About *Categories and Sheaves*

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The last version of this text is available at

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Contents

The purpose of this text is to make a few comments about the book

Categories and Sheaves by Kashiwara and Schapira, Springer 2006,

referred to as "the book" henceforth.

An important reference is

[GV] Grothendieck, A. and Verdier, J.-L. (1972). Préfaisceaux. In Artin, M., Grothendieck, A. and Verdier, J.-L., editors, Théorie des Topos et Cohomologie Étale des Schémas, volume 1 of Séminaire de géométrie algébrique du Bois-Marie, 4, pages 1-218. Springer.

http://www.normalesup.org/∼forgogozo/SGA4/01/01.pdf

Here are two useful links:

Schapira's Errata:
https://webusers.imj-prg.fr/∼pierre.schapira/books/Errata.pdf,

nLab entry: http://ncatlab.org/nlab/show/Categories+and+Sheaves.

The tex and pdf files for this text are available at

http://www.iecl.univ-lorraine.fr/∼Pierre-Yves.Gaillard/DIVERS/KS/

The tex file is available at

https://github.com/Pierre-Yves-Gaillard/acs

https://goo.gl/eJxVyj

More links are available at http://goo.gl/df2Xw.

I have rewritten some of the proofs in the book. Of course, I'm not suggesting that my wording is better than that of Kashiwara and Schapira! I just tried to make explicit a few points which are implicit in the book.

The notation of the book will be freely used. We will sometimes write $\mathcal{B}^{\mathcal{A}}$ for $\mathrm{Fct}(\mathcal{A},\mathcal{B})$, α_i for $\alpha(i)$, fg for $f\circ g$, and some parenthesis might be omitted. We write $\bigsqcup$ instead of $\coprod$ for the coproduct.

Following a suggestion of Pierre Schapira's, we shall denote projective limits by lim instead of $\varprojlim$, and inductive limits by colim instead of $\varinjlim$. We sometimes use the words *limit* and *colimit* instead of the phrases *projective limit* and *inductive limit.*

Thank you to Pierre Schapira and to Olaf Schnürer for their help and thei interest!

1 U-categories and U-small Categories

Here are a few comments about the definition of a $\mathcal{U}$-category on page 11 of the book.

§ 1. First of all let us insist on the fact that, in this text, the hom-sets of a category are **not** necessarily disjoint. For more on this disjointness issue, see §7 p. 11 and §21 p. 25 below.

Let $\mathcal{U}$ be a universe. Recall that an element of $\mathcal{U}$ is called a $\mathcal{U}$-set. The following definitions are used in the book:

Definition 2 ($\mathcal{U}$-category)**.** *A* $\mathcal{U}$-category *is a category* $\mathcal{C}$ *such that, for all objects* X, Y*, the set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *of morphisms from* X *to* Y *is equipotent to some* $\mathcal{U}$*-set.*

Definition 3 ($\mathcal{U}$-small category)**.** *The category* $\mathcal{C}$ *is* $\mathcal{U}$-small *if in addition the set of objects of* $\mathcal{C}$ *is equipotent to some* $\mathcal{U}$*-set.*

One could also consider the following variant:

Definition 4 ($\mathcal{U}$-category)**.** *A* $\mathcal{U}$-category *is a category* $\mathcal{C}$ *such that, for all objects* X, Y*, the set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is a* $\mathcal{U}$*-set.*

Definition 5 ($\mathcal{U}$-small category)**.** *The category* $\mathcal{C}$ *is* $\mathcal{U}$-small *if in addition the set of objects of* $\mathcal{C}$ *is a* $\mathcal{U}$*-set. More concisely:* $\mathcal{C}$ *is* $\mathcal{U}$*-small if and only if* $\mathcal{C} \in \mathcal{U}$*.*

Definition 6 (essentially $\mathcal{U}$-small category)**.** *The category* $\mathcal{C}$ *is* essentially $\mathcal{U}$-small *if it is equivalent to a* $\mathcal{U}$*-small category.*

Note that: (a) a category $\mathcal{C}$ is a $\mathcal{U}$-category in the sense of Definition 2 if and only if there is a $\mathcal{U}$-category in the sense of Definition 4 which is isomorphic to $\mathcal{C}$; (b) a similar statement holds for $\mathcal{U}$-small categories; (c) Statement (a) would have to be modified if the hom-sets were required to be disjoint.

In this text we shall always use Definitions 4 and 5.

We often assume implicitly that a universe $\mathcal{U}$ has been chosen, and we say "category" and "small category" instead of "$\mathcal{U}$-category" and "$\mathcal{U}$-small category".

See also Section 3.2 p. 24.

2 Typos and Details

∗ P. 11, Definition 1.2.1, Condition (b): $\mathrm{Hom}(X, X)$ should be $\mathrm{Hom}_{\mathcal{C}}(X, X)$.

§ 7. Page 14, definition of $\mathrm{Mor}(\mathcal{C})$. As the hom-sets of $\mathcal{C}$ are not assumed to be disjoint, it seems better to define $\mathrm{Mor}(\mathcal{C})$ as a category of functors. See §21 p. 25.

∗ P. 25, Corollary 1.4.6. Due to the definition of $\mathcal{U}$-small category used in this text (see Section 1 p. 10), the category $\mathcal{C}_A$ of the corollary is no longer $\mathcal{U}$-small, but only canonically isomorphic to some $\mathcal{U}$-small category.

∗ P. 25, Proof of Corollary 1.4.6 (second line): $\mathrm{h}_{\mathcal{C}}$ should be $\mathrm{h}_{\mathcal{C}'}$.

∗ P. 26, Proposition 1.4.10, end of the proof: $\mathrm{Hom}_{\mathcal{C}}(Y, X) \to F(X)$ should be $\mathrm{Hom}_{\mathcal{C}}(Y, X) \to F(Y)$.

∗ P. 33, Exercise 1.19: the arrow from $L_1 \circ R_1 \circ L_2$ to L_2 should be $\eta_1 \circ L_2$ instead of $\varepsilon_1 \circ L_2$.

∗ P. 37, Remark 2.1.5: "Let I be a small set" should be "Let I be a small category".

∗ P. 39, penultimate line "exits" should be "exists".

∗ P. 47, Proposition 2.2.4 (ii): "If $Y_0 \times Y_1$ and $Y_0 \times_X Y_1$ exist in $\mathcal{C}$" should be "If $X_0 \times X_1$ and $X_0 \times_Y X_1$ exist in $\mathcal{C}$".

∗ P. 41, sixth line: (i) should be (a).

∗ P. 52, fourth line: $\mathrm{Mor}(I, \mathcal{C})$ should be $\mathrm{Fct}(I, \mathcal{C})$.

∗ P. 53, Part (i) (c) of the proof of Theorem 2.3.3 (Line 2): "$\beta \in \mathrm{Fct}(J, \mathcal{A})$" should be "$\beta \in \mathrm{Fct}(J, \mathcal{C})$".

∗ P. 54, second display: we should have $i \to \varphi(j)$ instead of $\varphi(j) \to i$.

∗ P. 58, Corollary 2.5.3: The assumption that I and J are small is not necessary. (The statement does not depend on the Axiom of Universes.)

∗ P. 58, Proposition 2.5.4: Parts (i) and (ii) could be replaced with the statement: "If two of the functors φ, ψ and $\varphi \circ \psi$ are cofinal, so is the third one".

∗ Pp. 63-64, statement and proof of Corollary 2.7.4: all the h are slanted, but they should be straight.

∗ P. 65, Exercise 2.7: see Section 4.20 p. 93 below.

$*$ P. 74, first line of the proof of Theorem 3.1.6: $\varinjlim$ should be $\varinjlim\limits_i$.

$*$ P. 74, last four lines: α should be φ.

$*$ P. 79, proof of Proposition 3.2.5: the word "filtrant" should be replaced with the word "connected".

$*$ P. 80, last display: a $\varinjlim$ is missing.

$*$ P. 83, Statement of Proposition 3.3.7 (iv) and (v): k might be replaced with R.

$*$ Pp 83 and 85, Proof of Proposition 3.3.7 (iv): "Proposition 3.1.6" should be "Theorem 3.1.6". Same typo on p. 85, Line 6.

$*$ P. 84, Proposition 3.3.13. It is clear from the proof (I think) that the intended statement was the following one: If $\mathcal{C}$ is a category admitting finite inductive limits and if $A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a functor, then we have

$$\mathcal{C} \text{ small and } \mathcal{C}_A \text{ filtrant } \Rightarrow A \text{ left exact } \Rightarrow \mathcal{C}_A \text{ filtrant.}$$

$*$ P. 88, Proposition 3.4.3 (i). It would be better to assume that $\mathcal{C}$ admits small inductive limits.

$*$ P. 89, last sentence of the proof of Proposition 3.4.4. The argument is slightly easier to follow if ψ' is factored as

$$(J_1)^{j_2} \xrightarrow{a} (J_1)^{\psi_2(j_2)} \xrightarrow{b} (K_1)^{\psi_2(j_2)} \xrightarrow{c} (K_1)^{\varphi_2(i_2)}.$$

Then a, b and c are respectively cofinal by Parts (ii), (iii) and (iv) of Proposition 3.2.5 p. 79 of the book.

$*$ P. 90, Exercise 3.2: "Proposition 3.1.6" should be "Theorem 3.1.6".

$*$ P. 115, line 4: "two morphisms $i_1, i_2 : Y \to Y \sqcup_X Y$" should be "two morphisms $i_1, i_2 : Y \rightrightarrows Y \sqcup_X Y$".

$*$ P. 115, Line 8: $i_1 \circ g = i_2 \circ g$ should be $g \circ i_1 = g \circ i_2$.

$*$ P. 120, proof of Theorem 5.2.6. We define $u' : X' \to F$ as the element of $F(X')$ corresponding to the element (u, u_0) of $F(X) \times_{F(X_1)} F(Z_0)$ under the natural bijection. (Recall $X' := X \sqcup_{X_1} Z_0$.)

$*$ P. 121, proof of Proposition 5.2.9. The fact that, in Proposition 5.2.3 p. 118 of the book, only Part (iv) needs the assumption that $\mathcal{C}$ admits small coproducts is implicitly used in the sequel of the book.

∗ P. 128, proof of Theorem 5.3.9. Last display: $\sqcup$ should be $\cup$. It would be simpler in fact to put

$$\mathrm{Ob}(\mathcal{F}_n) := \{Y_1 \sqcup_X Y_2 \mid X \to Y_1 \text{ and } X \to Y_2 \text{ are morphisms in } \mathcal{F}_{n-1}\}.$$

∗ P. 128, proof of Theorem 5.3.9„ just before the "q.e.d.": Corollary 5.3.5 should be Proposition 5.3.5.

∗ P. 132, Line 2: It would be slightly better to replace "for small and filtrant categories I and J" with "for small and filtrant categories I and J and functors $\alpha : I \to \mathcal{C}, \beta : J \to \mathcal{C}$".

∗ P. 132, Line 3: $\mathrm{Hom}_{\mathcal{C}}(A, B)$ should be $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, B)$.

∗ P. 132, Lines 4 and 5: «We may replace "filtrant and small" by "filtrant and cofinally small" in the above definition»: see Proposition 158 p. 104.

∗ P. 132, Corollary 6.1.6: The following fact is implicit. Let $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$ be functors, let X' be in $\mathcal{C}'$, and assume that G is fully faithful. Then the functor $\mathcal{C}_{X'} \to \mathcal{C}_{G(X')}$ induced by G is an isomorphism.

∗ P. 133, Proposition 6.1.9. "There exists a unique functor ..." should be "There exists a functor ... Moreover, this functor is unique up to unique isomorphism."

§ 8. P. 133. In Part (ii) of Proposition 6.1.9 the authors, I think, intended to write

$$\text{“}\varinjlim\text{”}(IF \circ \alpha) \xrightarrow{\sim} IF(\text{“}\varinjlim\text{”}\alpha)$$

instead of

$$IF(\text{“}\varinjlim\text{”}\alpha) \xrightarrow{\sim} \text{“}\varinjlim\text{”}(IF \circ \alpha).$$

∗ P. 134, proof of Proposition 6.1.12: "$\mathcal{C}_A \times \mathcal{C}_{A'}$" should be "$\mathcal{C}_A \times \mathcal{C}'_{A'}$" (twice).

∗ P. 135, Corollary 6.1.14: $f = \text{“}\varinjlim\text{”}\varphi$ should be $f \simeq \text{“}\varinjlim\text{”}\varphi$. (This is an isomorphism in $\mathrm{Mor}(\mathrm{Ind}(\mathcal{C}))$.)

§ 9. ∗ P. 135, Corollary 6.1.15: $f = \text{“}\varinjlim\text{”}\varphi$ should be $f \simeq \text{“}\varinjlim\text{”}\varphi$ and $g = \text{“}\varinjlim\text{”}\psi$ should be $g \simeq \text{“}\varinjlim\text{”}\psi$. (See Section 8.5 p. 137 below.)

∗ P. 136, proof of Proposition 6.1.16: see §128 p. 96.

∗ P. 136, proof of Proposition 6.1.18. Second line of the proof: "Corollary 6.1.14" should be "Corollary 6.1.15".

∗ P. 136, last line: "the cokernel of $(\alpha(i), \beta(i))$" should be "the cokernel of (φ_i, ψ_i)". Moreover, the cokernel in question is denoted by λ_i on the last line of p. 136 and by $\lambda(i)$ on the first line of p. 137.

∗ P. 138, second line of Section 6.2: "the functor "$\varinjlim$" is representable in $\mathcal{C}$" should be "the functor "$\varinjlim$"α is representable in $\mathcal{C}$". Next line: "natural functor" should be "natural morphism".

∗ P. 138, Proposition 6.2.1. The assumption that I is small is not really necessary. (See Section 8.7 p. 141 below.)

∗ P. 141, Display (6.3.2): $\neq$ should be $\not\simeq$ (see Section 8.1 p. 130 below).

∗ P. 141, Corollary 6.3.7 (ii): id should be $\mathrm{id}_{\mathcal{C}}$.

∗ P. 143, third line of the proof of Proposition 6.4.2: $\{Y_i\}_{I\in I}$ should be $\{Y_i\}_{i\in I}$.

∗ P. 144, proof of Proposition 6.4.2, Step (ii), second sentence: It might be better to state explicitly the assumption that X^i_ν is in $\mathcal{C}_\nu$ for $\nu = 1, 2$.

∗ P. 146, Exercise 6.3. "Let $\mathcal{C}$ be a small category" should be "Let $\mathcal{C}$ be a category".

∗ P. 146, Exercise 6.8 (ii): $(\mathrm{Mod}(A))_M$ should be $(\mathrm{Mod}(R))_M$.

∗ P. 150, before Proposition 7.1.2. One could add after "This implies that $F_{\mathcal{S}}$ is unique up to unique isomorphism": Moreover we have $Q^\dagger F \simeq F_{\mathcal{S}} \simeq Q^\ddagger F$.

∗ P. 153, statement of Lemma 7.1.12. The readability might be improved by changing $s : X \to X' \in \mathcal{S}$ to $(s : X \to X') \in \mathcal{S}$. Same for Line 4 of the proof of Lemma 7.1.21 p. 157.

∗ P. 156, first line of the first display and first line after the first display: $\mathcal{C}_{\mathcal{S}}$ should be $\mathcal{C}^r_{\mathcal{S}}$.

∗ P. 160, second line after the diagram: "commutative" should be "commutative up to isomorphism".

§ 10. P. 160, proof of Proposition 7.3.2. "$F(s)$ is an isomorphism" should be "$Q_{\mathcal{S}}(s)$ is an isomorphism" or "$G(Q_{\mathcal{S}}(s))$ is an isomorphism". In fact I would replace

"Let us check that Lemma 7.1.3 applies to $\mathcal{I} \xrightarrow{\iota} \mathcal{C} \xrightarrow{Q_{\mathcal{S}}} \mathcal{C}_{\mathcal{S}}$ and hence to $\mathcal{I} \xrightarrow{\iota} \mathcal{C} \xrightarrow{G\circ Q_{\mathcal{S}}} \mathcal{A}$. Let $X \in \mathcal{C}$. By the hypothesis, there exist $Y \in \mathcal{I}$ and $s : X \to \iota(Y)$ with $s \in \mathcal{S}$.

Therefore, $F(s)$ is an isomorphism..."

with

"Let us check that Lemma 7.1.3 applies to $\mathcal{I} \xrightarrow{\iota} \mathcal{C} \xrightarrow{G \circ Q_{\mathcal{S}}} \mathcal{A}$. Let $X \in \mathcal{C}$. By the hypothesis, there exist $Y \in \mathcal{I}$ and $s : X \to \iota(Y)$ with $s \in \mathcal{S}$. Therefore, $G(Q_{\mathcal{S}}(s))$ is an isomorphism..."

∗ P. 163, last sentence of Remark 7.4.5: "right localizable" should be "universally right localizable".

∗ P. 168, Line 9: "$f : X \to Y$" should be "$f : Y \to X$".

∗ P. 170, Corollary 8.2.4. The period at the end of the last display should be moved to the end of the sentence.

∗ P. 172, proof of Lemma 8.2.10, first line: "composition morphism" should be "addition morphism".

∗ P. 179, about one third of the page: "a complex $X \xrightarrow{u} Y \overset{v}{\underset{w}{\rightrightarrows}} Z$ " should be "a sequence $X \xrightarrow{u} Y \overset{v}{\underset{w}{\rightrightarrows}} Z$ ".

∗ P. 180, Lemma 8.3.11 (b) (i): $\operatorname{Coker} f \xrightarrow{\sim} \operatorname{Coker} f'$ should be $\operatorname{Coker} f' \xrightarrow{\sim} \operatorname{Coker} f$. Proof of Lemma 8.3.11: The notation Hom for $\operatorname{Hom}_{\mathcal{C}}$ occurs eight times. Lemma 8.3.11 is stated below as Lemma 290 p. 178.

∗ P. 181, Lemma 8.3.13, second line of the proof: $h \circ f^2$ should be $f^2 \circ h$.

§ 11. P. 184, Definitions 8.3.21 (v) and (vi). Definition 8.3.21 (vi) says that a full subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is *generating* if any object of $\mathcal{C}$ is the target of some epimorphism whose source is in $\mathcal{S}$. It seems to me this definition might create confusion with Definition 192 p. 118. For want of a better idea, I suggest to say that $\mathcal{C}$ is *a-generating* if its satisfies the above condition. (The letter a stands for the word "abelian", the reason being that this notion seems to be only used for abelian categories.) The notion of co-a-generating is defined in the obvious way.

∗ P. 186, Corollary 8.3.26. The proof reads: "Apply Proposition 5.2.9". One could add: "and Proposition 5.2.3 (v)".

∗ P. 187, proof of Proposition 8.4.3. More generally, if F is a left exact additive functor between abelian categories, then, in view of the observations made on p. 183

of the book (and especially Exercise 8.17), F is exact if and only if it sends epimorphisms to epimorphisms. (A solution to the important Exercise 8.17 is given in Section 10.8.2 p. 203.)

∗ P. 188. In the second diagram $Y' \overset{l'}{\rightarrowtail} Z$ should be $Y' \overset{l'}{\rightarrowtail} X$. After the second diagram: "the set of isomorphism classes of Δ" should be "the set of isomorphism classes of objects of Δ".

∗ P. 190, proof of Proposition 8.5.5 (a) (i): all the R should be R^{op}, except for the last one.

∗ P. 191: The equality $\psi(M) = G \otimes_R M$ is used in the second display, whereas $\psi(M) = M \otimes_R G$ is used in the third display. It might be better to use $\psi(M) = M \otimes_{R^{\mathrm{op}}} G$ both times.

∗ P. 191, Proof of Theorem 8.5.8 (iii): "the product of finite copies of R" should be "the product of finitely many copies of R".

∗ P. 196, Proposition 8.6.9, last sentence of the proof of (i)⇒(ii): "Proposition 8.3.12" should be "Lemma 8.3.12".

∗ P. 201, proof of Lemma 8.7.7, first line: "we can construct a commutative diagram". I think the authors meant "we can construct an exact commutative diagram".

§ 12. P. 218, middle of the page: "$b := \inf(J \setminus A)$" should be "$b := \inf(J \setminus A')$" (the prime is missing).

∗ P. 218, proof of Lemma 9.2.5, first sentence: "Proposition 3.2.4" should be "Proposition 3.2.2".

∗ P. 220, part (ii) of the proof of Proposition 9.2.9, last sentence of the first paragraph: $s(j)$ should be $\tilde{s}(j)$. Moreover, in the last two paragraphs of the proof, it would be better to denote $j(u)$ by $i(u)$.

∗ P. 221, Lemma 9.2.15. "Let $A \in \mathcal{C}$" should be "Let $A \in \mathrm{Ind}(\mathcal{C})$".

∗ P. 224, proof of Proposition 9.3.2, line 2: "there exist maps $S \to A(G) \to S$ whose composition is the identity" should be "there exist maps $A(G) \to S$ such that the composition $S \to A(G) \to S$ is the identity of S".

∗ Pp 224-228, from Proposition 9.3.2 to the end of the section. The notation $G^{\sqcup S}$, where S is a set, is used twice (each time on p. 224), and the notation $G^{\coprod S}$ is used many times in the sequel of the section. I think the two pieces of notation have the same meaning. If so, it might be slightly better to uniformize the notation.

§ 13. P. 225, line 3: “Since N_s is a subobject of A and $\mathrm{card}(A(G)) < \pi$” should be “Since $\mathrm{card}(A(G)) < \pi$”.

$*$ P. 225, line 4: “there exists $i_0 \to i_1$ such that $N_{i_1} \to A$ is an epimorphism” should be “there exists $s : i_0 \to i$ such that $N_s \to A$ is an epimorphism”.

$*$ P. 226, four lines before the end: “By 9.3.4 (c)” should be “By (9.3.4) (c)” (the parenthesis are missing).

$*$ P. 227. The second sentence uses Proposition 165 p. 106.

$*$ P. 228, line 3: $\mathcal{C}$ should be $\mathcal{C}_\pi$.

$*$ P. 228, Corollary 9.3.6: $\varinjlim$ should be σ_π.

§ 14. P. 228: It might be better to state Part (iv) of Corollary 9.3.8 as “G is in $\mathcal{S}$”, instead of “there exists an object $G \in \mathcal{S}$ which is a generator of $\mathcal{C}$”. (Indeed, G is already mentioned in Condition (9.3.1), which is one of the assumptions of Corollary 9.3.8.)

$*$ P. 229, proof of 9.4.3 (i): it might be better to write “containing $\mathcal{S}$ strictly” (or “properly”), instead of just “containing $\mathcal{S}$”.

$*$ P. 229, proof of 9.4.4: “The category $\mathcal{C}^X$ is nonempty, essentially small ...”: the adverb “essentially” is not necessary since $\mathcal{C}$ is supposed to be small.

$*$ P. 237: “Proposition 9.6.3” should be “Theorem 9.6.3” (twice).

$*$ P. 237, proof of Corollary 9.6.6, first display: “$\psi : \mathcal{C} \to \mathcal{C}$” should be “$\psi : \mathcal{C} \to \mathcal{I}_{inj}$”.

$*$ P. 237, end of proof of Corollary 9.6.6: it might be slightly more precise to write “$X \to \iota(\psi(X)) = K^{\mathrm{Hom}_{\mathcal{C}}(X,K)}$” instead of “$X \to \psi(X) = K^{\mathrm{Hom}_{\mathcal{C}}(X,K)}$”.

$*$ P. 244, second diagram: the arrow from X' to Z' should be dotted. (For a nice picture of the octahedral diagram see p. 49 of Miličić’s text

http://www.math.utah.edu/∼milicic/Eprints/dercat.pdf.)

$*$ P. 245, beginning of the proof of Proposition 10.1.13: The letters f and g being used in the sequel, it would be better to write $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ instead of $X \to Y \to Z \to TX$.

$*$ P. 245, first display in the proof of Proposition 10.1.13: The subscript $\mathcal{D}$ is missing (three times) in $\mathrm{Hom}_{\mathcal{D}}$.

∗ P. 250, Line 1: "TR3" should be "TR2". After the second diagram, $s \circ f$ should be $f \circ s$.

∗ P. 251, right after Remark 10.2.5: "Lemma 7.1.10" should be "Proposition 7.1.10".

∗ P. 252, last five lines:

- "u is represented by morphisms $u' : \oplus_i X_i \xrightarrow{u'} Y' \xleftarrow{s} Y$" should be "$u$ is represented by morphisms $\oplus_i X_i \xrightarrow{u'} Y' \xleftarrow{s} Y$",
- v_i' should (I believe) be u_i',
- $Q(u)$ should be $Q(u')$.

∗ P. 253, Definition 10.3.1. It would be better (I think) to remove (or alter) the second sentence of the definition. (This sentence is supposed to recall Definition 7.3.1 p. 159 of the book, but it is not clear to me that the formulation in the reminder is equivalent to the one in Definition 7.3.1; moreover the formulation in Definition 7.3.1 is consistent with the way Kan extensions are defined in the book.)

∗ P. 253, sentence between Definition 10.3.2 and Proposition 10.3.3: "Note that if $F(\mathcal{N}) \subset \mathcal{N}'$, then $\mathcal{D}$ is both F-injective and F-projective." I don't understand why this is true.

∗ P. 254. The functor RF of Notation 10.3.4 coincides with the functor $R_{\mathcal{N}Q}F$ of Definition 7.3.1 p. 159 of the book.

∗ P. 257, first display: the expression $\mathcal{T}_{X_1} \oplus \mathcal{T}_{X_2}$, which occurs twice, should be replaced with $\mathcal{T}_{X_1} \times \mathcal{T}_{X_2}$.

∗ P. 266, Exercise 10.6. I think the authors forgot to assume that the top left square commutes.

∗ P. 278: The first display should start with $T''(s'')$ instead of $T''(s)$, and the second $F(X, d_Y)$ on the third line of the display should be $F(X', d_Y)$.

∗ P. 287, first display after Proposition 11.5.4: $v(X^{n,m})$ should be $v(X)^{n,m}$.

§ 15. P. 282, Definition 11.3.12. As indicated in Pierre Schapira's Errata,

$$d^n_{\mathrm{C}(F)(X)} = (-1)^n \, F(d_X^{-n-1})$$

should be replaced with

$$d^n_{\mathrm{C}(F)(X)} = (-1)^{n+1} \, F(d_X^{-n-1}).$$

The following is essentially a rewriting of the comment after Definition 11.3.12 taking the above correction into account:

We have

$$F(X[1])^n = F(X)^{n-1} = (F(X)[-1])^n,$$

$$d^n_{F(X[1])} = (-1)^{n+1}\,F(d^{-n-1}_{X[1]}) = (-1)^n\,F(d^{-n}_X),$$

$$d^n_{F(X)[-1]} = -d^{n-1}_{F(X)} = -(-1)^n\,F(d^{-n}_X) = (-1)^{n+1}\,F(d^{-n}_X),$$

$$d^n_{F(X)[-1]} = -d^n_{F(X[1])}.$$

§ 16. ∗ P. 290, Line 17: as indicated in Pierre Schapira's Errata, one should read

$$d''^{n,m} = \mathrm{Hom}_{\mathcal{C}}((-1)^{m+1}d_X^{-m-1}, Y^n).$$

∗ P. 290, Line -3: "We define the functor" should be "We define the isomorphisms of functors".

∗ P. 303, just after the diagram: "the exact sequence (12.2.2) give rise" should be "the exact sequence (12.2.2) gives rise".

∗ P. 313, third line from the bottom: it would be better to write "double complex" instead of "complex".

∗ P. 320, Display (13.1.2): we have Qis $= N^{\mathrm{ub}}(\mathcal{C})$.

∗ P. 321, Line 8: $\widetilde{\tau}^{\geq n}(X) \to \widetilde{\tau}^{\geq n}(X)$ should be $\widetilde{\tau}^{\geq n}(X) \to \tau^{\geq n}(X)$.

∗ P. 327, Lemma 13.2.4: $\mathcal{C}^+(\mathcal{I}_{\mathcal{C}})$ should be $\mathrm{C}^+(\mathcal{I}_{\mathcal{C}})$.

∗ P. 327, Proposition 13.2.46: $\mathcal{N}$ should be $\mathrm{N}(\mathcal{C})$.

∗ P. 328, Line 8: I think the authors meant "$X^i \to Z^i$ is an isomorphism for $i > n+d$" instead of "$i \geq n + d$".

∗ P. 328. After the second display the phrase "the natural isomorphism $\mathrm{Coker}\, d_M^{i-2} \to \mathrm{Ker}\, d_M^i$ is an isomorphism" should be "the natural morphism $\mathrm{Coker}\, d_M^{i-2} \to \mathrm{Ker}\, d_M^i$ is an isomorphism".

$*$ P. 330, right after Definition 13.3.1: "F admits a right derived functor on $\mathrm{K}^*(\mathcal{C})$" should be "$F$ admits a right derived functor on $\mathrm{D}^*(\mathcal{C})$".

$*$ P. 331, Remark 13.3.6 (iii): $\mathrm{C}^+(\mathcal{I})$ should be $\mathrm{C}^+(\mathcal{I}_{\mathcal{C}})$.

§ 17. P. 337, Theorem 13.4.1. The phrase "right localizable at (Y, X)" should be "universally right localizable at (Y, X), and let $\mathrm{RHom}_{\mathcal{C}}$ denote its right localization".

$*$ P. 348, proof of Lemma 14.1.2: $d_{M(X)}$ should be $d_{M(Z)}$.

$*$ P. 359, Line 3: σ should be sh.

$*$ P. 360, Line 5 of Step (ii) of the proof of Theorem 14.4.5: "Then X'' is an exact complex in $\mathrm{K}^-(\mathcal{P})$" should be (I think) "Then X'' is an exact complex in $\mathrm{K}^-(\mathcal{C})$".

$*$ P. 362, Line 8: $\mathrm{K}(G)$-projective should be G-projective (see Definition 13.4.2 p. 338 of the book).

$*$ P. 364, Step (g) of the proof of Theorem 14.4.8: $\mathcal{P}_1 = \mathrm{K}^-(\mathcal{C}_1)$ should be $\mathcal{P}_1 = \mathcal{C}_1$.

$*$ P. 365, line between the last two displays: "adjoint" should be "derived".

$*$ P. 392, Lemma 16.1.6 (ii). It would be better to write $v : C \to U$ instead of $u : C \to U$ and $t \circ v$ instead of $t \circ u$.

$*$ P. 396, proof of Lemma 16.2.4 (ii), last sentence of the proof: It would be better (I think) write "by LE2 and LE3" instead of "by Proposition 16.1.11 (ii)".

$*$ P. 401, Line 6: $B'' \to B$ should be $B'' \to B'$.

$*$ P. 406, first line of the second display: $(\mathcal{C}_Y)^\wedge$ should be $\mathcal{C}_Y$ (twice). (See §510 p. 295.)

$*$ P. 409, line 2: $\lambda \circ (\mathrm{h}^t_X)_A \simeq \mathrm{h}_A$ should be $\lambda \circ (\mathrm{h}^t_X)_A \simeq \mathrm{h}^t_A$.

§ 18. P. 410, Display (17.1.15): instead of

$$\mathrm{Hom}_{\mathrm{PSh}(X,\mathcal{A})}(F, G) \simeq \lim_{U \in \mathcal{C}_X} \mathcal{H}om_{\mathrm{PSh}(U,\mathcal{A})}(F, G)(U).$$

we should have

$$\mathrm{Hom}_{\mathrm{PSh}(X,\mathcal{A})}(F, G) \simeq \lim_{U \in \mathcal{C}_X} \mathcal{H}om_{\mathrm{PSh}(X,\mathcal{A})}(F, G)(U).$$

See §516 p. 297.

§ 19. P. 412, proof of Lemma 17.2.2 (ii), (b)⇒(a), Step (3). "Since $(f^t)^\frown(u_V)$ is an epimorphism by (2), $(f^t)^\frown(u_V)$ is a local isomorphism" should be "Since $(f^t)^\frown(u_V)$ is a local epimorphism by (2), $(f^t)^\frown(u_V)$ is a local isomorphism".

∗ P. 414, line before the last display: $h_X^\ddagger F$ should be $\mathrm{h}_X^\ddagger\, F$, *i.e.* the h should be straight, not slanted.

∗ P. 417, first sentence of the paragraph containing Display (17.4.2): $A, A' \in \mathcal{C}^\wedge$ should be $A, A' \in \mathcal{C}_X^\wedge$.

∗ P. 418, last display:

$$\varinjlim : \varinjlim_{(B\to A)\in\mathcal{LI}_A} F(B) \to \varinjlim_{(B\to A)\in\mathcal{LI}_A} F^b(B)$$

should be

$$\varinjlim_{(B\to A)\in\mathcal{LI}_A} : \varinjlim_{(B\to A)\in\mathcal{LI}_A} F(B) \to \varinjlim_{(B\to A)\in\mathcal{LI}_A} F^b(B).$$

∗ P. 419, second line: "applying Corollary 2.3.4 to $\theta = \mathrm{id}_{\mathcal{LI}_A}$" should be "applying Corollary 2.3.4 to $\varphi = \mathrm{id}_{\mathcal{LI}_A}$".

∗ P. 421, Theorem 17.4.7 (i): $(h_X^\ddagger F)^b \simeq (h_X^\ddagger F^a)$ should be $(\mathrm{h}_X^\ddagger\, F)^b \simeq (\mathrm{h}_X^\ddagger\, F^a)$, *i.e.* the h's should be straight, not slanted.

∗ P. 424, proof of Theorem 17.5.2 (iv). "The functor $f^\dagger$ is left exact" should be "The functor $f^\dagger$ is exact". (See §512 p. 296.)

∗ P. 426, Line 5: "morphism of sites by" should be "morphism of sites".

∗ P. 428, Notation 17.6.13 (i). "For $M \in \mathcal{A}$, let us denote by M_A the sheaf associated with the constant presheaf $\mathcal{C}_X \ni U \mapsto M$" should be

"For $M \in \mathcal{A}$, let us denote by M_A the sheaf over $\mathcal{C}_A$ associated with the constant presheaf $\mathcal{C}_A \ni (U \to A) \mapsto M$".

It might also be worth mentioning that M_A is called the *constant sheaf over A with stalk M*.

∗ P. 437, Line 3 of Step (ii) of the proof of Lemma 18.1.5: It might be better to write "$\bigoplus_{s\in A(U)} G(U \xrightarrow{s} A)$" instead of "$\coprod_{s\in A(U)} G(U \xrightarrow{s} A)$"; indeed $\bigoplus$ is more usual that $\coprod$ to denote the coproduct of k-modules.

$*$ P. 438, right after "q.e.d.": "Notations (17.6.13)" should be "Notations 17.6.13" (no parenthesis).

$*$ P. 438, bottom: One can add that we have $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, F) \simeq F$ for all F in $\mathrm{PSh}(\mathcal{R})$.

$*$ P. 439, after Definition 18.2.2: One can add that we have $F \overset{\mathrm{psh}}{\otimes}_{\mathcal{R}} \mathcal{R} \simeq F$ for F in $\mathrm{PSh}(\mathcal{R})$ and $F \otimes_{\mathcal{R}} \mathcal{R} \simeq F$ for F in $\mathrm{Mod}(\mathcal{R})$.

$*$ P. 439, Proposition 18.2.3 (ii). Here is a slightly stronger statement: If $\mathcal{R}, \mathcal{S}, \mathcal{T}$ are k_X-algebras, if F is a $(\mathcal{T} \otimes_{k_X} \mathcal{R}^{\mathrm{op}})$-module, if G is an $(\mathcal{R} \otimes_{k_X} \mathcal{S})$-module, and if H is an $(\mathcal{S} \otimes_{k_X} \mathcal{T})$-module, then there are isomorphisms

$$\mathrm{Hom}_{\mathcal{S} \otimes_{k_X} \mathcal{T}}(F \otimes_{\mathcal{R}} G, H) \simeq \mathrm{Hom}_{\mathcal{R} \otimes_{k_X} \mathcal{S}}(G, \mathcal{H}om_{\mathcal{T}}(F, H)),$$

$$\mathcal{H}om_{\mathcal{S} \otimes_{k_X} \mathcal{T}}(F \otimes_{\mathcal{R}} G, H) \simeq \mathcal{H}om_{\mathcal{R} \otimes_{k_X} \mathcal{S}}(G, \mathcal{H}om_{\mathcal{T}}(F, H)),$$

functorial with respect to F, G and H.

$*$ P. 440, last line of second display: $\mathrm{Hom}_{\mathcal{R}(U)}(G(U) \otimes_k F(U), H(U))$ should be $\mathrm{Hom}_k(F(U) \otimes_{\mathcal{R}(U)} G(U), H(U))$.

$*$ P. 440, first line of the fourth display, $\overset{\mathrm{psh}}{\otimes}_{\mathcal{R}(V)}$ should be $\otimes_{\mathcal{R}(V)}$.

$*$ P. 441. The proof of Proposition 18.2.5 uses Display (17.1.11) p. 409 of the book and Exercise 17.5 (i) p. 431 of the book (see §559 p. 317).

$*$ P. 442, first line of Step (ii) of the proof of Proposition 18.2.7: $\mathcal{H}om_{\mathcal{R}}(\mathcal{R} \otimes k_{XA}, F)$ should be $\mathcal{H}om_{\mathcal{R}}(\mathcal{R} \otimes_{k_X} k_{XA}, F)$.

$*$ P. 442, Line 3 of last display of Section 18.2: $\mathrm{j}_{A \to X!}\, \mathrm{j}_{A \to X}^{-1}$ should be $\mathrm{j}_{A \to X}^{\ddagger}\, \mathrm{j}_{A \to X*}$.

$*$ P. 442. Lemma 18.3.1 (i) follows from Proposition 17.5.1 p. 432 of the book.

$*$ P. 443, first display: On the third and fourth lines, $\mathcal{H}om_{k_X}$ should be $\mathcal{H}om_{k_Z}$.

$*$ P. 443, sentence preceding Lemma 18.3.2: $j_{A \to X}$ should be $\mathrm{j}_{A \to X}$ (the slanted j should be straight).

$*$ Pp 447-8, proof of Lemma 18.5.3: in (18.5.3) $M'|_U$ and $M|_U$ should be $M'(U)$ and $M(U)$, and, after the second display on p. 448, $s_1 \in ((\mathcal{R}^{\mathrm{op}})^{\oplus m} \otimes_{\mathcal{R}} P)(U)$ should be $s_1 \in ((\mathcal{R}^{\mathrm{op}})^{\oplus n} \otimes_{\mathcal{R}} P)(U)$.

$*$ P. 448, Proposition 18.5.4, Line 3 of the proof: $G^{\oplus I} \twoheadrightarrow M$ should be $\mathcal{G}^{\oplus I} \twoheadrightarrow M$.

$*$ P. 452, Part (i) (a) of the proof of Lemma 18.6.7. I think that $\mathcal{O}_U$ and $\mathcal{O}_V$ stand for $\mathcal{O}_X|_U$ and $\mathcal{O}_Y|_V$. (If this is so, it would be better, in the penultimate display of the page, to write $\mathcal{O}_V$ instead of $\mathcal{O}_Y|_V$.)

$*$ P. 452, a few lines before the penultimate display of the page, $f_W^{-1} : \mathcal{O}_U^{\oplus n} \xrightarrow{u} \mathcal{O}_U^{\oplus m}$ should be (I think) $f_W^{-1} : \mathcal{O}_W^{\oplus n} \to \mathcal{O}_W^{\oplus m}$.

$*$ P. 494, Index. I found useful to add the following subentries to the entry "injective": $\mathcal{F}$-injective, 231; F-injective, 253, 255, 330.

3 About Chapter 1

3.1 Universes (p. 9)

The book starts with a few statements which are not proved, a reference being given instead. Here are the proofs.

A **universe** is a set $\mathcal{U}$ satisfying

(i) $\varnothing \in \mathcal{U}$,

(ii) $u \in U \in \mathcal{U} \Rightarrow u \in \mathcal{U}$,

(iii) $U \in \mathcal{U} \Rightarrow \{U\} \in \mathcal{U}$,

(iv) $U \in \mathcal{U} \Rightarrow \mathcal{P}(U) \in \mathcal{U}$,

(v) $I \in \mathcal{U}$ and $U_i \in \mathcal{U}$ for all $i \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{U}$,

(vi) $\mathbb{N} \in \mathcal{U}$.

We want to prove:

(vii) $U \in \mathcal{U} \Rightarrow \bigcup_{u \in U} u \in \mathcal{U}$,

(viii) $U, V \in \mathcal{U} \Rightarrow U \times V \in \mathcal{U}$,

(ix) $U \subset V \in \mathcal{U} \Rightarrow U \in \mathcal{U}$,

(x) $I \in \mathcal{U}$ and $U_i \in \mathcal{U}$ for all $i \Rightarrow \prod_{i \in I} U_i \in \mathcal{U}$.

(We have kept Kashiwara and Schapira's numbering of Conditions (i) to (x).)

Obviously, (ii) and (v) imply (vii), whereas (iv) and (ii) imply (ix). Axioms (iii), (vi) and (v) imply

(a) $U, V \in \mathcal{U} \Rightarrow \{U, V\} \in \mathcal{U}$, and thus

(b) $U, V \in \mathcal{U} \Rightarrow (U, V) := \{\{U\}, \{U, V\}\} \in \mathcal{U}$.

Proof of (viii). If $u \in U$ and $v \in V$, then $\{(u, v)\} \in \mathcal{U}$ by (ii), (b) and (iii). Now (v) yields

$$U \times V = \bigcup_{u \in U} \bigcup_{v \in V} \{(u, v)\} \in \mathcal{U}. \text{ q.e.d.}$$

Assume $U, V \in \mathcal{U}$, and let V^U be the set of all maps from U to V. As $V^U \in \mathcal{P}(U \times V)$, Statements (viii), (iv) and (ii) give

(c) $U, V \in \mathcal{U} \Rightarrow V^U \in \mathcal{U}$.

Proof of (x). As

$$\prod_{i \in I} U_i \in \mathcal{P}\left(\left(\bigcup_{i \in I} U_i\right)^I\right),$$

(x) follows from (v), (c) and (iv). q.e.d.

3.2 Definition of a category (p. 11)

We slightly modify Definition 1.2.1 p. 11 as follows:

Definition 20. *A category $\mathcal{C}$ consists of:*

(i) *a set* $\mathrm{Ob}(\mathcal{C})$*,*

(ii) *for any X, Y in* $\mathrm{Ob}(\mathcal{C})$*, a set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$*,*

(iii) *for any X, Y, Z in* $\mathrm{Ob}(\mathcal{C})$*, a map:*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

called the composition and denoted by

$$(f, g) \mapsto g \circ_{ZYX} f,$$

these data satisfying:

(a) $\circ$ *is associative, i.e., for* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z), h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$*, we have*

$$(h \circ_{WZY} g) \circ_{WYX} f = h \circ_{WZX} (g \circ_{ZYX} f),$$

(b) *for each* X *in* $\mathrm{Ob}(\mathcal{C})$*, there exists* id_X *in* $\mathrm{Hom}_{\mathcal{C}}(X,X)$ *such that*

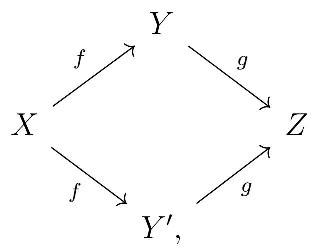
$$f \circ_{YXX} \mathrm{id}_X = f$$

for all f *in* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *and*

$$\mathrm{id}_Y \circ_{YYX} g = g$$

for all g *in* $\mathrm{Hom}_{\mathcal{C}}(Y,X)$*.*

The common practice is to abbreviated $\circ_{ZYX}$ by $\circ$. If one does that without any precaution, one may end up with an inequality of the form $g \circ f \neq g \circ f$, as suggested by the diagram



where we assume (as we may) $g \circ_{ZYX} f \neq g \circ_{ZY'X} f$. It is not clear to me which precautions one can take in order to avoid this problem. Also note that a phrase like "the morphism f is a monomorphism" doesn't make sense, and one should say instead something like "the morphism f is a monomorphism with respect to the pair of objects (X,Y)".

Another option (which would be simpler in my humble opinion) would be to impose, in the definition of a category, the condition that the Hom-sets are disjoint, and, for each category, to choose a universe $\mathcal{U}$ such that the (automatically disjoint) union of the Hom-sets is an element of $\mathcal{U}$. An argument in favor of this option would be to say that, as we want our statements to be compatible with universe enlargement, there is no harm in choosing a large enough universe at the outset.

See also Section 1 p. 10.

3.3 Brief comments

§ 21. Page 14, category of morphisms. Here are some comments about Definition 1.2.5 p. 14:

Notation 22. *For any category $\mathcal{C}$ define the category $\mathcal{C}^*$ as follows. The objects of $\mathcal{C}^*$ are the objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}^*}(X,Y)$ is defined by*

$$\mathrm{Hom}_{\mathcal{C}^*}(X,Y) := \{Y\} \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \{X\},$$

and the composition is defined by

$$(Z,g,Y) \circ (Y,f,X) := (Z, g \circ f, X).$$

Note that there are natural mutually inverse isomorphisms $\mathcal{C} \rightleftarrows \mathcal{C}^*$.

Notation 23. *Let $\mathcal{C}$ be a category. Define the category $\mathrm{Mor}(\mathcal{C})$ by*

$$\mathrm{Ob}(\mathrm{Mor}(\mathcal{C})) := \bigcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}^*}(X,Y),$$

$$\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((Y,f,X),(V,g,U)) :=$$

$$\{(a,b) \in \mathrm{Hom}_{\mathcal{C}}(X,U) \times \mathrm{Hom}_{\mathcal{C}}(Y,V) \mid g \circ a = b \circ f\},$$

i.e.

$$\begin{array}{ccc} X & \xrightarrow{a} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[b]{} & V, \end{array}$$

and the composition is defined in the obvious way.

Observe that a functor $\mathcal{A} \to \mathcal{B}$ is given by two maps

$$\mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B}), \quad \mathrm{Ob}(\mathrm{Mor}(\mathcal{A})) \to \mathrm{Ob}(\mathrm{Mor}(\mathcal{B}))$$

satisfying certain conditions.

When $\mathcal{C}$ is a small category (see Section 1 p. 10), we assume that the hom-sets of $\mathcal{C}$ are disjoint.

§ 24. P. 16, Definition 1.2.11 (iii). Note that fully faithful functors are conservative.

§ 25. P. 16. Here are some exercises.

(a) Let $\mathcal{U}$ be a universe and **Set** the category of $\mathcal{U}$-sets. Show that the only proper subfunctor of the identity functor $I : \mathbf{Set} \to \mathbf{Set}$ is the initial object of $\mathbf{Set}^{\mathbf{Set}}$.

(b) In the same setting, let $T : \mathbf{Set} \to \mathbf{Set}$ be a terminal object of $\mathbf{Set}^{\mathbf{Set}}$. Show that the only proper subfunctors of T are the initial object of $\mathbf{Set}^{\mathbf{Set}}$ and the image of the unique morphism $I \to T$.

§ 26. P. 18, Definition 1.2.16. If $F : \mathcal{C} \to \mathcal{C}'$ is a functor and X' an object of $\mathcal{C}'$, then we have natural isomorphisms

$$(\mathcal{C}_{X'})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{X'}, \quad (\mathcal{C}^{X'})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{X'}. \tag{1}$$

Also note that, if **Cat** is the category of small categories (Definition 5 p. 10), then the formula $X' \mapsto \mathcal{C}_{X'}$ defines a functor $\mathcal{C}' \to \mathbf{Cat}$, and the formula $X' \mapsto \mathcal{C}^{X'}$ defines a functor $\mathcal{C}'^{\,\mathrm{op}} \to \mathbf{Cat}$.

§ 27. P. 18, Definition 1.2.18. We define a subobject as being an element of the indicated equivalence class. Unless otherwise stated, we choose this element "at random". (Note that there a many cases in which an explicit choice is possible.) See §496 p. 290.

§ 28. P. 19. Let M be a monoid. Define the category $\mathcal{C}$ by the conditions $\mathrm{Ob}(\mathcal{C}) = \{\star\}$ and $\mathrm{End}_{\mathcal{C}}(\star) = M$.

We want to reconstruct the monoid M from the category $\mathcal{C}$.

Define

- the functor $A : \mathcal{C} \to \mathbf{Set}$ by $A(\star) := M$ and $A(m)(n) := mn$,

- the forgetful functor $U : \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}$ by $U(X) := X(\star)$ and $U(\alpha) := \alpha_\star$ for any morphism $\alpha : X \to Y$ in $\mathbf{Set}^{\mathcal{C}}$,

- the map $f : M \to \mathrm{End}(U)$ by $f(m)_X := X(m)$ for any $X : \mathcal{C} \to \mathbf{Set}$ and any m in M,

- the map $g : \mathrm{End}(U) \to M$ by $g(\theta) := \theta_A(1)$ for any endomorphism θ of U.

Then f and g are inverse monoid morphisms.

Proof. For m, n in M and $X : \mathcal{C} \to \mathbf{Set}$ we have

$$f(mn)_X = X(mn) = X(m) \circ X(n) = f(m)_X \circ f(n)_X = (f(m) \circ f(n))_X$$

This shows that f is a monoid morphism. Thus it suffices to prove that f and g are inverse bijections.

For m in M we have

$$g(f(m)) = f(m)_A(1) = A(m)(1) = m.$$

Let θ be an endomorphism of U and let's check $f(g(\theta)) = \theta$. Let $X : \mathcal{C} \to \mathbf{Set}$ be a functor and x an element of $X(\star)$. It suffices to prove

$$f(g(\theta))_X(x) = \theta_X(x).$$

We leave it to the reader to verify that the formula $\alpha_\star(m) := X(m)(x)$ defines a morphism $\alpha : A \to X$. We get

$$f(g(\theta))_X(x) = X(g(\theta))(x) = X(\theta_A(1))(x) = \alpha_\star(\theta_A(1)) = \theta_X(\alpha_\star(1)) = \theta_X(x).$$

□

§ 29. P. 19. We compute the endomorphisms of the *covariant* power set functor $P : \mathbf{Set} \to \mathbf{Set}$. Let $\varepsilon : P \to P$ be defined by $\varepsilon_A(Z) = \varnothing$ for all set A and all $Z \subset A$.

We claim

$$\text{(a)} \quad \boxed{\operatorname{End}(P) = \{\mathrm{id}_P, \varepsilon\}}$$

Set $\mathbf{1} := \{0\}, \mathbf{2} := \{0, 1\}$ and let θ be an endomorphism of P.

(b) We obviously have $\theta_\varnothing = \mathrm{id}_{P(\varnothing)}$.

(c) Using (b) and applying θ to $\varnothing \to \mathbf{1}$ we see that $\theta_\mathbf{1}(\varnothing) = \varnothing$.

• Case 1: $\theta_\mathbf{1}(\mathbf{1}) = \varnothing$.

We claim $\theta = \varepsilon$ for all set A and all $Z \subset A$, and prove the claim by applying θ to $A \to \mathbf{1}$.

• Case 2: $\theta_\mathbf{1}(\mathbf{1}) = \mathbf{1}$.

We claim

(d) $\theta = \mathrm{id}_P$.

This will imply (a).

(e) We have $\theta_\mathbf{2} = \mathrm{id}_{P(\mathbf{2})}$. [Left to the reader.]

(f) Let A be a set with at least three elements. It suffices to show $\theta_A = \mathrm{id}_{P(A)}$.

(g) If Z and Y are two distinct subsets of A, then there is an $f : A \to \mathbf{2}$ such that $P(f)(Z) \neq P(f)(Y)$.

Proof: We can assume that there is a y in $Y \setminus Z$. Letting f be the characteristic function of $Y \setminus Z$, we get $1 \in P(f)(Y) \setminus P(f)(Z)$.

(h) Let Z be a subset of A. Applying θ to all the maps from A to $\mathbf{2}$ and using (g), we see that $\theta_A(Z) = Z$. This proves (f), and thus (d), and thus (a).

§ 30. P. 19. Let A be a set and F the functor $\mathrm{Hom}(-, A)$. A *maximal subfunctor of* F shall mean a subfunctor of F which is maximal among the proper subfunctors of F. A *maximal quotient of* F shall mean a quotient of F which is maximal among the proper quotients of F. A *congruence on* F consists of an equivalence relation on each $F(X)$ such that each map $F(X) \to F(Y)$ induced by a morphism sends equivalent elements to equivalent elements. Such a congruence is called *minimal* if it is minimal among the non-discrete congruences on F. Minimal congruences correspond of course to maximal quotients. There are analogous definitions for the functor $\mathrm{Hom}(A, -)$; the details are left to the reader.

Here we want to classify the maximal subfunctors and quotients of $\mathrm{Hom}(-, A)$ and $\mathrm{Hom}(A, -)$.

The classification being easy when A has zero or one element, we assume that A has at least two elements.

There is only one maximal subfunctor of $\mathrm{Hom}(A, -)$ and of $\mathrm{Hom}(-, A)$. The maximal subfunctor of $\mathrm{Hom}(A, -)$ consists of all the *non-injective* maps $A \to X$, and the maximal subfunctor of $\mathrm{Hom}(-, A)$ consists of all the *non-surjective* maps $X \to A$. The maximal quotients of $\mathrm{Hom}(-, A)$ are attached to subsets $\{a, b\}$ of cardinality two of A by forming the least equivalence relation on $\mathrm{Hom}(X, A)$ which identifies the constant map with value a to the constant map with value b.

We now describe the maximal quotients of $\mathrm{Hom}(A, -)$, and we do this by describing the minimal congruences.

Let A, B, C, D, E and X be sets. Assume that $A = B \sqcup C \sqcup D \sqcup E$, that A has at least two elements, and that B is nonempty. Suppose also that at least one of the three sets C, D and E is nonempty. For each $x_1, x_2, x_3, x_4 \in X$ write $x_1x_2x_3x_4$ for the map from A to X which has the constant value x_1 on B, x_2 on C, x_3 on D, and x_4 on E.

Then there is a unique equivalence relation on $F(X) := \mathrm{Hom}(A, X)$ such that the equivalence class of f is $\{xyxy, yxxy\}$ if $f = xyxy$ for some distinct elements x and y of X, and is the singleton $\{f\}$ otherwise.

If C is nonempty, then the equivalence relation attached to (C, B, E, D) coincides with that attached to (B, C, D, E), and this is the only case in which there is a coincidence.

Let A, B, E and X be sets, and let y be in X. Assume that $A = B \sqcup E$, and that B is nonempty. For each u, v in X write uv for the map from A to X which has the constant value u on B and v on E.

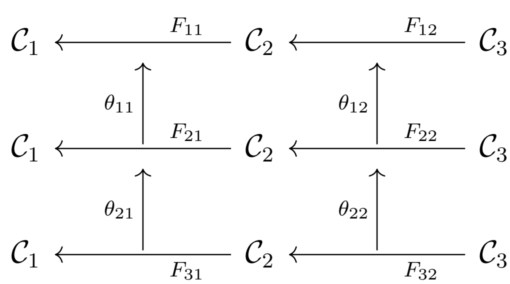
Then there is a unique equivalence relation on $F(X)$ such that the equivalence class of f is $\{zy \mid z \in X\}$ if $f = xy$ for some x in X, and is the singleton $\{f\}$ otherwise.

We claim that, when X varies, each of these equivalence relations on $F(X)$ defines a minimal congruence on F, and that there are no other minimal congruences on F.

We sketch the proof of the last statement. Let $\sim$ be a minimal congruence of F, and let $f, g \in F(X)$ satisfy $f \neq g$ and $f \sim g$. Pick an a in A such that $f(a) \neq g(a)$. Let $h : X \to \{f(a), g(a)\}$ fix $f(a)$ and $g(a)$. Then $h \circ f \neq h \circ g$ and $h \circ f \sim h \circ g$. By minimality, the congruence $\sim$ is generated by $h \circ f \sim h \circ g$. In other words, we can assume that $f(A) \cup g(A)$ has exactly two elements. From this point the proof is somewhat tedious, but straightforward. The proofs of the other statements are also straightforward.

3.4 Horizontal and vertical compositions (p. 19)

For each object X of $\mathcal{C}_3$ the diagram



of categories, functors and morphisms of functors yields the commutative diagram

$$\begin{array}{ccccc}
F_{11}F_{12}X & \xleftarrow{\theta_{11}F_{12}X} & F_{21}F_{12}X & \xleftarrow{\theta_{21}F_{12}X} & F_{31}F_{12}X \\
{\scriptstyle F_{11}\theta_{12}X}\uparrow & & {\scriptstyle F_{21}\theta_{12}X}\uparrow & & \uparrow{\scriptstyle F_{31}\theta_{12}X} \\
F_{11}F_{22}X & \xleftarrow{\theta_{11}F_{22}X} & F_{21}F_{22}X & \xleftarrow{\theta_{21}F_{22}X} & F_{31}F_{22}X \\
{\scriptstyle F_{11}\theta_{22}X}\uparrow & & {\scriptstyle F_{21}\theta_{22}X}\uparrow & & \uparrow{\scriptstyle F_{31}\theta_{22}X} \\
F_{11}F_{32}X & \xleftarrow[\theta_{11}F_{32}X]{} & F_{21}F_{32}X & \xleftarrow[\theta_{21}F_{32}X]{} & F_{31}F_{32}X
\end{array}$$

in $\mathcal{C}_1$. So, we get a well-defined morphism in $\mathcal{C}_1$ from $F_{31}F_{32}X$ to $F_{11}F_{12}X$, which is easily seen to define a morphism of functors from $F_{31}F_{32}$ to $F_{11}F_{12}$.

Notation 31. *We denote this morphism of functors by*

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} : F_{31}F_{32} \to F_{11}F_{12}.$$

If θ_{21} and θ_{22} are identity morphisms, we put

$$\theta_{11} \star \theta_{12} := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.$$

If θ_{12} and θ_{22} are identity morphisms, we put

$$\theta_{11} \circ \theta_{21} := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.$$

Let $m, n \geq 1$ be integers, let $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$ be categories, let

$$F_{i,j} : \mathcal{C}_{j+1} \to \mathcal{C}_j, \quad 1 \leq i \leq m+1,\ 1 \leq j \leq n$$

be functors, let

$$\theta_{i,j} : F_{i+1,j} \to F_{i,j}, \quad 1 \leq i \leq m,\ 1 \leq j \leq n$$

be morphisms of functors. For instance, if $m = 2, n = 4$, then we have

$$\begin{array}{ccccccccc}
\mathcal{C}_1 & \xleftarrow{F_{11}} & \mathcal{C}_2 & \xleftarrow{F_{12}} & \mathcal{C}_3 & \xleftarrow{F_{13}} & \mathcal{C}_4 & \xleftarrow{F_{14}} & \mathcal{C}_5 \\
 & {\scriptstyle \theta_{11}}\uparrow & & {\scriptstyle \theta_{12}}\uparrow & & {\scriptstyle \theta_{13}}\uparrow & & {\scriptstyle \theta_{14}}\uparrow & \\
\mathcal{C}_1 & \xleftarrow{F_{21}} & \mathcal{C}_2 & \xleftarrow{F_{22}} & \mathcal{C}_3 & \xleftarrow{F_{23}} & \mathcal{C}_4 & \xleftarrow{F_{24}} & \mathcal{C}_5 \\
 & {\scriptstyle \theta_{21}}\uparrow & & {\scriptstyle \theta_{22}}\uparrow & & {\scriptstyle \theta_{23}}\uparrow & & {\scriptstyle \theta_{24}}\uparrow & \\
\mathcal{C}_1 & \xleftarrow{F_{31}} & \mathcal{C}_2 & \xleftarrow{F_{32}} & \mathcal{C}_3 & \xleftarrow{F_{33}} & \mathcal{C}_4 & \xleftarrow{F_{34}} & \mathcal{C}_5.
\end{array}$$

The following proposition is clear

Proposition 32. *The operations $\star$ and $\circ$ are associative, and, in the above setting, we have the equality*

$$(\theta_{1,1}\star\cdots\star\theta_{1,n})\circ\cdots\circ(\theta_{m,1}\star\cdots\star\theta_{m,n})$$

$$=(\theta_{1,1}\circ\cdots\circ\theta_{m,1})\star\cdots\star(\theta_{1,n}\circ\cdots\circ\theta_{m,n}).$$

between functors from $F_{m+1,1}\cdots F_{m+1,n}$ to $F_{1,1}\cdots F_{1,n}$.

Notation 33. *We denote this morphism of functors by*

$$\begin{pmatrix}\theta_{1,1}&\cdots&\theta_{1,n}\\ \vdots&&\vdots\\ \theta_{m,1}&\cdots&\theta_{m,n}\end{pmatrix}:F_{m+1,1}\cdots F_{m+1,n}\to F_{1,1}\cdots F_{1,n}.$$

Proposition 34. *We have, in the above setting,*

$$(\theta_{1,1}\star\cdots\star\theta_{1,n})\circ\cdots\circ(\theta_{m,1}\star\cdots\star\theta_{m,n})=\begin{pmatrix}\theta_{1,1}\star\cdots\star\theta_{1,n}\\ \vdots\\ \theta_{m,1}\star\cdots\star\theta_{m,n}\end{pmatrix}$$

$$=\begin{pmatrix}\theta_{1,1}&\cdots&\theta_{1,n}\\ \vdots&&\vdots\\ \theta_{m,1}&\cdots&\theta_{m,n}\end{pmatrix}$$

$$=\begin{pmatrix}\theta_{1,1}\\ \vdots\\ \theta_{m,1}\end{pmatrix}\star\cdots\star\begin{pmatrix}\theta_{1,n}\\ \vdots\\ \theta_{m,n}\end{pmatrix}=(\theta_{1,1}\circ\cdots\circ\theta_{m,1})\star\cdots\star(\theta_{1,n}\circ\cdots\circ\theta_{m,n}).$$

Definition 35 (horizontal and vertical composition, Interchange Law)**.** *We call $\star$ the* horizontal composition*. We call $\circ$ the* vertical composition*. We call the equalities in Proposition 34 the* Interchange Law*.*

3.5 Brief comment

P. 19, Definition 1.3.16, notion of essentially small category. Here is a simple but crucial fact which is often left implicit:

The category of $\mathcal{U}$-sets is not *essentially small. More precisely, there is no $\mathcal{U}$-set which is equipotent to the set of cardinalities of $\mathcal{U}$-sets.*

Here is a sketch of a proof.

Let κ be the supremum of the cardinalities of the elements of $\mathcal{U}$. Then κ is strongly inaccessible. See

http://www.normalesup.org/∼forgogozo/SGA4/01/01.pdf

Section 5 of the appendix. Then $\kappa = \aleph_\kappa$. See

https://mathoverflow.net/a/117809/461

Hence the set of cardinals less than κ coincides with the set

$$\{\aleph_\alpha \mid \alpha \text{ ordinal less than } \kappa\},$$

whose cardinality is κ. □

For additional details, see also

https://en.wikipedia.org/wiki/Cofinality♯Regular_and_singular_ordinals

https://mathoverflow.net/a/117809/461

3.6 The Yoneda Lemma (p. 24)

We state the Yoneda Lemma for the sake of completeness:

Theorem 36 (Yoneda's Lemma). *Let $\mathcal{C}$ be a category.*

(a) *Let $h : \mathcal{C} \to \mathcal{C}^\wedge$ be the Yoneda embedding, let A be in $\mathcal{C}^\wedge$, let X be in $\mathcal{C}$, and define*

$$A(X) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{C}^\wedge}(h(X), A) \tag{2}$$

by

$$\varphi(x)_Y(f) := A(f)(x), \quad \psi(\theta) := \theta_X(\mathrm{id}_X) \tag{3}$$

for

$$x \in A(X), \quad Y \in \mathcal{C}, \quad f \in \operatorname{Hom}_{\mathcal{C}}(Y,X), \quad \theta \in \operatorname{Hom}_{\mathcal{C}^\wedge}(h(X),A):$$

$$f \in \operatorname{Hom}_{\mathcal{C}}(Y,X) \xrightarrow{\varphi(x)_Y} A(Y) \xleftarrow{A(f)} A(X) \ni x.$$

Then φ and ψ are mutually inverse bijections. In the particular case where A is equal to $h(Z)$ for some Z in $\mathcal{C}$, we get

$$\varphi(x) = h(x) \in \operatorname{Hom}_{\mathcal{C}^\wedge}(h(X),h(Z)).$$

This shows that h is fully faithful.

(b) *Let $k : \mathcal{C} \to \mathcal{C}^\vee$ be the Yoneda embedding, let A be in $\mathcal{C}^\vee$, let X be in $\mathcal{C}$, and define*

$$A(X) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \operatorname{Hom}_{\mathcal{C}^\vee}(A,k(X)) = \operatorname{Hom}_{\mathbf{Set}^{\mathcal{C}}}(k(X),A) \tag{4}$$

by (3) *for*

$$x \in A(X), \quad Y \in \mathcal{C}, \quad f \in \operatorname{Hom}_{\mathcal{C}}(X,Y), \quad \theta \in \operatorname{Hom}_{\mathbf{Set}^{\mathcal{C}}}(k(X),A):$$

$$f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\varphi(x)_Y} A(Y) \xleftarrow{A(f)} A(X) \ni x.$$

Then φ and ψ are mutually inverse bijections. In the particular case where A is equal to $k(Z)$ for some Z in $\mathcal{C}$, we get

$$\varphi(x) = k(x) \in \operatorname{Hom}_{\mathcal{C}^\vee}(k(Z),k(X)).$$

This shows that k is fully faithful.

(c) *The bijections* (2) *and* (4) *are functorial in A and X.*

Proof. (a) We have

$$\psi(\varphi(x)) = \varphi(x)_X(\mathrm{id}_X) = A(\mathrm{id}_X)(x) = x$$

and

$$\varphi(\psi(\theta))_Y(f) = A(f)(\psi(\theta)) = A(f)(\theta_X(\mathrm{id}_X)) = \theta_Y(f),$$

the last equality following from the commutativity of the square

$$\begin{array}{ccc} h(X)(Y) & \xrightarrow{\theta_Y} & A(Y) \\ {\scriptstyle h(X)(f)}\uparrow & & \uparrow{\scriptstyle A(f)} \\ h(X)(X) & \xrightarrow[\theta_X]{} & A(X), \end{array}$$

which is equal to the square

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(Y,X) & \xrightarrow{\theta_Y} & A(Y) \\ {\scriptstyle \circ f}\uparrow & & \uparrow{\scriptstyle A(f)} \\ \operatorname{Hom}_{\mathcal{C}}(X,X) & \xrightarrow[\theta_X]{} & A(X). \end{array}$$

(b) The proof of (b) is similar.

(c) Let $h:\mathcal{C}\to\mathcal{C}^\wedge$ be the Yoneda embedding, and, for X in $\mathcal{C}$ and A in $\mathcal{C}^\wedge$ let

$$\Phi_{X,A}:\operatorname{Hom}_{\mathcal{C}^\wedge}(h(X),A)\to A(X),\quad \theta\mapsto\theta_X(\mathrm{id}_X)$$

be the Yoneda bijection. We shall prove that $\Phi_{X,A}$ is functorial in X and A.

Functoriality in A: Let B be in $\mathcal{C}^\wedge$ and let $h(X)\xrightarrow{\theta}A\xrightarrow{\lambda}B$ be morphisms of functors. We must show $\lambda_X(\Phi_{X,A}(\theta))=\Phi_{X,B}((\lambda\circ)(\theta))$:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}^\wedge}(h(X),A) & \xrightarrow{\Phi_{X,A}} & A(X) \\ {\scriptstyle \lambda\circ}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ \operatorname{Hom}_{\mathcal{C}^\wedge}(h(X),B) & \xrightarrow[\Phi_{X,B}]{} & B(X). \end{array}$$

We have

$$\lambda_X(\Phi_{X,A}(\theta))=\lambda_X(\theta_X(\mathrm{id}_X)),$$

$$\Phi_{X,B}((\lambda\circ)(\theta))=\Phi_{X,B}(\lambda\circ\theta)=(\lambda\circ\theta)_X(\mathrm{id}_X)=(\lambda_X\circ\theta_X)(\mathrm{id}_X)=\lambda_X(\theta_X(\mathrm{id}_X)),$$

where the penultimate equality follows from the definition of the vertical composition of morphisms of functors (Definition 35 p. 32):

$$\operatorname{Hom}_{\mathcal{C}}(X,X)\xrightarrow{\theta_X}A(X)\xrightarrow{\lambda_X}B(X).$$

Functoriality in X: Let $f:X\to Y$ be a morphism in $\mathcal{C}$ and $\theta:h(Y)\to A$ be a morphism in $\mathcal{C}^\wedge$. We must show

$$\Phi_{X,A}\Big(\big(\circ\, h(f)\big)(\theta)\Big)=A(f)\big(\Phi_{Y,A}(\theta)\big):$$

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}^\wedge}(h(X),A) & \xrightarrow{\Phi_{X,A}} & A(X) \\ {\scriptstyle \circ h(f)}\uparrow & & \uparrow{\scriptstyle A(f)} \\ \operatorname{Hom}_{\mathcal{C}^\wedge}(h(Y),B) & \xrightarrow[\Phi_{Y,A}]{} & A(Y). \end{array}$$

We have

$$\Phi_{X,A}\Big(\big(\circ\, h(f)\big)(\theta)\Big) = \Phi_{X,A}(\theta \circ h(f)) = (\theta_X \circ h(f)_X)(\mathrm{id}_X)$$
$$= (\theta_X \circ (f\circ))(\mathrm{id}_X) = \theta_X(f),$$

where the second equality follows from the definition of the vertical composition of morphisms of functors:

$$h(X) \xrightarrow{h(f)} h(Y) \xrightarrow{\theta} A,$$
$$\mathrm{Hom}_{\mathcal{C}}(X,X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\theta_X} A(X)$$

because $h(f)_X = f\circ$. We also have

$$A(f)\big(\Phi_{Y,A}(\theta)\big) = A(f)\big(\theta_Y(\mathrm{id}_Y)\big) = \theta_X(f),$$

where the last equality follows from the naturality of θ:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \xrightarrow{\theta_X} & A(X) \\ {\scriptstyle \circ f}\uparrow & & \uparrow{\scriptstyle A(f)} \\ \mathrm{Hom}_{\mathcal{C}}(Y,Y) & \xrightarrow[\theta_Y]{} & A(Y). \end{array}$$

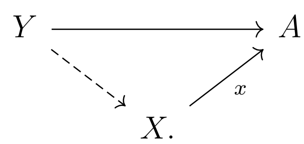
□

Corollary 37. *In the setting of Theorem 36* (a)*, p. 33, X represents A if and only if there is an x in $A(X)$ such that, for all Y in $\mathcal{C}$, the map $f \mapsto A(f)(x)$ from* $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ *to $A(Y)$ is bijective. In particular this condition does not depend on the universe $\mathcal{U}$ such that $\mathcal{C}, A \in \mathcal{U}$. (See Remark 1.4.13 p. 27 of the book.)*

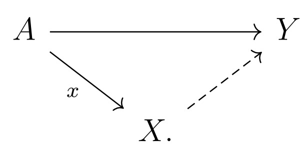
Convention 38. An object Y in a category $\mathcal{A}$ is *terminal* if all X in $\mathcal{A}$ admits a unique morphism $X \to Y$. Let $T_{\mathcal{A}}$ be the set of terminal objects of $\mathcal{A}$. If $Y, Z \in T_{\mathcal{A}}$, then there is a unique morphism $Y \to Z$, and this morphism is an isomorphism. For all category $\mathcal{A}$ such that $T_{\mathcal{A}} \neq \varnothing$ we choose an element in $T_{\mathcal{A}}$ and call it **the** terminal object of $\mathcal{A}$. Let us insist: we make a distinction between "*a* terminal object of $\mathcal{A}$" and "*the* terminal object of $\mathcal{A}$" (when they exist). Unless otherwise indicated, the choice of *the* terminal object of $\mathcal{A}$ is random (but there will be two exceptions to this rule: see Convention 55 p. 47 and Convention 57 p. 48).

