \, dx \\
&+ C \int_{B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} \frac{1}{|x|^{\alpha_-(\gamma)(2^\star(s)-p_\epsilon)+s}} \, dx \\
\leq\, & C \int_{B_{\frac{r_\epsilon}{k_{N,\epsilon}}}(0)\setminus \overline{B}_R(0)} \frac{1}{|x|^{n+2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon\alpha_+(\gamma)}} \, dx \\
&+ C \int_{B_1(0)\setminus \overline{B}_{\frac{Rk_{N,\epsilon}}{r_\epsilon}}(0)} \frac{1}{|x|^{n-2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon\alpha_-(\gamma)}} \, dx \\
\leq\, & C \left(R^{-2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon\alpha_+(\gamma)} + r_\epsilon^{2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)+p_\epsilon\alpha_-(\gamma)} \right).
\end{aligned}
$$

Therefore

$$
\lim_{R\to+\infty}\lim_{\epsilon\to 0} \int_{B_{r_\epsilon}(0)\setminus \overline{B}_{Rk_{N,\epsilon}}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx = 0. \tag{118}
$$

It follows from Proposition 2 with a change of variable $x = k_{i,\epsilon}y$, that for any $1 \leq i \leq N$

$$
\lim_{R\to+\infty}\lim_{\rho\to 0}\lim_{\epsilon\to 0} \int_{B_{Rk_{i,\epsilon}}(0)\setminus \overline{B}_{\rho k_{i,\epsilon}}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx = \frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s} \, dx. \tag{119}
$$

Let $1 \leq i \leq N-1$. In Proposition 3, we had obtained the following pointwise estimates: For any $R, \rho > 0$ and all $\epsilon > 0$ we have

$$
|u_\epsilon(x)| \leq C\, \frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{|x|^{\alpha_+(\gamma)}} + \frac{C}{\mu_{i+1,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}} |x|^{\alpha_-(\gamma)}}
$$

for all $x \in B_{\rho k_{i+1,\epsilon}}(0) \setminus \overline{B}_{Rk_{i,\epsilon}}(0)$.

Then we have as $\epsilon \to 0$

$$
\begin{aligned}
&\int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} \left| b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right| \, dx \\
&\leq C \int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} \left[\frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_+(\gamma)(2^\star(s)-p_\epsilon)+s}} + \frac{\mu_{i+1,\epsilon}^{-\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_-(\gamma)(2^\star(s)-p_\epsilon)+s}} \right] \, dx \\
&\leq C \int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} \left[\frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_+(\gamma)(2^\star(s)-p_\epsilon)+s}} + \frac{\mu_{i+1,\epsilon}^{-\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_-(\gamma)(2^\star(s)-p_\epsilon)+s}} \right] \, dx \\
&\leq C \int_{B_{\frac{\rho k_{i+1,\epsilon}}{k_{i,\epsilon}}}(0)\setminus \overline{B}_{R}(0)} \frac{1}{|x|^{n+2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon \alpha_+(\gamma)}} \, dx \\
&\quad + C \int_{B_{2\rho}(0)\setminus \overline{B}_{\frac{Rk_{i,\epsilon}}{k_{i+1,\epsilon}}}(0)} \frac{1}{|x|^{n-2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon \alpha_-(\gamma)}} \, dx \\
&\leq C \left(R^{-2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon \alpha_+(\gamma)} + \rho^{2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)+p_\epsilon \alpha_-(\gamma)} \right).
\end{aligned}
$$

And so

$$
\lim_{R\to+\infty}\lim_{\rho\to 0}\lim_{\epsilon\to 0} \int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{Rk_{i,\epsilon}}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx = 0. \tag{120}
$$

Again, from the pointwise estimates of Proposition 3, we have as $\epsilon \to 0$

$$
\begin{aligned}
&\int_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \left| b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right| \, dx \\
&\leq C \int_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \frac{\mu_{1,\epsilon}^{-\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_-(\gamma)(2^\star(s)-p_\epsilon)+s}} \, dx \\
&\leq C \int_{B_{\rho}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \frac{1}{|x|^{n-2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)-p_\epsilon \alpha_-(\gamma)}} \, dx \\
&\leq C \, \rho^{2^\star(s)\left(\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\right)+p_\epsilon \alpha_-(\gamma)}.
\end{aligned}
$$

Therefore

$$
\lim_{\rho\to 0}\lim_{\epsilon\to 0} \int_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s} \, dx = 0. \tag{121}
$$

Combining (118), (119), (120) and (121) we obtain (117).

We now prove (117) under the assumption that $u_0 \equiv 0$. We decompose the integral as

$$\int_{B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx = \int_{B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx + \int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx,$$

with $r_\epsilon := \sqrt{\mu_{N,\epsilon}}$. From the estimate (70) and $u_0 \equiv 0$, we get as $\epsilon \to 0$

$$\int_{B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)} \left|b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right|\,dx \le C \int_{B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)} \left[\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-p_\epsilon)}}{|x|^{\alpha_+(\gamma)(2^\star(s)-p_\epsilon)+s}}\right]\,dx$$

Since $\alpha_+(\gamma)2^\star(s)+s>n$, we then get that

$$\int_{B_{\delta_0}(0)\setminus \overline{B}_{r_\epsilon}(0)} \left|b(x)\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right|\,dx \le C\left(\frac{\mu_{N,\epsilon}}{r_\epsilon}\right)^{\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))} = o(1)$$

as $\epsilon \to 0$. Then with (117) we get (117) and this proves (117). This ends the proof of Step P2. □

Step P3. *Assuming that b satisfies (34) and taking (ρ_ϵ) as in (114), we obtain as $\epsilon \to 0$*

- *If $\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))>2$, then, we have that*

$$\int_{B_{\rho_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} (x,\nabla b(x))\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx = \mu_{N,\epsilon}^2\left(\frac{\partial_{ij}b(0)}{2t_N^{\frac{n}{2^\star(s)-2}}}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n} X^iX^j\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX\right) + o(\mu_{N,\epsilon}^2). \tag{122}$$

- *If $\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))=2$ then, we have that*

$$\int_{B_{\rho_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} (x,\nabla b(x))\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx = O\left(\mu_{N,\epsilon}^2\ln\frac{1}{\mu_{N,\epsilon}}\right). \tag{123}$$

- *If $\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))<2$, then*

$$\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} (x,\nabla b(x))\frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\,dx = o\left(\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\right) \tag{124}$$

Proof of Step P3: The proof is very similar to Step P4 and we only sketch it. For convenience, for any $i=1,...,N$, we define

$$B_{i,\epsilon}(x) := \frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}|x|^{\alpha_-(\gamma)}+|x|^{\alpha_+(\gamma)}}. \tag{125}$$

We have the following useful estimates:

Lemma 4. *For any $i \in \{1,...,N\}$ and (r_ϵ) such that $c\mu_{i,\epsilon} \leq r_\epsilon \leq \delta_0$, we have that for $k \geq 0$*

$$\int_{B_{r_\epsilon}(0)} \frac{|x|^k B_{i,\epsilon}^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx$$

$$\leq C \begin{cases} \mu_{i,\epsilon}^k & \text{if } \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) > k; \\ \mu_{i,\epsilon}^k \left(1+\left|\ln \frac{r_\epsilon}{\mu_{i,\epsilon}}\right|\right) & \text{if } \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) = k; \\ \mu_{i,\epsilon}^{\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))} r_\epsilon^{k-\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))} & \text{if } \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) < k. \end{cases}$$

and

$$\int_{B_{r_\epsilon}(0)} |x|^k \left(1+\left|\ln\frac{1}{|x|}\right|\right) \frac{B_{i,\epsilon}^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx$$

$$\leq C \begin{cases} \mu_{i,\epsilon}^k \ln \frac{1}{\mu_{i,\epsilon}} & \text{if } \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) > k; \\ \mu_{i,\epsilon}^k \ln\frac{1}{\mu_{i,\epsilon}} \left(1+\left|\ln \frac{r_\epsilon}{\mu_{i,\epsilon}}\right|\right) & \text{if } \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) = k; \\ \mu_{i,\epsilon}^{\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))} r_\epsilon^{k-\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))} \left(1+\left|\ln\frac{1}{|r_\epsilon|}\right|\right) & \text{if } \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) < k. \end{cases}$$

The proof follows from straightforward estimates and the change of variable $x = \mu_{i,\epsilon} z$. Note that we have used that $\mu_{i,\epsilon}^{p_\epsilon} \to t_i \neq 0$ as $\epsilon \to 0$.

We start with the case $\dfrac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) > 2$. With the same change of variable, we get that

$$\int_{\left(B_{r_\epsilon}(0)\setminus B_{Rk_{N,\epsilon}}(0)\right)\cup B_{R^{-1}k_{N,\epsilon}}(0)} \frac{|x|^2 B_{N,\epsilon}^{2^\star-p_\epsilon}}{|x|^s}\, dx \leq \epsilon(R)\mu_{N,\epsilon}^2, \tag{126}$$

where $\lim_{R\to\infty} \epsilon(R) = 0$. With the change of variables $x = k_{N,\epsilon} y$, the definition (A4) of $\tilde{u}_N$ and $k_{i,\epsilon}$ and the asymptotic (A9) in Proposition 2, we get that

$$\int_{B_{Rk_{N,\epsilon}}(0)\setminus B_{R^{-1}k_{N,\epsilon}}(0)} (x,\nabla b(x)) \frac{|u_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\, dx \tag{127}$$

$$= \mu_{N,\epsilon}^{2-\frac{n}{2^\star(s)-2}p_\epsilon} \int_{B_R(0)\setminus B_{R^{-1}}(0)} \left(x, \frac{\nabla b(k_{N,\epsilon}x)}{k_{\epsilon,N}}\right)) \frac{|\tilde{u}_{N,\epsilon}|^{2^\star(s)-p_\epsilon}}{|y|^s}\, dy. \tag{128}$$

Since $\nabla b(0) = 0$, we have that

$$(x,\nabla b(x)) = \partial_{ij} b(0) x^i x^j + o(|x|^2) \text{ as } x \to 0. \tag{129}$$

Finally, with $r_\epsilon := \sqrt{\mu_{\epsilon,N}}$, we have that

$$\int_{B_{r_\epsilon}(0)} \frac{|x|^2}{|x|^s} |x|^{-\alpha_-(\gamma)(2^\star(s)-p_\epsilon)}\, dx = O\left(\mu_{N,\epsilon}^{1+\frac{2^\star(s)}{4}(\alpha_+(\gamma)-\alpha_-(\gamma))}\right).$$

Now, for $R>0$, with the upper-bound (70), we get that

$$\left|\int_{(B_{r_\epsilon}(0)\setminus B_{Rk_{N,\epsilon}}(0))\cup B_{R^{-1}k_{N,\epsilon}}(0)}(x,\nabla b(x))\frac{|u_\epsilon|^{2^\star(s)}}{|x|^s}\,dx\right|$$
$$\leq C\int_{B_{r_\epsilon}(0)}\frac{|x|^2}{|x|^s}|x|^{-\alpha_-(\gamma)(2^\star(s)-p_\epsilon)}\,dx+C\sum_{i<N}\int_{B_{r_\epsilon}(0)}\frac{|x|^2B_{i,\epsilon}^{2^\star-p_\epsilon}}{|x|^s}\,dx$$
$$+C\int_{(B_{r_\epsilon}(0)\setminus B_{Rk_{N,\epsilon}}(0))\cup B_{R^{-1}k_{N,\epsilon}}(0)}\frac{|x|^2B_{N,\epsilon}^{2^\star-p_\epsilon}}{|x|^s}\,dx. \tag{130}$$

Using (129), letting $\epsilon\to 0$ and then $R\to+\infty$ in (128) and (130) in the estimates above, we get (122). This is similar for (122). The estimates for the case $\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))\leq 2$ are obtained via a rough upper-bound. This ends Step P3. □

Estimates of the L^2 term in the localized Pohozaev identity.

Step P4. *We have, as $\epsilon\to 0$,*

- *When $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta$,*

$$\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=\mu_{N,\epsilon}^{2-\theta}\left[\left(\frac{2-\theta}{2}\right)\frac{K_{h_0}}{t_N^{\frac{n-\theta}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{\tilde{u}_N^2}{|x|^\theta}\,dx+o(1)\right]. \tag{131}$$

  where K_{h_0} is as in (32).

- *When $\alpha_+(\gamma)-\alpha_-(\gamma)=2-\theta$,*

$$\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=O\left(\mu_{N,\epsilon}^{2-\theta}\ln\frac{1}{\mu_{\epsilon,N}}\right). \tag{132}$$

- *When $\alpha_+(\gamma)-\alpha_-(\gamma)<2-\theta$,*

$$\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=O\left(\mu_{N,\epsilon}^{\frac{2-\theta+(\alpha_+(\gamma)-\alpha_-(\gamma))}{2}}\right). \tag{133}$$

Proof of Step P4: Due to the hypothesis (32) and (33), we have that

$$\left|h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right|\leq C|x|^{-\theta}\text{ for all }x\in\Omega\setminus\{0\}\text{ and }\epsilon>0. \tag{134}$$

Case 1: $\alpha_+(\gamma)-\alpha_-(\gamma)<2-\theta$. We estimate roughly the integral with (70)

$$\begin{aligned}
\left|\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} |x|^{-\theta}u_\epsilon^2\,dx\right| &\leq C\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} |x|^{-\theta}\left(\sum_{i=1}^N \frac{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}}{|x|^{2\alpha_+(\gamma)}}+\frac{1}{|x|^{2\alpha_-(\gamma)}}\right)dx\\
&\leq C\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} |x|^{-\theta-2\alpha_+(\gamma)}\,dx+C\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} |x|^{-\theta-2\alpha_-(\gamma)}\,dx\\
&\leq C\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\int_0^{r_\epsilon} r^{n-\theta-2\alpha_+(\gamma)-1}\,dx+C\int_0^{r_\epsilon} r^{n-\theta-2\alpha_-(\gamma)-1}\,dx\\
&\leq C\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\int_0^{r_\epsilon} r^{2-\theta-(\alpha_+(\gamma)-\alpha_-(\gamma))-1}\,dr+C\int_0^{r_\epsilon} r^{\alpha_+(\gamma)-\alpha_+(\gamma)+2-\theta-1}\,dr\\
&\leq C\mu_{N,\epsilon}^{\frac{2-\theta+(\alpha_+(\gamma)-\alpha_-(\gamma))}{2}}
\end{aligned}$$

since $\alpha_+(\gamma)-\alpha_-(\gamma)<2-\theta$ and $r_\epsilon=\sqrt{\mu_{N,\epsilon}}$. This proves (133).

Case 2: $\alpha_+(\gamma)-\alpha_-(\gamma)=2-\theta$. Here again since $r_\epsilon=\sqrt{\mu_{N,\epsilon}}$, we estimate roughly the integral with (70) to get

$$\begin{aligned}
&\left|\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} |x|^{-\theta}u_\epsilon^2\,dx\right|\\
&\leq C\int_{B_{r_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} |x|^{-\theta}\left(\sum_{i=1}^N \frac{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}}{\mu_{i,\epsilon}^{2(\alpha_+(\gamma)-\alpha_-(\gamma))}|x|^{2\alpha_-(\gamma)}+|x|^{2\alpha_+(\gamma)}}+\frac{1}{|x|^{2\alpha_-(\gamma)}}\right)dx\\
&\leq C\sum_{i=1}^N \mu_{i,\epsilon}^{2-\theta}\int_{B_{\mu_{i,\epsilon}^{-1}r_\epsilon}(0)} \frac{1}{|x|^\theta\left(|x|^{2\alpha_-(\gamma)}+|x|^{2\alpha_+(\gamma)}\right)}\,dx+C\int_{B_{r_\epsilon}(0)} |x|^{-\theta-2\alpha_-(\gamma)}\,dx\\
&\leq C\sum_{i=1}^N \mu_{i,\epsilon}^{2-\theta}\ln\frac{r_\epsilon}{\mu_{i,\epsilon}}+C\mu_{N,\epsilon}^{2-\theta}\leq C\sum_{i=1}^N \mu_{i,\epsilon}^{2-\theta}\ln\frac{1}{\mu_{i,\epsilon}}\leq C\mu_{N,\epsilon}^{2-\theta}\ln\frac{1}{\mu_{N,\epsilon}}.
\end{aligned}$$

Since $\mu_{i,\epsilon}=o(\mu_{N,\epsilon})$ for $i<N$, this proves (132).

Case 3: $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta$. It follows from point (2) of Theorem 13.1 that the for all $1\leq i\leq N$, the function $\frac{\tilde{u}_i}{|x|^{\theta/2}}\in L^2(\mathbb{R}^n)$ in this case. For any $R,\rho>0$ we decompose the integral as

$$\int\limits_{B_{r_\epsilon}(0)\setminus B_{k^2_{1,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=$$

$$\int\limits_{B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx+\sum_{i=1}^{N}\int\limits_{B_{Rk_{i,\epsilon}}(0)\setminus\overline{B}_{\rho k_{i,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx$$
$$+\sum_{i=1}^{N-1}\int\limits_{B_{\rho k_{i+1,\epsilon}}(0)\setminus\overline{B}_{Rk_{i,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx+\int\limits_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k^2_{1,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx.$$

From the estimate (70), we get as $\epsilon\to0$

$$\begin{aligned}&\mu_{N,\epsilon}^{-(2-\theta)}\int\limits_{B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)}\left|h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right|u_\epsilon^2\,dx\\
&\leq C\,\mu_{N,\epsilon}^{-(2-\theta)}\int\limits_{B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)}\frac{1}{|x|^\theta}\left[\frac{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}}{|x|^{2\alpha_+(\gamma)}}+\frac{1}{|x|^{2\alpha_-(\gamma)}}\right]dx\\
&\leq C\int\limits_{B_{\frac{r_\epsilon}{k_{N,\epsilon}}}(0)\setminus\overline{B}_R(0)}\frac{1}{|x|^\theta}\frac{1}{|x|^{n+(\alpha_+(\gamma)-\alpha_-(\gamma)-2)}}\,dx\\
&+C\int\limits_{B_1(0)\setminus\overline{B}_{\frac{Rk_{N,\epsilon}}{r_\epsilon}}(0)}\frac{1}{|x|^\theta}\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)-(2-\theta)}{2}}}{|x|^{n-(\alpha_+(\gamma)-\alpha_-(\gamma)+2)}}\,dx\\
&\leq C\left(R^{-(\alpha_+(\gamma)-\alpha_-(\gamma)-(2-\theta))}+\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)-(2-\theta)}{2}}\right).\end{aligned}$$

Therefore when $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta$

$$\lim_{R\to+\infty}\lim_{\epsilon\to0}\mu_{N,\epsilon}^{-(2-\theta)}\int\limits_{B_{r_\epsilon}(0)\setminus\overline{B}_{Rk_{N,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=0.\tag{135}$$

Since in this case $\frac{\tilde{u}_i}{|x|^{\theta/2}}\in L^2(\mathbb{R}^n)$ for any $1\leq i\leq N$, it follows from Proposition 2 with a change of variable $x=k_{N,\epsilon}y$, that

$$\lim_{R\to+\infty}\lim_{\rho\to0}\lim_{\epsilon\to0}\left(\mu_{i,\epsilon}^{-(2-\theta)}\int\limits_{B_{Rk_{i,\epsilon}}(0)\setminus\overline{B}_{\rho k_{i,\epsilon}}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx\right)$$
$$=\left(\frac{2-\theta}{2}\right)\frac{K_{h_0}}{t_i^{\frac{n-\theta}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{\tilde{u}_i^2}{|x|^\theta}\,dx.\tag{136}$$

Let $1 \leq i \leq N-1$. Using the pointwise estimates (70), for any $R, \rho > 0$ and all $\epsilon > 0$ we have as $\epsilon \to 0$

$$\begin{aligned}
&\mu_{i+1,\epsilon}^{-(2-\theta)} \int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{R k_{i,\epsilon}}(0)} \left| h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2} \right| u_\epsilon^2 \, dx \\
&\leq C\, \mu_{i+1,\epsilon}^{-(2-\theta)} \int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{R k_{i,\epsilon}}(0)} \frac{1}{|x|^\theta} \left[\frac{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}}{|x|^{2\alpha_+(\gamma)}} + \frac{\mu_{i+1,\epsilon}^{-(\alpha_+(\gamma)-\alpha_-(\gamma))}}{|x|^{2\alpha_-(\gamma)}} \right] dx \\
&\leq C\, \mu_{i+1,\epsilon}^{-(2-\theta)} \int_{B_{\frac{\rho k_{i+1,\epsilon}}{k_{i,\epsilon}}}(0)\setminus \overline{B}_{R}(0)} \frac{1}{|x|^\theta} \frac{\mu_{i,\epsilon}^{2-\theta}}{|x|^{n+(\alpha_+(\gamma)-\alpha_-(\gamma)-2)}} \, dx \\
&\quad + C\, \mu_{i+1,\epsilon}^{-(2-\theta)} \int_{B_{\rho}(0)\setminus \overline{B}_{\frac{R k_{i,\epsilon}}{k_{i+1,\epsilon}}}(0)} \frac{1}{|x|^\theta} \frac{\mu_{i+1,\epsilon}^{2-\theta}}{|x|^{n-(\alpha_+(\gamma)-\alpha_-(\gamma)+2)}} \, dx \\
&\leq C\, \mu_{i+1,\epsilon}^{-(2-\theta)} \left(\mu_{i,\epsilon}^{2-\theta} R^{-(\alpha_+(\gamma)-\alpha_-(\gamma)-(2-\theta))} + \mu_{i+1,\epsilon}^{2-\theta} \rho^{\alpha_+(\gamma)-\alpha_-(\gamma)+(2-\theta)} \right).
\end{aligned}$$

And so as $\alpha_+(\gamma) - \alpha_-(\gamma) > 2$

$$\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\epsilon\to 0} \mu_{i+1,\epsilon}^{-(2-\theta)} \int_{B_{\rho k_{i+1,\epsilon}}(0)\setminus \overline{B}_{R k_{i,\epsilon}}(0)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2} \right) u_\epsilon^2 \, dx = 0. \tag{137}$$

Similarly from the pointwise estimates (70) of Theorem 3, we have as $\epsilon \to 0$

$$\begin{aligned}
&\mu_{1,\epsilon}^{-(2-\theta)} \int_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \left| h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2} \right| u_\epsilon^2 \, dx \\
&\leq C\, \mu_{1,\epsilon}^{-(2-\theta)} \int_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \frac{1}{|x|^\theta} \frac{\mu_{1,\epsilon}^{-(\alpha_+(\gamma)-\alpha_-(\gamma))}}{|x|^{2\alpha_-(\gamma)}} \, dx \\
&\leq C \int_{B_\rho(0)} \frac{1}{|x|^\theta} \frac{1}{|x|^{n-(\alpha_+(\gamma)-\alpha_-(\gamma)+2)}} \, dx \\
&\leq C\, \rho^{\alpha_+(\gamma)-\alpha_-(\gamma)+(2-\theta)}.
\end{aligned}$$

Therefore

$$\lim_{\rho\to 0} \lim_{\epsilon\to 0} \mu_{1,\epsilon}^{-(2-\theta)} \int_{B_{\rho k_{1,\epsilon}}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2} \right) u_\epsilon^2 \, dx = 0. \tag{138}$$

From (135), (136), (137), (138) and Proposition 2 we then obtain (131). This ends Step P4. □

Step P5. *Suppose that $u_0 \equiv 0$. We claim that, as $\epsilon \to 0$*

- *When* $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta$,

$$\int\limits_{B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^2}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=\mu_{N,\epsilon}^{2-\theta}\left[\left(\frac{2-\theta}{2}\right)\frac{K_{h_0}}{t_N^{\frac{n-\theta}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{\tilde{u}_N^2}{|x|^\theta}\,dx+o(1)\right].\tag{139}$$

where K_{h_0} *is as in* (32)*;*

- *When* $\alpha_+(\gamma)-\alpha_-(\gamma)=2-\theta$,

$$\int\limits_{B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^2}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=O\left(\mu_{N,\epsilon}^{2-\theta}\ln\frac{1}{\mu_{\epsilon,N}}\right).\tag{140}$$

- *When* $\alpha_+(\gamma)-\alpha_-(\gamma)<2-\theta$,

$$\int\limits_{B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^2}(0)}\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon,x)}{2}\right)u_\epsilon^2\,dx=O\left(\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\right).\tag{141}$$

Proof of Step P5*:* It follows from (70) that

$$\int\limits_{B_{\delta_0}(0)\setminus B_{r_\epsilon}(0)}\frac{u_\epsilon^2}{|x|^\theta}\,dx\leq C\sum_{i=1}^N\int\limits_{B_{\delta_0}(0)\setminus B_{r_\epsilon}(0)}\frac{1}{|x|^\theta}\frac{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}}{\mu_{i,\epsilon}^{2(\alpha_+(\gamma)-\alpha_-(\gamma))}|x|^{2\alpha_-(\gamma)}+|x|^{2\alpha_+(\gamma)}}\,dx$$

$$\leq C\sum_{i=1}^N\int\limits_{B_{\delta_0}(0)\setminus B_{r_\epsilon}(0)}\frac{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}}{|x|^{2\alpha_+(\gamma)+\theta}}\,dx\leq C\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\int\limits_{r_\epsilon}^{\delta_0}r^{2-\theta-(\alpha_+(\gamma)-\alpha_-(\gamma))-1}\,dr.$$

Combining this estimate with (131) and (132) yield (139) and (140): this proves Step P5 for $\alpha_+(\gamma)-\alpha_-(\gamma)\geq 2-\theta$. For the case $\alpha_+(\gamma)-\alpha_-(\gamma)<2-\theta$, the same estimate as above yields

$$\int\limits_{B_{\delta_0}(0)}\frac{u_\epsilon^2}{|x|^\theta}\,dx\leq C\mu_{\epsilon,N}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\int\limits_0^{\delta_0}r^{2-\theta-(\alpha_+(\gamma)-\alpha_-(\gamma))-1}\,dr\leq C\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)},$$

which gives us (141). This ends Step P5. □

Proof of the sharp blow-up rates when $u_0\equiv 0$ and $\alpha_+(\gamma)-\alpha_-(\gamma)=2-\theta$.

Step P6. *We let* (u_ϵ), (h_ϵ), (p_ϵ) *and* b *be such that* (E_ϵ), (33), (31), (34) *and* (36) *holds. Assume that blow-up occurs as in* (108)*. Suppose* $u_\epsilon>0$ *for all* $\epsilon>0$ *and* $u_0\equiv 0$*. We fix a family of parameters* $(\lambda_\epsilon)_{\epsilon>0}\in(0,+\infty)$ *such that*

$$\lim_{\epsilon\to 0}\lambda_\epsilon=0\ \text{and}\ \lim_{\epsilon\to 0}\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}=0.\tag{142}$$

Then, for all $x\in\mathbb{R}^n\setminus\{0\}$, $x\neq 0$*, we have that*

$$\lim_{\epsilon\to 0}\frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}u_\epsilon(\lambda_\epsilon x)=\frac{\mathcal{K}}{|x|^{\alpha_+(\gamma)}},$$

where

$$\mathcal{K}:=\frac{L_{\gamma,\Omega}}{t_N^{\frac{\alpha_+(\gamma)}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{\tilde{u}_N(z)^{2^\star(s)-1}}{|z|^{s+\alpha_-(\gamma)}}\,dz>0\tag{143}$$

and $L_{\gamma,\Omega}$ *is defined in* (207)*. Moreover, this limit holds in* $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$*.*

Proof of Step P6*:* We define

$$w_\epsilon(x) := \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} u_\epsilon(\lambda_\epsilon x)$$

for all $x \in \mathbb{R}^n \setminus \{0\} \cap B_{\delta\lambda_\epsilon^{-1}}(0)$. From (E_ϵ) it follows that for all $\epsilon > 0$, we have that

$$\left\{ \begin{array}{ll} -\Delta w_\epsilon - \frac{\gamma}{|x|^2} w_\epsilon - \lambda_\epsilon^2 \; h_\epsilon \circ (\lambda_\epsilon x) w_\epsilon = \Xi_\epsilon b(\lambda_\epsilon x) \frac{w_\epsilon^{2^\star(s)-1-p_\epsilon}}{|x|^s} & \text{in } \mathbb{R}^n \setminus \{0\} \cap B_{\delta\lambda_\epsilon^{-1}}(0) \\ w_\epsilon > 0 & \text{in } \mathbb{R}^n \setminus \{0\} \cap B_{\delta\lambda_\epsilon^{-1}}(0). \end{array} \right.$$

With

$$\Xi_\epsilon := \left(\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\lambda_\epsilon^{\alpha_+(\gamma)}} \right)^{2^\star-2-p_\epsilon} \lambda_\epsilon^{2-s}.$$

Since $\mu_{N,\epsilon}^{p_\epsilon} \to t_N > 0$ (see (A9) of Proposition 2) and

$$\alpha_+(\gamma)(2^\star(s)-2) - (2-s) = (2^\star(s)-2)\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2},$$

then using the hypothesis (142), we get that

$$\left(\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\lambda_\epsilon^{\alpha_+(\gamma)}} \right)^{2^\star-2-p_\epsilon} \lambda_\epsilon^{2-s} \leq C \left(\frac{\mu_{N,\epsilon}}{\lambda_\epsilon} \right)^{\alpha_+(\gamma)(2^\star(s)-2-p_\epsilon)-(2-s)} = o(1)$$

as $\epsilon \to 0$. Since $u_0 \equiv 0$, it follows from the pointwise control (70) that there exists $C > 0$ such that $0 < w_\epsilon(x) \leq C|x|^{-\alpha_+(\gamma)}$ for all $x \in \mathbb{R}^n \setminus \{0\} \cap B_{\delta\lambda_\epsilon^{-1}}(0)$. It then follows from standard elliptic theory that there exists $w \in C^2(\mathbb{R}^n \setminus \{0\})$ such that

$$\lim_{\epsilon \to 0} w_\epsilon = w \text{ in } C^2_{loc}(\mathbb{R}^n \setminus \{0\}) \tag{144}$$

with

$$\left\{ \begin{array}{ll} -\Delta w - \frac{\gamma}{|x|^2} w = 0 & \text{in } \mathbb{R}^n \setminus \{0\} \\ 0 \leq w(x) \leq C|x|^{-\alpha_+(\gamma)} & \text{in } \mathbb{R}^n \setminus \{0\}. \end{array} \right.$$

From Proposition 11 in Ghoussoub-Robert [19] it follows that there exists $\Lambda \geq 0$ such that $w(x) = \Lambda |x|^{-\alpha_+(\gamma)}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. We are left with proving that $\Lambda = \mathcal{K}$ defined in (143). We fix $x \in \mathbb{R}^n \setminus \{0\}$. Green's representation formula yields

$$\begin{aligned} w_\epsilon(x) &= \int_\Omega \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} G_\epsilon(\lambda_\epsilon x, y) \frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s} \, dy \\ &= \int_{B_{Rk_{N,\epsilon}}(0) \setminus B_{\delta k_{N,\epsilon}}(0)} + \int_{\Omega \setminus (B_{Rk_{N,\epsilon}}(0) \setminus B_{\delta k_{N,\epsilon}}(0))} \end{aligned} \tag{145}$$

Here G_ϵ is the Green's function for the coercive operator $-\Delta - \frac{\gamma}{|x|^2} - h_\epsilon$, for ϵ small, on Ω with Dirichlet boundary condition.

Step P6.*1:* We estimate the first term of the right-hand-side. A change of variable yields

$$\int_{B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0)} \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} G_\epsilon(\lambda_\epsilon x, y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy$$
$$= \Xi_\epsilon^{(1)} \int_{B_R(0)\setminus B_\delta(0)} G_\epsilon(\lambda_\epsilon x, k_{N,\epsilon} z)\frac{\tilde{u}_{N,\epsilon}(z)^{2^\star(s)-1-p_\epsilon}}{|z|^s}(1+o(1))\,dz$$

with

$$\Xi_\epsilon^{(1)} := \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} k_{N,\epsilon}^{n-s}\mu_{N,\epsilon}^{-\frac{n-2}{2}(2^\star(s)-1-p_\epsilon)}$$

It follows from (207) that for any $z\in\mathbb{R}^n\setminus\{0\}$, we have that

$$G_\epsilon(\lambda_\epsilon x, k_{N,\epsilon} z) = (L_{\gamma,\Omega}+o(1))\frac{1}{\lambda_\epsilon^{\alpha_+(\gamma)}|x|^{\alpha_+(\gamma)}}\cdot\frac{1}{k_{N,\epsilon}^{\alpha_-(\gamma)}|z|^{\alpha_-(\gamma)}},$$

and that the convergence is uniform with repect to $z\in B_R(0)\setminus B_\delta(0)$. Plugging this estimate in the above equality, using that $k_{N,\epsilon}=\mu_{N,\epsilon}^{1-p_\epsilon/(2^\star(s)-2)}$, $\mu_{N,\epsilon}^{p_\epsilon}\to t_N>0$ and the convergence of $\tilde{u}_{N,\epsilon}$ to $\tilde{u}_N$ (see Proposition 2), we get that

$$\int_{B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0)} \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} G_\epsilon(\lambda_\epsilon x, y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy$$
$$= L_{\gamma,\Omega}\frac{1}{|x|^{\alpha_+(\gamma)}}t_N^{-\frac{\alpha_+(\gamma)}{2^\star(s)-2}}\int_{B_R(0)\setminus B_\delta(0)}\frac{1}{|z|^{\alpha_-(\gamma)}}\frac{\tilde{u}_N(z)^{2^\star(s)-1}}{|z|^s}\,dz+o(1)$$

as $\epsilon\to 0$. Therefore,

$$\lim_{R\to+\infty,\delta\to 0}\lim_{\epsilon\to 0}\int_{B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0)} \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} G_\epsilon(\lambda_\epsilon x, y)\frac{u_\epsilon(y)^{2^\star(s)-1}}{|y|^s}\,dy$$
$$= \frac{\mathcal{K}}{|x|^{\alpha_+(\gamma)}} \tag{146}$$

where $\mathcal{K}$ is as in (143).