Convention 39. We often identify the source and the target of φ in (2) and (4), and we also often consider $\mathcal{C}$ as a full subcategory of $\mathcal{C}^\wedge$ and $\mathcal{C}^\vee$ thanks to the Yoneda embeddings. Let A be in $\mathcal{C}^\wedge$ and (X,x), with X in $\mathcal{C}$ and $x : X \to A$ a morphism in $\mathcal{C}^\wedge$, an object in the category $\mathcal{C}_A$ (see Definition 1.2.16 p. 18 of the book). Then (X, x) is terminal if and only if x is an isomorphism. If the category $\mathcal{C}_A$ admits a terminal object, we say that A is *representable*. Let (X, x) be a (resp. the) terminal object of $\mathcal{C}_A$. We say that the couple (X, x), or sometimes just the morphism x, is a (resp. the) *representation* of A, and that X is a (resp. the) *representative* of A (or that X, or x, *represents* A). We use a similar terminology if A in is in $\mathcal{C}^\vee$ instead of $\mathcal{C}^\wedge$, replacing the words *representable*, *representative*, *representation* with *co-representable*, *co-representative*, *co-representation*.

A morphism $x : X \to A$ in $\mathcal{C}^\wedge$ with X in $\mathcal{C}$ is a representation of A if and only if any morphism $Y \to A$ with Y in $\mathcal{C}$ factors uniquely through x:



A morphism $x : A \to X$ in $\mathcal{C}^\vee$ with X in $\mathcal{C}$ is a co-representation of A if and only if any morphism $A \to Y$ with Y in $\mathcal{C}$ factors uniquely through x:



Here are two corollaries to the Yoneda Lemma:

Corollary 40. *In the setting of the Yoneda Lemma (Theorem 36 p. 33), an element* $x \in A(X)$ *represents* A *if and only if, for any* Y *in* $\mathcal{C}$*, the map* $\operatorname{Hom}_{\mathcal{C}}(Y, X) \to A(Y)$*,* $f \mapsto A(f)(x)$ *is bijective.*

Corollary 41. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be universes, let* $\mathcal{C}$ *be a* $\mathcal{U}$*- and* $\mathcal{V}$*-category, let* A *be in* $\mathcal{C}^\wedge_{\mathcal{U}}$ *and* B *in* $\mathcal{C}^\wedge_{\mathcal{V}}$*, and assume* $A(X) = B(X)$ *for all object* X *in* $\mathcal{C}$*, and* $A(f) = B(f)$ *for all* $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ *and all* $X, Y \in \mathcal{C}$*. Then* A *is representable if and only if* B *is. Let* X *be in* $\mathcal{C}$*. Then* X *represents* A *if and only if* X *represents* B*. Let* x *be in* $A(X) = B(X)$*. Then* x *is a representation of* A *if and only if* x *is a representation of* B*.*

Proof. This follows from Corollary 40. □

Here is a typical situation where Corollary 41 applies: Let $L : \mathcal{C} \to \mathcal{C}'$ be an arbitrary functor, let X' be an object of $\mathcal{C}'$, let $\mathcal{U}$ and $\mathcal{V}$ be universes such that $\mathcal{C}$ and $\mathcal{C}'$ are $\mathcal{U}$- and $\mathcal{V}$-categories, and define $A \in \mathcal{C}^\wedge_\mathcal{U}$ and $B \in \mathcal{C}^\wedge_\mathcal{V}$ by

$$A(X) = B(X) = \mathrm{Hom}_{\mathcal{C}'}(L(X), X').$$

Then A is representable if and only if B is. More on this in §45 p. 39.

3.7 Brief comments

§ 42. P. 25, Corollary 1.4.7. A statement slightly stronger than Corollary 1.4.7 of the book can be proved more naively:

Proposition 43. *A morphism $f : A \to B$ in a category $\mathcal{C}$ is an isomorphism if and only if*

$$\mathrm{Hom}_\mathcal{C}(X, f) : \mathrm{Hom}_\mathcal{C}(X, A) \to \mathrm{Hom}_\mathcal{C}(X, B)$$

is (i) surjective for $X = B$ and (ii) injective for $X = A$.

Proof. By (i) there is a $g : B \to A$ satisfying $f \circ g = \mathrm{id}_B$, yielding $f \circ g \circ f = f$, and (ii) implies $g \circ f = \mathrm{id}_A$. □

§ 44. P. 26, Lemma 1.4.12. We can define the functors

$$(\mathcal{C}^\wedge)_A \underset{\mu}{\overset{\lambda}{\rightleftarrows}} (\mathcal{C}_A)^\wedge$$

as follows:

$$\lambda(B \xrightarrow{b} A)(X \xrightarrow{x} A) := b_X^{-1}(x),$$

$$\mu(C) := (\mu_0(C) \to A), \quad \mu_0(C)(X) := \bigsqcup_{x \in A(X)} C(X \xrightarrow{x} A),$$

$\mu_0(C)(X) \to A(X)$ being the obvious map.

3.8 Partially defined adjoints (Section 1.5, p. 28)

§ 45. Let $L : \mathcal{C} \to \mathcal{C}'$ be a functor and X' an object of $\mathcal{C}'$. If the functor

$$\mathrm{Hom}_{\mathcal{C}'}(L(\ ), X') : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

is representable, we denote its representative by $R(X')$ and its representation by $\eta_{X'} : L(R(X')) \to X'$ (see Convention 39 p. 36), and we say that

"the value of the right adjoint R to L at X' is defined and isomorphic to $R(X')$",

or, abusing the terminology, that

"$R(X')$ exists".

The following lemma will result from Lemma 49 below.

Lemma 46. *In the above setting, if $\eta_{X'} : L(R(X')) \to X'$ is a morphism in $\mathcal{C}'$, then the following two conditions are equivalent:*

(a) $\eta_{X'}$ *is a representation of* $\mathrm{Hom}_{\mathcal{C}'}(L(\ ), X')$,

(b) *for all X in $\mathcal{C}$ and all $g : L(X) \to X'$ there is a unique $f : X \to R(X')$ such that $\eta_{X'} \circ L(f) = g$:*

$$\begin{array}{ccc} X & \quad & L(X) \xrightarrow{\ g\ } X' \\ {\scriptstyle f}\,\downarrow & & {\scriptstyle L(f)}\downarrow \quad \nearrow {\scriptstyle \eta_{X'}} \\ R(X') & & L(R(X')). \end{array} \tag{5}$$

We call $\eta_{X'}$ the *unit* of the adjunction.

§ 47. Note that Condition (b) in Lemma 46 involves no universe. In the statement of Condition (a) it is implicitly assumed that a universe $\mathcal{U}$ such that $\mathcal{C}$ and $\mathcal{C}'$ are $\mathcal{U}$-categories (Definition 4 p. 10) has been chosen. In particular, if $\mathcal{V}$ is another such universe, then (a) holds for $\mathcal{V}$ if and only if it holds for $\mathcal{U}$. (See also Corollary 41 p. 37.)

§ 48. By definition, $\eta_{X'} : L(R(X')) \to X'$ in $\mathcal{C}'$ is a representation of $\mathrm{Hom}_{\mathcal{C}'}(L(\ ), X')$ if and only if, for all morphism $\theta : X \to \mathrm{Hom}_{\mathcal{C}'}(L(\ ), X')$ in $\mathcal{C}^\wedge$, there is a unique morphism $f : X \to R(X')$ in $\mathcal{C}$ such that $\eta_{X'} \circ f = \theta$:

$$\begin{array}{ccc} X & \xrightarrow{\ \theta\ } & \mathrm{Hom}_{\mathcal{C}'}(L(\ ), X') \\ {\scriptstyle f}\,\downarrow & \nearrow {\scriptstyle \eta_{X'}} & \\ R(X'). & & \end{array} \tag{6}$$

Even if it is straightforward, we state and prove formally the fact that the above condition is equivalent to the condition in Lemma 46. For the purpose of this proof, we prefer to rewrite (5) as

$$\begin{array}{ccc} X & \qquad & L(X) \xrightarrow{\theta_X(\mathrm{id}_X)} X' \\ {\scriptstyle f}\,\Big\downarrow & & {\scriptstyle L(f)}\Big\downarrow \quad \nearrow {\scriptstyle \eta_{X',R(X')}(\mathrm{id}_{R(X')})} \\ R(X') & & L(R(X')). \end{array} \tag{7}$$

Lemma 49. *If $L : \mathcal{C} \to \mathcal{C}'$ is a functor, if X and $R(X')$ are objects of $\mathcal{C}$, if $f : X \to R(X')$ is a morphism in $\mathcal{C}$, and if*

$$\eta_{X'} : R(X') \to \mathrm{Hom}_{\mathcal{C}'}(L(\ ), X') \quad \text{and} \quad \theta : X \to \mathrm{Hom}_{\mathcal{C}'}(L(\ ), X')$$

are morphisms of functors, then we have

$$\eta_{X'} \circ f = \theta \iff \eta_{X',R(X')}(\mathrm{id}_{R(X')}) \circ L(f) = \theta_X(\mathrm{id}_X)$$

(see (6) and (7)).

Proof. The equalities

$$(\eta_{X'} \circ f)_X(\mathrm{id}_X) = \eta_{X',X}(f) = \eta_{X',R(X')}(\mathrm{id}_{R(X')}) \circ L(f)$$

are respectively justified by the definition of the vertical composition of morphisms of functors and by the naturality of $\eta_{X'}$. As the Yoneda Lemma (Theorem 36 p. 33) implies

$$\eta_{X'} \circ f = \theta \iff (\eta_{X'} \circ f)_X(\mathrm{id}_X) = \theta_X(\mathrm{id}_X),$$

the lemma is proved. □

§ 50. Let T be a terminal object of **Set**. Then a functor $A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if the right adjoint of $A^{\mathrm{op}} : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$ is defined at T.

Indeed we have

$$\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(A^{\mathrm{op}}(\ ), T) \simeq \mathrm{Hom}_{\mathbf{Set}}(T, A(\ )) \simeq A.$$

§ 51. Let us spell out the statement dual to §45:

Let $R : \mathcal{C}' \to \mathcal{C}$ be a functor and X an object of $\mathcal{C}$. If the functor

$$\mathrm{Hom}_{\mathcal{C}'}(X, R(\ )) : \mathcal{C} \to \mathbf{Set}$$

is co-representable, we denote its co-representative by $L(X)$ and its co-representation by $\varepsilon_X : X \to R(L(X))$ (see Convention 39 p. 36), and we say that

"the value of the left adjoint L to R at X is defined and isomorphic to $L(X)$",

or, abusing the terminology, that

"$L(X)$ exists".

Concretely this means that, for all X' in $\mathcal{C}'$ and all $g : X \to R(X')$ there is a unique $f : L(X) \to X'$ such that $R(f) \circ \varepsilon_X = g$:

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & R(L(X)) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle R(f)} \\ & & R(X') \end{array} \qquad\qquad \begin{array}{c} L(X) \\ \downarrow{\scriptstyle f} \\ X'. \end{array}$$

We call ε_X the *co-unit* of the adjunction.

3.9 Commutativity of Diagram (1.5.6) p. 28

Let us prove the commutativity of the diagram (1.5.6) p. 28 of the book. Recall the setting: We have a pair (L, R) of adjoint functors:

$$\begin{array}{c} \mathcal{C} \\ {\scriptstyle L}\downarrow\uparrow{\scriptstyle R} \\ \mathcal{C}'. \end{array}$$

Let us denote the functorial bijection defining the adjunction by

$$\lambda_{X,X'} : \mathrm{Hom}_{\mathcal{C}}(X, RX') \to \mathrm{Hom}_{\mathcal{C}'}(LX, X')$$

for X in $\mathcal{C}$ and X' in $\mathcal{C}'$. The diagram (1.5.6) can be written as

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}'}(X', Y') & \xrightarrow{R} & \mathrm{Hom}_{\mathcal{C}}(RX', RY'), \\ & {\scriptstyle \circ\lambda_{RX',X'}(\mathrm{id}_{RX'})}\searrow & \downarrow{\scriptstyle \lambda_{RX',Y'}} \\ & & \mathrm{Hom}_{\mathcal{C}'}(LRX', Y'). \end{array} \tag{8}$$

As the diagram

$$\begin{array}{ccc}\operatorname{Hom}_{\mathcal{C}}(RX',RX') & \xrightarrow{\lambda_{RX',X'}} & \operatorname{Hom}_{\mathcal{C}'}(LRX',X'), \\ {\scriptstyle R(f)\circ}\downarrow & & \downarrow{\scriptstyle f\circ} \\ \operatorname{Hom}_{\mathcal{C}}(RX',RY') & \xrightarrow[\lambda_{RX',Y'}]{} & \operatorname{Hom}_{\mathcal{C}'}(LRX',Y')\end{array}$$

commutes for f in $\operatorname{Hom}_{\mathcal{C}'}(X',Y')$, we get in particular

$$f\circ\lambda_{RX',X'}(\operatorname{id}_{RX'})=\lambda_{RX',Y'}(R(f)\circ\operatorname{id}_{RX'})=\lambda_{RX',Y'}(R(f)).$$

This shows that (8) commutes, as required.

3.10 Equalities (1.5.8) and (1.5.9) p. 29

Warning: many authors designate ε by η and η by ε.

3.10.1 Statements

We have a pair (L,R) of adjoint functors:

$$\begin{array}{c}\mathcal{C}\\ {\scriptstyle L}\downarrow\uparrow{\scriptstyle R}\\ \mathcal{C}'.\end{array}$$

Recall that $\varepsilon_X\in\operatorname{Hom}_{\mathcal{C}}(X,RLX)$ and $\eta_{X'}\in\operatorname{Hom}_{\mathcal{C}'}(LRX',X')$ for all X in $\mathcal{C}$ and all X' in $\mathcal{C}'$:

$$\varepsilon_X:X\to RLX,\quad \eta_{X'}:LRX'\to X'.$$

Using Notation 31 p. 31, Equalities (1.5.8) and (1.5.9) become respectively

$$(\eta\star L)\circ(L\star\varepsilon)=L \tag{9}$$

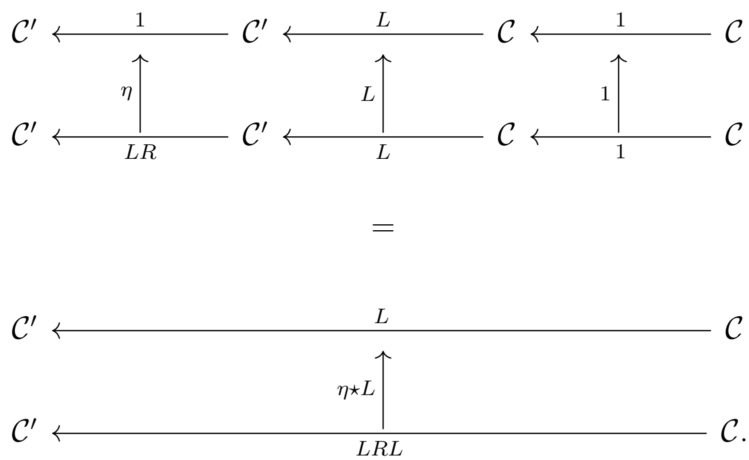
and

$$(R\star\eta)\circ(\varepsilon\star R)=R. \tag{10}$$

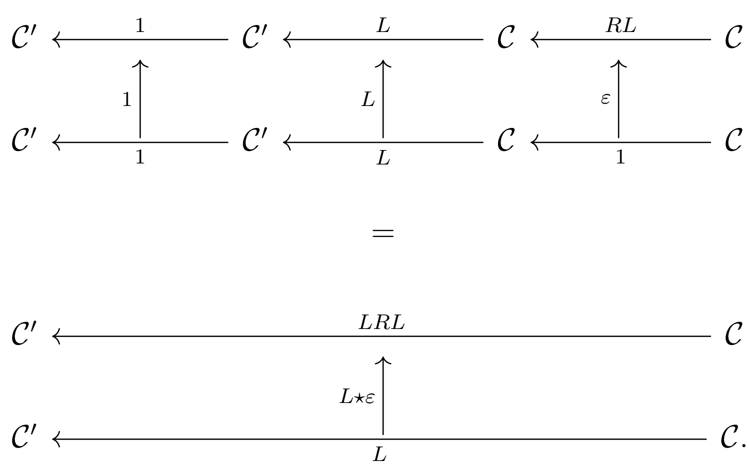
3.10.2 Pictures

Let us try to illustrate these two equalities by diagrams:

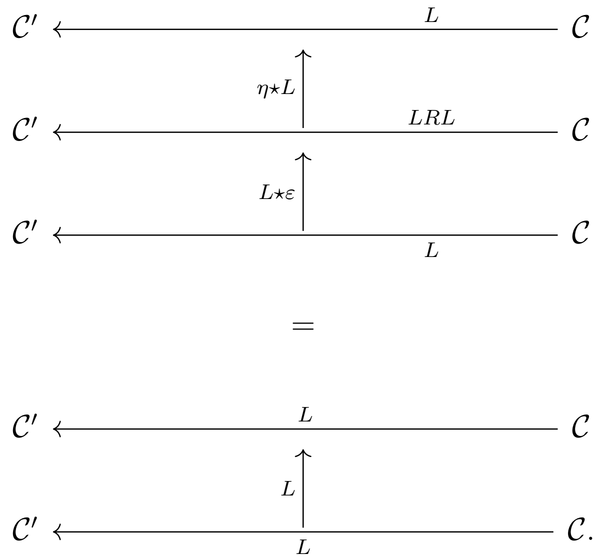
Picture of $L \xleftarrow{\eta \star L} LRL$:



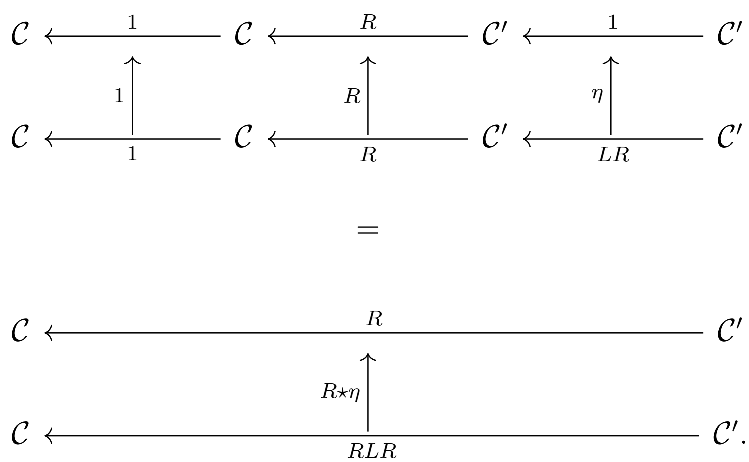
Picture of $LRL \xleftarrow{L \star \varepsilon} L$:



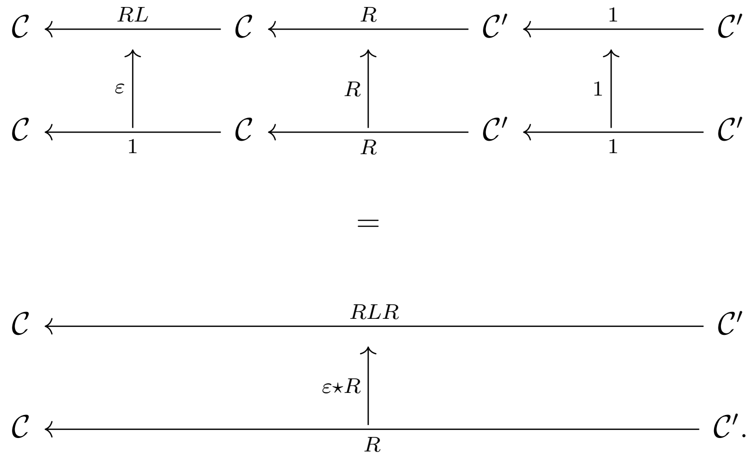
Picture of (9), that is, $(\eta \star L) \circ (L \star \varepsilon) = L$:



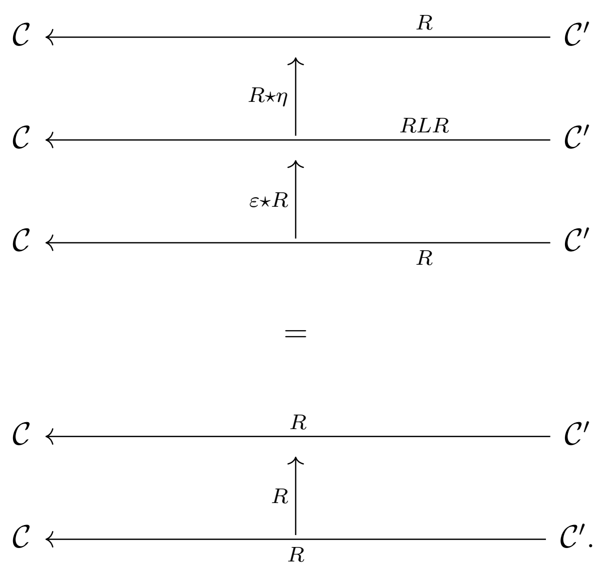
Picture of $R \xleftarrow{R\star\eta} RLR$:



Picture of $RLR \xleftarrow{\varepsilon \star R} R$:



Picture of (10), that is, $(R \star \eta) \circ (\varepsilon \star R) = R$:



3.10.3 Proofs

For the reader's convenience we prove (9) p. 42 and (10) p. 42. It clearly suffices to prove (9). Recall that (9) claims

$$(\eta \star L) \circ (L \star \varepsilon) = L.$$

Let us denote the functorial mutually inverse bijections defining the adjunction by

$$\mathrm{Hom}_{\mathcal{C}}(X, RX') \underset{\mu_{X,X'}}{\overset{\lambda_{X,X'}}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{C}'}(LX, X'), \tag{11}$$

and recall that ε_X and $\eta_{X'}$ are defined by

$$\varepsilon_X := \mu_{X,LX}(\mathrm{id}_{LX}), \quad \eta_{X'} := \lambda_{RX',X'}(\mathrm{id}_{RX'}). \tag{12}$$

Equality (9) p. 42 can be written

$$\lambda_{RLX,LX}(\mathrm{id}_{RLX}) \circ L(\varepsilon_X) = \mathrm{id}_{LX},$$

and we have

$$\begin{aligned} \mathrm{id}_{LX} &\overset{(a)}{=} \lambda_{X,LX}\big(\mu_{X,LX}(\mathrm{id}_{LX})\big) \overset{(b)}{=} \lambda_{X,LX}(\varepsilon_X) \overset{(c)}{=} \big(\lambda_{X,LX} \circ (\circ\varepsilon_X)\big)(\mathrm{id}_{RLX}) \\ &\overset{(d)}{=} \Big(\big(\circ L(\varepsilon_X)\big) \circ \lambda_{RLX,LX}\Big)(\mathrm{id}_{RLX}) \overset{(e)}{=} \lambda_{RLX,LX}(\mathrm{id}_{RLX}) \circ L(\varepsilon_X), \end{aligned}$$

the successive equalities being justified as follows:

(a) follows from (11),

(b) follows from (12),

(c) is obvious,

(d) follows from the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(RLX, RLX) & \xrightarrow{\lambda_{RLX,LX}} & \mathrm{Hom}_{\mathcal{C}'}(LRLX, LX) \\ {\scriptstyle \circ\varepsilon_X}\downarrow & & \downarrow{\scriptstyle \circ L(\varepsilon_X)} \\ \mathrm{Hom}_{\mathcal{C}}(X, RLX) & \xrightarrow[\lambda_{X,LX}]{} & \mathrm{Hom}_{\mathcal{C}'}(LX, LX), \end{array}$$

(e) is obvious.

4 About Chapter 2

4.1 Definition of limits (§2.1 p. 36)

Notation 52. *If I and $\mathcal{C}$ are categories, we denote by Δ the diagonal functor from $\mathcal{C}$ to $\mathcal{C}^I$. The categories I and $\mathcal{C}$ shall be explicitly indicated only when they are not* *clear from the context. Furthermore, we shall often write* ΔX *for* $\Delta(X)$*. To be more precise,* ΔX *is the constant functor from* I *to* $\mathcal{C}$ *with value* X*.*

Definition 53 ("projective limit" or "limit")**.** *Let* $\alpha : I^{\mathrm{op}} \to \mathcal{C}$ *be a functor. If the value at* α *of the right adjoint* lim *to* $\Delta : \mathcal{C} \to \mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C})$ *exists (see §45 p. 39), we denote it by* $\lim \alpha$ *and call it* the projective limit*, or just* the limit*, of* α*. Moreover, we say that the unit* $p : \Delta \lim \alpha \to \alpha$ *of the adjunction is* ***the*** projection*. More generally we say that* $q : \Delta X \to \alpha$ *(with* X *in* $\mathcal{C}$*) is* ***a*** projection *if the corresponding morphism*

$$X \to \mathrm{Hom}_{\mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C})}(\Delta(\ ), \alpha)$$

in $\mathcal{C}^\wedge$ *(see Convention 39 p. 36) is an isomorphism.*

The characteristic property of the pair $(\lim \alpha, p)$ can be described as follows: For each Y in $\mathcal{C}$ and each morphism of functors $\theta : \Delta Y \to \alpha$ there is a unique morphism $f : Y \to \lim \alpha$ satisfying $p \circ \Delta f = \theta$:

$$\begin{array}{ccccc} Y & \quad & \Delta Y & & \\ {\scriptstyle f}\downarrow & & {\scriptstyle \Delta f}\downarrow & \searrow^{\theta} & \\ \lim \alpha & & \Delta \lim \alpha & \xrightarrow[p]{} & \alpha. \end{array} \tag{13}$$

Remark 54. Note that this definition of limit involves no universe. This will be also the case for the notion of colimit that will be introduced shortly. This observation has already been made in §47 p. 39.

In Convention 38 p. 36 we stated a rule and indicated that we would make some exceptions to it. Here is the first such exception:

Convention 55. If $\alpha : I^{\mathrm{op}} \to \mathbf{Set}$ is a functor defined on a small category (Definition 5 p. 10), then we define its projective limit $\lim \alpha$ by

$$\lim \alpha := \left\{ x \in \prod_{i \in I} \alpha(i) \;\middle|\; x_i = \alpha(s)(x_j) \ \forall \ s : i \to j \right\} \in \mathbf{Set},$$

and we define *the* projection $p : \Delta \lim \alpha \to \alpha$ by $p_i(x) := x_i$. Then p is *a* projection in the sense of Definition 53. [Indeed, let θ in (13) be given. If such an f exists, it must satisfy $p_i(f(y)) = \theta_i(y)$ for all i in I and all y in Y. This implies $f(y) = (\theta_i(y))_{i \in I}$, and proves the uniqueness of f. It is straightforward to check that the map f defined by the above equality does the job.]

Note that the projective limit of $\alpha : I^{\text{op}} \to \mathbf{Set}$ does not depend on the universe which makes I a small category (Definition 5 p. 10).

Definition 56 ("inductive limit" or "colimit")**.** *Let $\alpha : I \to \mathcal{C}$ be a functor. If the value at α of the left adjoint* colim *to $\Delta : \mathcal{C} \to \mathcal{C}^I$ exists, we denote it by* $\operatorname{colim}\alpha$ *and call it* the inductive limit*, or* the colimit*, of α (see §51 p. 40). Moreover, we say that the co-unit $p : \alpha \to \Delta \operatorname{colim}\alpha$ of the adjunction is **the*** coprojection*. More generally we say that $q : \alpha \to \Delta X$ (with X in $\mathcal{C}$) is **a*** coprojection *if the corresponding morphism*

$$\operatorname{Hom}_{\mathcal{C}^I}(\alpha, \Delta(\ )) \to X$$

in $\mathcal{C}^\vee$ is an isomorphism.

The characteristic property of the pair $(\operatorname{colim}\alpha, p)$ can be described as follows: For each Y in $\mathcal{C}$ and each morphism of functors $\theta : \alpha \to \Delta Y$ there is a unique morphism $f : X \to Y$ satisfying $\Delta f \circ p = \theta$:

$$\begin{array}{ccc} \alpha & \xrightarrow{p} & \Delta \operatorname{colim}\alpha \\ & {\scriptstyle\theta}\searrow & \downarrow{\scriptstyle \Delta f} \\ & & \Delta Y \end{array} \qquad \begin{array}{c} \operatorname{colim}\alpha \\ \downarrow{\scriptstyle f} \\ Y. \end{array} \tag{14}$$

In Convention 38 p. 36 we stated a rule and indicated that we would make some exceptions to it. Here is the second such exception:

Convention 57. Let $\alpha : I \to \mathbf{Set}$ be a functor defined on a small category (Definition 5 p. 10), set

$$U := \{(i,x) \in \mathcal{U} \mid i \in I, x \in \alpha(i)\},$$

and let $\sim$ be the least equivalence relation on U satisfying $(i,x) \sim (j, \alpha(f)(x))$ for all morphisms $f : i \to j$. Then we define the inductive limit $\operatorname{colim}\alpha$ as the quotient $U/{\sim}$. Let $\pi : U \to \operatorname{colim}\alpha$ be the canonical projection, and, for all i in I, define $p_i : \alpha(i) \to \operatorname{colim}\alpha$ by $p_i(x) := \pi(i,x)$. We call the resulting morphism $p : \alpha \to \operatorname{colim}\alpha$ the coprojection. Then p is *a* coprojection in the sense of Definition 53. [Indeed, given θ in (14) let us prove the uniqueness of f. Any x in X is of the form $p_i(t)$ for some i in I and t in $\alpha(i)$, and we must have $f(x) = \theta_i(t)$. This proves the uniqueness. To verify the existence, we must assume $p_i(t) = p_j(u)$ (obvious notation), and derive $\theta_i(t) = \theta_j(u)$. We may assume that there is a morphism $s : i \to j$, and the verification is straightforward.]

Note that the inductive limit of $\alpha : I \to \mathbf{Set}$ does not depend on the universe which makes I a small category (Definition 5 p. 10).

4.2 Brief comments

§ 58. We shall spell out two wordings of a certain statement about the following setting: $\alpha : I \to \mathcal{C}$ is a functor and Z is an object of $\mathcal{C}$.

First wording: Assume that $\operatorname{colim} \alpha$ exists in $\mathcal{C}$ and, for each i in I, let $p_i : \alpha(i) \to \operatorname{colim} \alpha$ be the corresponding coprojection. Then the map

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} \alpha, Z) \to \prod_{i \in I} \operatorname{Hom}_{\mathcal{C}}(\alpha(i), Z), \quad f \mapsto (f \circ p_i)_{i \in I}$$

induces a bijection

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} \alpha, Z) \xrightarrow{\sim} \lim \operatorname{Hom}_{\mathcal{C}}(\alpha, Z).$$

The proof is left to the reader.

Second wording: Let X be an object of $\mathcal{C}$ and $p : \alpha \to \Delta X$ a coprojection in the sense of Definition 56 p. 48:

$$\begin{array}{ccc} \alpha & \xrightarrow{p} & \Delta X \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \Delta f} \\ & & \Delta Y \end{array} \qquad \begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y. \end{array} \tag{15}$$

We claim that

$$\circ p : \Delta \operatorname{Hom}_{\mathcal{C}}(X, Z) \to \operatorname{Hom}_{\mathcal{C}}(\alpha, Z)$$

is a projection in the sense of Definition 53 p. 47:

$$\begin{array}{c} S \\ {\scriptstyle g}\downarrow \\ \operatorname{Hom}_{\mathcal{C}}(X, Z) \end{array} \qquad \begin{array}{ccc} \Delta S & & \\ {\scriptstyle \Delta g}\downarrow & \searrow^{\mu} & \\ \Delta \operatorname{Hom}_{\mathcal{C}}(X, Z) & \xrightarrow[\circ p]{} & \operatorname{Hom}_{\mathcal{C}}(\alpha, Z). \end{array} \tag{16}$$

More precisely, assume we are given μ as above and s in S. Then we set $Y := Z$ and $\lambda_i := \mu_i(s)$ in (15). We get an $f : X \to Z$, and we set $g(s) := f$. We leave it to the reader to check that this process yields a solution to (16), and that this solution is unique.

§ 59. P. 38, Proposition 2.1.6. We want to find a setting where the isomorphism

$$\operatorname{colim} \alpha(j) \xrightarrow{\sim} (\operatorname{colim} \alpha)(j)$$

makes sense and is true.

Let $\alpha : I \to \mathcal{C}^J$ be a functor, and let us assume that for each j in J the functor $\alpha(\ )(j) : I \to \mathcal{C}$ admits a coprojection $p_j : \alpha(\ )(j) \to \Delta X_j$ in the sense of Definition 56 p. 48:

$$\begin{array}{ccccc} \alpha(\ )(j) & \xrightarrow{p_j} & \Delta X_j & & X_j \\ & \searrow & \downarrow & & \vdots \\ & & \Delta Y & & Y. \end{array} \tag{17}$$

We claim that there is a natural functor $\beta : J \to \mathcal{C}$ satisfying $\beta(j) = X_j$ for all j in J. Given $j \to j'$ we define $X_j \to X_{j'}$ as suggested by the commutative diagram

$$\begin{array}{ccccc} \alpha(\ )(j) & \xrightarrow{p_j} & \Delta X_j & & X_j \\ \downarrow & & \downarrow & & \vdots \\ \alpha(\ )(j') & \xrightarrow[p_{j'}]{} & \Delta X_{j'} & & X_{j'}. \end{array}$$

We leave it to the reader to verify that this construction does define our functor β.

We want to define a morphism $q : \alpha \to \Delta\beta$. Let i be in I. We must define $q_i : \alpha(i) \to \beta$, that is, given j in J we must define $q_{ij} : \alpha(i)(j) \to \beta(j)$. It suffices to set $q_{ij} := p_{ji}$.

Proposition 60. *In the above setting the morphism q is a coprojection in the sense of Definition 56 p. 48.*

Proof. Let $\gamma : J \to \mathcal{C}$ be a functor and $\lambda : \alpha \to \Delta\gamma$ a morphism of functors. We must solve the problem described by the commutative diagram

$$\begin{array}{ccccc} \alpha & \xrightarrow{q} & \Delta\beta & & \beta \\ & \overset{\lambda}{\searrow} & \downarrow \Delta\mu & & \vdots\ \mu \\ & & \Delta\gamma & & \gamma. \end{array}$$

Note that λ is given by a family of morphisms $\lambda_i : \alpha(i) \to \gamma$, morphisms given in turn by families $\lambda_{ij} : \alpha(i)(j) \to \gamma(j)$.

In view of (17) we can define $\mu_j : \beta(j) \to \gamma(j)$ as suggested by the commutative diagram

$$\begin{array}{ccccc} \alpha(\ )(j) & \xrightarrow{p_j} & \Delta\beta(j) & \qquad & \beta(j) \\ & \searrow^{\lambda_{\bullet j}} & \downarrow{\scriptstyle \Delta\mu_j} & & \vdots{\scriptstyle \mu_j} \\ & & \Delta\gamma(j) & & \gamma(j). \end{array}$$

It is straightforward to check that the morphisms $\mu_j : \beta(j) \to \gamma(j)$ give rise to a morphism $\mu : \beta \to \gamma$, and that this morphism satisfies $\Delta\mu \circ q = \lambda$, as required. □

§ 61. P. 38, Proposition 2.1.6. Here is an example of a functor $\alpha : I \to \mathcal{C}^J$ such that $\operatorname{colim} \alpha$ exists in $\mathcal{C}^J$ but there is a j in J such that colim $(\rho_j \circ \alpha)$ does not exist in $\mathcal{C}$. (Recall that $\rho_j : \mathcal{C}^J \to \mathcal{C}$ is the evaluation at $j \in J$.) This example is taken from Section 3.3 of the book **Basic Concepts of Enriched Category Theory** of G.M. Kelly:

http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html

The category J has two objects, 1, 2; it has exactly one nontrivial morphism; and this morphism goes from 1 to 2. The category $\mathcal{C}$ has exactly three objects, 1, 2, 3, and exactly four nontrivial morphisms, $f, g, h, g \circ f = h \circ f$, with

$$1 \xrightarrow{f} 2 \underset{h}{\overset{g}{\rightrightarrows}} 3.$$

Then $\mathcal{C}^J$ is the category of morphisms in $\mathcal{C}$. It is easy to see that the morphism $(f, h) : f \to g$, that is

$$\begin{array}{ccc} 1 & \xrightarrow{f} & 2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ 2 & \xrightarrow[h]{} & 3, \end{array}$$

in $\mathcal{C}^J$ is an epimorphism, and that this implies that the commutative square

$$\begin{array}{ccc} f & \xrightarrow{(f,h)} & g \\ {\scriptstyle (f,h)}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_g} \\ g & \xrightarrow[\mathrm{id}_g]{} & g \end{array}$$

in $\mathcal{C}^J$ is cocartesian. But it is also easy to see that the morphism f in $\mathcal{C}$ is *not* an epimorphism, and that this implies that the commutative square

$$\begin{array}{ccc} 1 & \xrightarrow{f} & 2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_2} \\ 2 & \xrightarrow[\mathrm{id}_2]{} & 2 \end{array}$$

in $\mathcal{C}$ is *not* cocartesian.

In Proposition 72 p. 59 we shall see a way to prevent the kind of pathology displayed by the above example.

§ 62. P. 39, Proposition 2.1.7. We want to find a setting where the isomorphism

$$\operatorname*{colim}_{i,j} \alpha(i,j) \simeq \operatorname*{colim}_{i} \operatorname*{colim}_{j} \alpha(i,j)$$

makes sense and is true.

Let $\alpha : I \times J \to \mathcal{C}$ be a bifunctor, and let $(X_i)_{i \in I}$ be a family of objects of $\mathcal{C}$. Assume that for any i in I there is some morphism $p_i : \alpha(i,\ ) \to \Delta X_i$ which is a coprojection in the sense of Definition 56 p. 48:

$$\begin{array}{ccccc} \alpha(i,\ ) & \xrightarrow{p_i} & \Delta X_i & \qquad & X_i \\ & \searrow & \downarrow & & \vdots \\ & & \Delta Y & & Y. \end{array}$$

By arguing as in §59 p. 49 we see that there is a natural functor $\beta : I \to \mathcal{C}$ such that $\beta(i) = X_i$ for all i. Let $q : \beta \to \Delta X$ be a coprojection:

$$\begin{array}{ccccc} \beta & \xrightarrow{q} & \Delta X & \qquad & X \\ & \searrow & \downarrow & & \vdots \\ & & \Delta Y & & Y. \end{array} \tag{18}$$

We claim that the obvious morphism of functors $r : \alpha \to \Delta X$ is a coprojection.

Let Y be in $\mathcal{C}$ and $\theta : \alpha \to \Delta Y$ a morphism of functors. We must solve the problem

$$\begin{array}{ccccc} \alpha & \xrightarrow{r} & \Delta X & \qquad & X \\ & {\scriptstyle \theta}\searrow & \downarrow & & \vdots \\ & & \Delta Y & & Y. \end{array}$$

Noting that θ induces, for all i, a morphism of functors $\alpha(i,\ ) \to \Delta Y$, we get firstly a morphism $\beta(i) \to Y$:

$$\begin{array}{ccc} \alpha(i,\ ) & \xrightarrow{p_i} & \Delta\beta(i) \\ & \searrow & \downarrow \\ & & \Delta Y \end{array} \qquad \begin{array}{c} \beta(i) \\ \downarrow \\ Y, \end{array}$$

secondly a morphism of functors $\beta \to \Delta Y$, and thirdly a morphism $X \to Y$ by (18). It is straightforward to check that this morphism $X \to Y$ does the job. q.e.d.

§ 63. The two propositions below are basic.

Proposition 64. *If $\alpha : I^{\mathrm{op}} \to \mathcal{C}$ is a functor defined on a small category (Definition 5 p. 10), if X is in $\mathcal{C}$, if $p : \Delta X \to \alpha$ is a morphism in* $\mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C})$*, and if $h : \mathcal{C} \to \mathcal{C}^\wedge$ is the Yoneda embedding, then the following conditions* (a), (b), (c) *are equivalent:*

(a) *p is a projection in the sense of Definition 53 p. 47,*

(b) *the morphism $h(p) : \Delta h(X) \to h \circ \alpha$ in* $\mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C}^\wedge)$ *induced by p is a projection,*

(c) *for all Y in $\mathcal{C}$ the morphism* $\mathrm{Hom}_{\mathcal{C}}(Y, p) : \Delta \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, \alpha)$ *in* $\mathrm{Fct}(I^{\mathrm{op}}, \mathbf{Set})$ *induced by p is a projection,*

Moreover, if (b) *holds for some universe $\mathcal{U}$ such that I is $\mathcal{U}$-small and $\mathcal{C}$ is a $\mathcal{U}$-category (Definitions 4 p. 10 and 5 p. 10), then it holds for any such universe; the same applies to* (c).

Condition (c) is often abridged by

$$\mathrm{Hom}_{\mathcal{C}}(Y, \lim \alpha) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathcal{C}}(Y, \alpha).$$

Proof. Conditions (b) and (c) are equivalent by Proposition 60 p. 50. We sketch the proof that (a) and (c) are equivalent. Let us summarize (a) and (c) by the following self-explanatory commutative diagrams:

$$\begin{array}{c} Z \\ {\scriptstyle f}\downarrow \\ X \end{array} \qquad \begin{array}{ccc} \Delta Z & & \\ {\scriptstyle \Delta f}\downarrow & \overset{\lambda}{\searrow} & \\ \Delta X & \xrightarrow[p]{} & \alpha, \end{array} \tag{19}$$

$$\begin{array}{ccc} S & \qquad & \Delta S \xrightarrow{\quad\mu\quad} \operatorname{Hom}_{\mathcal{C}}(Y,\alpha) \\ {\scriptstyle g}\downarrow & & {\scriptstyle \Delta g}\downarrow \qquad\qquad \nearrow \\ \operatorname{Hom}_{\mathcal{C}}(Y,X) & & \Delta\operatorname{Hom}_{\mathcal{C}}(Y,X) \xrightarrow[\operatorname{Hom}_{\mathcal{C}}(Y,p)]{} \operatorname{Hom}_{\mathcal{C}}(Y,\alpha). \end{array} \tag{20}$$

To prove (c)⇒(a), we suppose Z and λ given in (19), and in (20) we let S be a singleton, we set $Y := Z$, we define μ by the formula $\mu_i(s) := \lambda_i$, we get a g as above, we set $f := g(s)$, and we check that this works.

To prove (a)⇒(c), we suppose S, Y and μ given in (20), and we let s be in S. We must define $g(s) : Y \to X$. We set $Z := Y$ in (19). We must define $\lambda : \Delta Y \to \alpha$. Letting i be in I, it suffices to define $\lambda_i : Y \to \alpha(i)$. To do this we set $\lambda_i := \mu_i(s)$, we get an f (depending on s) as above, we set $g(s) := f$, and we check that this works.

The last sentence is obvious (see Remark 54 p. 47). □

The proof of the following proposition is similar to the previous one and is left to the reader as an easy exercise.

Proposition 65. *If $\alpha : I \to \mathcal{C}$ is a functor defined on a small category (Definition 5 p. 10), if X is in $\mathcal{C}$, if $p : \alpha \to \Delta X$ is a morphism in $\mathcal{C}^I$, and if $k : \mathcal{C} \to \mathcal{C}^\vee$ is the Yoneda embedding, then the following conditions* (a), (b), (c), (d) *are equivalent:*

(a) *p is a coprojection in the sense of Definition 56 p. 48,*

(b) *the morphism $k(p) : k \circ \alpha \to \Delta k(X)$ in $\operatorname{Fct}(I, \mathcal{C}^\vee)$ induced by p is a coprojection,*

(c) *the morphism $k(p) : \Delta k(X) \to k \circ \alpha^{\mathrm{op}}$ in $\operatorname{Fct}(I^{\mathrm{op}}, \mathbf{Set}^{\mathcal{C}})$ induced by p is a projection in the sense of Definition 53 p. 47,*

(d) *for all Y in $\mathcal{C}$ the morphism $\operatorname{Hom}_{\mathcal{C}}(p, Y) : \Delta \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{C}}(\alpha, Y)$ in $\operatorname{Fct}(I^{\mathrm{op}}, \mathbf{Set})$ is a projection.*

Morevover, if (b) *holds for some universe $\mathcal{U}$ such that I is $\mathcal{U}$-small and $\mathcal{C}$ is a $\mathcal{U}$-category (Definitions 4 p. 10 and 5 p. 10), then it holds for any such universe; the same applies to* (c) *and* (d).

Condition (d) is often abridged by

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} \alpha, Y) \xrightarrow{\sim} \lim \operatorname{Hom}_{\mathcal{C}}(\alpha, Y).$$

In §58 p. 49 we proved that (a) implies (d).

4.3 Proposition 2.1.10 p. 40

4.3.1 A first generalization

Here is a mild generalization of Proposition 2.1.10 p. 40 of the book (stated below as Corollary 67):

Proposition 66. *Let* $\mathcal{C} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{B}$ *be functors and* I *a small category (Definition 5 p. 10). Assume that* $\mathcal{A}$ *admits inductive limits indexed by* I*, that* G *commutes with such limits, and that for each* Y *in* $\mathcal{B}$ *there is a* Z *in* $\mathcal{C}$ *and an isomorphism*

$$\mathrm{Hom}_{\mathcal{B}}(F(\ ), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(G(\ ), Z)$$

in $\mathcal{A}^\wedge$*. Then* F *commutes with inductive limits indexed by* I*.*

Proof. Let θ be the isomorphism $\mathrm{Hom}_{\mathcal{B}}(F(\ ), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(G(\ ), Z)$, let $\alpha : I \to \mathcal{A}$ be a functor and let $p : \alpha \to \Delta \operatorname{colim} \alpha$ be the coprojection. Note that $\operatorname{colim} F \circ \alpha$ exists in $\mathcal{B}^\vee$. Consider the self-explanatory commutative diagrams

$$\begin{array}{ccccc} F \circ \alpha & \xrightarrow{q} & \Delta \operatorname{colim} F \circ \alpha & \qquad & \operatorname{colim} F \circ \alpha \\ & \searrow^{F(p)} & \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle f} \\ & & \Delta F(\operatorname{colim} \alpha) & & F(\operatorname{colim} \alpha) \end{array} \tag{21}$$

and

$$\begin{array}{ccccc} G \circ \alpha & \xrightarrow{r} & \Delta \operatorname{colim} G \circ \alpha & \qquad & \operatorname{colim} G \circ \alpha \\ & \searrow^{G(p)} & \downarrow{\scriptstyle \Delta g} & & \downarrow{\scriptstyle g} \\ & & \Delta G(\operatorname{colim} \alpha) & & G(\operatorname{colim} \alpha), \end{array}$$

where q and r are the coprojections. Note that g is an isomorphism by assumption. Our goal is to prove that f is an isomorphism too. Let Y be in $\mathcal{B}$. It suffices to show that the map

$$\mathrm{Hom}_{\mathcal{B}^\vee}(f, Y) : \mathrm{Hom}_{\mathcal{B}}(F(\operatorname{colim} \alpha), Y) \to \mathrm{Hom}_{\mathcal{B}^\vee}(\operatorname{colim} F \circ \alpha, Y)$$

is bijective. Form the commutative diagram

$$\begin{CD}
\Delta \operatorname{Hom}_{\mathcal{B}}(F(\operatorname{colim}\alpha),Y) @>{\operatorname{Hom}_{\mathcal{B}}(F(p),Y)}>> \operatorname{Hom}_{\mathcal{B}}(F\circ\alpha,Y)\\
@V{\Delta\theta_{\operatorname{colim}\alpha}}V{\sim}V @V{\sim}V{\theta_\alpha}V\\
\Delta \operatorname{Hom}_{\mathcal{C}}(G(\operatorname{colim}\alpha),Z) @>{\operatorname{Hom}_{\mathcal{C}}(r,Z)}>> \operatorname{Hom}_{\mathcal{C}}(G\circ\alpha,Z)\\
@V{\Delta\operatorname{Hom}_{\mathcal{C}}(g,Z)}V{\sim}V @|\\
\Delta \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} G\circ\alpha,Y) @>{\operatorname{Hom}_{\mathcal{C}}(G(p),Z)}>> \operatorname{Hom}_{\mathcal{C}}(G\circ\alpha,Z)\\
@. @A{\sim}A{\theta_\alpha}A\\
\Delta \operatorname{Hom}_{\mathcal{B}^\vee}(\operatorname{colim} F\circ\alpha,Y) @>{\operatorname{Hom}_{\mathcal{B}^\vee}(q,Y)}>> \operatorname{Hom}_{\mathcal{B}}(F\circ\alpha,Y).
\end{CD}$$

The last three horizontal arrows are projections. The bottom horizontal arrow being a projection, there is a unique map

$$h:\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} G\circ\alpha,Y)\to\operatorname{Hom}_{\mathcal{B}^\vee}(\operatorname{colim} F\circ\alpha,Y)$$

making the diagram

$$\begin{CD}
\Delta \operatorname{Hom}_{\mathcal{B}}(F(\operatorname{colim}\alpha),Y) @>{\operatorname{Hom}_{\mathcal{B}}(F(p),Y)}>> \operatorname{Hom}_{\mathcal{B}}(F\circ\alpha,Y)\\
@V{\Delta\theta_{\operatorname{colim}\alpha}}V{\sim}V @V{\sim}V{\theta_\alpha}V\\
\Delta \operatorname{Hom}_{\mathcal{C}}(G(\operatorname{colim}\alpha),Z) @>{\operatorname{Hom}_{\mathcal{C}}(r,Z)}>> \operatorname{Hom}_{\mathcal{C}}(G\circ\alpha,Z)\\
@V{\Delta\operatorname{Hom}_{\mathcal{C}}(g,Z)}V{\sim}V @|\\
\Delta \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} G\circ\alpha,Y) @>{\operatorname{Hom}_{\mathcal{C}}(G(p),Z)}>> \operatorname{Hom}_{\mathcal{C}}(G\circ\alpha,Z)\\
@V{\Delta h}VV @A{\sim}A{\theta_\alpha}A\\
\Delta \operatorname{Hom}_{\mathcal{B}^\vee}(\operatorname{colim} F\circ\alpha,Y) @>{\operatorname{Hom}_{\mathcal{B}^\vee}(q,Y)}>> \operatorname{Hom}_{\mathcal{B}}(F\circ\alpha,Y)
\end{CD}\tag{22}$$

commute. Moreover h is bijective because $\operatorname{Hom}_{\mathcal{C}}(G(p),Z)$ is a projection. Define the bijection

$$k:\operatorname{Hom}_{\mathcal{B}}(F(\operatorname{colim}\alpha),Y)\to\operatorname{Hom}_{\mathcal{B}^\vee}(\operatorname{colim} F\circ\alpha,Y)$$

by $k:=h\circ\operatorname{Hom}_{\mathcal{C}}(g,Z)\circ\theta_{\operatorname{colim}\alpha}$. It is enough to check that we have $\operatorname{Hom}_{\mathcal{B}^\vee}(f,Y)=k$. As $\operatorname{Hom}_{\mathcal{B}^\vee}(q,Y)$ is a projection, this equality follows from the commutativity of (22). □

Corollary 67 (Proposition 2.1.10 p. 40)**.** *Let $F:\mathcal{A}\to\mathcal{B}$ be a functor and I a small category (Definition 5 p. 10). Assume that $\mathcal{A}$ admits inductive limits indexed by I*

and that F admits a right adjoint. Then F commutes with inductive limits indexed by I.

Proof. Let $R : \mathcal{B} \to \mathcal{A}$ be right adjoint to F, and in Proposition 66, let $\mathcal{C}$ be $\mathcal{A}$, G be $\mathrm{id}_{\mathcal{A}}$ and Z be $R(Y)$. □

4.3.2 A second generalization

Here is another mild generalization of Proposition 2.1.10.

Proposition 68. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor admitting a left adjoint and α a functor from I^{op} to $\mathcal{C}$, where I is a small category, such that $\lim \alpha$ exists in $\mathcal{C}$. Then $F(\lim \alpha)$ is a limit of $F \circ \alpha$. Moreover, if $p : \Delta \lim \alpha \to \alpha$ is the old projection, then $F(p) : \Delta F(\lim \alpha) \to F \circ \alpha$ is the new projection.*

Proof. Let $\theta' : \Delta X' \to F \circ \alpha$ be a morphism. It suffices to solve the problem

$$\begin{array}{ccccc} X' & & \Delta X' & \xrightarrow{\theta'} & \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \Delta f'} & & \\ F(\lim \alpha) & & \Delta F(\lim \alpha) & \xrightarrow[F(p)]{} & F \circ \alpha. \end{array}$$

By adjunction we get a morphism $\theta : \Delta G(X') \to \alpha$. Write $f : G(X') \to \lim \alpha$ for the solution to the new problem

$$\begin{array}{ccccc} G(X') & & \Delta G(X') & \xrightarrow{\theta} & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Delta f} & & \\ \lim \alpha & & \Delta \lim \alpha & \xrightarrow[p]{} & \alpha, \end{array}$$

and define $f' : X' \to F(\lim \alpha)$ as the morphism attached to $f : G(X') \to \lim \alpha$ by adjunction. Let i be in I. We are left with checking that $F(p_i) \circ f' = \theta'_i$ knowing that $p_i \circ f = \theta_i$. Let

$$\lambda_{X',X} : \mathrm{Hom}_{\mathcal{C}'}(X', F(X)) \to \mathrm{Hom}_{\mathcal{C}}(G(X'), X).$$

be the bijection given by the adjunction. Set $X := \lim \alpha$. The equality $F(p_i) \circ f' = \theta'_i$ follows from the commutativity of

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}'}(X', F(X)) & \xrightarrow{\lambda_{X',X}} & \mathrm{Hom}_{\mathcal{C}}(G(X'), X) \\ {\scriptstyle F(p_i)\circ}\downarrow & & \downarrow{\scriptstyle p_i\circ} \\ \mathrm{Hom}_{\mathcal{C}'}(X', F(\alpha(i))) & \xrightarrow[\lambda_{X',\alpha(i)}]{} & \mathrm{Hom}_{\mathcal{C}}(G(X'), \alpha(i)). \end{array}$$

□

4.4 Universal limits

Proposition 69. *Let $\alpha : I \to \mathcal{C}$ be an arbitrary functor. For each universe $\mathcal{U}$ such that I is $\mathcal{U}$-small and $\mathcal{C}$ is a $\mathcal{U}$-category (Definitions 5 p. 10 and 4 p. 10), let $h_{\mathcal{U}} : \mathcal{C} \to \mathcal{C}^\wedge_{\mathcal{U}}$ be the Yoneda embedding and define $A_{\mathcal{U}} \in \mathcal{C}^\wedge_{\mathcal{U}}$ by $A_{\mathcal{U}}(X) := \operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha)$, where $\operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha)$ is defined as in Convention 57 p. 48. Let $\mathcal{U}$ and $\mathcal{V}$ be two such universes. Then $A_{\mathcal{U}}$ is representable if and only if $A_{\mathcal{V}}$ is. Let X be in $\mathcal{C}$. Then X represents $A_{\mathcal{U}}$ if and only if X represents $A_{\mathcal{V}}$. Assume that such is the case. Then X is a colimit of α in $\mathcal{C}$. Moreover, for any functor $F : \mathcal{C} \to \mathcal{C}'$, the natural morphism $\operatorname{colim} F \circ \alpha \to F(X)$ is an isomorphism.*

Proof. Let us prove the last sentence, the others following from Corollary 41 p. 37. Let $\mathcal{U}$ be a universe such that I is $\mathcal{U}$-small and $\mathcal{C}$ and $\mathcal{C}'$ are $\mathcal{U}$-categories, let X' be in $\mathcal{C}'$ and define $A \in \mathcal{C}^\wedge_{\mathcal{U}}$ by $A := \mathrm{Hom}_{\mathcal{C}'}(F(\ ), X')$. Let $p : \alpha \to \Delta X$ be the coprojection. By Proposition 65 p. 54, it suffices to show that the morphism

$$\mathrm{Hom}_{\mathcal{C}'}(F(p), X') : \Delta \mathrm{Hom}_{\mathcal{C}'}(F(X), X') \to \mathrm{Hom}_{\mathcal{C}'}(F \circ \alpha, X')$$

in $\mathrm{Fct}(I^{\mathrm{op}}, \mathbf{Set}_{\mathcal{U}})$ is a projection (see Definition 53 p. 47). Consider the commutative diagram

$$\begin{array}{ccc} \Delta A(X) & \xrightarrow{A(p)} & A \circ \alpha \\ {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} \\ \Delta \mathrm{Hom}_{\mathcal{C}^\wedge_{\mathcal{U}}}(X, A) & \xrightarrow[\mathrm{Hom}_{\mathcal{C}^\wedge_{\mathcal{U}}}(h_{\mathcal{U}}(p), A)]{} & \mathrm{Hom}_{\mathcal{C}^\wedge_{\mathcal{U}}}(\alpha, A) \end{array}$$

in $\mathrm{Fct}(I^{\mathrm{op}}, \mathbf{Set}_{\mathcal{V}})$, where $\mathcal{V}$ is a universe such that $\mathcal{C}^\wedge_{\mathcal{U}}$ is a $\mathcal{V}$-category, the vertical isomorphisms being given by the Yoneda Lemma. The bottom horizontal arrow being a projection by assumption, the top horizontal arrow is also a projection. □

Definition 70 (universal inductive limit)**.** *Let $\alpha : I \to \mathcal{C}$ be an arbitrary functor and let X be an object of $\mathcal{C}$. If, in the notation of Proposition 69, the functor $A_{\mathcal{U}}$ is representable for some $\mathcal{U}$ such that I is $\mathcal{U}$-small and $\mathcal{C}$ is a $\mathcal{U}$-category (Definitions 4 p. 10 and 5 p. 10), we say that X is a* universal *inductive limit of α, and that* $\operatorname{colim} \alpha$ *exists* universally *in $\mathcal{C}$.*

There is of course an analogous notion of universal projective limit.

Here is the classic example of a non-universal inductive limit. Letting α be the unique functor from the empty category to **Set**, we get $\operatorname{colim} \alpha = \varnothing$. Writing $h : \mathbf{Set} \to \mathbf{Set}^\wedge$ for the Yoneda embedding yields $\operatorname{colim} h \circ \alpha = \Delta\varnothing$. But we have, on the one hand $(\Delta\varnothing)(\varnothing) = \varnothing$, and on the other hand $(h(\varnothing))(\varnothing) \not\simeq \varnothing$, implying

$$\operatorname{colim} h \circ \alpha \not\simeq h(\operatorname{colim} \alpha).$$

This shows that the inductive limit of α does not exist universally in **Set**.

4.5 Brief comments

§ 71.

Proposition 72. *If I and J are big categories, if $\mathcal{C}$ is a $\mathcal{U}$-category (Definition 4 p. 10), if $\alpha : I \times J \to \mathcal{C}$ is a functor and if* $\operatorname{colim}_i \alpha(i, \ )$ *exists universally in the sense of Definition 70 p. 59, then* $\operatorname{colim}_i \alpha(i, j)$ *exists universally for all j.*

In §61 p. 51 we saw that, without the adverb "universally", the claim is false.

Proof. To prove the proposition, we may assume that I is small (Definition 5 p. 10). Then the statement follows from Proposition 60 p. 50. □

§ 73. P. 40, proof of Lemma 2.1.11 (minor variant).

Lemma 74. *If T is an object of a category $\mathcal{C}$, then*

$$T \text{ is terminal } \Leftrightarrow T \simeq \operatorname{colim} \operatorname{id}_{\mathcal{C}} .$$

Proof. $\Rightarrow$: Straightforward.

$\Leftarrow$: Let $p : \mathrm{id}_{\mathcal{C}} \to \Delta T$ be a coprojection (see Definition 56 p. 48) and let X be in $\mathcal{C}$. For all morphism of functors $\theta : \mathrm{id}_{\mathcal{C}} \to \Delta X$ there is a unique morphism $f : T \to X$ satisfying $\Delta f \circ p = \theta$:

$$\begin{array}{ccccc} \mathrm{id}_{\mathcal{C}} & \xrightarrow{p} & \Delta T & \qquad & T \\ & \searrow^{\theta} & \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle f} \\ & & \Delta X & & X. \end{array}$$

We claim

$$\mathrm{id}_T = p_T. \tag{23}$$

We have indeed $(\Delta p_T \circ p)_X = p_T \circ p_X = p_X = (\Delta \, \mathrm{id}_T \circ p)_X$. This proves (23). If $f : X \to T$ is a morphism in $\mathcal{C}$, then we have $f = \mathrm{id}_T \circ f = p_T \circ f = p_X$, the second equality following from (23). This shows that T is terminal. $\square$

Corollary 75. *If $\mathcal{C}$ is a category and A an object of $\mathcal{C}^\wedge$, then the following conditions are equivalent:*

(a) *A is representable,*

(b) *$\mathcal{C}_A$ has a terminal object,*

(c) *the identity of $\mathcal{C}_A$ has an inductive limit in $\mathcal{C}_A$.*

Proof. This follows from Lemma 74 above and Convention 39 p. 36. $\square$

§ 76. P. 41, Lemma 2.1.12. The following variant will be useful to prove Proposition 2.5.2 p. 57 of the book (see §105 p. 79 below).

Lemma 77. *If I and $\mathcal{C}$ are categories, if X is in $\mathcal{C}$, if $\Delta X : I \to \mathcal{C}$ is the constant functor with value X, and if I is connected, then*

(a) *$\mathrm{id}_{\Delta X} : \Delta X \to \Delta X$ is a coprojection in the sense of Definition 56 p. 48,*

(b) *if i is in I, Y in $\mathcal{C}$, $f : X \to Y$ and $\theta : \Delta X \to \Delta Y$, then the equalities $\Delta f = \theta$ and $f = \theta_i$ are equivalent:*

$$\begin{array}{ccccc} \Delta X & \xrightarrow{\mathrm{id}} & \Delta X & \qquad & X \\ & \searrow^{\theta} & \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle f} \\ & & \Delta Y & & Y. \end{array}$$

(c) *we have $\theta_i = \theta_j$ for all $\theta : \Delta X \to \Delta Y$ with Y in $\mathcal{C}$ and all i, j in I.*

Proof. To prove (c) we can assume that there is a morphism $i \to j$, in which case the claim is obvious. Clearly (c) implies (a) and (b). □

§ 78. P. 42, proof of Lemma 2.1.15. Here are some additional details about the last diagram on p. 42:

To the commutative diagram

$$\begin{array}{ccc} i & \xrightarrow{f} & j \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle f} \\ i & \xrightarrow[\mathrm{id}]{} & i \end{array}$$

in I we attach the commutative diagram

$$\begin{array}{ccc} \alpha(i) & \xrightarrow{\varphi(f)} & \beta(j) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle \beta(f)} \\ \alpha(i) & \xrightarrow[\varphi(\mathrm{id}_i)]{} & \beta(i) \end{array} \qquad (24)$$

in $\mathcal{C}$. Turning (24) upside down we get

$$\begin{array}{ccc} \alpha(i) & \xrightarrow{\varphi(\mathrm{id}_i)} & \beta(i) \\ {\scriptstyle \mathrm{id}}\big\uparrow & & \big\downarrow{\scriptstyle \beta(f)} \\ \alpha(i) & \xrightarrow[\varphi(f)]{} & \beta(j). \end{array} \qquad (25)$$

To the commutative diagram

$$\begin{array}{ccc} i & \xrightarrow{f} & j \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ j & \xrightarrow[\mathrm{id}]{} & j \end{array}$$

in I we attach the commutative diagram

$$\begin{array}{ccc} \alpha(i) & \xrightarrow{\varphi(f)} & \beta(j) \\ {\scriptstyle \alpha(f)}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ \alpha(j) & \xrightarrow[\varphi(\mathrm{id}_j)]{} & \beta(j) \end{array} \qquad (26)$$

in $\mathcal{C}$. Splicing (25) and (26) we get

$$\begin{array}{ccc}
\alpha(i) & \xrightarrow{\varphi(\mathrm{id}_i)} & \beta(i) \\
{\scriptstyle \mathrm{id}}\uparrow & & \downarrow{\scriptstyle \beta(f)} \\
\alpha(i) & \xrightarrow{\varphi(f)} & \beta(j) \\
{\scriptstyle \alpha(f)}\downarrow & & \uparrow{\scriptstyle \mathrm{id}} \\
\alpha(j) & \xrightarrow[\varphi(\mathrm{id}_j)]{} & \beta(j).
\end{array}$$

Reversing the identity arrows, we get

$$\begin{array}{ccc}
\alpha(i) & \xrightarrow{\varphi(\mathrm{id}_i)} & \beta(i) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \beta(f)} \\
\alpha(i) & \xrightarrow{\varphi(f)} & \beta(j) \\
{\scriptstyle \alpha(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\alpha(j) & \xrightarrow[\varphi(\mathrm{id}_j)]{} & \beta(j),
\end{array}$$

as desired.

§ 79. Lemma 2.1.15 p. 42. Here is a complement which will be used in §516 p. 297.

Theorem 80. *Let I be a small (Definition 5 p. 10) category; let $\alpha, \beta : I \to \mathcal{C}$ be two functors; for each i in I, let $U^i : I^i \to I$ be the forgetful functor; and set*

$$S := \mathrm{Hom}_{\mathrm{Fct}(I,\mathcal{C})}(\alpha, \beta), \quad T := \varprojlim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Fct}(I^i,\mathcal{C})}(\alpha \circ U^i, \beta \circ U^i).$$

Then there is a unique map $f : T \to S$ satisfying $f(t)_i = t_{\mathrm{id}_i}$ for all t in T. Moreover f is inverse to the natural map from S to T.

Proof. Left to the reader. □

§ 81. P. 44, Definition 2.2.2 (iii). A sequence $X \to Y \rightrightarrows Z$ in a category $\mathcal{C}$ is exact if and only if its image in $\mathcal{C}^\wedge$ is exact.

4.6 Stability by base change

§ 82. (See also Section 4.20 p. 93.) Recall the following definition:

Definition 83 (Definition 2.2.6 p. 47, stability by base change)**.** *Let $\mathcal{C}$ be a category which admits fiber products and inductive limits indexed by a category I.*

(i) *We say that inductive limits in $\mathcal{C}$ indexed by I are* stable by base change *if for any morphism $Y \to Z$ in $\mathcal{C}$, the base change functor $\mathcal{C}_Z \to \mathcal{C}_Y$ given by*

$$\mathcal{C}_Z \ni (X \to Z) \mapsto (X \times_Z Y \to Y) \in \mathcal{C}_Y$$

commutes with inductive limits indexed by I.

This is equivalent to saying that for any inductive system $(X_i)_{i \in I}$ in $\mathcal{C}$ and any pair of morphisms $Y \to Z$ and $\operatorname{colim}_i X_i \to Z$ in $\mathcal{C}$, we have the isomorphism

$$\operatorname*{colim}_i (X_i \times_Z Y) \xrightarrow{\sim} \left(\operatorname*{colim}_i X_i\right) \times_Z Y.$$

(ii) *If $\mathcal{C}$ admits small inductive limits and* (i) *holds for any small category I (Definition 5 p. 10), we say that* small inductive limits in $\mathcal{C}$ are stable by base change*.*

The following lemma is implicit:

Lemma 84. *Let I and $\mathcal{C}$ be categories, let Y be an object of $\mathcal{C}$, let $U : \mathcal{C}_Y \to \mathcal{C}$ be the forgetful functor, and let $\alpha : I \to \mathcal{C}_Y$ be a functor such that $\operatorname{colim} U \circ \alpha$ exists in $\mathcal{C}$. Then $\operatorname{colim} \alpha$ exists in $\mathcal{C}_Y$ and is given by the natural morphism $\operatorname{colim} U \circ \alpha \to Y$. More precisely, let $X \to Y$ be a morphism in $\mathcal{C}$, let $p : \alpha \to \Delta(X \to Y)$ be a morphism in $\operatorname{Fct}(I, \mathcal{C}_Y)$, and let $U \star p : U \circ \alpha \to \Delta X$ be the corresponding morphism in $\operatorname{Fct}(I, \mathcal{C})$. [Recall that $\star$ denotes the horizontal composition defined in Definition 35 p. 32.] If $U \star p$ is a coprojection (see Definition 56 p. 48), then so is p.*

Proof. Let $Z \to Y$ be a morphism in $\mathcal{C}$ and $\lambda : \alpha \to \Delta(Z \to Y)$ be a morphism in $\operatorname{Fct}(I, \mathcal{C}_Y)$. We must show that there is a unique morphism $f : X \to Z$ in $\mathcal{C}_Y$ such that $\Delta f \circ p = \lambda$:

$$\begin{array}{ccccc} \alpha & \xrightarrow{p} & \Delta(X \to Y) & \qquad & (X \to Y) \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle f} \\ & & \Delta(Z \to Y) & & (Z \to Y). \end{array}$$

Let $\mu : U \circ \alpha \to \Delta Z$ be the morphism in $\text{Fct}(I, \mathcal{C})$ induced by λ. Then there is a unique morphism $f : X \to Z$ such that $\Delta f \circ (U \star p) = \mu$:

$$\begin{array}{ccccc} U \circ \alpha & \xrightarrow{U\star p} & \Delta X & \qquad & X \\ & \searrow^{\mu} & \downarrow{\scriptstyle \Delta f} & & \vdots{\scriptstyle f} \\ & & \Delta Z & & Z. \end{array}$$

It remains to check that f is a morphism in $\mathcal{C}_Y$, that is, we must prove

$$(X \xrightarrow{f} Z \to Y) = (X \to Y).$$

Let i be in I. As $U \star p$ is a coprojection, it suffices to show

$$\Big(U(\alpha(i)) \to X \xrightarrow{f} Z \to Y\Big) = \Big(U(\alpha(i)) \to X \to Y\Big).$$

But we have

$$\begin{aligned} (U(\alpha(i)) &\xrightarrow{(U\star p)_i} X \xrightarrow{f} Z \longrightarrow Y) &= \\ (U(\alpha(i)) &\xrightarrow{\lambda_i} Z \longrightarrow Y) &= \\ (U(\alpha(i)) &\longrightarrow Y) &= \\ (U(\alpha(i)) &\xrightarrow{(U\star p)_i} X \longrightarrow Y). \end{aligned}$$

Indeed, the first and third equalities follow from the fact that $U \star p$ is a coprojection, and the second equality follows from the fact that λ_i is a morphism in $\mathcal{C}_Y$. $\square$

§ 85. We make an easy but useful observation. Let I, J and $\mathcal{C}$ be three categories.

If $\mathcal{C}$ admits fiber products, and if inductive limits indexed by I exist in $\mathcal{C}$ and are stable by base change, then $\mathcal{C}^J$ admits fiber products, and inductive limits indexed by I exist in $\mathcal{C}^J$ and are stable by base change.