Step P6.*2:* With the control (205) on the Green's function and the pointwise control (70) on u_ϵ, we get that

$$\int_{\Omega\setminus(B_{Rk_{N,\epsilon}}(0)\setminus B_{\delta k_{N,\epsilon}}(0)} \frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}} G_\epsilon(\lambda_\epsilon x, y)\frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\,dy$$
$$\leq \sum_{i=1}^{N-1} A_{i,\epsilon}+B_\epsilon(R)+C_\epsilon(\delta) \tag{147}$$

where

$$A_{i,\epsilon} := C\frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}\int_{B_{R_0}(0)}\frac{\ell_\epsilon(x,y)^{\alpha_-(\gamma)}}{|\lambda_\epsilon x-y|^{n-2}|y|^s}\left(\frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}|y|^{\alpha_-(\gamma)}+|y|^{\alpha_+(\gamma)}}\right)^{2^\star(s)-1-p_\epsilon}\,dy$$

$$B_\epsilon(R) := C\frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}\int_{B_{R_0}(0)\setminus B_{Rk_{N,\epsilon}}(0)}\frac{\ell_\epsilon(x,y)^{\alpha_-(\gamma)}}{|\lambda_\epsilon x-y|^{n-2}}\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-1-p_\epsilon)}}{|y|^{\alpha_+(\gamma)(2^\star(s)-1-p_\epsilon)+s}}\,dy$$

$$C_\epsilon(\delta) := C(x)\frac{\lambda_\epsilon^{\alpha_+(\gamma)+2-n+\alpha_-(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}\cdot(2^\star(s)-p_\epsilon)}}\int_{B_{\delta k_{N,\epsilon}}(0)}\frac{dy}{|y|^{\alpha_-(\gamma)(2^\star(s)-1-p_\epsilon)+s+\alpha_-(\gamma)}}$$

where $\ell_\epsilon(x,y) := \frac{\max\{\lambda_\epsilon|x|,|y|\}}{\min\{\lambda_\epsilon|x|,|y|\}}$,

Step P6.3. We first estimate $C_\epsilon(\delta)$. Since $n > s+2^\star(s)\alpha_-(\gamma)$ (this is a consequence of $\alpha_-(\gamma) < (n-2)/2$), straightforward computations yield

$$C_\epsilon(\delta) \le C(x)\delta^{\frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))},$$

and therefore

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} C_\epsilon(\delta) = 0. \tag{148}$$

Step P6.4. We estimate $B_\epsilon(R)$. We split the integral as

$$B_\epsilon(R) = \int_{Rk_{\epsilon,N}<|y|<\frac{\lambda_\epsilon|x|}{2}} I_\epsilon(y)\,dy + \int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|} I_\epsilon(y)\,dy + \int_{|y|>2\lambda_\epsilon|x|} I_\epsilon(y)\,dy$$

where $I_\epsilon(y)$ is the integrand. Since

$$n-(s+\alpha_+(\gamma)(2^\star(s)-1)+\alpha_-(\gamma)) = -\frac{2^\star(s)-2}{2}(\alpha_+(\gamma)-\alpha_-(\gamma)) < 0,$$

straightforward computations yield

$$\begin{aligned}
&\int_{Rk_{N,\epsilon}<|y|<\frac{\lambda_\epsilon|x|}{2}} I_\epsilon(y)\,dy\\
&\le C(x)\frac{\lambda_\epsilon^{\alpha_+(\gamma)+\alpha_-(\gamma)+2-n}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}\int_{Rk_{N,\epsilon}<|y|<\frac{\lambda_\epsilon|x|}{2}}\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-1-p_\epsilon)}}{|y|^{\alpha_+(\gamma)(2^\star(s)-1-p_\epsilon)+s+\alpha_-(\gamma)-1}}\,dy\\
&\le C(x)R^{-\frac{2^\star(s)-2}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))},
\end{aligned}$$

For the next term, a change of variable yields

$$\begin{aligned}
&\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|} I_\epsilon(y)\,dy\\
&\le C(x)\frac{\lambda_\epsilon^{\alpha_+(\gamma)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}\int_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|}|\lambda_\epsilon x-y|^{2-n}\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-1)}}{|y|^{\alpha_+(\gamma)(2^\star(s)-1)+s}}\,dy\\
&\le C(x)\left(\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}\right)^{\frac{2^\star(s)-2}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))}\int_{\frac{|x|}{2}<|z|<2|x|}|x-z|^{2-n}\,dz = o(1)
\end{aligned}$$

as $\epsilon \to 0$. Finally, since $\alpha_+(\gamma)+\alpha_-(\gamma) = n-2$ and $n-s-\alpha_+(\gamma)2^\star(s) = \frac{2^\star(s)}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))$, we estimate the last term

$$\begin{aligned}&\int\limits_{|y|>2\lambda_\epsilon|x|} I_\epsilon(y)\,dy\\ &\leq C(x)\mu_{N,\epsilon}^{\frac{2^\star(s)-2}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))}\lambda_\epsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)}\int\limits_{|y|>2\lambda_\epsilon|x|}\frac{|y|^{\alpha_-(\gamma)+1-n-s}\,dy}{|y|^{\alpha_+(\gamma)(2^\star(s)-1)}}\\ &\leq C(x)\left(\frac{\mu_{N,\epsilon}}{\lambda_\epsilon}\right)^{\frac{2^\star(s)-2}{2}(\alpha_+(\gamma)-\alpha_-(\gamma))} = o(1)\end{aligned}$$

as $\epsilon \to 0$. All these inequalities yield

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0} B_\epsilon(R) = 0. \tag{149}$$

Step P6.5. We fix $i \in \{1,...,N-1\}$ and estimate $A_{i,\epsilon}$. As above, we split the integral as

$$A_{i,\epsilon} = \int\limits_{|y|<\frac{\lambda_\epsilon|x|}{2}} J_{i,\epsilon}(y)\,dy + \int\limits_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|} J_{i,\epsilon}(y)\,dy + \int\limits_{|y|>2\lambda_\epsilon|x|} J_{i,\epsilon}(y)\,dy,$$

where $J_{i,\epsilon}$ is the integrand. Since $\mu_{i,\epsilon} \leq \mu_{N,\epsilon}$, as one checks, the second and the third integral of the right-hand-side are controled from above respectively by $\int\limits_{\frac{\lambda_\epsilon|x|}{2}<|y|<2\lambda_\epsilon|x|} I_\epsilon(y)\,dy$ and $\int\limits_{|y|>2\lambda_\epsilon|x|} I_\epsilon(y)\,dy$ that have been computed just above and go to 0 as $\epsilon \to 0$. We are then left with the first term. With a change of variables, we have that

$$\begin{aligned}&\int\limits_{|y|<\frac{\lambda_\epsilon|x|}{2}} J_{i,\epsilon}(y)\,dy\\ &\leq C(x)\frac{\lambda_\epsilon^{\alpha_+(\gamma)+\alpha_-(\gamma)+2-n}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}\\ &\times\int\limits_{|y|<\frac{\lambda_\epsilon|x|}{2}}\left(\frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}|y|^{\alpha_-(\gamma)}+|y|^{\alpha_+(\gamma)}}\right)^{2^\star(s)-1-p_\epsilon}\frac{dy}{|y|^{s+\alpha_-(\gamma)}}\\ &\leq C(x)\frac{\mu_{i,\epsilon}^{n-s-\alpha_-(\gamma)-\frac{n-2}{2}(2^\star(s)-1-p_\epsilon)}}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}\\ &\times\int\limits_{|z|<\frac{\lambda_\epsilon|x|}{2\mu_{i,\epsilon}}}\frac{1}{|z|^{\alpha_-(\gamma)+s}}\left(\frac{1}{|z|^{\alpha_-(\gamma)}+|z|^{\alpha_+(\gamma)}}\right)^{2^\star(s)-1-p_\epsilon}dz\\ &\leq C(x)\left(\frac{\mu_{i,\epsilon}}{\mu_{N,\epsilon}}\right)^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}\end{aligned}$$

since $n > s+2^\star(s)\alpha_-(\gamma)$ and $n < \alpha_-(\gamma)+s+(2^\star(s)-1)\alpha_+(\gamma)$. Since $\mu_{i,\epsilon} = o(\mu_{N,\epsilon})$ as $\epsilon \to 0$, we get that

$$\lim_{\epsilon\to 0} A_{i,\epsilon} = 0. \tag{150}$$

Step P6.6: Plugging (146), (148), (149) and (150) into (145) and (147) yields $\lim_{\epsilon\to 0} w_\epsilon(x) = \frac{\mathcal{K}}{|x|^{\alpha_+(\gamma)}}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. With (144), we then get that $\Lambda = \mathcal{K}$. This proves Step P6.

Now we get the optimal asymptotic when $\alpha_+(\gamma) - \alpha_-(\gamma) = 2 - \theta$:

Step P7. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs as in (108) and $\alpha_+(\gamma) - \alpha_-(\gamma) = 2 - \theta$. Suppose $u_\epsilon > 0$ for all $\epsilon > 0$ and $u_0 \equiv 0$. Then*

$$\int_{B_{\delta_0}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2} \right) u_\epsilon^2 \, dx = \left(\frac{2-\theta}{2} \mathcal{K}^2 \omega_{n-1} K_{h_0} + o(1) \right) \mu_{N,\epsilon}^{2-\theta} \ln \frac{1}{\mu_{N,\epsilon}}. \tag{151}$$

Proof of Step P7: Note that it follows from (33) that

$$\left(h_\epsilon(x) + \frac{(\nabla h_\epsilon, x)}{2} \right) = \frac{(2-\theta)K_{h_0} + o(1)}{2|x|^\theta} \text{ as } x \to 0 \text{ and } \epsilon \to 0. \tag{152}$$

We define $\theta_\epsilon := \frac{1}{\sqrt{|\ln \mu_{N,\epsilon}|}}$, $\alpha_\epsilon := \mu_{N,\epsilon}^{\theta_\epsilon}$ and $\beta_\epsilon := \mu_{N,\epsilon}^{1-\theta_\epsilon}$. As one checks, we have that

$$\left.\begin{array}{lll} \mu_{N,\epsilon} = o(\beta_\epsilon) & \beta_\epsilon = o(\alpha_\epsilon) & \alpha_\epsilon = o(1) \\ \ln \frac{\alpha_\epsilon}{\beta_\epsilon} \simeq \ln \frac{1}{\mu_{N,\epsilon}} & \ln \frac{\beta_\epsilon}{\mu_{N,\epsilon}} = o\left(\ln \frac{1}{\mu_{N,\epsilon}} \right) & \ln \alpha_\epsilon = o(\ln \mu_{N,\epsilon}) \end{array}\right\} \tag{153}$$

as $\epsilon \to 0$. It then follows from (70) and the properties (153) that

$$\left.\begin{array}{l} \displaystyle\int_{B_{\delta_0}(0)\setminus B_{\alpha_\epsilon}(0)} \frac{u_\epsilon^2}{|x|^\theta} \, dx = O\left(\mu_{N,\epsilon}^{2-\theta} \ln \frac{1}{\alpha_\epsilon} \right) = o\left(\mu_{N,\epsilon}^{2-\theta} \ln \frac{1}{\mu_{N,\epsilon}} \right); \\ \displaystyle\int_{B_{\beta_\epsilon}(0)\setminus B_{k_{1,\epsilon}^2}(0)} \frac{u_\epsilon^2}{|x|^\theta} \, dx = o\left(\mu_{N,\epsilon}^{2-\theta} \ln \frac{1}{\mu_{N,\epsilon}} \right). \end{array}\right\} \tag{154}$$

Since $\mu_{N,\epsilon} = o(\beta_\epsilon)$ and $\alpha_\epsilon = o(1)$ as $\epsilon \to 0$, it follows from Proposition P6 that

$$\lim_{\epsilon\to 0} \sup_{x \in B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \left| \frac{|x|^{2\alpha_+(\gamma)} u_\epsilon^2(x)}{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}} - \mathcal{K}^2 \right| = 0 \tag{155}$$

and therefore

$$\begin{aligned} \int_{B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \frac{u_\epsilon^2}{|x|^\theta} \, dx &= (\mathcal{K}^2 + o(1)) \mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)} \int_{B_{\alpha_\epsilon}(0)\setminus B_{\beta_\epsilon}(0)} \frac{dx}{|x|^{\theta+2\alpha_+(\gamma)}} \\ &= (\mathcal{K}^2 \omega_{n-1} + o(1)) \mu_{N,\epsilon}^{2-\theta} \ln \frac{1}{\mu_{N,\epsilon}}. \end{aligned}$$

Plugging this last estimate into (152) yields (151). This ends the proof of Step P7. □

Estimates of the boundary terms.

Step P8. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. We assume that blow-up occurs as in (108). For any $\epsilon > 0$, we define*

$$\tilde{v}_\epsilon(x) := r_\epsilon^{\alpha_-(\gamma)} u_\epsilon(r_\epsilon x) \qquad \text{for } x \in B_{2\delta_0 r_\epsilon^{-1}}(0) \setminus \{0\},$$

with $r_\epsilon := \sqrt{\mu_{N,\epsilon}}$. We claim that there exists $\tilde{v} \in C^1(\mathbb{R}^n)$ such that

$$\lim_{\epsilon\to 0} \tilde{v}_\epsilon(x) = \tilde{v} \quad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\})$$

where $\tilde{v}$ is a solution of

$$\left\{ \begin{array}{rcll} -\Delta \tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} & = & 0 & \text{in } \mathbb{R}^n \setminus \{0\} \end{array} \right. \tag{156}$$

Proof of Step P8: From (E_ϵ) it follows that for all $\epsilon > 0$, the rescaled functions $\tilde{v}_\epsilon$ weakly satisfies the equation

$$-\Delta \tilde{v}_\epsilon - \frac{\gamma}{|x|^2}\tilde{v}_\epsilon - r_\epsilon^2 \; h_\epsilon \circ (r_\epsilon x) \; \tilde{v}_\epsilon = r_\epsilon^{\vartheta + p_\epsilon \alpha_-(\gamma)} b(r_\epsilon x) \frac{|\tilde{v}_\epsilon|^{2^\star(s)-2-p_\epsilon}\tilde{v}_\epsilon}{|x|^s}. \tag{157}$$

with $\vartheta := (2^\star(s) - 2)\dfrac{\alpha_+(\gamma) - \alpha_-(\gamma)}{2} > 0$.

Using the pointwise estimates (70) we obtain the bound, that as $\epsilon \to 0$ we have for $x \in \mathbb{R}^n$

$$\begin{aligned}
|\tilde{v}_\epsilon(x)| &\leq C \; r_\epsilon^{\alpha_-(\gamma)} \sum_{i=1}^{N} \frac{\mu_{i,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\mu_{i,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}|x|^{\alpha_-(\gamma)} + |x|^{\alpha_+(\gamma)}} \\
&\quad + \; C \; r_\epsilon^{\alpha_-(\gamma)} \frac{\||x|^{\alpha_-(\gamma)}u_0\||_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}} \\
&\leq C \; \sum_{i=1}^{N} \frac{\left(\frac{\mu_{i,\epsilon}}{\mu_{N,\epsilon}}\right)^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\left(\frac{\mu_{i,\epsilon}}{\sqrt{\mu_{N,\epsilon}}}\right)^{\alpha_+(\gamma)-\alpha_-(\gamma)} \left|\frac{x}{r_\epsilon}\right|^{\alpha_-(\gamma)} + \left|\frac{x}{r_\epsilon}\right|^{\alpha_+(\gamma)}} \\
&\quad + \; C \; \frac{\||x|^{\alpha_-(\gamma)}u_0\||_{L^\infty(\Omega)}}{\left|\frac{x}{r_\epsilon}\right|^{\alpha_-(\gamma)}} \\
&\leq C \left(\sum_{i=1}^{N} \frac{\left(\frac{\mu_{i,\epsilon}}{\mu_{N,\epsilon}}\right)^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}{\left(\frac{\mu_{i,\epsilon}}{\sqrt{\mu_{N,\epsilon}}}\right)^{\alpha_+(\gamma)-\alpha_-(\gamma)} |x|^{\alpha_-(\gamma)} + |x|^{\alpha_+(\gamma)}} + \frac{\||x|^{\alpha_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}} \right) \\
&\leq \; C \; \left(\frac{1}{|x|^{\alpha_+(\gamma)}} + \frac{\||x|^{\alpha_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}} \right).
\end{aligned}$$

Then passing to limits in the equation (157), standard elliptic theory yields the existence of $\tilde{v} \in C^2(\mathbb{R}^n \setminus \{0\})$ such that $\tilde{v}_\epsilon \to \tilde{v}$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ and $\tilde{v}$ satisfies the equation:

$$\left\{ \begin{array}{rcll} -\Delta \tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} & = & 0 & \text{in } \mathbb{R}^n \setminus \{0\} \end{array} \right.$$

and we have the following bound on $\tilde{v}$

$$|\tilde{v}(x)| \leq C \left(\frac{1}{|x|^{\alpha_+(\gamma)}} + \frac{\||x|^{\alpha_-(\gamma)}u_0\||_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}} \right) \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

This ends the proof of Step P8. □

Step P9. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. We assume that blow-up occurs as in (108). We claim that, as $\epsilon \to 0$,*

$$\int_{\partial B_{r_\epsilon}(0)} F_\epsilon(x)\, d\sigma = \mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}} (\mathcal{F}_0 + o(1))$$
$$\text{and} \int_{\partial B_{k_{1,\epsilon}^2}(0)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}\right), \tag{158}$$

with

$$\mathcal{F}_0 := \int_{\partial B_1(0)} (x,\nu)\left(\frac{|\nabla \tilde{v}|^2}{2} - \frac{\gamma}{2}\frac{\tilde{v}^2}{|x|^2}\right) - \left(x^i\partial_i \tilde{v} + \frac{n-2}{2}\tilde{v}\right)\partial_\nu \tilde{v}\, d\sigma \tag{159}$$

Proof of Step P9*:* We keep the notations of Step P8. With a change of variable and the definition of $\tilde{v}_\epsilon$, and $\vartheta := (2^\star(s)-2)\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2} > 0$, we get

$$\int_{\partial B_{r_\epsilon}(0)} F_\epsilon(x)\, d\sigma =$$
$$r_\epsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)} \int_{\partial B_1(0)} (x,\nu)\left(\frac{|\nabla \tilde{v}_\epsilon|^2}{2} - \frac{\gamma}{2}\frac{\tilde{v}_\epsilon^2}{|x|^2}\right) - \left(x^i\partial_i \tilde{v}_\epsilon + \frac{n-2}{2}\tilde{v}_\epsilon\right)\partial_\nu \tilde{v}_\epsilon\, d\sigma$$
$$- r_\epsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)} \int_{\partial B_1(0)} (x,\nu)\left(r_\epsilon^2\frac{h_\epsilon(r_\epsilon x)}{2}\tilde{v}_\epsilon^2 - b(r_\epsilon x)\frac{r_\epsilon^{\vartheta+\alpha_-(\gamma)p_\epsilon}}{2^\star(s)-p_\epsilon}\frac{|\tilde{v}_\epsilon|^{2^\star(s)-p_\epsilon}}{|x|^s}\right) d\sigma.$$