4.7 Brief comments

§ 86. P. 50, Corollary 2.2.11. We also have:

A category admits finite projective limits if and only if it admits a terminal object and binary fibered products.

Indeed, if $f, g : X \rightrightarrows Y$ is a pair of parallel arrows, and if the square

$$\begin{array}{ccc} K & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

is cartesian, then $K \simeq \operatorname{Ker}(f, g)$. (As usual, Δ is the diagonal morphism.)

§ 87. P. 50, Definition 2.3.1. The three pieces of notation $\varphi_*, \varphi^\dagger$ and $\varphi^\ddagger$ are justified by Notation 17.1.5 p. 407 (see also (199) p. 295).

§ 88. P. 50, Definition 2.3.1. Let $\varphi : J \to I$ be a functor of small categories (Definition 5 p. 10), let $\mathcal{C}$ be a category, and consider the functor

$$\varphi_* := \circ\varphi : \mathcal{C}^I \to \mathcal{C}^J. \tag{27}$$

The following fact results from Proposition 2.1.6 p. 38 of the book (see §59 p. 49):

If $\mathcal{C}$ admits small inductive (resp. projective) limits, then so do $\mathcal{C}^I$ and $\mathcal{C}^J$, and φ_* commutes with such limits.

Recall that we denote horizontal composition of morphisms of functors by $\star$ (see Definition 35 p. 32).

The effect of the functor (27) on morphisms can be described as follows: If $\theta : \alpha \to \beta$ is a morphism in $\mathcal{C}^I$, then the morphism $\varphi_*\theta : \varphi_*\alpha \to \varphi_*\beta$ in $\mathcal{C}^J$ is defined by $\varphi_*\theta := \theta \star \varphi$, which is in turn defined by $(\theta \star \varphi)_j := \theta_{\varphi(j)}$.

§ 89. P. 51, Definition 2.3.2. Recall that we have functors $I \xleftarrow{\varphi} J \xrightarrow{\beta} \mathcal{C}$. We spell out Definition 2.3.2 using the terminology of Section 3.8 p. 39. Let $\varphi^\dagger\beta$ and $\varphi^\ddagger\beta$ be in $\mathcal{C}^I$.

(a) We say that "$\varphi^\dagger\beta$ exists" if there is a co-unit $\varepsilon_\beta : \beta \to \varphi_*\varphi^\dagger\beta$, that is, for all $\alpha : I \to \mathcal{C}$ and all $w : \beta \to \varphi_*\alpha$ there is a unique $v : \varphi^\dagger\beta \to \alpha$ such that $(v \star \varphi) \circ \varepsilon_\beta = w$:

$$\begin{array}{ccccc} \beta & \xrightarrow{\varepsilon_\beta} & \varphi_*\varphi^\dagger\beta & \qquad & \varphi^\dagger\beta \\ & {\scriptstyle w}\searrow & \big\downarrow{\scriptstyle v\star\varphi} & & \big\downarrow{\scriptstyle v} \\ & & \varphi_*\alpha & & \alpha. \end{array}$$

(b) We say that "$\varphi^\ddagger\beta$ exists" if there is a unit $\eta_\beta : \varphi_*\varphi^\ddagger\beta \to \beta$, that is, for all $\alpha : I \to \mathcal{C}$ and all $w : \varphi_*\alpha \to \beta$ there is a unique $v : \alpha \to \varphi^\ddagger\beta$ such that $\eta_\beta \circ (v \star \varphi) = w$:

$$\begin{array}{ccc} \alpha & \qquad & \varphi_*\alpha & & \\ {\scriptstyle v}\,\vdots\downarrow & & {\scriptstyle v\star\varphi}\downarrow & \searrow^{w} & \\ \varphi^\ddagger\beta & & \varphi_*\varphi^\ddagger\beta & \xrightarrow[\eta_\beta]{} & \beta \end{array}$$

(c) Let $\gamma : I \to \mathcal{C}$ and let u be an endomorphism of the functor $\varphi_*\gamma : J \to \mathcal{C}$. The phrase "$\varphi^\ddagger\varphi_*\gamma$ exists and is isomorphic to γ via u" shall mean that for all $\alpha : I \to \mathcal{C}$ and all $w : \varphi_*\alpha \to \varphi_*\gamma$ there is a unique $v : \alpha \to \gamma$ such that $u \circ (v \star \varphi) = w$:

$$\begin{array}{ccc} \alpha & \qquad & \varphi_*\alpha & & \\ {\scriptstyle v}\,\vdots\downarrow & & {\scriptstyle v\star\varphi}\downarrow & \searrow^{w} & \\ \gamma & & \varphi_*\gamma & \xrightarrow[u]{} & \varphi_*\gamma. \end{array}$$

In particular, the phrase "$\varphi^\ddagger\varphi_*\gamma$ exists and is isomorphic to γ via the identity of $\varphi_*\gamma : J \to \mathcal{C}$" shall mean that, for all $\alpha : I \to \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}^I}(\alpha, \gamma) \to \mathrm{Hom}_{\mathcal{C}^J}(\varphi_*\alpha, \varphi_*\gamma), \quad v \mapsto v \star \varphi$$

is bijective. (See §246 p. 156 below.)

§ 90. Let $I \xleftarrow{\varphi} J \xrightarrow{\beta} \mathcal{C} \xrightarrow{F} \mathcal{C}'$ be functors, let β be in $\mathcal{C}^J$ and assume that $\varphi^\dagger(\beta)$ and $\varphi^\dagger(F \circ \beta)$ exist:

$$\begin{array}{ccc} J & \xrightarrow{\varphi} & I \\ {\scriptstyle\beta}\downarrow & \swarrow_{\varphi^\dagger(\beta)} & \downarrow{\scriptstyle\varphi^\dagger(F\circ\beta)} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}'. \end{array}$$

We claim that there is a natural morphism $\varphi^\dagger(F \circ \beta) \to F \circ \varphi^\dagger(\beta)$.

As $F \circ \varphi^\dagger(\beta) \circ \varphi = \varphi_*(F \circ \varphi^\dagger(\beta))$, it suffices to define a natural morphism $F \circ \beta \to F \circ \varphi^\dagger(\beta) \circ \varphi$:

$$\begin{array}{ccc} F \circ \beta & \xrightarrow{\varepsilon_{F\circ\beta}} & \varphi_*\varphi^\dagger(F \circ \beta) \\ & \searrow & \downarrow \\ & & \varphi_*(F \circ \varphi^\dagger(\beta)) \end{array} \qquad \begin{array}{c} \varphi^\dagger(F \circ \beta) \\ \vdots\downarrow \\ F \circ \varphi^\dagger(\beta). \end{array}$$

As we have $\varepsilon_\beta : \beta \to \varphi^\dagger(\beta) \circ \varphi$, we can take $F \star \varepsilon_\beta : F \circ \beta \to F \circ \varphi^\dagger(\beta) \circ \varphi$.

Definition 91. *If the above morphism $\varphi^\dagger(F \circ \beta) \to F \circ \varphi^\dagger(\beta)$ is an isomorphism, we say that F commutes with $\varphi^\dagger$ at β.*

§ 92. P. 51, Definition 2.3.2 (minor variant). We assume that no underlying universe has been given. Let $I \xleftarrow{\varphi} J \xrightarrow{\beta} \mathcal{C}$ be functors, let β be in $\mathcal{C}^J$, and let $\varphi^\dagger\beta$ be in $\mathcal{C}^I$. The following conditions are equivalent:

(a) $\varphi^\dagger\beta$ represents $\mathrm{Hom}_{\mathcal{C}^J}(\beta, \varphi_*(\ )) \in (\mathcal{C}^I)^\vee_{\mathcal{U}}$ for *some* universe $\mathcal{U}$ such that $\mathcal{C}^J$ is a $\mathcal{U}$-category (Definition 4 p. 10),

(b) $\varphi^\dagger\beta$ represents $\mathrm{Hom}_{\mathcal{C}^J}(\beta, \varphi_*(\ )) \in (\mathcal{C}^I)^\vee_{\mathcal{U}}$ for *any* universe $\mathcal{U}$ such that $\mathcal{C}^J$ is a $\mathcal{U}$-category.

Definition 93 (Universal Kan extension)**.** *If the above equivalent conditions hold, we say that $\varphi^\dagger\beta$* exists *(this is compatible with §89 (a) p. 65). If, in addition, $\varphi^\dagger(F \circ \beta)$ exists and the natural morphism $\varphi^\dagger(F \circ \beta) \to F \circ \varphi^\dagger(\beta)$ is an isomorphism for all big category $\mathcal{C}'$ and all functor $F : \mathcal{C} \to \mathcal{C}'$, we say that $\varphi^\dagger\beta$ exists* universally*.*

4.8 Theorem 2.3.3 (i) p. 52

Note that projective and inductive limits are particular cases of Kan extensions.

Recall the statement:

Theorem 94 (Theorem 2.3.3 (i) p. 52)**.** *Let $I \xleftarrow{\varphi} J \xrightarrow{\beta} \mathcal{C}$ be functors. Assume that*

$$\operatorname*{colim}_{(\varphi(j)\to i)\in J_i} \beta(j)$$

exists in $\mathcal{C}$ for all i in I. Then $\varphi^\dagger(\beta)$ exists and we have

$$\varphi^\dagger(\beta)(i) \simeq \operatorname*{colim}_{(\varphi(j)\to i)\in J_i} \beta(j) \tag{28}$$

for all i in I. In particular, if $\mathcal{C}$ admits small inductive limits and J is small, then $\varphi^\dagger$ exists. If moreover φ is fully faithful, then $\varphi^\dagger$ is fully faithful and the co-unit $\varepsilon_\beta : \mathrm{id}_{\mathcal{C}^J} \to \varphi_* \circ \varphi^\dagger$ (see §51 p. 40) is an isomorphism.

Note that, as observed by Kelly (see Section 4.2 of the book quoted above in §61 p. 51), the above sufficient condition for $\varphi^\dagger(\beta)$ to exist is not necessary. This non-necessity results from §61 and the following remark. If $\pi : I \times J \to I$ is the projection, and $\alpha : I \times J \to \mathcal{C}$ is a functor, then $\pi^\dagger(\alpha) \simeq \operatorname{colim}_{j\in J} \alpha(\ ,j)$ (in the strong sense that one exists if and only if the other exists); in contrast we have $\operatorname{colim}_{(\varphi(i',j)\to i)\in(I\times J)_i} \alpha(j) \simeq \operatorname{colim}_{j\in J} \alpha(i,j)$ (in the same strong sense); and we saw that the existence of $\operatorname{colim}_{j\in J} \alpha(i,j)$ for all i implies that of $\operatorname{colim}_{j\in J} \alpha(\ ,j)$, but the converse is not true. We have however

Theorem 95 (Universal Kan Extension Theorem)**.** *If $I \xleftarrow{\varphi} J \xrightarrow{\beta} \mathcal{C}$ are arbitrary functors and*

$$\operatorname*{colim}_{(\varphi(j)\to i)\in J_i} \beta(j) \tag{29}$$

exists in $\mathcal{C}$ for all i in I, then $\varphi^\dagger\beta$ exists, and $(\varphi^\dagger\beta)(i)$ is isomorphic to (29)*. Moreover, the following conditions are equivalent*

(a) *the colimit* (29) *exists universally for all i in the sense of Definition 70 p. 59,*

(b) *$\varphi^\dagger\beta$ exists universally in the sense of Definition 93 p. 67,*

(c) *$\varphi^\dagger\beta$ exists and there is a universe $\mathcal{U}$ such that J is $\mathcal{U}$-small, $\mathcal{C}$ is a $\mathcal{U}$-category (Definitions 4 p. 10 and 5 p. 10), the functor $\varphi^\dagger(h \circ \beta)$ (where $h : \mathcal{C} \to \mathcal{C}^\wedge$ is the Yoneda embedding) exists, and the natural morphism $\varphi^\dagger(h \circ \beta) \to h \circ \varphi^\dagger(\beta)$ is an isomorphism.*

Proof. This follows from Theorem 94 p. 67 and Proposition 69 p. 58. □

In the book the proof of Theorem 2.3.3 (i) is divided into three steps, called (a), (b) and (c). We shall follow this subdivision.

4.8.1 Step (a)

We define $\varphi^\dagger(\beta)$ by (28). The purpose of Step (a) is to show that $\varphi^\dagger(\beta)$ is indeed a functor.

For the reader's convenience we reproduce the argument in the book:

Let $i \to i'$ be a morphism in I. It is easily checked that there is a unique morphism

$\varphi^\dagger(\beta)(i) \to \varphi^\dagger(\beta)(i')$ which make the diagrams

$$\begin{array}{ccc} \varphi^\dagger(\beta)(i) = \operatorname*{colim}_{(\varphi(j)\to i)\in J_i} \beta(j) & \longrightarrow & \operatorname*{colim}_{(\varphi(j)\to i')\in J_{i'}} \beta(j) = \varphi^\dagger(\beta)(i') \\ {\scriptstyle p[\varphi(j)\to i]}\nwarrow & & \nearrow{\scriptstyle q[\varphi(j)\to i\to i']} \\ & \beta(j) & \end{array}$$

commute, where $p[\varphi(j) \to i]$ and $q[\varphi(j) \to i \to i']$ are the coprojections, and that the assignment

$$(i \to i') \mapsto \Big(\varphi^\dagger(\beta)(i) \to \varphi^\dagger(\beta)(i')\Big)$$

is functorial.

4.8.2 Step (b)

The purpose of Step (b) is to prove

$$\operatorname{Hom}_{\mathcal{C}^I}(\varphi^\dagger(\beta), \alpha) \simeq \operatorname{Hom}_{\mathcal{C}^J}(\beta, \varphi_*(\alpha)) \tag{30}$$

for all $\alpha : I \to \mathcal{C}$. As pointed out in the book, this can also be achieved by using Lemma 2.1.15 p. 42. Here is a sketch of the argument. We start with a reminder of Lemma 2.1.15.

To any category $\mathcal{A}$ we attach the category $\operatorname{Mor}_0(\mathcal{A})$ defined as follows. The objects of $\operatorname{Mor}_0(\mathcal{A})$ are the triples (X, f, Y) such that f is a morphism in $\mathcal{C}$ from X to Y. The morphisms in $\operatorname{Mor}_0(\mathcal{A})$ from (X, f, Y) to (X', f', Y') are the pairs (u, v) with $u : X \to X'$, $v : Y' \to Y$, and $f = v \circ f' \circ u$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \uparrow{\scriptstyle v} \\ X' & \xrightarrow[g]{} & Y'. \end{array}$$

The composition of morphisms is the obvious one. Lemma 2.1.15 can be stated as follows:

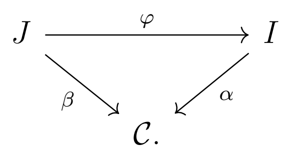
If I and $\mathcal{A}$ are categories, and $a, b : I \rightrightarrows \mathcal{A}$ are functors, then

$$(i, i \to j, j) \mapsto \operatorname{Hom}_{\mathcal{A}}(a(i), b(j))$$

is a functor from $\mathrm{Mor}_0(I)^{\mathrm{op}}$ to **Set**, and there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}^I}(a,b) \xrightarrow{\sim} \lim_{(i\to j)\in\mathrm{Mor}_0(I)} \mathrm{Hom}_{\mathcal{A}}(a(i),b(j)). \tag{31}$$

Returning to (30), we have functors



Let us define the categories M and N as follows: an object of M is a pair

$$(j,\varphi(j)\to i\to i')$$

with j in J and i,i' in I. A morphism

$$\Big(j_1,\varphi(j_1)\to i_1\to i_1'\Big)\to\Big(j_2,\varphi(j_2)\to i_2\to i_2'\Big)$$

is given by a triple of morphisms $j_1\to j_2, i_1\to i_2, i_1'\leftarrow i_2'$ such that the obvious diagram commutes. The category N is $\mathrm{Mor}_0(J)$. Consider the functors

$$\gamma: M^{\mathrm{op}}\to\mathbf{Set},\quad \Big(j,\varphi(j)\to i\to i'\Big)\mapsto \mathrm{Hom}_{\mathcal{C}}\big(\beta(j),\alpha(i')\big),$$

$$\delta: N^{\mathrm{op}}\to\mathbf{Set},\quad (j\to j')\mapsto \mathrm{Hom}_{\mathcal{C}}\big(\beta(j),\alpha(\varphi(j'))\big).$$

The existence of a natural bijection

$$\mathrm{Hom}_{\mathcal{C}^J}\big(\beta,\varphi_*(\alpha)\big)\xrightarrow{\sim}\lim\delta \tag{32}$$

follows immediately from (31). Using (31) again, it is easy to see that we also have a natural bijection

$$\mathrm{Hom}_{\mathcal{C}^I}\big(\varphi^\dagger(\beta),\alpha\big)\xrightarrow{\sim}\lim\gamma. \tag{33}$$

By (32) and (33), it suffices to show

Lemma 96. *There is a natural bijection* $\lim\gamma\simeq\lim\delta$.

Proof. To define a map $\lim\gamma\to\lim\delta$, we attach, to a family

$$\big(\beta(j)\to\alpha(i')\big)_{\varphi(j)\to i\to i'}\in\lim\gamma$$

and to a morphism $j \to j'$, a morphism $\beta(j) \to \alpha(\varphi(j'))$ by setting

$$i := i' := \varphi(j'), \quad (i \to i') := \mathrm{id}_{\varphi(j')},$$

and by taking as $\beta(j) \to \alpha(\varphi(j'))$ the corresponding member of our family. We leave it to the reader to check that this defines indeed a map $\lim \gamma \to \lim \delta$. To define a map $\lim \delta \to \lim \gamma$, we attach, to a family

$$\big(\beta(j) \to \alpha(\varphi(j'))\big)_{j \to j'} \in \lim \delta$$

and to a chain of morphisms $\varphi(j) \to i \to i'$, a morphism $\beta(j) \to \alpha(i')$ by setting

$$j' := j, \quad (j \to j') := \mathrm{id}_j,$$

and by taking as $\beta(j) \to \alpha(i')$ the composition

$$\beta(j) \to \alpha(\varphi(j)) \to \alpha(i) \to \alpha(i').$$

We leave it to the reader to check that this defines indeed a map $\lim \delta \to \lim \gamma$, and that this map is inverse to the map constructed above. □

4.8.3 Step (c)

The purpose of Step (c) is to prove the last two sentences of the statement of Theorem 2.3.3 (i) p. 52 of the book (stated above as Theorem 94 p. 67). We assume that $\mathcal{C}$ admits small inductive limits, that J is small (in particular $\varphi^\dagger$ exists), and that φ is fully faithful. We must show that $\varphi^\dagger$ is fully faithful, and that the co-unit $\varepsilon_\beta : \mathrm{id}_{\mathcal{C}^J} \to \varphi_* \circ \varphi^\dagger$ (see §51 p. 40) is an isomorphism. Recall that we have

$$\varphi^\dagger(\beta)(i) := \operatorname*{colim}_{(\varphi(j) \to i) \in J_i} \beta(j),$$

and let

$$p[\varphi(j) \to i] : \beta(j) \to \varphi^\dagger(\beta)(i)$$

be the coprojections. By the proof of Step (b) in the book, the co-unit

$$\varepsilon_{\beta,j} : \beta(j) \to \varphi^\dagger(\beta)(\varphi(j))$$

coincides with the coprojection $p[\varphi(j) \xrightarrow{\mathrm{id}} \varphi(j)]$. We shall define a morphism

$$\varphi^\dagger(\beta)(\varphi(j)) \to \beta(j)$$

and leave it to the reader to check that it is inverse to $\varepsilon_{\beta,j}$. It suffices to define a functorial family of morphisms

$$f(\varphi(j') \to \varphi(j)) : \beta(j') \to \beta(j)$$

indexed by $(\varphi(j') \to \varphi(j)) \in J_{\varphi(j)}$. Let such a morphism $\varphi(j') \to \varphi(j)$ be given. As φ is fully faithful, we get a well-defined morphism $j' \to j$, and we set

$$f(\varphi(j') \to \varphi(j)) := \beta(j' \to j).$$

It is straightforward to verify that the family of morphisms defined this way is functorial.

4.8.4 A Corollary

Here is a corollary to Theorem 94 p. 67 (which is Theorem 2.3.3 (i) p. 52 of the book):

Corollary 97. *If, in the setting of Theorem 94, we have $\mathcal{C} = \mathbf{Set}$ and J is small (Definition 5 p. 10), then $\varphi^\dagger(\beta)(i)$ is (in natural bijection with) the quotient of*

$$\bigsqcup_{j \in J} \beta(j) \times \operatorname{Hom}_I(\varphi(j), i)$$

by the smallest equivalence relation $\sim$ satisfying the following condition: If $j \to j'$ is a morphism in J, if x is in $\beta(j)$, and if $\varphi(j') \to i$ is a morphism in I, then

$$p_j(x, \varphi(j) \to \varphi(j') \to i) \sim p_{j'}(\beta(j \to j')(x), \varphi(j') \to i),$$

where p_j is the j-coprojection.

Proof. Recall that Theorem 94 p. 67 states the existence of an isomorphism

$$\varphi^\dagger(\beta)(i) \simeq \operatorname*{colim}_{(\varphi(j) \to i) \in J_i} \beta(j).$$

By Proposition 2.4.1 p. 54 of the book, the right-hand side is, in a natural way, the quotient of

$$\bigsqcup_{(\varphi(j) \to i) \in J_i} \beta(j)$$

by a certain equivalence relation. We have

$$\bigsqcup_{(\varphi(j)\to i)\in J_i}\beta(j)=\bigsqcup_{j\in J}\ \bigsqcup_{u\in \mathrm{Hom}_I(\varphi(j),i)}\beta(j)\simeq\bigsqcup_{j\in J}\beta(j)\times\mathrm{Hom}_I(\varphi(j),i),$$

and it easy to see that the three data of the above bijection, of the equivalence relation in Proposition 2.4.1 of the book, and of the equivalence relation in Corollary 97 above are compatible. □

Under the same assumptions $\varphi^\ddagger(\beta)(i)$ is (in natural bijection with) the set of all x in

$$\prod_{(i\to\varphi(j))\in J^i}\beta(j)$$

such that $x_{i\to\varphi(j)\to\varphi(j')}=\beta(j\to j')(x_{i\to\varphi(j)})$ for all morphism $j\to j'$ in J.

4.9 Brief comments

§ 98. P. 53, Corollary 2.3.4. Recall that we have functors $\mathcal{C}\xleftarrow{\beta}J\xrightarrow{\varphi}I$, where I and J are small (Definition 5 p. 10) and $\mathcal{C}$ admits small inductive limits, and that Corollary 2.3.4 says that we have a natural isomorphism $\operatorname{colim}\beta\simeq\operatorname{colim}\varphi^\dagger\beta$, that is

$$\operatorname{colim}\beta\simeq\operatorname*{colim}_i\ \operatorname*{colim}_{(j,u)\in J_i}\beta(j),\tag{34}$$

where (j,u) runs over J_i, with $u:\varphi(j)\to i$.

Proof of (34). We define morphisms

$$\operatorname{colim}\beta\ \underset{g}{\overset{f}{\rightleftarrows}}\ \operatorname*{colim}_i\ \operatorname*{colim}_{(j,u)\in J_i}\beta(j),$$

and claim that f and g are inverse isomorphisms. We have the coprojections

$$\beta(j)\xrightarrow{p_j}\operatorname{colim}\beta,\qquad \beta(j)\xrightarrow{q_{i,j,u}}\operatorname*{colim}_{(j',u')\in J_i}\beta(j')\xrightarrow{r_i}\operatorname*{colim}_{i'}\ \operatorname*{colim}_{(j'',u'')\in J_{i'}}\beta(j'').$$

We define f by the condition that we have

$$f\circ p_j=r_{\varphi(j)}\circ q_{\varphi(j),j,\mathrm{id}_{\varphi(j)}}$$

for all j in J:

$$\begin{array}{ccc}
\beta(j) & \xrightarrow{\mathrm{id}} & \beta(j) \\
 & & \downarrow{\scriptstyle q_{\varphi(j),j,\mathrm{id}_{\varphi(j)}}} \\
\downarrow{\scriptstyle p_j} & & \operatorname*{colim}_{(j',u)\in J_{\varphi(j)}} \beta(j') \\
 & & \downarrow{\scriptstyle r_{\varphi(j)}} \\
\operatorname{colim}\beta & \xrightarrow[f]{} & \operatorname*{colim}_i \operatorname*{colim}_{(j',u)\in J_i} \beta(j').
\end{array}$$

To define g, we form the commutative diagram

$$\begin{array}{ccc}
\beta(j) & \xrightarrow{\mathrm{id}} & \beta(j) \\
\downarrow{\scriptstyle q_{i,j,u}} & & \downarrow{\scriptstyle p_j} \\
\operatorname*{colim}_{(j',u')\in J_i} \beta(j') & \xrightarrow{g_i} & \operatorname{colim}\beta \\
\downarrow{\scriptstyle r_i} & & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname*{colim}_{i'} \operatorname*{colim}_{(j',u')\in J_{i'}} \beta(j') & \xrightarrow[g]{} & \operatorname{colim}\beta.
\end{array} \tag{35}$$

as follows: We let i be in I and define g_i by the condition that the top square of (35) commutes for all $(j,u) \in J_i$. Then we define g by the condition that the bottom square of (35) commutes for all i.

Let us prove that $f \circ g$ is the identity of $\operatorname{colim}_i \operatorname{colim}_{(j,u)\in J_i} \beta(j)$. We have

$$f \circ g \circ r_i \circ q_{i,j,u} = f \circ p_j = r_{\varphi(j)} \circ q_{\varphi(j),j,\mathrm{id}_{\varphi(j)}}$$

for all $i \in J, (j,u) \in J_i$. Let $i \in J, (j,u) \in J_i$. It suffices to show

$$r_i \circ q_{i,j,u} = r_{\varphi(j)} \circ q_{\varphi(j),j,\mathrm{id}_{\varphi(j)}}, \tag{36}$$

that is, it suffices to show that the diagram

$$\begin{array}{ccc}
\beta(j) & \xrightarrow{\text{id}} & \beta(j) \\
{\scriptstyle q_{\varphi(j),j,\text{id}_{\varphi(j)}}}\downarrow & & \downarrow{\scriptstyle q_{i,j,u}} \\
\operatorname*{colim}\limits_{(j',u')\in J_{\varphi(j)}} \beta(j') & \xrightarrow{u_*} & \operatorname*{colim}\limits_{(j',u')\in J_i} \beta(j') \\
{\scriptstyle r_{\varphi(j)}}\downarrow & & \downarrow{\scriptstyle r_i} \\
\operatorname*{colim}\limits_{i'} \operatorname*{colim}\limits_{(j',u')\in J_i} \beta(j') & \xrightarrow[\text{id}]{} & \operatorname*{colim}\limits_{i'} \operatorname*{colim}\limits_{(j',u')\in J_i} \beta(j'),
\end{array}$$

where u_* denotes the morphism induced by u, commutes. The top square commutes by definition of u_*, and the bottom square commutes for obvious reasons.

We leave the proof of the fact that $g \circ f$ is the identity of $\operatorname{colim} \beta$ to the reader. $\square$

Display (34) p. 73 above just says that there exists an isomorphism between two given objects of $\mathcal{C}$. But the proof proves much more than that! The proof indeed exhibits a morphism from the first object to the second, a morphism from the second to the first, and a proof that these two morphisms are inverse isomorphisms. When we invoke (34) in a subsequent argument, we shall often tacitly refer not only to the mere display (34), but also to the two morphisms involved in its proof. In many cases it will be clear that the mere existential statement (34) wouldn't suffice to make the argument in question work, and that the invocation of the *proof* of (34) is crucial. Such a situation will happen so often that we think it advisable to issue a general warning:

Warning 99. When we invoke a previously proved statement, we tacitly understand once and for all that the *proof* of the statement in question is also implicitly invoked.

4.10 Kan extensions of modules

Let R be a ring, let $\mathcal{U}$ and $\mathcal{V}$ be universes such that $R \in \mathcal{U} \in \mathcal{V}$, put, with self-explanatory notation,

$$I := \text{Mod}^{\mathcal{U}}(R), \quad \mathcal{C} := \text{Mod}^{\mathcal{V}}(R),$$

let J be the full subcategory of I whose single object is R, and let $\mathcal{C} \xleftarrow{\beta} J \xrightarrow{\varphi} I$ be the inclusion functors. We identify $\text{Hom}_R(R, M)$ to M whenever convenient.

We claim that the functor $\varphi^\dagger(\beta) : I \to \mathcal{C}$ satisfies

$$\varphi^\dagger(\beta)(M) \simeq M. \tag{37}$$

To prove (37), set

$$M' := \operatorname*{colim}_{(x:R\to M)\in J_M} R \in \mathcal{C},$$

and let $p_x : R \to M'$ be the coprojections. As Theorem 2.3.3 (i) p. 52 of the book (stated above as Theorem 94 p. 67) implies $M' \simeq \varphi^\dagger(\beta)(M)$, it suffices to prove $M' \simeq M$. We define a family of linear maps $\Phi_x : R \to M$, indexed by $x : R \to M$, by setting $\Phi_x := x$, and leave it to the reader to check that the Φ_x induce a linear map $\Phi : M' \to M$. We define the set theoretic map $\Psi : M \to M'$ by putting $\Psi(x) := p_x(1)$, and leave it to the reader to verify that Φ and Ψ are mutually inverse bijections. This proves (37).

We claim that the functor $\varphi^\ddagger(\beta) : I \to \mathcal{C}$ satisfies

$$\varphi^\ddagger(\beta)(M) \simeq M^{**}, \tag{38}$$

where M^{**} is the double of M.

To prove (38), set

$$M' := \lim_{(f:M\to R)\in J^M} R \in \mathcal{C},$$

and let $p_f : M' \to R$ be the projections. As Theorem 2.3.3 (ii) p. 52 of the book implies $M' \simeq \varphi^\ddagger(\beta)(M)$, it suffices to prove $M' \simeq M^{**}$. We define a family of linear maps $\Phi_f : M^{**} \to R$, indexed by $f : M \to R$, by setting $\Phi_f(F) := F(f)$, and leave it to the reader to check that the Φ_f induce a linear map $\Phi : M^{**} \to M'$. We define the linear map $\Psi : M' \to M^{**}$ by putting $\Psi((\lambda_f))(g) := \lambda_g$, and leave it to the reader to verify that Φ and Ψ are mutually inverse linear bijections. This proves (38).

Let $R, \mathcal{U}$ and $\mathcal{V}$ be as above, put, with self-explanatory notation,

$$I := \operatorname{Mod}^{\mathcal{U}}(R)^{\mathrm{op}}, \quad \mathcal{C} := \operatorname{Mod}^{\mathcal{V}}(R^{\mathrm{op}}),$$

let J be the full subcategory of I whose single object is R, let $\varphi : J \to I$ be the inclusion functor, and let $\beta : J \to \mathcal{C}$ be the obvious functor satisfying $\beta(R) = R^{\mathrm{op}}$.

We claim that the functor $\varphi^\dagger(\beta) : I \to \mathcal{C}$ satisfies

$$\varphi^\dagger(\beta)(M) \simeq M^*, \tag{39}$$

where M^* is the of M.

To prove (39), set

$$M' := \operatorname*{colim}_{(R\to M)\in J_M} R^{\mathrm{op}} = \operatorname*{colim}_{(f:M\to R)\in (J^{\mathrm{op}})^M} R^{\mathrm{op}} \in \mathcal{C},$$

and let $p_f : R^{\mathrm{op}} \to M'$ be the coprojections. As Theorem 2.3.3 (i) p. 52 of the book (stated above as Theorem 94 p. 67) implies $M' \simeq \varphi^\dagger(\beta)(M)$, it suffices to prove $M' \simeq M^*$. We define a family of linear maps $\Phi_f : R^{\mathrm{op}} \to M^*$, indexed by $f : M \to R$, by setting $\Phi_f(1) := f$, and leave it to the reader to check that the Φ_f induce a linear map $\Phi : M' \to M^*$. We define the set theoretic map $\Psi : M^* \to M'$ by putting $\Psi(f) := p_f(1)$, and leave it to the reader to verify that Φ and Ψ are mutually inverse bijections. This proves (39).

We claim that the functor $\varphi^\ddagger(\beta) : I \to \mathcal{C}$ satisfies

$$\varphi^\ddagger(\beta)(M) \simeq M^*, \tag{40}$$

where M^* is the of M.

To prove (40), set

$$M' := \lim_{(M\to R)\in J^M} R^{\mathrm{op}} = \lim_{(x:R\to M)\in (J^{\mathrm{op}})_M} R^{\mathrm{op}} \in \mathcal{C},$$

and let $p_x : M' \to R^{\mathrm{op}}$ be the projections. As Theorem 2.3.3 (ii) p. 52 of the book implies $M' \simeq \varphi^\ddagger(\beta)(M)$, it suffices to prove $M' \simeq M^*$. We define a family of linear maps $\Phi_x : M^* \to R^{\mathrm{op}}$, indexed by $x : R \to M$, by setting $\Phi_x(f) := f(x)$, and leave it to the reader to check that the Φ_x induce a linear map $\Phi : M^* \to M'$. We define the linear map $\Psi : M' \to M^*$ by putting $\Psi((\lambda_x))(y) := \lambda_y$, and leave it to the reader to verify that Φ and Ψ are mutually inverse linear bijections. This proves (40).

4.11 Brief comments

§ 100. P. 55, proof of Corollary 2.4.4 (iii) (minor variant).

Proposition 101. *If I is a small category (Definition 5 p. 10), if S is in* **Set** *and $\Delta S : I \to$* **Set** *is the corresponding constant functor, then there is a canonical bijection*

$$\operatorname{colim} \Delta S \simeq \pi_0(I) \times S.$$

(See Notation 52 p. 46.)

Proof. On the one hand we have

$$\pi_0(I) := \mathrm{Ob}(I)/\!\sim\ ,$$

where $\sim$ is the equivalence relation defined on p. 18 of the book. On the other hand we have by Proposition 2.4.1 p. 54 of the book

$$\operatorname{colim} \Delta S \simeq (\mathrm{Ob}(I) \times S)/\!\approx\ ,$$

where $\approx$ is the equivalence relation described in the proposition. In view of the definition of $\approx$ and $\sim$, we get

$$(i, s) \approx (j, t) \ \Leftrightarrow\ [i \sim j \text{ and } s = t].$$

□

4.12 Corollary 2.4.6 p. 56

Recall the statement:

Proposition 102 (Corollary 2.4.6 p. 56)**.** *If X' and X'' are objects, if $\mathcal{C}$ and $\mathcal{C}'$ are categories, and if F and G are functors satisfying*

$$X' \in \mathcal{C}' \xleftarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{C}'' \ni X'', \tag{41}$$

then we have

$$\operatorname*{colim}_{(G(X)\to X'')\in\mathcal{C}_{X''}} \mathrm{Hom}_{\mathcal{C}'}(X', F(X)) \simeq \operatorname*{colim}_{(X'\to F(X))\in(\mathcal{C}^{X'})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}''}(G(X), X''). \tag{42}$$

Proof. Consider the diagram

$$\begin{array}{ccc}
\operatorname*{colim}_{(G(X)\to X'')\in\mathcal{C}_{X''}} \mathrm{Hom}_{\mathcal{C}'}(X', F(X)) & \underset{g}{\overset{f}{\rightleftarrows}} & \operatorname*{colim}_{(X'\to F(X))\in(\mathcal{C}^{X'})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}''}(G(X), X'') \\
{\scriptstyle p[G(X)\to X'']}\uparrow & & \uparrow{\scriptstyle q[X'\to F(X)]} \\
\mathrm{Hom}_{\mathcal{C}'}(X', F(X)) & & \mathrm{Hom}_{\mathcal{C}''}(G(X), X''),
\end{array}$$

where the vertical arrows are the coprojections. We leave it to the reader to check firstly that there are maps f and g as in the above diagram satisfying

$$f\Big(p\big[G(X) \to X''\big]\big(X' \to F(X)\big)\Big) := q\big[X' \to F(X)\big]\big(G(X) \to X''\big),$$

$$g\Big(q\big[X' \to F(X)\big]\big(G(X) \to X''\big)\Big) := p\big[G(X) \to X''\big]\big(X' \to F(X)\big)$$

for all morphism $G(X) \to X''$ in $\mathcal{C}''$ and all morphism $X' \to F(X)$ in $\mathcal{C}'$, and secondly that f and g are inverse bijections. □

4.13 Brief comments

§ 103. P. 56, proof of Lemma 2.4.7 (minor variant).

Lemma 104. *If I is a small category (Definition 5 p. 10), i_0 is in I, and $\alpha : I \to$ **Set** is the functor* $\mathrm{Hom}_I(i_0,\ )$*, then* $\operatorname{colim}\alpha$ *is a terminal object of* **Set***.*

Proof. We shall use (14) p. 48. Let $X = \{x\}$ be a terminal object of **Set**, let $p : \alpha \to \Delta X$ be the unique morphism from α to ΔX, let $\theta : \alpha \to \Delta Y$ be a morphism in $\mathbf{Set}^I$ (with Y in **Set**), and let us show that there is a unique map $f : X \to Y$ such that $\Delta f \circ p = \theta$:

$$\begin{array}{ccccc} \alpha & \xrightarrow{p} & \Delta X & \qquad & X \\ & \searrow^{\theta} & \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle f} \\ & & \Delta Y & & Y. \end{array}$$

Any such f must satisfy $f(x) = \theta_{i_0}(\mathrm{id}_{i_0})$. This proves the uniqueness. For the existence, it is easy to see that the map f defined by the above equality does the job. □

Here is a second proof:

Proof. Each element of $\operatorname{colim}_{i\in I}\mathrm{Hom}_I(i_0, i)$ is represented by some morphism $i_0 \to i$ in I. Moreover, a composition of the form $i_0 \to i \to j$ represents the same element as $i_0 \to i$. In particular $i_0 \to i$ represents the same element as id_{i_0}. □

§ 105. P. 57, proof of Proposition 2.5.2 (minor variant). Instead of proving (i)⇒(v), we prove (i)⇒(ii), that is, we prove the following statement:

Lemma 106. *If $\varphi : J \to I$ and $\beta : I^{\mathrm{op}} \to$ **Set** are functors defined on small categories (Definition 5 p. 10) and if the category J^i is connected for all i in I, then the natural map*

$$f : \lim\beta \to \lim\beta\circ\varphi^{\mathrm{op}}$$

is bijective.

Proof. We shall define a map

$$g : \lim \beta \circ \varphi^{\mathrm{op}} \to \lim \beta$$

and leave it to the reader to check that f and g are inverse. Let y be in $\lim \beta \circ \varphi^{\mathrm{op}}$. In particular y is of the form $(y_j)_{j \in J}$ with $y_j \in \beta(\varphi(j))$. We must define the element $g(y)_i$ in $\beta(i)$, where i is an arbitrary element in I. Let us choose a morphism $i \to \varphi(j)$ in I. It suffices to show that the element $\beta(i \to \varphi(j))(y_j)$ in $\beta(i)$ does not depend on the choice of $i \to \varphi(j)$, enabling us to set

$$g(y)_i := \beta(i \to \varphi(j))(y_j).$$

Given another choice $i \to \varphi(j')$, we must prove

$$\beta(i \to \varphi(j))(y_j) = \beta(i \to \varphi(j'))(y_{j'}).$$

As J^i is connected, we may assume that there is a morphism $j \to j'$ in J such that $(i \to \varphi(j')) = (i \to \varphi(j) \to \varphi(j'))$, and the proof is straightforward. □

§ 107. P. 58, implication (vi)⇒(i) of Proposition 2.5.2. Here is a slightly stronger statement:

Proposition 108. *If $\varphi : J \to I$ is a functor, then the obvious map*

$$\operatorname{colim} \operatorname{Hom}_I(i, \varphi) \to \pi_0(J^i) \tag{43}$$

is bijective.

Proof. Let L_i be the left-hand side of (43), and, for j in J, let

$$p_j : \operatorname{Hom}_I(i, \varphi(j)) \to L_i$$

be the coprojection. It is easy to check that the map

$$\operatorname{Ob}(J^i) \to L_i, \quad (j, i \to \varphi(j)) \mapsto p_j(i \to \varphi(j))$$

factors through $\pi_0(J^i)$, and that the induced map $\pi_0(J^i) \to L_i$ is inverse to (43). □

4.14 Proposition 2.6.3 (i) p. 61

Let $\mathcal{C}$ be a category and let A be in $\mathcal{C}^\wedge$. Consider the statements

$$\underset{(X\to A)\in\mathcal{C}_A}{\text{“colim”}}\ X \xrightarrow{\sim} A, \tag{44}$$

$$\underset{(X\to A)\in\mathcal{C}_A}{\operatorname{colim}} \operatorname{Hom}_{\mathcal{C}}(Y,X) \xrightarrow{\sim} A(Y) \text{ for all } Y\in\mathcal{C}, \tag{45}$$

$$\operatorname{Hom}_{\mathcal{C}^\wedge}(A,B) \xrightarrow{\sim} \lim_{(X\to A)\in\mathcal{C}_A} B(X) \text{ for all } B\in\mathcal{C}^\wedge. \tag{46}$$

We prove (44), (45) and (46) in §109 p. 81 below. [Note that Warning 99 p. 75 applies particularly well to (44), (45) and (46).]

Note that (44) can be stated as follows: If $h:\mathcal{C}\to\mathcal{C}^\wedge$ is the Yoneda embedding, then the natural morphism $h^\dagger(h)\to\operatorname{id}_{\mathcal{C}^\wedge}$ is an isomorphism. This implies in particular that (44) is functorial in A.

§ 109. We shall prove (44), (45) and (46). More precisely, we shall spell out these three isomorphisms in terms of Diagram (14) p. 48 and Diagram (13) p. 47).

Warning: In this proof the symbols X and Y will designate either two objects of $\mathcal{C}$ or the image of these objects in $\mathcal{C}^\wedge$. The context only will tell which interpretation is the good one. (It seems to me the choice of the correct interpretation will always be obvious.)

• Isomorphism (44) can be decoded as follows: Consider the functor

$$\alpha:\mathcal{C}_A\to\mathcal{C}^\wedge,\quad (X\to A)\mapsto X,$$

and let $p:\alpha\to\Delta A$ be the tautological morphism in $(\mathcal{C}^\wedge)^{\mathcal{C}_A}$ defined by

$$p_{X\to A}:=(X\to A) \tag{47}$$

for all $X\to A$ in $\mathcal{C}_A$. Let B be in $\mathcal{C}^\wedge$ and $\theta:\alpha\to\Delta B$. Diagram (14) p. 48 becomes

$$\begin{array}{ccccc} \alpha & \xrightarrow{p} & \Delta A & \qquad & A \\ & \searrow^{\theta} & \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle f} \\ & & \Delta B & & B. \end{array}$$

The uniqueness of f follows from the fact that the equality

$$\Delta f \circ p = \theta \tag{48}$$

implies

$$(X \to A \xrightarrow{f} B) = \theta_{X \to A} \quad \forall \quad X \to A, \tag{49}$$

and the existence of f follows from the fact that (49) implies (48).

• Isomorphism (45) can be decoded as follows: Let Y be in $\mathcal{C}$, consider the functor

$$\beta : \mathcal{C}_A \to \mathbf{Set}, \quad (X \to A) \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X),$$

and let $q : \beta \to \Delta A(Y)$ be the morphism in $\mathbf{Set}^{\mathcal{C}_A}$ defined by

$$q_{X \to A}(Y \to X) := (Y \to X \to A)$$

for all $X \to A$ in $\mathcal{C}_A$. Let S be in $\mathbf{Set}$ and Y in $\mathcal{C}$. Diagram (14) p. 48 becomes

$$\begin{array}{ccccc} \beta & \xrightarrow{q} & \Delta A(Y) & \qquad & A(Y) \\ & \searrow^{\theta} & \downarrow \Delta f & & \downarrow f \\ & & \Delta S & & S. \end{array}$$

The equality $\Delta f \circ q = \theta$ is equivalent to the condition

$$f(Y \to X \to A) = \theta_{X \to A}(Y \to X) \quad \forall \quad Y \to X \to A. \tag{50}$$

Consider the condition

$$f(Y \to A) = \theta_{Y \to A}(\mathrm{id}_Y) \quad \forall \quad Y \to A. \tag{51}$$

The uniqueness of f follows from the fact that (50) implies (51), and the existence of f follows from the fact that (51) implies (50).

• Isomorphism (46) can be decoded as follows: Let B be in $\mathcal{C}^\wedge$, consider the functor

$$\gamma : (\mathcal{C}_A)^{\mathrm{op}} \to \mathbf{Set}, \quad (X \to A) \mapsto B(X),$$

let $r : \Delta B(A) \to \gamma$ be the morphism in $\mathbf{Set}^{(\mathcal{C}_A)^{\mathrm{op}}}$ defined by

$$r_{X \to A}(A \to B) := (X \to A \to B)$$

for all $X \to A$ in $\mathcal{C}_A$ and all $A \to B$ in $B(A)$, and let S be in **Set**. Diagram (13) p. 47 becomes

$$\begin{array}{ccccc} S & \qquad & \Delta S & & \\ {\scriptstyle f}\downarrow & & {\scriptstyle \Delta f}\downarrow & \searrow^{\theta} & \\ B(A) & & \Delta B(A) & \xrightarrow[r]{} & \gamma. \end{array}$$

The uniqueness of f follows from the fact that the equality

$$r \circ \Delta f = \theta \tag{52}$$

implies

$$(X \to A \xrightarrow{f(s)} B) = \theta_{X \to A}(s) \quad \text{for all } s \in S \text{ and all } X \to A, \tag{53}$$

and the existence of f follows from the fact that (53) implies (52). q.e.d.

4.15 Brief comments

§ 110. P. 62, Proposition 2.7.1. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\mathrm{h}_{\mathcal{C}}} & \mathcal{C}^\wedge & \xleftarrow{\alpha} & I \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \widetilde{F}} & & \\ & & \mathcal{A}, & & \end{array}$$

where I is a small category (Definition 5 p. 10) and $\widetilde{F}$ satisfies

$$\widetilde{F}(A) \simeq \operatorname*{colim}_{(X \to A) \in \mathcal{C}_A} F(X)$$

for all A in $\mathcal{C}^\wedge$. Let us rewrite the proof of the fact that the natural morphism $\operatorname{colim} \widetilde{F} \circ \alpha \to \widetilde{F}\,(\operatorname{colim} \alpha)$ is an isomorphism.

By Proposition 2.1.10 p. 40 of the book (stated on p. 56 above as Corollary 67), it suffices to check that the functor $G : \mathcal{A} \to \mathcal{C}^\wedge$ defined by

$$G(X')(X) := \operatorname{Hom}_{\mathcal{A}}(F(X), X').$$

is right adjoint to $\widetilde{F}$. This results from the following computation:

$$\operatorname{Hom}_{\mathcal{A}}\left(\widetilde{F}(A), X'\right) \simeq \operatorname{Hom}_{\mathcal{A}}\left(\operatorname*{colim}_{(X \to A) \in \mathcal{C}_A} F(X), X'\right) \simeq \lim_{(X \to A) \in \mathcal{C}_A} \operatorname{Hom}_{\mathcal{A}}(F(X), X')$$

$$= \lim_{(X \to A) \in \mathcal{C}_A} G(X')(X) \simeq \mathrm{Hom}_{\mathcal{C}^\wedge}(A, G(X')),$$

the last isomorphism following from (46) p. 81. q.e.d.

§ 111. P. 62. In the setting of Proposition 2.7.1, the functors

$$\mathcal{A}^{\mathcal{C}} \to \mathcal{A}, \quad F \mapsto (\mathrm{h}^\dagger_{\mathcal{C}} F)(A) \quad \text{and} \quad \mathcal{C}^\wedge \to \mathcal{A}, \quad A \mapsto (\mathrm{h}^\dagger_{\mathcal{C}} F)(A)$$

commute with small inductive limits.

Indeed, for the first functor the conclusion follows from the isomorphism

$$(\mathrm{h}^\dagger_{\mathcal{C}} F)(A) \simeq \operatorname*{colim}_{(U \to A) \in \mathcal{C}_A} F(U), \tag{54}$$

and, for the second functor it follows from Proposition 2.7.1 p. 62 of the book.

4.16 Three formulas

Here is a complement to Section 2.3 pp 52-54 of the book, complement which will be used in §515 p. 297 to prove Proposition 17.1.9 p. 409 of the book.

In this section we shall use the following notation: The Yoneda embedding $\mathcal{C} \to \mathcal{C}^\wedge$ will be denoted by $h[\mathcal{C}]$, and the forgetful functor $\mathcal{C}_A \to \mathcal{C}$ by $j[\mathcal{C}_A]$:

$$h[\mathcal{C}] : \mathcal{C} \to \mathcal{C}^\wedge, \qquad j[\mathcal{C}_A] : \mathcal{C}_A \to \mathcal{C}.$$

4.16.1 Preliminaries

Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}^\wedge$. Recall that there is a unique functor

$$\lambda : (\mathcal{C}^\wedge)_A \to (\mathcal{C}_A)^\wedge$$

such that

$$\lambda(B \to A)(U \to A) = \mathrm{Hom}_{(\mathcal{C}^\wedge)_A}(U \to A, B \to A) \tag{55}$$

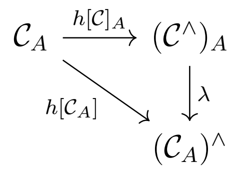
for all $(B \to A)$ in $(\mathcal{C}^\wedge)_A$ and all $(U \to A)$ in $\mathcal{C}_A$. Moreover

$$\lambda \text{ is an equivalence,} \tag{56}$$

and we have

$$\lambda \circ h[\mathcal{C}]_A \simeq h[\mathcal{C}_A], \tag{57}$$

that is, the diagram



quasi-commutes. (See Lemma 1.4.12 p. 26 of the book.)

The statement below follows from Proposition 2.7.1 p. 62 of the book:

Proposition 112. *Let $F:\mathcal{C}\to\mathcal{A}$ be a functor, assume that $\mathcal{C}$ is small (Definition 5 p. 10) and that $\mathcal{A}$ admits small inductive limits. Then the functor $h[\mathcal{C}]^\dagger(F):\mathcal{C}^\wedge\to\mathcal{A}$ exists, commutes with small inductive limits and satisfies $h[\mathcal{C}]^\dagger(F)\circ h[\mathcal{C}]\simeq F$.*

Let $F:\mathcal{C}\to\mathcal{A}$ be a functor and A an object of $\mathcal{C}^\wedge$, and assume that $\mathcal{C}$ is small (Definition 5 p. 10) and that $\mathcal{A}$ admits small inductive and projective limits.

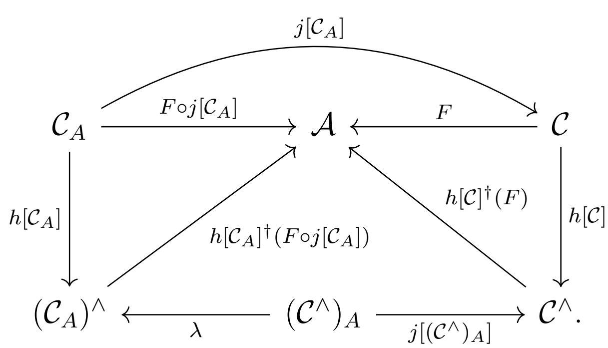
4.16.2 First Formula

We claim

$$h[\mathcal{C}_A]^\dagger(F\circ j[\mathcal{C}_A])\circ\lambda\simeq h[\mathcal{C}]^\dagger(F)\circ j[(\mathcal{C}^\wedge)_A] \tag{58}$$

(see the diagram (59) below).

Proof. Consider the diagram



(59)

For B in $\mathcal{C}^\wedge$ and $B\to A$ in $(\mathcal{C}^\wedge)_A$, setting

$$X := h[\mathcal{C}_A]^\dagger\Big(F \circ j[\mathcal{C}_A]\Big)\Big(\lambda(B \to A)\Big),$$

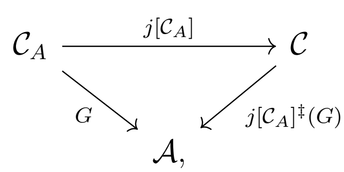
we get

$$\begin{aligned}
X &\simeq h[\mathcal{C}_A]^\dagger(F \circ j[\mathcal{C}_A])\left(\lambda\left(\left(\operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} h[\mathcal{C}](U)\right) \to A\right)\right) && \text{by (44)}\\
&\simeq h[\mathcal{C}_A]^\dagger(F \circ j[\mathcal{C}_A])\left(\lambda\left(\operatorname*{colim}_{(U\to B)\in\mathcal{C}_B}\Big(h[\mathcal{C}](U) \to A\Big)\right)\right) && \text{by Lemma 84}\\
&\simeq \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} h[\mathcal{C}_A]^\dagger\Big(F \circ j[\mathcal{C}_A]\Big)\Big(\lambda\Big(h[\mathcal{C}](U) \to A\Big)\Big) && \text{by (56) \& Prop. 112}\\
&\simeq \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} h[\mathcal{C}_A]^\dagger\Big(F \circ j[\mathcal{C}_A]\Big)\Big(\lambda\Big(h[\mathcal{C}]_A(U \to A)\Big)\Big)\\
&\simeq \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} h[\mathcal{C}_A]^\dagger\Big(F \circ j[\mathcal{C}_A]\Big)\Big(h[\mathcal{C}_A](U \to A)\Big) && \text{by (57)}\\
&\simeq \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} \big(F \circ j[\mathcal{C}_A]\big)(U \to A) && \text{by Prop. 112}\\
&\simeq \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} F(U)\\
&\simeq \left(h[\mathcal{C}]^\dagger(F) \circ j\Big((\mathcal{C}^\wedge)_A\Big)\right)(B \to A) && \text{by (54).}
\end{aligned}$$

□

4.16.3 Second Formula

Proposition 113. *Consider the quasi-commutative diagram*



and let U be in $\mathcal{C}$. Then we have

$$j[\mathcal{C}_A]^\ddagger(G)(U) \simeq \prod_{U \to A} G(U \to A). \tag{60}$$

Lemma 114. *The discrete category $A(U)$ is cocofinal in $(\mathcal{C}_A)^U$.*

Proof of Lemma 114. We probably give too many details, and the reader may want to skip this proof. An object a of $(\mathcal{C}_A)^U$ is given by a triple

$$a = (U_a, U_a \xrightarrow{u_a} A; U \xrightarrow{s_a} U_a),$$

and a morphism from a to

$$b = (U_b, U_b \xrightarrow{u_b} A; U \xrightarrow{s_b} U_b) \in (\mathcal{C}_A)^U$$

is given by a commutative diagram

$$\begin{array}{ccc} & U & \\ {}^{s_a}\swarrow & & \searrow^{s_b} \\ U_a & \xrightarrow{\quad c \quad} & U_b \\ {}_{u_a}\searrow & & \swarrow_{u_b} \\ & A. & \end{array}$$

The embedding $\varphi : A(U) \to (\mathcal{C}_A)^U$ implicit in the statement of Lemma 114 is given by

$$\varphi(u) = (U, U \xrightarrow{u} A; U \xrightarrow{\mathrm{id}_U} U).$$

It is easy to see that, for any b in $(\mathcal{C}_A)^U$, there is precisely one pair (u, c) such that u is in $A(U)$ and c is a morphism from U to U_b making the diagram

$$\begin{array}{ccc} & U & \\ {}^{\mathrm{id}_U}\swarrow & & \searrow^{s_b} \\ U & \xrightarrow{\quad c \quad} & U_b \\ {}_{u}\searrow & & \swarrow_{u_b} \\ & A & \end{array}$$

commute. This implies the lemma. □

Proof of Proposition 113. We have

$$j[\mathcal{C}_A]^\ddagger(G)(U) \simeq \lim_{(U \to j[\mathcal{C}_A](V \to A)) \in (\mathcal{C}_A)^U} G(V \to A)$$

$$\simeq \lim_{U \to A} G(U \to A) \simeq \prod_{U \to A} G(U \to A),$$

the penultimate isomorphism following from Lemma 114. □

4.16.4 Third Formula

Put $j := j[\mathcal{C}_A]$, $h := h[\mathcal{C}]$, $h_A := h[\mathcal{C}_A]$, and consider the diagram

$$\begin{array}{ccccccccc} \mathcal{C}^\wedge & \xleftarrow{h} & \mathcal{C} & \xleftarrow{j} & \mathcal{C}_A & \xrightarrow{h_A} & (\mathcal{C}_A)^\wedge & \xleftarrow{\lambda} & (\mathcal{C}^\wedge)_A \\ {\scriptstyle h^\dagger(j^\dagger(G))}\downarrow & & {\scriptstyle j^\dagger(G)}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle (h_A)^\dagger(G)} & & \\ \mathcal{A} & = & \mathcal{A} & = & \mathcal{A} & = & \mathcal{A}. & & \end{array}$$

(See (55) p. 84 for the definition of λ.) Let B be in $\mathcal{C}^\wedge$. We claim

$$h^\dagger(j^\dagger(G))(B) \simeq (h_A)^\dagger(G)(\lambda(B \times A \to A)). \tag{61}$$

Proof. We have, for U in $\mathcal{C}$,

$$j^\dagger(G)(U) \simeq \operatorname*{colim}_{(j(V\to A)\to U)\in(\mathcal{C}_A)_U} G(V \to A) \simeq \operatorname*{colim}_{(A\leftarrow V\to U)\in(\mathcal{C}_A)_U} G(V \to A)$$

$$\simeq \operatorname*{colim}_{((V\to A)\to(U\times A\to A))\in(\mathcal{C}_A)_{U\times A\to A}} G(V \to A) \simeq (h_A)^\dagger(G)(\lambda(U \times A \to A)),$$

that is:

$$j^\dagger(G)(U) \simeq (h_A)^\dagger(G)(\lambda(U \times A \to A)). \tag{62}$$

For B in $\mathcal{C}^\wedge$ we get

$$h^\dagger(j^\dagger(G))(B) \simeq \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} j^\dagger(G)(U) \overset{\text{(a)}}{\simeq} \operatorname*{colim}_{(U\to B)\in\mathcal{C}_B} (h_A)^\dagger(G)(\lambda(U \times A \to A))$$

$$\overset{\text{(b)}}{\simeq} (h_A)^\dagger(G)\left(\lambda\left(\operatorname*{colim}_{(U\to B)\in\mathcal{C}_B}(U \times A \to A)\right)\right)$$

$$\overset{\text{(c)}}{\simeq} (h_A)^\dagger(G)\left(\lambda\left(\left(\operatorname*{colim}_{(U\to B)\in\mathcal{C}_B}(U \times A)\right) \to A\right)\right)$$

$$\overset{\text{(d)}}{\simeq} (h_A)^\dagger(G)\left(\lambda\left(\left(\operatorname*{``colim"}_{(U\to B)\in\mathcal{C}_B} U\right) \times A \to A\right)\right) \overset{\text{(e)}}{\simeq} (h_A)^\dagger(G)(\lambda(B \times A \to A)),$$

where (a) follows from (62); (b) follows from Proposition 112 p. 85 and (56) p. 84; (c) follows from Lemma 84 p. 63; (d) follows from the fact that small inductive limits in **Set** are stable by base change (see Section 4.6 p. 63 above and Section 4.20 p. 93) below; (e) follows from (44) p. 81. □

4.17 Notation 2.7.2 p. 63

Recall that $F : \mathcal{C} \to \mathcal{C}'$ is a functor of small categories (Definition 5 p. 10). The formula

$$\widehat{F}(A)(X') = \operatorname*{colim}_{(X \to A) \in \mathcal{C}_A} \mathrm{Hom}_{\mathcal{C}'}(X', F(X))$$

may also be written as

$$\widehat{F}(A) = \operatorname*{“colim”}_{(X \to A) \in \mathcal{C}_A} F(X). \tag{63}$$

It might be worth stating explicitly the isomorphism

$$\widehat{F} \circ \mathrm{h}_{\mathcal{C}} \xrightarrow{\sim} \mathrm{h}_{\mathcal{C}'} \circ F,$$

which says that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ {\scriptstyle \mathrm{h}_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathrm{h}_{\mathcal{C}'}} \\ \mathcal{C}^\wedge & \xrightarrow[\widehat{F}]{} & \mathcal{C}'^\wedge. \end{array}$$

quasi-commutes.

Remark 115. Recall that $F : \mathcal{C} \to \mathcal{C}'$ is a functor of small categories (Definition 5 p. 10). Let A' be in $\mathcal{C}'^\wedge$, and let $\mathcal{C}_{A' \circ F} \xrightarrow{\varphi} \mathcal{C}'_{A'} \xrightarrow{\psi} \mathcal{C}'^\wedge$ be the natural functors. The natural morphism $\operatorname{colim} \psi \circ \varphi \to \operatorname{colim} \psi$ induces a morphism $f : \widehat{F}(A' \circ F) \to A'$ functorial in A':

$$\widehat{F}(A' \circ F) = \operatorname{colim} \psi \circ \varphi \to \operatorname{colim} \psi \simeq A',$$

the equality $F(A' \circ F) = \operatorname{colim} \psi \circ \varphi$ and the isomorphism $\operatorname{colim} \psi \simeq A'$ following respectively from (63) and (44) p. 81. Moreover f is an isomorphism whenever φ is cofinal. Note that the condition that f is an isomorphism means that, for each X' in $\mathcal{C}'$, the natural map

$$\operatorname*{colim}_{(X \to A' \circ F) \in \mathcal{C}_{A' \circ F}} \mathrm{Hom}_{\mathcal{C}'}(X', F(X)) \to A'(X')$$

is bijective. This condition depends only on the functor $F : \mathcal{C} \to \mathcal{C}'$ and the projective system of sets $(A'(X'))_{X' \in \mathcal{C}'}$. (This remark will be used to prove Proposition 242 p. 154.)

The proof is obvious.

Remark 116. If F is fully faithful, then there is an isomorphism $\widehat{F}(A) \circ F \xrightarrow{\sim} A$ functorial in $A \in \mathcal{C}^\wedge$.

Proof. We have

$$\widehat{F}(A)(F(X)) = \operatorname*{colim}_{(Y\to A)\in\mathcal{C}_A} \operatorname{Hom}_{\mathcal{C}'}(F(X),F(Y))$$

$$\simeq \operatorname*{colim}_{(Y\to A)\in\mathcal{C}_A} \operatorname{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\sim} A(X),$$

the last isomorphism following from (45) p. 81. □

As observed in the book (see also §110 p. 83):

Remark 117. The functor $\widehat{F}$ commutes with small inductive limits.

Let X be in $\mathcal{C}$ and A a terminal object of $\mathcal{C}^\wedge$. We have

$$\widehat{F}(A)(F(X)) \simeq \bigsqcup_{Y\in\mathcal{C}} \operatorname{Hom}_{\mathcal{C}'}(F(X),F(Y)).$$

Let us identify these two sets.

Remark 118. Assume A is a terminal object of $\mathcal{C}^\wedge$, and define, using the above identification, $G:\mathcal{C}\to\mathcal{C}'_{\widehat{F}(A)}$ by

$$G(X) := (F(X), p_X(\mathrm{id}_{F(X)})),$$

where $p_X : \operatorname{Hom}_{\mathcal{C}'}(F(X),F(X)) \to \widehat{F}(A)(F(X))$ is the coprojection. Then the composition of G with the forgetful functor $\mathcal{C}'_{\widehat{F}(A)} \to \mathcal{C}'$ is F.

The proof is obvious.

4.18 Brief comments

§ 119. P. 63, Corollary 2.7.4. Here is a variant:

Let $\mathcal{C}$ be a category and $\mathcal{A}$ a category admitting small projective limits, let $h:\mathcal{C}\to\mathcal{C}^\wedge$ the Yoneda embedding, and let $\mathrm{Fct}^{p\ell}((\mathcal{C}^\wedge)^{\mathrm{op}},\mathcal{A})$ be the category of functors from $(\mathcal{C}^\wedge)^{\mathrm{op}}$ to $\mathcal{A}$ commuting with small projective limits. Then the functors

$$\mathrm{Fct}^{p\ell}((\mathcal{C}^\wedge)^{\mathrm{op}},\mathcal{A}) \underset{(h^{\mathrm{op}})^\ddagger}{\overset{(h^{\mathrm{op}})_*}{\rightleftarrows}} \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$$

are mutually quasi-inverse equivalences.

Let F be in $\mathrm{Fct}((\mathcal{C}^\wedge)^{\mathrm{op}}, \mathcal{A})$. Assume (A_i) is a projective system in $(\mathcal{C}^\wedge)^{\mathrm{op}}$, or, equivalently, (A_i) is an inductive system in $\mathcal{C}^\wedge$. In particular $(F(A_i))$ is a projective system in $\mathcal{A}$.

Then F is in $\mathrm{Fct}^{p\ell}((\mathcal{C}^\wedge)^{\mathrm{op}}, \mathcal{A})$ if and only if the following condition holds:

For any system (A_i) as above, the natural morphism

$$F\left(\operatorname*{colim}_i A_i\right) \to \lim_i F(A_i)$$

is an isomorphism.

The functor $(h^{\mathrm{op}})^\ddagger$ is given by

$$(h^{\mathrm{op}})^\ddagger(F)(A) = \lim_{(U \to A) \in \mathcal{C}_A} F(U).$$

The functors

$$\mathcal{A}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{A}, \quad F \mapsto (h^{\mathrm{op}})^\ddagger(F)(A) \quad \text{and} \quad \mathcal{C}^\wedge \to \mathcal{A}, \quad A \mapsto (h^{\mathrm{op}})^\ddagger(F)(A)$$

commute with small projective limits. (For a justification, see §111 p. 84.)

§ 120. P. 64. It might be worth displaying the formula

$$\widehat{F}(A)(X') \simeq \operatorname*{colim}_{(X \to A) \in \mathcal{C}_A} \mathrm{Hom}_{\mathcal{C}'}(X', F(X)) \simeq \operatorname*{colim}_{(X' \to F(X)) \in \mathcal{C}^{X'}} A(X), \tag{64}$$

which is contained in the proof of Proposition 2.7.5 p. 64 of the book, and which follows from Corollary 2.4.6 p. 56 of the book (see Proposition 102 p. 78). Recall that $F : \mathcal{C} \to \mathcal{C}'$ is a functor of small categories (Definition 5 p. 10), that A is in $\mathcal{C}^\wedge$, and that X' is in $\mathcal{C}'$.

For the reader's convenience we reproduce the statement of Proposition 2.7.5:

Proposition 121 (Proposition 2.7.5 p. 64)**.** *If $F : \mathcal{C} \to \mathcal{C}'$ is a functor of small categories, then the functors $\widehat{F}$ and $(F^{\mathrm{op}})^\dagger$ from $\mathcal{C}^\wedge$ to $\mathcal{C}'^\wedge$ are isomorphic.*

This follows from (64).

§ 122. P. 64, end of Chapter 2. One could add the following observation:

If $\mathcal{C}$ is a small category (Definition 5 p. 10), if A is in $\mathcal{C}^\wedge$, if B is a terminal object of $(\mathcal{C}_A)^\wedge$, and if $F : \mathcal{C}_A \to \mathcal{C}$ is the forgetful functor, then we have

$$\widehat{F}(B) \simeq A.$$

Indeed, we have

$$\widehat{F}(B)(X) \simeq \operatorname*{colim}_{((Y \to A) \to B) \in (\mathcal{C}_A)_B} \mathrm{Hom}_{\mathcal{C}}(X, F(Y \to A))$$

$$\simeq \operatorname*{colim}_{(Y \to A) \in \mathcal{C}_A} \mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq A(X),$$

the last isomorphism following from (45) p. 81.

4.19 Exercise 2.4

P. 64, Exercise 2.4. Here is (with some minor changes) the statement of Exercise 2.4.

Let $f : X \to Y$ be a morphism in a category admitting fiber products. Set $P := X \times_Y X$; let $p_1, p_2 : P \to X$ be the projections; and let $\delta : X \to P$ be the diagonal morphism.

(i) We have $p_1 \circ \delta = \mathrm{id}_X = p_2 \circ \delta$. In particular p_1 and p_2 are epimorphisms and δ is a monomorphism.

(ii) We have: f monomorphism $\Leftrightarrow p_1 = p_2 \Leftrightarrow \delta$ isomorphism $\Leftrightarrow \delta$ epimorphism.

Solution: Claim (i) is obvious. Let us prove (ii):

f monomorphism $\Rightarrow p_1 = p_2$: we have $f \circ p_1 = f \circ p_2$;

$p_1 = p_2 \Rightarrow \delta$ isomorphism: $p_i \circ \delta \circ p_j = p_i \circ \mathrm{id}_P$ for all i, j, and thus $\delta \circ p_j = \mathrm{id}_P$ for all j;

δ isomorphism $\Rightarrow \delta$ epimorphism: obvious;

δ epimorphism $\Rightarrow$ f monomorphism: let $g, h : Z \rightrightarrows X$ satisfy $f \circ g = f \circ h$; let $k : Z \to P$ satisfy $p_1 \circ k = g$, $p_2 \circ k = h$; then the assumption that δ is an epimorphism and the equality $p_1 \circ \delta = p_2 \circ \delta$ observed in (i) imply $p_1 = p_2$, and thus $g = h$.

4.20 Exercise 2.7

P. 65, Exercise 2.7, Line 3: "that the functor $\cdot \times_Z Y : \mathbf{Set}_Z \to \mathbf{Set}_Z$" should be "that, given $Y \in \mathbf{Set}_Z$, the functor $\cdot \times_Z Y : \mathbf{Set}_Z \to \mathbf{Set}_Y$". For the reader's convenience we paste the exercise below:

Exercise 2.7. Let $Z \in \mathbf{Set}$.

(i) Prove that the category $\mathbf{Set}_Z$ admits products (denoted here by $X \times_Z Y$) and that, given $Y \in \mathbf{Set}_Z$, the functor $\cdot \times_Z Y : \mathbf{Set}_Z \to \mathbf{Set}_Y$ is left adjoint to the functor $\mathcal{H}om_Z(Y, \,\cdot\,)$ given by

$$\mathcal{H}om_Z(Y, X) = \bigsqcup_{z \in Z} \mathrm{Hom}_{\mathbf{Set}}(Y_z, X_z),$$

where X_z is the fiber of $X \to Z$ over $z \in Z$.

(ii) Deduce that small inductive limits in $\mathbf{Set}$ are stable by base change (see Definition 83 p. 63).

Here is a solution: (i) The fact that $\mathbf{Set}_Z$ admits products is clear. The bijective correspondence between

$$f \in \mathrm{Hom}_{\mathbf{Set}_Y}(U \times_Z Y, X)$$

and

$$g \in \mathrm{Hom}_{\mathbf{Set}_Z}\left(U \ , \ \bigsqcup_{z \in Z} \mathrm{Hom}_{\mathbf{Set}}(Y_z, X_z)\right)$$

is given by

$$(\forall\ z \in Z)\ \ (\forall\ u \in U_z)\ \ (\forall\ y \in Y_z)\ \ (f(u, y) = g(u)(y)).$$

(ii) The statement follows from (i) and Proposition 2.1.10 p. 40 of the book.

(See also Section 4.6 p. 63.)

Note that (ii) can also be proved directly by observing that the category $\mathbf{Set}_Z$ is canonically isomorphic to

$$\prod_{z \in Z} \mathbf{Set},$$

and that, given $f : Y \to Z$, if we identify $\mathbf{Set}_Z$ to $\prod_{z \in Z} \mathbf{Set}$ and $\mathbf{Set}_Y$ to $\prod_{y \in Y} \mathbf{Set}$, then the change of base functor, viewed as a functor

$$\prod_{z \in Z} \mathbf{Set} \to \prod_{y \in Y} \mathbf{Set},$$

maps $(X_z)_{z \in Z}$ to $(X_{f(y)})_{y \in Y}$.

5 About Chapter 3

5.1 Brief comments

§ 123. P. 72, proof of Lemma 3.1.2. Here is a minor variant of the proof of the following statement:

If $\varphi : J \to I$ *is a functor with* I *filtrant and* J *finite, then* $\lim \mathrm{Hom}_I(\varphi, i) \neq \varnothing$ *for some* i *in* I*.*

Indeed, let S be a set of morphisms in J. It is easy to prove

$$(\exists\ i \in I) \left(\exists\ a \in \prod_{j \in J} \mathrm{Hom}_I(\varphi(j), i)\right)\ (\forall\ (s : j \to j') \in S)\ (a_{j'} \circ \varphi(s) = a_j)$$

by induction on the cardinal of S, and to see that this implies the claim. q.e.d.

§ 124. P. 74, Theorem 3.1.6. The proof of Theorem 3.1.6 implies:

Proposition 125. *Let* I *be a (not necessarily small) filtrant* $\mathcal{U}$*-category (Definitions 4 p. 10 and 5 p. 10),* J *a finite category, and* $\alpha : I \times J^{\mathrm{op}} \to \mathbf{Set}$ *a functor such that* $\mathrm{colim}_i\, \alpha(i, j)$ *exists in* **Set** *for all* j*. Then* $\mathrm{colim}_i \lim_j \alpha(i, j)$ *exists in* **Set***, and the natural map*

$$\operatorname*{colim}_i \lim_j \alpha(i, j) \to \lim_j \operatorname*{colim}_i \alpha(i, j)$$

is bijective.

This corollary is implicitly used in the proof of Proposition 3.3.13 p. 84 (see Proposition 150 p. 101 below).

§ 126. P. 75, Proposition 3.1.8 (i). In the proof of Proposition 3.3.15 p. 85 of the book, a slightly stronger result is needed (see §151 p. 101). We state and prove this stronger result.

Proposition 127. *Let*

$$J \xrightarrow{\varphi} I \xrightarrow{\theta} L \xleftarrow{\psi} K$$

be a diagram of categories. Assume that ψ *is cofinal, and that the obvious functor* $\varphi_k : J_{\psi(k)} \to I_{\psi(k)}$ *is cofinal for all* k *in* K*. Then* φ *is cofinal.*

Proof. Pick a universe making I, J, K and L small (Definition 5 p. 10), let $\alpha : I \to$ **Set** be a functor, and consider the commutative diagrams

$$\begin{array}{ccccc}
\alpha(\varphi(j)) & \xrightarrow{\mathrm{id}} & \alpha(\varphi(j)) & \xrightarrow{\mathrm{id}} & \alpha(\varphi(j)) \\
\Big\downarrow p_j & & \Big\downarrow & & \Big\downarrow \\
 & & \operatorname*{colim}\limits_{\theta(\varphi(j))\to\ell} \alpha(\varphi(j)) & \longrightarrow & \operatorname*{colim}\limits_{\theta(\varphi(j))\to\psi(k)} \alpha(\varphi(j)) \\
 & & \Big\downarrow & & \Big\downarrow \\
\operatorname{colim} \alpha \circ \varphi & \xrightarrow[a]{} & \operatorname*{colim}\limits_{\ell} \operatorname*{colim}\limits_{\theta(\varphi(j))\to\ell} \alpha(\varphi(j)) & \xrightarrow[b]{} & \operatorname*{colim}\limits_{k} \operatorname*{colim}\limits_{\theta(\varphi(j))\to\psi(k)} \alpha(\varphi(j)),
\end{array}$$

$$\begin{array}{ccccc}
\alpha(\varphi(j)) & \xrightarrow{\mathrm{id}} & \alpha(\varphi(j)) & \xrightarrow{\mathrm{id}} & \alpha(\varphi(j)) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\operatorname*{colim}\limits_{\theta(\varphi(j))\to\psi(k)} \alpha(\varphi(j)) & \longrightarrow & \operatorname*{colim}\limits_{\theta(i)\to\psi(k)} \alpha(i) & \xrightarrow{\mathrm{id}} & \operatorname*{colim}\limits_{\theta(i)\to\psi(k)} \alpha(i) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\operatorname*{colim}\limits_{k} \operatorname*{colim}\limits_{\theta(\varphi(j))\to\psi(k)} \alpha(\varphi(j)) & \xrightarrow[c]{} & \operatorname*{colim}\limits_{k} \operatorname*{colim}\limits_{\theta(i)\to\psi(k)} \alpha(i) & \xrightarrow[d]{} & \operatorname*{colim}\limits_{\ell} \operatorname*{colim}\limits_{\theta(i)\to\ell} \alpha(i),
\end{array}$$

$$\begin{array}{ccc}
\alpha(\varphi(j)) & \xleftarrow{\mathrm{id}} & \alpha(\varphi(j)) \\
\Big\downarrow & & \Big\downarrow q_j \\
\operatorname*{colim}\limits_{\theta(i)\to\psi(k)} \alpha(i) & & \\
\Big\downarrow & & \\
\operatorname*{colim}\limits_{\ell} \operatorname*{colim}\limits_{\theta(i)\to\ell} \alpha(i) & \xleftarrow[e]{} & \operatorname{colim} \alpha.
\end{array}$$

Note that the last row of the first (resp. second) diagram coincides with the first row of the second (resp. third) diagram. Moreover the vertical arrows are coprojections, the squares above a and e result from the proof of (34) p. 73, the squares above b and d result from the cofinality of ψ, and the squares above c result from the cofinality of φ_k. In particular, the maps a, b, c, d and e are bijective. As the bijection $f := e^{-1} \circ d \circ c \circ b \circ a$ satisfies $f \circ p_j = q_j$, it is the natural map from $\operatorname{colim} \alpha \circ \varphi$ to $\operatorname{colim} \alpha$. □

§ 128. P. 75. Throughout the section about the IPC Property, one can assume that $\mathcal{A}$ is a big category. This applies in particular to Corollary 3.1.12 p. 77, corollary used in this generalized form at the end of the proof of Proposition 6.1.16 p. 136 of the book.

§ 129. P. 77, Proposition 3.1.11 (ii). This proposition says that **Set** has the IPC property. Recall the setting:

Let S be a small set, for each s in S let I_s be a small set and $\alpha_s : I_s \to \mathbf{Set}$ a functor, put $I := \prod_{s\in S} I_s$, let

$$p_j : \prod_{s\in S} \alpha_s(j_s) \to \operatorname*{colim}_{i\in I} \prod_{s\in S} \alpha_s(i_s), \quad q_{j_s} : \alpha_s(j_s) \to \operatorname*{colim}_{i_s\in I_s} \alpha_s(i_s)$$

be the coprojections, and define

$$f : \operatorname*{colim}_{i\in I} \prod_{s\in S} \alpha_s(i_s) \to \prod_{s\in S} \operatorname*{colim}_{i_s\in I_s} \alpha_s(i_s)$$

by $(f(p_j(x)))_s := q_{j_s}(x_s)$. Let

$$g : \prod_{s\in S} \operatorname*{colim}_{i_s\in I_s} \alpha_s(i_s) \to \operatorname*{colim}_{i\in I} \prod_{s\in S} \alpha_s(i_s) \tag{65}$$

and consider the following condition on g:

Condition 130. We have

$$g((q_{j_s}(y_s))_{s\in S}) = p_j(y)$$

for all j in I and all y in $\prod_{s\in S} \alpha_s(j_s)$.

Clearly the proposition below implies Proposition 3.1.11 (ii) in the book.

Proposition 131. *If g satisfies Condition 130, then f and g are inverse bijections. If I_s is filtrant for all s in S, then there is a g as in* (65) *satisfying Condition 130.*

Proof. The proof of the first sentence is straightforward. To prove the second sentence, let j and k be in I, let y in $\prod_{s\in S} \alpha_s(j_s)$ and z in $\prod_{s\in S} \alpha_s(k_s)$ satisfy $q_{j_s}(y_s) = q_{k_s}(z_s)$ for all s in S. It suffices to show $p_j(y) = p_k(z)$. By Corollary 3.1.4 (ii) p. 73 in the book, for each s in S there is a diagram

$$j_s \xrightarrow{u_s} \ell_s \xleftarrow{v_s} k_s$$

in I_s and an element w_s in $\alpha_s(\ell_s)$ such that $\alpha_s(u_s)(y_s) = w_s = \alpha_s(v_s)(z_s)$. This implies $p_j(y) = p_\ell(w) = p_k(z)$, and thus $p_j(y) = p_k(z)$ as requested. □

Proposition 132. *Let $\mathcal{U}$ be a universe, let* **Set** *be the category of $\mathcal{U}$-sets, let Z be in* **Set***, and let Z' be the discrete category whose set of objects is Z. Then there are canonical isomorphisms*

$$\mathbf{Set}_Z \simeq \prod_{z\in Z} \mathbf{Set} \simeq \mathrm{Fct}(Z', \mathbf{Set}).$$

Propositions 131 and 132 imply

Proposition 133. *If $Z \in$ **Set**, then $\mathbf{Set}_Z$ has the IPC property.*

§ 134. P. 78, Proposition 3.2.2. It is easy to see that Condition (iii) is equivalent to

$$\operatorname{colim}\ \mathrm{Hom}_I(i, \varphi) \simeq \{\mathrm{pt}\} \quad \text{for all } i \in I, \tag{66}$$

which is Condition (vi) in Proposition 2.5.2 p. 57 of the book. (Proposition 2.5.2 states, among other things, that (66) is equivalent to the cofinality of φ.)

§ 135. P. 79, proof of Corollary 3.2.3 (ii). Here are more details: For $(i, j) \in I \times J$ we have $I^{(i,j)} \simeq (I^i)^j$. Part (i) implies that I^i is filtrant and the forgetful functor $I^i \to I$ is cofinal. Then Proposition 3.2.2, (i) $\Rightarrow$ (ii), p. 78 of the book implies that $(I^i)^j$ is filtrant. Finally, Proposition 3.2.2, (ii) $\Rightarrow$ (i) implies that the diagonal functor $I \to I \times I$ is cofinal.

§ 136. P. 79, Proposition 3.2.5. It is claimed that (ii) is a particular case of (iv). More precisely, (ii) is obtained from (iv) by replacing the setting

$$I \xrightarrow{\varphi} J \xrightarrow{\psi} K, \quad u : k \to \psi(j)$$

with

$$I \xrightarrow{\mathrm{id}_I} I \xrightarrow{\varphi} J, \quad \mathrm{id}_{\varphi(i)} : \varphi(i) \to \varphi(i).$$

§ 137. P. 80. Propositions 3.2.4 and 3.2.6 can be combined as follows.

Proposition 138. *Let $\varphi : J \to I$ be fully faithful. Assume that I is filtrant and cofinally small, and that for each i in I there is a morphism $i \to \varphi(j)$ for some j in J. Then φ is cofinal and J is filtrant and cofinally small.*

Proof. In view of Proposition 3.2.4 it suffices to show that J is cofinally small. By Proposition 3.2.6, there is a small full subcategory (Definition 5 p. 10) $S \subset I$ cofinal to I. For each s in S pick a morphism $s \to \varphi(j_s)$ with j_s in J. Then, for each j in J there are morphisms $\varphi(j) \to s \to \varphi(j_s)$ with s in S. As φ is full there is a morphism $j \to j_s$, and we conclude by using again Proposition 3.2.6. □

§ 139. P. 80, proof of Lemma 3.2.8 (minor variant). As already pointed out, a $\varinjlim$ is missing in the last display. Recall the statement:

Lemma 140. *Let I be a small ordered set, let $\alpha : I \to \mathcal{C}$ be a functor, let $\mathcal{J}$ be the set of finite subsets of I ordered by inclusion, and for each J in $\mathcal{J}$ let $\alpha_J : \mathcal{J} \to \mathcal{C}$ be the restriction of α to $\mathcal{J}$. Then $\mathcal{J}$ is small (Definition 5 p. 10) and filtrant, and we have*

$$\operatorname{colim} \alpha \simeq \operatorname*{colim}_{J \in \mathcal{J}} \operatorname{colim} \alpha_J.$$

in $\mathcal{C}^\vee$.

Proof. Set

$$A := \operatorname{colim} \alpha, \quad \beta(J) := \operatorname{colim} \alpha_J, \quad B := \operatorname{colim} \beta.$$

Let

$$p_i : \alpha(i) \to A, \quad p_{i,J} : \alpha(i) \to \beta(J), \quad p_J : \beta(J) \to B$$

be the coprojections. Note that $p_{i,J}$ is defined only for i in J. We easily check that

- the morphisms$f_i := p_{\{i\}} \circ p_{i,\{i\}} : \alpha(i) \to B$ induce a morphism $f : A \to B$,
- the morphisms $g_{i,J} := p_i : \alpha(i) \to A$ (with i in J) induce a morphism $g_J : \beta(J) \to A$,
- the morphisms g_J induce a morphism $g : B \to A$,
- f and g are mutually inverse isomorphisms. □

For the reader's convenience we reproduce Definition 3.3.1 p. 81.

Definition 141 (Definition 3.3.1, exactness)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor.*

(i) *We say that F is* right exact *if the category $\mathcal{C}_{X'}$ is filtrant for all X' in $\mathcal{C}'$.*

(ii) *We say that F is* left exact *if $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$ is right exact, or equivalently if the category $\mathcal{C}^{X'}$ is cofiltrant for all X' in $\mathcal{C}'$.*

(iii) *We say that F is* exact *if it is both right and left exact.*

§ 142. P. 81, proof of Proposition 3.3.2 (minor variant). Recall the statement:

Proposition 143 (Proposition 3.3.2 p. 81)**.** *Consider functors $I \xrightarrow{\alpha} \mathcal{C} \xrightarrow{F} \mathcal{C}'$, and assume that I is finite, that F is right exact, and that* $\operatorname{colim} \alpha$ *exists in $\mathcal{C}$. Then* $\operatorname{colim} F \circ \alpha$ *exists in $\mathcal{C}'$, and the natural morphism* $\operatorname{colim} F \circ \alpha \to F(\operatorname{colim} \alpha)$ *is an isomorphism.*

Proof. Let X' be in $\mathcal{C}'$. It suffices to show that the natural map

$$\mathrm{Hom}_{\mathcal{C}'}(F(\operatorname{colim}\alpha),X')\to\lim\mathrm{Hom}_{\mathcal{C}'}(F\circ\alpha,X')$$

is bijective. We claim

$$\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\mathrm{Hom}_{\mathcal{C}}(X,Y)\simeq\operatorname*{colim}_{(X\to Y)\in(\mathcal{C}^X)^{\mathrm{op}}}\mathrm{Hom}_{\mathcal{C}'}(F(Y),X') \tag{67a}$$

$$\simeq\mathrm{Hom}_{\mathcal{C}'}(F(X),X'). \tag{67b}$$

Indeed, we obtain (67a) by replacing the setting (41) p. 78 with

$$X\in\mathcal{C}\xleftarrow{\mathrm{id}_{\mathcal{C}}}\mathcal{C}\xrightarrow{F}\mathcal{C}'\ni X'$$

in the isomorphism (42) p. 78, and we prove (67b) by noting that the identity of X is an initial object of $\mathcal{C}^X$. We have five sets and four bijections:

$$\mathrm{Hom}_{\mathcal{C}'}(F(\operatorname{colim}\alpha),X')\xrightarrow{\sim}\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}\alpha,Y)$$

$$\xrightarrow{\sim}\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\lim\mathrm{Hom}_{\mathcal{C}}(\alpha,Y)\xrightarrow{\sim}\lim\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\mathrm{Hom}_{\mathcal{C}}(\alpha,Y)$$

$$\xrightarrow{\sim}\lim\mathrm{Hom}_{\mathcal{C}'}(F\circ\alpha,X').$$

The first and last bijections follow from (67), the second one is clear, and the third one can be justified as follows: Set-valued inductive limits over the category $\mathcal{C}_{X'}$, which is filtrant because F is right exact, commute with set-valued projective limits over the finite category I (Theorem 3.1.6 p. 74 of the book).

Let us denote these five sets and four bijections by

$$S_1\xrightarrow{f_1}S_2\xrightarrow{f_2}S_3\xrightarrow{f_3}S_4\xrightarrow{f_4}S_5,$$

and let

$$f:\mathrm{Hom}_{\mathcal{C}'}(F(\operatorname{colim}\alpha),X')\to\lim\mathrm{Hom}_{\mathcal{C}'}(F\circ\alpha,X').$$

be the natural map. It remains to show

$$f_4\circ f_3\circ f_2\circ f_1=f. \tag{68}$$

Let Y be in $\mathcal{C}$, let $F(Y)\to X'$ be a morphism in $\mathcal{C}'$, and let

$$p[F(Y)\to X']:\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}\alpha,Y)\to\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}\alpha,Y),$$

$$q[F(Y)\to X']:\lim\mathrm{Hom}_{\mathcal{C}}(\alpha,Y)\to\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\lim\mathrm{Hom}_{\mathcal{C}}(\alpha,Y),$$
$$r[F(Y)\to X']:\mathrm{Hom}_{\mathcal{C}}(\alpha,Y)\to\operatorname*{colim}_{(F(Y)\to X')\in\mathcal{C}_{X'}}\mathrm{Hom}_{\mathcal{C}}(\alpha,Y)$$

be the coprojections.

We shall use implicitly, not only the statements of the bijections (67a) and (67b), but also their proofs (see Warning 99 p. 75).

For $F(\operatorname{colim}\alpha)\to X'$ in $\mathrm{Hom}_{\mathcal{C}'}(F(\operatorname{colim}\alpha),X')$, we have (omitting most of the parenthesis)

$$\begin{aligned}
&f_4f_3f_2f_1(F(\operatorname{colim}\alpha)\to X')\\
&=f_4f_3f_2\left(p[F(\operatorname{colim}\alpha)\to X']\left(\operatorname{colim}\alpha\xrightarrow{\mathrm{id}}\operatorname{colim}\alpha\right)\right)\\
&=f_4f_3\left(q[F(\operatorname{colim}\alpha)\to X']\left(\left(\alpha(i)\to\operatorname{colim}\alpha\right)_i\right)\right)\\
&=f_4\left(\left(r[F(\operatorname{colim}\alpha)\to X']\big(\alpha(i)\to\operatorname{colim}\alpha\big)\right)_i\right)\\
&=\left(\left(F(\alpha(i))\to F(\operatorname{colim}\alpha)\to X'\right)_i\right).
\end{aligned}$$

This proves (68). □

§ 144. P. 83, Proposition 3.3.6. Here is a mild generalization:

Proposition 145. *Let $\mathcal{C}\xleftarrow{G}\mathcal{A}\xrightarrow{F}\mathcal{B}$ be functors. Assume that for each Y in $\mathcal{B}$ there is a Z in $\mathcal{C}$ and an isomorphism*

$$\mathrm{Hom}_{\mathcal{B}}(F(\ ),Y)\simeq\mathrm{Hom}_{\mathcal{C}}(G(\ ),Z)$$

in $\mathcal{A}^\wedge$. If G is right exact, then so if F.

Proof. The proof is similar to that of Proposition 3.3.6 in the book. The details are left to the reader. □

§ 146. P. 83, proof of Proposition 3.3.7 (i). The proof uses Proposition 3.3.3.

§ 147. Some more details in the proof of Proposition 3.3.12 p. 84:

Proposition 148 (Proposition 3.3.12 p. 84)**.** *Let $F:\mathcal{C}\to\mathcal{C}'$ and $G:\mathcal{C}'\to\mathcal{C}''$ be two functors. If F and G are right exact, then $G\circ F$ is right exact.*

Proof. Since G is right exact, $\mathcal{C}'_{X''}$ is filtrant for any X'' in $\mathcal{C}''$. The obvious functor $\mathcal{C}_{X''} \to \mathcal{C}'_{X''}$ is again right exact. Indeed, for any $G(X') \to X''$ in $\mathcal{C}'_{X''}$, the category $(\mathcal{C}_{X''})_{G(X') \to X''} \simeq \mathcal{C}_{X'}$ is filtrant because F is right exact. Hence, Proposition 3.3.11 implies that $\mathcal{C}_{X''}$ is filtrant. $\square$

§ 149. P. 84, Proposition 3.3.13. Recall the statement:

Proposition 150 (Proposition 3.3.13 p. 84)**.** *Let $\mathcal{C}$ be a category admitting finite inductive limits, and let A be in $\mathcal{C}^\wedge$. Then A is left exact if and only if $\mathcal{C}_A$ is filtrant.*

We spell out the details of the proof of the implication $\mathcal{C}_A$ is filtrant $\Rightarrow$ A left exact.

By Proposition 3.3.3 of the book, stated as Proposition 161 p. 105 below, it suffices to show that A commutes with finite projective limits. Let (X_i) be a finite inductive system in $\mathcal{C}$. We must check that the natural map

$$e : A\left(\operatorname*{colim}_i X_i\right) \to \lim_i A(X_i)$$

is bijective. Let us abbreviate $(Y \to A) \in \mathcal{C}_A$ by Y, and consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{colim}_Y \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_i X_i, Y) & \xrightarrow{a} & \operatorname{colim}_Y \lim_i \operatorname{Hom}_{\mathcal{C}}(X_i, Y) \\
 & & \Big\downarrow b \\
\Big\downarrow d & & \lim_i \operatorname{colim}_Y \operatorname{Hom}_{\mathcal{C}}(X_i, Y) \\
 & & \Big\downarrow c \\
A(\operatorname{colim}_i X_i) & \xrightarrow[e]{} & \lim_i A(X_i),
\end{array}$$

where a is defined by §58 p. 49, c and d are defined by (45) p. 81 and b is defined by Proposition 125 p. 94 (see Warning 99 p. 75). These four maps are clearly bijective. We leave it to the reader to check that this diagram commutes. This implies that e is bijective.

§ 151. P. 85, proof of Proposition 3.3.15. To prove that $\mathcal{A} \to \mathcal{C}$ is cofinal, one can apply Proposition 127 p. 94 with $J = \mathcal{A}, I = \mathcal{C}, L = \mathcal{C}', K = \mathcal{S}$.

§ 152. P. 86, proof of Theorem 3.3.18 (b). The proof uses the following fact, whose proof is straightforward:

Let $\alpha : I \times J^{\mathrm{op}} \to \mathcal{C}$ be a functor. Assume that $\mathcal{C}$ admits inductive limits indexed by I and projective limits indexed by J^{op}. Then the morphism obtained by composing the canonical morphism

$$\operatorname*{colim}_{i\in I} \lim_{j\in J^{\mathrm{op}}} \alpha(i,j) \to \lim_{j\in J^{\mathrm{op}}} \operatorname*{colim}_{i\in I} \alpha(i,j)$$

with the projection

$$\lim_{j\in J^{\mathrm{op}}} \operatorname*{colim}_{i\in I} \alpha(i,j) \to \operatorname*{colim}_{i\in I} \alpha(i,j)$$

coincides with the morphism obtained by applying the functor $\operatorname{colim}_{i\in I}$ to the projection

$$\lim_{j\in J^{\mathrm{op}}} \alpha(i,j) \to \alpha(i,j).$$

5.2 Proposition 3.4.3 (i) p. 88

Lemma 153. *If $I \xrightarrow{\varphi} K \xleftarrow{\psi} J$ are functors between small categories (Definition 5 p. 10), if*

$$M := M[I \xrightarrow{\varphi} K \xleftarrow{\psi} J]$$

is the category defined in Definition 3.4.1 p. 87 of the book, if $\alpha : M \to \mathcal{C}$ is a functor, and if $\mathcal{C}$ admits small inductive limits, then there is a natural functor (described in the proof) from J to $\mathcal{C}$ mapping $j \in J$ to

$$\operatorname*{colim}_{(i,u)\in I_{\psi(j)}} \alpha(i,j,u)$$

(u being a morphism in K from $\varphi(i)$ to $\psi(j)$).

Proof. Let $j \to j'$ be a morphism in J. It is easily checked that there is a unique dashed arrow which make all diagrams

$$\begin{array}{ccc} \operatorname*{colim}_{(i,u)\in I_{\psi(j)}} \alpha(i,j,u) & \dashrightarrow & \operatorname*{colim}_{(i,u)\in I_{\psi(j')}} \alpha(i,j',u) \\ \uparrow{\scriptstyle p_{iu}} & & \uparrow{\scriptstyle q_{iu'}} \\ \alpha(i,j,u) & \xrightarrow[\alpha(\mathrm{id}_i, j\mapsto j')]{} & \alpha(i,j',u') \end{array}$$

commute, where p_{iu} and $q_{iu'}$ are the coprojections and u' is the obvious composition

$$\varphi(i) \xrightarrow{u} \psi(j) \to \psi(j'),$$

and that the assignment

$$(j \to j') \mapsto \left(\operatorname*{colim}_{(i,u) \in I_{\psi(j)}} \alpha(i,j,u) \to \operatorname*{colim}_{(i,u) \in I_{\psi(j')}} \alpha(i,j',u) \right)$$

is functorial. □

Proposition 154 (Proposition 3.4.3 (i) p. 88)**.** *We have an isomorphism*

$$\operatorname{colim} \alpha \simeq \operatorname*{colim}_j \ \operatorname*{colim}_{i,u} \alpha(i,j,u),$$

where (i,u) *runs over* $I_{\psi(j)}$ *(with* $u : \varphi(i) \to \psi(j)$*). This isomorphism is explicitly described in the proof.*

Proof. Let

$$\alpha(i,j,u) \xrightarrow{p_{iju}} \operatorname{colim} \alpha, \qquad \alpha(i,j,u) \xrightarrow{q_{iju}} \operatorname*{colim}_{i,u} \alpha(i,j,u) \xrightarrow{r_j} \operatorname*{colim}_j \ \operatorname*{colim}_{i,u} \alpha(i,j,u)$$

be the coprojections. There is a unique morphism

$$f : \operatorname{colim} \alpha \to \operatorname*{colim}_j \ \operatorname*{colim}_{i,u} \alpha(i,j,u)$$

such that $f \circ p_{iju} = r_j \circ q_{iju}$ for all i,j,u:

$$\begin{array}{ccc} \alpha(i,j,u) & \xrightarrow{\text{id}} & \alpha(i,j,u) \\ & & \downarrow q_{iju} \\ \downarrow p_{iju} & & \operatorname*{colim}_{i,u} \alpha(i,j,u) \\ & & \downarrow r_j \\ \operatorname{colim} \alpha & \xrightarrow[f]{} & \operatorname*{colim}_j \operatorname*{colim}_{i,u} \alpha(i,j,u). \end{array}$$

We construct the commutative diagram

$$\begin{array}{ccc} \alpha(i,j,u) & \xrightarrow{\text{id}} & \alpha(i,j,u) \\ q_{iju} \downarrow & & \downarrow p_{iju} \\ \operatorname*{colim}_{i,u} \alpha(i,j,u) & \xrightarrow{g_j} & \operatorname{colim} \alpha \\ r_j \downarrow & & \downarrow \text{id} \\ \operatorname*{colim}_j \operatorname*{colim}_{i,u} \alpha(i,j,u) & \xrightarrow[g]{} & \operatorname{colim} \alpha. \end{array} \tag{69}$$

as follows: We fix j and define g_j by the condition that the top square of (69) commutes for all (i, u). Then we define g by the condition that the bottom square of (69) commutes for all j. We leave it to the reader to check that f and g are inverse isomorphisms. □

In view of Proposition 125 p. 94, Proposition 154 implies

Proposition 155. *If J and $I_{\psi(j)}$ are filtrant for all j in J, then M is filtrant.*

5.3 Brief comments

§ 156. We prove Proposition 3.4.3 (ii) p. 88. Recall the statement:

If ψ is cofinal, then $M[I \to K \leftarrow J] \to I$ is cofinal.

To prove this, we let $\alpha : I \to \mathbf{Set}$ be a functor, we denote by β the composition $M[I \to K \leftarrow J] \to I \to \mathbf{Set}$, and we verify that the natural map $\operatorname{colim} \beta \to \operatorname{colim} \alpha$ is bijective as follows.

In the commutative diagram below we write u for a generic morphism $\varphi(i) \to \psi(j)$ and v for a generic morphism $\varphi(i) \to k$, with $i \in I, j \in J, k \in K$, and we abbreviate $\operatorname{colim}_{(i,u)\in I_{\psi(j)}}$ by $\operatorname{colim}_{i,u}$ and $\operatorname{colim}_{(i,v)\in I_k}$ by $\operatorname{colim}_{i,v}$:

$$\begin{array}{ccccccc}
\alpha(i) & \xrightarrow{\text{id}} & \alpha(i) & \xrightarrow{\text{id}} & \alpha(i) & \xrightarrow{\text{id}} & \alpha(i) \\
\Big\downarrow a_{iju} & & \Big\downarrow b_{iju} & & \Big\downarrow b_{i\psi(j)u} & & \Big\downarrow e_i \\
 & & \operatorname*{colim}_{i,u} \alpha(i) & \xrightarrow{\text{id}} & \operatorname*{colim}_{i,u} \alpha(i) & & \\
 & & \Big\downarrow c_j & & \Big\downarrow d_{\psi(j)} & & \\
\operatorname{colim} \beta & \xrightarrow{f} & \operatorname*{colim}_{j} \operatorname*{colim}_{i,u} \alpha(i) & \xrightarrow{g} & \operatorname*{colim}_{k} \operatorname*{colim}_{i,v} \alpha(i) & \xrightarrow{h} & \operatorname{colim} \alpha.
\end{array}$$

(The vertical arrows are the various coprojections.) The diagram being commutative, $h \circ g \circ f$ is the natural map $\operatorname{colim} \beta \to \operatorname{colim} \alpha$. Moreover f is bijective by the proof of Proposition 154, g is bijective because ψ is cofinal and h is bijective by the proof of (34) p. 73.

§ 157. P. 89, Proposition 3.4.5 (iii). The proof uses implicitly the following fact:

Proposition 158. *If F is a cofinally small filtrant category, then there is a small* filtrant *full subcategory of F cofinal to F.*

This results immediately from Corollary 2.5.6 p. 59 and Proposition 3.2.4 p. 79 (see Proposition 138 p. 97). This fact also justifies the sentence "We may replace 'filtrant and small' by 'filtrant and cofinally small' in the above definition" p. 132, Lines 4 and 5 of the book.

5.4 Five closely related statements

For the reader's convenience we collect five statements closely related to Exercise 3.4 (i) p. 90 of the book.

5.4.1 Proposition 2.1.10 p. 40

Proposition 159 (Proposition 2.1.10 p. 40)**.** *If $F : \mathcal{C} \to \mathcal{C}'$ is a functor admitting a left adjoint, if I is a category, and if $\mathcal{C}$ admits projective limits indexed by I, then F commutes with such limits.*

(This fact has already been stated as Corollary 67 p. 56.)

5.4.2 Exercise 2.7 (ii) p. 65

Proposition 160 (Exercise 2.7 (ii) p. 65)**.** *The base change functors (see Section 4.6 p. 63) in* **Set** *commute with small inductive and projective limits. In particular, small inductive limits in* **Set** *are stable by base change.*

Note that Proposition 160 generalizes the distributivity of multiplication over addition in $\mathbb{N}$.

5.4.3 Proposition 3.3.3 p. 82

Proposition 161 (Proposition 3.3.3 p. 82)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and assume that $\mathcal{C}$ admits finite projective limits. Then F is left exact if and only if it commutes with such limits.*

Corollary 162. *In the setting of Proposition 2.7.1 p. 62 of the book, the functors*

$$\mathcal{A}^{\mathcal{C}} \to \mathcal{A}, \quad F \mapsto (\mathrm{h}^{\dagger}_{\mathcal{C}}\, F)(A) \quad \textit{and} \quad \mathcal{C}^{\wedge} \to \mathcal{A}, \quad A \mapsto (\mathrm{h}^{\dagger}_{\mathcal{C}}\, F)(A)$$

are right exact.

Proof. This follows from Proposition 161 and §111 p. 84. □

Corollary 163. *In the setting of §119 p. 90, the functors*

$$\mathcal{A}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{A}, \quad F \mapsto (h^{\mathrm{op}})^{\ddagger}(F)(A) \quad \textit{and} \quad \mathcal{C}^{\wedge} \to \mathcal{A}, \quad A \mapsto (h^{\mathrm{op}})^{\ddagger}(F)(A)$$

are left exact.

5.4.4 Proposition 3.3.6 p. 83

Proposition 164 (Proposition 3.3.6 p. 83)**.** *A functor admitting a left adjoint is left exact.*

5.4.5 Exercise 3.4 (i) p. 90

Proposition 165 (Exercise 3.4 (i) p. 90)**.** *If $F : \mathcal{C} \to \mathcal{C}'$ is a right exact functor and $f : X \twoheadrightarrow Y$ is an epimorphism in $\mathcal{C}$, then $F(f) : F(X) \to F(Y)$ is an epimorphism in $\mathcal{C}'$.*

(This exercise is used in the second sentence of p. 227 of the book.)

Proof. Let $f_1', f_2' : F(Y) \rightrightarrows X'$ be morphisms in $\mathcal{C}'$ satisfying

$$f_1' \circ F(f) = f_2' \circ F(f) =: f'.$$

This is visualized by the diagram

$$\Big(F(X) \xrightarrow{f} F(Y) \underset{f_2'}{\overset{f_1'}{\rightrightarrows}} X'\Big) = \Big(F(X) \xrightarrow{f'} X'\Big).$$

It suffices to prove $f_1' = f_2'$. For $i = 1, 2$ let f_i be the morphism f viewed as a morphism from (X, f') to (Y, f_i') in $\mathcal{C}_{X'}$:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & {}_{f'}\searrow \quad \swarrow_{f_i'} & \\ & X'. & \end{array}$$

As $\mathcal{C}_{X'}$ is filtrant, there are morphisms $\gamma_i : (Y, f'_i) \to (Z, g')$, defined by morphisms $g_i : Y \to Z$, such that $\gamma_1 \circ f_1 = \gamma_2 \circ f_2$:

$$\begin{array}{ccccc} F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g_i)} & F(Z) \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f'_i}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ X' & = & X' & = & X'. \end{array}$$

As f is an epimorphism, the equality $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2 =: g$, and thus $f'_1 = g' \circ F(g) = f'_2$. □

Corollary 166. *Let $\mathcal{C}$ be a category, let $\mathcal{C}'$ be a category admitting finite inductive limits, and let $\theta : F \to G$ be a morphism in ${\mathcal{C}'}^{\mathcal{C}}$. Then θ is an epimorphism if and only if $\theta_X : F(X) \to G(X)$ is an epimorphism for all X in $\mathcal{C}$.*

Proof. This follows from Proposition 161 p. 105 and Proposition 165 just above. □

6 About Chapter 4

§ 167. P. 93, Lemma 4.1.2. Here is a slightly more general statement:

Lemma 168. *Let $\mathcal{C}$ be a category, let $P : \mathcal{C} \to \mathcal{C}$ be a functor, let $\varepsilon : \mathrm{id}_{\mathcal{C}} \to P$ be a morphism of functors, and let X be an object of $\mathcal{C}$. Then the following conditions are equivalent:*

(a) *$\varepsilon_{P(X)}$ is an isomorphism and $P(\varepsilon_X)$ is an epimorphism,*

(b) *$P(\varepsilon_X)$ is an isomorphism and $\varepsilon_{P(X)}$ is a monomorphism,*

(c) *$\varepsilon_{P(X)}$ and $P(\varepsilon_X)$ are equal isomorphisms.*

Proof. It is enough to prove (a)⇒(c)⇐(b).

(a)⇒(c): Put $u := (\varepsilon_{P(X)})^{-1} \circ P(\varepsilon_X)$. It suffices to show

$$u = \mathrm{id}_{P(X)} \,. \tag{70}$$

We have

$$u \circ \varepsilon_X = (\varepsilon_{P(X)})^{-1} \circ P(\varepsilon_X) \circ \varepsilon_X = (\varepsilon_{P(X)})^{-1} \circ \varepsilon_{P(X)} \circ \varepsilon_X = \varepsilon_X,$$

and thus
$$P(u) \circ P(\varepsilon_X) = P(\varepsilon_X) = \mathrm{id}_{P^2(X)} \circ P(\varepsilon_X).$$
As $P(\varepsilon_X)$ is an epimorphism, this implies $P(u) = \mathrm{id}_{P^2(X)}$, and thus
$$\varepsilon_{P(X)} \circ u = P(u) \circ \varepsilon_{P(X)} = \varepsilon_{P(X)}.$$
As $\varepsilon_{P(X)}$ is an isomorphism, this implies (70), as required.

(b)⇒(c): We shall use several times the assumption that $P(\varepsilon_X)$ is an isomorphism. Put $v := P(\varepsilon_X)^{-1} \circ \varepsilon_{P(X)}$. It suffices to show
$$v = \mathrm{id}_{P(X)}\,. \tag{71}$$
We have
$$v \circ \varepsilon_X = P(\varepsilon_X)^{-1} \circ \varepsilon_{P(X)} \circ \varepsilon_X = P(\varepsilon_X)^{-1} \circ P(\varepsilon_X) \circ \varepsilon_X = \varepsilon_X,$$
$$P(v) \circ P(\varepsilon_X) = P(\varepsilon_X),$$
$$P(v) = \mathrm{id}_{P^2(X)},$$
$$\varepsilon_{P(X)} \circ v = P(v) \circ \varepsilon_{P(X)} = \varepsilon_{P(X)} = \varepsilon_{P(X)} \circ \mathrm{id}_{P(X)}\,.$$
As $\varepsilon_{P(X)}$ is a monomorphism, this implies (71), as required. □

Definition 4.1.1 p. 93 of the book can be stated as follows:

Definition 169 (Definition 4.1.1 p. 93, projector)**.** *Let $\mathcal{C}$ be a category. A* projector *on $\mathcal{C}$ is the data of a functor $P : \mathcal{C} \to \mathcal{C}$ and a morphism $\varepsilon : \mathrm{id}_{\mathcal{C}} \to P$ such that each object X of $\mathcal{C}$ satisfies the equivalent conditions of Lemma 168.*

§ 170. P. 94, proof of (a)⇒(b) in Proposition 4.1.3 (ii) (additional details): In the commutative diagram
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(P(Y), X) & \xrightarrow{\circ\varepsilon_Y} & \mathrm{Hom}_{\mathcal{C}}(Y, X) \\ {\scriptstyle \varepsilon_X\circ}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle \varepsilon_X\circ} \\ \mathrm{Hom}_{\mathcal{C}}(P(Y), P(X)) & \xrightarrow[\circ\varepsilon_Y]{\sim} & \mathrm{Hom}_{\mathcal{C}}(Y, P(X)), \end{array}$$
the vertical arrows are bijective by (a), and the bottom arrow is bijective by (i).

§ 171. P. 95, end of the proof of Proposition 4.1.3. Recall that we have functors

$$\mathcal{C}_0 \underset{P}{\overset{\iota}{\rightleftarrows}} \mathcal{C}.$$

The last sentence of the proof says that P "is a left adjoint to ι by (i)". One could also write that P "is a left adjoint to ι by Condition (b) in Part (ii)". Indeed, Condition (b) in Part (ii) asserts that the map

$$\operatorname{Hom}_{\mathcal{C}}(P(Y), X) \xrightarrow{\circ\varepsilon_Y} \operatorname{Hom}_{\mathcal{C}}(Y, X),$$

that is

$$\operatorname{Hom}_{\mathcal{C}_0}(P(Y), X) \xrightarrow{\circ\varepsilon_Y} \operatorname{Hom}_{\mathcal{C}}(Y, \iota(X)),$$

is bijective for all Y in $\mathcal{C}$.

§ 172. P. 95, Proposition 4.1.4.

• Proof of (i) (additional details). The authors write: "The two compositions

$$P \underset{P\circ\varepsilon}{\overset{\varepsilon\circ P}{\rightrightarrows}} P^2 \xrightarrow{R\eta L} P$$

are equal to id_P". If we translate this statement into the language of Notation 31 p. 31 and Notation 33 p. 32, we get

$$\begin{pmatrix} R\star\eta\star L \\ \varepsilon\star R\star L \end{pmatrix} = RL = \begin{pmatrix} R\star\eta\star L \\ R\star L\star\varepsilon \end{pmatrix}. \tag{72}$$

To prove (72), write

$$\begin{pmatrix} R\star\eta\star L \\ \varepsilon\star R\star L \end{pmatrix} = \begin{pmatrix} R\star\eta & L \\ \varepsilon\star R & L \end{pmatrix} = \begin{pmatrix} R\star\eta \\ \varepsilon\star R \end{pmatrix} \star \begin{pmatrix} L \\ L \end{pmatrix} \overset{\text{(a)}}{=} RL$$

$$\overset{\text{(b)}}{=} \begin{pmatrix} R \\ R \end{pmatrix} \star \begin{pmatrix} \eta\star L \\ L\star\varepsilon \end{pmatrix} = \begin{pmatrix} R & \eta\star L \\ R & L\star\varepsilon \end{pmatrix} = \begin{pmatrix} R\star\eta\star L \\ R\star L\star\varepsilon \end{pmatrix},$$

Equalities (a) and (b) resulting respectively from (10) p. 42 and (9) p. 42, and the other equalities following from Proposition 34 p. 32.

• Statement of (iii): As explained in the proof, the phrase "$\mathcal{C}'$ is equivalent to $\mathcal{C}_0$" really means "R induces an equivalence from $\mathcal{C}'$ to $\mathcal{C}_0$".

§ 173. Definition 4.2.1 p. 96. It is important for aesthetic reasons to note that tensor products can be transported along equivalences. We sketch a proof of this fact. In this section we will use a notation which very different from the one used in the rest of this text (and in the book).

Let $f : A \to B$ and $g : B \to A$ be quasi-inverse equivalences. If b is an object of B, we write b^g for the image of b under g, and gf for the functor $b \mapsto b^{gf}$, etc. We also write xy for $x \otimes y$. Let us assume that A is a tensor category.

We define b_1b_2 for b_1, b_2 in B by

$$b_1b_2 := (b_1^g b_2^g)^f.$$

Let α be the associator of A. We define

$$\beta(b_1, b_2, b_3) : (b_1b_2)b_3 \to b_1(b_2b_3),$$

that is

$$\beta(b_1, b_2, b_3) : ((b_1^g b_2^g)^{fg} b_3^g)^f \to (b_1^g (b_2^g b_3^g)^{fg})^f,$$

as being the composite of the obvious isomorphisms

$$((b_1^g b_2^g)^{fg} b_3^g)^f \to ((b_1^g b_2^g) b_3^g)^f \xrightarrow{\alpha(b_1,b_2,b_3)^f} (b_1^g (b_2^g b_3^g))^f \to (b_1^g (b_2^g b_3^g)^{fg})^f.$$

Let b_1, b_2, b_3, b_4 be in B. We must check that the pentagon build from b_1, b_2, b_3, b_4 commutes.

Pick one edge of this pentagon, say the edge

$$\beta(b_1b_2, b_3, b_4) : \big((b_1b_2)b_3\big)b_4 \to (b_1b_2)(b_3b_4),$$

that is

$$\beta(b_1b_2, b_3, b_4) : \Big(\big((b_1^g b_2^g)^{fg} b_3^g\big)^{fg} b_4^g\Big)^f \to \Big((b_1^g b_2^g)^{fg} (b_3^g b_4^g)^{fg}\Big)^f.$$

We complete this edge to the obvious square of isomorphisms

$$\begin{array}{ccc} \Big(\big((b_1^g b_2^g)^{fg} b_3^g\big)^{fg} b_4^g\Big)^f & \xrightarrow{\beta(b_1b_2,b_3,b_4)} & \Big((b_1^g b_2^g)^{fg} (b_3^g b_4^g)^{fg}\Big)^f \\ \downarrow & & \downarrow \\ \Big(\big((b_1^g b_2^g) b_3^g\big) b_4^g\Big)^f & \xrightarrow[\alpha(b_1^g b_2^g, b_3^g, b_4^g)^f]{} & \Big((b_1^g b_2^g)(b_3^g b_4^g)\Big)^f. \end{array} \tag{73}$$

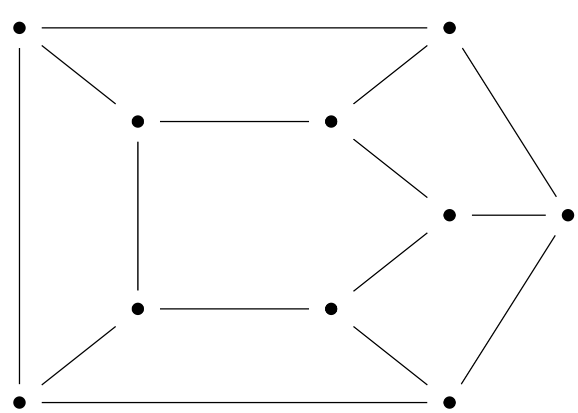
We claim that (73) commutes. Consider the diagram of isomorphisms

$$\begin{array}{ccc}
\Big(\big((b_1^g b_2^g)^{fg} b_3^g\big)^{fg} b_4^g\Big)^f & \xrightarrow{\beta(b_1b_2,b_3,b_4)} & \Big((b_1^g b_2^g)^{fg}(b_3^g b_4^g)^{fg}\Big)^f \\
\downarrow & & \downarrow \\
\Big(\big((b_1^g b_2^g)^{fg} b_3^g\big) b_4^g\Big)^f & \xrightarrow{\alpha((b_1^g b_2^g)^{fg},b_3^g,b_4^g)^f} & \Big((b_1^g b_2^g)^{fg}(b_3^g b_4^g)\Big)^f \\
\downarrow & & \downarrow \\
\Big(\big((b_1^g b_2^g) b_3^g\big) b_4^g\Big)^f & \xrightarrow[\alpha(b_1^g b_2^g,b_3^g,b_4^g)^f]{} & \Big((b_1^g b_2^g)(b_3^g b_4^g)\Big)^f.
\end{array}$$

The top square commutes by definition of β, whereas the bottom square commutes by functoriality of α in its first variable. This proves that (73) commutes.

There is a commutative square of isomorphisms similar to (73) for each edge of the pentagon build from b_1, b_2, b_3, b_4, so that we get two pentagons, one over the other, each vertex of the top pentagon being linked by an edge to the corresponding vertex of the bottom pentagon. The bottom pentagon commutes because A is a tensor category. We've just verified that one of the vertical squares commutes. The other vertical squares commute for similar reasons (the details are left to the reader). So, the top pentagon commutes, as was to be shown.

We can also reason on the planar figure

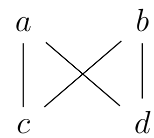


(All the morphisms under consideration being isomorphisms, it is not necessary to orient the edges.) The argument is this: Assuming that all the quadrilaterals commute, if one of the pentagons commutes, so does the other.

§ 174. Definition 4.2.5 p. 98. Here is an example of a category $\mathcal{C}$ admitting no tensor product with unit. More precisely $\mathcal{C}$ is an ordered set X which admits no ordered monoid structure. This example is taken from Comments 2 and 13 in

Azimuth Forum, Applied Category Theory Course, Lecture 21 - Chapter 2: Monoidal Preorders, p. 1, https://tinyurl.com/y4onm8kk

Our ordered set X is



Suppose X has an ordered monoid structure with $d = 1$. Then

$$a \otimes b \geq a \otimes d = a \otimes 1 = a$$

and

$$a \otimes b \geq d \otimes b = 1 \otimes b = b.$$

This is a contradiction since a and b do not have a common upper bound. Similar arguments show the unit can't be a, b or c either, so there is no ordered monoid structure on X.

7 About Chapter 5

7.1 Beginning of Section 5.1 p. 113

We want to define the notions of coimage (denoted by Coim) and image (denoted by Im) in a slightly more general way than in the book. To this end we start by defining these notions in a particular context in which they coincide. To avoid confusions we (temporarily) use the notation IM for these particular cases. The proof of the following lemma is obvious.

Lemma 175. *For any set theoretical map $g : U \to V$ we have natural bijections*

$$\operatorname{Coker}(U \times_V U \rightrightarrows U) \simeq \operatorname{IM} g \simeq \operatorname{Ker}(V \rightrightarrows V \sqcup_U V),$$

where $\operatorname{IM} g$ *denotes the image of* g.

Let $\mathcal{C}$ be a $\mathcal{U}$-small category (Definition 5 p. 10), and let us denote by $\mathrm{h} : \mathcal{C} \to \mathcal{C}^\wedge$ and $\mathrm{k} : \mathcal{C} \to \mathcal{C}^\vee$ the Yoneda embeddings. For any morphism $f : X \to Y$ in $\mathcal{C}$ define $\mathrm{IM}\,\mathrm{h}(f)$ in $\mathcal{C}^\wedge$ and $\mathrm{IM}\,\mathrm{k}(f)$ in $\mathcal{C}^\vee$ by

$$(\mathrm{IM}\,\mathrm{h}(f))(Z) := \mathrm{IM}\ \mathrm{h}(f)_Z, \quad (\mathrm{IM}\,\mathrm{k}(f))(Z) := \mathrm{IM}\ \mathrm{k}(f)_Z$$

for any Z in $\mathcal{C}$. Note the equalities

$$\mathrm{IM}\ \mathrm{h}(f)_Z = f \circ \mathrm{Hom}_{\mathcal{C}}(Z, X) = \{f \circ x \mid x \in \mathrm{Hom}_{\mathcal{C}}(Z, X)\},$$

$$\mathrm{IM}\ \mathrm{k}(f)_Z = \mathrm{Hom}_{\mathcal{C}}(Y, Z) \circ f = \{y \circ f \mid y \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)\}.$$

Lemma 175 implies

$$\begin{aligned} \mathrm{IM}\,\mathrm{h}(f) &\simeq \mathrm{Coker}(\mathrm{h}(X) \times_{\mathrm{h}(Y)} \mathrm{h}(X) \rightrightarrows \mathrm{h}(X)), \\ \mathrm{IM}\,\mathrm{k}(f) &\simeq \mathrm{Ker}(\mathrm{k}(Y) \rightrightarrows \mathrm{k}(Y) \sqcup_{\mathrm{k}(X)} \mathrm{k}(Y)). \end{aligned} \tag{74}$$

Definition 176 (coimage, image)**.** *In the above setting, the* coimage *of f is the object* $\mathrm{Coim}\, f$ *of $\mathcal{C}^\vee$ defined by*

$$(\mathrm{Coim}\, f)(Z) := \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{IM}\,\mathrm{h}(f), \mathrm{h}(Z))$$

for all Z in $\mathcal{C}$, and the image *of f is the object* $\mathrm{Im}\, f$ *of $\mathcal{C}^\wedge$ defined by*

$$(\mathrm{Im}\, f)(Z) := \mathrm{Hom}_{\mathcal{C}^\vee}(\mathrm{k}(Z), \mathrm{IM}\,\mathrm{k}(f))$$

for all Z in $\mathcal{C}$.

Proposition 177. *We may regard* $(\mathrm{Coim}\, f)(Z)$ *as a subset of* $\mathrm{Hom}_{\mathcal{C}}(X, Z)$*, and* $(\mathrm{Im}\, f)(Z)$ *as a subset of* $\mathrm{Hom}_{\mathcal{C}}(Z, Y)$*. (These subsets will be spelled out by Proposition 179 below.)*

Proof. We prove that $(\mathrm{Coim}\, f)(Z)$ is naturally embedded in $\mathrm{Hom}_{\mathcal{C}}(X, Z)$. The morphisms

$$\mathrm{h}(X) \to \mathrm{IM}\,\mathrm{h}(f) \to \mathrm{h}(Y)$$

are given by the maps

$$\mathrm{Hom}_{\mathcal{C}}(Z, X) = \mathrm{h}(X)(Z) \to \mathrm{IM}\,\mathrm{h}(f)_Z \to \mathrm{h}(Y)(Z) = \mathrm{Hom}_{\mathcal{C}}(Z, Y).$$

In view of the definition of $\mathrm{Coim}(f)$, it suffices to check that $\mathrm{h}(X) \to \mathrm{IM}\,\mathrm{h}(f)$ is an epimorphism in $\mathcal{C}^\wedge$, that is, it suffices, by Corollary 166 p. 107, to check that the map $\mathrm{h}(X)(Z) \to \mathrm{IM}\,\mathrm{h}(f)_Z$ is surjective for all Z in $\mathcal{C}$. But this is clear.

We prove that $(\mathrm{Im}\, f)(Z)$ is naturally embedded in $\mathrm{Hom}_{\mathcal{C}}(Z, Y)$. The morphisms

$$\mathrm{k}(X) \to \mathrm{IM}\,\mathrm{k}(f) \to \mathrm{k}(Y)$$

in $\mathcal{C}^\vee$ are given by the morphisms

$$\mathrm{k}(Y) \to \mathrm{IM}\,\mathrm{k}(f) \to \mathrm{k}(X)$$

in $\mathbf{Set}^{\mathcal{C}}$, which are, in turn, given by the maps

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) = \mathrm{k}(Y)(Z) \to \mathrm{IM}\,\mathrm{k}(f)_Z \to \mathrm{k}(X)(Z) = \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

In view of the definition of $\mathrm{Im}(f)$, it suffices to check that $\mathrm{k}(Y) \to \mathrm{IM}\,\mathrm{k}(f)$ is an epimorphism in $\mathbf{Set}^{\mathcal{C}}$, that is, it suffices, by Corollary 166 p. 107, to check that the map $\mathrm{k}(Y)(Z) \to \mathrm{IM}\,\mathrm{k}(f)_Z$ is surjective for all Z in $\mathcal{C}$. But this is clear. □

According to Proposition 177 we regard from now on $(\mathrm{Coim}\, f)(Z)$ as a subset of $\mathrm{Hom}_{\mathcal{C}}(X, Z)$ and $(\mathrm{Im}\, f)(Z)$ as a subset of $\mathrm{Hom}_{\mathcal{C}}(Z, Y)$.

Convention 178. If $A \rightrightarrows B \to C$ is a diagram in a given category, then the notation $[A \rightrightarrows B \to C]$ shall mean that the two compositions coincide.

Proposition 179. *If $f : X \to Y$ is a morphism in a category $\mathcal{C}$, and if Z is an object of $\mathcal{C}$, then we have*

$$(\mathrm{Coim}\, f)(Z) = \left\{ x : X \to Z \;\middle|\; \left[W \rightrightarrows X \xrightarrow{f} Y\right] \Rightarrow \left[W \rightrightarrows X \xrightarrow{x} Z\right] \ \forall\, W \in \mathcal{C} \right\},$$

$$(\mathrm{Im}\, f)(Z) = \left\{ y : Z \to Y \;\middle|\; \left[X \xrightarrow{f} Y \rightrightarrows W\right] \Rightarrow \left[Z \xrightarrow{y} Y \rightrightarrows W\right] \ \forall\, W \in \mathcal{C} \right\}.$$

In particular, these two sets do not depend on the universe $\mathcal{U}$ making $\mathcal{C}$ a $\mathcal{U}$-category (Definition 4 p. 10). There are natural morphisms

$$\mathrm{k}(X) \to \mathrm{Coim}\, f \to \mathrm{k}(Y), \quad \mathrm{h}(X) \to \mathrm{Im}\, f \to \mathrm{h}(Y)$$

in $\mathcal{C}^\vee$ and $\mathcal{C}^\wedge$ respectively. Moreover, $\mathrm{k}(X) \to \mathrm{Coim}\, f$ is an epimorphism, and $\mathrm{Im}\, f \to \mathrm{h}(Y)$ is a monomorphism.

For the sake of emphasis we write

$$\mathrm{k}(X) \twoheadrightarrow \operatorname{Coim} f \to \mathrm{k}(Y), \quad \mathrm{h}(X) \to \operatorname{Im} f \rightarrowtail \mathrm{h}(Y).$$

Proof. To prove the first equality, let $x : W \to X$ be a morphism in $\mathcal{C}$ and consider the condition

(a) there is a map $u : f \circ \operatorname{Hom}_{\mathcal{C}}(W, X) \to \operatorname{Hom}_{\mathcal{C}}(W, Z)$ such that $u(g) = x \circ g$ for all g in $\operatorname{Hom}_{\mathcal{C}}(W, X)$:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(W,X) & \xrightarrow{\quad x\circ \quad} & \operatorname{Hom}_{\mathcal{C}}(W,Z) \\ & \searrow \qquad \qquad \nearrow_{u} & \\ & f \circ \operatorname{Hom}_{\mathcal{C}}(W,X). & \end{array}$$

It suffices to show that (a) is equivalent to

(b) $\left[W \rightrightarrows X \xrightarrow{f} Y\right] \Rightarrow \left[W \rightrightarrows X \xrightarrow{x} Z\right]$.

To show (a)⇒(b), let g_1 and g_2 in $\operatorname{Hom}_{\mathcal{C}}(W, X)$ satisfy $f \circ g_1 = f \circ g_2$. This yields $x \circ g_1 = u(f \circ g_1) = u(f \circ g_2) = x \circ g_2$.

To show (b)⇒(a), given g in $\operatorname{Hom}_{\mathcal{C}}(W, X)$ we must prove that the morphism $x \circ g$ does depends only on $f \circ g$, and not on g itself. But this is precisely what (b) says. This proves the first equality in the statement of the proposition.

Let us show that the natural morphism $\mathrm{k}(X) \to \operatorname{Coim} f$ is an epimorphism. As $\mathrm{k}(X) \to \operatorname{Coim} f$ is a morphism in $\mathcal{C}^\vee$, it is given by a morphism $\operatorname{Coim} f \to \mathrm{k}(X)$ in $\mathbf{Set}^{\mathcal{C}}$, and we must check that $\operatorname{Coim} f \to \mathrm{k}(X)$ is a monomorphism in $\mathbf{Set}^{\mathcal{C}}$. But in Proposition 177, we noticed that $(\operatorname{Coim} f)(Z)$ could be viewed as a subset of $\operatorname{Hom}_{\mathcal{C}}(X, Z) = \mathrm{k}(X)(Z)$ for any Z in $\mathcal{C}$.

Let us show that the natural morphism $\operatorname{Im} f \to \mathrm{h}(Y)$ is an monomorphism. But in Proposition 177, we noticed that $(\operatorname{Im} f)(Z)$ could be viewed as a subset of $\operatorname{Hom}_{\mathcal{C}}(Z, Y) = \mathrm{h}(Y)(Z)$ for any Z in $\mathcal{C}$.

The rest of the proof is left to the reader. □

By (74) we have

$$(\operatorname{Coim} f)(Z) \simeq \operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{C}}(X, Z) \rightrightarrows \operatorname{Hom}_{\mathcal{C}^\wedge}\big(\mathrm{h}(X) \times_{\mathrm{h}(Y)} \mathrm{h}(X), \mathrm{h}(Z)\big)\Big),$$

$$(\operatorname{Im} f)(Z) \simeq \operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{C}}(Z,Y) \rightrightarrows \operatorname{Hom}_{\mathcal{C}^\vee}\big(\mathrm{k}(Z), \mathrm{k}(Y) \sqcup_{\mathrm{k}(X)} \mathrm{k}(Y)\big)\Big).$$

This implies

Proposition 180. *If* $P := X \times_Y X$ *exists in* $\mathcal{C}$*, then* $\operatorname{Coim} f$ *is naturally isomorphic to* $\operatorname{Coker}(P \rightrightarrows X) \in \mathcal{C}^\vee$*. If* $S := Y \sqcup_X Y$ *exists in* $\mathcal{C}$*, then* $\operatorname{Im} f$ *is naturally isomorphic to* $\operatorname{Ker}(Y \rightrightarrows S) \in \mathcal{C}^\wedge$*.*

In view of Lemma 175 and Proposition 180 we can replace the notation IM with Im (or Coim). The following proposition is obvious:

Proposition 181. *We have:*

$f \mapsto \operatorname{Im} \mathrm{h}(f)$ *and* Im *are functors from* $\operatorname{Mor}(\mathcal{C})$ *to* $\mathcal{C}^\wedge$*,*

$f \mapsto \operatorname{Im} \mathrm{k}(f)$ *and* Coim *are functors from* $\operatorname{Mor}(\mathcal{C})$ *to* $\mathcal{C}^\vee$*.*

Definition 182 (strict epimorphism)**.** *A morphism* $f : X \to Y$ *in a category* $\mathcal{C}$ *is a* strict epimorphism *if the morphism* $\operatorname{Coim} f \to \mathrm{k}(Y)$ *in* $\mathcal{C}^\vee$ *is an isomorphism.*

The proposition below is obvious:

Proposition 183. *A morphism* $f : X \to Y$ *in a category* $\mathcal{C}$ *is a strict epimorphism if and only if, for all* Z *in* $\mathcal{C}$*, the map*

$$\circ f : \operatorname{Hom}_{\mathcal{C}}(Y,Z) \to \operatorname{Hom}_{\mathcal{C}}(X,Z)$$

induces a bijection

$$\operatorname{Hom}_{\mathcal{C}}(Y,Z) \xrightarrow{\sim} (\operatorname{Coim} f)(Z).$$

By Proposition 179 p. 114, this condition does not depend on the universe $\mathcal{U}$ *making* $\mathcal{C}$ *a* $\mathcal{U}$*-category (Definition 4 p. 10). Moreover, a strict epimorphism is an epimorphism.*

7.2 Brief comments

§ 184. P. 115, Proposition 5.1.5 (i). For the sake of completeness we spell out some details, and, for the reader's convenience we reproduce Proposition 5.1.5 (i) p. 115 of the book.

Proposition 185 (Proposition 5.1.5 (i) p. 115)**.** *If $\mathcal{C}$ is a category admitting finite inductive and projective limits, then the following five conditions on a morphism $f : X \to Y$ are equivalent:*

(a) *f is an epimorphism and* $\operatorname{Coim} f \to \operatorname{Im} f$ *is an isomorphism,*

(b) $\operatorname{Coim} f \xrightarrow{\sim} Y$*,*

(c) *the sequence $X \times_Y X \rightrightarrows X \to Y$ is exact,*

(d) *there exists a pair of parallel arrows $g, h : Z \rightrightarrows X$ such that $f \circ g = f \circ h$ and* $\operatorname{Coker}(g, h) \to Y$ *is an isomorphism,*

(e) *for any Z in $\mathcal{C}$, the set* $\operatorname{Hom}_{\mathcal{C}}(Y, Z)$ *is isomorphic to the set of morphisms $u : X \to Z$ satisfying $u \circ v_1 = u \circ v_2$ for any pair of parallel morphisms $v_1, v_2 : W \rightrightarrows X$ such that $f \circ v_1 = f \circ v_2$.*

Here are the additional details:

(b)⇒(a): The composition $\operatorname{Coim} f \to \operatorname{Im} f \to Y$ being an isomorphism by assumption, $\operatorname{Im} f \to Y$ is an epimorphism. Then Proposition 5.1.2 (iv) of the book implies that f is an epimorphism and that $\operatorname{Im} f \to Y$ is an isomorphism, from which we conclude that $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

§ 186. Proposition 5.1.5 p. 115. Here is a corollary to Proposition 5.1.5 and to Proposition 165 p. 106:

Corollary 187. *Let F and G be functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$, let $\theta : F \to G$ be a morphism of functors, and consider the following conditions:*

(a) *$\mathcal{C}'$ admits finite inductive and projective limits,*

(b) *θ is an epimorphism,*

(c) *θ is a strict epimorphism,*

(d) *$\theta_X : F(X) \to G(X)$ is an epimorphism for all X in $\mathcal{C}$,*

(e) *$\theta_X : F(X) \to G(X)$ is a strict epimorphism for all X in $\mathcal{C}$,*

(f) *θ is a monomorphism,*

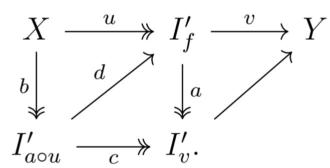
(g) *$\theta_X : F(X) \to G(X)$ is a monomorphism for all X in $\mathcal{C}$.*

Then (d) ⇒ (b), (g) ⇒ (f), (a) *and* (b) *imply* (d), (a) *and* (f) *imply* (g), (a) *implies that* (c) *and* (e) *are equivalent.*

§ 188. P. 116, proof of Proposition 5.1.7 (i) (minor variant). Recall the statement:

Proposition 189 (Proposition 5.1.7 (i) p. 116)**.** *Let $\mathcal{C}$ be a category admitting finite inductive and projective limits in which epimorphisms are strict. Let us denote by I'_g the coimage of any morphism g in $\mathcal{C}$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and $X \xrightarrow{u} I'_f \xrightarrow{v} Y$ its factorization through I'_f. Then v is a monomorphism.*

Proof. Consider the commutative diagram



(We first form a, then b and c, and finally d; the existence of d is a very particular case of Proposition 181 p. 116.) By (the of) Proposition 5.1.2 (iv) p. 114 of the book, it suffices to show that a is an isomorphism. As $a \circ u$ is a strict epimorphism, Proposition 185, (a)⇒(b), p. 117, implies that c is an isomorphism. We claim that $d \circ c^{-1}$ is inverse to a. We have

$$a \circ d \circ c^{-1} = c \circ c^{-1} = \mathrm{id}_{I'_v}$$

and

$$d \circ c^{-1} \circ a \circ u = d \circ c^{-1} \circ c \circ b = d \circ b = u = \mathrm{id}_{I'_f} \circ u,$$

and the conclusion follows from the fact that u is an epimorphism. □

§ 190. P. 116, Proposition 5.1.7 (ii). The proof shows that the natural morphism $I \to \mathrm{Coim}\, f$ is an isomorphism.

§ 191. P. 117, Definition 5.2.1 (definition of a system of generators). There is an important comment about this in Pierre Schapira's Errata

https://webusers.imj-prg.fr/~pierre.schapira/books/Errata.pdf.

As observed at the bottom of p. 121 of the book, the definition can be stated as follows:

Definition 192 (generator, system of generators)**.** *Let S be a set of objects of a category $\mathcal{C}$ and $\mathcal{S}$ the corresponding full subcategory. We say that S is a* system of generators *if the functor $\varphi : \mathcal{C} \to \mathcal{S}^\wedge$, $X \mapsto \mathrm{Hom}_{\mathcal{C}}(\ , X)$ is conservative. The notions of co-generator and system of co-generators are defined in the obvious way.*

§ 193. P. 118, second display: the isomorphism

$$\mathrm{Hom}_{\mathbf{Set}}\Big(\mathrm{Hom}_{\mathcal{C}}(G,X),\mathrm{Hom}_{\mathcal{C}}(G,X)\Big)\simeq\mathrm{Hom}_{\mathcal{C}^\vee}(G^{\sqcup\,\mathrm{Hom}_{\mathcal{C}}(G,X)},X)$$

is a particular case of the following isomorphism, which holds for any $\mathcal{U}$-set S and any objects G and X of $\mathcal{C}$:

$$\mathrm{Hom}_{\mathbf{Set}}(S,\mathrm{Hom}_{\mathcal{C}}(G,X))\simeq\mathrm{Hom}_{\mathcal{C}^\vee}(G^{\sqcup S},X).$$

§ 194. P. 118, proof of Proposition 5.2.3 (v): Writing Z' for $\varphi_G(Z)$, observe that, for $i=1,2$, the composition of the natural isomorphisms

$$(Y_1\times_X Y_2)'\xrightarrow{\sim}Y_1'\times_{X'}Y_2'\to Y_i'$$

is the natural isomorphism $(Y_1\times_X Y_2)'\xrightarrow{\sim}Y_i'$. Moreover, the phrase "$Y_1$ and Y_2 are isomorphic" should be understood as "there is an isomorphism $Y_1\xrightarrow{\sim}Y_2$ whose composition with the natural morphism $Y_2\to X$ is the natural morphism $Y_1\to X$".

§ 195. P. 119, Theorem 5.2.5: see Corollary 75 p. 60.

§ 196. P. 119. Proposition 5.2.8 will be used to prove Proposition 8.3.27 p. 186 of the book.

§ 197. P. 121, proof of Proposition 5.2.9. The last words are "by Proposition 5.2.3 (v)". A more precise wording would be "by the proof of Proposition 5.2.3 (v)".

§ 198. P. 121. Corollary 5.2.10 follows from Theorem 5.2.6 p. 119 and Proposition 5.2.9 p. 121 of the book. Corollary 5.2.10 will be used to prove Proposition 8.3.27 p. 186 of the book.

§ 199. P. 122, sentence following Definition 5.3.1. This sentence is "Note that if $\mathcal{F}$ is strictly generating, then $\mathrm{Ob}(\mathcal{F})$ is a system of generators". See §24 p. 26.

7.3 Lemma 5.3.2 p. 122

Here is a minor variant of the proof of Lemma 5.3.2.