Then from the convergence result of Step P8, we then get (158). This ends Step P9. □

Step P10. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs as in (108). Suppose $u_0 \equiv 0$. We define*

$$\bar{u}_\epsilon := \frac{u_\epsilon}{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}}. \tag{160}$$

We claim that there exists $\bar{u} \in C^2(\overline{\Omega}\setminus\{0\})$ such that

$$\lim_{\epsilon\to 0} \bar{u}_\epsilon = \bar{u} \text{ in } C^2_{loc}(\overline{\Omega}\setminus\{0\}) \text{ with } \begin{cases} -\Delta \bar{u} - \left(\frac{\gamma}{|x|^2} + h_0\right)\bar{u} = 0 & \text{in } \Omega\setminus\{0\} \\ \bar{u} = 0 & \text{in } \partial\Omega \end{cases} \tag{161}$$

Proof of Step P10*:* Since $u_0 \equiv 0$, it follows from (70) that there exists $C > 0$ such that

$$|\bar{u}_\epsilon(x)| \leq C|x|^{-\alpha_+(\gamma)} \text{ for all } x \in \Omega\setminus\{0\} \text{ and } \epsilon > 0. \tag{162}$$

Moreover, equation (E_ϵ) rewrites

$$-\Delta \bar{u}_\epsilon - \left(\frac{\gamma}{|x|^2} + h_\epsilon\right)\bar{u}_\epsilon = \mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star(s)-2-p_\epsilon)} b(x)\frac{|\bar{u}_\epsilon|^{2^\star(s)-2-p_\epsilon}\bar{u}_\epsilon}{|x|^s} \text{ in } \Omega,$$

and $\bar{u}_\epsilon = 0$ on $\partial\Omega$. It then follows from standard elliptic theory that the claim holds. This ends Step P10. □

Step P11. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs as in (108). Suppose that $u_0 \equiv 0$. We claim that*

$$\int_{\partial B_{\delta_0}(0)} F_\epsilon(x)\, d\sigma = (\mathcal{F}_{\delta_0} + o(1))\, \mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}$$

$$\text{and} \quad \int_{\partial B_{k_{1,\epsilon}^2}(0)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\right), \tag{163}$$

where

$$\mathcal{F}_{\delta_0} := \int_{\partial B_{\delta_0}(0)} (x,\nu)\left(\frac{|\nabla \bar{u}|^2}{2} - \left(\frac{\gamma}{|x|^2} + h_0\right)\frac{\bar{u}^2}{2}\right) - \left(x^i\partial_i \bar{u} + \frac{n-2}{2}\bar{u}\right)\partial_\nu \bar{u}\, d\sigma. \tag{164}$$

Proof of Step P11: With a change of variable, the definition of $\bar{u}_\epsilon$ and the convergence (161), we get

$$\int_{\partial B_{\delta_0}(0)} F_\epsilon(x)\, d\sigma \tag{165}$$

$$= \mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)} \int_{\partial B_{\delta_0}(0)} (x,\nu)\left(\frac{|\nabla \bar{u}_\epsilon|^2}{2} - \left(\frac{\gamma}{|x|^2} + h_\epsilon\right)\frac{\bar{u}_\epsilon^2}{2}\right) d\sigma$$

$$-\frac{\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}(2^\star - p_\epsilon)}}{2^\star - p_\epsilon} \int_{\partial B_{\delta_0}(0)} (x,\nu) b(x) \frac{|\bar{u}_\epsilon|^{2^\star-\epsilon}\bar{u}_\epsilon}{|x|^2}\, d\sigma$$

$$- \int_{\partial B_{\delta_0}(0)} \left(x^i\partial_i \bar{u}_\epsilon + \frac{n-2}{2}\bar{u}_\epsilon\right)\partial_\nu \bar{u}_\epsilon\, d\sigma$$

$$= \mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)} \left(\mathcal{F}_{\delta_0} + o(1)\right). \tag{166}$$

where $\mathcal{F}_{\delta_0}$ is as above. And directly from the bound (70) we obtain

$$\int_{\partial B_{k_{1,\epsilon}^2}(0)} F_\epsilon(x)\, d\sigma = o\left(\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\right)$$

This ends Step P11. □

Step P12. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs as in (108). Suppose that $u_\epsilon > 0$ for all $\epsilon > 0$. We have $\mathcal{F}_0 \geq 0$ and*

$$\mathcal{F}_0 > 0 \iff u_0 > 0.$$

where $\mathcal{F}_0$ is as in (159).

Proof of Step P12: We let $\tilde{v}$ be defined as in Step P8. It then follows from Step P8 that $\tilde{v} \geq 0$ and $\tilde{v}$ satisfies (156) and we have the following bound on $\tilde{v}$

$$|\tilde{v}(x)| \leq C\left(\frac{1}{|x|^{\alpha_+(\gamma)}} + \frac{\||x|^{\alpha_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\alpha_-(\gamma)}}\right) \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{167}$$

Given $\alpha \in \mathbb{R}$, we define $v_\alpha(x) := |x|^{-\alpha}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Since $\tilde{v} \geq 0$, it follows from Proposition 6.4 in Ghoussoub-Robert [19] that there exists $A, B \geq 0$ such that

$$\tilde{v} := A v_{\alpha_+(\gamma)} + B v_{\alpha_-(\gamma)}. \tag{168}$$

Step P12.1: We claim that $B=0$ when $u_0\equiv 0$.
This is a direct consequence of controling (168) with (167) when $u_0\equiv 0$ and letting $|x|\to\infty$.

Step P12.2: We claim that $B>0$ when $u_0>0$.
We prove the claim. We fix $x\in\mathbb{R}^n\setminus\{0\}$. Green's representation formula yields

$$\tilde{v}_\epsilon(x) = \int_\Omega r_\epsilon^{\alpha_-(\gamma)} G_\epsilon(r_\epsilon x,y)\frac{u_\epsilon^{2^\star(s)-1-p_\epsilon}(y)}{|y|^s}\,dy.$$

We fix $\mathcal{D}\subset\subset\Omega\setminus\{0\}$. Then there exists $\tilde{c}(\mathcal{D})>0$ such that $|y|\geq\tilde{c}(\mathcal{D})$ for all $y\in\mathcal{D}$. Moreover, the control (206) of the Green's function yields

$$\tilde{v}_\epsilon(x) \geq C\int_{\mathcal{D}} r_\epsilon^{\alpha_-(\gamma)}\frac{1}{r_\epsilon^{\alpha_-(\gamma)}|x|^{\alpha_-(\gamma)}}|\tilde{c}(\mathcal{D})-r_\epsilon|x||^{-(n-2)}\frac{u_\epsilon^{2^\star(s)-1-p_\epsilon}(y)}{|y|^s}\,dy,$$

and then, passing to the limit $\epsilon\to 0$, we get that

$$\tilde{v}(x) \geq \frac{C}{|x|^{\alpha_-(\gamma)}}\int_{\mathcal{D}}\frac{u_0^{2^\star(s)-1}(y)}{|y|^s}\,dy,$$

for all $x\in\mathbb{R}^n\setminus\{0\}$. As one checks, this yields $B\geq C\int_{\mathcal{D}}\frac{u_0^{2^\star(s)-1}(y)}{|y|^s}\,dy>0$ when $u_0>0$. This ends Step P12.2.

Step P12.3: We claim that $A>0$.
The proof is similar to Step P12.2. We fix $x\in\mathbb{R}^n\setminus\{0\}$ and $\tilde{\mathcal{D}}\subset\subset\mathbb{R}^n\setminus\{0\}$. Green's representation formula and the pointwise control (206) yields

$$\begin{aligned}
\tilde{v}_\epsilon(x) &\geq \int_{k_{N,\epsilon}\tilde{\mathcal{D}}} r_\epsilon^{\alpha_-(\gamma)}G_\epsilon(r_\epsilon x,y)\frac{u_\epsilon^{2^\star(s)-1}(y)}{|y|^s}\,dy\\
&\geq \tilde{c}(\tilde{\mathcal{D}})\int_{\tilde{\mathcal{D}}} r_\epsilon^{\alpha_-(\gamma)}G_\epsilon(r_\epsilon x,k_{N,\epsilon}y)k_{N,\epsilon}^n\frac{u_\epsilon(k_{N,\epsilon}y)^{2^\star(s)-1}}{|k_{N,\epsilon}y|^s}\,dy\\
&\geq \tilde{c}(\tilde{\mathcal{D}})\int_{\tilde{\mathcal{D}}} r_\epsilon^{\alpha_-(\gamma)}\left(\frac{r_\epsilon|x|}{k_{N,\epsilon}|y|}\right)^{\alpha_-(\gamma)}|r_\epsilon x-k_{N,\epsilon}y|^{2-n}k_{N,\epsilon}^{\frac{n-2}{2}}\frac{\tilde{u}_{\epsilon,N}(y)^{2^\star(s)-1}}{|y|^s}\,dy
\end{aligned}$$

Since $r_\epsilon:=\sqrt{\mu_{N,\epsilon}}$, letting $\epsilon\to 0$, we get with the convergence (A4) of Proposition 2 that

$$\begin{aligned}
\tilde{v}_\epsilon(x) &\geq \tilde{c}(\tilde{\mathcal{D}})\int_{\tilde{\mathcal{D}}} r_\epsilon^{2\alpha_-(\gamma)-(n-2)}|x|^{\alpha_-(\gamma)}\left|x-\frac{k_{N,\epsilon}}{r_\epsilon}y\right|^{2-n}k_{N,\epsilon}^{\frac{n-2}{2}-\alpha_-(\gamma)}\frac{\tilde{u}_{\epsilon,N}(y)^{2^\star(s)-1}}{|y|^s}\,dy\\
&\geq \frac{\tilde{c}(\tilde{\mathcal{D}})}{|x|^{\alpha_+(\gamma)}}\int_{\tilde{\mathcal{D}}}\frac{\tilde{u}_N(y)^{2^\star(s)-1}}{|y|^s}\,dy
\end{aligned}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$. Therefore, as one checks, $A\geq c(\omega)\int_{\tilde{\mathcal{D}}}\frac{\tilde{u}_N(y)^{2^\star(s)-1}}{|y|^s}\,dy>0$. This ends Step P12.3.

Step P12.4: We claim that

$$\mathcal{F}_0 = 2\omega_{n-1}\left(\frac{(n-2)^2}{4}-\gamma\right)\cdot AB. \tag{169}$$

We prove the claim. Definition (159) reads

$$\mathcal{F}_0 := \int_{\partial B_1(0)} (x,\nu)\left(\frac{|\nabla\tilde{v}|^2}{2} - \frac{\gamma}{2}\frac{\tilde{v}^2}{|x|^2}\right) - \left(x^i\partial_i\tilde{v} + \frac{n-2}{2}\tilde{v}\right)\partial_\nu\tilde{v}\, d\sigma \tag{170}$$

For simplicity, we define the bilinear form:

$$\begin{aligned}\mathcal{H}_\delta(u,v) &= \int_{\partial B_\delta(0)} \left[(x,\nu)\left((\nabla u,\nabla v) - \gamma\frac{uv}{|x|^2}\right) - \left(x^i\partial_i u + \frac{n-2}{2}u\right)\partial_\nu v\right.\\ &\qquad \left. - \left(x^i\partial_i v + \frac{n-2}{2}v\right)\partial_\nu u\right] d\sigma\end{aligned}$$

As one checks, using (168)

$$\begin{aligned}\mathcal{F}_0 &= \frac{1}{2}\mathcal{H}_1(Av_{\alpha_+(\gamma)} + Bv_{\alpha_-(\gamma)}, Av_{\alpha_+(\gamma)} + Bv_{\alpha_-(\gamma)})\\ &= \frac{A^2}{2}\mathcal{H}_1(v_{\alpha_+(\gamma)}, v_{\alpha_+(\gamma)}) + AB\mathcal{H}_1(v_{\alpha_+(\gamma)}, v_{\alpha_-(\gamma)})\\ &\quad + \frac{B^2}{2}\mathcal{H}_1(v_{\alpha_-(\gamma)}, v_{\alpha_-(\gamma)})\end{aligned}$$

In full generality, we compute $\mathcal{H}_\delta(v_\alpha, v_\beta)$ for all $\alpha,\beta\in\mathbb{R}$ and all $\delta>0$. Consequently, straightforward computations yield

$$\left(x^i\partial_i v_\alpha + \frac{n-2}{2}v_\alpha\right)\partial_\nu v_\beta = -\beta\left(\frac{n-2}{2} - \alpha\right)\frac{v_\alpha v_\beta}{|x|}$$

and

$$(x,\nu)\left((\nabla v_\alpha, \nabla v_\beta) - \frac{\gamma}{|x|^2}v_\alpha v_\beta\right) = (\alpha\beta - \gamma)\frac{v_\alpha v_\beta}{|x|}$$

and then

$$\mathcal{H}_\delta(v_\alpha, v_\beta) = \int_{\partial B_\delta(0)} \left(\alpha\beta - \gamma + \beta\left(\frac{n-2}{2} - \alpha\right) + \alpha\left(\frac{n-2}{2} - \beta\right)\right)\frac{v_\alpha v_\beta}{|x|}\, d\sigma$$

Plugging all these identities together yields

$$\mathcal{H}_\delta(v_\alpha, v_\beta) = \left(-\alpha\beta - \gamma + (\alpha+\beta)\frac{n-2}{2}\right)\delta^{-\alpha-\beta-1+n-1}\omega_{n-1}$$

Since $\alpha_+(\gamma), \alpha_-(\gamma)$ are solutions to $X^2 - (n-2)X + \gamma = 0$, we get that

$$\mathcal{H}_\delta(v_{\alpha_-(\gamma)}, v_{\alpha_-(\gamma)}) = \mathcal{H}_\delta(v_{\alpha_+(\gamma)}, v_{\alpha_+(\gamma)}) = 0.$$

Since $\alpha_+(\gamma) + \alpha_-(\gamma) = n-2$ and $\alpha_+(\gamma)\alpha_-(\gamma) = \gamma$, we get that

$$\mathcal{H}_\delta(v_{\alpha_-(\gamma)}, v_{\alpha_+(\gamma)}) = 2\omega_{n-1}\left(\frac{(n-2)^2}{4} - \gamma\right).$$

Plugging all these results together yields (169). This ends Step P12.4.

These substeps end the proof of Step P12. □

Step P13. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. Assume that blow-up occurs as in (108). If $u_\epsilon > 0$ for all $\epsilon > 0$ then $u_0 \equiv 0$, when $\alpha_+(\gamma) - \alpha_-(\gamma) < 4 - 2\theta$.*

Proof of Step P13: Since $\dfrac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}<2-\theta$, plugging (117), (131), (132), (133) and (158) and (123) and (123) into the Pohozaev identity (116), we get as $\epsilon\to 0$

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^N \frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)=-\left(\mathcal{F}_0+o(1)\right)\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}, \tag{171}$$

where $\mathcal{F}_0$ is as in (170). Therefore $\mathcal{F}_0\leq 0$. Since $u_\epsilon>0$, it then follows from (169) of Step P12 that $u_0\equiv 0$. This proves Step P13. □

8. Proof of the sharp blow-up rates

We now prove the sharp blow-up rates claimed in Propositions 4 and 5.

The case $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta$. It follows from the Pohozaev identity (116), (117), (122), (131) and Step P9 when $u_0\not\equiv 0$, and the Pohozaev identity (116), (119), (122), (139) and Step P11 when $u_0\equiv 0$ that

$$\begin{aligned}&\mu_{N,\epsilon}^{2-\theta}\left[\left(\frac{2-\theta}{2}\right)\frac{K_{h_0}}{t_N^{\frac{n-\theta}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{\tilde{u}_N^2}{|x|^\theta}\,dx+o(1)\right].\\
&+\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^N\frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)\\
&+\frac{\mathbf{1}_{\theta=0}}{2^\star(s)}\mu_{\epsilon,N}^2\left(\frac{\partial_{ij}b(0)}{t_N^{\frac{n}{2^\star(s)-2}}}\int_{\mathbb{R}^n}X^iX^j\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX\right)+o(\mu_{\epsilon,N}^2).\\
&=\begin{cases}-\mu_{N,\epsilon}^{\frac{\alpha_+(\gamma)-\alpha_-(\gamma)}{2}}\left(\mathcal{F}_0+o(1)\right)\\ -\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\left(\mathcal{F}_{\delta_0}+o(1)\right) & \text{if } u_0\equiv 0.\end{cases}\end{aligned} \tag{172}$$

We will use the following lemma:

Lemma 5. *Suppose we have $\alpha_+(\gamma)-\alpha_-(\gamma)>2$, $\gamma\geq 0$ and $\tilde{u}_N>0$. Then*

$$\frac{\partial_{ij}b(0)}{2^\star(s)}\frac{\int_{\mathbb{R}^n}X^iX^j\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}\frac{\tilde{u}_N^2}{|x|^\theta}\,dx}=\frac{\Delta b(0)}{b(0)}\frac{(n-2)[(\alpha_+(\gamma)-\alpha_-(\gamma))^2-4]}{4n(2n-2-s)}. \tag{173}$$

This lemma will be proved at the end of the present section.

Case 1: $\alpha_+(\gamma)-\alpha_-(\gamma)>2(2-\theta)$ or $\{u_0\equiv 0$ and $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta\}$. Then (172) yields (110) and Proposition 4 is proved. When $\tilde{u}_\epsilon>0$, then $\tilde{u}_N\geq 0$ and $\tilde{u}_N\not\equiv 0$: it then follows from the strong comparison principle that $\tilde{u}_N>0$ and then Lemma 5 holds. This yields Proposition 5 for $\alpha_+(\gamma)-\alpha_-(\gamma)>2(2-\theta)$ or $\{u_0\equiv 0$ and $\alpha_+(\gamma)-\alpha_-(\gamma)>2-\theta\}$.

Case 2: $u_\epsilon>0$ and $\alpha_+(\gamma)-\alpha_-(\gamma)=2(2-\theta)$. It then follows from (172) that

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}}=-\frac{\left(\frac{2-\theta}{2}\right)\frac{K_{h_0}}{t_N^{\frac{n-\theta}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{\tilde{u}_N^2}{|x|^\theta}\,dx+\frac{\mathbf{1}_{\theta=0}}{2^\star(s)}\frac{\partial_{ij}b(0)}{t_N^{\frac{n}{2^\star(s)-2}}}\int_{\mathbb{R}^n}X^iX^j\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX+\mathcal{F}_0}{\frac{1}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^N\frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx\right)}.$$

Then Step P12 and (173) yield Proposition 5 when $\alpha_+(\gamma)-\alpha_-(\gamma)=2(2-\theta)$.

Case 3: $u_\epsilon > 0$ and $2-\theta < \alpha_+(\gamma)-\alpha_-(\gamma) < 2(2-\theta)$. It follows from Step P13 that $u_0 \equiv 0$. Then (172) and (173) yields Proposition 4 for $2-\theta < \alpha_+(\gamma)-\alpha_-(\gamma) < 2(2-\theta)$.

The case $u_\epsilon > 0$ and $\alpha_+(\gamma)-\alpha_-(\gamma) = 2-\theta$. The Pohozaev identity (116), (119), (122), StepP7 and Step P11 yields Proposition 4 for $\alpha_+(\gamma)-\alpha_-(\gamma) = 2-\theta$.

The case $u_\epsilon > 0$ and $\alpha_+(\gamma)-\alpha_-(\gamma) < 2-\theta$.

Step P14. *We let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), (34) and (36) holds. We assume that blow-up occurs as in (108). Suppose $u_\epsilon > 0$ for all $\epsilon > 0$ and $\alpha_+(\gamma)-\alpha_-(\gamma) < 2-\theta$. Then (112) holds, that is*

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}} = -\frac{2\omega_{n-1}2^\star(s)^2\left(\frac{(n-2)^2}{4}-\gamma\right)A^2}{(n-s)\sum\limits_{i=1}^{N}\frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx}\cdot m_{\gamma,h}(\Omega) \tag{174}$$

for some $A > 0$, where $m_{\gamma,h}(\Omega)$ is the mass.

Proof of Step P14: It follows from Step P13 that $u_0 \equiv 0$.

Step P14.1: We now claim that

$$\frac{p_\epsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)}\right)\left(\sum_{i=1}^{N}\frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx + o(1)\right) = \mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\left(M_{\delta_0}+o(1)\right)$$

where

$$M_{\delta_0} \quad := \quad -\int\limits_{B_{\delta_0}(0)}\left(h_0(x)+\frac{(\nabla h_0, x)}{2}\right)\bar{u}^2\,dx - \mathcal{F}_{\delta_0} \tag{175}$$

with $\mathcal{F}_{\delta_0}$ is as in (164) and $\bar{u}$ is as in (161).

Indeed the convergence (160), (161), (162) and $\alpha_+(\gamma)-\alpha_-(\gamma) < 2-\theta$ gives

$$\begin{aligned}\int\limits_{B_{\delta_0}(0)}&\left(h_\epsilon(x)+\frac{(\nabla h_\epsilon, x)}{2}\right)u_\epsilon^2\,dx \\ &= \mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}\left(\int\limits_{B_{\delta_0}(0)}\left(h_0(x)+\frac{(\nabla h_0, x)}{2}\right)\bar{u}^2\,dx + o(1)\right)\end{aligned} \tag{176}$$

Then from the Pohozaev identity (116), (119), (122), (139) and Step P11 we obtain (175), which proves the claim and ends Step P14.1.

Fix $\delta < \delta'$. Taking $U := B_{\delta'}(0)\setminus B_\delta(0)$, $b \equiv 0$ and $u = \bar{u}$ in (194), and using (161), we get that M_δ is independent of the choice of $\delta > 0$ small enough.

Step P14.2: We claim that $\bar{u} > 0$.

We prove this claim. Since $\bar{u} \geq 0$ is a solution to (161), it is enough to prove that $\bar{u} \not\equiv 0$. We argue as in the proof of Step P12. We fix $x \in \Omega \setminus \{0\}$. Green's identity and $u_\epsilon > 0$ gives

$$\overline{u}_\epsilon(x) = \mu_{N,\epsilon}^{-(\alpha_+(\gamma)-\alpha_-(\gamma))/2} \int_\Omega G_\epsilon(x,y) \frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\, dy$$

$$\geq \mu_{N,\epsilon}^{-(\alpha_+(\gamma)-\alpha_-(\gamma))/2} \int_{\mathcal{D}_\epsilon} G_\epsilon(x,y) \frac{u_\epsilon(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\, dy$$

$$\geq C\mu_{N,\epsilon}^{n-s-(\alpha_+(\gamma)-\alpha_-(\gamma))/2} \int_{\mathcal{D}} G_\epsilon(x,\mu_{N,\epsilon} y) \frac{u_\epsilon(\mu_{N,\epsilon} y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\, dy,$$

where $\mathcal{D}_\epsilon := B_{2\mu_{N,\epsilon}}(0) \setminus B_{\mu_{N,\epsilon}}(0)$, $\mathcal{D} := B_2(0) \setminus B_1(0)$. With the pointwise control (205), we get

$$\overline{u}_\epsilon(x) \geq$$

$$C(x) \int_{\mathcal{D}} \left(\frac{|x|}{|y|}\right)^{\alpha_-(\gamma)} |x - \mu_{N,\epsilon} y|^{2-n} \frac{u_{\epsilon,N}(y)^{2^\star(s)-1-p_\epsilon}}{|y|^s}\, dy$$

where $u_{\epsilon,N}$ is as in Proposition 2. Letting $\epsilon \to 0$ and using the convergence (A4) of Proposition 2, we get that

$$\bar{u}(x) \geq C(x) \frac{1}{|x|^{\alpha_+(\gamma)}} \text{ for all } x \in \Omega.$$

And then $\bar{u} > 0$ in Ω. This proves the claim and Step P14.2.

We fix $r_0 > 0$ and $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ in $B_{r_0}(0)$ and $\eta(x) = 0$ in $\mathbb{R}^n \setminus B_{2r_0}(0)$. It then follows from [18, 20] that, for $r_0 > 0$ small enough, there exists $A > 0$ and $\beta \in H_0^1(\Omega)$ such that

$$\bar{u}(x) = A\left(\frac{\eta(x)}{|x|^{\alpha_+(\gamma)}} + \beta(x)\right) \text{ for all } x \in \Omega$$

with

$$\beta(x) = m_{\gamma,h}(\Omega) \frac{\eta(x)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{\eta(x)}{|x|^{\alpha_-(\gamma)}}\right)$$

as $\epsilon \to 0$. Here, $m_{\gamma,h_0}(\Omega)$ is the mass of Ω associated with the operator $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$, defined in Theorem 1.7..

Step P14.3: We claim that

$$\lim_{\delta\to 0} M_\delta = -2\omega_{n-1}\left(\frac{(n-2)^2}{4} - \gamma\right) A^2 \cdot m_{\gamma,h}(\Omega) \tag{177}$$

We prove the claim. Since $\bar{u}$ is a solution to (161), it follows from standard elliptic theory that there exists $C > 0$ such that $\bar{u}(x) + |x||\nabla \bar{u}(x)| \leq C|x|^{-\alpha_+(\gamma)}$ for all $x \in B_{2\delta_0}(0)$. Therefore, since $\alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta$, we get that

$$\lim_{\delta\to 0} \left(\int_{B_\delta(0)} \bar{u}^2\, dx + \int_{\partial B_\delta(0)} (x,\nu)\bar{u}^2\, d\sigma \right) = 0.$$

And therefore,

$$M_\delta = -\frac{A^2}{2} \bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_+(\gamma)} + \bar{v}_{\alpha_-(\gamma)}, \bar{v}_{\alpha_+(\gamma)} + \bar{v}_{\alpha_-(\gamma)}) + o(1)$$

as $\delta \to 0$, where

$$\begin{aligned}\bar{\mathcal{H}}_\delta(u,v) \quad := \quad & \int_{\partial B_{\delta_0}(0)} \left[(x,\nu)\left((\nabla u, \nabla v) - \frac{\gamma}{|x|^2}uv\right) - \left(x^i\partial_i u + \frac{n-2}{2}u\right)\partial_\nu v \right.\\ & \left. - \left(x^i\partial_i v + \frac{n-2}{2}v\right)\partial_\nu u\right] d\sigma\end{aligned}$$

with

$$\bar{v}_{\alpha_+(\gamma)}(x) := \frac{\eta(x)}{|x|^{\alpha_+(\gamma)}} \text{ and } \bar{v}_{\alpha_-(\gamma)}(x) = \beta(x) \text{ for all } x \in \Omega.$$

We then get that

$$\begin{aligned} M_\delta \quad = \quad & -\frac{A^2}{2}\bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_+(\gamma)}, \bar{v}_{\alpha_+(\gamma)}) - A^2\bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_+(\gamma)}, \bar{v}_{\alpha_-(\gamma)}) \\ & -\frac{A^2}{2}\bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_-(\gamma)}, \bar{v}_{\alpha_-(\gamma)}) + o(1)\end{aligned}$$

as $\delta \to 0$. Since $\alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta$ and $\alpha_+(\gamma) + \alpha_-(\gamma) = n-2$, we get with a change of variable that as $\delta \to 0$,

$$\begin{aligned}\bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_+(\gamma)}, \bar{v}_{\alpha_+(\gamma)}) \quad &= \quad \mathcal{H}_\delta(v_{\alpha_+(\gamma)}, v_{\alpha_+(\gamma)}) + O(\delta^{2-(\alpha_+(\gamma)-\alpha_-(\gamma))})\\ \bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_+(\gamma)}, \bar{v}_{\alpha_-(\gamma)}) \quad &= \quad m_{\gamma,h}(\Omega)\cdot \mathcal{H}_\delta(v_{\alpha_+(\gamma)}, v_{\alpha_-(\gamma)}) + o(\delta^2)\\ \bar{\mathcal{H}}_\delta(\bar{v}_{\alpha_-(\gamma)}, \bar{v}_{\alpha_-(\gamma)}) \quad &= \quad O(\delta^{2+(\alpha_+(\gamma)-\alpha_-(\gamma))}).\end{aligned}$$

Using the computations performed in the proof of Step P12, we then get (177). This proves the claim and ends Step P14.3.

End of the proof of Step P14*:* Since M_δ is independent of δ small, we then get that $M_{\delta_0} = -2\omega_{n-1}\left(\frac{(n-2)^2}{4} - \gamma\right)A^2m_{\gamma,h}($
Putting this estimate in (175), we then get (174). This end Step P14. □

Proof of Lemma 5. We assume that $\tilde{u}_N > 0$. We define

$$U(x) := \left(\frac{1}{|x|^{\frac{2-s}{n-2}\alpha_-(\gamma)} + |x|^{\frac{2-s}{n-2}\alpha_+(\gamma)}}\right)^{\frac{n-2}{2-s}}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. As one checks, we have that

$$\begin{aligned} -\Delta U - \frac{\gamma}{|x|^2}U \quad &= \quad \frac{4(n-s)}{n-2}\left(\frac{(n-2)^2}{4} - \gamma\right)\frac{U^{2^\star(s)-1}}{|x|^s}\\ &= \quad \frac{(n-s)(\alpha_+(\gamma)-\alpha_-(\gamma))^2}{n-2}\frac{U^{2^\star(s)-1}}{|x|^s}.\end{aligned}$$

We define

$$\mu^{2^\star(s)-2} = \frac{(n-s)(\alpha_+(\gamma)-\alpha_-(\gamma))^2}{(n-2)b(0)}.$$

so that μU is a solution to (55).

We claim that there exists $\nu > 0$ such that $\tilde{u}_N = \mu\nu^{-\frac{n-2}{2}}U(\nu^{-1}\cdot)$.

We prove the claim. It follows from the convergence (A4) and the pointwise control (70) that

$$\tilde{u}_N(x) \leq \frac{C}{|x|^{\alpha_-(\gamma)} + |x|^{\alpha_+(\gamma)}} \text{ for } x \in \mathbb{R}^n \setminus \{0\}.$$

Since $\tilde{u}_N$ solves (55), one has that

$$-\text{div}(|x|^{-2\alpha_-(\gamma)}\nabla(|x|^{\alpha_-(\gamma)}\tilde{u}_N)) = |x|^{-\alpha_-(\gamma)2^\star(s)-s}(|x|^{\alpha_-(\gamma)}\tilde{u}_N)^{2^\star(s)-1} \text{ in } \mathbb{R}^n \setminus \{0\}$$

with $|x|^{\alpha_-(\gamma)}\tilde{u}_N \in C^2(\mathbb{R}^n\setminus\{0\})$ and $|x|^{\alpha_-(\gamma)}\tilde{u}_N \leq C(1+|x|^{\alpha_+(\gamma)-\alpha_-(\gamma)})^{-1}$ in $\mathbb{R}^n\setminus\{0\}$. Since $s>0$, it then follows from Chou-Chu [7] that $\tilde{u}_N$ is radially symmetrical when $\gamma\geq 0$.

We define $\varphi(t) := e^{-\frac{n-2}{2}t}\tilde{u}_N(e^{-t})$ for $t\in\mathbb{R}$. We get that $-\varphi" + \left(\frac{(n-2)^2}{4}-\gamma\right)\varphi = \varphi^{2^\star(s)-1}$ in $\mathbb{R}$, $\varphi>0$ and $\lim_{t\to\pm\infty}\varphi(t)=0$. The ODE yields the existence of $\mathcal{S}\in\mathbb{R}$ such that

$$\frac{(\varphi')^2}{2} + \left(\frac{(n-2)^2}{4}-\gamma\right)\frac{\varphi^2}{2} - \frac{\varphi^{2^\star(s)}}{2^\star(s)} = \mathcal{S}. \tag{178}$$

Letting $t\to+\infty$ yields $\mathcal{S}=0$. Since $\lim_{t\to\pm\infty}\varphi(t)=0$, there exists $t_0\in\mathbb{R}$ such that $\varphi'(t_0)=0$, and (178) yields a unique possible value for $\varphi(t_0)$. Therefore, the theory of ODEs yields uniqueness of φ up to translation. Since λU is also such a positive solution, we then get that there exists $\nu>0$ such that $\tilde{u}_N = \mu\nu^{-\frac{n-2}{2}}U(\nu^{-1}\cdot)$. This proves the claim.

With a change of variables and symmetry, we get that

$$\frac{\partial_{ij}b(0)}{2^\star(s)}\frac{\int_{\mathbb{R}^n} X^iX^j\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}\tilde{u}_N^2\,dx} = \frac{\Delta b(0)}{n\,2^\star(s)}\frac{\int_{\mathbb{R}^n}|X|^2\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}\tilde{u}_N^2\,dx}$$

$$= \frac{\Delta b(0)\mu^{2^\star(s)-2}(n-2)}{2n(n-s)}\frac{\int_{\mathbb{R}^n}|X|^2\frac{U^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}U^2\,dx} = \frac{\Delta b(0)}{b(0)}\frac{(\alpha_+(\gamma)-\alpha_-(\gamma))^2}{2n}\frac{\int_{\mathbb{R}^n}|X|^2\frac{U^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}U^2\,dx}$$

We estimate this ratio as in Jaber [21]. Passing to radial coordinates and taking the change of variable $t = r^{\frac{2-s}{n-2}(\alpha_+(\gamma)-\alpha_-(\gamma))}$, we get that

$$\frac{\int_{\mathbb{R}^n}|X|^2\frac{U^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}U^2\,dx} = \frac{\int_0^\infty \frac{r^{2-s+n-1}}{\left(r^{\frac{2-s}{n-2}\alpha_-(\gamma)}+r^{\frac{2-s}{n-2}\alpha_+(\gamma)}\right)^{\frac{2(n-s)}{2-s}}}\,dr}{\int_0^\infty \frac{r^{n-1}}{\left(r^{\frac{2-s}{n-2}\alpha_-(\gamma)}+r^{\frac{2-s}{n-2}\alpha_+(\gamma)}\right)^{\frac{2(n-2)}{2-s}}}\,dr} = \frac{I_{\mathcal{Q}+2}^{\mathcal{P}+1}}{I_{\mathcal{Q}}^{\mathcal{P}}}$$

where

$$I_{\mathcal{Q}}^{\mathcal{P}} = \int_0^\infty \frac{t^{\mathcal{P}}}{(1+t)^{\mathcal{Q}}}\,dt\,;\ \mathcal{Q} = \frac{2(n-2)}{2-s}\text{ and }\mathcal{P} = \frac{(n-2)(n-2\alpha_-(\gamma))}{(2-s)(\alpha_+(\gamma)-\alpha_-(\gamma))}-1$$

Integrarting by parts, see Jaber [21], we have that

$$I_{\mathcal{Q}+1}^{\mathcal{P}+1} = \frac{\mathcal{P}+1}{\mathcal{Q}}I_{\mathcal{Q}}^{\mathcal{P}}\text{ and }I_{\mathcal{Q}+1}^{\mathcal{P}} = \frac{\mathcal{Q}-\mathcal{P}-1}{\mathcal{Q}}I_{\mathcal{Q}}^{\mathcal{P}}.$$

We then get that

$$\frac{\int_{\mathbb{R}^n}|X|^2\frac{U^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}U^2\,dx} = \frac{(\mathcal{P}+1)(\mathcal{Q}-\mathcal{P}-1)}{\mathcal{Q}(\mathcal{Q}+1)} = \frac{(n-2)(\alpha_+(\gamma)-\alpha_-(\gamma)+2)(\alpha_+(\gamma)-\alpha_-(\gamma)-2)}{2(\alpha_+(\gamma)-\alpha_-(\gamma))^2(2n-2-s)},$$

so that

$$\frac{\partial_{ij}b(0)}{2^\star(s)}\frac{\int_{\mathbb{R}^n} X^iX^j\frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\,dX}{\int_{\mathbb{R}^n}\tilde{u}_N^2\,dx} = \frac{\Delta b(0)}{b(0)}\frac{(n-2)((\alpha_+(\gamma)-\alpha_-(\gamma))^2-4)}{4n(2n-2-s)}\text{ when }\tilde{u}_N>0.$$

This completes Lemma 5.