Lemma 200 (Lemma 5.3.2 p. 122)**.** *If $\mathcal{F}\subset\mathcal{G}$ are full subcategories of a category $\mathcal{C}$, and if $\mathcal{F}$ is strictly generating, then $\mathcal{G}$ is strictly generating.*

Proof. Let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\gamma} & \mathcal{G}^\wedge \\ & \searrow^{\varphi} & \downarrow \rho \\ & & \mathcal{F}^\wedge \end{array}$$

be the natural functors (ρ being the restriction), and let X and Y be in $\mathcal{C}$. We have

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(X,Y) & \xrightarrow{\gamma'} & \operatorname{Hom}_{\mathcal{G}^\wedge}(\gamma(X),\gamma(Y)) \\ & \underset{\varphi'}{\overset{\sim}{\searrow}} & \downarrow \rho' \\ & & \operatorname{Hom}_{\mathcal{F}^\wedge}(\varphi(X),\varphi(Y)). \end{array}$$

We want to prove that γ' is bijective. As φ' is bijective, it suffices to show that γ' is surjective. Let ξ be in $\operatorname{Hom}_{\mathcal{G}^\wedge}(\gamma(X),\gamma(Y))$. There is a (unique) f in $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ such that

$$\rho(\xi) = \varphi(f), \tag{75}$$

and it suffices to prove $\xi = \gamma(f)$. Let Z be in $\mathcal{G}$ and z be in $\operatorname{Hom}_{\mathcal{C}}(Z,X)$. It suffices to show that the morphisms

$$Z \underset{f\circ z}{\overset{\xi_Z(z)}{\rightrightarrows}} Y$$

coincide. As $\mathcal{F}$ is strictly generating, it suffices to show that the morphisms

$$\varphi(Z) \underset{\varphi(f\circ z)}{\overset{\varphi(\xi_Z(z))}{\rightrightarrows}} \varphi(Y)$$

coincide. Let W be in $\mathcal{F}$. It suffices to show that the maps

$$\varphi(Z)(W) \underset{\varphi(f\circ z)_W}{\overset{\varphi(\xi_Z(z))_W}{\rightrightarrows}} \varphi(Y)(W)$$

coincide, that is, it suffices to show that the maps

$$\operatorname{Hom}_{\mathcal{C}}(W,Z) \underset{f\circ z\circ}{\overset{\xi_Z(z)\circ}{\rightrightarrows}} \operatorname{Hom}_{\mathcal{C}}(W,Y)$$

coincide. We have, for w in $\operatorname{Hom}_{\mathcal{C}}(W,Z)$,

$$\xi_Z(z)\circ w \overset{\text{(a)}}{=} \xi_W(z\circ w) \overset{\text{(b)}}{=} \rho(\xi)_W(z\circ w) \overset{\text{(c)}}{=} \varphi(f)_W(z\circ w) \overset{\text{(d)}}{=} f\circ z\circ w,$$

Equality (a) following from the functoriality of ξ (see diagram below), Equality (b) following from the definition of ρ, Equality (c) following from (75), and Equality (d) following from the definition of φ. □

For the reader's convenience, we add the commutative diagram

$$\begin{array}{ccccc} Z & \quad & z \in \operatorname{Hom}_{\mathcal{C}}(Z,X) & \xrightarrow{\xi_Z} & \operatorname{Hom}_{\mathcal{C}}(Z,Y) \\ {\scriptstyle w}\uparrow & & \downarrow{\scriptstyle \circ w} & & \downarrow{\scriptstyle \circ w} \\ W & & \operatorname{Hom}_{\mathcal{C}}(W,X) & \xrightarrow[\xi_W]{} & \operatorname{Hom}_{\mathcal{C}}(W,X). \end{array}$$

7.4 Brief comments

§ 201. P. 122. The proof of Lemma 5.3.3 proves a statement that is much stronger than Lemma 5.3.3. This stronger statement can be phrased as follows:

Lemma 202. *Let $\mathcal{C}$ be a category which admits small inductive limits, let $\mathcal{F}$ be a small (Definition 5 p. 10) full subcategory of $\mathcal{C}$, let F be in $\mathcal{F}^\wedge$, set*

$$\psi(F) := \operatorname*{colim}_{(Y\to F)\in\mathcal{F}_F} Y,$$

let X be in $\mathcal{C}$, let $f : \psi(F) \to X$ be a morphism in $\mathcal{C}$ and, for each $(Y \to F) \in \mathcal{F}_F$, let $f_{Y\to F} : Y \to X$ be the composition of f with the coprojection $Y \to \psi(F)$. Then there is a unique morphism $\theta : F \to \varphi(X)$ in $\mathcal{F}^\wedge$ such that

$$\theta_Y(Y \to F) = f_{Y\to F}$$

for all Y in $\mathcal{F}$. Moreover, the map

$$\operatorname{Hom}_{\mathcal{C}}(\psi(F), X) \to \operatorname{Hom}_{\mathcal{F}^\wedge}(F, \varphi(X)), \quad f \mapsto \theta$$

is bijective and functorial in F and X. In particular $\psi : \mathcal{F}^\wedge \to \mathcal{C}$ is left adjoint to $\varphi : \mathcal{C} \to \mathcal{F}^\wedge$.

Proof. We have, for X in $\mathcal{C}$ and A in $\mathcal{F}^\wedge$,

$$\operatorname{Hom}_{\mathcal{C}}\left(\operatorname*{colim}_{(Y\to A)\in\mathcal{F}_A} Y, X\right) \xrightarrow{\sim} \lim_{(Y\to A)\in\mathcal{F}_A} \operatorname{Hom}_{\mathcal{C}}(Y,X)$$

$$\xrightarrow{\sim} \lim_{(Y\to A)\in\mathcal{F}_A} \varphi(X)(Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{F}^\wedge}(A,\varphi(X)),$$

the last isomorphism following from (46) p. 81, and it is straightforward to check that the composition of these bijections coincides with the map $f \mapsto \theta$ in the statement of our lemma. □

§ 203. P. 123, proof of Theorem 5.3.4. The following fact is implicit in the proof: The map

$$f : \operatorname{Hom}_{\mathcal{F}^\wedge}(\varphi\psi(G),F) \to \operatorname{Hom}_{\mathcal{F}^\wedge}(G,F),$$

obtained by composing the chain of isomorphisms in the proof, is equal to $\circ\varepsilon_G$. This equality is easily checked using Lemma 202 and the commutativity of the obvious diagram

$$\begin{array}{ccc}
\varphi(X) & \longrightarrow & \operatorname{colim}\limits_{X\to G} \varphi(X) \\
\downarrow & & \downarrow{\scriptstyle\sim} \\
\varphi\left(\operatorname{colim}\limits_{X\to G} X\right) & \xleftarrow[\varepsilon_G]{} & G.
\end{array}$$

in $\mathcal{F}^\wedge$ (the isomorphism on the right following from (44)).

7.5 Theorem 5.3.6 p. 124

Theorem 204 (Theorem 5.3.6 p. 124)**.** *Let $\mathcal{C}$ be a category such that*

(a) *$\mathcal{C}$ admits small inductive limits and finite projective limits,*

(b) *small filtrant inductive limits are stable by base change (see Section 4.6 p. 63),*

(c) *epimorphisms are strict.*

Let $\mathcal{F}$ be an essentially small (Definition 6 p. 10) full subcategory of $\mathcal{C}$ such that

(d) *the functor $\varphi : \mathcal{C} \to \mathcal{F}^\wedge$ is defined by $\varphi(X)(Y) := \operatorname{Hom}_{\mathcal{C}}(Y,X)$ is faithful,*

(e) *$\mathcal{F}$ is closed by finite coproducts in $\mathcal{C}$.*

Then φ is full, or, in other words, $\mathcal{F}$ is strictly generating.

Proof. We may assume from the beginning that $\mathcal{F}$ is small (Definition 5 p. 10).

Step 1. We have slightly changed the statement of Theorem 5.3.6 p. 124 of the book, but we want to keep the division of the proof in six steps used in the book.

In the book φ is supposed to be conservative, and Step 1 consists in invoking the proof of Proposition 5.2.3 (i) p. 118 of the book to conclude that φ is faithful. In the present setting, Step 1 can be ignored thanks to Assumption (d).

Step 2. By Proposition 1.2.12 p. 16 of the book, a morphism f in $\mathcal{C}$ is an epimorphism as soon as $\varphi(f)$ is an epimorphism.

Step 3. Let X be in $\mathcal{C}$, and let $(Y_i \to X)_{i\in I}$ be a small filtrant inductive system in $\mathcal{C}_X$. We claim that the natural morphism

$$\operatorname*{colim}_i \operatorname{Coim}(Y_i \to X) \to \operatorname{Coim}\left(\operatorname*{colim}_i Y_i \to X\right) \tag{76}$$

is an isomorphism. As $\mathcal{F}^\wedge$ satisfies Assumptions (a), (b) and (c), the above statement also applies to $\mathcal{F}^\wedge$.

Let X and Y be in $\mathcal{C}$.

Step 4. If $z : Z \to X$ is in $\mathcal{F}_X$, then the natural map

$$\operatorname{Hom}_{\mathcal{C}}(Z, Y) \to \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z), \varphi(Y)\big), \tag{77}$$

which is bijective by the Yoneda Lemma, induces a bijection

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{Coim} z, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{F}^\wedge}(\operatorname{Coim} \varphi(z), \varphi(Y)) \tag{78}$$

in the following sense:

There are natural bijections

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{Coim} z, Y) \simeq \operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}}(Z, Y) \rightrightarrows \operatorname{Hom}_{\mathcal{C}}(Z \times_X Z, Y)\big),$$

$$\operatorname{Hom}_{\mathcal{F}^\wedge}(\operatorname{Coim} \varphi z, \varphi Y) \simeq \operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi Z, \varphi Y\big) \rightrightarrows \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi Z \times_{\varphi X} \varphi Z, \varphi Y\big)\Big).$$

(We have omitted some parenthesis to save space.) Let $Z \to Y$ be a morphism in $\mathcal{C}$. Then $Z \to Y$ is in

$$\operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}}(Z, Y) \rightrightarrows \operatorname{Hom}_{\mathcal{C}}(Z \times_X Z, Y)\big)$$

if and only if its image $\varphi(Z) \to \varphi(Y)$ is in

$$\operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z), \varphi(Y)\big) \rightrightarrows \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z) \times_{\varphi(X)} \varphi(Z), \varphi(Y)\big)\Big).$$

[To make our argument work, it is not enough that the natural bijection (78) exist; the fact that it is induced by (77) will be crucial.]

Let us denote by I the set of finite subsets of $\mathrm{Ob}(\mathcal{F}_X)$, ordered by inclusion. Regarding I as a category, it is small (Definition 5 p. 10). Assumption (e) implies that I is filtrant. For any A in I set $Z_A := \bigsqcup_{a\in A}\zeta(a)$, where $\zeta : \mathcal{C}_X \to \mathcal{C}$ is the forgetful functor.

Step 5. We claim that the natural morphism

$$\operatorname*{colim}_{A\in I} \varphi(Z_A) \to \varphi(X)$$

is an epimorphism.

Step 6. We claim that the natural morphism

$$\operatorname*{colim}_{A\in I} \mathrm{Coim}(Z_A \to X) \to X$$

is an isomorphism.

Lemma 205 below will show that these steps imply the theorem. We have, in the above setting,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X,Y) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}\left(\operatorname*{colim}_{A\in I} \mathrm{Coim}(Z_A \to X), Y\right) && \text{by Step 6}\\
&\xrightarrow{\sim} \lim_{A\in I} \mathrm{Hom}_{\mathcal{C}}(\mathrm{Coim}(Z_A \to X), Y)\\
&\xrightarrow{\sim} \lim_{A\in I} \mathrm{Hom}_{\mathcal{F}^\wedge}\Big(\mathrm{Coim}(\varphi(Z_A) \to \varphi(X)), \varphi(Y)\Big) && \text{by Step 4}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}^\wedge}\left(\operatorname*{colim}_{A\in I} \mathrm{Coim}(\varphi(Z_A) \to \varphi(X)), \varphi(Y)\right)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}^\wedge}\left(\mathrm{Coim}\left(\operatorname*{colim}_{A\in I} \varphi(Z_A) \to \varphi(X)\right), \varphi(Y)\right) && \text{by Step 3}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \varphi(Y)) && \text{by Step 5.}
\end{aligned}$$

Lemma 205. *Taking Steps 1 to 6 for granted, the composition of the six above bijections coincides with the natural map* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \varphi(Y))$.

Proof. Let us denote these six bijections by $f_1, \dots, f_6$; let $u : X \to Y$ be a morphism in $\mathcal{C}$; and let us compute successively $f_1u, f_2f_1u, \dots, f_6 \dots f_1u$. We have

$$f_1u = \left(\operatorname*{colim}_{A\in I} \operatorname{Coim}(Z_A \to X) \xrightarrow{\sim} X \xrightarrow{u} Y\right) \in \operatorname{Hom}_{\mathcal{C}}\left(\operatorname*{colim}_{A\in I} \operatorname{Coim}(Z_A \to X), Y\right).$$

Then f_2f_1u is the obvious family

$$(\operatorname{Coim}(Z_A \to X) \to X \xrightarrow{u} Y)_A \in \lim_{A\in I} \operatorname{Hom}_{\mathcal{C}}(\operatorname{Coim}(Z_A \to X), Y). \tag{79}$$

Let us compute $f_3f_2f_1u$ thanks to Step 4. Firstly, to the family (79) we attach the obvious family $(Z_A \to Y)_A$, each of whose member $Z_A \to Y$ is in

$$\operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}}(Z_A, Y) \rightrightarrows \operatorname{Hom}_{\mathcal{C}}(Z_A \times_X Z_A, Y)\big).$$

Secondly, applying the functor φ to the family $(Z_A \to Y)_A$ we get the obvious family $(\varphi(Z_A) \to \varphi(Y))_A$, each of whose member $\varphi(Z_A) \to \varphi(Y)$ is in

$$\operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z_A), \varphi(Y)\big) \rightrightarrows \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z_A) \times_{\varphi(X)} \varphi(Z_A), \varphi(Y)\big)\Big).$$

Thirdly, to the family $(\varphi(Z_A) \to \varphi(Y))_A$ we attach the family of morphisms

$$\operatorname{Coim}(\varphi(Z_A) \to \varphi(X)) \to \varphi(X) \xrightarrow{\varphi(u)} \varphi(Y), \tag{80}$$

family which makes up the sought-for morphism $f_3f_2f_1u$. The morphisms (80) give rise to a morphism

$$\operatorname*{colim}_{A\in I} \operatorname{Coim}(\varphi(Z_A) \to \varphi(X)) \to \varphi(X) \xrightarrow{\varphi(u)} \varphi(Y), \tag{81}$$

morphism equals to $f_4f_3f_2f_1u$. The morphism (81) induces a morphism

$$\operatorname{Coim}\left(\operatorname*{colim}_{A\in I} \varphi(Z_A) \to \varphi(X)\right) \to \varphi(X) \xrightarrow{\varphi(u)} \varphi(Y),$$

morphism equals to $f_5f_4f_3f_2f_1u$. This shows that $f_6f_5f_4f_3f_2f_1u$ is indeed equal to $\varphi(X) \xrightarrow{\varphi(u)} \varphi(Y)$. □

It remains to prove Steps 3, 4, 5 and 6.

Proof of Step 3. Set $Y := \operatorname{colim}_i Y_i$.

Lemma 206. *The natural morphism* $\operatorname{colim}_i Y_i \times_X Y_i \to Y \times_X Y$ *is an isomorphism.*

In the diagrams used to prove Step 3, the undefined arrows are the obvious ones.

Proof of Lemma 206. Consider the commutative diagrams

$$\begin{array}{ccc}
\operatorname*{colim}_i Y_i \times_X Y_i & \xrightarrow{a} & \operatorname*{colim}_{i,j} Y_i \times_X Y_j \\
\uparrow & & \uparrow \\
Y_i \times_X Y_i & \xrightarrow[\mathrm{id}]{} & Y_i \times_X Y_i,
\end{array}$$

$$\begin{array}{ccccc}
\operatorname*{colim}_{i,j} Y_i \times_X Y_j & \xrightarrow{b} & \operatorname*{colim}_i \operatorname*{colim}_j Y_i \times_X Y_j & \xrightarrow{c} & \operatorname*{colim}_i Y_i \times_X Y \\
\uparrow & & \uparrow & \nearrow & \uparrow \\
 & & \operatorname*{colim}_j Y_i \times_X Y_j & & \\
 & & \uparrow & & \\
Y_i \times_X Y_j & \xrightarrow[\mathrm{id}]{} & Y_i \times_X Y_j & \longrightarrow & Y_i \times_X Y,
\end{array}$$

$$\begin{array}{ccc}
\operatorname*{colim}_i Y_i \times_X Y & \xrightarrow{d} & Y \times_X Y \\
\uparrow & \nearrow & \\
Y_i \times_X Y. & &
\end{array}$$

The composition $d \circ c \circ b \circ a$ equals (76), and a is an isomorphism by Corollary 3.2.3 (ii) p. 79 of the book, b is an isomorphism by §62 p. 52, c and d are isomorphisms by Assumption (b).This proves Lemma 206. □

Taking the definition of Coim into account, we have

$$\operatorname{Coim}(Y_i \to X) = \operatorname{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i),$$

$$\operatorname{Coker}(Y \times_X Y \rightrightarrows Y) = \operatorname{Coim}(Y \to X).$$

Moreover, there is an obvious commutative diagram

$$\begin{array}{ccc} \operatorname*{colim}_i \operatorname{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i) & \xrightarrow{e} & \operatorname{Coker}(\operatorname*{colim}_i(Y_i \times_X Y_i) \rightrightarrows Y) \\ \uparrow & & \uparrow \\ \operatorname{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i), & \xrightarrow[\mathrm{id}]{} & \operatorname{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i), \end{array}$$

where e is an isomorphism, and Lemma 206 yields a commutative diagram

$$\begin{array}{ccc} \operatorname{Coker}(\operatorname*{colim}_i(Y_i \times_X Y_i) \rightrightarrows Y) & \xrightarrow{f} & \operatorname{Coker}(Y \times_X Y \rightrightarrows Y) \\ \uparrow & & \uparrow \\ \operatorname{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i) & \xrightarrow[\mathrm{id}]{} & \operatorname{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i), \end{array}$$

where f is an isomorphism. This implies that (76) is an isomorphism, completing the proof of Step 3. □

Proof of Step 4. We have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(\operatorname{Coim} z, Y) &= \operatorname{Hom}_{\mathcal{C}}\big(\operatorname{Coker}(Z \times_X Z \rightrightarrows Z), Y\big) \\ &\simeq \operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}}(Z, Y) \rightrightarrows \operatorname{Hom}_{\mathcal{C}}(Z \times_X Z, Y)\big), \end{aligned}$$

and similarly

$$\operatorname{Hom}_{\mathcal{F}^\wedge}(\operatorname{Coim} \varphi z, \varphi Y) \simeq \operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi Z, \varphi Y\big) \rightrightarrows \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi Z \times_{\varphi X} \varphi Z, \varphi Y\big)\Big).$$

The natural map

$$\operatorname{Hom}_{\mathcal{C}}(Z, Y) \to \operatorname{Hom}_{\mathcal{F}^\wedge}(\varphi(Z), \varphi(Y))$$

is bijective by the Yoneda Lemma. As φ is faithful by Assumption (d), the natural map

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(Z \times_X Z, Y) &\to \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z \times_X Z), \varphi(Y)\big) \\ &\simeq \operatorname{Hom}_{\mathcal{F}^\wedge}\big(\varphi(Z) \times_{\varphi(X)} \varphi(Z), \varphi(Y)\big). \end{aligned}$$

is injective. This implies our claims. □

Proof of Step 5. Let Z be in $\mathcal{F}$. We must show that the natural map

$$\operatorname*{colim}_{A\in I}\ \varphi(Z_A)(Z)\to\varphi(X)(Z):=\operatorname{Hom}_{\mathcal{C}}(Z,X)$$

is surjective. Let z be in $\operatorname{Hom}_{\mathcal{C}}(Z,X)$. It suffices to check that z is in the image of the natural map

$$\varphi(Z_{\{z\}})(Z)=\operatorname{Hom}_{\mathcal{C}}(Z,Z)\xrightarrow{z\circ}\operatorname{Hom}_{\mathcal{C}}(Z,X),$$

which is obvious. □

Proof of Step 6. As Step 3 implies

$$\operatorname*{colim}_{A\in I}\operatorname{Coim}(Z_A\to X)\simeq\operatorname{Coim}\left(\operatorname*{colim}_{A\in I}Z_A\to X\right),$$

it suffices to prove

$$\operatorname{Coim}\left(\operatorname*{colim}_{A\in I}Z_A\to X\right)\simeq X. \tag{82}$$

Epimorphisms being strict by Assumption (c), it is enough, in view of Proposition 185, (a)⇒(b), p. 117 to check that

$$\operatorname*{colim}_{A\in I}Z_A\to X \tag{83}$$

is an epimorphism. Let

$$\operatorname*{colim}_{A\in I}\varphi(Z_A)\xrightarrow{b}\varphi\left(\operatorname*{colim}_{A\in I}Z_A\right)\xrightarrow{a}\varphi(X)$$

be the natural morphisms. As $a\circ b$ is an epimorphism by Step 5, a is an epimorphism, and Step 2 implies that (83) is also an epimorphism. □

□

7.6 Brief comments

§ 207. P. 127, proof of Theorem 5.3.8.

The sentence "Since φ is conservative by (a), it remains to show that $\varphi(u)$ is a monomorphism" is justified by Proposition 5.1.5 (ii) p. 115 of the book and Corollary 187 p. 117 above.

The phrase "the two arrows $\varphi(X_{i_1} \times_X X_{i_1}) \rightrightarrows \operatorname{colim}_i \varphi(X_i)$ coincide" can be justified as follows: The two compositions

$$X_{i_1} \times_X X_{i_1} \underset{\xi_2}{\overset{\xi_1}{\rightrightarrows}} X_0 \xrightarrow{u} X$$

coincide by definition. Thus the two compositions

$$\varphi(X_{i_1} \times_X X_{i_1}) \rightrightarrows \varphi(X_0) \to \varphi(X),$$

which can be written as

$$\varphi(X_{i_1} \times_X X_{i_1}) \rightrightarrows \operatorname*{colim}_i \varphi(X_i) \to \varphi(X_0) \to \varphi(X),$$

coincide. The composition $\operatorname{colim}_i \varphi(X_i) \to \varphi(X_0) \to \varphi(X)$ being an isomorphism, the two morphisms $\varphi(X_{i_1} \times_X X_{i_1}) \rightrightarrows \operatorname{colim}_i \varphi(X_i)$ coincide.

§ 208. P. 128, Theorem 5.3.9. To prove the existence of $\mathcal{F}$, one can also argue as follows.

Lemma 209. *Let $\mathcal{C}$ be a category admitting finite inductive limits, and let $\mathcal{A}$ be a small (Definition 5 p. 10) full subcategory of $\mathcal{C}$. Then:*

(a) *There is a small full subcategory $\mathcal{B}$ of $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ and that $\mathcal{B}$ is closed by finite inductive limits in the following sense: if (X_i) is a finite inductive system in $\mathcal{B}$ and X is an inductive limit of (X_i) in $\mathcal{C}$, then X is isomorphic to some object of $\mathcal{B}$.*

(b) *There is a small full subcategory $\mathcal{A}'$ of $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{C}$ and that each finite inductive system in $\mathcal{A}$ has a limit in $\mathcal{A}'$.*

Proof. Since there are only countably many finite categories up to isomorphism, (b) is clear. To prove (a), let $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{A}'' \subset \cdots$ be a tower of full subcategories obtained by iterating the argument used to prove (b), and let $\mathcal{B}$ be the union of the $\mathcal{A}^{(n)}$. □

8 About Chapter 6

8.1 Definition 6.1.1 p. 131

Here is an example of an object of Ind(**Set**) which is isomorphic (in Ind(**Set**)) to no object of **Set**.

For each n in $\mathbb{N}$ set

$$\mathbf{n} := \{0, \dots, n-1\} \subset \mathbb{N},$$

let $\mathbb{N} \to \mathbf{Set}, n \mapsto \mathbf{n}$ be the obvious functor, and set

$$\mathbb{N}' := \text{“}\underset{n}{\mathrm{colim}}\text{”}\, \mathbf{n}.$$

(Note that $A(X)$ can be identified to the set of bounded maps from X to $\mathbb{N}$.) We clearly have $\mathbb{N}' \in \mathrm{Ind}(\mathbf{Set})$. Let X be in **Set** and let $u : X \to \mathbb{N}'$ be a morphism. To prove that $\mathbb{N}'$ is isomorphic to no object of **Set**, it suffices to show that u is *not* an epimorphism. As u factors through some coprojection $p_n : \mathbf{n} \to \mathbb{N}'$, if u were an epimorphism, so would be p_n.

Claim: p_n is *not* an epimorphism.

Proof: Let $f : \mathbb{N}' \to \mathbb{N}$ be the natural morphism. There is a morphism $g : \mathbb{N}' \to \mathbb{N}$ such that

$$g(p_i(j)) = \begin{cases} j & \text{if } j \le n \\ n & \text{if } j \ge n \end{cases}$$

whenever $0 \le j < i$, and we have $g \circ p_n = f \circ p_n$ but $g \ne f$. This proves the claim, and, thus, the fact that $\mathbb{N}'$ is isomorphic to no object of **Set**.

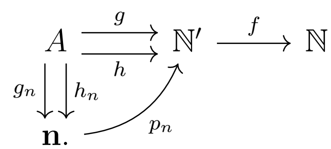
Claim: The natural morphism $f : \mathbb{N}' \to \mathbb{N}$ is a monomorphism.

Proof: Let

$$A \underset{h}{\overset{g}{\rightrightarrows}} \mathbb{N}' \xrightarrow{f} \mathbb{N}$$

be a diagram in Ind(**Set**) with $g \ne h$. It suffices to prove $f \circ g \ne f \circ h$. We can assume that A is in **Set**. There is an a in A satisfying $g(a) \ne h(a)$. Recall that $p_n : \mathbf{n} \to \mathbb{N}'$ is the n-th coprojection. There is an n in $\mathbb{N}$ and there are $g_n, h_n : A \rightrightarrows \mathbf{n}$

such that $g = p_n \circ g_n,\ h = p_n \circ h_n$:



The map $f \circ p_n$ being injective, this yields

$$(f \circ g)(a) = (f \circ p_n)(g_n(a)) \neq (f \circ p_n)(h_n(a)) = (f \circ h)(a),$$

and thus $f \circ g \neq f \circ h$. Hence, f is a monomorphism.

Claim: f is *not* an epimorphism.

Proof 1: Use Proposition 221 p. 139 below.

Proof 2: (Proof 2 is more direct.) Define $u : \mathbb{N} \to \mathbb{N}$ by

$$u(i) := \begin{cases} i-1 & \text{if } i \neq 0 \\ 0 & \text{if } i = 0, \end{cases}$$

and define the functor $\alpha : \mathbb{N} \to \mathbf{Set}$ as follows. To the object i of $\mathbb{N}$ we attach the object $\mathbb{N}$ of **Set**, and to the inequality $i \leq j$ in $\mathbb{N}$ we attach the endomap u^{j-i} of $\mathbb{N}$. Set $A :=$ "colim" α, let $q_i : \mathbb{N} \to A$ be the i-th coprojection and define $g : \mathbb{N} \to \mathbb{N}$ by $g(i) = 0$ for all $i \in \mathbb{N}$. It is easy to check that we have $q_0 \neq q_0 \circ g$ and $q_0 \circ f = q_0 \circ g \circ f$. This proves the claim.

8.2 Theorem 6.1.8 p. 132

Recall the statement:

Theorem 210 (Theorem 6.1.8 p. 132)**.** *If $\mathcal{C}$ is a category, then the category* $\mathrm{Ind}(\mathcal{C})$ *admits small filtrant inductive limits and the natural functor* $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ *commutes with such limits.*

Here is a minor variant of Step (i) of the proof of Theorem 6.1.8. We must show:

Lemma 211. *If $\alpha : I \to \mathrm{Ind}(\mathcal{C})$ is a functor, if I is small (Definition 5 p. 10) and filtrant, and if we define $A \in \mathcal{C}^\wedge$ by $A =$* "colim" *α, then $\mathcal{C}_A$ is filtrant.*

Proof. Let M be the category attached by Definition 3.4.1 p. 87 of the book to the functors

$$\mathcal{C} \xrightarrow{h} \mathcal{C}^\wedge \xleftarrow{\iota\circ\alpha} I,$$

where $h : \mathcal{C} \to \mathcal{C}^\wedge$ and $\iota : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ are the natural embeddings. Proposition 155 p. 104 implies that M is filtrant, and that it suffices to check that Conditions (iii) (a) and (iii) (b) of Proposition 3.2.2 p. 78 of the book hold for the obvious functor $\varphi : M \to \mathcal{C}_A$. Let us do it for Condition (iii) (b), the case of Condition (iii) (a) being similar and simpler.

For all i in I and all X in $\mathcal{C}$ let

$$p_i : \alpha(i) \to A \quad \text{and} \quad p_i(X) : \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i)) \to A(X)$$

be the coprojections. Note that $p_i(X) = p_i \circ$.

Given an object c of $\mathcal{C}_A$, and object m of M, and a pair of parallel morphisms $\sigma, \sigma' : c \rightrightarrows \varphi(m)$ in $\mathcal{C}_A$, we must find a morphism $\tau : m \to n$ in M satisfying $\varphi(\tau) \circ \sigma = \varphi(\tau) \circ \sigma'$.

Let c be given by the morphism $X \to A$ in $\mathcal{C}^\wedge$, let m be given by the morphism $Y \to \alpha(i)$ in $\mathrm{Ind}(\mathcal{C})$, and let σ and σ' be given by the morphisms $s, s' : X \rightrightarrows Y$ making the diagram below commute:

$$\begin{array}{ccc} X & \overset{s}{\underset{s'}{\rightrightarrows}} & Y \\ \downarrow & & \downarrow y \\ & & \alpha(i) \\ & & \downarrow p_i \\ A & = & A. \end{array}$$

Then we are looking for and object n of M given by a morphism $z : Z \to \alpha(j)$, and for a morphism $t : Y \to Z$ defining the sought-for morphism τ.

As $p_i(X)(y \circ s)$ equals $p_i(X)(y \circ s')$ in $A(X) \simeq \mathrm{colim}\, \mathrm{Hom}_{\mathcal{C}}(X, \alpha)$ and I is filtrant, there is a morphism $t : i \to j$ in I such that $\alpha(t) \circ y \circ s = \alpha(t) \circ y \circ s'$, and we can set

$Z := \alpha(j)$ and $z := \mathrm{id}_{\alpha(j)}$. The situation is depicted by the commutative diagram

$$\begin{array}{ccccc}
X & \overset{s}{\underset{s'}{\rightrightarrows}} & Y & \longrightarrow & \alpha(j) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle y} & & \Big\| \\
 & & \alpha(i) & \xrightarrow{\alpha(t)} & \alpha(j) \\
 & & \Big\downarrow{\scriptstyle p_i} & & \Big\downarrow{\scriptstyle p_j} \\
A & = & A & = & A.
\end{array}$$

□

8.3 Proposition 6.1.9 p. 133

8.3.1 Proof of Proposition 6.1.9

The following point is implicit in the book, and we give additional details for the reader's convenience. Proposition 6.1.9 results immediately from the statement below:

Proposition 212. *Let $\mathcal{A}$ be a category which admits small filtrant inductive limits, let $F : \mathcal{C} \to \mathcal{A}$ be a functor, and let $\mathcal{C} \xrightarrow{i} \mathrm{Ind}(\mathcal{C}) \xrightarrow{j} \mathcal{C}^\wedge$ be the natural embeddings. Then the functor $i^\dagger(F) : \mathrm{Ind}(\mathcal{C}) \to \mathcal{A}$ exists, commutes with small filtrant inductive limits, and satisfies $i^\dagger(F) \circ i \simeq F$. Conversely, any functor $\widetilde{F} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{A}$ commuting with small filtrant inductive limits with values in $\mathcal{C}$, and satisfying $\widetilde{F} \circ i \simeq F$, is isomorphic to $i^\dagger(F)$.*

Proof. The proof is essentially the same as that of Proposition 2.7.1 on p. 62 of the book. (See also §110 p. 83.) Again, we give some more details about the proof of the fact that $i^\dagger(F)$ commutes with small filtrant inductive limits. Put $\widetilde{F} := i^\dagger(F)$.

Let us attach the functor $B := \mathrm{Hom}_{\mathcal{A}}(F(\ ), Y) \in \mathcal{C}^\wedge$ to the object Y of $\mathcal{A}$. To apply Proposition 66 p. 55 to the diagram

$$\begin{array}{ccccc}
I & \xrightarrow{\alpha} & \mathrm{Ind}(\mathcal{C}) & \xrightarrow{\widetilde{F}} & \mathcal{A} \\
 & & {\scriptstyle j}\Big\downarrow & & \\
 & & \mathcal{C}^\wedge & &
\end{array}$$

(where I is a small filtrant category — Definition 5 p. 10), it suffices to check that there is an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}\left(\widetilde{F}(\ ), Y\right) \simeq \mathrm{Hom}_{\mathcal{C}^\wedge}(\ \ , B)$$

in $\mathrm{Ind}(\mathcal{C})^\wedge_{\mathcal{V}}$, where $\mathcal{V}$ is a universe containing $\mathcal{U}$ such that $\mathcal{C}^\wedge$ is a $\mathcal{V}$-category (Definition 4 p. 10). We have

$$\widetilde{F}(A) := \operatorname*{colim}_{(X \to A) \in \mathcal{C}_A} F(X),$$

as well as the following bijections functorial in $A \in \mathrm{Ind}(\mathcal{C})$:

$$\mathrm{Hom}_{\mathcal{A}}\left(\widetilde{F}(A), Y\right) = \mathrm{Hom}_{\mathcal{A}}\left(\operatorname*{colim}_{(X \to A) \in \mathcal{C}_A} F(X), Y\right) \simeq \lim_{(X \to A) \in \mathcal{C}_A} B(X)$$

$$\simeq \lim_{(X \to A) \in \mathcal{C}_A} \mathrm{Hom}_{\mathcal{C}^\wedge}((j \circ i)(X), B) \simeq \mathrm{Hom}_{\mathcal{C}^\wedge}\left(\operatorname*{``colim"}_{(X \to A) \in \mathcal{C}_A} X, B\right) \simeq \mathrm{Hom}_{\mathcal{C}^\wedge}(j(A), B).$$

□

8.3.2 Comments about Proposition 6.1.9

Let us record Part (i) of the proposition as

$$IF \circ \iota_{\mathcal{C}} \simeq \iota_{\mathcal{C}'} \circ F, \tag{84}$$

and note that we have, in the setting of Corollary 6.3.2 p. 140,

$$\operatorname{colim} F \circ \alpha \xrightarrow{\sim} (JF)(\text{``colim''}\ \alpha). \tag{85}$$

Let us also record Part (ii) of the proposition as

$$\text{``colim''}(IF \circ \alpha) \xrightarrow{\sim} IF(\text{``colim''}\ \alpha). \tag{86}$$

(See §8 p. 13.)

Also note that the proof of Proposition 6.1.9 shows

Proposition 213. *If $F : \mathcal{C} \to \mathcal{C}'$ is a functor of small categories (Definition 5 p. 10), then the functor $\widehat{F} : \mathcal{C}^\wedge \to \mathcal{C}'^\wedge$ defined in Notation 2.7.2 p. 63 of the book induces the functor $IF : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$.*

8.4 Proposition 6.1.12 p. 134

We give some more details about the proof. Recall the setting: We have two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, and we shall define functors

$$\operatorname{Ind}(\mathcal{C}_1 \times \mathcal{C}_2) \underset{\mu}{\overset{\theta}{\rightleftarrows}} \operatorname{Ind}(\mathcal{C}_1) \times \operatorname{Ind}(\mathcal{C}_2),$$

and prove that they are mutually quasi-inverse equivalences. (In fact, we shall only define the effect of θ and μ on objects, leaving also to the reader the definition of the effect of these functors on morphisms.) But first let us introduce some notation. We shall consider functors

$$A \in \operatorname{Ind}(\mathcal{C}_1 \times \mathcal{C}_2); \quad A_i, B_i \in \operatorname{Ind}(\mathcal{C}_i);$$

objects $X_i, Y_i, \dots$ in $\mathcal{C}_i$; and elements

$$x \in A(X_1, X_2),\ y \in A(Y_1, Y_2), \dots, \quad x_i \in A_i(X_i),\ y_i \in A_i(Y_i), \dots$$

When we write

$$\operatorname*{colim}_{x} \cdots, \quad \operatorname*{colim}_{x_i} \cdots, \quad \operatorname*{colim}_{x_1, x_2} \cdots,$$

we mean, in the first case, not only that x runs over the elements of $A(X_1, X_2)$, but also that X_1 and X_2 themselves run over the objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, so that we are taking the inductive limit of some functor defined over $(\mathcal{C}_1 \times \mathcal{C}_2)_A$. In the other cases, the interpretation is similar.

Let us define θ and μ: We define θ by setting $\theta(A) = (A_1, A_2)$ with

$$A_i := \text{“}\operatorname*{colim}_{x}\text{”}\ X_i, \tag{87}$$

and we define μ by putting $\mu(A_1, A_2) := A_1 \times A_2$ with

$$(A_1 \times A_2)(X_1, X_2) := A_1(X_1) \times A_2(X_2)$$

for all X_i in $\mathcal{C}_i$ $(i = 1, 2)$.

Proposition 214 (Proposition 6.1.12 p. 134)**.** *The functors θ and μ are mutually quasi-inverse.*

Proof. Let us prove

$$\theta \circ \mu \simeq \mathrm{id}_{\mathrm{Ind}(\mathcal{C}_1)\times\mathrm{Ind}(\mathcal{C}_2)}\,. \tag{88}$$

If A_i is in $\mathrm{Ind}(\mathcal{C}_i)$ for $i = 1, 2$; if A is $A_1 \times A_2$; and if (B_1, B_2) is $\theta(A)$, then we have

$$B_1 \overset{\text{(a)}}{\simeq} \text{“}\underset{x}{\mathrm{colim}}\text{”}\ X_1 \overset{\text{(b)}}{\simeq} \text{“}\underset{x_1,x_2}{\mathrm{colim}}\text{”}\ X_1 \overset{\text{(c)}}{\simeq} \text{“}\underset{x_1}{\mathrm{colim}}\text{”}\ X_1 \overset{\text{(d)}}{\simeq} A_1.$$

Indeed, Isomorphism (a) follows from (87), Isomorphism (b) from the definition of A, Isomorphism (c) from the fact that the projection

$$(\mathcal{C}_1)_{A_1} \times (\mathcal{C}_2)_{A_2} \to (\mathcal{C}_1)_{A_1}$$

is cofinal by Lemma 215 below coupled with the fact that $(\mathcal{C}_2)_{A_2}$ is connected, and Isomorphism (d) from our old friend (44) p. 81. (By the way, in this proof we are using (44) a lot without explicit reference.)

Lemma 215. *If I and J are categories and if J is connected, then the projection $I \times J \to I$ is cofinal.*

Proof. Let i_0 be in I. We must check that $(I \times J)^{i_0}$ is connected. We have $(I \times J)^{i_0} \simeq I^{i_0} \times J$, and it is easy to see that a product of two connected categories is connected. □

This ends the proof of (88).

Let us prove

$$\mu \circ \theta \simeq \mathrm{id}_{\mathrm{Ind}(\mathcal{C}_1\times\mathcal{C}_2)}\,. \tag{89}$$

Let A be in $\mathrm{Ind}(\mathcal{C}_1 \times \mathcal{C}_2)$ and set $(A_1, A_2) := \theta(A)$. We shall define morphisms $A \to A_1 \times A_2$ and $A_1 \times A_2 \to A$, and leave it to the reader to check that these morphisms are mutually inverse isomorphisms of functors.

• Definition of the morphism $A \to A_1 \times A_2$: Let X_i be in $\mathcal{C}_i$ $(i = 1, 2)$. We must define a map $A(X_1, X_2) \to A_1(X_1) \times A_2(X_2)$. We shall define firstly a map

$$A(X_1, X_2) \to A_1(X_1). \tag{90}$$

This will enable us to define a map $A(X_1, X_2) \to A_2(X_2)$ similarly, yielding our map $A(X_1, X_2) \to A_1(X_1) \times A_2(X_2)$. As we have an isomorphism

$$A_1(X_1) \simeq \underset{y}{\mathrm{colim}}\, \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1)$$

and coprojections

$$p_{1y}:\mathrm{Hom}_{\mathcal{C}_1}(X_1,Y_1)\to A_1(X_1),$$

we can define (90) by $x\mapsto p_{1x}(\mathrm{id}_{X_1})$. We leave it to the reader to check that this does define a morphism $A\to A_1\times A_2$.

• Definition of the morphism $A_1\times A_2\to A$. Letting X_i be in $\mathcal{C}_i$ as above, we must define a map $A_1(X_1)\times A_2(X_2)\to A(X_1,X_2)$. Letting x_i be in $A_i(X_i)$, we must define an element x in $A(X_1,X_2)$. We have $x_1=p_{1y}(f_1)$ and $x_2=p_{2z}(f_2)$ for some y and z in $(\mathcal{C}_1\times\mathcal{C}_2)_A$, some f_1 in $\mathrm{Hom}_{\mathcal{C}_1}(X_1,Y_1)$ and some f_2 in $\mathrm{Hom}_{\mathcal{C}_2}(X_2,Z_2)$. The category $(\mathcal{C}_1\times\mathcal{C}_2)_A$ being filtrant, we can assume $z=y$. We have an isomorphism

$$A(X_1,X_2)\simeq\operatorname*{colim}_w \mathrm{Hom}_{\mathcal{C}_1}(X_1,W_1)\times\mathrm{Hom}_{\mathcal{C}_2}(X_2,W_2)$$

and coprojections $q_w:\mathrm{Hom}_{\mathcal{C}_1}(X_1,W_1)\times\mathrm{Hom}_{\mathcal{C}_2}(X_2,W_2)\to A(X_1,X_2)$, we can define x by $x:=q_y(f_1,f_2)$. We leave it to the reader to check that this does define a morphism $A_1\times A_2\to A$, and that this morphism is an inverse to the morphism $A\to A_1\times A_2$ defined above.

This ends the proofs of Isomorphism (89) p. 136 and Proposition 214 p. 135. □

8.5 Corollary 6.1.15 p. 135

Recall the statement (see §9 p. 13 above):

Corollary 216 (Corollary 6.1.15 p. 135)**.** *Let $f,g:A\rightrightarrows B$ be two morphisms in* $\mathrm{Ind}(\mathcal{C})$*. Then there exist a small (Definition 5 p. 10) and filtrant category I and morphisms $\varphi,\psi:\alpha\rightrightarrows\beta$ of functors from I to $\mathcal{C}$ such that $A\simeq$ "colim" α, $B\simeq$ "colim" β, $f\simeq$ "colim" φ, $g\simeq$ "colim" ψ. (The last two isomorphisms take place in* $\mathrm{Mor}(\mathrm{Ind}(\mathcal{C}))$*.)*

Lemma 217. *Let $\alpha_1:I\to\mathcal{C}_1$ and $\alpha_2:I\to\mathcal{C}_2$ be functors defined on a small filtrant category I. Define $\alpha:I\to\mathcal{C}_1\times\mathcal{C}_2$ by $\alpha(i):=(\alpha_1(i),\alpha_2(i))$ and let X_k be in $\mathcal{C}_k$ $(k=1,2)$. Then the natural map*

$$(\text{“colim”}\,\alpha)(X_1,X_2)\to(\text{“colim”}\,\alpha_1)(X_1)\times(\text{“colim”}\,\alpha_2)(X_2)\tag{91}$$

is bijective.

Proof of Lemma 217. This follows from Corollary 3.2.3 (ii) p. 79 and Proposition 3.1.11 (ii) of the book. Let us just add that (91) is the natural composition

$$\operatorname*{colim}_{i\in I}\Big(\operatorname{Hom}_{\mathcal{C}_1}(X_1,\alpha_1(i))\times\operatorname{Hom}_{\mathcal{C}_2}(X_2,\alpha_2(i))\Big)\to$$

$$\operatorname*{colim}_{(i,j)\in I\times I}\Big(\operatorname{Hom}_{\mathcal{C}_1}(X_1,\alpha_1(i))\times\operatorname{Hom}_{\mathcal{C}_2}(X_2,\alpha_2(j))\Big)\to$$

$$\Big(\operatorname*{colim}_{i\in I}\operatorname{Hom}_{\mathcal{C}_1}(X_1,\alpha_1(i))\Big)\times\Big(\operatorname*{colim}_{j\in I}\operatorname{Hom}_{\mathcal{C}_2}(X_2,\alpha_2(j))\Big).$$

□

Proof of Corollary 216. Let I and J be small filtrant categories and let $\alpha : I \to \mathcal{C}$ and $\beta : J \to \mathcal{C}$ be two functors such that $A \simeq$ “colim” α and $B \simeq$ “colim” β. Denote by $\widetilde{\alpha} : I \to \mathcal{C}\times\mathcal{C}$ the functor $i \mapsto (\alpha(i),\alpha(i))$, and similarly with β.

Recall that there are quasi-inverse equivalences

$$\operatorname{Ind}(\mathcal{C}\times\mathcal{C}) \underset{\mu}{\overset{\theta}{\rightleftarrows}} \operatorname{Ind}(\mathcal{C})\times\operatorname{Ind}(\mathcal{C}),$$

that we sometimes write $A_1 \times A_2$ for $\mu(A_1, A_2)$, and that we have

$$(A_1\times A_2)(X_1,X_2) := A_1(X_1)\times A_2(X_2)$$

for all X_i in $\mathcal{C}$ $(i = 1, 2)$. (See Section 8.4 p. 135.)

Then $A \times A \simeq$ “colim” $\widetilde{\alpha}$ and $B \times B \simeq$ “colim” $\widetilde{\beta}$. By Lemma 217 and the above reminder, the morphism $(f, g) : (A, A) \to (B, B)$ in $\operatorname{Ind}(\mathcal{C}) \times \operatorname{Ind}(\mathcal{C})$ defines a morphism $f \times g : A \times A \to B \times B$ in $\operatorname{Ind}(\mathcal{C} \times \mathcal{C})$. Applying Proposition 6.1.13 p. 134 in the book, we find a small (Definition 5 p. 10) and filtrant category K, functors $p_I : K \to I, p_J : K \to J$ and a morphism of functors (φ, ψ) from $\alpha \circ p_I$ to $\beta \circ p_J$ such that $f \times g \simeq$ “colim”(φ, ψ). It follows that $f \simeq$ “colim” φ and $g \simeq$ “colim” ψ. □

8.6 Brief comments

§ 218. The proofs of Propositions 6.1.16 and 6.1.18 p. 136 in the book use the following lemma

Lemma 219. *Let I be a small (Definition 5 p. 10) filtrant category, let $\mathcal{C}$ be category, and let $F : \mathcal{C}^I \to \mathrm{Ind}(\mathcal{C})$ be the functor $\alpha \mapsto$ "colim" α.*

(a) *If $\mathcal{C}$ admits finite projective limits, F is left exact.*

(b) *If $\mathcal{C}$ admits finite inductive limits, F is right exact.*

Proof. Let $h : \mathcal{C} \to \mathcal{C}^\wedge$ be the Yoneda embedding and $\iota : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ the natural embedding. If α is in $\mathcal{C}^I$, then "colim" α can be defined as $\operatorname{colim} h \circ \alpha$ or as $\operatorname{colim} \iota \circ \alpha$ (Theorem 6.1.8 p. 132 in the book). Let J be a finite category.

(a) Let $\alpha : I \times J^{\mathrm{op}} \to \mathcal{C}$ be a functor. We claim

$$\text{“}\operatorname*{colim}_i\text{”} \lim_j \alpha(i,j) \xrightarrow{\sim} \operatorname*{colim}_i h \left(\lim_j \alpha(i,j) \right).$$

Clearly $\lim\limits_j$ commutes with h. As $\lim\limits_j$ commutes also with colim_i as far as $\mathcal{C}^\wedge$-valued functors are concerned, "$\operatorname*{colim}\limits_i$" commutes with $\lim\limits_j$, and the claim is proved.

(b) Let $\alpha : I \times J \to \mathcal{C}$ be a functor. We claim

$$\text{“}\operatorname*{colim}_i\text{”} \operatorname*{colim}_j \alpha(i,j) \xrightarrow{\sim} \operatorname*{colim}_i \iota \left(\operatorname*{colim}_j \alpha(i,j) \right).$$

The functor ι, being right exact (Corollary 6.1.6 p. 132 in the book), commutes with $\operatorname*{colim}\limits_j$. As $\operatorname*{colim}\limits_j$ commutes with $\operatorname*{colim}\limits_i$ for obvious reasons, it commutes with "$\operatorname*{colim}\limits_i$", and the claim is proved. □

§ 220.

Proposition 221. *If $\mathcal{C}$ is a category admitting finite inductive and projective limits, and if $\mathcal{C}$ is strict, then $\mathrm{Ind}(\mathcal{C})$ is strict.*

Lemma 222. *If $F : \mathcal{C} \to \mathcal{C}'$ is an exact functor between categories admitting finite inductive and projective limits, and if f is a strict morphism in $\mathcal{C}$, then $F(f)$ is a strict morphism in $\mathcal{C}'$.*

Proof. This is obvious. □

Proof of Proposition 221. Let $f : A \to B$ be a morphism in $\mathrm{Ind}(\mathcal{C})$. By Corollary 6.1.14 p. 135 in the book, there is a small filtrant category I and a morphism φ in $\mathcal{C}^I$ such that "colim" $\varphi \simeq f$. Clearly $\mathcal{C}^I$ is strict, and the theorem follows now from Lemmas 219 and 222. □

§ 223. P. 136, Corollary 6.1.17 (i). If $\mathcal{C}$ admits finite projective limits, then the natural functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is exact. (Recall that it is right exact by Corollary 6.1.6 p. 132 of the book.)

§ 224. P. 137, proof of Proposition 6.1.19. We just add a few references in the proof. Recall the statement:

Proposition 225. *If a category $\mathcal{C}$ admits finite inductive limits and finite projective limits, then small filtrant inductive limits are exact in* $\mathrm{Ind}(\mathcal{C})$.

Proof. By Proposition 161 p. 105 it suffices to check that small filtrant inductive limits commute with finite projective limits in $\mathrm{Ind}(\mathcal{C})$. Since the embedding $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ commutes with small filtrant inductive limits by Theorem 6.1.8 p. 132, and with finite projective limits by Corollary 6.1.17 (i) p. 136, this follows from the fact that small filtrant inductive limits are exact in $\mathcal{C}^\wedge$ (see Exercise 3.2 p. 90). □

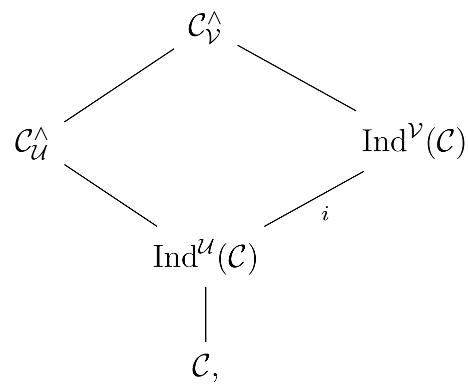
§ 226. P. 137, table. In view of Corollary 6.1.17 p. 136, one can add two lines to the table:

		$\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$	$\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$
1	finite inductive limits	$\circ$	$\times$
2	finite coproducts	$\circ$	$\times$
3	small filtrant inductive limits	$\times$	$\circ$
4	small coproducts	$\times$	$\times$
5	small inductive limits	$\times$	$\times$
6	finite projective limits	$\circ$	$\circ$
7	small projective limits	$\circ$	$\circ$

(In Line 6 we assume that $\mathcal{C}$ admits finite projective limits, whereas in Line 7 we assume that $\mathcal{C}$ admits small projective limits.)

§ 227. P. 138, Corollary 6.1.17. If $\mathcal{C}$ admits finite projective limits, then $\mathcal{C}$ is exact in $\mathrm{Ind}(\mathcal{C})$. This follows from Corollary 6.1.17, Corollary 6.1.6 p. 132 and Proposition 161 p. 105.

§ 228. P. 138, proof of Proposition 6.1.21. One can also argue as follows. Assume $\mathcal{C}$ admits finite projective limits. By Remark 2.6.5 p. 62 and Corollary 6.1.17 p. 136, all inclusions represented in the diagram



except perhaps inclusion i, commute with finite projective limits. Thus inclusion i commutes with finite projective limits. The argument for $\mathcal{U}$-small projective limits is the same. q.e.d.

8.7 Proposition 6.2.1 p. 138

We add a few details to the proof. Recall the statement:

Proposition 229 (Proposition 6.2.1 p. 138)**.** *Let $\alpha : I \to \mathcal{C}$ be a functor with I filtrant and let $Z \in \mathcal{C}$. The conditions below are equivalent:*

(i) *Z is a universal inductive limit of α in the sense of Definition 70 p. 59.*

(ii) *there exist an object $i_0 \in I$ and a morphism $\tau : Z \to \alpha(i_0)$ satisfying the property:*

for any morphism $s : i_0 \to i$, there exist a morphism $p_i : \alpha(i) \to Z$ and a morphism $t : i \to j$ satisfying

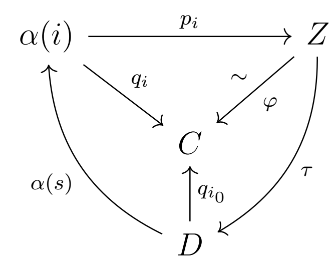
(a) $p_i \circ \alpha(s) \circ \tau = \mathrm{id}_Z$,

(b) $\alpha(t) \circ \alpha(s) \circ \tau \circ p_i = \alpha(t)$.

Proof. Choose a universe making I and $\mathcal{C}$ small (Definition 5 p. 10), set $A :=$ "colim" α and let $q_i : \alpha(i) \to A$ be the coprojections.

• (i) implies (ii). Let $\varphi : Z \to A$ be an isomorphism. By definition of A the isomorphism φ factors as $Z \xrightarrow{\tau} \alpha(i_0) \xrightarrow{q_{i_0}} A$. Let $s : i_0 \to i$. We define $p_i : \alpha(i) \to Z$

as being the composition $\alpha(i) \xrightarrow{q_i} A \underset{\sim}{\xleftarrow{\varphi}} Z$. As the three small triangles in the diagram



commute, we get

$$\varphi \circ p_i \circ \alpha(s) \circ \tau = q_i \circ \alpha(s) \circ \tau = \varphi,$$

which implies (a). The coprojection

$$\mathrm{Hom}_{\mathcal{C}}(\alpha(i), \alpha(i)) \to \operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \alpha) = \mathrm{Hom}_{\mathcal{C}}(\alpha(i), A)$$

being the map $q_i \circ$, and I being filtrant, the equalities

$$q_i \circ \alpha(s) \circ \tau \circ p_i = q_{i_0} \circ \tau \circ p_i = \varphi \circ p_i = q_i = q_i \circ \mathrm{id}_{\alpha(i)}$$

imply the existence of a morphism $t : i \to j$ satisfying (b).

• (ii) implies (i). Let $\varphi : Z \to A$ be the composition $Z \xrightarrow{\tau} \alpha(i_0) \xrightarrow{q_{i_0}} A$. It suffices to show that φ is an isomorphism. Let X be an object of $\mathcal{C}$. It suffices to show that the map

$$\varphi_X : \mathrm{Hom}_{\mathcal{C}}(X, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, A) \simeq \operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha), \quad u \mapsto \varphi \circ u$$

is bijective.

$\star$ φ_X is injective. Let $u, v \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ satisfy $\varphi \circ u = \varphi \circ v$, that is,

$$q_{i_0} \circ \tau \circ u = q_{i_0} \circ \tau \circ v.$$

As I is filtrant and as $\mathrm{Hom}_{\mathcal{C}}(X, A) \simeq \operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha)$, there is a morphism $s : i_0 \to i$ such that $\alpha(s) \circ \tau \circ u = \alpha(s) \circ \tau \circ v$. We have $q_i \circ \alpha(s) \circ \tau \circ u = q_{i_0} \circ \tau \circ u = \varphi \circ u$, and, similarly, $q_i \circ \alpha(s) \circ \tau \circ v = \varphi \circ v$, yielding

$$\varphi \circ u = q_i \circ \alpha(s) \circ \tau \circ u = q_i \circ \alpha(s) \circ \tau \circ v = \varphi \circ v,$$

and thus $u = v$.

$\star$ φ_X is surjective. Let $w : X \to \alpha(i)$ be a morphism. It suffices to show that there is a morphism $u : X \to Z$ such that $\varphi \circ u = w$. We may assume that there is a morphism $i_0 \to i$. If $p_i : \alpha(i) \to Z$ and $t : i \to j$ are as in (ii), we get

$$q_i \circ w = q_j \circ \alpha(t) \circ w = q_j \circ \alpha(t) \circ \alpha(s) \circ \tau \circ p_i \circ w = q_{i_0} \circ \tau \circ p_i \circ w = \varphi \circ p_i \circ w.$$

$\square$

Corollary 230. *Let $\alpha : I \to X$ be a functor from a filtrant category I to an ordered set X, let $f : \mathrm{Ob}(I) \to X$ be the obvious map, and let x_0 be in X. Then $x_0 = \operatorname{colim} \alpha$ if and only if $x_0 = \sup\ \mathrm{Im}\, f$. Moreover this inductive limit is universal in the sense of Definition 70 p. 59 if and only the supremum x_0 is reached by f.*

8.8 Brief comments

§ 231. P. 140, proof of Corollary 6.3.2. For X in $\mathcal{J}$ we have

$$F(X) \simeq \sigma_{\mathcal{C}}(\iota_{\mathcal{C}}(F(X))) \simeq \sigma_{\mathcal{C}}(IF(\iota_{\mathcal{J}}(X))).$$

§ 232. P. 140, Definition 6.3.3. Recall this definition:

Definition 233 (Definition 6.3.3. p. 140)**.** *Assume that $\mathcal{C}$ admits small filtrant inductive limits. We say that an object X of $\mathcal{C}$ is* of finite presentation *if for any $\alpha : I \to \mathcal{C}$ with I small and filtrant, the natural morphism*

$$\operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \operatorname{colim} \alpha)$$

is an isomorphism, that is, if

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, \sigma_{\mathcal{C}}(A))$$

is an isomorphism for any $A \in \mathrm{Ind}(\mathcal{C})$.

We spell out some details. Recall that the embedding functor $\iota : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ has a left adjoint functor σ:

$$\begin{array}{c} \mathcal{C} \\ {\scriptstyle\sigma}\uparrow\downarrow{\scriptstyle\iota} \\ \mathrm{Ind}(\mathcal{C}). \end{array}$$

In particular, for each A in $\mathrm{Ind}(\mathcal{C})$ we have a morphism $\varepsilon_A : A \to \iota(\sigma(A))$. Recall also that $\mathcal{C}$ is a category admitting small filtrant limits. Consider the following conditions on an object X in $\mathcal{C}$:

(a) The natural map $\operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \operatorname{colim} \alpha)$ is bijective for all functor $\alpha : I \to \mathcal{C}$ with I small (Definition 5 p. 10) and filtrant.

(b) The map $\varepsilon_A \circ : \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\iota(X), A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\iota(X), \iota(\sigma(A)))$ is bijective for all A in $\mathrm{Ind}(\mathcal{C})$.

Lemma 234. *The above conditions are equivalent.*

Proof. If $\alpha : I \to \mathcal{C}$ is a functor with I small and filtrant, then the obvious square

$$\begin{array}{ccc} \operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, \alpha) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X, \operatorname{colim} \alpha) \\ {\sim}\downarrow & & \downarrow{\sim} \\ \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, \text{“colim”}\ \alpha) & \xrightarrow[\varepsilon_{\text{“colim”}\ \alpha}]{} & \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, \operatorname{colim} \alpha) \end{array}$$

commutes. □

Definition 235. *We say that X is of* finite presentation *if the above conditions are satisfied.*

§ 236. P. 140, proof of Proposition 6.3.4. The authors construct a bijection

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{J})}(\text{“}\operatorname*{colim}_j\text{”}\ \beta(j), \text{“}\operatorname*{colim}_i\text{”}\ \alpha(i))$$
$$\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(JF(\text{“}\operatorname*{colim}_j\text{”}\ \beta(j)), JF(\text{“}\operatorname*{colim}_i\text{”}\ \alpha(i))).$$

We leave it to the reader to check that this bijection coincides with the natural map

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{J})}(\text{“}\operatorname*{colim}_j\text{”}\ \beta(j), \text{“}\operatorname*{colim}_i\text{”}\ \alpha(i))$$
$$\to \mathrm{Hom}_{\mathcal{C}}(JF(\text{“}\operatorname*{colim}_j\text{”}\ \beta(j)), JF(\text{“}\operatorname*{colim}_i\text{”}\ \alpha(i))).$$

Here is a consequence of Proposition 6.3.4 (see Corollary 6.3.5 p. 141 in the book):

Let $\mathcal{C}$ be a category admitting small (Definition 5 p. 10) filtrant inductive limits, and let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ whose objects are isomorphic to small filtrant inductive limits of objects of $\mathcal{C}^{fp}$. Then $\mathcal{C}'$ is equivalent to $\mathrm{Ind}(\mathcal{C}^{fp})$*. In particular $\mathcal{C}'$ admits small filtrant inductive limits. Moreover the inclusion $\mathcal{C}' \to \mathcal{C}$ commutes with such limits.*

Proof. Let $\iota : \mathcal{C}^{fp} \to \mathcal{C}$ be the inclusion functor. By Corollary 6.3.2 and Proposition 6.3.4 p. 140 of the book, the functor $J\iota : \mathrm{Ind}(\mathcal{C}^{fp}) \to \mathcal{C}$ is fully faithful and commutes with small filtrant inductive limits, and ι factors through $J\iota$. By Lemma 1.3.11 p. 21 of the book, $J\iota$ induces an equivalence $\mathrm{Ind}(\mathcal{C}^{fp}) \xrightarrow{\sim} \mathcal{C}'$. The claims above follow easily from these observations. □

§ 237. P. 143, proof of Proposition 6.4.1. The authors construct a bijection

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))}(\psi, \alpha(i)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))}\left(\psi, \operatorname*{colim}_i \alpha(i)\right).$$

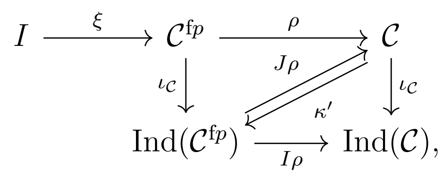
We leave it to the reader to check that this bijection coincides with the natural map

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))}(\psi, \alpha(i)) \to \mathrm{Hom}_{\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))}\left(\psi, \operatorname*{colim}_i \alpha(i)\right).$$

§ 238. P. 142, proof of Corollary 6.3.7. Let us check the isomorphism

$$\kappa(X) \simeq \text{“colim”}\ \rho \circ \xi. \tag{92}$$

Recall the setting:



κ' being quasi-inverse to $J\rho$ (for more details, see p. 141 of the book), κ is defined by $\kappa := I\rho \circ \kappa'$, and $X \simeq \operatorname{colim} \rho \circ \xi$. We have

$$\kappa(X) \simeq I\rho(\kappa'(\operatorname{colim} \rho \circ \xi)) \simeq I\rho(\text{“colim”}\ \xi)$$

$$\simeq \text{“colim”}(I\rho \circ \iota_{\mathcal{C}} \circ \xi) \simeq \text{“colim”}(\rho \circ \xi),$$

the second, third and fourth isomorphisms being respectively justified by (85) p. 134, (86) p. 134 and (84) p. 134. This proves (92).

Parts (ii) and (iii) of Corollary 6.3.7 are equivalent by Proposition 1.5.6 (ii) p. 29 of the book. To prove (ii) note that we have

$$\sigma(\kappa(\operatorname{colim} \rho \circ \xi)) \simeq \sigma(\text{“colim”}\ \rho \circ \xi) \simeq \operatorname{colim} \rho \circ \xi$$

by Corollary 6.3.7 (i) p. 141 and Proposition 6.3.1 (i) p. 139 of the book.

8.9 Theorem 6.4.3 p. 144

Notational convention for this section, *and for this section only!* Superscripts will *never* be used to designate a category of the form $\mathcal{C}^{X'}$ attached to a functor $\mathcal{C} \to \mathcal{C}'$ and to an object X' of $\mathcal{C}'$. Only two categories of the form $\mathcal{C}_{X'}$ (again attached to a functor $\mathcal{C} \to \mathcal{C}'$ and to an object X' of $\mathcal{C}'$) will be considered in this section. As a lot of subscripts will be used, we shall denote these categories by

$$\mathcal{C}/G(a) \text{ and } L/a \tag{93}$$

instead of $\mathcal{C}_{G(a)}$ and L_a, to avoid confusion. Superscripts will *always* be used to designate categories of functors, like the category $\mathcal{B}^{\mathcal{A}}$ of functors from $\mathcal{A}$ to $\mathcal{B}$.

Let $\mathcal{C}$ be a category and K a small category (Definition 5 p. 10). Recall that, by Corollary 6.3.2 p. 140 of the book (see (85) p. 134 above), there is a functor $\Phi : \mathrm{Ind}(\mathcal{C}^K) \to \mathrm{Ind}(\mathcal{C})^K$ such that, if $F : N \to \mathcal{C}^K$ is a functor defined on a small filtrant category and if k is in K, then we have

$$\Phi(\text{“colim” } F)(k) \simeq \underset{n \in N}{\text{“colim”}}(F(n)(k)) = \text{“colim”}(F(\ )(k)).$$

Theorem 239 (Theorem 6.4.3 p. 144)**.** *If $\mathcal{C}$ is a category and if K is a finite category such that* $\mathrm{Hom}_K(k,k) = \{\mathrm{id}_k\}$ *for all k in K, then the functor*

$$\Phi : \mathrm{Ind}(\mathcal{C}^K) \to \mathrm{Ind}(\mathcal{C})^K,$$

whose existence is recalled above, is an equivalence.

The key point is to check that

$$\Phi \text{ is essentially surjective.} \tag{94}$$

(The fact that Φ is fully faithful is proved as Proposition 6.4.1 p. 142 of the book.)

In the book (94) is proved by an inductive argument. The limited purpose of this section is to attach, in an “explicit” way (in the spirit of the proof of Proposition 6.1.13 p. 134 of the book), to an object G of $\mathrm{Ind}(\mathcal{C})^K$ a small (Definition 5 p. 10) filtrant category N and a functor $F : N \to \mathcal{C}^K$ such that

$$\Phi(\text{“colim” } F) \simeq G,$$

that is, we want isomorphisms

$$\text{“colim” } F(\ )(k) \simeq G(k)$$

functorial in $k \in K$.

As in the book we assume, as we may, that any two isomorphic objects of K are equal.

Let $\mathcal{C}, K$ and G be as above. We consider $\mathcal{C}$ as being given once and for all, so that, in the notation below, the dependence on $\mathcal{C}$ will be implicit. For each k in K, let I_k be a small (Definition 5 p. 10) filtrant category and let

$$\alpha_k : I_k \to \mathcal{C}$$

be a functor such that

$$G(k) = \text{“colim”}\ \alpha_k.$$

We define the category

$$N := N\{K, G, (\alpha_k)\}$$

as follows:

[Beginning of the definition of the category $N := N\{K, G, (\alpha_k)\}$.] An *object* of N is a pair $((i_k), P)$, where each i_k is in I_k and P is a functor from K to $\mathcal{C}$, subject to the conditions

- $\alpha_k(i_k) = P(k)$ for all k,
- the coprojections $u_k(i_k) : \alpha_k(i_k) = P(k) \to G(k)$ induce a morphism of functors

$$u' : P \to G. \tag{95}$$

(We regard $\mathcal{C}$ as a subcategory of $\text{Ind}(\mathcal{C})$.) The picture is very similar to the second diagram of p. 135 of the book: For each morphism $f : k \to \ell$ in K we have the commutative square

$$\begin{array}{ccccccc}
\alpha_k(i_k) & = & P(k) & \xrightarrow{P(f)} & P(\ell) & = & \alpha_\ell(i_\ell) \\
 & & {\scriptstyle u_k(i_k)}\downarrow & & \downarrow{\scriptstyle u_\ell(i_\ell)} & & \\
 & & G(k) & \xrightarrow[G(f)]{} & G(\ell) & &
\end{array}$$

in $\text{Ind}(\mathcal{C})$.

A *morphism* from $((i_k), P)$ to $((j_k), Q)$ is a pair $((f_k), \theta)$, where each f_k is a morphism $f_k : i_k \to j_k$ in I_k, and $\theta : P \to Q$ is a morphism of functors, subject to the condition $\theta_k = \alpha_k(f_k)$ for all k:

$$\begin{array}{ccc} \alpha_k(i_k) & \xrightarrow{\alpha_k(f_k)} & \alpha_k(j_k) \\ \| & & \| \\ P(k) & \xrightarrow[\theta_k]{} & Q(k). \end{array}$$

[End of the definition of the category $N := N\{K, G, (\alpha_k)\}$.]

Let $p_k : N \to I_k$ be the natural projection. Then the functor $F : N \to \mathcal{C}^K$ is given by

$$F(\ )(k) = \alpha_k \circ p_k \quad \forall\ k \in K :$$

$$N \xrightarrow{p_k} I_k \xrightarrow{\alpha_k} \mathcal{C}.$$

In other words, we set

$$F\big((i_k), P\big)(k_0) := \alpha_{k_0}(i_{k_0}).$$

Lemma 240. *The category N is small (Definition 5 p. 10) and filtrant, and the functor p_k is cofinal.*

Clearly, Lemma 240 implies Theorem 239.

Proof of Lemma 240. We start as in the proof of Theorem 6.4.3 p. 144 of the book:

We order $\mathrm{Ob}(K)$ be decreeing $k \le \ell$ if and only if $\mathrm{Hom}_K(k, \ell) \neq \varnothing$, and argue by induction on the cardinal n of $\mathrm{Ob}(K)$.

If $n = 0$ the result is clear.

Otherwise, let a be a maximal object of K; let L be the full subcategory of K such that

$$\mathrm{Ob}(L) = \mathrm{Ob}(K) \setminus \{a\};$$

let $G_L : L \to \mathrm{Ind}(\mathcal{C})$ be the restriction of G to L; let

$$\widetilde{\alpha_a} : I_a \to \mathcal{C}/G(a)$$

(see (93) p. 146 for the definition of $\mathcal{C}/G(a)$) be the functor defined by

$$\widetilde{\alpha_a}(i_a) := \Big(u'(a) : \alpha_a(i_a) \to G(a)\Big);$$

and put
$$N' := N\{L, G_L, (\alpha_\ell)\}.$$

We define the functor
$$\varphi : N' \to \big(\mathcal{C}/G(a)\big)^{L/a}$$
(see (93) p. 146 for the definition of L/a) as follows. Let $((i_\ell), Q)$ be in N'. In particular, Q is a functor from L to $\mathcal{C}$, and we have, for each ℓ in L, a morphism

$$Q(\ell) = \alpha_\ell(i_\ell) \xrightarrow{u'(\ell)} \text{“colim”}\ \alpha_\ell = G(\ell)$$

in $\mathcal{C}$ (see (95) p. 147). Letting $\ell \xrightarrow{f} a$ be a morphism in K viewed as an object of L/a, we put

$$\varphi\big((i_\ell), Q\big)\left(\ell \xrightarrow{f} a\right) := \left(Q(\ell) \xrightarrow{u'(\ell)} G(\ell) \xrightarrow{G(f)} G(a)\right) \in \mathcal{C}/G(a).$$

Letting
$$\Delta : \mathcal{C}/G(a) \to \big(\mathcal{C}/G(a)\big)^{L/a}$$
be the diagonal functor (see Notation 52 p. 46), we can form the category

$$M := M\left[N' \xrightarrow{\varphi} \big(\mathcal{C}/G(a)\big)^{L/a} \xleftarrow{\Delta\circ\widetilde{\alpha_a}} I_a\right].$$

Concretely, an object of M is a triple

$$\Big(\big((i_\ell), Q\big), i_a, \big(\xi_f : Q(\ell) \to \alpha_a(i_a)\big)_{f:\ell\to a}\Big), \tag{96}$$

where $((i_\ell), Q)$ is an object of N', where i_a is an object of I_a, where f runs over the morphisms from ℓ to a in K, and where ξ_f is a morphism from $Q(\ell)$ to $\alpha_a(i_a)$ which makes the square

$$\begin{array}{ccc} Q(\ell) & \xrightarrow{\xi_f} & \alpha_a(i_a) \\ {\scriptstyle u'(\ell)}\downarrow & & \downarrow{\scriptstyle u'(a)} \\ G(\ell) & \xrightarrow[G(f)]{} & G(a) \end{array}$$

in $\mathcal{C}$ commute, and a morphism from (96) to

$$\Big(\big((i'_\ell), Q'\big), i'_a, \big(\xi'_f : Q'(\ell) \to \alpha_a(i'_a)\big)_{f:\ell\to a}\Big)$$

is given by a family $(f_k : i_k \to i'_k)_{k\in K}$ of morphisms in I_k making the squares

$$\begin{array}{ccc} Q(\ell) & \xrightarrow{\alpha_\ell(f_\ell)} & Q'(\ell) \\ {\scriptstyle \xi_f}\downarrow & & \downarrow{\scriptstyle \xi'_f} \\ \alpha_a(i_a) & \xrightarrow[\alpha_a(f_a)]{} & \alpha_a(i'_a) \end{array}$$

in $\mathcal{C}$ commute. (Recall $Q(\ell) = \alpha_\ell(i_\ell)$, $Q'(\ell) = \alpha_\ell(i'_\ell)$.)