9. Blow-up rates when $b \notin C^2(\overline{\Omega})$

The function b that arises from the hyperbolic model (see Lemma 1) is not in C^2 when $n = 3, 4$, and therefore, the asymptotic rates of Propositions 4 and 5 do not apply. However, due to the behavior of $(x, \nabla b(x))$ in (13) and (14), we are in position to get optimal rates.

The case of behavior like (14). When the expressions make sense, we define

$$\mathcal{C}_{n,s} := \frac{\left(\frac{2-\theta}{2}\right) \frac{1}{t_N^{\frac{n-\theta}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|^\theta}\, dx}{\frac{n-s}{2^\star(s)^2} \sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx} \text{ and } \mathcal{D}_{n,s} := \frac{\left(\frac{1}{2^\star(s)}\right) \frac{1}{t_N^{\frac{n}{2^\star(s)-2}}} \int_{\mathbb{R}^n} |x|^2 \frac{|\tilde{u}_N|^{2^\star(s)}}{|x|^s}\, dx}{\frac{n-s}{2^\star(s)^2} \sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx} \quad (179)$$

The proof of compactness rely on the following two key propositions.

Proposition 6 ($b \in C^1 \setminus C^2$)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $\gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), and (36) holds. We assume that $b \in C^1(\overline{\Omega})$ and that there exists $C_1 \in \mathbb{R}$ such that*

$$(x, \nabla b(x)) = C_1 |x|^2 \ln \frac{1}{|x|} + O(|x|^2) \text{ as } x \to 0.$$

Assume that blow-up occurs, that is

$$\lim_{\epsilon \to 0} \||x|^\tau u_\epsilon\|_{L^\infty(\Omega)} = +\infty \text{ for some } \alpha_-(\gamma) < \tau < \frac{n-2}{2}.$$

Consider the $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Then, we have the following blow-up rates:

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)} \cdot K_{h_0} \text{ if } \theta > 0 \text{ and } \begin{cases} \text{either } \alpha_+(\gamma) - \alpha_-(\gamma) > 4 - 2\theta \\ \text{or } \{\alpha_+(\gamma) - \alpha_-(\gamma) > 2 - \theta \text{ and } u_0 \equiv 0\} \end{cases}$$

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^2 \ln \frac{1}{\mu_{N,\epsilon}}} = -\frac{\mathcal{D}_{n,s}}{b(0)} \cdot C_1 \text{ if } \theta = 0 \text{ and } \begin{cases} \text{either } \alpha_+(\gamma) - \alpha_-(\gamma) \geq 4 \\ \text{or } \{\alpha_+(\gamma) - \alpha_-(\gamma) > 2 \text{ and } u_0 \equiv 0\} \end{cases}.$$

Proposition 7 ($b \in C^1 \setminus C^2$, *The positive case*)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $0 \leq \gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be as in Proposition 6. In particular*

$$(x, \nabla b(x)) = C_1 |x|^2 \ln \frac{1}{|x|} + O(|x|^2) \text{ as } x \to 0.$$

Assume that blow-up occurs as in (108). Consider $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Suppose in addition that

$$u_\epsilon > 0 \qquad \text{for all } \epsilon > 0.$$

Then, we have the following blow-up rates:

(1) When $\alpha_+(\gamma) - \alpha_-(\gamma) = 4 - 2\theta$ and $\theta > 0$, we have

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta}} = -\frac{\mathcal{C}_{n,s}}{b(0)} \left(K_{h_0} + \tilde{K}\right),$$

for some $\tilde{K} \geq 0$ such that $\tilde{K} > 0$ iff $u_0 > 0$. For $\theta = 0$, see Proposition 6.

(2) When $2 - \theta < \alpha_+(\gamma) - \alpha_-(\gamma) < 4 - 2\theta$, we have $u_0 \equiv 0$ and Proposition 6 applies.

(3) *When $\alpha_+(\gamma) - \alpha_-(\gamma) = 2 - \theta$, we have $u_0 \equiv 0$ and*

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{2-\theta} \ln \frac{1}{\mu_{N,\epsilon}}} = -\frac{\mathcal{C}'_{n,s}}{b(0)} \cdot K_{h_0} - \mathbf{1}_{\theta=0} \frac{\mathcal{D}_{n,s}}{b(0)} \cdot C_1 \tag{180}$$

where

$$\mathcal{C}'_{n,s} := \frac{\left(\frac{2-\theta}{2}\right) \mathcal{K}^2 \omega_{n-1}}{\frac{n-s}{2^\star(s)^2} \sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx},$$

with $\mathcal{K}$ as defined in (143).

(4) *When $\alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta$, then $u_0 \equiv 0$ and*

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}} = -\chi \cdot m_{\gamma,h_0}(\Omega) \quad \text{if } \alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta, \tag{181}$$

where $\chi > 0$ is a constant and $m_{\gamma,h_0}(\Omega)$ is the mass of Ω associated with the operator $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$, defined in Theorem 1.7.

The proof of the propositions goes as the proof of Propositions 4 and 5, with the use of the pointwise control of Lemma 4.

The case of behavior like (13)**.** In this case, when writing the Pohozaev identity there are several terms to compare. For the sake of simplicity, we only deal here with the case $\theta = 1$ that corresponds to the hyperbolic case when $n = 3$. When the expression makes sense, we define

$$\mathcal{E}_{n,s} := \frac{\left(\frac{1}{2}\right) \frac{1}{t_N^{\frac{n-1}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|}\, dx}{\frac{n-s}{2^\star(s)^2} \sum_{i=1}^N \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\, dx} \tag{182}$$

The proof of compactness rely on the following two key propositions.

Proposition 8 ($b \in C^0 \setminus C^1$)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $\gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be such that (E_ϵ), (33), (31), and (36) holds. We assume that $\theta = 1$, that $b \in C^{0,1}(\overline{\Omega})$ and that there exists $C_2 \in \mathbb{R}$ such that*

$$(x, \nabla b(x)) = C_2|x| + O(|x|^2) \text{ as } x \to 0.$$

Assume that blow-up occurs, that is

$$\lim_{\epsilon\to 0} \||x|^\tau u_\epsilon\|_{L^\infty(\Omega)} = +\infty \quad \text{for some} \quad \alpha_-(\gamma) < \tau < \frac{n-2}{2}.$$

Consider the $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Assume that

$$\{\alpha_+(\gamma) - \alpha_-(\gamma) > 2\} \text{ or } \{\alpha_+(\gamma) - \alpha_-(\gamma) > 1 \text{ and } u_0 \equiv 0\}.$$

Then, we have the following blow-up rates:

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = -\frac{\mathcal{E}_{n,s}}{b(0)} \cdot \left(K_{h_0} + \frac{2}{2^\star(s)} \frac{\int\limits_{\mathbb{R}^n} |x| \frac{|\tilde{u}_N|^{2^\star(s)}}{|x|^s}\, dx}{\int\limits_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|}\, dx} \cdot C_2 \right)$$

Proposition 9 *($b \in C^0 \setminus C^1$, The positive case for $\theta = 1$)*. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $0 \leq \gamma < \frac{(n-2)^2}{4}$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be as in Proposition 8. In particular $\theta = 1$ and*

$$(x, \nabla b(x)) = C_2|x| + O(|x|^2) \text{ as } x \to 0.$$

Assume that blow-up occurs as in (108). Consider $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Suppose in addition that

$$u_\epsilon > 0 \text{ for all } \epsilon > 0 \text{ and } C_2 > 0$$

Then, we have the following blow-up rates:

(1) *When $\alpha_+(\gamma) - \alpha_-(\gamma) = 2$, we have that*

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = -\frac{\mathcal{E}_{n,s}}{b(0)} \cdot \left(K_{h_0} + \frac{(n-2)^2}{2(2n+2-s)} \left((n-2)^2 - 4\gamma - 1 \right) \cdot \frac{C_2}{b(0)} + \tilde{K} \right)$$

for some $\tilde{K} \geq 0$ such that $\tilde{K} > 0$ iff $u_0 > 0$.

(2) *When $1 < \alpha_+(\gamma) - \alpha_-(\gamma) < 2$, we have $u_0 \equiv 0$ and Proposition 8 applies.*

(3) *When $\alpha_+(\gamma) - \alpha_-(\gamma) = 1$, we have $u_0 \equiv 0$ and*

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon} \ln \frac{1}{\mu_{N,\epsilon}}} = -\frac{\mathcal{E}'_{n,s}}{b(0)} \cdot K_{h_0}$$

where

$$\mathcal{E}'_{n,s} := \frac{\left(\frac{1}{2}\right) \mathcal{K}^2 \omega_{n-1}}{\frac{n-s}{2^\star(s)^2} \sum_{i=1}^{N} \frac{1}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s} \, dx},$$

with $\mathcal{K}$ as defined in (143).

(4) *When $\alpha_+(\gamma) - \alpha_-(\gamma) < 1$, then $u_0 \equiv 0$ and*

$$\lim_{\epsilon \to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{\alpha_+(\gamma) - \alpha_-(\gamma)}} = -\chi \cdot m_{\gamma, h_0}(\Omega) \tag{183}$$

where $\chi > 0$ is a constant and $m_{\gamma,h_0}(\Omega)$ is the mass of Ω associated with the operator $-\Delta - \dfrac{\gamma}{|x|^2} - h_0(x)$, defined in Theorem 1.7.

The proof of the propositions goes as the proof of Propositions 4 and 5, with the use of the pointwise control of Lemma 4. We have also used that when $\tilde{u}_N > 0$, then

$$\frac{\int_{\mathbb{R}^n} |x| \frac{|\tilde{u}_N|^{2^\star(s)}}{|x|^s} \, dx}{\int_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|} \, dx} = \frac{(n-2)(n-s)}{2b(0)(2n+2-s)} \left((\alpha_+(\gamma) - \alpha_-(\gamma))^2 - 1 \right),$$

which is proved like Lemma 5.

For the proof of compactness in dimension 3, the following proposition will be useful: its proof is similar to the other ones:

Proposition 10 *($b \in C^0 \setminus C^1$, The positive case with $\theta = \gamma = 0$)*. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n = 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $\gamma = \theta = 0$. Let (u_ϵ), (h_ϵ), (p_ϵ) and b be as in Proposition 8. Assume that*

$$(x, \nabla b(x)) = C_2|x| + O(|x|^2) \text{ as } x \to 0.$$

Assume that blow-up occurs as in (108). Consider $\mu_{1,\epsilon}, ..., \mu_{N,\epsilon}$ and $t_1, ..., t_N$ from Proposition 2. Suppose in addition that $u_\epsilon > 0$ for all $\epsilon > 0$ and that $C_2 > 0$. Then, $\alpha_+(\gamma) - \alpha_-(\gamma) = 1$, $u_0 \equiv 0$ and

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}} = -\chi_1 \cdot m_{0,h_0}(\Omega) - \chi_2 \cdot C_2 \tag{184}$$

where $\chi_1, \chi_2 > 0$ are constants and $m_{0,h_0}(\Omega)$ is the mass of Ω associated with the operator $-\Delta - h_0(x)$, defined in Theorem 1.7.

10. Proof of Multiplicity

Proof of Theorem 1.6: We fix $\gamma < (n-2)^2/4$. Let $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$ be such that $-\Delta - \frac{\gamma}{|x|^2} - h_0(x)$ is coercive and let b satisfy (34). For each $2 < p \leq 2^\star(s)$, we consider the C^2-functional

$$I_{p,\gamma}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2\,dx - \gamma\frac{|u|^2}{|x|^2} - h_0 u^2\right)dx - \frac{1}{p}\int_\Omega b(x)\frac{|u|^p}{|x|^s}\,dx$$

on $H_{1,0}^2(\Omega)$, whose critical points are the weak solutions of

$$\begin{cases} -\Delta u - \dfrac{\gamma}{|x|^2}u - h_0 u &= b(x)\dfrac{|u|^{p-2}u}{|x|^s} \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{185}$$

For a fixed $u \in H_{1,0}^2(\Omega)$, $u \not\equiv 0$, we have that

$$I_{p,\gamma}(\lambda u) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx - \gamma\frac{\lambda^2}{2}\int_\Omega \frac{|u|^2}{|x|^2}\,dx - \frac{\lambda^2}{2}\int_\Omega h_0 u^2\,dx - \frac{\lambda^p}{p}\int_\Omega b(x)\frac{|u|^p}{|x|^s}\,dx.$$

Then, by coercivity, we have $\lim_{\lambda\to\infty} I_{p,\gamma}(\lambda u) = -\infty$, which means that for each finite dimensional subspace $E_k \subset E := H_{1,0}^2(\Omega)$, there exists $R_k > 0$ such that

$$\sup\{I_{p,\gamma}(u); u \in E_k, \|u\|_{H_1^2} > R_k\} < 0 \tag{186}$$

as $p \to 2^\star(s)$. Let $(E_k)_{k=1}^\infty$ be an increasing sequence of subspaces of $H_{1,0}^2(\Omega)$ such that $\dim E_k = k$ and $\overline{\cup_{k=1}^\infty E_k} = E := H_{1,0}^2(\Omega)$ and define the min-max values:

$$c_{p,k} = \inf_{g\in \mathbf{H}_k} \sup_{x\in E_k} I_{p,\gamma}(g(x)),$$

where

$$\mathbf{H}_k = \{g \in C(E,E);\ g \text{ is odd and } g(v) = v \text{ for } \|v\| > R_k \text{ for some } R_k > 0\}.$$

Proposition 11. *With the above notation and assuming $n \geq 3$, we have:*

(1) For each $k \in \mathbb{N}$, $c_{p,k} > 0$ and $\lim_{p\to 2^\star(s)} c_{p,k} = c_{2^\star(s),k} := c_k$.

(2) If $2 < p < 2^\star(s)$, there exists for each k, functions $u_{p,k} \in H_{1,0}^2(\Omega)$ such that $I'_{p,\gamma}(u_{p,k}) = 0$, and $I_{p,\gamma}(u_{p,k}) = c_{p,k}$.

(3) For each $2 < p < 2^\star(s)$, we have $c_{p,k} \geq D_{n,p} k^{\frac{p+1}{p-1}\frac{2}{n}}$ where $D_{n,p} > 0$ is such that $\lim_{p\to 2^\star(s)} D_{n,p} = 0$.

(4) $\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_{2^\star(s),k} = +\infty$.

Proof: (1) Coercivity yields the existence of $\Lambda_0 > 0$ such that

$$\int_\Omega \left(|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2 - h_0 u^2\right)dx \geq \Lambda_0 \int_\Omega |\nabla u|^2\,dx \text{ for all } u \in H_{1,0}^2(\Omega). \tag{187}$$

With coercivity, the Hardy and the Hardy-Sobolev inequality, there exists $C > 0$ and $\alpha > 0$ such that

$$I_{p,\gamma}(u) \geq \frac{\Lambda_0}{2}\|\nabla u\|_2^2 - C\|\nabla u\|_2^p = \|\nabla u\|_2^2\left(\frac{\Lambda_0}{2} - C\|\nabla u\|_2^{p-2}\right) \geq \alpha > 0$$

for all $u \in H^2_{1,0}(\Omega)$ such that $\|\nabla u\|_2 = \rho > 0$ is small enough. Then the sphere $S_\rho = \{u \in E; \|u\|_{H^2_{1,0}(\Omega)} = \rho\}$ intersects every image $g(E_k)$ by an odd continuous function g. It follows that

$$c_{p,k} \geq \inf\{I_{p,\gamma}(u); u \in S_\rho\} \geq \alpha > 0.$$

In view of (186), it follows that for each $g \in \mathbf{H}_k$, we have that

$$\sup_{x \in E_k} I_{p_i,\gamma}(g(x)) = \sup_{x \in D_k} I_{p,\gamma}(g(x))$$

where D_k denotes the ball in E_k of radius R_k. Consider now a sequence $p_i \to 2^\star(s)$ and note first that for each $u \in E$, we have that $I_{p_i,\gamma}(u) \to I_{2^\star(s),\gamma}(u)$. Since $g(D_k)$ is compact and the family of functionals $(I_{p,\gamma})_p$ is equicontinuous, it follows that $\sup\limits_{x \in E_k} I_{p,\gamma}(g(x)) \to \sup\limits_{x \in E_k} I_{2^\star(s),\gamma}(g(x))$, from which follows that $\limsup\limits_{i \in \mathbb{N}} c_{p_i,k} \leq \sup\limits_{x \in E_k} I_{2^\star(s),\gamma}(g(x))$. Since this holds for any $g \in \mathbf{H}_k$, it follows that

$$\limsup_{i \in \mathbb{N}} c_{p_i,k} \leq c_{2^\star(s),k} = c_k.$$

On the other hand, the function $f(r) = \frac{1}{p}r^p - \frac{1}{2^\star(s)}r^{2^\star(s)}$ attains its maximum on $[0, +\infty)$ at $r = 1$ and therefore $f(r) \leq \frac{1}{p} - \frac{1}{2^\star(s)}$ for all $r > 0$. It follows

$$\begin{aligned} I_{2^\star(s),\gamma}(u) &= I_{p,\gamma}(u) + \int_\Omega \frac{b(x)}{|x|^s}\left(\frac{1}{p}|u(x)|^p - \frac{1}{2^\star(s)}|u(x)|^{2^\star(s)}\right) dx \\ &\leq I_{p,\gamma}(u) + \int_\Omega \frac{b(x)}{|x|^s}\left(\frac{1}{p} - \frac{1}{2^\star(s)}\right) dx \end{aligned}$$

from which follows that $c_k \leq \liminf\limits_{i \in \mathbb{N}} c_{p_i,k}$, and claim (1) is proved.