We shall define functors

$$N \underset{\mu}{\overset{\lambda}{\rightleftarrows}} M$$

and leave it to the reader to check that they are mutually inverse isomorphisms. (In fact, we shall only define the effect of λ and μ on objects, leaving also to the reader the definition of the effect of these functors on morphisms.)

We shall define maps

$$\mathrm{Ob}(N) \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \mathrm{Ob}(M).$$

To define λ let $((i_k), P)$ be in N, and let Q be the restriction of P to L. Then $\lambda((i_k), P)$ will be of the form

$$\Big(\big((i_\ell), Q\big), i_a, \big(\xi_f : Q(\ell) \to \alpha_a(i_a)\big)_{f:\ell\to a}\Big).$$

As $Q(\ell) = P(\ell)$ and $\alpha_a(i_a) = P(a)$, we can (and do) put $\xi_f := P(f)$.

To define μ let

$$\Xi := \Big(\big((i_\ell), Q\big)\ ,\ i_a\ ,\ \varphi((i_\ell), Q) \to \Delta\widetilde{\alpha_a}(i_a)\Big)$$

be in M. The object $\mu(\Xi)$ of N will be of the form $((i_k), P)$, so that we must define a functor $P : K \to \mathcal{C}$.

We define $P(k)$ by putting $P(\ell) := Q(\ell)$ for ℓ in L and $P(a) := \alpha_a(i_a)$.

If $f : \ell \to m$ is a morphism in L, then we set $P(f) := Q(f) : P(\ell) \to P(m)$. Let ℓ be in L. There is at most one morphism $f : \ell \to a$. If this morphism does exist, then we put $P(f) := \xi_f$.

We leave it to the reader to check that λ and μ are mutually inverse bijections.

We also leave it to the reader to check that the set of morphisms in M from $\lambda((i_k),P)$ to $\lambda((i'_k),P')$ is *equal* (in the strictest sense of the word) to the set of morphisms in N from $((i_k),P)$ to $((i'_k),P')$, so that we get an isomorphism

$$N \simeq M\left[N' \xrightarrow{\varphi} (\mathcal{C}/G(a))^{L/a} \xleftarrow{\Delta\circ\widetilde{\alpha_a}} I_a\right].$$

By induction hypothesis,

$$N' \text{ is small and filtrant} \tag{97}$$

and the projection $N' \to I_\ell$ is cofinal for all ℓ in L. It follows from Proposition 2.6.3 (ii) p. 61 of the book that $\widetilde{\alpha_a}$ is cofinal. By assumption $\mathcal{C}/G(a)$ is filtrant, and Lemma 241 below will imply that Δ is cofinal. Thus,

$$\Delta\circ\widetilde{\alpha_a} \text{ is cofinal.} \tag{98}$$

Taking Lemma 241 below for granted, Lemma 240 p. 148 now follows from (97), (98) and Proposition 3.4.5 p. 89 of the book. □

As already observed, Lemma 240 implies Theorem 239 p. 146. The only remaining task is to prove

Lemma 241. *If I is a finite category and $\mathcal{C}$ a filtrant category, then the diagonal functor $\Delta:\mathcal{C}\to\mathcal{C}^I$ is cofinal.*

Proof. It suffices to verify Conditions (a) and (b) of Proposition 3.2.2 (iii) p. 78 of the book. Condition (b) is clear. To check Condition (a), let α be in $\mathcal{C}^I$. We must show that there is pair (X,λ), where X is in $\mathcal{C}$ and λ is a morphism of functors from α to ΔX. Let S be a set of morphisms in I. It is easy to prove

$$(\exists\ Y\in\mathcal{C})\left(\exists\ \mu\in\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(\alpha(i),Y)\right)\Big(\forall\ (s:i\to j)\in S\Big)\Big(\mu_j\circ\alpha(s)=\mu_i\Big)$$

by induction on the cardinal of S, and to see that this implies the existence of (X,λ). □

8.10 Exercise 6.8 p. 146

Recall the statement:

Let R be a ring.

(i) Prove that $M \in \mathrm{Mod}(R)$ is of finite presentation in the sense of Definition 233 p. 143 if and only if it is of finite presentation in the classical sense (see Examples 1.2.4 (iv)), that is, if there exists an exact sequence $R^m \to R^n \to M \to 0$.

(ii) Prove that any R-module M is a small filtrant inductive limit of modules of finite presentation. (Hint: consider the full subcategory of $(\mathrm{Mod}(R))_M$ consisting of modules of finite presentation and prove it is essentially small and filtrant.)

(iii) Deduce that the functor $J\rho$ defined in Diagram (6.3.1) induces an equivalence $J\rho : \mathrm{Ind}(\mathrm{Mod}^{fp}(R)) \to \mathrm{Mod}(R)$.

Solution. We shall freely use Proposition 3.1.3 p. 73 of the book, which describes the inductive limit of a set-valued functor defined on a small (Definition 5 p. 10) filtrant category, as well as Corollary 3.1.5 (same page), which says that the forgetful functor $\mathrm{Mod}(R) \to \mathbf{Set}$ commutes with small filtrant inductive limits.

(i) (a) Let $R^m \to R^n \to M \to 0$ be exact, and let us show that $M \in \mathrm{Mod}(R)$ is of finite presentation in the sense of Definition 233 p. 143.

Let $(N_i)_{i\in I}$ be an inductive system in $\mathrm{Mod}(R)$ indexed by a small (Definition 5 p. 10) filtrant category I, let N be its inductive limit, and, for each i, let

$$p_i : N_i \to N \quad \text{and} \quad q_i : \mathrm{Hom}_R(M, N_i) \to \operatorname*{colim}_i \mathrm{Hom}_R(M, N_i)$$

be the coprojections, and consider the map

$$\operatorname{colim} \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, N) \tag{99}$$

induced by the

$$p_i\circ : \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, N).$$

(i) (a1) The map (99) is injective. (This part of the proof also works if M is just finitely *generated*, without being finitely presented.) Let i be an object of I and $f : M \to N_i$ an R-linear map such that $q_i(f)$ is in the kernel of (99). It suffices to show $q_i(f) = 0$. Let F be the subset of M formed by the images of the elements of the canonical basis of R^n. For each x in F, the element $f(x)$ is annihilated by p_i. As F is finite, there is a j in I and a morphism $s : i \to j$ such that $N_s(f(x)) = 0$ for all

x in F, and thus for all x in M. This implies $q_i(f) = 0$, as required. This ends the proof of the injectivity of (99).

(i) (a2) The map (99) is surjective. Let $f : M \to N$ be R-linear. It suffices to show that f factors through $p_i : N_i \to N$ for some i in I. Let $a_j \in M$ be the image of the j-th element of the canonical basis of R^n, and let (λ_{jk}) be the matrix of our map $R^m \to R^n$, so that we have

$$\sum_{j=1}^{n} \lambda_{jk}\, a_j = 0 \quad \text{for} \quad k = 1, \dots, m.$$

There is an i' in I and there are $b_1, \dots, b_n$ in $N_{i'}$ such that $p_{i'}(b_j) = f(a_j)$ for all j. This yields

$$p_{i'}\left(\sum_{j=1}^{n} \lambda_{jk}\, b_j\right) = 0 \quad \text{for} \quad k = 1, \dots, m.$$

As a result, there is a i in I and there are $c_1, \dots, c_n$ in N_i such that $p_i(c_j) = f(a_j)$ for all j and

$$\sum_{j=1}^{n} \lambda_{jk}\, c_j = 0 \quad \text{for} \quad k = 1, \dots, m.$$

Hence there is an R-linear map $g : M \to N_i$ such that $g(a_j) = c_j$ for all j, and thus $p_i \circ g = f$. This ends the proof of the surjectivity of (99), and also the proof of the fact that any R-module which is of finite presentation in the classical sense is of finite presentation in the sense of Definition 233 p. 143.

(i) (b) We assume now that $M \in \operatorname{Mod}(R)$ is of finite presentation in the sense of Definition 233 p. 143, and we prove that M is of finite presentation in the classical sense.

(i) (b1) The R-module M is finitely generated. Let I be the set of all finitely generated submodules of M. Then I is a small filtrant ordered set. For each N in I let $q_N : \operatorname{Hom}_R(M, N) \to \operatorname{colim}_N \operatorname{Hom}_R(M, N)$ be the coprojection. Then the identity of M is the image of $q_N(f)$ for some N in I and some R-linear $f : M \to N$. This implies $N = M$.

(i) (b2) The R-module M is finitely presented in the classical sense. The argument is similar to the one in (i) (b1) above. There is a small set K, a positive integer n and an exact sequence $R^{\oplus K} \xrightarrow{f} R^n \to M \to 0$. Let I be the set of the finite subsets of K. Then I is a small filtrant ordered set. For each F in I set $M_F := R^n / f(R^{\oplus F})$.

Then $(M_F)_{F\in I}$ is, in a natural way, an inductive system whose colimit is M, the coprojections being the obvious maps $p_F : M_F \to M$. Let $q_F : \mathrm{Hom}_R(M, M_F) \to \mathrm{Hom}_R(M, M)$ be the coprojections. The identity of M factors through $p_F : M_F \to M$ for some F in I. This implies $M \simeq M_F$, ending the proof that M is finitely presented in the classical sense, and thus the proof of (i).

(ii) (I don't understand the hint.) Clearly any R-module is a small filtrant inductive limit of finitely generated R-modules. Hence, in view of §236 p. 144, it suffices to show that any finitely generated R-module is a small filtrant inductive limit of finitely presented R-modules. But this follows from (i) (b2) above.

There is also a more direct way to prove that any R-module M is a small filtrant inductive limit of finitely presented R-modules: Let F be a finite subset of M, let K_F be the kernel the natural map $R^{\oplus F} \to M$, let G be a finite subset of K_F, and let $M_{F,G}$ be the cokernel of the obvious map $R^{\oplus G} \to R^{\oplus F}$. Then the $M_{F,G}$ form a small filtrant inductive system of R-modules whose colimit is M.

(iii) Claim (iii) follows from §236 p. 144.

8.11 Exercise 6.11 p. 147

We prove the following slightly more precise statement:

Proposition 242. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor, let A' be in $\mathrm{Ind}(\mathcal{C}')$, and let S be the set of objects A of $\mathrm{Ind}(\mathcal{C})$ such that $IF(A) \simeq A'$. Then the following conditions are equivalent:*

(a) *$S \neq \varnothing$,*

(b) *all morphism $X' \to A'$ in $\mathrm{Ind}(\mathcal{C}')$ with X' in $\mathcal{C}'$ factors through $F(X)$ for some X in $\mathcal{C}$,*

(c) *the natural functor $\mathcal{C}_{A'\circ F} \to \mathcal{C}'_{A'}$ is cofinal,*

(d) *$A' \circ F$ is in S.*

Proof.

(a)⇒(b). Let $f : X' \to IF(A)$ be a morphism in $\mathrm{Ind}(\mathcal{C}')$ with X' in $\mathcal{C}'$ and A in $\mathrm{Ind}(\mathcal{C})$, let $\beta_0 : I \to \mathcal{C}$ be a functor with I small (Definition 5 p. 10) and filtrant and "colim" $\beta_0 \simeq A$; in particular "colim"$(F \circ \beta_0) \simeq IF(A)$. By Proposition 6.1.13 p. 134

of the book there is a functor $\beta : J \to \mathcal{C}$ and a morphism of functors $\varphi : \Delta X' \to F \circ \beta$, where $\Delta X' : J \to \mathcal{C}'$ is the constant functor equal to X', such that

J is small and filtrant,

"colim"$(F \circ \beta) \simeq IF(A)$,

"colim" $\varphi \simeq f$.

Then f factors as $X' \xrightarrow{\varphi_j} F(\beta(j)) \xrightarrow{p_j} IF(A)$, where p_j is the coprojection.

(b)⇒(c). This follows from Proposition 138 p. 97.

(c)⇒(d). This follows from Remark 115 p. 89 and Proposition 213 p. 134.

(d)⇒(a). This is obvious. □

9 About Chapter 7

§ 243. P. 149, Definition 7.1.1. We define the localization of a category $\mathcal{C}$ with respect to a set $\mathcal{S}$ of morphisms in a way that is slightly different from the one used in the book. It is obvious that a localization as defined here is also a localization as defined in the book.

Definition 244. *A localization of a category $\mathcal{C}$ with respect to a set $\mathcal{S}$ of morphisms is a category $\mathcal{C}_\mathcal{S}$ equipped with a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ such that*

(a) *$\mathcal{C}_\mathcal{S}$ has the same objects as $\mathcal{C}$,*

(b) *we have $Q(X) = X$ for all object X in $\mathcal{C}$,*

(c) *if $F : \mathcal{C} \to \mathcal{A}$ is a functor turning the elements of $\mathcal{S}$ into isomorphisms, then there is a unique functor $F_\mathcal{S} : \mathcal{C}_\mathcal{S} \to \mathcal{A}$ such that $F_\mathcal{S} \circ Q = F$,*

(d) *if G_1 and G_2 are two functors from $\mathcal{C}_\mathcal{S}$ to $\mathcal{A}$, then the natural map*

$$\mathrm{Hom}_{\mathrm{Fct}(\mathcal{C}_\mathcal{S},\mathcal{A})}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{A})}(G_1 \circ Q, G_2 \circ Q)$$

is bijective.

Proposition 245. *Let $\mathcal{C}$ be a category and $\mathcal{S}$ a set of morphisms in $\mathcal{C}$.*

(a) *There is a category $\mathcal{C}_\mathcal{S}$ equipped with a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ satisfying conditions (a), (b) and (c) of Definition 244. Moreover, the pair $(\mathcal{C}_\mathcal{S}, Q)$ is unique up to unique isomorphism.*

(b) *The category $\mathcal{C}_\mathcal{S}$ satisfies also condition (d) of Definition 244, and is thus a localization of $\mathcal{C}$ with respect to $\mathcal{S}$.*

Proof. (a) We define the objects of $\mathcal{C}_\mathcal{S}$ as being the objects of $\mathcal{C}$, and we construct the morphisms of $\mathcal{C}_\mathcal{S}$ by inverting formally the elements of $\mathcal{S}$. The details are left to the reader.

(b) Let F, G be in $\mathrm{Fct}(\mathcal{C}_\mathcal{S}, \mathcal{A})$ and consider the map

$$\star Q : \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C}_\mathcal{S},\mathcal{A})}(F, G) \to \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F \circ Q, G \circ Q),$$

where $\star$ denotes the horizontal composition (see Definition 35 p. 32). We shall define a putative inverse

$$\Phi : \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F \circ Q, G \circ Q) \to \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C}_\mathcal{S},\mathcal{A})}(F, G)$$

to $\star Q$. For λ in $\mathrm{Fct}(\mathcal{C}, \mathcal{A})(F \circ Q, G \circ Q)$ and X in $\mathcal{C}$ we define set $\Phi(\lambda)_X := \lambda_X : F(X) \to G(X)$. We must show that the square

$$\begin{array}{ccc} F(X) & \xrightarrow{\lambda_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\lambda_Y]{} & G(Y) \end{array}$$

commutes for all $f \in \mathrm{Hom}_{\mathcal{C}_\mathcal{S}}(X, Y)$. It suffices to check that this square commutes when $f = Q(s)^{-1}$ for s in $\mathcal{S} \cap \mathrm{Hom}_\mathcal{C}(Y, X)$, which is straightforward. □

§ 246. P. 150, Lemma 7.1.3. The last sentence says that, in terms of §89 (c) p. 65, the functor $Q^\ddagger Q_* G$ exists and is isomorphic to G via the identity of $G \circ Q$, or, more explicitly, that for all $F : \mathcal{C}' \to \mathcal{A}$, the map

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{C}'}}(F, G) \to \mathrm{Hom}_{\mathcal{A}^\mathcal{C}}(F \circ Q, G \circ Q), \quad v \mapsto v \star Q$$

is bijective. (Recall that $v \star Q$ denotes the horizontal composition of v and Q; see Definition 35 p. 32.)

Lemma 7.1.3 is used on p. 160 of the book to prove Theorem 7.1.16.

§ 247. P. 151, last sentence of the proof of Lemma 7.1.3. Omitting most of the parenthesis, we have

$$\begin{aligned} Gf \circ \widetilde{\theta} X_1 &= (Gs_2)^{-1} \circ (GQt_2)^{-1} \circ GQt_1 \circ Gs_1 \circ \widetilde{\theta} X_1 \\ &= (Gs_2)^{-1} \circ (GQt_2)^{-1} \circ \theta Y_3 \circ FQt_1 \circ Fs_1 \\ &= (Gs_2)^{-1} \circ (GQt_2)^{-1} \circ \theta Y_3 \circ FQt_2 \circ Fs_2 \circ Ff \\ &= \widetilde{\theta} X_2 \circ Ff. \end{aligned}$$

§ 248. P. 155, Theorem 7.1.16. If we define $\mathcal{C}_{\mathcal{S}}$ as in the proof of Proposition 245 p. 155, it is easy to check that, for X and Y in $\mathcal{C}$, the natural map

$$\operatorname*{colim}_{(Y \to Y') \in \mathcal{S}^Y} \operatorname{Hom}_{\mathcal{C}}(X, Y') \to \operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y)$$

is bijective.

§ 249. About the proof of Remark 7.1.18 (ii) p. 156. The following is almost a copy and paste of the display in the proof of Remark 7.1.18 (ii):

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}^{\ell}_{\mathcal{S}}}(X, Y) &\simeq \operatorname*{colim}_{(X' \to X) \in \mathcal{S}_X} \operatorname{Hom}_{\mathcal{C}}(X', Y) \\ &\xrightarrow{\sim} \operatorname*{colim}_{(X' \to X) \in \mathcal{S}_X, (Y \to Y') \in \mathcal{S}^Y} \operatorname{Hom}_{\mathcal{C}}(X', Y') \\ &\xleftarrow{\sim} \operatorname*{colim}_{(Y \to Y') \in \mathcal{S}^Y} \operatorname{Hom}_{\mathcal{C}}(X, Y') \simeq \operatorname{Hom}_{\mathcal{C}^{r}_{\mathcal{S}}}(X, Y). \end{aligned}$$

Let us describe the implicit map

$$\operatorname*{colim}_{(X' \to X) \in \mathcal{S}_X, (Y \to Y') \in \mathcal{S}^Y} \operatorname{Hom}_{\mathcal{C}}(X', Y') \to \operatorname*{colim}_{(X' \to X) \in \mathcal{S}_X} \operatorname{Hom}_{\mathcal{C}}(X', Y). \tag{100}$$

Given a diagram $X \xleftarrow{s} X' \xrightarrow{f} Y' \xleftarrow{t} Y$ with s and t in $\mathcal{S}$, we must first concoct a diagram $X \xleftarrow{u} X'' \xrightarrow{g} Y$ with u in $\mathcal{S}$. As $\mathcal{S}$ is a left multiplicative system, the solid diagram

$$\begin{array}{ccc} X'' & \overset{g}{\dashrightarrow} & Y \\ {\scriptstyle v}\, \downarrow & & \downarrow\, {\scriptstyle t} \\ X' & \underset{f}{\longrightarrow} & Y' \end{array}$$

can be completed to a commutative square as indicated, with v in $\mathcal{S}$, and it suffices to set $u := s \circ v$. We leave the proof of the fact that the element in the right hand side of (100) so obtained does not depend on the choice of the object X'' and the morphisms g and v. From this point the proof of Remark 7.1.18 (ii) is straightforward.

Unsolved Problem 250. P. 157, proof of Proposition 7.1.20. In the sentence "Since $t \circ s \in \mathcal{S}$, we have thus proved that, for $f : X \to Y$ in $\mathcal{C}$, if $Q(f)$ is an isomorphism, then there exists $g : Y \to Z$ such that $g \circ f \in \mathcal{S}$", I don't understand why $g \circ f \in \mathcal{S}$. (Proposition 7.1.20 doesn't seem to be used elsewhere in the book.)

§ 251. P. 159, Definition 7.3.1 (i). Recall the definition:

Let $\mathcal{C}$ be a $\mathcal{U}$-small category (Definition 5 p. 10), let $\mathcal{S}$ be a right multiplicative system, and let $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ be the right the localization of $\mathcal{C}$ by $\mathcal{S}$. A functor $F : \mathcal{C} \to \mathcal{A}$ is said to be *right localizable* if $Q^\dagger F$ exists, in which case we say that $Q^\dagger F$ a *right localization* of F, and denote this functor by $R_{\mathcal{S}} F$.

In terms of §89 (a) p. 65, the condition is that there is a morphism of functors $\tau : F \to R_{\mathcal{S}} F \circ Q$ such that for all $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ and all $w : F \to G \circ Q$ there is a unique $v : R_{\mathcal{S}} F \to G$ such that $(v \star Q) \circ \tau = w$:

$$\begin{array}{ccc} F & \xrightarrow{\tau} & R_{\mathcal{S}}F \circ Q \\ & \searrow^{w} & \downarrow{\scriptstyle v\star Q} \\ & & G \circ Q \end{array} \qquad \begin{array}{c} R_{\mathcal{S}}F \\ \downarrow{\scriptstyle v} \\ G. \end{array}$$

(Recall that $\star$ denotes the horizontal composition of morphisms of functors; see Definition 35 p. 32.)

§ 252. P. 159, Definition 7.3.1 (ii). The following proposition is obvious:

Proposition 253. *In the setting of §251, assume that $F(s)$ is an isomorphism for all s in $\mathcal{S}$. Recall that $\alpha^X : \mathcal{S}^X \to \mathcal{C}$ is the forgetful functor $(X', s) \mapsto X'$ (see Definition 7.1.9 p. 153 in the book). Define $p : F \circ \alpha^X \to \Delta F(X)$ by $p_{X',s} := F(s)^{-1}$. Then $F(X)$ is a universal inductive limit of $F \circ \alpha^X$ in $\mathcal{A}$ in the sense of Definition 70 p. 59. In particular F is universally right localizable, $R_{\mathcal{S}} F \simeq F_{\mathcal{S}}$ (for the definition of $F_{\mathcal{S}}$, see condition (c) in Definition 244 p. 155), and for any functor $K : \mathcal{A} \to \mathcal{A}'$ the diagram below commutes*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{A} \\ {\scriptstyle Q}\downarrow & \nearrow_{F_{\mathcal{S}}} & \downarrow{\scriptstyle K} \\ \mathcal{C}_{\mathcal{S}} & \xrightarrow[(K\circ F)_{\mathcal{S}}]{} & \mathcal{A}'. \end{array}$$

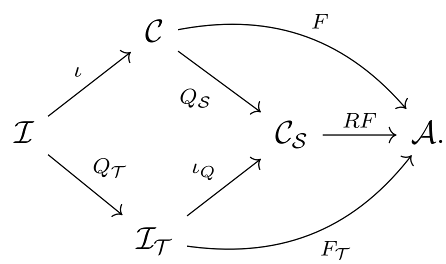
§ 254. P. 160, Proposition 7.3.2. (See also §10 p. 14.)

(a) The second sentence of the proof reads: "By hypothesis (i) and Corollary 7.2.2, $\iota_Q : \mathcal{I}_\mathcal{T} \to \mathcal{C}_\mathcal{S}$ is an equivalence". It is also worth noting that $\mathcal{T}$ is a right multiplicative system in $\mathcal{I}$.

(b) The third sentence of the proof reads: "By hypothesis (ii) the localization $F_\mathcal{T}$ of $F \circ \iota$ exists". See Proposition 253. By Proposition 245 p. 155 there is a unique functor $F_\mathcal{T} : I_\mathcal{T} \to \mathcal{A}$ such that

$$F_\mathcal{T} \circ Q_\mathcal{T} = F \circ \iota. \tag{101}$$

(c) Recall the diagram



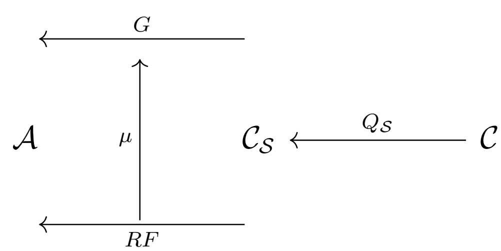
To each functor $G : \mathcal{C}_\mathcal{S} \to \mathcal{A}$ the book attaches a bijection

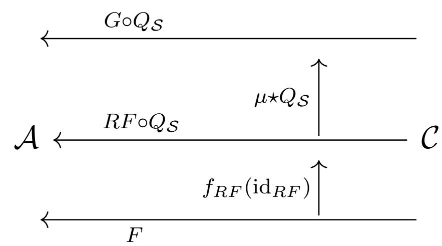
$$f_G : \mathrm{Hom}_{\mathcal{A}^{\mathcal{C}_\mathcal{S}}}(RF, G) \to \mathrm{Hom}_{\mathcal{A}^\mathcal{C}}(F, G \circ Q_\mathcal{S}).$$

We must verify that we have

$$f_G(\mu) = (\mu \star Q_\mathcal{S}) \circ f_{RF}(\mathrm{id}_{RF}) \tag{102}$$

for all $G : \mathcal{C}_\mathcal{S} \to \mathcal{A}$ and all $\mu : RF \to G$ (see Definition 35 p. 32 and § 251 p. 158). Here is a picture:





We can assume that we have the following equalities between functors:

$$RF = F_{\mathcal{T}} \circ \iota_Q^{-1}, \quad \iota_Q^{-1} \circ \iota_Q = \mathrm{id}_{\mathcal{I}_{\mathcal{T}}}, \quad F \circ \iota = F_{\mathcal{T}} \circ Q_{\mathcal{T}}, \quad Q_{\mathcal{S}} \circ \iota = \iota_Q \circ Q_{\mathcal{T}}.$$

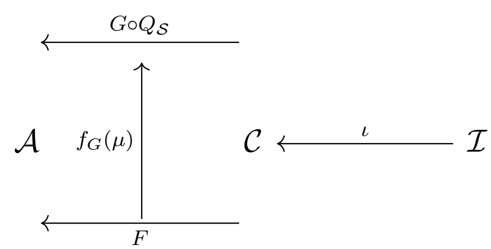
This gives in particular:

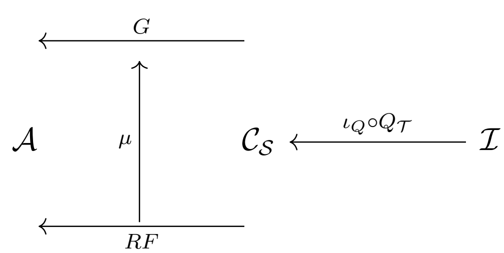
$$RF \circ Q_{\mathcal{S}} \circ \iota = F_{\mathcal{T}} \circ \iota_Q^{-1} \circ Q_{\mathcal{S}} \circ \iota = F_{\mathcal{T}} \circ \iota_Q^{-1} \circ \iota_Q \circ Q_{\mathcal{T}} = F_{\mathcal{T}} \circ Q_{\mathcal{T}} = F \circ \iota.$$

Note that f_G is characterized by the equality

$$(\star\, \iota) \circ f_G = \star\, (\iota_Q \circ Q_{\mathcal{T}}).$$

Here is a picture:





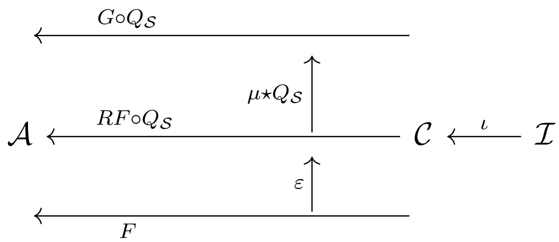
Setting $\varepsilon := f_{RF}(\mathrm{id}_{RF})$, we get

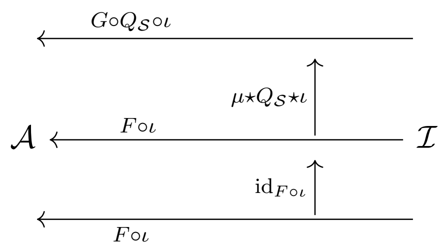
$$\varepsilon \star \iota = \mathrm{id}_{F \circ \iota}\,.$$

For μ in $\operatorname{Hom}_{\mathcal{A}^{\mathcal{C}_\mathcal{S}}}(RF, G)$, Equality (102) is then equivalent to the equality

$$((\mu \star Q_\mathcal{S}) \circ \varepsilon) \star \iota = \mu \star \iota_Q \star Q_\mathcal{T},$$

which is straightforward:





(d) The last claim in Proposition 7.3.2 is the existence of an isomorphism

$$RF \circ Q_\mathcal{S} \circ \iota \simeq F \circ \iota.$$

This can be proved as follows:

$$RF \circ Q_\mathcal{S} \circ \iota \simeq RF \circ \iota_Q \circ Q_\mathcal{T} \simeq F_\mathcal{T} \circ \iota_Q^{-1} \circ \iota_Q \circ Q_\mathcal{T} \simeq F_\mathcal{T} \circ Q_\mathcal{T} \simeq F \circ \iota,$$

the last isomorphism following from (101).

(e) For each X in $\mathcal{C}$ let us denote by $s_X : X \to \iota(W_X)$ the morphism in $\mathcal{S}$ with W_X in $\mathcal{I}$ which exists by assumption. Then we can define RF by

$$RF(Q_\mathcal{S}(X)) := F(\iota(W_X)).$$

Moreover, the structural morphism $F \to RF \circ Q_\mathcal{S}$ is given by

$$F(X) \xrightarrow{F(s_X)} F(\iota(W_X)) = RF(Q_\mathcal{S}(X)).$$

(f) If we set $W(X) := W_X$, then the functor

$$\underset{(W(X)\to W)\in\mathcal{T}^{W(X)}}{\text{“colim”}} F(\iota(W)) \in \mathcal{A}^\wedge$$

is represented by $RF(X)$.

§ 255. P. 161. We paste Display (7.3.7), which appears in Proposition 7.3.3 (iii) p. 161 of the book:

$$(R_{\mathcal{S}}F)(Q(X)) \simeq \operatorname*{colim}_{(X\to Y)\in\mathcal{S}^X} F(Y). \tag{103}$$

Let $\mathcal{C}$ be a $\mathcal{U}$-category (Definition 4 p. 10), and let $\mathcal{V}$ be a universe such that $\mathcal{U} \in \mathcal{V}$ and $\mathcal{C}$ is a $\mathcal{V}$-small category (Definition 5 p. 10). Writing $\mathcal{A}$ for the category of $\mathcal{V}$-sets, Proposition 7.3.3 (iii) of the book implies the following:

Let X and Y be two objects of $\mathcal{C}$.

If $\mathcal{S}$ is a right multiplicative system in $\mathcal{C}$, then the functor

$$R_{\mathcal{S}} \operatorname{Hom}_{\mathcal{C}}(X, \quad)$$

exists and is isomorphic to $\operatorname{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, \quad)$.

Similarly, if $\mathcal{S}$ is a left multiplicative system in $\mathcal{C}$, then the functor

$$R_{\mathcal{S}^{\text{op}}} \operatorname{Hom}_{\mathcal{C}}(\quad, Y)$$

exists and is isomorphic to $\operatorname{Hom}_{\mathcal{C}^\ell_{\mathcal{S}}}(\quad, Y)$.

§ 256. P. 161. We prove the isomorphism at the bottom of p. 161.

Recall the setting: $\mathcal{S}$ is a right multiplicative system in a category $\mathcal{C}$ such that $\mathcal{S}^X$ is cofinally small for all X in $\mathcal{C}$. Let X and Y be in $\mathcal{C}$. It is claimed in the book that there is a natural isomorphism

$$\operatorname*{colim}_{(Y\to Y')\in\mathcal{S}^Y} \operatorname{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\sim} \lim_{(X\to X')\in\mathcal{S}^X} \operatorname*{colim}_{(Y\to Y')\in\mathcal{S}^Y} \operatorname{Hom}_{\mathcal{C}}(X', Y'). \tag{104}$$

We can rewrite (104) as

$$\operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(X), Q(Y)) \xrightarrow{\sim} \lim_{(X\to X')\in\mathcal{S}^X} \operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(X'), Q(Y)). \tag{105}$$

For $X \to X'$ in $\mathcal{S}$ let

$$p[X \to X'] : \lim_{(X\to X'')\in\mathcal{S}^X} \operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(X''), Q(Y)) \to \operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(X'), Q(Y))$$

be the projection, and define the maps

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(X),Q(Y)) \underset{g}{\overset{f}{\rightleftarrows}} \lim_{(X\to X')\in\mathcal{S}^X} \mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(Q(X'),Q(Y))$$

as follows: We define f by

$$p[X\to X']\Big(f\big(Q(X)\to Q(Y)\big)\Big) := \big(Q(X')\to Q(X)\to Q(Y)\big)$$

for $X\to X'$ in $\mathcal{S}$, where $Q(X')\to Q(X)$ is the inverse of $Q(X\to X')$, and we define g by

$$g := p[X \xrightarrow{\mathrm{id}} X].$$

To show that $f\circ g$ is the identity of the right-hand side of (105), note that we have in the above notation

$$p[X\to X']\left(f\left(g\Big(\big(Q(X'')\to Q(Y)\big)_{X\to X''}\Big)\right)\right)$$

$$= p[X\to X']\Big(f\big(Q(X)\to Q(Y)\big)\Big)$$

$$= \big(Q(X')\to Q(X)\to Q(Y)\big) = p[X\to X']\Big(\big(Q(X'')\to Q(Y)\big)_{X\to X''}\Big).$$

The proof that $g\circ f$ is the identity of the left-hand side of (105) is similar and easier.

§ 257. P. 162, Display (7.4.3). We must prove $R_{\mathcal{S}}(\iota_{\mathcal{A}}\circ F)\simeq IF\circ\alpha_{\mathcal{S}}$. Let X be in $\mathcal{C}$. It suffices to show $R_{\mathcal{S}}(\iota_{\mathcal{A}}\circ F)(Q(X))\simeq IF(\alpha_{\mathcal{S}}(Q(X)))$. We have

$$R_{\mathcal{S}}(\iota_{\mathcal{A}}\circ F)(Q(X)) \simeq \operatorname*{colim}_{(X\to X')\in\mathcal{S}^X} \iota_{\mathcal{A}}(F(X'))$$

$$\simeq IF\left(\operatorname*{colim}_{(X\to X')\in\mathcal{S}^X} \iota_{\mathcal{C}}(X')\right) \simeq IF(\alpha_{\mathcal{S}}(Q(X))),$$

the isomorphisms following respectively from (7.3.7) p. 161 of the book, Proposition 6.1.9 p. 133 of the book and Proposition 7.4.1 p. 162 of the book.

§ 258. P. 162, Definition 7.4.2. If $F:\mathcal{C}\to\mathcal{A}$ is a functor and X an object of $\mathcal{C}$, then the condition that F is right localizable at X does not depend on the choice of a universe $\mathcal{U}$ such that $\mathcal{C}$ and $\mathcal{A}$ are $\mathcal{U}$-categories (Definition 4 p. 10).

§ 259. P. 162, proof of Lemma 7.4.3. Recall the statement:

Lemma 260 (Lemma 7.4.3 p. 162)**.** *If $G : \mathcal{A} \to \mathcal{A}'$ is a functor and F is right localizable at X, then $G \circ F$ is right localizable at X.*

Proof. This follows from Proposition 69 p. 58. □

§ 261. P. 163, Remark 7.4.5. In this § we adhere to Convention 11.7.1 of the book, according to which, paradoxically, in the expression $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, the variable Y is considered as the *first* variable and X as the *second* variable.

Let $\mathcal{S}$ be a left and right multiplicative system in $\mathcal{C}$, and let X and Y be two objects of $\mathcal{C}$. §255 p. 162 implies that the functors

$$R_{\mathcal{S}} \,\mathrm{Hom}_{\mathcal{C}}(X, \ \ ), \quad R_{\mathcal{S}^{\mathrm{op}}} \,\mathrm{Hom}_{\mathcal{C}}(\ \ , Y), \quad R_{\mathcal{S}\times\mathcal{S}^{\mathrm{op}}} \,\mathrm{Hom}_{\mathcal{C}}$$

exist and satisfy

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y) \simeq R_{\mathcal{S}}\big(\mathrm{Hom}_{\mathcal{C}}(X, \ \ )\big)(Y)$$

$$\simeq R_{\mathcal{S}^{\mathrm{op}}}\big(\mathrm{Hom}_{\mathcal{C}}(\ \ , Y)\big)(X) \simeq R_{\mathcal{S}\times\mathcal{S}^{\mathrm{op}}} \,\mathrm{Hom}_{\mathcal{C}}(X, Y).$$

More precisely, if, in the diagram

$$\begin{array}{ccccc} R_{\mathcal{S}} \,\mathrm{H}_{\mathcal{C}}(X, \ \ )(Y) & \longrightarrow & R_{\mathcal{S}\times\mathcal{S}^{\mathrm{op}}} \,\mathrm{H}_{\mathcal{C}}(X, Y) & \longleftarrow & R_{\mathcal{S}^{\mathrm{op}}} \,\mathrm{H}_{\mathcal{C}}(\ \ , Y)(X) \\ & \nwarrow\!\!\searrow & \updownarrow & \swarrow\!\!\nearrow & \\ & & \mathrm{H}_{\mathcal{C}_{\mathcal{S}}}(X, Y), & & \end{array} \tag{106}$$

where we have written H for Hom to save space, the horizontal arrows are the natural maps, and the other arrows are the above bijections, then (106) commutes and all its arrows are bijective.

§ 262. Exercise 7.4 p. 164.

Statement: In a category endowed with a right multiplicative system $\mathcal{S}$, if there is a diagram

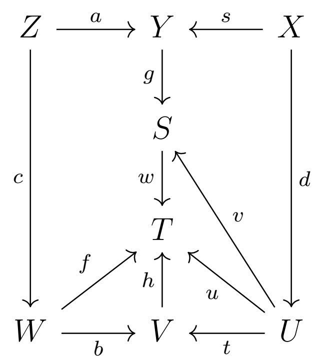
$$\begin{array}{ccccc} Z & \xrightarrow{a} & Y & \xleftarrow{s} & X \\ {\scriptstyle c}\downarrow & & & & \downarrow{\scriptstyle d} \\ W & \xrightarrow[b]{} & V & \xleftarrow[t]{} & U \end{array}$$

with $s, t \in \mathcal{S}$ and $Q(d) \circ Q(s)^{-1} \circ Q(a) = Q(t)^{-1} \circ Q(b \circ c)$, then there is a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{a} & Y & \xleftarrow{s} & X \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle d} \\ W & \xrightarrow[f]{} & T & \xleftarrow[u]{} & U \end{array}$$

with $u \in \mathcal{S}$ and $Q(u)^{-1} \circ Q(f) = Q(t)^{-1} \circ Q(b)$. [This statement solves clearly the exercise.]

Proof: We build a commutative diagram



with $u, v, w \in \mathcal{S}$ by forming firstly g and v, secondly h and w, and thirdly f and u.

10 About Chapter 8

10.1 About Section 8.1

The following definition of a commutative group object is much less general and much less elegant than the one in the book (p. 168), but it is slightly simpler and seems sufficient in this context.

Let $\mathcal{C}$ be a category with finite products; let 0 be the terminal object of $\mathcal{C}$; let X be in $\mathcal{C}$; let $p_1, p_2 : X \times X \to X$ be the projections; and let $v : X \times X \to X \times X$ be defined by the equalities

$$p_i \circ v = p_j$$

for all i, j such that $\{i, j\} = \{1, 2\}$.

A structure of *commutative group object* on an object X of $\mathcal{C}$ is a triple (α, e, a) satisfying the following conditions:

We have

$$\alpha : X \times X \to X, \quad e : 0 \to X, \quad a : X \to X,$$

and the following diagrams commute:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mathrm{id}\times\alpha} & X \times X \\ {\scriptstyle \alpha\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X \times X & \xrightarrow[\alpha]{} & X, \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id},e)} & X \times X \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha} \\ & & X, \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{(e,\mathrm{id})} & X \times X \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha} \\ & & X, \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id},a)} & X \times X \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ 0 & \xrightarrow[e]{} & X, \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{(a,\mathrm{id})} & X \times X \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ 0 & \xrightarrow[e]{} & X, \end{array}$$

$$\begin{array}{ccc} X \times X & \xrightarrow{v} & X \times X \\ & {\scriptstyle \alpha}\searrow & \downarrow{\scriptstyle \alpha} \\ & & X. \end{array}$$

10.2 About Section 8.2

10.2.1 Definition 8.2.1 p. 169

The proposition and lemma below are obvious.

Proposition 263. *Let $\mathcal{C}$ be a pre-additive category, let $\mathcal{A}$ be the category of additive functors from $\mathcal{C}^{\text{op}}$ to $\text{Mod}(\mathbb{Z})$, let $h : \mathcal{C} \to \mathcal{A}$ be the obvious functor satisfying $h(X)(Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ for all X and Y in $\mathcal{C}$, let X be in $\mathcal{C}$ and A in $\mathcal{A}$, and let*

$$\text{Hom}_{\mathcal{A}}(h(X), A) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} A(X)$$

be defined by

$$\Phi(\theta) = \theta_X(\text{id}_X), \quad \Psi(x)(f) = A(f)(x).$$

Then Φ and Ψ are mutually inverse abelian group isomorphisms.

(See Theorem 36 p. 33.)

Convention 264. In the above setting we denote $\mathcal{A}$ by $\mathcal{C}^\wedge$ and h by $\text{h}_{\mathcal{C}}$. (This abuse is justified by Proposition 263.)

Lemma 265. *Let $\mathcal{C}$ and $\mathcal{C}'$ be pre-additive categories, let $\mathcal{A}$ be the category of additive functors from $\mathcal{C}$ to $\mathcal{C}'$, and let $\alpha : I \to \mathcal{A}$ be a functor such that $\text{colim}\, \alpha(X)$ exists in $\mathcal{C}'$ for all X in $\mathcal{C}$. Then $\text{colim}\, \alpha$ exists in $\mathcal{A}$ and satisfies*

$$(\text{colim}\, \alpha)(X) \simeq \text{colim}\, \alpha(X)$$

for all X in $\mathcal{C}$. (There is a similar statement for projective limits.)

10.2.2 Lemma 8.2.3 p. 169

Here is a statement contained in Lemma 8.2.3:

Corollary 266. *Let $\mathcal{C}$ be a pre-additive category, let X_1 and X_2 be two objects of $\mathcal{C}$ such that the product $X = X_1 \times X_2$ exists in $\mathcal{C}$, let $p_a : X \to X_a$ be the projection, and define $i_a : X_a \to X$ by*

$$p_a \circ i_b = \begin{cases} \text{id}_{X_a} & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

Then X is a coproduct of X_1 and X_2 by i_1 and i_2. Moreover we have

$$i_1 \circ p_1 + i_2 \circ p_2 = \text{id}_{X_1 \times X_2}.$$

For the reader's convenience we reproduce the statement and the proof of Lemma 8.2.3 (ii) p. 169 of the book:

Lemma 267 (Lemma 8.2.3 (ii) p. 169)**.** *Let $\mathcal{C}$ be a pre-additive category; let X, X_1, and X_2 be objects of $\mathcal{C}$; and, for $a = 1, 2$, let $X_a \xrightarrow{i_a} X \xrightarrow{p_a} X_a$ be morphisms satisfying*

$$p_a \circ i_b = \delta_{ab}\ \mathrm{id}_{X_a}, \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_X .$$

Then X is a product of X_1 and X_2 by p_1 and p_2 and a coproduct of X_1 and X_2 by i_1 and i_2.

Proof. For any Y in $\mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(Y, p_a) \circ \mathrm{Hom}_{\mathcal{C}}(Y, i_b) = \delta_{ab}\ \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(Y,X_a)},$$

$$\mathrm{Hom}_{\mathcal{C}}(Y, i_1) \circ \mathrm{Hom}_{\mathcal{C}}(Y, p_1) + \mathrm{Hom}_{\mathcal{C}}(Y, i_2) \circ \mathrm{Hom}_{\mathcal{C}}(Y, p_2) = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(Y,X)} .$$

This implies that $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a product of $\mathrm{Hom}_{\mathcal{C}}(Y, X_1)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, X_2)$ by $\mathrm{Hom}_{\mathcal{C}}(Y, p_1)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, p_2)$, and thus, Y being arbitrary, that X is a product of X_1 and X_2 by p_1 and p_2, and we conclude by applying this observation to the opposite category. □

Note also the following corollary to Lemma 8.2.3 (ii) (stated above as Lemma 267).

Corollary 268. *Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor of pre-additive categories; let X, X_1 and X_2 be objects of $\mathcal{C}$; and, for $a = 1, 2$, let $X_a \xrightarrow{i_a} X \xrightarrow{p_a} X_a$ be morphisms such that X is a product of X_1 and X_2 by p_1, p_2 and a coproduct of X_1 and X_2 by i_1, i_2. Then $F(X)$ is a product of $F(X_1)$ and $F(X_2)$ by $F(p_1), F(p_2)$ and a coproduct of $F(X_1)$ and $F(X_2)$ by $F(i_1), F(i_2)$.*

10.2.3 Brief comments

§ 269. P. 170, Corollary 8.2.4. Recall the statement:

Corollary 270 (Corollary 8.2.4 p. 170)**.** *Let $\mathcal{C}$ be a pre-additive category and let $X_1, X_2 \in \mathcal{C}$. If $X_1 \times X_2$ exists in $\mathcal{C}$, then $X_1 \sqcup X_2$ also exists. Moreover denoting by*

$i_j : X_j \to X_1 \sqcup X_2$ and $p_j : X_1 \times X_2 \to X_j$ the j-th co-projection and projection, the morphism $r : X_1 \sqcup X_2 \to X_1 \times X_2$ given by

$$p_j \circ r \circ i_k = \begin{cases} \mathrm{id}_{X_k} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

is an isomorphism.

Convention 271. Let X_1 and X_2 be two objects of a category $\mathcal{C}$. Assume that the product $X_1 \times X_2$ and the coproduct $X_1 \sqcup X_2$ exist in $\mathcal{C}$ and are isomorphic. In such a situation, we make a new exception to Convention 38 p. 36: we set $X_1 \oplus X_2 := X_1 \times X_2$, we transport the coprojections of $X_1 \sqcup X_2$ to $X_1 \oplus X_2$ and redefine $X_1 \sqcup X_2$ by setting

$$X_1 \sqcup X_2 := X_1 \oplus X_2 := X_1 \times X_2,$$

so that $X_1 \oplus X_2$ is at the same time a product and a coproduct of X_1 and X_2.

The following lemma, whose proof is left to the reader, is implicit in the book.

Lemma 272. *For $a = 1, 2$ let $f_a : X_a \to Y_a$ be a morphism in a pre-additive category $\mathcal{C}$. Assume that $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ exist in $\mathcal{C}$ (see Convention 271 above). Then we have*

$$f_1 \times f_2 = f_1 \sqcup f_2$$

(equality in $\mathrm{Hom}_{\mathcal{C}}(X_1 \oplus X_2, Y_1 \oplus Y_2)$).

We denote this morphism by $f_1 \oplus f_2$.

§ 273. P. 171, Corollary 8.2.6. Recall the statement:

Corollary 274 (Corollary 8.2.6 p. 171)**.** *Let $\mathcal{C}$ be a pre-additive category, $X, Y \in \mathcal{C}$ and $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Assume that the direct sums $X \oplus X$ and $Y \oplus Y$ exist (see Convention 271 p. 169). Then $f_1 + f_2 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ coincides with the composition*

$$X \xrightarrow{\delta_X} X \oplus X \xrightarrow{f_1 \oplus f_2} Y \oplus Y \xrightarrow{\sigma_Y} Y.$$

Here $\delta_X : X \to X \times X = X \oplus X$ is the diagonal morphism and $\sigma_Y : Y \oplus Y = Y \sqcup Y \to Y$ is the codiagonal morphism.

Proof. For $a = 1, 2$ let

$$X \oplus X \underset{i_a}{\overset{p_a}{\rightleftarrows}} X \qquad Y \oplus Y \underset{j_a}{\overset{q_a}{\rightleftarrows}} Y$$

be the projections and coprojections. Writing xy for $x \circ y$ we have

$$\sigma_Y \; (f_1 \oplus f_2) \; \delta_X = \sum_{a,b} \; \sigma_Y \; j_a \; q_a \; (f_1 \oplus f_2) \; i_b \; p_b \; \delta_X$$

$$= \sum_{a,b} \; q_a \; (f_1 \oplus f_2) \; i_b = \sum_{a} \; q_a \; (f_1 \oplus f_2) \; i_a = f_1 + f_2,$$

the second equality following from the definitions of σ_Y and δ_X, and the third and fourth equalities following from the definitions of $f_1 \oplus f_2$. (The justification of the first equality is left to the reader.) □

§ 275. P. 172, Lemma 8.2.9. Recall the statement:

Lemma 276 (Lemma 8.2.9 p. 172)**.** *Let $\mathcal{C}$ be a pre-additive category which admits finite products. Then $\mathcal{C}$ is additive.*

Let us check that $\mathcal{C}$ has a zero object. (This part of the proof is left to the reader by the authors.)

Let X and Y be in $\mathcal{C}$. By Lemma 8.2.3 p. 169 of the book, the product $X \times Y$ is also a coproduct of X and Y. Let us denote this object by $X \oplus Y$. Let T be a terminal object of $\mathcal{C}$. For any X in $\mathcal{C}$ we have a natural isomorphism $X \oplus T \simeq X$. In particular T can be viewed as $T \sqcup T$ via the morphisms $T \xrightarrow{\text{id}} T \xleftarrow{\text{id}} T$. This implies successively that, for X in $\mathcal{C}$, the diagonal map

$$\mathrm{Hom}_{\mathcal{C}}(T, X) \to \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, X)$$

is bijective, that the set $\mathrm{Hom}_{\mathcal{C}}(T, X)$ has at most one element, and that it has exactly one element. As X is arbitrary, this entails that T is a zero object. q.e.d.

Also note that Corollary 8.2.4 p. 170 of the book is useful to prove Lemma 8.2.9.

§ 277. P. 172, Lemma 8.2.10. Let me state the result in a more explicit way:

Lemma 278 (Lemma 8.2.10 p. 172)**.** *If X is an object of an additive category $\mathcal{C}$, then the morphism*

$$X \times X = X \sqcup X \xrightarrow{\sigma_X} X$$

defines a structure of a commutative group object on X.

The associativity of the addition can also be proved as follows:

Put $X^n := X \oplus \cdots \oplus X$ (n factors), and let $X \xrightarrow{i_a} X^n \xrightarrow{\sigma_n} X$ be respectively the a-th coprojection and the codiagonal morphism. It clearly suffices to show that the composition

$$X^3 \xrightarrow{\sigma_2 \oplus X} X^2 \xrightarrow{\sigma_2} X$$

is equal to σ_3. This follows from the fact that the composition

$$X \xrightarrow{i_a} X^3 \xrightarrow{\sigma_2 \oplus X} X^2$$

is equal to i_b with

$$b = \begin{cases} 1 & \text{if } a = 1, 2 \\ 2 & \text{if } a = 3. \end{cases}$$

q.e.d.

§ 279. P. 173, proof of Proposition 8.2.13. The fact that any X in $\mathcal{C}$ has a structure of commutative group object follows from Lemma 8.2.10 p. 172 of the book.

§ 280. Proposition 8.2.13 p. 173. Let $\mathcal{C}$ be an additive category, let $\mathcal{C}'$ be the category of finite product preserving functors from $\mathcal{C}$ to $\text{Mod}(\mathbb{Z})$, let $\mathcal{C}''$ be the category of finite product preserving functors from $\mathcal{C}$ to **Set**, let U be the forgetful functor from $\text{Mod}(\mathbb{Z})$ to **Set** and define the functor $V : \mathcal{C}'' \to \mathcal{C}'$ by the formula $(V(F))(X) := (F(X), +)$, where $+$ is the addition defined in the proof of Proposition 8.2.13.

Proposition 281. *In the above setting, the functors V and $U\circ : \mathcal{C}' \to \mathcal{C}''$ are mutually inverse isomorphisms (not just mutually quasi-inverse equivalences).*

Proof. For F in $\mathcal{C}''$, for G in $\mathcal{C}'$ and for X in $\mathcal{C}$ we have

$$\Big(\big((U\circ) \circ V\big)(F)\Big)(X) = U\Big(\big(V(F)\big)(X)\Big) = F(X)$$

and

$$\Big(\big(V \circ (U\circ)\big)(G)\Big)(X) = \big(V(U \circ G)\big)(X) = G(X).$$

Indeed, the last equality follows from Lemma 8.2.11 p. 172 of the book, and the others are straightforward. □

§ 282. P. 173, Theorem 8.2.14. Let me state the result in a more explicit way:

Theorem 283 (Theorem 8.2.14)**.** *If $\mathcal{C}$ is an additive category, then $\mathcal{C}$ has a unique structure of a pre-additive category. More precisely, for f and g in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, the morphism $f+g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is given by the composition*

$$\begin{array}{ccccccc} X & \xrightarrow{\delta_X} & X\times X & \xrightarrow{f\times g} & Y\times Y & & \\ & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \\ & & X\sqcup X & \xrightarrow[f\sqcup g]{} & Y\sqcup Y & \xrightarrow{\sigma_Y} & Y. \end{array}$$

Let me also try to rewrite the beginning of the proof:

Let $X \in \mathcal{C}$. By applying Proposition 8.2.13 p. 173 of the book (see §286 p. 173) to the functor $F := \mathrm{Hom}_{\mathcal{C}}(X,\ )$, we obtain that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has a structure of an additive group for all Y in $\mathcal{C}$. Then Lemma 8.2.11 p. 172 of the book implies that the addition on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is given by the above commutative diagram.

We complete the proof by showing as in the book that this addition does define a pre-additive structure on $\mathcal{C}$.

§ 284. P. 173, Theorem 8.2.14 (stated above as Theorem 283). Consider the following claims:

(a) the fields $\mathbb{Q}$ and $\mathbb{F}_3(X)$ have isomorphic multiplicative groups,

(b) there is a category $\mathcal{C}$ admitting two pre-additive structures p and q such that there is no additive equivalence from $(\mathcal{C},p)$ to $(\mathcal{C},q)$.

We leave it to the reader to prove (a) and to show that (a) implies (b).

§ 285. P. 173, Proposition 8.2.15. Recall the setting: $F : \mathcal{C} \to \mathcal{C}'$ is a functor between additive categories, and the claim is:

$$F \text{ is additive } \Leftrightarrow \ F \text{ commutes with finite products.}$$

I think the authors forgot to prove implication $\Rightarrow$. Let us do it. It suffices to show that F commutes with n-fold products for $n = 0$ or $n = 2$.

Case $n = 0$: Put $X := F(0)$. We must prove $X \simeq 0$. The equality $0 = 1$ holds in the ring $\mathrm{Hom}_{\mathcal{C}}(X,X)$ because it holds in the ring $\mathrm{Hom}_{\mathcal{C}}(0,0)$. As a result, the morphisms $0 \to X$ and $X \to 0$ are mutually inverse isomorphisms.

Case $n = 2$: Let X_1, X_2 be in $\mathcal{C}$. The natural morphisms

$$F(X_1 \oplus X_2) \rightleftarrows F(X_1) \oplus F(X_2)$$

are mutually inverse isomorphisms by Corollary 268 p. 168 above. q.e.d.

§ 286. Let us insist on the main point.

If $\mathcal{C}$ is an additive category, then the following categories are canonically isomorphic:

the category $\mathcal{C}'$ of finite product preserving functors from $\mathcal{C}$ to $\text{Mod}(\mathbb{Z})$*,*

the category $\mathcal{C}''$ of finite product preserving functors from $\mathcal{C}$ to **Set***,*

the category $\mathcal{C}'''$ of additive functors from $\mathcal{C}$ to $\text{Mod}(\mathbb{Z})$*.*

Moreover $\mathcal{C}'$ and $\mathcal{C}'''$ are equal.

This follows from §280 p. 171 and §285 p. 172.

10.3 About Section 8.3

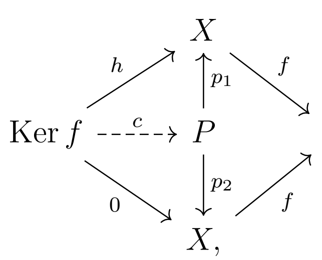
10.3.1 Proposition 8.3.4 p. 176

Here are a few more details about the proof of Proposition 8.3.4. Recall the setting: We have a morphism $f : X \to Y$ in an abelian category $\mathcal{C}$. Let P be the fiber product $X \times_Y X$; let $p_1, p_2 : P \rightrightarrows X$ be the projections; let p be the morphism $p_1 - p_2$ from P to X; and consider the diagram

$$\begin{array}{ccccccc} \operatorname{Ker} f & \xrightarrow{h} & X & \xrightarrow{a} & \operatorname{Coker} h & & \\ & & \| & & & & \\ P & \xrightarrow[p]{} & X & \xrightarrow[b]{} & \operatorname{Coker} p & = & \operatorname{Coim} f, \end{array}$$

where h, a, and b are the natural morphisms.

We claim $b \circ h = 0$. Indeed, we define $c : \operatorname{Ker} f \to P$ by the condition $p_1 \circ c = h, p_2 \circ c = 0$:



and we get $b \circ h = b \circ p \circ c = 0 \circ c = 0$. This proves the claim. Hence, there is a unique morphism $d : \operatorname{Coker} h \to \operatorname{Coim} f$ making the diagram

$$\begin{array}{ccccccc} \operatorname{Ker} f & \xrightarrow{h} & X & \xrightarrow{a} & \operatorname{Coker} h & & \\ & & \| & & \downarrow{\scriptstyle d} & & \\ P & \xrightarrow[p]{} & X & \xrightarrow[b]{} & \operatorname{Coim} f & = & \operatorname{Coker} p \end{array}$$

commute.

As p factors through h, we have $a \circ p = 0$, and there is a unique morphism $e : \operatorname{Coim} f \to \operatorname{Coker} h$ making the diagram

$$\begin{array}{ccccccc} \operatorname{Ker} f & \xrightarrow{h} & X & \xrightarrow{a} & \operatorname{Coker} h & & \\ & & \| & & \uparrow{\scriptstyle e} & & \\ P & \xrightarrow[p]{} & X & \xrightarrow[b]{} & \operatorname{Coim} f & = & \operatorname{Coker} p \end{array}$$

commute.

It is easy to see that d and e are mutually inverse isomorphisms. In short, there is a natural isomorphism $\operatorname{Coker} h \simeq \operatorname{Coim} f$ which makes the diagram

$$\begin{array}{ccccccc} \operatorname{Ker} f & \xrightarrow{h} & X & \xrightarrow{a} & \operatorname{Coker} h & & \\ & & \| & & \updownarrow{\scriptstyle \sim} & & \\ P & \xrightarrow[p]{} & X & \xrightarrow[b]{} & \operatorname{Coim} f & = & \operatorname{Coker} p \end{array} \tag{107}$$

commute.

Dually, let S (for "sum") be the fiber coproduct $Y \oplus_X Y$, let $i_a : Y \to S$ be the coprojection, let i be the morphism $i_1 - i_2$ from Y to S, and consider the diagram

$$\begin{array}{ccccc} \operatorname{Im} f & \xrightarrow{a} & Y & \xrightarrow{i} & S \\ & & \| & & \\ \operatorname{Ker} k & \xrightarrow{b} & Y & \xrightarrow[k]{} & \operatorname{Coker} f \end{array}$$

where a, b, and k are the natural morphisms. Then there is a natural isomorphism $\operatorname{Im} f \simeq \operatorname{Ker} k$ which makes the diagram

$$\begin{array}{ccccc} \operatorname{Im} f & \longrightarrow & Y & \xrightarrow{i} & S \\ {\scriptstyle \sim}\updownarrow & & \| & & \\ \operatorname{Ker} k & \longrightarrow & Y & \xrightarrow[k]{} & \operatorname{Coker} f \end{array} \tag{108}$$

commute. Let us record these observations:

Proposition 287. *In the above setting there are natural isomorphisms*

$$\operatorname{Coker} h \simeq \operatorname{Coim} f, \quad \operatorname{Im} f \simeq \operatorname{Ker} k$$

which make Diagrams (107) *and* (108) *commute.*

Note that we can splice Diagrams (107) and (108):

$$\begin{array}{ccccccc}
\operatorname{Ker} f & \xrightarrow{h} & X & \longrightarrow & \operatorname{Coker} h & & & & \\
& & \| & & \updownarrow\wr & & & & \\
P & \xrightarrow[p]{} & X & \longrightarrow & \operatorname{Coim} f & & & & \\
& & & & \downarrow\wr & & & & \\
& & & & \operatorname{Im} f & \longrightarrow & Y & \xrightarrow{i} & S \\
& & & & \wr\updownarrow & & \| & & \\
& & & & \operatorname{Ker} k & \longrightarrow & Y & \xrightarrow[k]{} & \operatorname{Coker} f.
\end{array}$$

10.3.2 Definition 8.3.5 p. 177

The following definitions and observations are implicit in the book. Let $\mathcal{A}$ be a subcategory of a pre-additive category $\mathcal{B}$, and let $\iota : \mathcal{A} \to \mathcal{B}$ be the inclusion. If $\mathcal{A}$ is pre-additive and ι is additive, we say that $\mathcal{A}$ is a *pre-additive subcategory* of $\mathcal{B}$. If, moreover, $\mathcal{A}$ and $\mathcal{B}$ are additive (resp. abelian), we say that $\mathcal{A}$ is *an additive (resp. abelian) subcategory* of $\mathcal{B}$. Now let $\mathcal{A}$ and $\mathcal{B}$ be categories. If $\mathcal{B}$ is pre-additive (resp. additive, abelian), then so is the category $\mathcal{C} := \mathcal{B}^{\mathcal{A}}$ of functors from $\mathcal{A}$ to $\mathcal{B}$. Assume furthermore that $\mathcal{A}$ is pre-additive. If $\mathcal{B}$ is pre-additive (resp. additive, abelian), then the full subcategory $\mathcal{D} := \operatorname{Add}(\mathcal{A}, \mathcal{B})$ of $\mathcal{C}$ whose objects are the additive functors from $\mathcal{A}$ to $\mathcal{B}$ is a pre-additive (resp. additive, abelian) subcategory of $\mathcal{C}$.

10.3.3 The Complex (8.3.3) p. 178

Let us just add a few details about the proof of the isomorphisms

$$\begin{aligned}
\operatorname{Im} u &\simeq \operatorname{Coker}(\varphi : \operatorname{Im} f \to \operatorname{Ker} g) \simeq \operatorname{Coker}(X' \to \operatorname{Ker} g) \\
&\simeq \operatorname{Ker}(\psi : \operatorname{Coker} f \to \operatorname{Im} g) \simeq \operatorname{Ker}(\operatorname{Coker} f \to X''),
\end{aligned} \tag{109}$$

labeled (8.3.4) in the book. Recall that the underlying category $\mathcal{C}$ is abelian, and that the complex in question is denoted

$$X' \xrightarrow{f} X \xrightarrow{g} X''. \qquad (110)$$

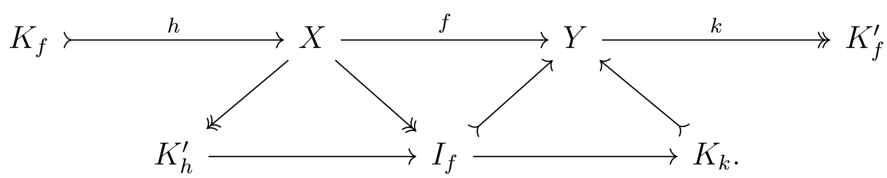
We shall freely use the isomorphism between image and coimage, as well as the abbreviations

$$K_v := \operatorname{Ker} v, \quad K'_v := \operatorname{Coker} v, \quad I_v := \operatorname{Im} v.$$

Let us also write "$A \xrightarrow{\sim} B$" for "the natural morphism $A \to B$ is an isomorphism".

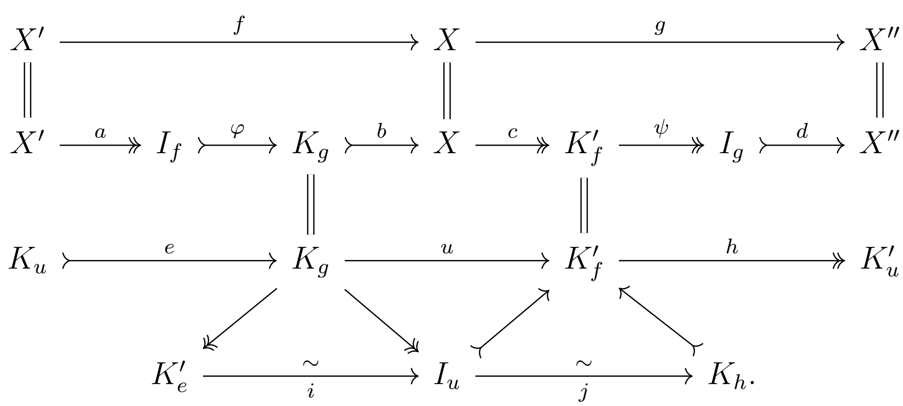
Proposition 287 p. 175 can be stated as follows.

Proposition 288. *Let $f : X \to Y$ be a morphism, and consider the commutative diagram*



Then the bottom arrows are isomorphisms.

Going back to our complex (110) p. 176, let us introduce the notation



By Proposition 287 p. 175

$$i \text{ and } j \text{ are isomorphisms.} \qquad (111)$$

We shall prove

$$K'_{\varphi\circ a} \xrightarrow[\sim]{k} K'_\varphi \xrightarrow[\sim]{\ell} K'_e \xrightarrow[\sim]{i} I_u \xrightarrow[\sim]{j} K_h \xrightarrow[\sim]{m} K_\psi \xrightarrow[\sim]{n} K_{d\circ\psi}.$$

This will imply (109) p. 175.

The morphisms k and n are isomorphisms because a is an epimorphism and d a monomorphism. Thus, in view of (111), it only remains to prove that

$$\ell \text{ and } m \text{ are isomorphisms.} \tag{112}$$

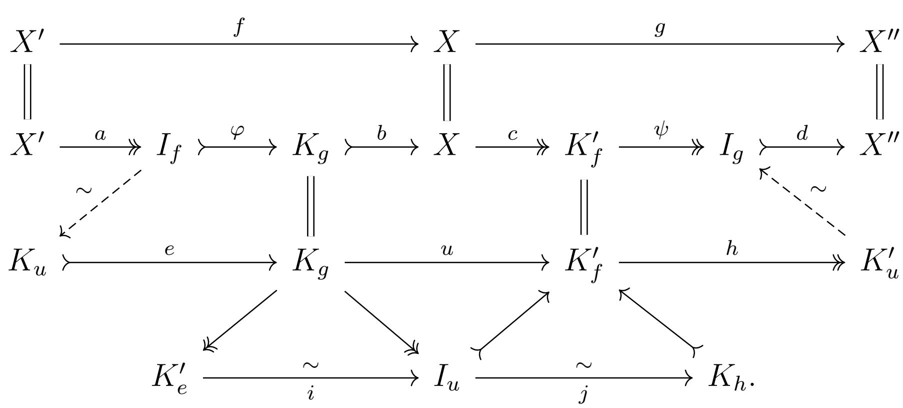
There is a natural monomorphism from I_f to K_u. Indeed, we have

$$u \circ \varphi \circ a = c \circ f = 0.$$

As a is an epimorphism, this implies $u \circ \varphi = 0$.

It is easy to see that there is a natural monomorphisms from K_u to K_c. By Proposition 287 p. 175, we have $I_f \xrightarrow{\sim} K_c$, and it is easy to see that this implies $I_f \xrightarrow{\sim} K_u$. Similarly we prove $K'_u \xrightarrow{\sim} I_g$.

We can thus complete our previous diagram as follows:



(The two dashed arrows have been added.) Now (112) is clear.

10.3.4 Brief comments

§ 289. For the reader's convenience we state Lemma 8.3.11 p. 180. Consider the commutative square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{113}$$

in the abelian category $\mathcal{C}$.

Lemma 290 (Lemma 8.3.11 p. 180)**.** *We have:*

(a) *Assume that* (113) *is cartesian.*

  (i) *We have* $\operatorname{Ker} f' \xrightarrow{\sim} \operatorname{Ker} f$.

  (ii) *If f is an epimorphism, then* (113) *is cocartesian and f' is an epimorphism.*

(b) *Assume that* (113) *is cocartesian.*

  (i) *We have* $\operatorname{Coker} f' \xrightarrow{\sim} \operatorname{Coker} f$.

  (ii) *If f' is a monomorphism, then* (113) *is cartesian and f is a monomorphism.*

§ 291. P. 180, Lemma 8.3.12. Here is a minor variant:

Lemma 292. *For a complex $Z \to Y \to X$ in some abelian category, the following conditions are equivalent:*

(a) *the complex is exact,*

(b) *any commutative diagram of solid arrows*

$$\begin{array}{ccccc} V & \dashrightarrow\!\!\!\!\rightarrow & W & & \\ \vdots\downarrow & & \downarrow & \searrow^{0} & \\ Z & \longrightarrow & Y & \longrightarrow & X \end{array}$$

can be completed as indicated ($V \to W$ being an epimorphism),

(c) *any commutative diagram of solid arrows*

$$\begin{array}{ccccc} Z & \longrightarrow & Y & \longrightarrow & X \\ & \searrow_{0} & \downarrow & & \downarrow\vdots \\ & & W & \rightarrowtail\!\!\dashrightarrow & V \end{array}$$

can be completed as indicated ($W \to V$ being a monomorphism).

Proof. Equivalence (a) $\Leftrightarrow$ (b) is proved in the book, and Equivalence (a) $\Leftrightarrow$ (c) follows by reversing arrows. $\square$

§ 293. Here are two lemmas which be used in the sequel:

Lemma 294. *Let*

$$\begin{array}{ccccc} Z & \xrightarrow{a} & Y & \xrightarrow{b} & X \\ \| & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ Z & \xrightarrow[e]{} & W & \xrightarrow[f]{} & V \end{array}$$

be a commutative diagram of complexes in an abelian category, and assume that the right square is cartesian.

(a) *If e is a monomorphism, so is a.*

(b) *If f is an epimorphism, so is b.*

(c) *If the bottom row is exact, so is the top row.*

Proof. Part (a) is obvious, and (b) follows from Lemma 290 (a) (ii) p. 178. To prove (c), let $g : U \to Y$ be a morphism satisfying $bg = 0$ (in this proof we write uv for $u \circ v$). By Lemma 292, (a) $\Leftrightarrow$ (b), p. 178, it suffices to complete the commutative diagram

$$\begin{array}{ccccc} V & \overset{h}{\dashrightarrow\!\!\!\!\rightarrow} & U & & \\ {\scriptstyle i}\,\vdots & & {\scriptstyle g}\downarrow & \searrow{\scriptstyle 0} & \\ Z & \xrightarrow[a]{} & Y & \xrightarrow[b]{} & X \end{array}$$

as indicated. Consider the larger commutative diagram

$$\begin{array}{ccccc} V & \overset{h}{\dashrightarrow\!\!\!\!\rightarrow} & U & & \\ {\scriptstyle i}\,\vdots & & {\scriptstyle g}\downarrow & \searrow{\scriptstyle 0} & \\ Z & \xrightarrow{a} & Y & \xrightarrow{b} & X \\ \| & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ Z & \xrightarrow[e]{} & W & \xrightarrow[f]{} & V. \end{array}$$

Invoking again Lemma 292, (a) $\Leftrightarrow$ (b), p. 178, we find an epimorphism $h : V \twoheadrightarrow U$ and a morphism $i : V \to Z$ such that $cgh = ei$, and it suffices to prove $gh = ai$. But the equality $gh = ai$ is equivalent to the conjunction of the equalities $cgh = cai$ and $bgh = bai$, equalities whose proof is straightforward. □

Lemma 295. *Let*

$$\begin{array}{ccccc} Z & \xrightarrow{a} & Y & \xrightarrow{b} & X \\ {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow & & \| \\ W & \xrightarrow[e]{} & V & \xrightarrow[f]{} & X \end{array}$$

be a commutative diagram of complexes in an abelian category, and assume that the left square is cartesian.

(a) *If e is a monomorphism, so is a.*

(b) *If the bottom row is exact, so is the top row.*

Proof. Part (a) follows from Lemma 290 (a) (i) p. 178. Let us prove (b). By Lemma 292, (a) ⇔ (b), p. 178, it suffices, given a morphism $g : U \to Y$ such that $bg = 0$, to complete the commutative diagram

$$\begin{array}{ccccc} T & \overset{h}{\dashrightarrow\!\!\!\!\rightarrow} & U & & \\ {\scriptstyle i}\,\vdots & & {\scriptstyle g}\downarrow & \overset{0}{\searrow} & \\ Z & \xrightarrow[a]{} & Y & \xrightarrow[b]{} & X \end{array}$$

as indicated. (In this proof we write uv for $u \circ v$.) Invoking again Lemma 292, (a) ⇔ (b), p. 178, we find $h : T \twoheadrightarrow U$ and $j : T \to W$ such that $ej = dgh$:

$$\begin{array}{ccccc} T & \overset{h}{\twoheadrightarrow} & U & & \\ {\scriptstyle j}\downarrow & & {\scriptstyle dg}\downarrow & \overset{0}{\searrow} & \\ W & \xrightarrow[e]{} & V & \xrightarrow[f]{} & X. \end{array}$$

As $ej = dgh$, there is a unique morphism $i : T \to Z$ such that $ci = j$ and $ai = gh$. □

§ 296. Page 181, the Five Lemma (minor variant of the proof).

Theorem 297 (Lemma 8.3.13 p. 181, Five Lemma)**.** *Consider the commutative diagram of complexes*

$$\begin{array}{ccccccc} X^0 & \xrightarrow{a^0} & X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 \\ {\scriptstyle f^0}\big\downarrow\!\!\!\downarrow & & {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} \\ Y^0 & \xrightarrow[b^0]{} & Y^1 & \xrightarrow[b^1]{} & Y^2 & \xrightarrow[b^2]{} & Y^3, \end{array}$$

where f^0 is an epimorphism, f^1 and f^3 are monomorphisms, and $X^1 \to X^2 \to X^3$ and $Y^0 \to Y^1 \to Y^2$ are exact. Then f^2 is a monomorphism.

Proof. Note that Equivalence (a)⇔(b) in Lemma 292 p. 178 can be stated as follows:

$(*)$ $f : X \to Y$ is an epimorphism if and only if any subobject of Y is the image of some subobject of X.

We write fx for the image of a subobject x of X, and fg for $f \circ g$.

Put $x^2 := \operatorname{Ker} f^2$. Using $(*)$ we see that there is:

- a subobject x^1 of X^1 such that $x^2 = a^1x^1$ (because f^3 is a monomorphism, $f^3a^2x^2 = 0$, and $X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} X^3$ is exact),
- a subobject y^0 of Y^0 such that $f^1x^1 = b^0y^0$ (because $b^1f^1x^1 = 0$ and $Y^0 \xrightarrow{b^0} Y^1 \xrightarrow{b^1} Y$ is exact), and
- a subobject x^0 of X^0 such that $y^0 = f^0x^0$ (because f^0 is an epimorphism).

This yields

$$f^1a^0x^0 = b^0f^0x^0 = b^0y^0 = f^1x^1,$$

implying $a^0x^0 = x^1$ (because f^1 is a monomorphism), and thus

$$0 = a^1a^0x^0 = a^1x^1 = x^2.$$

□

§ 298. P. 181, Lemma 8.3.13 (Five Lemma). We spell out the dual of Theorem 297 above.

Theorem 299. *Consider the commutative diagram of complexes*

$$\begin{array}{ccccccc}
X^0 & \xrightarrow{a^0} & X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 \\
{\scriptstyle f^0}\big\downarrow\!\!\downarrow & & {\scriptstyle f^1}\big\downarrow & & \big\downarrow\!\!\downarrow{\scriptstyle f^2} & & \big\downarrow{\scriptstyle f^3} \\
Y^0 & \xrightarrow[b^0]{} & Y^1 & \xrightarrow[b^1]{} & Y^2 & \xrightarrow[b^2]{} & Y^3,
\end{array}$$

where f^0 and f^2 are epimorphisms, f^3 is a monomorphism, and $X^0 \to X^1 \to X^2$ and $Y^1 \to Y^2 \to Y^3$ are exact. Then f^1 is an epimorphism.

§ 300. P. 182, proof of the equivalence (iii)⇔(iv) in Proposition 8.3.14. Here is the statement of the proposition:

Proposition 301 (Proposition 8.3.14 p. 182)**.** *Let* $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ *be a short exact sequence in an abelian category* $\mathcal{C}$*. Then the conditions below are equivalent:*

(i) *there exits* $h : X'' \to X$ *such that* $g \circ h = \mathrm{id}_{X''}$*,*

(ii) *there exits* $k : X \to X'$ *such that* $k \circ f = \mathrm{id}_{X'}$*,*

(iii) *there exits* $h : X'' \to X$ *and* $k : X \to X'$ *such that* $\mathrm{id}_X = f \circ k + h \circ g$*,*

(iv) *there exits* $\varphi = (k, g)$ *and* $\psi = (f, h)$ *such that* $X \xrightarrow{\varphi} X' \oplus X''$ *and* $X' \oplus X'' \xrightarrow{\psi} X$ *are mutually inverse isomorphisms,*

(v) *for any* Y *in* $\mathcal{C}$*, the map* $\mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{C}}(Y, X'')$ *is surjective,*

(vi) *for any* Y *in* $\mathcal{C}$*, the map* $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X', Y)$ *is surjective.*

The authors say that the equivalence (iii)⇔(iv) is obvious. I agree, but here are a few more details. Implication (iv)⇒(iii) is indeed obvious in the strongest sense of the word. Implication (iii)⇒(iv) can be proved as follows.

Assume (iii), that is, we have morphisms $h : X'' \to X$ and $k : X \to X'$ such that

$$f \circ k + h \circ g = \mathrm{id}_X . \tag{114}$$

As $g \circ f = 0$, this implies

$$g \circ h \circ g = g \circ f \circ k + g \circ h \circ g = g \circ \mathrm{id}_X = g.$$

Since g is an epimorphism, this entails $g \circ h = \mathrm{id}_{X''}$. We prove similarly $k \circ f = \mathrm{id}_{X'}$. Let us record the two above equalities:

$$g \circ h = \mathrm{id}_{X''}, \quad k \circ f = \mathrm{id}_{X'} . \tag{115}$$

Now (114) and (115) imply

$$k \circ h = k \circ (f \circ k + h \circ g) \circ h = k \circ f \circ k \circ h + k \circ h \circ g \circ h = k \circ h + k \circ h,$$

and thus

$$k \circ h = 0, \tag{116}$$

and (iv) follows from (114), (115) and (116). q.e.d.

§ 302. P. 183. Here is an example showing that filtrant and cofiltrant small projective limits of R-modules are not exact in general:

$$\lim_{n\in\mathbb{N}} \big(\mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z} \to 0\big) = \big(\mathbb{Z} \to \mathbb{Z}_2 \to 0\big).$$

§ 303. P. 184, Definitions 8.3.21 (v) and (vi). See § 11 p. 15.

10.3.5 Proof of Lemma 8.3.23 p. 184

In the book, the proofs of the two lemmas below are left to the reader.

Lemma 304. *If*

$$\begin{array}{ccccccccc}
 & & & & Y & \xrightarrow{u} & Y'' & \longrightarrow & 0 \\
 & & & & {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & Y' & \xrightarrow[a]{} & X & \xrightarrow[b]{} & Y'' & \longrightarrow & 0
\end{array}$$

is an exact commutative diagram in an abelian category, then $Y \oplus Y' \xrightarrow{(c,a)} X$ *is an epimorphism.*

Proof. In this proof below we write $\psi\varphi$ for $\psi \circ \varphi$, and we tacitly use Lemma 292 p. 178.

Let $x : Z \to X$ and let us show that the solid diagram

$$\begin{array}{ccc}
W & \overset{d}{\dashrightarrow\!\!\!\!\to} & Z \\
{\scriptstyle e}\,\vdots\downarrow & & \downarrow{\scriptstyle x} \\
Y \oplus Y' & \xrightarrow[(c,a)]{} & X
\end{array}$$

may be completed as indicated. We get a commutative square

$$\begin{array}{ccc}
V & \overset{f}{\twoheadrightarrow} & Z \\
{\scriptstyle y}\downarrow & & \downarrow{\scriptstyle bx} \\
Y & \underset{bc}{\twoheadrightarrow} & Y''
\end{array}$$

and then a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{g} & V & & \\ {\scriptstyle y'}\downarrow & & \downarrow{\scriptstyle xf-cy} & & \\ Y' & \xrightarrow[a]{} & X & \xrightarrow[b]{} & Y''. \end{array}$$

Setting $d := fg, e := (yg, y')$ yields

$$(c, a)e = (c, a)(yg, y') = cyg + ay' = cyg + xfg - cyg = xfg = xd.$$

□

Lemma 305. *If*

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Z & \xrightarrow{a} & Y & & \\ & & & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & X & \xrightarrow{d} & W & \xrightarrow{e} & V & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \xrightarrow[h]{} & U & \xrightarrow[i]{} & T & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

is an exact commutative diagram in an abelian category, then a is an isomorphism.

Proof. In this proof below we write $\psi\varphi$ for $\psi \circ \varphi$, and we tacitly use Lemma 292 p. 178.

We claim

$$a \text{ is a monomorphism.} \tag{117}$$

Let $z : S \to Z$ satisfy $az = 0$, and let us show $z = 0$. We have $ebz = 0$, and the solid diagram

$$\begin{array}{ccccc} R & \overset{j}{\dashrightarrow} & S & & \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle bz} & & \\ X & \xrightarrow[d]{} & W & \xrightarrow[e]{} & V \end{array}$$

may be completed as indicated, yielding successively $hx = fdx = fbzj = 0$, $x = 0$, $bzj = 0$, $bz = 0$, $z = 0$. This proves (117).

We claim

$$a \text{ is an epimorphism.} \tag{118}$$

Let $y : S \to Y$ and let us show that the solid diagram

$$\begin{array}{ccc} R & \overset{j}{\dashrightarrow\!\!\!\!\to} & S \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle y} \\ Z & \underset{a}{\longrightarrow} & Y \end{array}$$

may be completed as indicated. We get successively: a commutative square

$$\begin{array}{ccc} Q & \overset{k}{\twoheadrightarrow} & S \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle cy} \\ W & \underset{e}{\twoheadrightarrow} & V; \end{array}$$

equalities $ifw = gew = gcyk = 0$; an exact commutative diagram

$$\begin{array}{ccccc} P & \overset{\ell}{\twoheadrightarrow} & Q & & \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle fw} & & \\ W & \underset{h}{\longrightarrow} & V & \underset{i}{\longrightarrow} & T; \end{array}$$

equalities $fdx = hx = fw\ell$, $f(w\ell - dx) = 0$; an exact commutative diagram

$$\begin{array}{ccccc} R & \overset{m}{\twoheadrightarrow} & P & & \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle w\ell - dx} & & \\ Z & \underset{b}{\longrightarrow} & W & \underset{f}{\longrightarrow} & U; \end{array}$$

equalities $caz = ebz = ew\ell m - edxm = ew\ell m = cyk\ell m$, $az = yk\ell m$; and it suffices to set $j := k\ell m$. This proves (118), and thus our lemma. □

10.3.6 Brief comments

§ 306. On p. 185 we read:

"Recall (see Proposition 5.2.4) that in an abelian category, the conditions below are equivalent:

(i) G is a generator, that is, the functor $\varphi_G = \mathrm{Hom}_{\mathcal{C}}(G,\ )$ is conservative,

(ii) The functor φ_G is faithful.

...

Moreover, if $\mathcal{C}$ admits small inductive limits, the conditions above are equivalent to:

(iii) for any $X \in \mathcal{C}$, there exist a small set I and an epimorphism $G^{\sqcup I} \twoheadrightarrow X$."

It would be better (I think) to refer to Proposition 2.2.3 p. 45 of the book for the equivalence between (i) and (ii).

§ 307. P. 186, Definition 8.3.24 (definition of a Grothendieck category). The condition that small filtrant inductive limits are exact is not automatic. I know no entirely elementary proof of this important fact. Here is a proof using a little bit of sheaf theory. To show that there is an abelian category where small filtrant inductive limits exist but are not exact, it suffices to prove that there is an abelian category $\mathcal{C}$ where small filtrant *projective* limits exist but are not exact. It is even enough to show that small products are not exact in $\mathcal{C}$. Let X be a topological space, and let $U_0 \supset U_1 \supset \cdots$ be a decreasing sequence of open subsets whose intersection is a non-open closed singleton $\{a\}$. We can take for $\mathcal{C}$ the category of small abelian sheaves on X. To see this, let G be the abelian presheaf over X such that $G(U)$ is $\mathbb{Z}$ if a is in U and 0 otherwise, and, for each n in $\mathbb{N}$, let F_n be the abelian presheaf over X such that $F_n(U)$ is $\mathbb{Z}$ if $U \subset U_n$ and 0 otherwise. These presheaves are easily seen to be sheaves. For each n in $\mathbb{N}$ and each open set U let $F_n(U) \to G(U)$ be the identity if a is in $U \subset U_n$ and 0 otherwise. This family of morphisms defines, when U varies, an epimorphism $\varphi_n : F_n \twoheadrightarrow G$. Put

$$F := \prod_{n\in\mathbb{N}} F_n, \quad H := \prod_{n\in\mathbb{N}} G, \quad \varphi := \prod_{n\in\mathbb{N}} \varphi_n : F \to H.$$

It suffices to show that the morphism $\varphi(a) : F(a) \to H(a)$ between the stalks at a induced by φ is not an epimorphism. This is clear because $\varphi(a)$ is the natural morphism

$$\bigoplus_{n\in\mathbb{N}} \mathbb{Z} \to \prod_{n\in\mathbb{N}} \mathbb{Z}.$$

q.e.d.

§ 308. Recall the statement of Corollary 8.3.26 p. 186 of the book:

Let $\mathcal{C}$ be a Grothendieck category and let $X \in \mathcal{C}$. Then the set of quotients of X and the the set of subobjects of X are small.

The proof is phrased as follows: "Apply Proposition 5.2.9". One could add "...and Proposition 5.2.3 (v)".

§ 309. P. 186. Proposition 8.3.27 will be used to prove Corollary 381 p. 238 below (which is Corollary 9.6.6 p. 237 of the book), Corollary 14.4.6 (i) p. 361 and Corollary 14.4.9 p. 365 of the book. See also §310 below.

§ 310. P. 186. By Proposition 8.3.27 (i) and Lemma 3.3.9 p. 83 of the book, in a Grothendieck $\mathcal{U}$-category $\mathcal{U}$-small (Definition 5 p. 10) filtrant inductive limits are stable by base change (see Section 4.6 p. 63).

10.4 About Section 8.4

This is about Proposition 8.4.7 p. 187. Let us just rewrite in a slightly less concise way the part of the proof on p. 188 which starts with the sentence "Define $Y := Y_0 \times_X G_i$" at the fifth line of the last paragraph of the proof, and goes to the end of the proof.

It suffices to show that there is a morphism $a_0 : G_i \to Y_0$ satisfying $l_0 \circ a_0 = \varphi$:

$$\begin{array}{ccccc} X' & \xrightarrow{k_0} & Y_0 & \xrightarrow{l_0} & X \\ {\scriptstyle h}\downarrow & \swarrow_{g_0} & & \nwarrow^{a_0} & \uparrow{\scriptstyle \varphi} \\ Z & & & & G_i. \end{array}$$

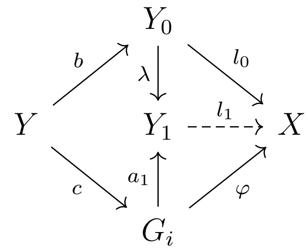
Form the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{b} & Y_0 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle l_0} \\ G_i & \xrightarrow[\varphi]{} & X, \end{array}$$

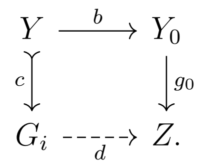
and the cocartesian square

$$\begin{array}{ccc} Y & \xrightarrow{b} & Y_0 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \lambda} \\ G_i & \xrightarrow[a_1]{} & Y_1. \end{array}$$

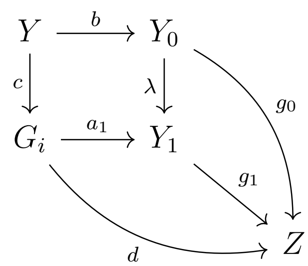
Let $l_1 : Y_1 \to X$ be the morphism which makes the diagram



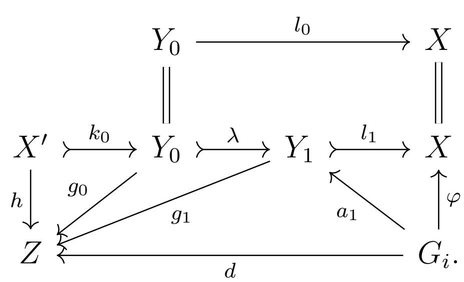
commutative. By Lemma 290 (a) (i) p. 178, c is a monomorphism, and, by Part (b) (ii) of the same lemma, λ is also a monomorphism. As Z is injective, there is a morphism $d : G_i \to Z$ satisfying $d \circ c = g_0 \circ b$:



By the definition of Y_1 there is a morphism $g_1 : Y_1 \to Z$ such that



commutes. We get the commutative diagram



As λ is an isomorphism by maximality of (Y_0, g_0, l_0), we can set $a_0 := \lambda^{-1} \circ a_1$, and we get

$$l_0 \circ a_0 = l_0 \circ \lambda^{-1} \circ a_1 = l_1 \circ \lambda \circ \lambda^{-1} \circ a_1 = l_1 \circ a_1 = \varphi.$$

q.e.d.

10.5 About Section 8.5

10.5.1 Brief comments

§ 311. P. 190, Proposition 8.5.5. It might be worth writing explicitly the formulas (for $X, Y \in \mathrm{Mod}(R, \mathcal{C})$):

$$\mathrm{Hom}_{R^{\mathrm{op}}}(N, \mathrm{Hom}_{\mathcal{C}}(X, Y)) \simeq \mathrm{Hom}_{\mathcal{C}}\left(N \otimes_R X, Y\right),$$

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{C}}(Y, X)) \simeq \mathrm{Hom}_{\mathcal{C}}\left(Y, \mathrm{Hom}_R(M, X)\right),$$

$$R^{\mathrm{op}} \otimes_R X \simeq X,$$

$$\mathrm{Hom}_R(R, X) \simeq X.$$

One could also mention explicitly the adjunctions

$$\begin{array}{ccc} \mathrm{Mod}(R^{\mathrm{op}}) & \qquad & \mathrm{Mod}(R)^{\mathrm{op}} \\ {\scriptstyle -\otimes_R X}\downarrow\uparrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(X,-)} & & {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(-,X)}\downarrow\uparrow{\scriptstyle \mathrm{Hom}_R(-,X)} \\ \mathcal{C} & & \mathcal{C}, \end{array}$$

where, we hope, the notation is self-explanatory.

§ 312. P. 191, proof of Theorem 8.5.8 (iii) (minor variant). Recall the statement:

Proposition 313 (Theorem 8.5.8 (iii) p. 191)**.** *Let $\mathcal{C}$ be a Grothendieck category, let G be a generator, let R be the ring $\mathrm{End}_{\mathcal{C}}(G)^{\mathrm{op}}$, put $\mathcal{M} := \mathrm{Mod}(R)$, let $\varphi : \mathcal{C} \to \mathcal{M}$ be the functor defined by $\varphi(X) := \mathrm{Hom}_{\mathcal{C}}(G, X)$. Then φ is fully faithful.*

Proof. Let $\psi : \mathcal{M} \to \mathcal{C}$ be the functor defined by $\psi(M) := G \otimes_R M$, let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ whose objects are

$$0, \quad G, \quad G \oplus G, \quad G \oplus G \oplus G, \quad \dots,$$

and let $\mathcal{M}_0$ be the full subcategory of $\mathcal{M}$ whose objects are

$$0, \quad R, \quad R \oplus R, \quad R \oplus R \oplus R, \quad \dots$$

Then φ and ψ induce mutually quasi-inverse equivalences

$$\mathcal{C}_0 \underset{\psi_0}{\overset{\varphi_0}{\rightleftarrows}} \mathcal{M}_0.$$

We can assume that $\mathcal{C}_0$ and $\mathcal{M}_0$ are small (Definition 5 p. 10). If $\lambda : \mathcal{C} \to (\mathcal{C}_0)^\wedge$ and $\lambda' : \mathcal{M} \to (\mathcal{M}_0)^\wedge$ are the obvious functors, then the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \mathcal{M} \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda'} \\ (\mathcal{C}_0)^\wedge & \xrightarrow[\widehat{\varphi}_0]{} & (\mathcal{M}_0)^\wedge \end{array}$$

quasi-commutes. The functors λ and λ' are fully faithful by §310 p. 187 and Theorem 204 p. 122 above. As $\widehat{\varphi}_0$ is an equivalence (a quasi-inverse being $\widehat{\psi}_0$, see Proposition 2.7.1 p. 62 in the book), the proof is complete. □

10.5.2 Theorem 8.5.8 (iv) p. 191

Here is a minor variant of Step (a) of the proof of Theorem 8.5.8 (iv). Recall the statement:

Lemma 314. *In the setting of Proposition 313, assume that there is a finite set F, an epimorphism $R^F \twoheadrightarrow M$ in $\mathcal{M}$, a small set S, and a monomorphism $M \rightarrowtail R^{\oplus S}$. Let $\psi : \mathcal{M} \to \mathcal{C}$ be the functor defined by $\psi(M) := G \otimes_R M$. Then $\psi(M) \to \psi(R^{\oplus S})$ is a monomorphism.*

Proof. There is a finite subset F' of S such that $M \rightarrowtail R^{\oplus S}$ factors as

$$M \rightarrowtail R^{F'} \rightarrowtail R^{\oplus S}.$$

As $R^{F'}$ is a direct summand of $R^{\oplus S}$, the morphism $\psi(R^{F'}) \to \psi(R^{\oplus S})$ is a monomorphism. In other words, we may assume $S = F'$, and it suffices to check that $\psi(M) \to \psi(R^{F'})$ is a monomorphism, or, more explicitly, that

$$f : \psi(M) \to G^{F'} \text{ is a monomorphism.} \tag{119}$$

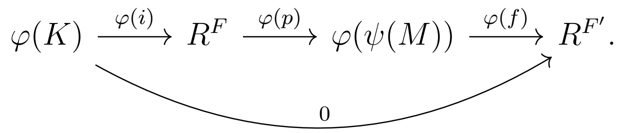
Applying the right exact functor ψ to

$$R^F \twoheadrightarrow M \rightarrowtail R^{F'},$$

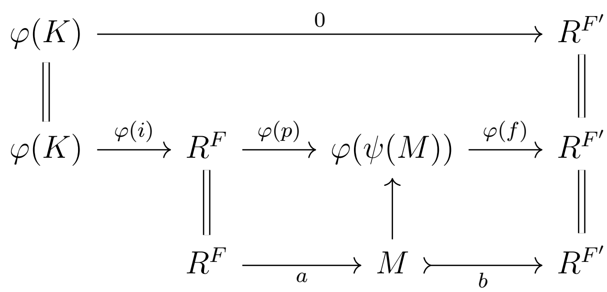
we get

$$K \overset{i}{\rightarrowtail} G^F \overset{p}{\twoheadrightarrow} \psi(M) \xrightarrow{f} G^{F'}, \qquad (K \xrightarrow{0} G^{F'})$$

where $K := \mathrm{Ker}(f \circ p)$. Applying φ we obtain



The commutative diagram



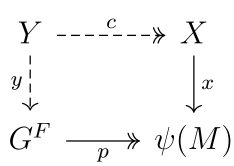
yields $b \circ a \circ \varphi(i) = 0$. As b is a monomorphism, we get $a \circ \varphi(i) = 0$, and thus $\varphi(p) \circ \varphi(i) = 0$. Since φ is faithful by Proposition 313 p. 189, this implies

$$p \circ i = 0. \tag{120}$$

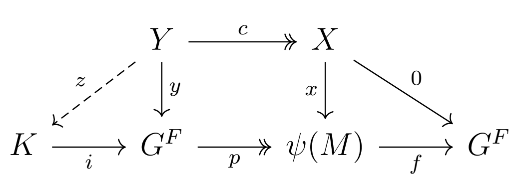
Let us prove (119). Let $x : X \to \psi(M)$ be a morphism in $\mathcal{C}$ satisfying $f \circ x = 0$. It suffices to prove

$$x = 0. \tag{121}$$

As p is an epimorphism, the diagram of solid arrows



can be completed, by Lemma 290 (b) (i) p. 178, to a commutative square as indicated, c being an epimorphism. The commutative diagram of solid arrows



can in turn be completed to a commutative diagram as indicated, and we get

$$x \circ c = p \circ i \circ z = 0$$

by (120). As c is an epimorphism, this implies successively (121), (119) and the lemma. □

10.6 About Section 8.6

§ 315. P. 193, second sentence of Section 8.6. The proof of the following statement is straightforward.

Let $\mathcal{C}$ and $\mathcal{C}'$ be pre-additive categories, let $\mathcal{C}'^{\mathcal{C}}$ be the category of functors from $\mathcal{C}$ to $\mathcal{C}'$, and let $\alpha : I \to \mathcal{C}'^{\mathcal{C}}$ be a functor. Assume that $\alpha(i)$ is additive for all i in I, and that the colimit $\operatorname{colim} \alpha$ exists in $\mathcal{C}'^{\mathcal{C}}$. Then $\operatorname{colim} \alpha$ is additive. There is a similar statement for limits.

§ 316. P. 193. Just before the statement of Proposition 8.6.2 it is claimed that the inclusion

$$\operatorname{Ind}(\mathcal{C}) \subset \mathcal{C}^{\wedge,add}$$

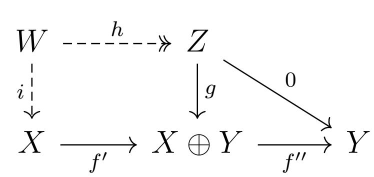
holds. This inclusion follows from Propositions 3.3.3 p. 82 (see Proposition 161 p. 105), 6.1.7 p. 132, 8.2.13 p. 173 (see §286 p. 173) 8.2.15 p. 173 in the book.

§ 317. P. 194, Theorem 8.6.5 (ii). See §223 p. 140.

§ 318. Proof of Lemma 8.6.7 p. 195. The proof uses the following lemma:

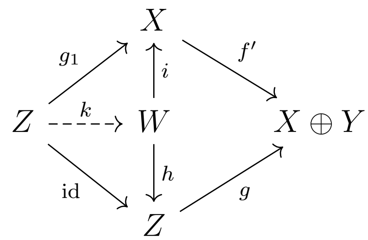
Lemma 319. *Let $f : X \to Y$ be a morphism in an abelian category, define $f' : X \to X \oplus Y$ and $f'' : X \oplus Y \to Y$ by $f' := \left[\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right]$, $f'' := [f \ \ -1]$ (obvious notation). Then the sequence $X \xrightarrow{f'} X \oplus Y \xrightarrow{f''} Y$ is exact.*

Proof. It suffices to show that an arbitrary solid commutative diagram



may be completed as indicated. Let the above square be cartesian. Note that g is of the form $\left[\begin{smallmatrix} g_1 \\ f \circ g_1 \end{smallmatrix}\right]$ with $g_1 : Z \to X$. It suffices to show that h is an epimorphism. Let

$j : Z \to V$ be a morphism such that $j \circ h = 0$. It suffices to prove $j = 0$. The solid commutative diagram



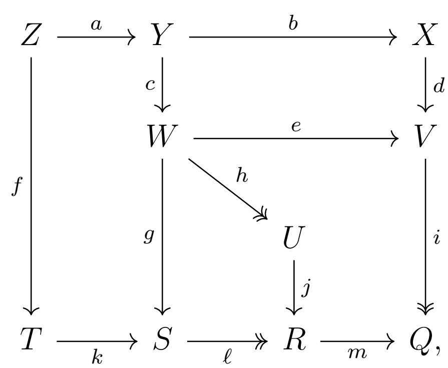
may be completed as indicated, yielding $0 = j \circ h \circ k = j$. □

§ 320. P. 197, proof of Proposition 8.6.12. The existence of the epimorphism $X_1 \to Y''$ follows from Proposition 8.6.9 in the book, and the statement "Since the top square on the left is co-Cartesian, the middle row is exact" follows from the Lemma 294 p. 179.

10.7 About Section 8.7

10.7.1 Lemma 8.7.3 p. 198

Let us spell out the proof of the fact that $K(\alpha)$ is a monomorphism. Consider the commutative diagram



where the five rectangles are cartesian, and the three sequences

$$T \to S \to R \to 0, \quad X \to V \to Q \to 0, \quad Z \to W \to U \to 0$$

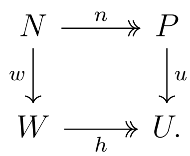
are exact.

We must check the j is a monomorphism.

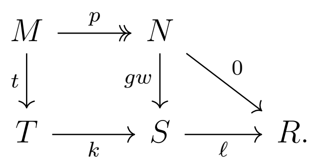
(In the proof below we omit the composition symbols $\circ$ and most of the parenthesis. We shall freely use the equivalence (a)$\Leftrightarrow$(b) in Lemma 292 p. 178.)

Let $u : P \to U$ satisfy $ju = 0$, and let us show $u = 0$.

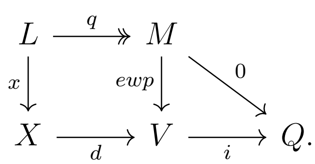
There is a commutative square



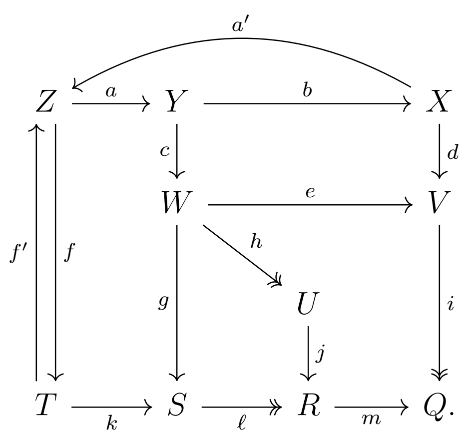
As $\ell gw = jhw = jun = 0$, there is a commutative diagram



As $iewp = m\ell gwp = m\ell kt = 0$, there is a commutative diagram



As $Z \simeq T \oplus X$, we can introduce the coprojections a' and f' indicated below:



Define $z : L \to Z$ by $z := f'tq + a'x$. We claim

$$caz = wpq. \tag{122}$$

It suffices to verify

$$gcaz = gwpq, \quad ecaz = ewpq.$$

The proof of the two above equalities is straightforward and left to the reader, so that we consider that (122) has been proved. We get

$$unpq = hwpq = hcaz = 0.$$

As pq is an epimorphism, this implies $u = 0$, as desired. q.e.d.

10.7.2 Lemma 8.7.5 (i) p. 199

Let us spell out the proof of the fact that $\operatorname{Coker}(u) \to \operatorname{Coker}(v)$ is a monomorphism. We shall use the same notation and arguments as in Section 10.7.1 p. 193.

In the commutative diagram

$$\begin{array}{ccc}
Y' & & \\
{\scriptstyle a}\big\downarrow & & \\
Z & \xrightarrow{\;q\;} & Y \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
X' & \xrightarrow{\;b\;} & X \\
{\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle d} \\
W' & \xrightarrow[\;e\;]{} & W,
\end{array}$$

the square $ZYXX'$ is cartesian and the sequences

$$Y \to X \to W \to 0, \quad Y' \to X' \to W' \to 0$$

are exact.

We must show that e is an isomorphism. Clearly e is an epimorphism. It suffices to prove that e is a monomorphism. Let $w' : V \to W'$ satisfy $ew' = 0$. It suffices to prove $w' = 0$.

Form the commutative square

$$\begin{array}{ccc} U & \overset{f}{\twoheadrightarrow} & V \\ {\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle w'} \\ X' & \underset{c}{\twoheadrightarrow} & W'. \end{array}$$

As we have $dbx' = ecx' = ew'f = 0$, we can form the commutative diagram

$$\begin{array}{ccccc} T & \overset{g}{\twoheadrightarrow} & U & & \\ {\scriptstyle y}\downarrow & & {\scriptstyle bx'}\downarrow & \overset{0}{\searrow} & \\ Y & \underset{u}{\longrightarrow} & X & \underset{d}{\longrightarrow} & W. \end{array}$$

As we have $bx'g = uy$, we get a morphism $z : T \to Z$ such that $pz = x'g$ and $qz = y$, and we can form the commutative square

$$\begin{array}{ccc} S & \overset{h}{\twoheadrightarrow} & T \\ {\scriptstyle y'}\downarrow & & \downarrow{\scriptstyle z} \\ Y' & \underset{a}{\twoheadrightarrow} & Z. \end{array}$$

This yields $w'fgh = cx'gh = cpzh = cpay' = 0$. As f, g and h are epimorphisms, this implies $w' = 0$, as desired. q.e.d.

Here is a second version:

P. 199, proof of Lemma 8.7.5 (i). As we have

$$\mathrm{Coker}(Y \times_X X' \to X') \xrightarrow{\sim} \mathrm{Coker}(u)$$

by Lemma 113 p. 178, we can assume $X \in \mathcal{J}$. Let $b : Y' \twoheadrightarrow Y$ be an epimorphism with $Y' \in \mathcal{J}$, and set $v := ub$, $W := \mathrm{Coker}(v)$, $Z := \mathrm{Coker}(u)$:

$$\begin{array}{ccccccc} Y' & \overset{v}{\longrightarrow} & X & \overset{a}{\longrightarrow} & W & \longrightarrow & 0 \\ {\scriptstyle b}\downarrow\!\!\downarrow & & \| & & \downarrow{\scriptstyle c} & & \\ Y & \underset{u}{\longrightarrow} & X & \underset{d}{\longrightarrow} & Z & \longrightarrow & 0. \end{array}$$

(The above diagram commutes and the rows are exact.) We shall use Lemma 292 p. 178. We must show that c is an isomorphism. Clearly c is an epimorphism. It suffices to prove that c is a monomorphism. Let $w : T \to W$ satisfy $cw = 0$. It suffices to show $w = 0$. There are commutative diagrams with exact rows and equalities

$$\begin{array}{ccccc} R & \xrightarrow{t} & T & & \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle w} & & \\ X & \xrightarrow[a]{} & W & \longrightarrow & 0, \end{array}$$

$$0 = cwt = cax = dx,$$

$$\begin{array}{ccccc} Q & \xrightarrow{r} & R & & \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} & & \\ Y & \xrightarrow[u]{} & X & \xrightarrow[d]{} & Z, \end{array}$$

$$\begin{array}{ccccc} P & \xrightarrow{q} & Q & & \\ {\scriptstyle y'}\downarrow & & \downarrow{\scriptstyle y} & & \\ Y' & \xrightarrow[b]{} & Y & \longrightarrow & 0, \end{array}$$

$$wtrq = axrq = auyq = auby' = avy' = 0.$$

As t, r and q are epimorphisms, this implies $w = 0$, as desired. q.e.d.

10.7.3 Proof of (8.7.3) p. 200

Right after (8.7.4) we read

"The condition that $K(\alpha)$ is an isomorphism is equivalent to the fact that the sequence $Y \to X \oplus Y' \to X' \to 0$ is exact."

It seems to me we get a counterexample by setting $0 \simeq Y \simeq X \simeq X' \not\simeq Y'$, but we can prove

(8.7.3) for $\alpha : u \to v$ in $\mathrm{Mor}(\mathcal{D}_0)$, if $K(\alpha)$ is an isomorphism, then $A'(\alpha)$ is an isomorphism

as follows:

Let $\alpha : u \to v$ in $\mathrm{Mor}(\mathcal{D}_0)$ be such that $K(\alpha)$ is an isomorphism. By Proposition 8.3.18 p. 183, the diagram below commutes:

$$\begin{array}{ccc}
A'(u) & \xrightarrow{A'(\alpha)} & A'(v) \\
\| & & \| \\
\operatorname{Coker} A(u) & \xrightarrow{\operatorname{Coker} A(\alpha)} & \operatorname{Coker} A(v) \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
A(K(u)) & \xrightarrow[\sim]{A(K(\alpha))} & A(K(v)).
\end{array}$$

10.7.4 Proof of (8.7.2) p. 200

Let us spell out the proof of the claim

"The condition $K(\alpha) = 0$ implies that $X \times_{X'} Y' \to X$ is an epimorphism."

(This is the third sentence of the last paragraph.)

We shall use the same notation and arguments as in Section 10.7.1 p. 193.

We have the commutative square with exact columns

$$\begin{array}{ccc}
Y & \xrightarrow{\alpha_0} & Y' \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
X & \xrightarrow{\alpha_1} & X' \\
{\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v'} \\
Ku & \xrightarrow{0} & Kv \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

Set $Z := X \times_{X'} Y'$ and write $p : Z \to X$ and $q : Z \to Y'$ for the projections, so that we must show that p is an epimorphism. Let $x : W \to X$ be given. It suffices to complete the solid diagram

$$\begin{array}{ccc}
V & \overset{a}{\dashrightarrow\!\!\!\!\rightarrow} & W \\
{\scriptstyle z}\,\vdots\downarrow & & \downarrow{\scriptstyle x} \\
Z & \xrightarrow[p]{} & X
\end{array}$$

as indicated. As $v'\alpha_1 x = 0$, we get the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{a} & W & & \\ {\scriptstyle y'}\downarrow & & \downarrow{\scriptstyle \alpha_1 x} & \searrow^{0} & \\ Y' & \xrightarrow[v]{} & X' & \xrightarrow[v']{} & Kv. \end{array}$$

As $\alpha_1 xa = vy'$ we get the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{y'} & Y' \\ {\scriptstyle xa}\downarrow & \searrow^{z} & \uparrow{\scriptstyle q} \\ X & \xleftarrow[p]{} & Z. \end{array}$$

10.7.5 Commutativity of the last diagram p. 200

I failed to prove that the triangle $Y_1 Y' X_1$ commutes, but it seems to me that this is not needed.

10.7.6 Proof of Lemma 8.7.7 p. 201

The last sentence of the proof of Lemma 8.7.7 uses Exercise 8.19 p. 204 (see Section 10.8.3 p. 207 below).

(In the rest of this section we omit the composition symbols $\circ$ and most of the parenthesis, we freely use Lemma 290 p. 178 and Lemma 292 p. 178, and we let the setting of Lemma 8.7.7 of the book be in force.)

Lemma 321. *If $Z \xrightarrow{a} Y \xrightarrow{b} X \to 0$ is an exact sequence in $\mathcal{C}$, then there is an exact commutative diagram*

$$\begin{array}{ccccccc} T & \xrightarrow{k} & S & \xrightarrow{\ell} & R & \longrightarrow & 0 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\ W & \xrightarrow{f} & V & \xrightarrow{g} & U & \longrightarrow & 0 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} & & \\ Z & \xrightarrow[a]{} & Y & \xrightarrow[b]{} & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array} \qquad (123)$$

with R, S, T, U, V, W in $\mathcal{J}$.

Lemma 322. *The solid diagram in $\mathcal{C}$ below can be completed as indicated to a commutative diagram in $\mathcal{C}$ with Z in $\mathcal{J}$:*

$$\begin{array}{ccccc} \operatorname{Ker} b & \overset{a}{\dashrightarrow} & Z & \overset{b}{\dashrightarrow} & Y \\ {\scriptstyle c}\,\vdots\!\downarrow & & \vdots\!\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ \operatorname{Ker} g & \underset{f}{\rightarrowtail} & X & \underset{g}{\twoheadrightarrow} & W. \end{array}$$

Proof of Lemma 322. Form the cartesian square

$$\begin{array}{ccc} V & \overset{h}{\longrightarrow} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle e} \\ X & \underset{g}{\twoheadrightarrow} & W. \end{array}$$

Note that h and i are epimorphisms. Let $Z \overset{j}{\twoheadrightarrow} V$ be an epimorphism in $\mathcal{C}$ with Z in $\mathcal{J}$. We get the commutative diagram

$$\begin{array}{ccccc} \operatorname{Ker} b & \overset{a}{\rightarrowtail} & Z & & \\ & & {\scriptstyle j}\downarrow & \overset{b}{\searrow} & \\ {\scriptstyle c}\downarrow & & V & \underset{h}{\twoheadrightarrow} & Y \\ & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle e} \\ \operatorname{Ker} g & \underset{f}{\rightarrowtail} & X & \underset{g}{\twoheadrightarrow} & W. \end{array}$$

Set $d := ij$. It only remains to check that c is an epimorphism. Let $x : U \to \operatorname{Ker} g$. It suffices to complete the solid diagram

$$\begin{array}{ccc} T & \overset{k}{\dashrightarrow} & U \\ {\scriptstyle z}\,\vdots\!\downarrow & & \downarrow{\scriptstyle x} \\ \operatorname{Ker} b & \underset{c}{\longrightarrow} & \operatorname{Ker} g \end{array}$$

as indicated. There is a morphism $v : U \to V$ such that

$$\begin{array}{ccc} & & Y \\ & \overset{0}{\nearrow} & \uparrow{\scriptstyle h} \\ U & \overset{v}{\longrightarrow} & V \\ & \underset{fx}{\searrow} & \downarrow{\scriptstyle i} \\ & & X \end{array}$$

commutes, and there are morphisms k and z' such that

$$\begin{array}{ccc} T & \overset{k}{\twoheadrightarrow} & U \\ {\scriptstyle z'}\downarrow & & \downarrow{\scriptstyle v} \\ Z & \underset{j}{\twoheadrightarrow} & V \end{array}$$

commutes. As we have $bz' = hjz' = hvk = 0$, there is a morphism $z : T \to \operatorname{Ker} b$ such that $az = z'$, and we get $fcz = ijaz = ijz' = ivk = fxk$. Since f is a monomorphism, this yields $cz = xk$, as desired. □

The above proof shows that we have in fact:

Lemma 323. *The solid diagram in $\mathcal{C}$ below can be completed as indicated to a commutative diagram in $\mathcal{C}$ with Z in $\mathcal{J}$:*

$$\begin{array}{ccccc} & & \operatorname{Ker} b & \dashrightarrow\!\!\!\!\rightarrow & \operatorname{Ker} c \\ & & \text{⤚}\!\!\downarrow & & \text{⤚}\!\!\downarrow \\ \operatorname{Ker} a & \rightarrowtail\!\!\dashrightarrow & Z & \overset{a}{\dashrightarrow\!\!\!\!\rightarrow} & Y \\ \downarrow\!\downarrow & & {\scriptstyle b}\downarrow\!\downarrow & & \downarrow\!\downarrow{\scriptstyle c} \\ \operatorname{Ker} d & \rightarrowtail & X & \underset{d}{\twoheadrightarrow} & W. \end{array}$$

Proof of Lemma 321. Let $e : U \twoheadrightarrow X$ be an epimorphism in $\mathcal{C}$ with U in $\mathcal{J}$. A first application of Lemma 323 gives a commutative diagram

$$\begin{array}{ccccccc} & & & & \operatorname{Ker} d & \overset{m}{\twoheadrightarrow} & \operatorname{Ker} e \\ & & & & \downarrow & & \downarrow \\ & & \operatorname{Ker} g & \rightarrowtail & V & \overset{g}{\twoheadrightarrow} & U \\ & & \downarrow\!\downarrow & & \downarrow\!\downarrow{\scriptstyle d} & & \downarrow\!\downarrow{\scriptstyle e} \\ Z & \twoheadrightarrow & \operatorname{Ker} b & \rightarrowtail & Y & \underset{b}{\twoheadrightarrow} & X \end{array}$$

with V in $\mathcal{J}$. A second application of Lemma 323 gives a commutative diagram

$$\begin{array}{ccccccc} & & & & \operatorname{Ker} d & \overset{m}{\twoheadrightarrow} & \operatorname{Ker} e \\ & & & & \downarrow & & \downarrow \\ W & \twoheadrightarrow & \operatorname{Ker} g & \rightarrowtail & V & \overset{g}{\twoheadrightarrow} & U \\ {\scriptstyle c}\downarrow\!\downarrow & & \downarrow\!\downarrow & & \downarrow\!\downarrow{\scriptstyle d} & & \downarrow\!\downarrow{\scriptstyle e} \\ Z & \twoheadrightarrow & \operatorname{Ker} b & \rightarrowtail & Y & \underset{b}{\twoheadrightarrow} & X \end{array}$$

with W in $\mathcal{J}$. Let $R \twoheadrightarrow \operatorname{Ker} e$ be an epimorphism in $\mathcal{C}$ with R in $\mathcal{J}$. A third application of Lemma 323 gives a commutative diagram

$$\begin{array}{ccccccc}
 & & \operatorname{Ker} \ell & \rightarrowtail & S & \overset{\ell}{\twoheadrightarrow} & R \\
 & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\
 & & \operatorname{Ker} m & \rightarrowtail & \operatorname{Ker} d & \overset{m}{\twoheadrightarrow} & \operatorname{Ker} e \\
 & & & & \downarrow & & \downarrow \\
W & \twoheadrightarrow & \operatorname{Ker} g & \rightarrowtail & V & \overset{g}{\twoheadrightarrow} & U \\
\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow{\scriptstyle d} & & \downarrow\!\!\!\downarrow{\scriptstyle e} \\
Z & \twoheadrightarrow & \operatorname{Ker} b & \rightarrowtail & Y & \underset{b}{\twoheadrightarrow} & X
\end{array}$$

with S in $\mathcal{J}$. Let $\operatorname{Ker} c \twoheadleftarrow T \twoheadrightarrow \operatorname{Ker} \ell$ be a diagram in $\mathcal{C}$ with T in $\mathcal{J}$. We finally get

$$\begin{array}{ccccccc}
T & \twoheadrightarrow & \operatorname{Ker} \ell & \rightarrowtail & S & \overset{\ell}{\twoheadrightarrow} & R \\
\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\
\operatorname{Ker} c & & \operatorname{Ker} m & \rightarrowtail & \operatorname{Ker} d & \overset{m}{\twoheadrightarrow} & \operatorname{Ker} e \\
\downarrow & & & & \downarrow & & \downarrow \\
W & \twoheadrightarrow & \operatorname{Ker} g & \rightarrowtail & V & \overset{g}{\twoheadrightarrow} & U \\
{\scriptstyle c}\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow{\scriptstyle d} & & \downarrow\!\!\!\downarrow{\scriptstyle e} \\
Z & \twoheadrightarrow & \operatorname{Ker} b & \rightarrowtail & Y & \underset{b}{\twoheadrightarrow} & X,
\end{array}$$

as required. □

10.8 About the exercises

10.8.1 Exercise 8.4 p. 202

Recall the statement:

Let $\mathcal{C}$ be an additive category and $\mathcal{S}$ a right multiplicative system. Prove that the localization $\mathcal{C}_{\mathcal{S}}$ is an additive category and $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is an additive functor.

It is easy to equip $\mathcal{C}_{\mathcal{S}}$ with a pre-additive structure making Q additive. Then the result follows from Corollary 268 p. 168.

The pre-additive structure on $\mathcal{C}_{\mathcal{S}}$ is described in a very detailed way at the beginning of the following text of Dragan Miličić:

http://www.math.utah.edu/∼milicic/Eprints/dercat.pdf

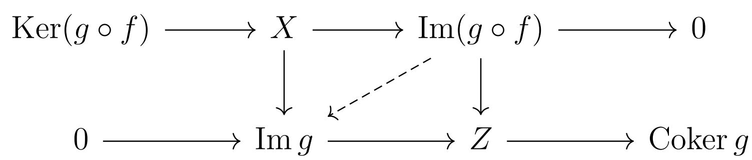
10.8.2 Exercise 8.17 p. 204

Preliminaries

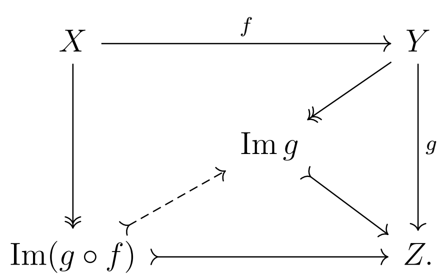
Lemma 324. *If*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{124}$$

are morphisms in an abelian category $\mathcal{C}$ *(we do not assume* $g \circ f = 0$*), then the commutative diagram*

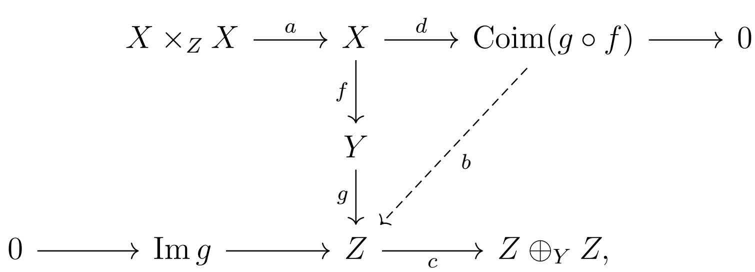


of solid arrows, whose rows are exact sequences, can be completed as indicated. The situation can also be represented as follows:



In particular $\operatorname{Im}(g \circ f) \to \operatorname{Im} g$ *is a monomorphism.*

Proof. We claim that the diagram of solid arrows



whose rows are exact sequences, can be completed as indicated. Indeed, the existence of b follows from the equality $g \circ f \circ a = 0$. To prove the lemma, it is enough to check that b factors through $\operatorname{Im} g$, or, equivalently, that $c \circ b = 0$. As d is an epimorphism, the vanishing of $c \circ b$ is equivalent to the vanishing of $c \circ b \circ d$. But we have $c \circ b \circ d = c \circ g \circ f = 0 \circ f = 0$. $\square$

Lemma 325. *If, in the setting of Lemma 326, f is an epimorphism, then*

$$\operatorname{Im}(g \circ f) \to \operatorname{Im} g$$

is an isomorphism.

Proof. Consider the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle a} \\ \operatorname{Im}(g \circ f) & \xrightarrow[b]{} & \operatorname{Im} g, \end{array}$$

where a and b are the natural morphisms. As f and a are epimorphisms, so is b. $\square$

Exercise 8.17 The exercise follows easily from Lemmas 326 and 327 below.

Let us denote the cokernel of any morphism $h : Y \to Z$ in any abelian category by $Z/\operatorname{Im} h$.

Recall that, by Proposition 8.3.18 p. 183 of the book, an additive functor between abelian categories $F : \mathcal{C} \to \mathcal{C}'$ is left exact if and only if

$$\left.\begin{array}{c} 0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \text{ exact} \\ \Rightarrow \\ 0 \to F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \text{ exact} \end{array}\right\} \tag{125}$$

Consider the condition

$$\left.\begin{array}{c} 0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \text{ exact} \\ \Rightarrow \\ 0 \to F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \text{ exact} \end{array}\right\} \tag{126}$$

Lemma 326. *We have (125)$\Leftrightarrow$(126).*

Proof. Implication $\Rightarrow$ is clear. To prove $\Leftarrow$, let

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X''$$

be exact. We must check that

$$0 \to F(X') \to F(X) \to F(X'') \tag{127}$$

is exact. Let I be the image of g. The sequence

$$0 \to X' \to X \to I \to 0$$

being exact, so is

$$0 \to F(X') \to F(X) \to F(I). \tag{128}$$

This implies that (127) is exact at $F(X')$. The sequence

$$0 \to I \to X'' \to X''/I \to 0$$

being exact, so is

$$0 \to F(I) \to F(X''),$$

and we have

$$\text{Ker}\big(F(X) \to F(I)\big) \xrightarrow{\sim} \text{Ker}\big(F(X) \to F(X'')\big). \tag{129}$$

The exactness of (128) implies

$$\text{Im}\big(F(X') \to F(X)\big) \xrightarrow{\sim} \text{Ker}\big(F(X) \to F(I)\big), \tag{130}$$

and the exactness of (127) at $F(X)$ follows from (129) and (130). □

Consider the conditions below on our additive functor $F : \mathcal{C} \to \mathcal{C}'$:

$$\left.\begin{array}{c} 0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \text{ exact} \\ \Rightarrow \\ 0 \to F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \to 0 \text{ exact} \end{array}\right\} \tag{131}$$

$$\left.\begin{array}{c} X' \xrightarrow{f} X \xrightarrow{g} X'' \text{ exact} \\ \Rightarrow \\ F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \text{ exact} \end{array}\right\} \tag{132}$$

Lemma 327. *We have (131)⇔(132).*

Proof. Implication $\Leftarrow$ is clear. To prove $\Rightarrow$, let

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

be exact. We must show that

$$F(X') \to F(X) \to F(X'') \tag{133}$$

is exact. Let K_g, K_f and I_g denote the indicated kernels and image. The sequence

$$0 \to I_g \to X'' \to X''/I_g \to 0$$

being exact, so is

$$0 \to F(I_g) \to F(X''),$$

and we get

$$\text{Ker}\big(F(X) \to F(I_g)\big) \xrightarrow{\sim} \text{Ker}\big(F(X) \to F(X'')\big). \tag{134}$$

The sequence

$$0 \to K_g \to X \to I_g \to 0$$

being exact, so is

$$F(K_g) \to F(X) \to F(I_g),$$

and we get

$$\text{Im}\big(F(K_g) \to F(X)\big) \xrightarrow{\sim} \text{Ker}\big(F(X) \to F(I_g)\big). \tag{135}$$

The sequence

$$0 \to K_f \to X' \to K_g \to 0$$

being exact, so is

$$F(X') \to F(K_g) \to 0,$$

and the isomorphism

$$\text{Im}\big(F(X') \to F(X)\big) \xrightarrow{\sim} \text{Im}\big(F(K_g) \to F(X)\big) \tag{136}$$

results from Lemma 325 p. 204 with $F(X') \to F(K_g) \to F(X)$ instead of (124) p. 203. The exactness of (133) follows from (134), (135) and (136). □

10.8.3 Exercise 8.19 p. 204

Let

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{a} & Y & \xrightarrow{b} & X \\ & & c\downarrow & & \downarrow d & & \downarrow e \\ 0 & \longrightarrow & W & \xrightarrow{f} & V & \xrightarrow{g} & U \\ & & h\downarrow & & \downarrow i & & \\ & & T & \xrightarrow[j]{} & S & & \end{array}$$

be a commutative diagram in an abelian category. If the first two rows and the last two columns are exact, then the first column is exact.

Proof. (In this proof we omit the composition symbols $\circ$ and most of the parenthesis, and we freely use Lemma 290 p. 178 and Lemma 292 p. 178.)

Exactness at Z: If $z : R \to Z$ satisfies $cz = 0$, we get $daz = fcz = 0$, and thus $z = 0$.

Exactness at W: Let $w : Q \to W$ satisfy $hw = 0$:

$$\begin{array}{ccccc} & & Q & & \\ & & w\downarrow & \searrow^{0} & \\ Z & \xrightarrow[c]{} & W & \xrightarrow[h]{} & T. \end{array}$$

The equalities $ifw = jhw = 0$ yield the commutative diagram

$$\begin{array}{ccccc} P & \overset{k}{\twoheadrightarrow} & Q & & \\ y\downarrow & & fw\downarrow & \searrow^{0} & \\ Y & \xrightarrow[d]{} & V & \xrightarrow[i]{} & S, \end{array}$$

and the equalities $eby = gdy = gfwk = 0$ and thus $by = 0$ yield the commutative diagram

$$\begin{array}{ccccc} N & \overset{\ell}{\twoheadrightarrow} & P & & \\ z\downarrow & & y\downarrow & \searrow^{0} & \\ Z & \xrightarrow[a]{} & Y & \xrightarrow[b]{} & X. \end{array}$$

This implies $fcz = daz = dy\ell = fwk\ell$, and thus $cz = wk\ell$:

$$\begin{array}{ccccc}
N & \overset{k\ell}{\twoheadrightarrow} & Q & & \\
{\scriptstyle z}\downarrow & & {\scriptstyle w}\downarrow & \overset{0}{\searrow} & \\
Z & \xrightarrow[c]{} & W & \xrightarrow[h]{} & T.
\end{array}$$

□

For the reader's convenience we spell out the statement:

Let

$$\begin{array}{ccccccc}
 & & U & \xrightarrow{e} & X & \longrightarrow & 0 \\
 & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle b} & & \\
S & \xrightarrow{i} & V & \xrightarrow{d} & Y & \longrightarrow & 0 \\
{\scriptstyle j}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle a} & & \\
T & \xrightarrow[h]{} & W & \xrightarrow[c]{} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram in an abelian category. If the last two columns and the first two rows are exact, then the last row is exact.

10.8.4 Exercise 8.37 p. 211

It will be convenient to denote the identity morphisms by 1 and the shift morphisms by s. We shall often write fg for $f \circ g$.

(i) We follow the hint and make the

Claim. There is a split exact sequence

$$0 \longrightarrow \bigoplus_{n=0}^{m} X_n \xrightarrow{1-s} \bigoplus_{n=0}^{m+1} X_n \longrightarrow X_{m+1} \longrightarrow 0,$$

that is, there is a diagram

$$0 \longrightarrow \bigoplus_{n=0}^{m} X_n \underset{b}{\overset{1-s}{\rightleftarrows}} \bigoplus_{n=0}^{m+1} X_n \underset{c}{\overset{a}{\rightleftarrows}} X_{m+1} \longrightarrow 0$$

such that

$$a\,(1-s)=0,\quad b\,(1-s)=1,\quad a\,c=1,\quad (1-s)\,b+c\,a=1.$$

We shall only define a, b, c for $m = 2$, leaving the rest of the proof of the claim to the reader. So we assume $m = 2$. The morphism $1 - s$ is given by the matrix

$$1-s=\begin{pmatrix}1&0&0\\-s&1&0\\0&-s&1\\0&0&-s\end{pmatrix},$$

and we define a, b, c by the matrices

$$a:=\begin{pmatrix}s^3&s^2&s&1\end{pmatrix},\quad b:=\begin{pmatrix}1&0&0&0\\s&1&0&0\\s^2&s&1&0\end{pmatrix},\quad c:=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Part (i) of the exercise follows clearly from the claim.

(ii) This part follows from the previous one together with Theorem 6.1.8 p. 132 and Proposition 6.1.19 p. 137 in the book.

(iii) I haven't been able to solve this part of the exercise.

11 About Chapter 9

I find Chapter 9 especially beautiful!

11.1 Brief comments

§ 328. P. 217, beginning of Section 9.2.

Proposition 329. *Let π be an infinite cardinal. The following conditions on a small category I (Definition 5 p. 10) are equivalent:*

(a) *For any category J with* $\operatorname{card}(\operatorname{Mor}(J)) < \pi$ *and any functor*

$$\alpha : I \times J^{\mathrm{op}} \to \mathbf{Set}$$

the natural map

$$\operatorname*{colim}_{i\in I}\lim_{j\in J}\alpha(i,j)\to\lim_{j\in J}\operatorname*{colim}_{i\in I}\alpha(i,j)$$

is bijective.

(b) *For any category J with* $\mathrm{card}(\mathrm{Mor}(J))<\pi$ *and any functor*

$$\alpha : I\times J^{\mathrm{op}}\to\mathbf{Set}$$

the natural map

$$\operatorname*{colim}_{i\in I}\lim_{j\in J}\alpha(i,j)\to\lim_{j\in J}\operatorname*{colim}_{i\in I}\alpha(i,j)$$

is surjective.

(c) *The following conditions hold:*

(c1) *for any* $A\subset\mathrm{Ob}(I)$ *such that* $\mathrm{card}(A)<\pi$ *there is a j in I such that for any a in A there is a morphism $a\to j$ in I,*

(c2) *for any i and j in I and for any* $B\subset\mathrm{Hom}_I(i,j)$ *such that* $\mathrm{card}(B)<\pi$ *there is a morphism $j\to k$ in I such that the composition $i\xrightarrow{s}j\to k$ does not depend on $s\in B$.*

(d) *For any category J such that* $\mathrm{card}(\mathrm{Mor}(J))<\pi$ *and any functor $\varphi: J\to I$ there is an i in I such that* $\lim\mathrm{Hom}_I(\varphi,i)\neq\varnothing$.

Proof. Implications (c) $\Leftrightarrow$ (d) $\Rightarrow$ (a) are proved in Proposition 9.2.1 p. 217 and Proposition 9.2.9 p. 219 of the book. Implication (a)$\Rightarrow$(b) is obvious. The proof of Implication (b)$\Rightarrow$(d) is the same as the proof of Implication (b)$\Rightarrow$(a) in Theorem 3.1.6 p. 74 of the book. □

Definition 330 (π-filtrant category)**.** *Let π be an infinite cardinal and I a category. Then I is* π-filtrant *if (and only if) the equivalent conditions of Proposition 329 are satisfied.*

§ 331. P. 218. One can make the following observation after Definition 9.2.2:

If I admits inductive limits indexed by categories J such that $\mathrm{card}(\mathrm{Mor}(J))<\pi$, *then I is π-filtrant.*

Proof. For $\varphi: J\to I$ we have

$$\lim\mathrm{Hom}_{\mathcal{C}}(\varphi,\operatorname{colim}\varphi)\xleftarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}\varphi,\operatorname{colim}\varphi)\neq\varnothing.$$

□

§ 332. P. 218, Example 9.2.3. See §12 p. 16.

§ 333. P. 218, Lemma 9.2.5.

Lemma 334 (Lemma 9.2.5 p. 218)**.** *Let $\varphi : J \to I$ be a cofinal functor. If J is π-filtrant, so is I.*

Clearly, I satisfies conditions (a) and (b) in Proposition 329 p. 209.

§ 335. P. 219, proof of Remark 9.2.6. To prove that I' is π-filtrant it is straightforward to check that I' satisfies Conditions (c1) and (c2) in Proposition 329 p. 209.

§ 336. Definition 9.2.7 p. 219. For the reader's convenience we paste the definition in question:

Definition 337. *Let π be an infinite cardinal and let $\mathcal{C}$ be a category which admits π-filtrant small inductive limits.*

An object $X \in \mathcal{C}$ is π-accessible if for any π-filtrant small category I and any functor $\alpha : I \to \mathcal{C}$ the natural map

$$\operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{C}}(X, \alpha(i)) \to \operatorname{Hom}_{\mathcal{C}}\left(X\ ,\ \operatorname*{colim}_{i\in I} \alpha(i)\right)$$

is bijective.

We denote by $\mathcal{C}_\pi$ the full subcategory of $\mathcal{C}$ consisting of π-accessible objects.

§ 338. P. 220, proof of Proposition 9.2.9. We add a few details to the argument in the book. Recall the statement:

Proposition 339. *Let π be an infinite cardinal. Let J be a category such that* $\operatorname{card}(\operatorname{Mor}(J)) < \pi$ *and let I be a small (Definition 5 p. 10) π-filtrant category. Consider a functor $\alpha : I \times J^{\mathrm{op}} \to \mathbf{Set}$, $(i, j) \mapsto \alpha_{ij}$. Then the natural map λ below is bijective:*

$$\lambda : \operatorname*{colim}_i \lim_j \alpha_{ij} \to \lim_j \operatorname*{colim}_i \alpha_{ij}.$$

Proof. Let

$$\begin{array}{ccccc}
\operatorname*{colim}_i \lim_j \alpha_{ij} & & \xrightarrow{\lambda} & & \lim_j \operatorname*{colim}_i \alpha_{ij} \\
{\scriptstyle p_i}\uparrow & & & & \downarrow{\scriptstyle q'_j} \\
\lim_j \alpha_{ij} & \xrightarrow[p'_{ij}]{} & \alpha_{ij} & \xrightarrow[q_{ij}]{} & \operatorname*{colim}_i \alpha_{ij}
\end{array}$$

be the obvious commutative diagram.

(i) Injectivity. Let i be in I and let $x, y \in \lim_j \alpha_{ij}$ satisfy $q_{ij}\, x_j = q_{ij}\, y_j$ for all j. (In this proof we omit almost all parenthesis.) It suffices to prove $p_i\, x = p_i\, y$. For each j there is a morphism $i \to i(j)$ in I such that $\alpha_{i\to i(j),j}\, x_j = \alpha_{i\to i(j),j}\, y_j$. Since I is π-filtrant and $\operatorname{card}(J) < \pi$, there is an i' in I and there are morphisms $i(j) \to i'$ in I such that the composition $i \to i(j) \to i'$ does not depend on j. We get $\alpha_{i\to i',j}\, x_j = \alpha_{i\to i',j}\, y_j$ for all j, and thus $p_i\, x = p_i\, y$.

(i) Surjectivity. Let y be in $\lim_j \operatorname{colim}_i \alpha_{ij}$. Each y_j is of the form $q_{ij}\, z_j$. *A priori* i depends on j, but it is easy to see that we can assume that i independent of j. Let $j \to j'$ be a morphism in J^{op}. We have $q_{ij'}\, z_{j'} = q_{ij'}\, \alpha_{i,j\to j'}\, z_j$, and there is a morphism $i \to i(j \to j')$ in I such that

$$\alpha_{i\to i(j\to j'),j'}\, z_{j'} = \alpha_{i\to i(j\to j'),j'}\, \alpha_{i,j\to j'}\, z_j.$$

Since $\operatorname{card}(\operatorname{Mor}(J)) < \pi$, there are morphisms $i(j \to j') \to i'$ in I such that the composition $i \to i(j \to j') \to i'$ does not depend on $j \to j'$. For each j set $x_j := \alpha_{i\to i',j}\, z_j \in \alpha_{i'j}$.

We claim $x \in \lim_j \alpha_{ij}$. Indeed, for any morphism $j \to j'$ in J^{op} we have

$$\alpha_{i,j\to j'}\, x_j = \alpha_{i,j\to j'}\, \alpha_{i\to i',j}\, z_j = \alpha_{i(j\to j')\to i',j'}\, \alpha_{i\to i(j\to j'),j'}\, \alpha_{i,j\to j'}\, z_j$$

$$= \alpha_{i(j\to j')\to i',j'}\, \alpha_{i\to i(j\to j'),j'}\, z_{j'} = \alpha_{i\to i',j'}\, z_{j'} = x_{j'}.$$

We claim $\lambda\, p_i\, x = y$. Let j be in J. It suffices to show $q_{i'j}\, x_j = q_{ij}\, z_j$. We have $q_{i'j}\, x_j = q_{i'j}\, \alpha_{i\to i',j}\, z_j = q_{ij}\, z_j$. □

§ 340. For the reader's convenience we state and prove Proposition 9.2.10 p. 220.

Proposition 341 (Proposition 9.2.10 p. 220)**.** *If $\mathcal{C}$ and J are categories, if $\mathcal{C}$ admits small π-filtrant inductive limits, if J satisfies* $\operatorname{card}(\operatorname{Mor}(J)) < \pi$*, if $\beta : J \to \mathcal{C}_\pi$ is a functor and if* $\operatorname{colim}\beta$ *exists in $\mathcal{C}$, then it belongs to $\mathcal{C}_\pi$.*

Proof. Let $\alpha : I \to \mathcal{C}$ be a functor with I small (Definition 5 p. 10) and π-filtrant, and consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{colim}_i \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_j \beta(j), \alpha(i)) & \xrightarrow{a} & \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_j \beta(j), \operatorname{colim} \alpha) \\
\sim\downarrow b & & \\
\operatorname{colim}_i \lim_j \operatorname{Hom}_{\mathcal{C}}(\beta(j), \alpha(i)) & & \sim\downarrow e \\
\sim\downarrow c & & \\
\lim_j \operatorname{colim}_i \operatorname{Hom}_{\mathcal{C}}(\beta(j), \alpha(i)) & \xrightarrow[\sim]{d} & \lim_j \operatorname{Hom}_{\mathcal{C}}(\beta(j), \operatorname{colim} \alpha).
\end{array}$$

The maps b and e are bijective for obvious reasons. The map c is bijective because of our assumptions on I and J. The map d is bijective because $\beta(j)$ is in $\mathcal{C}_\pi$ for all j. Thus, the map a is bijective. □

§ 342. P. 220, proof of Corollary 9.2.11.

Corollary 343 (Corollary 9.2.11 p. 220)**.** *If $\mathcal{C}$ admits small inductive limits and if X is an object of $\mathcal{C}$, then $\mathcal{C}_\pi$ and $(\mathcal{C}_\pi)_X$ are π-filtrant.*

This follows from §331 p. 210. Note that it suffices to assume that $\mathcal{C}$ admits inductive limits indexed by categories J such that $\operatorname{card}(\operatorname{Mor}(J)) < \pi$. (For the case of $(\mathcal{C}_\pi)_X$, see Lemma 84 p. 63.)

§ 344. P. 222, Proposition 9.2.17.

• Proof of implication (ii)⇒(i). I suspect that the argument of the book is better than the one given here, but, unfortunately, I don't understand it. Here is a less concise wording:

Recall the setting: $\mathcal{C}$ is a category admitting inductive limits indexed by any category J such that $\operatorname{card}(\operatorname{Mor}(J)) < \pi$, and A is in $\operatorname{Ind}(\mathcal{C})$. Conditions (i) and (ii) are as follows:

(i) $\mathcal{C}_A$ is π-filtrant,

(ii) for any category J such that $\operatorname{card}(\operatorname{Mor}(J)) < \pi$ and any functor $\varphi : J \to \mathcal{C}$, the natural map $A(\operatorname{colim} \varphi) \to \lim A(\varphi)$ is surjective.