If now $p < 2^\star(s)$, we are in the subcritical case, that is we have compactness in the Sobolev embedding $H^2_{1,0}(\Omega) \to L^p(\Omega; |x|^{-s}dx)$ and therefore $I_{p,\gamma}$ satisfies the Palais-Smale condition. It is then standard to find critical points $u_{p,k}$ for $I_{p,\gamma}$ at each level $c_{p,k}$ (see for example the book [14]). Consider now the functional

$$I_{p,0}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p}\int_\Omega b(x)\frac{|u|^p}{|x|^s}\, dx$$

and its critical values

$$c^0_{p,k} = \inf_{g \in \mathbf{H}_k} \sup_{x \in E_k} I_{p,0}(g(x)).$$

In [17] it has been shown that (1), (2) and (3) of Proposition 11 holds, with $c^0_{p,k}$ and c^0_k replacing $c_{p,k}$ and c_k respectively and $b(x) \equiv 1$. By similar arguments, $\lim\limits_{k\to\infty} c^0_k = \lim\limits_{k\to\infty} c^0_{2^\star(s),k} = +\infty$. On the other hand, with the coercivity (187), we obtain

$$I_{p,\gamma}(u) \geq \Lambda_0^{\frac{p}{p-2}} I_{p,0}(v) \quad \text{for every } u \in H^2_{1,0}(\Omega),$$

where $v = \Lambda_0^{-\frac{1}{p-2}}u$. It then follows that $\lim\limits_{k\to\infty} c_k = \lim\limits_{k\to\infty} c_{2^\star(s),k} = +\infty$.

To complete the proof of Theorem 1.6, notice that since for each k, we have

$$\lim_{p_i \to 2^\star(s)} I_{p_i,\gamma}(u_{p_i,k}) = \lim_{p_i \to 2^\star(s)} c_{p_i,k} = c_k,$$

it follows that the sequence $(u_{p_i,k})_i$ is uniformly bounded in $H^2_{1,0}(\Omega)$. Moreover, since $I'_{p_i}(u_{p_i,k}) = 0$, letting $p_i \to 2^\star(s)$ and using the compactness Theorem 1.6 we get a solution u_k of (185) in such a way that $I_{2^\star(s),\gamma}(u_k) = \lim_{p\to 2^\star(s)} I_{p,\gamma}(u_{p,k}) = \lim_{p\to 2^\star(s)} c_{p,k} = c_k$. Since the latter sequence goes to infinity, it follows that (185) has an infinite number of critical levels with $2 < p \le 2^\star(s)$.

11. Proof of Theorems 1.1, 1.2 and 1.4

We reduce the problem to a smooth bounded domain of the Euclidean space $\mathbb{R}^n$.

Case 1: $n \geq 5$. It follows from Lemma 1 that finding solutions to (9) amounts to finding solutions to

$$\begin{cases} -\Delta v - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right) v &= b(x)\dfrac{v^{2^\star(s)-1}}{|x|^s} \quad \text{in } \Omega \\ \qquad\qquad v &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

where $h_{\gamma,\lambda} \in C^1(\overline{\Omega})$ with $h_{\gamma,\lambda}(0) = 4\lambda + \dfrac{4(n-2)}{n-4}\left(\gamma - \dfrac{n(n-4)}{4}\right)$. Moreover in this case, $b \in C^2(\overline{\Omega})$, $\nabla b(0) = 0$ and

$$\frac{\Delta b(0)}{b(0)} = \frac{4n(2n-2-s)}{n-4}.$$

Since b is radial, we have that $\partial_{ij} b(0) = \dfrac{\Delta b(0)}{n}\delta_{ij}$. Therefore with $\theta = 0$, the limit (110) in Proposition 5 reads

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^2} = -\frac{\mathcal{C}_{n,s}}{b(0)}\left(4\lambda + \frac{4(n-2)}{n-4}\left(\gamma - \frac{n(n-4)}{4}\right) + \frac{\Delta b(0)}{n\ 2^\star(s)} \frac{\int_{\mathbb{R}^n} |X|^2 \frac{|\tilde{u}_N|^{2^\star(s)}}{|X|^s}\, dX}{\int_{\mathbb{R}^n} \tilde{u}_N^2\, dx}\right). \tag{188}$$

when $\gamma < \dfrac{(n-2)^2}{4} - 4$. Since $\Delta b(0) > 0$, then arguing as in the proof of Theorem 1.6, we get multiplicity of solutions as soon as we have: $4\lambda + \dfrac{4(n-2)}{n-4}\left(\gamma - \dfrac{n(n-4)}{4}\right) \geq 0$.

We now consider the case of positive solutions. We have here $\theta = 0$. Using the value of $\Delta b(0)$, it then follows from Lemma 5 and Proposition 5 that for $\gamma \geq 0$

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^2} \leq -4\lambda \frac{\mathcal{C}_{n,s}}{b(0)} \quad \text{when } \alpha_+(\gamma) - \alpha_-(\gamma) > 2. \tag{189}$$

Therefore, for $\lambda > 0$ we get a contradiction and hence compactness for positive subcritical solutions. This yields compactness of minimizers and the existence of minimisers when $\lambda > 0$ for $\alpha_+(\gamma) - \alpha_-(\gamma) > 2$ and $\gamma \geq 0$. When $\alpha_+(\gamma) - \alpha_-(\gamma) = 2$ and $\gamma \geq 0$, we get the same result by using (111). When $\alpha_+(\gamma) - \alpha_-(\gamma) < 2$ and $\gamma \geq 0$, we apply (112) to get existence of minimizers when the mass $m_{\gamma,\lambda}(\Omega_{\mathbb{B}^n})$ is positive.

Case 2: $n = 4$. We prove point (2) of Theorem 1.4 for $n = 4$. By Lemma 1 here $h_{\gamma,\lambda} \in C^1\left(\overline{\Omega} \setminus \{0\}\right)$ and $h_{\gamma,\lambda}(r) = 8\gamma \ln \frac{1}{r} + c_4 + O(r)$, as $r \to 0$. In particular $h_{\gamma,\lambda}(x) = o(|x|^{-\theta})$ as $x \to 0$ for all $\theta > 0$.

- Assume that $\gamma < 0$, so that $\alpha_+(\gamma) - \alpha_-(\gamma) > 2$. We fix $\theta \in (0,2)$ small, so that $\alpha_+(\gamma) - \alpha_-(\gamma) > 2 - \theta$ and $K_{h_{\gamma,\lambda}} = 0$. Then Propositions 6 and 7 do not permit to conclude.
- Assume that $\gamma = 0$, so that $\alpha_+(\gamma) - \alpha_-(\gamma) = 2$ and we have that $h_{\gamma,\lambda}(r) = c_4 + O(r)$. Therefore we take $\theta = 0$ and $K_{h_{\gamma,\lambda}} = c_4 = 2(\lambda - 2)$. Then point (3) of Proposition 7 with $C_1 = 4(2^\star(s) + 2)$ yields

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^2 \ln \frac{1}{\mu_{N,\epsilon}}} = -\left(\frac{\mathcal{C}'_{n,s}}{b(0)} \cdot c_4 + \frac{\mathcal{D}_{n,s}}{b(0)} \cdot 4(2^\star(s)+2)\right) \text{ when } u_\epsilon > 0.$$

With the definitions of $\mathcal{C}'_{n,s}$ and $\mathcal{D}_{n,s}$, we have that

$$\frac{\mathcal{C}'_{n,s}}{b(0)}\cdot c_4+\frac{\mathcal{D}_{n,s}}{b(0)}\cdot 4(2^\star(s)+2)=\frac{\mathcal{K}^2\omega_3 c_4+\left(\frac{4(2^\star(s)+2)}{2^\star(s)}\right)\frac{1}{t_N^{\frac{4}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^4}|x|^2\frac{|\tilde{u}_N|^{2^\star(s)}}{|x|^s}\,dx}{\frac{4-s}{2^\star(s)^2}b(0)\sum\limits_{i=1}^N\frac{1}{t_i^{\frac{2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^4}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx}$$

with, according to (143),

$$\mathcal{K}:=\frac{L_{0,\Omega}}{t_N^{\frac{\alpha_+(\gamma)}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^4}\frac{\tilde{u}_N(z)^{2^\star(s)-1}}{|z|^{s+\alpha_-(\gamma)}}\,dz=\frac{L_{0,\Omega}}{t_N^{\frac{2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^4}\frac{\tilde{u}_N(z)^{2^\star(s)-1}}{|z|^s}\,dz$$

since $\alpha_+(\gamma)=n-2=2$ and $\alpha_-(\gamma)=0$. We first compute $L_{0,\Omega}$. Since for $x,y\to 0$, we have that the Green's function for $-\Delta-h_{0,\lambda}$ behaves like $((n-2)\omega_{n-1})^{-1}|x-y|^{2-n}$ (see Robert [24]), we get that

$$L_{0,\Omega}=\frac{1}{(n-2)\omega_{n-1}}=\frac{1}{2\omega_3}.$$

Arguing as in the proof of Lemma 5 and using the same notations, we get that

$$\int\limits_{\mathbb{R}^4}|x|^2\frac{|\tilde{u}_N|^{2^\star(s)}}{|x|^s}\,dx=\frac{\mu^{2^\star(s)}\nu^2\omega_3}{2-s}I_{p+2}^p\text{ with }p=\frac{4}{2-s}$$

Integrating by parts as in Jaber [21], we get that for any $q\geq 0$

$$\left(1-\frac{q}{p+1}\right)\int_0^R\frac{t^{p+1}\,dt}{(1+t)^{q+1}}+\int_0^R\frac{t^p\,dt}{(1+t)^{q+1}}=\frac{R^{p+1}}{(p+1)(1+R)^{p+1}}\text{ for all }R>0.$$

Taking $q=p+1$ and letting $R\to+\infty$ yields $I_{p+2}^p=\frac{1}{p+1}$. With the same notations and the value of $-\Delta U$ given in the proof of Lemma 5, we have that

$$\begin{aligned}\int\limits_{\mathbb{R}^4}\frac{\tilde{u}_N(z)^{2^\star(s)-1}}{|z|^s}\,dz &= \mu\nu^{\frac{n-2}{2}}\int_{\mathbb{R}^4}(-\Delta U)\,dz=\lim_{R\to+\infty}\mu\nu^{\frac{n-2}{2}}\int_{B(0,R)}(-\Delta U)\,dz\\ &= \lim_{R\to+\infty}\mu\nu^{\frac{n-2}{2}}\int_{B(0,R)}-\partial_\nu U\,d\sigma=\mu\nu^{\frac{n-2}{2}}(n-2)\omega_{n-1}\\ &= 2\mu\nu\omega_3\end{aligned}$$

Putting these expressions together, we get that

$$\frac{\mathcal{C}'_{n,s}}{b(0)}\cdot c_4+\frac{\mathcal{D}_{n,s}}{b(0)}\cdot 4(2^\star(s)+2)=\frac{4\mu^2\nu^2\omega_3}{t_N^{\frac{4}{2-s}}\frac{4-s}{2^\star(s)^2}b(0)\sum\limits_{i=1}^N\frac{1}{t_i^{\frac{2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^4}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx}\cdot\lambda$$

and then there exists a constant $\kappa_1>0$ such that

$$\lim_{\epsilon\to 0}\frac{p_\epsilon}{\mu_{N,\epsilon}^2\ln\frac{1}{\mu_{N,\epsilon}}}=-\kappa\cdot\lambda.$$

Therefore we get compactness when $\lambda>0$.

• Assume that $\gamma>0$, so that $\alpha_+(\gamma)-\alpha_-(\gamma)<2$. We choose $\theta\in(0,2)$ such that $\alpha_+(\gamma)-\alpha_-(\gamma)<2-\theta$. Then point (4) of Proposition 7 applies. Compactness for the sequence positive subcritical solutions then follows from the positivity of the mass as in the proof of Theorem 1.8. So in this case we get existence of minimizers when $\gamma>0$ and the mass $m_{\gamma,\lambda}(\Omega_{\mathbb{B}^n})$ is positive. This proves point (2) of Theorem 1.4 for $n=4$.

Case 3: $n = 3$**.** We prove point (2) of Theorem 1.4 for $n = 3$. In this case, by Lemma 1 we have $h_{\gamma,\lambda} \in C^1\left(\overline{\Omega} \setminus \{0\}\right)$ and $h_{\gamma,\lambda}(r) = \frac{4\gamma}{r} + c_3 + O(r)$ as $r \to 0$. Then $\theta = 1$ and $K_{h_{\gamma,\lambda}} = 4\gamma$ here. Moreover, we have $(x, \nabla b(x)) = C_2|x| + O(|x|^2)$ with $C_2 > 0$.

• Assume that $\gamma < 0$, so that $\alpha_+(\gamma) - \alpha_-(\gamma) > 1$. Since $\theta = 1$, it follows from Propositions 8 and 9 and $C_2 := (2^\star(s) + 2)b(0) > 0$ that

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}} = -\frac{\mathcal{E}_{n,s}}{b(0)} \cdot \left(\frac{16\gamma}{8-s} + \tilde{K}\right)$$

for some $\tilde{K} \geq 0$. Since $\gamma < 0$, we cannot conclude.

• Assume that $\gamma = 0$, so that $\alpha_+(\gamma) - \alpha_-(\gamma) = 1$. It follows from Lemma 1 that we can take $\theta = 0$. It then follows from Proposition 10 that

$$\lim_{\epsilon\to 0} \frac{p_\epsilon}{\mu_{N,\epsilon}^{\alpha_+(\gamma)-\alpha_-(\gamma)}} = -\chi_1 \cdot m_{0,h_0}(\Omega) - \chi_2 \cdot C_2$$

Compactness for the sequence positive subcritical solutions then follows from the positivity of the mass.

• Assume that $\gamma > 0$, so that $\alpha_+(\gamma) - \alpha_-(\gamma) < 1$. Then point (4) of Proposition 9 applies. Compactness for the sequence positive subcritical solutions then follows from the positivity of the mass as in the proof of Theorem 1.8. So in this case we get existence of minimizers when $\gamma > 0$ and the mass $m_{\gamma,\lambda}(\Omega_{\mathbb{B}^n})$ is positive. This proves point (2) of Theorem 1.4 for $n = 3$.

A quick analysis as above yields no compactness result for sign-changing solutions for $n = 3, 4$. To be more precise, when estimating the different terms in the Pohozaev identity, we get compactness when a combination of several terms is positive: however, it is not possible to give an apriori criterion to get the positivity.

12. Proof of the Non-existence result

Proof of Theorem 1.9: We argue by contradiction. Let $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$ be such that $-\Delta - \gamma|x|^{-2} - h_0$ is coercive and let b be such that (34) holds. We fix $\gamma < \frac{(n-2)^2}{4}$ and $\Lambda > 0$. We assume that there is a family $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$ of solutions to

$$\left\{\begin{array}{ll} -\Delta u_\epsilon - \gamma\frac{u_\epsilon}{|x|^2} - h_\epsilon u_\epsilon = b(x)\frac{u_\epsilon^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u_\epsilon > 0 & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{array}\right. \tag{190}$$

with $\|\nabla u_\epsilon\|_2 \leq \Lambda$, and (32), (33) holds.

We claim that $(u_\epsilon)_{\epsilon>0}$ is not pre-compact in $H^2_{1,0}(\Omega)$. Otherwise, up to extraction, there would be $u_0 \in H^2_{1,0}(\Omega)$, $u_0 \geq 0$, such that $u_\epsilon \to u_0$ in $H^2_{1,0}(\Omega)$ as $\epsilon \to 0$. Passing to the limit in the equation, we get that $u_0 \geq 0$ and

$$\left\{\begin{array}{ll} -\Delta u_0 - \gamma\frac{u_0}{|x|^2} - h_0 u_0 = b(x)\frac{u_0^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u_0 \geq 0 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{191}$$

The coercivity of $-\Delta u_0 - \gamma|x|^{-2} - h_0$ and the convergence of $(h_\epsilon)_\epsilon$ yields

$$C\left(\int_\Omega \frac{u_\epsilon^{2^\star(s)}}{|x|^s}dx\right)^{2/2^\star(s)} \leq \int_\Omega |\nabla u_\epsilon|^2\,dx - \int_\Omega \left(\frac{\gamma}{|x|^2} + h_\epsilon\right)u_\epsilon^2\,dx \leq \int_\Omega b(x)\frac{u_\epsilon^{2^\star(s)}}{|x|^s}dx,$$

for some constant $C > 0$ and $\epsilon > 0$ small, and then, since $u_\epsilon > 0$, there exists $c_0 > 0$ such that

$$\int_\Omega \frac{u_\epsilon^{2^\star(s)}}{|x|^s} dx \geq c_0$$

for all $\epsilon > 0$ small. Passing to the limit yields $u_0 \not\equiv 0$. Therefore, $u_0 > 0$ is a solution to (190) with $\epsilon = 0$. This is not possible by the hypothesis.

The family $(u_\epsilon)_\epsilon$ is therefore not pre-compact and hence it blows-up with bounded energy. Let $u_0 \in H^2_{1,0}(\Omega)$ be its weak limit, which is necessarily a solution to (191), and hence must be the trivial solution $u_0 \equiv 0$. Proposition 5 then yields that either

$$\begin{aligned} & \alpha_+(\gamma) - \alpha_-(\gamma) \geq 2 - \theta \text{ and therefore } K_{h_0} = 0, \\ \text{or} \quad & \alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta \text{ and therefore } m_{\gamma,h_0}(\Omega) = 0. \end{aligned} \tag{192}$$

Since $\gamma > \frac{(n-2)^2}{4} - \left(1 - \frac{\theta}{2}\right)^2$ here, this contradicts our assumption on the mass. Therefore no such a family of positive solutions $(u_\epsilon)_{\epsilon>0}$ exists, which proves the theorem. □

Proof of Corollary 1: First note that if h_0 satisfies

$$h_0(x) + \frac{1}{2}(\nabla h_0(x), x) \leq 0 \text{ for all } x \in \Omega, \tag{193}$$

then by differentiating for any $x \in \Omega$, the function $t \mapsto t^2 h_0(tx)$ (which is well defined for $t \in [0,1]$ since Ω is starshaped), we get that $h_0 \leq 0$. Therefore $-\Delta - \gamma|x|^{-2} - h_0$ is coercive.
Assume now there is a positive variational solution u_0 of (190) corresponding to $\epsilon = 0$. The Pohozaev identity (194) then gives

$$\int_{\partial\Omega} (x,\nu)\frac{(\partial_\nu u_0)^2}{2}\, d\sigma - \int_\Omega \left(h_0 + \frac{1}{2}(\nabla h_0, x)\right) u_0^2 \, dx = 0.$$

Hopf's strong comparison principle yields $\partial_\nu u_0 < 0$. Since Ω is starshaped with respect to 0, we get that $(x,\nu) \geq 0$ on $\partial\Omega$. Therefore, with (193), we get that $(x,\nu) = 0$ for all $x \in \Omega$, which is a contradiction since Ω is smooth and bounded. Since $\alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta$, we use Theorem 7.1 in Ghoussoub-Robert [18] to find $\mathcal{K} \in C^2(\overline{\Omega} \setminus \{0\})$ and $A > 0$ such that

$$\begin{cases} -\Delta\mathcal{K} - \frac{\gamma}{|x|^2}\mathcal{K} - h_0\mathcal{K} = 0 & \text{in } \Omega \\ \mathcal{K} > 0 & \text{in } \Omega \\ \mathcal{K} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

and such that

$$\mathcal{K}(x) = A\left(\frac{\eta(x)}{|x|^{\alpha_+(\gamma)}} + \beta(x)\right) \text{ for all } x \in \Omega,$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$ and $\beta \in H^2_{1,0}(\Omega)$ are as in Step P14. We now apply the Pohozaev identity (194) to $\mathcal{K}$ on the domain $U := B_\delta(0)$. Using that $\mathcal{K}^2 \in L^1(\Omega)$ and $(\cdot,\nu)(\partial_\nu\mathcal{K})^2 \in L^1(\partial\Omega)$ when $\alpha_+(\gamma) - \alpha_-(\gamma) < 2 - \theta$, we get that

$$\int_{\partial\Omega} (x,\nu)\frac{(\partial_\nu \mathcal{K})^2}{2}\, d\sigma - \int_\Omega \left(h_0 + \frac{1}{2}(\nabla h_0, x)\right) \mathcal{K}^2 \, dx = M_\delta$$

where M_δ is defined in (175). With (177), we then get

$$\int_{\partial\Omega} (x,\nu)\frac{(\partial_\nu \mathcal{K})^2}{2}\, d\sigma - \int_\Omega \left(h_0 + \frac{1}{2}(\nabla h_0, x)\right) \mathcal{K}^2 \, dx = -2\omega_{n-1}\left(\frac{(n-2)^2}{4} - \gamma\right) A^2 \cdot m_{\gamma,h_0}(\Omega).$$

Since Ω is star-shaped and h_0 satisfies (193), it follows that $m_{\gamma,h_0}(\Omega)<0$ and Theorem 1.9 then applies to complete our corollary.

13. Appendices

Appendix A: Pohozaev identity. We let $U\subset\mathbb{R}^n$ be a smooth bounded domain in $\mathbb{R}^n$. Let $h\in C^1(\overline{U})$ and let $b\in C^1(\overline{U})$. We suppose $u\in C^2(\overline{U})$ and $p\in[0,2^\star(s)-2)$. The classical Pohozaev identity and integration by parts yields for any $y_0\in\mathbb{R}^n$

$$\begin{aligned}
&\int_U\left((x-y_0)^i\partial_i u+\frac{n-2}{2}u\right)\left(-\Delta u-\gamma\frac{u}{|x|^2}-hu-b(x)\frac{|u|^{2^\star(s)-2-p}}{|x|^s}u\right)\,dx \\
&-\int_U h(x)u^2\,dx-\frac{1}{2}\int_U(\nabla h,x-y_0)\,u^2\,dx-\frac{p}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p}\right)\int_U b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\,dx \\
&-\frac{1}{2^\star(s)-p}\int_U(x-y_0,\nabla b(x))\,\frac{|u|^{2^\star(s)-p}}{|x|^s}\,dx-\gamma\int_U\frac{(x,y_0)}{|x|^4}u^2dx \\
&-\frac{s}{2^\star(s)-p}\int_U\frac{(x,y_0)}{|x|^{s+2}}b(x)|u|^{2^\star(s)-p}\,dx \\
&=\int_{\partial U}\left[(x-y_0,\nu)\left(\frac{|\nabla u|^2}{2}-\frac{\gamma}{2}\frac{u^2}{|x|^2}-\frac{h(x)}{2}u^2-\frac{b(x)}{2^\star(s)-p}\frac{|u|^{2^\star(s)-p}}{|x|^s}\right)\right]d\sigma \\
&\quad-\int_{\partial U}\left[\left((x-y_0)^i\partial_i u+\frac{n-2}{2}u\right)\partial_\nu u\right]d\sigma,
\end{aligned}\tag{194}$$

where ν is the outer normal to the boundary ∂U.

Appendix B: Regularity. As to the regularity of the solutions, this will follow from the following result established by Ghoussoub-Robert in [18, 19]. Assuming that $\gamma<\frac{(n-2)^2}{4}$, note that the function $x\mapsto|x|^{-\alpha}$ is a solution of

$$(-\Delta-\frac{\gamma}{|x|^2})u=0\qquad\text{on }\mathbb{R}^n\setminus\{0\},\tag{195}$$

if and only if $\alpha\in\{\alpha_+(\gamma),\alpha_-(\gamma)\}$, where

$$\alpha_\pm(\gamma):=\frac{n-2}{2}\pm\sqrt{\frac{(n-2)^2}{4}-\gamma}.\tag{196}$$

Actually, one can show that any non-negative solution $u\in C^2(\mathbb{R}^n\setminus\{0\})$ of (195) is of the form

$$u(x)=C_-|x|^{-\alpha_-(\gamma)}+C_+|x|^{-\alpha_+(\gamma)}\text{ for all }x\in\mathbb{R}^n\setminus\{0\},\tag{197}$$

where $C_-,C_+\geq 0$.

We collect the following important results from the papers [18, 19] which we shall use repeatedly in our work.

Theorem 13.1 (Optimal regularity and Hopf Lemma). *Let $\gamma<\frac{(n-2)^2}{4}$ and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function such that*

$$|f(x,v)|\leq C|v|\left(1+\frac{|v|^{2^\star(s)-2}}{|x|^s}\right)\text{ for all }x\in\Omega\text{ and }v\in\mathbb{R}.$$

(1) Let $u \in H^2_{1,0}(\Omega)$ *be a weak solution of*

$$\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x,u), \tag{198}$$

for some $\tau > 0$. *Then, there exists* $K \in \mathbb{R}$ *such that*

$$\lim_{x\to 0} \frac{u(x)}{|x|^{-\alpha_-(\gamma)}} = K. \tag{199}$$

Moreover, if $u \geq 0$ *and* $u \not\equiv 0$, *we have that* $K > 0$.

(2) As a consequence, one gets that if $u \in D^{1,2}(\mathbb{R}^n)$ *is a weak solution for*

$$-\Delta u - \frac{\gamma}{|x|^2} u = \frac{|u|^{2^\star(s)-2}u}{|x|^s} \qquad \text{in } \mathbb{R}^n \setminus \{0\},$$

then there exists $K_1, K_2 \in \mathbb{R}$ *such that*

$$u(x) \sim_{|x|\to 0} \frac{K_1}{|x|^{\alpha_-(\gamma)}} \qquad \text{and} \qquad u(x) \sim_{|x|\to+\infty} \frac{K_1}{|x|^{\alpha_+(\gamma)}},$$

and therefore there exists a constant $C > 0$ *such that for all* x *in* $\mathbb{R}^n \setminus \{0\}$,

$$|u(x)| \leq \frac{C}{|x|^{\alpha_-(\gamma)} + |x|^{\alpha_+(\gamma)}}. \tag{200}$$

Via the conformal transformation, the above regularity result will yield that the corresponding solutions for equation (9) in the Hyperbolic Sobolev Space $H^1_0(\Omega_{\mathbb{B}^n})$ will satisfy

$$\lim_{|x|\to 0} \frac{u(x)}{G(|x|)^{\alpha_-}} = K \in \mathbb{R}, \tag{201}$$

where

$$\alpha_-(\gamma) = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\gamma}{(n-2)^2}}, \tag{202}$$

which amounts to the regularity claimed in Theorem 1.4.

Appendix C: Green's function. The next theorem describes the properties of the Green's function of the Hardy-Schrödinger operator in a bounded smooth domain. To prove this theorem one argue as in the case $\theta = 0$ in [19].

Theorem 13.2 (Green's function)**.** *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^n$ *such that* $0 \in \Omega$ *is an interior point. Let* $\gamma < \frac{(n-2)^2}{4}$, $0 \leq \theta < 2$ *and let* $h_0 \in C^1(\overline{\Omega}, |x|^{-\theta})$ *be such that the operator* $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ *is coercive in* Ω. *Then there exists*

$$G : (\Omega \setminus \{0\})^2 \setminus \{(x,x)/\, x \in \Omega \setminus \{0\}\} \to \mathbb{R}$$

such that

(i) *For any* $p \in \Omega \setminus \{0\}$, $G_p := G(p,\cdot) \in H^2_1(\Omega \setminus B_\delta(p))$ *for all* $\delta > 0$, $G_p \in C^{2,\theta}(\overline{\Omega} \setminus \{0,p\})$

(ii) *For all* $f \in L^p(\Omega)$, $p > n/2$, *and all* $\varphi \in H^2_{1,0}(\Omega)$ *such that*

$$-\Delta\varphi - \left(\frac{\gamma}{|x|^2} + h(x)\right)\varphi = f \text{ in } \Omega \,;\; \varphi_{|\partial\Omega} = 0,$$

we have

$$\varphi(p) = \int_\Omega G(p,x) f(x)\, dx \tag{203}$$

In addition, $G > 0$ *is unique and*

(iii) *For all $p\in\Omega\setminus\{0\}$, there exists $C_0(p)>0$ such that*

$$G_p(x)\sim_{x\to 0}\frac{C_0(p)}{|x|^{\alpha_-(\gamma)}}\text{ and }G_p(x)\sim_{x\to p}\frac{1}{(n-2)\omega_{n-1}|x-p|^{n-2}} \tag{204}$$

(iv) *There exists $C>0$ such that*

$$0<G_p(x)\leq C\left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}}\right)^{\alpha_-(\gamma)}|x-p|^{2-n} \tag{205}$$

(v) *For all $\omega\Subset\Omega$, there exists $C(\omega)>0$ such that*

$$C(\omega)\left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}}\right)^{\alpha_-(\gamma)}|x-p|^{2-n}\leq G_p(x)\text{ for all }p,x\in\omega\setminus\{0\}. \tag{206}$$

(v) *There exists $L_{\gamma,\Omega}>0$ such that for any $(h_i)\in C^1(\overline{\Omega},|x|^{-\theta})$ satisfying*

$$\lim_{i\to+\infty}\sup_{x\in\Omega}|x|^\theta|h(x)-h_0(x)|+|x|^{\theta+1}|\nabla(h-h_0)(x)|<\epsilon,$$

and for any sequences of points $(x_i)_i,(y_i)_i\in\Omega$ with

$$y_i=o(|x_i|)\text{ and }x_i=o(1)\text{ as }i\to+\infty,$$

we have as $i\to+\infty$

$$G_{h_i}(x_i,y_i)=\frac{L_{\gamma,\Omega}+o(1)}{|x_i|^{\alpha_+(\gamma)}|y_i|^{\alpha_-(\gamma)}}. \tag{207}$$

References

[1] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), no. 1, 47–92.

[2] Haïm Brezis and Louis Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical exponents*, Comm. Pure Appl. Math **36** (1983), 437-477.

[3] D. Cao, S. Peng, *A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms.* J. Differential Equations **193** (2003), 424-434.

[4] D. Cao, S. Yan, *Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential.* Calc. Var. Partial Differential Equations **38** (2010), 471-501.

[5] F. Catrina, R. Lavine, *Radial solutions for weighted semilinear equations*, Commun. Contemp. Math. 4 (2002), no. 3, 529–545.

[6] Hardy Chan, Nassif Ghoussoub, Saikat Mazumdar, Shaya Shakerian, and Luiz Fernando de Oliveira Faria, *Mass and extremals associated with the Hardy-Schrödinger operator on hyperbolic space*, Adv. Nonlinear Stud. **18** (2018), no. 4, 671–689.

[7] K.S. Chou and C. W. Chu, *On the best constant for a weighted Sobolev-Hardy inequality*, J. London Math. Soc. (2) **48** (1993), 137–151.

[8] G. Devillanova, S. Solimini, *Concentration estimates and muliple solutions to ellipic problems at critical growth.* Adv. Differential Equations **7** (2002),1257-1280.

[9] Olivier Druet, *Elliptic equations with critical Sobolev exponents in dimension 3*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), no. 2, 125–142.

[10] ———, *Optimal Sobolev inequalities and extremal functions. The three-dimensional case*, Indiana Univ. Math. J. **51** (2002), no. 1, 69–88.

[11] Olivier Druet, Emmanuel Hebey, and Paul Laurain, *Stability of elliptic PDEs with respect to perturbations of the domain*, J. Differential Equations **255** (2013), no. 10, 3703–3718.

[12] Olivier Druet and Paul Laurain, *Stability of the Pohožaev obstruction in dimension 3*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1117–1149.

[13] P. Esposito, N. Ghoussoub, A. Pistoia, G. Vaira, *Sign-changing solutions for critical equations with Hardy potential*, Analysis and PDE, Vol 14, No 2 (2021) p. 1-38

[14] Nassif Ghoussoub, *Duality and Perturbation Methods in Critical Point Theory*, Cambridge Tracts in Mathematics, Cambridge University Press, 1993.

[15] Nassif Ghoussoub, Saikat Mazumdar, and Frédéric Robert, *Multiplicity and stability of the Pohozaev obstruction for Hardy-Schrödinger equations with boundary singularity*, Memoirs of the AMS (to appear in 2021).

[16] Nassif Ghoussoub and Frédéric Robert, *The effect of curvature on the best constant in the Hardy-Sobolev inequalities*, Geom. Funct. Anal. **16** (2006), no. 6, 1201–1245.

[17] Nassif Ghoussoub and Frédéric Robert, *Concentration estimates for Emden-Fowler equations with boundary singularities and critical growth*, International Mathematics Research Papers **2006** (2006), no. 2187, 1– 86.

[18] Nassif Ghoussoub and Frédéric Robert, *Hardy-Singular Boundary Mass and Sobolev-Critical Variational Problems*, Analysis and PDE **10** (2016), no. 5, 1017-1079.

[19] Nassif Ghoussoub and Frédéric Robert, *The Hardy-Schrödinger operator with interior singularity: The remaining cases*, Calculus of Variations and Partial Differential Equations **56** (2017), no. 5, 1-49.

[20] Nassif Ghoussoub and Frédéric Robert, *Sobolev inequalities for the Hardy–Schrödinger operator: extremals and critical dimensions*, Bull. Math. Sci. **6** (2016), no. 1, 89–144.

[21] Hassan Jaber, *Hardy-Sobolev equations on compact Riemannian manifolds*, Nonlinear Anal. **103** (2014), 39–54.

[22] Enrico Jannelli, *The Role played by Space Dimensions in Elliptic Critical Problems*, Jour. Diff. Eq. **156** (1999), 407426.

[23] S. Peng, C. Wang, *Infinitely many solutions for Hardy-Sobolev equation involving critical growth.* Math. Meth. Appl. Sci. 2013. DOI: 10.1002/mma.3060

[24] Frédéric Robert, *Existence et asymptotiques optimales des fonctions de Green des opérateurs elliptiques d'ordre deux (Existence and optimal asymptotics of the Green's functions of second-order elliptic operators)* (2010). Unpublished notes.

[25] Kunnath Sandeep and Cyril Tintarev, *A subset of Caffarelli-Kohn-Nirenberg inequalities in the hyperbolic space* $\mathbb{H}^N$, Ann. Mat. Pura Appl. (4) **196** (2017), no. 6, 2005–2021.

[26] C. Wang, J. Wang, *Infinitely many solutions for Hardy-Sobolev-Maz'ya equation involving critical growth.* Commun. Contemp. Math. **14** (2012), 1250044, 38 pp.

[27] C. Wang, J. Yang, *Infinitely many solutions for an elliptic problem with double critical Hardy-Sobolev-Maz'ya terms.* accepted by Discrete and Continuous Dynamical Systems. Series A

[28] S. Yan, J. Yang, *Infinitely many solutions for an elliptic problem involving critical Sobolev and Hardy-Sobolev exponents.* Calc. Var. Partial Differential Equations **48** (2013), 587-610.

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