To prove (ii)⇒(i), let J be a category satisfying

$$\operatorname{card}(\operatorname{Mor}(J)) < \pi,$$

and let $\psi : J \to \mathcal{C}_A$ be a functor. It suffices find a ξ in $\mathcal{C}_A$ satisfying

$$\lim \operatorname{Hom}_{\mathcal{C}_A}(\psi, \xi) \neq \varnothing$$

(see Condition (d) in Proposition 329 p. 209). Let $\varphi : J \to \mathcal{C}$ be the composition of ψ with the forgetful functor $\mathcal{C}_A \to \mathcal{C}$, and write

$$\psi(j) = \Big(\varphi(j)\ ,\ \varphi(j) \xrightarrow{y_j} A\Big) \in \mathcal{C}_A.$$

In particular the family (y_j) belongs to $\lim A(\varphi)$. Our assumption about $\mathcal{C}$ implies that $\operatorname{colim} \varphi$ exists in $\mathcal{C}$. Let $p_j : \varphi(j) \to \operatorname{colim} \varphi$ be the coprojection. By surjectivity of the map $A(\operatorname{colim} \varphi) \to \lim A(\varphi)$ in (ii), there is an $x : \operatorname{colim} \varphi \to A$ such that $x \circ p_j = y_j$ for all j. Setting

$$\xi := \Big(\operatorname{colim} \varphi, \operatorname{colim} \varphi \xrightarrow{x} A\Big) \in \mathcal{C}_A,$$

and letting $f_j : \psi(j) \to \xi$ be the obvious morphism, we get $(f_j) \in \lim \operatorname{Hom}_{\mathcal{C}_A}(\psi, \xi)$. q.e.d.

• Proof of implication (i)⇒(iii). The proof of the book exhibits a bijection

$$A\left(\operatorname*{colim}_{j \in J} \varphi(j)\right) \to \lim_{j \in J} A(\varphi(j)).$$

To prove that it coincides with the natural morphism, it suffices to check that the obvious diagram

$$\begin{array}{ccc} A(\operatorname{colim}_{j \in J} \varphi(j)) & \longrightarrow & \lim_{j \in J} A(\varphi(j)) \\ & \searrow \quad \swarrow & \\ & A(\varphi(j)) & \end{array}$$

commutes for all j in J. The details are left to the reader.

11.2 Section 9.3 pp 223–228

Here is a slightly different wording.

11.2.1 Conditions (9.3.1) p. 223

Recall Conditions (9.3.1) of the book: $\mathcal{C}$ is a category satisfying

(i) $\mathcal{C}$ admits small inductive limits,

(ii) $\mathcal{C}$ admits finite projective limits,

(iii) small filtrant inductive limits are exact,

(iv) there exists a generator G,

(v) epimorphisms are strict.

11.2.2 Summary of Section 9.3

The main purpose of Section 9.3 of the book is to prove Corollaries 9.3.7 and 9.3.8 p. 228 of the book, and these corollaries could be stated immediately after Conditions (9.3.1) above. For the reader's convenience we recall the definition of a regular cardinal and state Corollary 9.3.7:

Definition 345 (regular cardinal)**.** *A cardinal π is* regular *if for any family of sets $(B_i)_{i\in I}$ we have*

$$\operatorname{card}(I) < \pi, \quad \operatorname{card}(B_i) < \pi \ \forall\ i \quad \Rightarrow \quad \operatorname{card}\left(\bigsqcup_i B_i\right) < \pi.$$

Corollary 346 (Corollary 9.3.7 p. 228)**.** *Assume (9.3.1). Then for any small subset S of $\operatorname{Ob}(\mathcal{C})$ there exists an infinite cardinal π such that $S \subset \operatorname{Ob}(\mathcal{C}_\pi)$.*

We make a few comments about Corollary 9.3.8. Firstly, it would be simpler (I think) to replace $\mathcal{S}$ with $\mathcal{C}_\pi$ in the statement, since in the first sentence of the proof one sets $\mathcal{S} := \mathcal{C}_\pi$. Secondly, in view of the way Theorem 9.6.1 p. 235 of the book is phrased, it would be better, even if it is a repetition, to incorporate Part (iv) of Corollary 9.3.5 (which says that $\mathcal{C}_\pi$ is closed by finite projective limits) into Corollary 9.3.8. Then, Corollary 9.3.8 would read as follows:

Corollary 347 (Corollary 9.3.8 p. 227)**.** *Assume (9.3.1) and let κ be a cardinal. Then there exists an infinite regular cardinal $\pi > \kappa$ such that*

(i) *$\mathcal{C}_\pi$ is essentially small,*

(ii) *if* $X \twoheadrightarrow Y$ *is an epimorphism and* X *is in* $\mathcal{C}_\pi$*, then* Y *is in* $\mathcal{C}_\pi$*,*

(iii) *if* $X \rightarrowtail Y$ *is a monomorphism and* Y *in* $\mathcal{C}_\pi$*, then* X *is in* $\mathcal{C}_\pi$*,*

(iv) G *is in* $\mathcal{C}_\pi$*,*

(v) *for any epimorphism* $f : X \twoheadrightarrow Y$ *in* $\mathcal{C}$ *with* Y *in* $\mathcal{C}_\pi$*, there exists* Z *in* $\mathcal{C}_\pi$ *and a monomorphism* $g : Z \rightarrowtail X$ *such that* $f \circ g : Z \to Y$ *is an epimorphism,*

(vi) $\mathcal{C}_\pi$ *is closed by inductive limits indexed by categories* J *which satisfy*

$$\operatorname{card}(\operatorname{Mor}(J)) < \pi,$$

(vii) $\mathcal{C}_\pi$ *is closed by finite projective limits.*

See also Theorem 376 p. 237 below.

11.2.3 Lemma 9.3.1 p. 224

For the reader's convenience we state the lemma:

Lemma 348 (Lemma 9.3.1 p. 224)**.** *Assume that Conditions (9.3.1) p. 223 of the book (see §11.2.1 p. 215) hold, let* π *be an infinite regular cardinal, let* I *be a* π*-filtrant small category (Definition 5 p. 10), let* $\alpha : I \to \mathcal{C}$ *be a functor, and let* $\operatorname{colim}\alpha \to Y$ *be an epimorphism in* $\mathcal{C}$*. Assume either* $\operatorname{card}(Y(G)) < \pi$ *or* $Y \in \mathcal{C}_\pi$*. Then there is an* i_0 *in* I *such that the obvious morphism* $\alpha(i_0) \to Y$ *is an epimorphism.*

The proof of Lemma 348 in the book uses twice the following lemma:

Lemma 349. *Let* $\mathcal{C}$ *be a category, let* π *be an infinite cardinal, and let* $\alpha : I \to \mathcal{C}$ *be a functor admitting an inductive limit* X *in* $\mathcal{C}$*. Assume that the coprojections* $p_i : \alpha(i) \to X$ *are monomorphisms, and consider the conditions below:*

(a) I *is* π*-filtrant and* X *is* π*-accessible,*

(b) *the identity of* X *factors through the coprojection* p_i *for some* i*,*

(c) *the coprojection* p_i *is an isomorphism for some* i*,*

(d) *there is an* i *in* I *such that* $\alpha(s) : \alpha(i) \to \alpha(j)$ *is an isomorphism for all morphism* $s : i \to j$ *in* I*.*

Then we have (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

Proof. This follows immediately from Exercise 1.7 p. 31 of the book. □

We add a few details to the beginning of the proof of Lemma 9.3.1.

Set $X_i = \alpha(i)$ and $Y_i = \operatorname{Im}(X_i \to Y) = \operatorname{Ker}(Y \rightrightarrows Y \sqcup_{X_i} Y)$. In particular the natural morphism $Y_i \to Y$ is a monomorphism. Since small π-filtrant inductive limits are exact, we have

$$\operatorname*{colim}_i Y_i \xrightarrow{\sim} \operatorname{Ker}(Y \rightrightarrows Y \sqcup_{\operatorname{colim}_i X_i} Y) \xrightarrow{\sim} \operatorname{Im}(\operatorname*{colim}_i X_i \to Y) \xrightarrow{\sim} Y, \tag{137}$$

where, in view of the hypothesis that $\operatorname{colim}_i X_i \to Y$ is an epimorphism, the last isomorphism follows from Proposition 5.1.2 (iv) p. 114 of the book. It is easy to see that the chain of isomorphisms (137) coincides with the natural morphism $\operatorname{colim}_i Y_i \to Y$, and that the coprojections $Y_i \to Y$ are monomorphisms. In particular the maps $Y_i(G) \to Y(G)$ are injective by Proposition 161 p. 105 and Proposition 165 p. 106. This is easily seen to imply that $\operatorname{colim}_i Y_i(G) \to Y(G)$ is also injective.

(a) Assume that $\operatorname{card}(Y(G)) < \pi$. Then $\operatorname{card}(S) < \pi$, where $S := \operatorname{colim}_i Y_i(G)$. By Corollary 9.2.12 p. 221, $S \in \mathbf{Set}_\pi$ and this implies

$$\operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathbf{Set}}(S, Y_i(G)) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{Set}}(S, S).$$

Hence, there exist i_0 and a map $S \to Y_{i_0}(G)$ such that the composition $S \to Y_{i_0}(G) \rightarrowtail S$ is the identity. Therefore $Y_{i_0}(G) \xrightarrow{\sim} S$ and hence, $Y_{i_0}(G) \to Y_i(G)$ is bijective for any $i_0 \to i$ by Lemma 349. Hence $Y_{i_0} \to Y_i$ is an isomorphism, which implies, again by Lemma 349, that $Y_{i_0} \to Y$ is an isomorphism. Applying Proposition 5.1.2 (iv), we find that $X_{i_0} \to Y$ is an epimorphism.

11.2.4 Proposition 9.3.2 p. 224

Proposition 350 (Proposition 9.3.2 p. 224)**.** *Let $\mathcal{C}$ be a category satisfying Conditions (9.3.1) of the book, conditions stated in Section 11.2.1 p. 215 above. If π is an infinite regular cardinal, if A is in $\mathcal{C}$, and if*

$$\operatorname{card}(A(G)) < \pi, \quad \operatorname{card}\left(G^{\sqcup A(G)}(G)\right) < \pi,$$

then A is in $\mathcal{C}_\pi$.

Here is a rewriting of the proof with a few more details:

Proof of Proposition 350.

• Step 1. Note that **Set** $\ni S \mapsto G^{\sqcup S} \in \mathcal{C}$ is a well-defined covariant functor. Also note that $\operatorname{card}(G^{\sqcup S}(G)) < \pi$ for any $S \subset A(G)$. Indeed, there are maps

$$S \to A(G) \to S$$

whose composition is the identity. Hence, the composition

$$G^{\sqcup S}(G) \to G^{\sqcup A(G)}(G) \to G^{\sqcup S}(G)$$

is the identity.

• Step 2. Let I be a small (Definition 5 p. 10) π-filtrant category, let $(X_i)_{i\in I}$ be an inductive system in $\mathcal{C}$, and let X be its inductive limit. Claim 351 below will imply Proposition 350.

Claim 351. The map

$$\lambda_A : \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{C}}(A, X_i) \to \operatorname{Hom}_{\mathcal{C}}(A, X).$$

is bijective.

Claim 352. The map λ_A is injective.

Proof of Claim 352. (We shall only use $\operatorname{card}(A(G)) < \pi$.) Suppose that $f, g : A \rightrightarrows X_{i_0}$ have same image in $\operatorname{Hom}_{\mathcal{C}}(A, X)$. This just means that the two compositions

$$A \rightrightarrows X_{i_0} \to X$$

coincide. We must show that f and g have already same image in

$$\operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{C}}(A, X_i),$$

that is, we must show that there is a morphism $s_1 : i_0 \to i_1$ in I such that the two compositions $A \rightrightarrows X_{i_0} \to X_{i_1}$ coincide. For each $s : i_0 \to i$, set

$$N_s := \operatorname{Ker}(A \rightrightarrows X_i).$$

By Corollary 3.2.3 (i) p. 79 of the book, I^{i_0} is filtrant and the forgetful functor $I^{i_0} \to I$ is cofinal. One of our assumptions, namely Condition (9.3.1) (iii) in Section 11.2.1

p. 215, says that small filtrant inductive limits are exact in $\mathcal{C}$. In particular, $\operatorname{colim}_{s\in I^{i_0}}$ is exact in $\mathcal{C}$, and we get

$$\operatorname*{colim}_{s\in I^{i_0}} N_s \simeq \operatorname{Ker}\left(A \rightrightarrows \operatorname*{colim}_{s\in I^{i_0}} X_i\right) \simeq \operatorname{Ker}(A \rightrightarrows X) \simeq A.$$

As $\operatorname{card}(A(G)) < \pi$ by assumption, Lemma 348 p. 216 implies that there is a morphism $s_1 : i_0 \to i_1$ in I such that $N_{s_1} \to A$ is an epimorphism. Hence, the two compositions $A \rightrightarrows X_{i_0} \to X_{i_1}$ coincide, as was to be shown. This proves Claim 352. □

It only remains, in order to prove Proposition 350, to check that λ_A is surjective. Let $f : A \to X$ be a morphism. Claim 353 below will imply the surjectivity of λ_A, and thus the truth of Proposition 350.

Claim 353. There is an i in I and a morphism $g : A \to X_i$ such that $p_i \circ g = f$, where $p_i : X_i \to X$ is the coprojection.

• Step 3. Consider the following conditions:

(a) there is an i_0 in I such that the diagram of solid arrows

$$\begin{array}{ccc} B & \dashrightarrow\!\!\!\!\to & A \\ \downarrow & & \downarrow f \\ X_{i_0} & \xrightarrow[p_{i_0}]{} & X \end{array}$$

can be completed to a commutative diagram as indicated (the morphism $B \to A$ being an epimorphism),

(b) there is an i_0 in I such that the diagram of solid arrows

$$\begin{array}{ccc} C & \overset{a}{\dashrightarrow\!\!\!\!\to} & A \\ x \downarrow & & \downarrow f \\ X_{i_0} & \xrightarrow[p_{i_0}]{} & X \end{array} \tag{138}$$

can be completed to a commutative diagram as indicated, with $\operatorname{card}(C(G)) < \pi$ (the morphism $C \to A$ being an epimorphism).

We shall show that (a) holds, that (a) implies (b), and that (b) implies Claim 353, and thus Proposition 350 p. 217.

• Step 4: (a) holds. For each i in I define $Y_i := A \times_X X_i$. As colim_i is exact in $\mathcal{C}$, we have $\operatorname{colim}_i Y_i \simeq A$. As $\operatorname{card}(A(G)) < \pi$, Lemma 9.3.1 p. 224 of the book (stated above as Lemma 348 p. 216) implies that there is an i_0 in I such that $B := Y_{i_0} \to A$ is an epimorphism.

• Step 5: (a) implies (b). Assuming (a), we consider the commutative square

$$\begin{array}{ccc} B & \twoheadrightarrow & A \\ \downarrow & & \downarrow f \\ X_{i_0} & \xrightarrow[p_{i_0}]{} & X, \end{array} \tag{139}$$

and we put $S := \operatorname{Im}(B(G) \to A(G)) \subset A(G)$, so that $B(G) \twoheadrightarrow S \rightarrowtail A(G)$ is the "epi-mono" factorization of $B(G) \to A(G)$. Let $S \rightarrowtail B(G)$ be a section of $B(G) \twoheadrightarrow S$. By Step 1 we have $\operatorname{card}(C(G)) < \pi$. We set $C := G^{\sqcup S}$. The vertical arrows of the commutative diagram

$$\begin{array}{ccccc} G^{\sqcup B(G)} & \longrightarrow & C & \longrightarrow & G^{\sqcup A(G)} \\ \downarrow & & & & \downarrow \\ B & & \twoheadrightarrow & & A \end{array} \tag{140}$$

being epimorphisms by Proposition 5.2.3 (iv) p. 118 of the book, so is $C \to A$. From the commutative diagram

$$\begin{array}{ccccc} S & & = & & S \\ \| & & & & \| \\ S & \longrightarrow & B(G) & \longrightarrow & S, \end{array}$$

we get, by Step 1, the commutative diagram

$$\begin{array}{ccccc} C & & = & & C \\ \| & & & & \| \\ C & \longrightarrow & G^{\sqcup B(G)} & \longrightarrow & C. \end{array} \tag{141}$$

Splicing (139), (140) and (141) gives

$$\begin{array}{ccccc} C & = & = & = & C \\ \| & & & & \| \\ C & \longrightarrow & G^{\sqcup B(G)} & \longrightarrow & C \\ & \searrow & \downarrow & & \downarrow\!\!\!\downarrow a \\ & x\ \searrow & B & \longrightarrow & A \\ & & \downarrow & & \downarrow \\ & & X_{i_0} & \xrightarrow[p_{i_0}]{} & X. \end{array}$$

This proves (b).

• Step 6: (b) implies Claim 353 p. 219, and thus Proposition 350 p. 217. Assuming (b), form the cartesian square

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow & & \downarrow\!\!\!\downarrow a \\ C & \underset{a}{\twoheadrightarrow} & A. \end{array}$$

Epimorphisms in $\mathcal{C}$ being strict, the sequence $P \rightrightarrows C \xrightarrow{a} A$ is exact (see Proposition 5.1.5, (i), (a)⇒(c), p. 115 of the book). As

$$P(G) \le \operatorname{card}(C(G))^2 < \pi,$$

Claim 352 implies that the natural map

$$\lambda_P : \operatorname*{colim}_{i \in I} \operatorname{Hom}_{\mathcal{C}}(P, X_i) \to \operatorname{Hom}_{\mathcal{C}}(P, X)$$

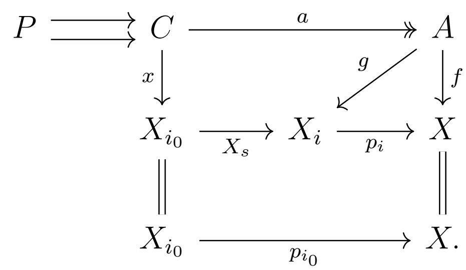
is injective. Consider the commutative diagram

$$\begin{array}{ccccc} P & \rightrightarrows & C & \xrightarrow{a} & A \\ & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle f} \\ & & X_{i_0} & \xrightarrow[p_{i_0}]{} & X. \end{array}$$

As λ_P is injective, and as the compositions $P \rightrightarrows C \xrightarrow{x} X_{i_0} \xrightarrow{p_{i_0}} X$ are equal, there is a morphism $s : i_0 \to i$ such that the compositions $P \rightrightarrows C \xrightarrow{x} X_{i_0} \xrightarrow{X_s} X_i$ are equal. The exactness of $P \rightrightarrows C \xrightarrow{a} A$ implies the existence of a morphism $g : A \to X_i$ such that

$$X_s \circ x = g \circ a. \tag{142}$$

Proof of Claim 353. It suffices to show that the above morphism g satisfies $f = p_i \circ g$. Consider the diagram



We have

$$\begin{aligned} f \circ a &= p_{i_0} \circ x && \text{by (138) p. 219} \\ &= p_i \circ X_s \circ x \\ &= p_i \circ g \circ a && \text{by (142) p. 221.} \end{aligned}$$

As a is an epimorphism, this forces $f = p_i \circ g$, and the proof of Claim 353 is complete. □

As already indicated, Claim 353 implies Proposition 350 p. 217. □

11.2.5 Definition of two infinite regular cardinals

(See (9.3.4) p. 226 of the book.) Let $\mathcal{C}$ be a category satisfying Conditions (9.3.1) in Section 11.2.1 p. 215 above. Let π_0 be an infinite regular cardinal such that

$$\operatorname{card}\big(G(G)\big) < \pi_0, \quad \operatorname{card}\big(G^{\sqcup G(G)}(G)\big) < \pi_0.$$

Now choose a cardinal $\pi_1 \geq \pi_0$ such that we have $\operatorname{card}\big(X(G)\big) < \pi_1$ for all set A with $\operatorname{card}(A) < \pi_0$ and all quotient X of $G^{\sqcup A}$. (Since the set of quotients of $G^{\sqcup A}$ is small by Proposition 5.2.9 p. 121 of the book, such a cardinal π_1 exists.) In the sequel of Section 11.2 we assume

Condition 354. Conditions (i)–(v) of Section 11.2.1 p. 215 of the book hold; π_0 and π_1 are as above; and π is the successor of 2^{π_1}.

The cardinals π and π_0 satisfy

(a) π and π_0 are infinite regular cardinals,

(b) G is in $\mathcal{C}_{\pi_0}$,

(c) $\pi'^{\pi_0} < \pi$ for any $\pi' < \pi$,

(d) if X is a quotient of $G^{\sqcup A}$ with $\operatorname{card}(A) < \pi_0$, then $\operatorname{card}\big(X(G)\big) < \pi$,

(e) if A is a set with $\operatorname{card}(A) < \pi_0$, then $\operatorname{card}\big(G^{\sqcup A}(G)\big) < \pi$.

Condition (a) holds because π_0 is infinite regular by assumption, and π is infinite regular by Statement (iv) p. 217 of the book. Condition (b) holds by Proposition 350 p. 217. Condition (c) is proved as follows: if $\pi' < \pi$, then $\pi' \le 2^{\pi_1}$ and

$$\pi'^{\pi_0} \le (2^{\pi_1})^{\pi_0} = 2^{\pi_0\pi_1} = 2^{\pi_1} < \pi.$$

Conditions (d) and (e) are clear.

11.2.6 Lemma 9.3.3 p. 226

We state Lemma 9.3.3 for the reader's convenience:

Lemma 355 (Lemma 9.3.3 p. 226)**.** *If Condition 354 holds, if A is a set of cardinal $< \pi$, and if X is a quotient of $G^{\sqcup A}$, then* $\operatorname{card}(X(G)) < \pi$*.*

The beginning of the proof of Lemma 9.3.3 in the book uses implicitly the following two lemmas, which we prove for the sake of completeness.

Lemma 356. *If α is a cardinal, then the cardinal of the set of those cardinals β such that $\beta < \alpha$ does not exceed α.*

Lemma 357. *If π_0, π and A are as above, and if $I := \{B \subset A \mid \operatorname{card}(B) < \pi_0\}$, then we have* $\operatorname{card}(I) < \pi$*.*

Proof of Lemma 356. Recall that a subset S of an ordered set X is a *segment* if

$$X \ni x < s \in S \quad \Longrightarrow \quad x \in S.$$

In particular $X_{<x}$ (obvious notation) is a segment of X for any x in X. We take for granted the following well-known facts:

• every set can be well-ordered,

• if T is a set of two non-isomorphic well-ordered sets, then there is a unique triple (W_1, W_2, S) such that $T = \{W_1, W_2\}$ and S is a proper segment of W_2 isomorphic to W_1,

• if W is a well-ordered set, then the assignment $w \mapsto W_{<w}$ is an isomorphism of well-ordered sets from W onto the set of proper segments of W.

Let A be a well-ordered set of cardinal α, and, for each cardinal β with $\beta < \alpha$, let B be a well-ordered set of cardinal β. Then B is isomorphic to $A_{<a}$ for a unique a in A, and the map $\beta \mapsto a$ is injective. □

Proof of Lemma 357. Putting $\alpha := \mathrm{card}(A)$ we have

$$\mathrm{card}(I) = \sum_{\pi' < \pi_0} \binom{\alpha}{\pi'} \le \sum_{\pi' < \pi_0} \alpha^{\pi_0} < \pi,$$

the last inequality following from Lemma 356, (c) and (a). □

11.2.7 Theorem 9.3.4 p. 227

Theorem 358 (Theorem 9.3.4 p. 227)**.** *Assume Condition 354 p. 222 holds and let X be an object of $\mathcal{C}$. Then we have*

$$X \in \mathcal{C}_\pi \Leftrightarrow \mathrm{card}(X(G)) < \pi.$$

Proof of Theorem 358.

$\Rightarrow$: We prove this implication as in the book. For the reader's convenience we reproduce the argument: Set $I := \{A \subset X(G) \mid \mathrm{card}(A) < \pi\}$. By Example 9.2.4 p. 218 of the book, I is π-filtrant. We get the morphisms

$$G^{\sqcup A} \to G^{\sqcup X(G)} \to X$$

for A in I, and

$$\operatorname*{colim}_{A \in I} G^{\sqcup A} \xrightarrow{\sim} G^{\sqcup X(G)} \to X.$$

Then we see that $G^{\sqcup X(G)} \to X$ is an epimorphism by Proposition 5.2.3 (iv) p. 118 of the book, that $G^{\sqcup A} \to X$ is an epimorphism for some A in I by Lemma 348 p. 216, and that $\mathrm{card}(X(G)) < \pi$ by Lemma 355 p. 223.

$\Leftarrow$: In view of Proposition 350 p. 217, it suffices to prove

$$\operatorname{card}\left(G^{\sqcup X(G)}(G)\right) < \pi. \tag{143}$$

To verify this inequality, we argue as in the proof of Lemma 9.3.3 p. 226 of the book (stated on p. 223 above as Lemma 355). (Conditions (b), (c) and (e) referred to below are stated in Section 11.2.5 p. 222.)

Let I be the ordered set of all subsets of $X(G)$ of cardinal $< \pi_0$. Then I is π_0-filtrant by Example 9.2.4 p. 218 of the book, and we have

$$G^{\sqcup X(G)} \simeq \operatorname*{colim}_{B \in I} G^{\sqcup B}.$$

As G is π_0-accessible by (b), we get

$$G^{\sqcup X(G)}(G) \simeq \operatorname*{colim}_{B \in I}\ G^{\sqcup B}(G).$$

By Lemma 357 p. 223 we have $\operatorname{card}(I) < \pi$. Since $\operatorname{card}(G^{\sqcup B}(G)) < \pi$ for all B in I by (e), this implies (143). □

11.2.8 Brief comments

∗ P. 227, Corollary 9.3.5. In the proof of (i) we use Propositions 5.2.3 (iv) p. 118 and 5.2.9 p. 121 of the book. As already pointed out, in the proof of (iv), $\mathcal{C}$ should be $\mathcal{C}_\pi$.

∗ P. 228, Corollary 9.3.6. As already pointed out, $\varinjlim$ in the statement should be σ_π. As for the proof, Conditions (i), (ii) and (iii) of Proposition 9.2.19 p. 223 of the book follow respectively from (9.3.1) (i) (see (i) at the beginning of Section 11.2 p. 214), (9.3.4) (b) (see (b) right after Condition 354 p. 222), and Corollary 9.3.5 (i) p. 227 of the book.

∗ P. 228, Corollary 9.3.7. As $\{\operatorname{card}(X(G)) \mid X \in S\}$ is a small set of cardinals, we may assume in Condition 354 p. 222 that we have $\pi > \operatorname{card}(X(G))$ for all X in S, and apply Theorem 358 p. 224.

∗ P. 228, Corollary 9.3.8. The proof uses implicitly Proposition 5.2.3 (iv) p. 118 of the book and Example 9.2.4 p. 218 of the book.

11.3 Quasi-Terminal Object Theorem

Recall the following result:

Theorem 359 (Zorn's Lemma)**.** *If X is an ordered set such that each well-ordered subset of X has an upper bound, then X has a maximal element.*

The purpose of this section is to prove a common generalization of Theorem 359 above and of Theorem 9.4.2 p. 229 of the book, stated below as Theorem 361. We start with a reminder:

Definition 360 (Definition 9.4.1 p. 228, quasi-terminal object)**.** *An object X of a category $\mathcal{C}$ is* quasi-terminal *if any morphism $u : X \to Y$ admits a left inverse.*

Theorem 361 (Theorem 9.4.2 p. 229)**.** *Any essentially small nonempty category admitting small filtrant inductive limits has a quasi-terminal object.*

Here is a weakening of the notion of inductive limit:

Definition 362 (small well-ordered upper bounds)**.** *Let I be a nonempty well-ordered small set and $\alpha : I \to \mathcal{C}$ a functor. An* upper bound *for α is a morphism of functors $a : \alpha \to \Delta X$ (see Notation 52 p. 46). If $\mathcal{C}$ has the property that any such functor admits some upper bound, we say that $\mathcal{C}$* admits small well-ordered upper bounds*.*

Definition 363 (special well-ordered small set)**.** *Let $\mathcal{C}$ be a category. A nonempty well-ordered small set I is $\mathcal{C}$-*special *if it has no largest element and if, for any functor $\alpha : I \to \mathcal{C}$, there is some upper bound $(a_i : \alpha(i) \to X)_{i \in I}$ and some element i_0 in I such that a_{i_0} is an epimorphism.*

Our goal is to prove:

Theorem 364 (Quasi-Terminal Object Theorem)**.** *If $\mathcal{C}$ is a nonempty essentially small category (Definition 5 p. 10) $\mathcal{C}$ admitting small well-ordered upper bounds and a $\mathcal{C}$-special well-ordered set, then $\mathcal{C}$ has a quasi-terminal object.*

Theorem 364 clearly implies Zorn's Lemma (Theorem 359). Lemma 370 below will show that Theorem 361 follows also from Theorem 364. Theorem 361 will be used in the book to prove Theorem 9.5.5 p. 233.

The proof of Theorem 364 is essentially the same as the proof of Theorem 361 given in the book. For the reader's convenience, we spell out the details.

Lemma 365. *If $\mathcal{C}$ is a nonempty small category (Definition 5 p. 10) admitting small well-ordered upper bounds, then there is an X in $\mathcal{C}$ such that, for all morphism $X \to Y$, there is a morphism $Y \to X$.*

Proof. Let $\mathcal{F}$ be the set of well-ordered subcategories of $\mathcal{C}$. For I and J in $\mathcal{F}$ we decree that $I \le J$ if and only if I is an initial segment of J. This order is clearly inductive. Let $\mathcal{S}$ be a maximal element of $\mathcal{F}$. As $\mathcal{S}$ is small, it admits an upper bound $(a_S : S \to X)_{S \in \mathcal{S}}$.

We shall prove that X satisfies the conditions in the statement. Let $u : X \to Y$ be a morphism in $\mathcal{C}$.

(i) The object Y is in $\mathcal{S}$. Otherwise, we can form the well-ordered subcategory $\widetilde{\mathcal{S}}$ of $\mathcal{C}$ by appending the element Y to $\mathcal{S}$ and making it the largest element of $\widetilde{\mathcal{S}}$, the morphism $S \to Y$ being $u \circ a_S$. We have $\widetilde{\mathcal{S}} \in \mathcal{F}$ and $\mathcal{S} < \widetilde{\mathcal{S}}$, contradicting the maximality of $\mathcal{S}$.

(ii) As Y is in $\mathcal{S}$, there is a morphism $Y \to X$, namely a_Y. □

Definition 366 (Property (P))**.** *We say that a morphism $a : A \to B$ in a given category has* Property (P) *if for any morphism $b : B \to C$ there is a morphism $c : C \to B$ satisfying $c \circ b \circ a = a$.*

Lemma 367 (Sublemma 9.4.4 p. 229)**.** *If $\mathcal{C}$ is a small (Definition 5 p. 10) nonempty category admitting small well-ordered upper bounds, and if X is an object of $\mathcal{C}$, then there is a morphism $f : X \to Y$ having Property (P).*

Proof. The category $\mathcal{C}^X$ is again nonempty, small (Definition 5 p. 10), and admits small well-ordered upper bounds, so that Lemma 365 applies to it. Let $f : X \to Y$ be to $\mathcal{C}^X$ what X is to $\mathcal{C}$ in Lemma 365. Then it is easy to see that f has Property (P). □

We recall the notion of *construction by transfinite induction.*

Theorem 368 (Construction by Transfinite Induction)**.** *Let $\mathcal{U}$ be a universe, let $F : \mathcal{U} \to \mathcal{U}$ be a map, and let I be a well-ordered $\mathcal{U}$-set. Then there is a unique pair (S, f) such that S is a $\mathcal{U}$-set, $f : I \to S$ is a surjection, and we have*

$$f(i) = F(f(j)_{j<i})$$

for all i in I, where $f(j)_{j<i}$ is viewed as a family of elements of $\{f(j) \mid j \in I, j < i\}$ indexed by $\{j \in I \mid j < i\}$.

Proof. Uniqueness: Assume that (S, f) and (T, g) have the indicated properties. It suffices to prove $f(i) = g(i)$ for all i in I. Suppose this is false, and let i be the least element of I such that $f(i) \neq g(i)$. We have

$$f(i) = F(f(j)_{j<i}) = F(g(j)_{j<i}) = g(i),$$

a contradiction.

Existence: Recall that a subset J of I is called a *segment* if $I \ni i < j \in J$ implies $i \in J$. Let Z be the set of all triples (J, S_J, f_J), where J is a segment of I, where $f : J \to S_J$ is a surjection, and where we have $f_J(j) = F(f_J(k)_{k<j})$ for all j in J. Decree that

$$Z \ni (J, S_J, f_J) \leq (K, S_K, f_K) \in Z$$

if and only if J is a segment of K. By the uniqueness part, $(Z, \leq)$ is inductive. Let (J, S_J, f_J) be a maximal element of Z. It suffices to assume that J is a proper segment of I and to derive a contradiction. Let k be the minimum of $I \setminus J$, put

$$K := J \cup \{k\}, \quad f_K(j) := f_J(j) \ \forall\ j \in J,$$

$$f_K(k) := F(f_K(j)_{j<k}), \quad S_K := S_J \cup \{f_K(k)\}.$$

Then (K, S_K, f_K) contradicts the maximality if (J, S_J, f_J). □

Proof of the Quasi-Terminal Object Theorem (Theorem 364 p. 226). Let $\mathcal{C}$ be as in the statement. We assume (as we may) that $\mathcal{C}$ is small (Definition 5 p. 10). Let us choose a $\mathcal{C}$-special well-ordered set I, and let us define an inductive system $(X_i)_{i\in I}$ by transfinite induction as follows: For the least element 0 of I we choose an arbitrary object X_0 of $\mathcal{C}$. Let $i > 0$ and assume that X_j and $u_{jk} : X_k \to X_j$ have been constructed for $k \leq j < i$.

(a) If $i = j + 1$ for some j, take $u_{ij} : X_j \to X_i$ with Property (P), and put $u_{ik} := u_{ij} \circ u_{jk}$ for any $k \leq j$.

(b) If $i = \sup\{j \mid j < i\}$, let $(a_j : X_j \to X_i)_{j<i}$ be some upper bound for $(X_j)_{j<i}$ and put $u_{ij} := a_j$.

(Recall that, by Definition 366 p. 227, the condition "$u_{ij} : X_j \to X_i$ has Property (P)" means that for any morphism $b : X_i \to C$ there is a morphism $c : C \to X_i$ satisfying $c \circ b \circ u_{ij} = u_{ij}$. Recall also that such a u_{ij} exists by Lemma 367 p. 227.)

Then $(X_i)_{i\in I}$ is indeed an inductive system in $\mathcal{C}$. As I is $\mathcal{C}$-special, there is an upper bound $(b_i : X_i \to X)_{i\in I}$ for $(X_i)_{i\in I}$, and there is an i_0 in I such that $b_{i_0} : X_{i_0} \to X$ is an epimorphism.

We claim that X is quasi-terminal. Let $u : X \to Y$ be a morphism. It suffices to prove the claim below:

Claim 369. There is a morphism $v : Y \to X$ such that $v \circ u = \mathrm{id}_X$.

Consider the morphisms

$$X_{i_0} \xrightarrow{u_{i_0+1,i_0}} X_{i_0+1} \xrightarrow{u \circ b_{i_0+1}} Y.$$

As u_{i_0+1,i_0} has Property (P), there is a morphism $w : Y \to X_{i_0+1}$ satisfying

$$w \circ u \circ b_{i_0+1} \circ u_{i_0+1,i_0} = u_{i_0+1,i_0}. \tag{144}$$

Put

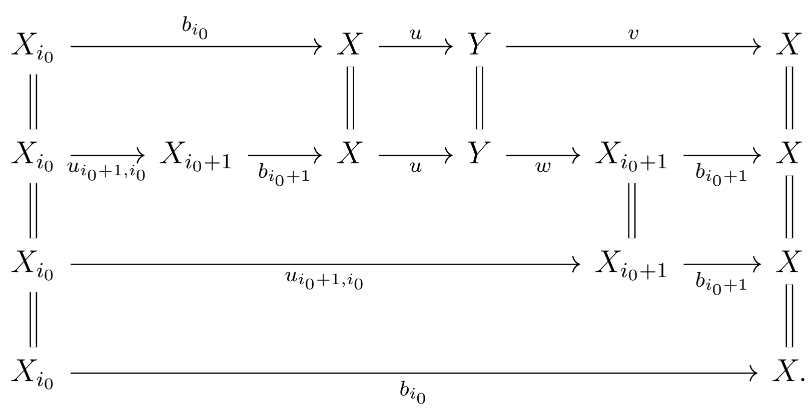
$$v := b_{i_0+1} \circ w : Y \to X. \tag{145}$$

It suffices to show that v satisfies the equality $v \circ u = \mathrm{id}_X$ in Claim 369 p. 229. We have

$$\begin{aligned} v \circ u \circ b_{i_0} &= b_{i_0+1} \circ w \circ u \circ b_{i_0} && \text{by (145)} \\ &= b_{i_0+1} \circ w \circ u \circ b_{i_0+1} \circ u_{i_0+1,i_0} \\ &= b_{i_0+1} \circ u_{i_0+1,i_0} && \text{by (144)} \\ &= b_{i_0} \\ &= \mathrm{id}_X \circ b_{i_0}. \end{aligned}$$

As b_{i_0} is an epimorphism, this implies $v \circ u = \mathrm{id}_X$, proving Claim 369 p. 229, and thus the Quasi-Terminal Object Theorem (Theorem 364 p. 226). □

Here is a diagrammatic illustration of the above computation:



For the reader's convenience we state and prove Sublemma 9.4.5 p. 229 of the book.

Lemma 370 (Sublemma 9.4.5 p. 229)**.** *If $\mathcal{C}$ is a small (Definition 5 p. 10) nonempty category admitting small filtrant inductive limits, if π is an infinite regular cardinal such that* $\operatorname{card}(\operatorname{Mor}(\mathcal{C})) < \pi$, *if I is a π-filtrant small category, and if $(X_i)_{i \in I}$ is an inductive system in $\mathcal{C}$ indexed by I, then there is an i_0 in I such that the coprojection $X_{i_0} \to \operatorname{colim}_i X_i$ is an epimorphism.*

Proof. Set $X := \operatorname{colim}_i X_i$ and let $a_i : X_i \to X$ be the coprojection. For any Y in $\mathcal{C}$ let

$$b_i^Y : \operatorname{Hom}_{\mathcal{C}}(Y, X_i) \to \operatorname*{colim}_j \operatorname{Hom}_{\mathcal{C}}(Y, X_j)$$

be the coprojection, let $F(Y)$ be the image of the natural map

$$\operatorname*{colim}_j \operatorname{Hom}_{\mathcal{C}}(Y, X_j) \to \operatorname{Hom}_{\mathcal{C}}(Y, X),$$

and define φ^Y by the commutative diagram

$$\begin{array}{ccccc}
\operatorname{colim}_j \operatorname{Hom}_{\mathcal{C}}(Y, X_j) & \xrightarrow{\varphi^Y} & F(Y) & \hookrightarrow & \operatorname{Hom}_{\mathcal{C}}(Y, X) \\
{\scriptstyle b_i^Y}\uparrow & \nearrow{\scriptstyle \varphi_i^Y := a_i \circ} & & & \\
\operatorname{Hom}_{\mathcal{C}}(Y, X_i). & & & &
\end{array}$$

Claim: There is an i_0 in I such that $\varphi_{i_0}^Y := a_{i_0} \circ : \operatorname{Hom}_{\mathcal{C}}(Y, X_{i_0}) \to F(Y)$ is surjective for all Y in $\mathcal{C}$.

As $\operatorname{card}(\operatorname{Hom}_{\mathcal{C}}(Y, X)) < \pi$, we have $F(Y) \in \mathbf{Set}_\pi$ by Corollary 9.2.12 p. 221 of the book. By Lemma 9.3.1 p. 224 of the book (stated above as Lemma 348 p. 216), there is an i_Y in I such that

$$a_{i_Y} \circ : \operatorname{Hom}_{\mathcal{C}}(Y, X_{i_Y}) \to F(Y)$$

is surjective. As $\operatorname{card}(\{i_Y \mid Y \in \operatorname{Ob}(\mathcal{C})\}) < \pi$ and I is π-filtrant, there is an i_0 in I such that, for any Y in $\mathcal{C}$, there is a morphism $i_Y \to i_0$. This implies the claim.

Let i be in I. In particular, $a_i = \varphi_i^{X_i}(\operatorname{id}_{X_i})$ is in $F(X_i)$. As

$$\varphi_{i_0}^{X_i} := a_{i_0} \circ : \operatorname{Hom}_{\mathcal{C}}(X_i, X_{i_0}) \to F(X_i)$$

is surjective by the claim, there is a morphism $h_i : X_i \to X_{i_0}$ such that $a_{i_0} \circ h_i = a_i$.

Let us show that $a_{i_0} : X_{i_0} \to X$ is an epimorphism. Let $f_1, f_2 : X \rightrightarrows Y$ be a pair of parallel arrows such that $f_1 \circ a_{i_0} = f_2 \circ a_{i_0}$. Then, for any i in I, we have

$$f_1 \circ a_i = f_1 \circ a_{i_0} \circ h_i = f_2 \circ a_{i_0} \circ h_i = f_2 \circ a_i.$$

This implies $f_1 = f_2$. □

We give again a diagrammatic illustration of the above computation:

$$\begin{array}{ccccccc}
X_i & & \xrightarrow{\quad a_i \quad} & & X & \xrightarrow{f_1} & Y \\
\| & & & & \| & & \| \\
X_i & \xrightarrow{h_i} & X_{i_0} & \xrightarrow{a_{i_0}} & X & \xrightarrow{f_1} & Y \\
\| & & \| & & & & \| \\
X_i & \xrightarrow{h_i} & X_{i_0} & \xrightarrow{a_{i_0}} & X & \xrightarrow{f_2} & Y \\
\| & & & & \| & & \| \\
X_i & & \xrightarrow[\quad a_i \quad]{} & & X & \xrightarrow[f_2]{} & Y.
\end{array}$$

11.4 Lemma 9.5.3 p. 231

We give more details about the proof, but first let us recall the setting:

Let $\mathcal{C}$ be a $\mathcal{U}$-category (Definition 4 p. 10), let $\mathcal{C}_0$ be a subcategory of $\mathcal{C}$, and assume

(9.5.2) (i) $\mathcal{C}_0$ admits small filtrant inductive limits and $\mathcal{C}_0 \to \mathcal{C}$ commutes with them.

(9.5.2) (ii) Any diagram of solid arrows

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow[u']{} & Y',
\end{array} \tag{146}$$

with u in $\mathrm{Mor}(\mathcal{C}_0)$ and f in $\mathrm{Mor}(\mathcal{C})$, can be completed to a commutative diagram with dashed arrows u' in $\mathrm{Mor}(\mathcal{C}_0)$ and g in $\mathrm{Mor}(\mathcal{C})$.

Lemma 371 (Lemma 9.5.3 p. 231)**.** *If X' is in $\mathcal{C}_0$, if I is a small set, and if*

$$(u_i : X_i \to Y_i)_{i \in I}, \quad (f_i : X_i \to X')_{i \in I}$$

are families of morphisms in $\mathcal{C}_0$ and $\mathcal{C}$ respectively, then there is an object Y' of $\mathcal{C}_0$, a morphism $u' : X' \to Y'$ in $\mathcal{C}_0$, and a family $(g_i : Y_i \to Y')_{i \in I}$ of morphisms in $\mathcal{C}$ such that $g_i \circ u_i = u' \circ f_i$ for all i:

$$\begin{array}{ccc} X_i & \xrightarrow{u_i} & Y_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle g_i} \\ X' & \underset{u'}{\dashrightarrow} & Y'. \end{array}$$

Proof. We assume, as we may, that I is nonempty, well-ordered, and admits a maximum m. Let 0 be the least element of I. We shall complete the following Task (T_i) by transfinite induction on $i \in I$ (see Theorem 368 p. 227):

[Beginning of the description of Task (T_i).] Construct, for each $j \leq i$ in I, a commutative diagram

$$\begin{array}{ccccc} X_j & & \xrightarrow{\quad u_j \quad} & & Y_j \\ {\scriptstyle f_j}\downarrow & & & & \downarrow{\scriptstyle h_j} \\ X' & \underset{v_j}{\dashrightarrow} & Y'_{<j} & \underset{w_j}{\dashrightarrow} & Y'_j, \end{array}$$

with v_j, w_j in $\operatorname{Mor}(\mathcal{C}_0)$, and construct, for each (i, j, k) in I^3 with $i \geq j > k$, a commutative diagram

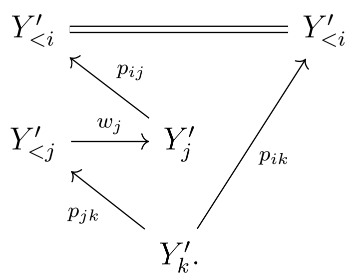
$$\begin{array}{ccccc} X' & \xrightarrow{v_j} & Y'_{<j} & \xrightarrow{w_j} & Y'_j \\ \| & & & \nwarrow{\scriptstyle p_{jk}} & \\ X' & \xrightarrow[v_k]{} & Y'_{<k} & \xrightarrow[w_k]{} & Y'_k, \end{array}$$

with p_{jk} in $\operatorname{Mor}(\mathcal{C}_0)$, in such a way that we have

$$p_{ij} \circ w_j \circ p_{jk} = p_{ik} \quad \forall \quad i > j > k, \tag{147}$$

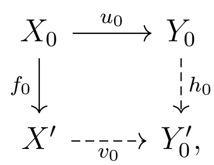
$$w_0 = \mathrm{id}_{Y'_0}. \tag{148}$$

Here is a diagrammatic illustration of (147):

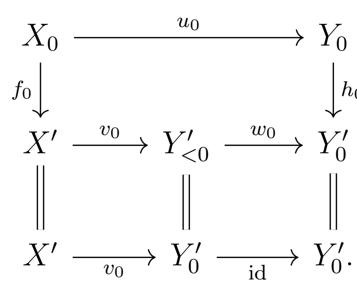


[End of the description of Task (T_i).]

[Beginning of the accomplishment of Task (T_i) for all i.] To handle Task (T_0), we define Y'_0, v_0 and h_0 by (9.5.2) (ii):



and we define $Y'_{<0}$ and w_0 by the commutative diagram



Let i in I satisfy $i > 0$, and let us tackle Task (T_i).

We assume (as we may) that Task (T_j) has already been achieved for $j < i$, *i.e.* that the $Y'_{<j}, Y'_j, h_j, v_j, w_j$ have already been constructed for $j < i$, that the p_{jk} have already been constructed for $k < j < i$, and that all these morphisms satisfy the required conditions.

It suffices to define $Y'_{<i}, Y'_i, h_i, v_i, w_i$, and p_{ij} for $j < i$, in such a way that the required conditions are still satisfied.

For $k < j < i$ we define $u_{jk} : Y'_k \to Y'_j$ by

$$u_{jk} := w_j \circ p_{jk}. \tag{149}$$

By (147) we have $u_{jk} \circ u_{k\ell} = u_{j\ell}$ for all $\ell < k < j < i$. In particular,

$$(Y'_j)_{j<i} \text{ is an inductive system in } \mathcal{C}_0. \tag{150}$$

We denote its limit (which exists in $\mathcal{C}_0$ thanks to (9.5.2) (i)) by $Y'_{<i}$, and we write p_{ij} for the coprojection $Y'_j \to Y'_{<i}$. We also set

$$v_i := p_{i0} \circ v_0, \tag{151}$$

and we define

$$Y'_{<i} \xrightarrow{w_i} Y'_i \xleftarrow{h_i} Y_i$$

by (9.5.2) (ii):

$$\begin{array}{ccc} X_i & \xrightarrow{u_i} & Y_i \\ {\scriptstyle v_i \circ f_i}\downarrow & & \downarrow{\scriptstyle h_i} \\ Y'_{<i} & \underset{w_i}{\dashrightarrow} & Y'_i, \end{array}$$

so that we have

$$h_i \circ u_i = w_i \circ v_i \circ f_i. \tag{152}$$

We must check

$$p_{ik} \circ w_k \circ v_k = v_i \ \forall\ k < i, \tag{153}$$

$$p_{ij} \circ w_j \circ p_{jk} = p_{ik} \ \forall\ k < j < i. \tag{154}$$

To prove (153), first note that we have

$$v_k = p_{k0} \circ w_0 \circ v_0$$

by induction hypothesis, $w_0 = \mathrm{id}_{Y'_0}$ by (148), and thus

$$v_k = p_{k0} \circ v_0. \tag{155}$$

We get

$$\begin{aligned} p_{ik} \circ w_k \circ v_k &= p_{ik} \circ w_k \circ p_{k0} \circ v_0 && \text{by (155)} \\ &= p_{i0} \circ v_0 && \text{by (147)} \\ &= v_i && \text{by (151).} \end{aligned}$$

This proves (153). We have

$$\begin{aligned} p_{ij} \circ w_j \circ p_{jk} &= p_{ij} \circ u_{jk} && \text{by (149)} \\ &= p_{ik} && \text{by (150).} \end{aligned}$$

This proves (154).

Task (T_i) has been performed for the specific i we have been considering, and thus Task (T_i) has been completed for all i in I. [End of the accomplishment of Task (T_i) for all i.]

In particular Task (T_m), where, remember, m is the maximum of I, has also been achieved. Putting $Y' := Y'_m$ and

$$g_i := u_{mi} \circ h_i \ \forall \ i < m, \tag{156}$$

$$g_m := h_m, \tag{157}$$

$$u' := w_m \circ v_m, \tag{158}$$

we get

$$\begin{aligned} g_i \circ u_i &= u_{mi} \circ h_i \circ u_i && \text{by (156)} \\ &= u_{mi} \circ w_i \circ v_i \circ f_i && \text{by (152)} \\ &= w_m \circ p_{mi} \circ w_i \circ v_i \circ f_i && \text{by (149)} \\ &= w_m \circ v_m \circ f_i && \text{by (153)} \\ &= u' \circ f_i && \text{by (158)} \end{aligned}$$

for $i < m$, and

$$\begin{aligned} g_m \circ u_m &= h_m \circ u_m && \text{by (157)} \\ &= w_m \circ v_m \circ f_m && \text{by (152)} \\ &= u' \circ f_m && \text{by (158).} \end{aligned}$$

□

11.5 Theorems 9.5.4 and 9.5.5 pp 232-234

The purpose of this section is to give a combined statement of Theorems 9.5.4 and 9.5.5.

Remark 372. In Definition 9.5.1 (i) p. 231 we read "Let $\mathcal{F} \subset \mathrm{Mor}(\mathcal{C})$ be a family of morphisms in $\mathcal{C}$". To be completely clear we mentally replace the above quote with "Let $\mathcal{F}$ be a full subcategory of $\mathrm{Mor}(\mathcal{C})$".

Let $\mathcal{C}$ be a $\mathcal{U}$-category (Definition 4 p. 10), let $\mathcal{C}_0$ be a subcategory of $\mathcal{C}$, let $\mathcal{F}$ be an essentially small full subcategory of $\mathrm{Mor}(\mathcal{C}_0)$ (see Remark 374 below), let π be an infinite cardinal such that X is in $\mathcal{C}_\pi$ for any $X \to Z$ in $\mathcal{F}$, and assume

(9.5.2) (i) $\mathcal{C}_0$ admits small filtrant inductive limits and $\mathcal{C}_0 \to \mathcal{C}$ commutes with them;

(9.5.2) (ii) any diagram of solid arrows

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[u']{} & Y', \end{array}$$

with u in $\mathrm{Mor}(\mathcal{C}_0)$ and f in $\mathrm{Mor}(\mathcal{C})$, can be completed as indicated to a commutative diagram with dashed arrows u' in $\mathrm{Mor}(\mathcal{C}_0)$ and g in $\mathrm{Mor}(\mathcal{C})$;

(9.5.6) for any X in $\mathcal{C}_0$, the category $(\mathcal{C}_0)_X$ is essentially small;

(9.5.7) any cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in $\mathcal{C}$ with f, f' in $\mathrm{Mor}(\mathcal{C}_0)$ decomposes into a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & & \\ {\scriptstyle u}\downarrow & & \downarrow & \overset{v}{\searrow} & \\ X & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & Y \end{array}$$

such that the square (X', Y', Z, X) is cocartesian, g and h are in $\mathrm{Mor}(\mathcal{C}_0)$, and $f = h \circ g$;

(9.5.8) if a morphism $f : X \to Y$ in $\mathcal{C}_0$ is such that any cartesian square of solid arrows

$$\begin{array}{ccc} U & \xrightarrow{s} & V \\ {\scriptstyle u}\downarrow & \overset{\xi}{\swarrow} & \downarrow{\scriptstyle v} \\ X & \xrightarrow[f]{} & Y \end{array}$$

can be completed as indicated to a commutative diagram in $\mathcal{C}$ with the dashed arrow ξ, then f is an isomorphism.

Theorem 373. *If the above conditions hold, then, for any X in $\mathcal{C}_0$, there is a* $\mathrm{Mor}(\mathcal{C}_0)$*-injective object Y of $\mathcal{C}$, and morphism $f : X \to Y$ in $\mathcal{C}_0$. If (9.5.2) holds, but (9.5.6), (9.5.7) and (9.5.8) do not necessarily hold, then there is an $\mathcal{F}$-injective object Y of $\mathcal{C}$, and a morphism $f : X \to Y$ in $\mathcal{C}_0$.*

Remark 374. In the book $\mathcal{F}$ is supposed to be small, but the proof clearly works if $\mathcal{F}$ is only essentially small. (See Remark 372 p. 236 above and §377 below.)

11.6 Brief comments

§ 375. P. 235, Theorem 9.6.1. In view of the comments made before Corollary 347 p. 215, Theorem 9.6.1 could be stated as follows:

Theorem 376 (Theorem 9.6.1 p. 235)**.** *Let $\mathcal{C}$ be a Grothendieck category. Then, for any small subset E of* $\mathrm{Ob}(\mathcal{C})$*, there exists an infinite cardinal π such that*

(i) $\mathrm{Ob}(\mathcal{C}_\pi)$ *contains E,*

(ii) $\mathcal{C}_\pi$ *is a fully abelian subcategory of $\mathcal{C}$,*

(iii) $\mathcal{C}_\pi$ *is essentially small,*

(iv) $\mathcal{C}_\pi$ *contains a generator of $\mathcal{C}$,*

(v) $\mathcal{C}_\pi$ *is closed by subobjects and quotients in $\mathcal{C}$,*

(vi) *for any epimorphism $f : X \twoheadrightarrow Y$ in $\mathcal{C}$ with Y in $\mathcal{C}_\pi$, there exists Z in $\mathcal{C}_\pi$ and a monomorphism $g : Z \rightarrowtail X$ such that $f \circ g : Z \to Y$ is an epimorphism,*

(vii) $\mathcal{C}_\pi$ *is closed by countable direct sums.*

§ 377. P. 236, proof of Theorem 9.6.2.

Line 3: One could change “Let $\mathcal{F} \subset \mathrm{Mor}(\mathcal{C}_0)$ be the set of monomorphisms $N \hookrightarrow G$. This is a small set by Corollary 8.3.26” to “Let $\mathcal{F}$ be the full subcategory of $\mathrm{Mor}(\mathcal{C}_0)$ whose objects are the monomorphisms $N \rightarrowtail G$. Then $\mathcal{F}$ is essentially small by Corollary 8.3.26” (see Remark 372 p. 236). In view of Remark 374, we can still apply Theorem 9.5.4.

Line 6: Condition (9.5.2) (i) (see Section 11.5 p. 235 above) follows from

Lemma 378. *Let $\mathcal{C}$ be a category. Assume that small filtrant inductive limits exist in $\mathcal{C}$ and are exact. Let $\alpha : I \to \mathcal{C}$ be a functor such that I is small and filtrant, and $\alpha(s) : \alpha(i) \to \alpha(j)$ is a monomorphism for all morphism $s : i \to j$ in I. Then the coprojection $p_i : \alpha(i) \to \operatorname{colim} \alpha$ is a monomorphism.*

Proof. By Corollary 3.2.3 p. 79 of the book, I^i is filtrant and the forgetful functor $\varphi : I^i \to I$ is cofinal. Define the morphism of functors

$$\theta \in \operatorname{Hom}_{\operatorname{Fct}(I^i,\mathcal{C})}(\Delta\alpha(i), \alpha \circ \varphi)$$

(see Notation 52 p. 46) by

$$\theta_{(s:i\to j)} := \big(\alpha(s) : \alpha(i) \to \alpha(j)\big).$$

As θ is a monomorphism, Proposition 165 p. 106 implies that $\operatorname{colim}\theta$ is also a monomorphism. Then the conclusion follows from the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{colim} \Delta\alpha(i) & \xrightarrow{\operatorname{colim}\theta} & \operatorname{colim} \alpha \circ \varphi \\ \sim\downarrow & & \downarrow\sim \\ \alpha(i) & \xrightarrow[p_i]{} & \operatorname{colim} \alpha. \end{array}$$

□

§ 379. Pp 237-239. For the reader's convenience we first reproduce (with minor changes) two corollaries with their proof.

Corollary 380 (Corollary 9.6.5 p. 237)**.** *If $\mathcal{C}$ is a small (Definition 5 p. 10) abelian category, then $\operatorname{Ind}(\mathcal{C})$ admits an injective cogenerator.*

Proof. Apply Theorem 8.6.5 (vi) p. 194 and Theorem 9.6.3 p. 236 of the book. □

Corollary 381 (Corollary 9.6.6 p. 237)**.** *Let $\mathcal{C}$ be a Grothendieck category. Denote by $\mathcal{I}$ the full additive subcategory of $\mathcal{C}$ consisting of injective objects, and by $\iota : \mathcal{I} \to \mathcal{C}$ the inclusion functor. Then there exist a (not necessarily additive) functor $\Psi : \mathcal{C} \to \mathcal{I}$ and a morphism of functors $\operatorname{id}_{\mathcal{C}} \to \iota \circ \Psi$ such that $X \to \Psi(X)$ is a monomorphism for any X in $\mathcal{C}$.*

Proof. The category $\mathcal{C}$ admits an injective cogenerator K by Theorem 9.6.3 p. 236 of the book, and admits small products by Proposition 8.3.27 (i) p. 186 of the book. Consider the (non additive) functor

$$\Psi : \mathcal{C} \to \mathcal{I}, \quad X \mapsto K^{\operatorname{Hom}_{\mathcal{C}}(X,K)}.$$

The identity of

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{C}}(X,K),\mathrm{Hom}_{\mathcal{C}}(X,K)) \simeq \mathrm{Hom}_{\mathcal{C}}(X,K^{\mathrm{Hom}_{\mathcal{C}}(X,K)})$$

defines a morphism $X \to \iota(\Psi(X)) = K^{\mathrm{Hom}_{\mathcal{C}}(X,K)}$, and this morphism is a monomorphism by Proposition 5.2.3 (iv) p. 118 of the book. □

The first sentence of the proof of Lemma 9.6.8 p. 238 of the book follows from Proposition 5.2.3 (iv) p. 118 of the book.

The third sentence of the proof of Lemma 9.6.9 p. 238 of the book follows from Proposition 5.2.3 (i) p. 118 of the book (the assumption that $\mathcal{C}$ admits small coproducts is not used in the proof of Proposition 5.2.3 (i)).

In the proof of Theorem 9.6.10 p. 238 of the book, the exactness of $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ follows from Theorem 8.6.5 (ii) p. 194 of the book (see also §223 p. 140).

12 About Chapter 10

12.1 Definition of a triangulated category

The purpose of this Section is to spell out the observation made by J. P. May that, in the definition of a triangulated category, Axiom TR4 of the book (p. 243) follows from the other axioms. See Section 1 of *The axioms for triangulated categories* by J. P. May:

http://www.math.uchicago.edu/∼may/MISC/Triangulate.pdf

Various related links are given in the document http://goo.gl/df2Xw.

To make things as clear as possible, we remove TR4 from the definition of a triangulated category and prove that any triangulated category satisfies TR4:

Definition 382 (triangulated category)**.** *A* triangulated category *is an additive category $(\mathcal{D},T)$ with translation endowed with a set of triangles satisfying Axioms* TR0, TR1, TR2, TR3 *and* TR5 *on p. 243 of the book.*

Let $(\mathcal{D},T)$ be a triangulated category. In the book the theorem below is stated as Exercise 10.6 p. 266 and is used at the top of p. 251 within the proof of Theorem 10.2.3 p. 249.

Theorem 383. *Let*

$$\begin{array}{ccccccc}
X^0 & \xrightarrow{u} & X^1 & \xrightarrow{v} & X^2 & \xrightarrow{w} & TX^0 \\
{\scriptstyle f}\downarrow & & \downarrow & & \vdots & & \downarrow{\scriptstyle Tf} \\
Y^0 & \longrightarrow & Y^1 & \longrightarrow & Y^2 & \longrightarrow & TY^0 \\
{\scriptstyle g}\downarrow & & \downarrow & & \vdots & & \downarrow{\scriptstyle Tg} \\
Z^0 & \dashrightarrow & Z^1 & \dashrightarrow & Z^2 & \dashrightarrow & TZ^0 \\
{\scriptstyle h}\downarrow & & \downarrow & & \vdots & & \downarrow{\scriptstyle -Th} \\
TX^0 & \xrightarrow[Tu]{} & TX^1 & \xrightarrow[Tv]{} & TX^2 & \xrightarrow[-Tw]{} & T^2X^0
\end{array}$$

be a diagram of solid arrows in $\mathcal{D}$. Assume that the first two rows and columns are distinguished triangles, and the top left square commutes[1]*. Then the dotted arrows may be completed in order that the bottom right square anti-commutes, the eight other squares commute, and all rows and columns are distinguished triangles.*

Corollary 384. *Any category which is triangulated in the sense of Definition 382 satisfies* TR4.

Recall Axiom TR5: If the diagram

$$\begin{array}{ccccccccc}
U & \longrightarrow & V & \longrightarrow & W' & \longrightarrow & TU & & \\
\| & & \| & & & & & & \\
\| & & V & \longrightarrow & W & \longrightarrow & U' & \longrightarrow & TV \\
\| & & & & \| & & & & \\
U & & \longrightarrow & & W & \longrightarrow & V' & \longrightarrow & TU
\end{array}$$

commutes, and if the rows are distinguished triangles, then there is a distinguished

[1] I think the assumption that the top left square commutes is implicit in the book.

triangle $W' \to V' \to U' \to TW'$ such that the diagram

$$\begin{CD}
U @>>> V @>>> W' @>>> TU \\
@| @VVV @VVV @| \\
U @>>> W @>>> V' @>>> TU \\
@VVV @| @VVV @VVV \\
V @>>> W @>>> U' @>>> TV \\
@VVV @VVV @| @VVV \\
W' @>>> V' @>>> U' @>>> TW'
\end{CD}$$

commutes.

Proof of Theorem 383. From

$$\begin{array}{ccccccccc}
X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & TX^0 & & \\
\| & & \| & & & & & & \\
 & & X^1 & \longrightarrow & Y^1 & \longrightarrow & Z^1 & \longrightarrow & TX^1 \\
\| & & & & \| & & & & \\
X^0 & & \longrightarrow & & Y^1 & \longrightarrow & W & \longrightarrow & TX^0,
\end{array}$$

where the last row is obtained by TR2, we get by TR5

$$\begin{CD}
X^0 @>>> X^1 @>>> X^2 @>>{w}> TX^0 \\
@| @VVV @VV{a}V @| \\
X^0 @>>> Y^1 @>>> W @>{d}>> TX^0 \\
@VVV @| @VV{b}V @VVV \\
X^1 @>>> Y^1 @>>> Z^1 @>>> TX^1 \\
@VVV @VVV @| @VVV \\
X^2 @>>{a}> W @>>{b}> Z^1 @>>{c}> TX^2.
\end{CD} \tag{159}$$

From

$$\begin{array}{cccccccccc}
X^0 & \longrightarrow & Y^0 & \longrightarrow & Z^0 & \longrightarrow & TX^0 & & & \\
\| & & \| & & & & & & & \\
 & & Y^0 & \longrightarrow & Y^1 & \longrightarrow & Y^2 & \longrightarrow & TY^0 & \\
\| & & & & \| & & & & & \\
X^0 & & \longrightarrow & & Y^1 & & \longrightarrow & W & \longrightarrow & TX^0,
\end{array}$$

we get by TR5

$$\begin{CD}
X^0 @>>> Y^0 @>>> Z^0 @>>h> TX^0\\
@| @VVV @VVeV @|\\
X^0 @>>> Y^1 @>>> W @>d>> TX^0\\
@VVV @| @VVV @VVV\\
Y^0 @>>> Y^1 @>>> Y^2 @>>> TY^0\\
@VVV @VVV @| @VVV\\
Z^0 @>>e> W @>>> Y^2 @>>> TZ^0.
\end{CD} \tag{160}$$

From

$$\begin{array}{cccccccccc}
Z^0 & \xrightarrow{e} & W & \longrightarrow & Y^2 & \longrightarrow & TZ^0 & & & \\
\| & & \| & & & & & & & \\
 & & W & \xrightarrow{b} & Z^1 & \xrightarrow{c} & TX^2 & \xrightarrow{-Ta} & TW & \\
\| & & & & \| & & & & & \\
Z^0 & & \longrightarrow & & Z^1 & & \longrightarrow & Z^2 & \xrightarrow{\ell} & TZ^0,
\end{array}$$

where the second row is obtained from $X^2 \xrightarrow{a} W \xrightarrow{b} Z^1 \xrightarrow{c} TX^2$ in (159) by TR3 and TR0, and

$$\text{the last row is obtained by TR2,} \tag{161}$$

we get by TR5

$$\begin{CD}
Z^0 @>>> W @>>> Y^2 @>>> TZ^0 \\
@| @VVbV @VVjV @| \\
Z^0 @>>> Z^1 @>>> Z^2 @>>\ell> TZ^0 \\
@VVV @| @VVkV @VTeVV \\
W @>b>> Z^1 @>c>> TX^2 @>-Ta>> TW \\
@VVV @VVV @| @VVV \\
Y^2 @>>j> Z^2 @>>k> TX^2 @>>-Ti> TY^2,
\end{CD} \tag{162}$$

where

$$X^2 \xrightarrow{i} Y^2 \xrightarrow{j} Z^2 \xrightarrow{k} TX^2 \text{ is a distinguished triangle.} \tag{163}$$

We want to prove that the bottom right square of

$$\begin{CD}
X^0 @>u>> X^1 @>v>> X^2 @>w>> TX^0 \\
@VfVV @VVV @VViV @VVTfV \\
Y^0 @>>> Y^1 @>>> Y^2 @>>> TY^0 \\
@VgVV @VVV @VVjV @VVTgV \\
Z^0 @>>> Z^1 @>>> Z^2 @>\ell>> TZ^0 \\
@VhVV @VVV @VVkV @VV-ThV \\
TX^0 @>>Tu> TX^1 @>>Tv> TX^2 @>>-Tw> T^2X^0
\end{CD} \tag{164}$$

anti-commutes, that the eight other squares commute, and that all rows and columns are distinguished triangles.

We list the nine squares of each of the diagrams (159), (160), (162), (164) as follows:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

and we denote the j-th square of Diagram (i) by $(i)j$.

The commutativity of (159)2 and (160)5 implies that of (164)2.

The commutativity of (159)3 and (160)6 implies that of (164)3.

The commutativity of (160)7 and (162)1 implies that of (164)4.

The commutativity of (160)8 and (162)2 implies that of (164)5.

The commutativity of (160)9 and (162)3 implies that of (164)6.

The commutativity of (160)3 and (159)6 implies that of (164)7.

The commutativity of (159)9 and (162)8 implies that of (164)8.

To prove the anti-commutativity of the bottom right square of (164), note

$$\begin{aligned} Th \circ \ell &= Td \circ Te \circ \ell && \text{by (160)} \\ &= -Td \circ Ta \circ k && \text{by (162)} \\ &= -Tw \circ k && \text{by (159).} \end{aligned}$$

The third row and column are distinguished triangles by (161) and (163) respectively. It is easy to check that the other rows and columns are distinguished triangles too. □

12.2 Brief comments

§ 385. Definition 10.1.9 (i) p. 244. It is written:

"A triangulated functor of triangulated categories $F : (\mathcal{D}, T) \to (\mathcal{D}', T')$ is a functor of additive categories with translation sending distinguished triangles to distinguished triangles. If moreover F is an equivalence of categories, F is called an equivalence of triangulated categories."

This terminology is justified by the fact that, in the above setting, any quasi-inverse to F is triangulated. This point is implicit in the proof of Proposition 10.3.3 p. 253 of the book. Here are more details:

Lemma 386. *If $(\mathcal{D}, T)$ is an additive category with translation, if Δ and Δ' are two sets of triangles in $(\mathcal{D}, T)$ such that $(\mathcal{D}, T, \Delta)$ and $(\mathcal{D}, T, \Delta')$ are triangulated categories, and if $\Delta \subset \Delta'$, then $\Delta = \Delta'$.*

Proof. Left to the reader. □

Lemma 387. *If $(\mathcal{D}, T, \Delta)$ and $(\mathcal{D}', T', \Delta')$ are two triangulated categories, if $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ are two quasi-inverse equivalences, and if F is triangulated, then so is G.*

Proof. It is easy to see that the functor G is additive and commutes with the translations (up to isomorphism). Let $G\Delta'$ be the set of those triangles in $\mathcal{D}$ which are isomorphic to the image under G of some d.t. in $\mathcal{D}'$. It is straightforward to check that $(\mathcal{D}, T, G\Delta')$ is triangulated, that $\Delta \subset G\Delta'$, and that Lemma 386 implies $G\Delta' = \Delta$, that is, G is triangulated. □

§ 388. P. 248, Lemma 10.2.1. The proof shows also the following: If $\mathcal{D}$ is a triangulated category and $\mathcal{N}$ is a full saturated triangulated subcategory of $\mathcal{D}$, then a triangle in $\mathcal{N}$ is a d.t. in $\mathcal{D}$ if and only if it is a d.t. in $\mathcal{N}$.

§ 389. P. 249, Theorem 10.2.3 (v). After "Then F factors uniquely through Q" one could add "and the induced functor $\mathcal{D}/\mathcal{N} \to \mathcal{D}'$ is triangulated".

§ 390. P. 250, proof of Theorem 10.2.3 (iii). In view of Corollary 384 p. 240, it is not necessary to prove TR4.

§ 391. P. 253, Definition 10.3.1. The definition is stated as follows:

We say that a triangulated functor $F : \mathcal{D} \to \mathcal{D}'$ is right (resp. left) localizable with respect to $(\mathcal{N}, \mathcal{N}')$ if $Q' \circ F : \mathcal{D} \to \mathcal{D}'/\mathcal{N}'$ is universally right (resp. left) localizable with respect to the multiplicative system $\mathcal{N}Q$ (see Definition 7.3.1). Recall that it means that, for any $X \in \mathcal{D}$,

$$\underset{(X\to Y)\in\mathcal{N}Q^X}{\text{“colim”}}\, Q'F(Y),$$

respectively

$$\underset{(Y\to X)\in\mathcal{N}Q_X}{\text{“lim”}}\, Q'F(Y),$$

is representable in $\mathcal{D}'/\mathcal{N}'$. If there is no risk of confusion, we simply say that F is right (resp. left) localizable or that RF exists.

It is implicitly assumed that the underlying universe $\mathcal{U}$ has been chosen so that $\mathcal{D}$ is $\mathcal{U}$-small (Definition 5 p. 10). The second sentence in the above quote is justified by Theorem 95 p. 68 (the "Universal Kan Extension Theorem").

§ 392. P. 254. The fact that the morphism $R_{\mathcal{N}}^{\mathcal{N}'}F$ in the commutative diagram is triangulated follows from Lemma 387 p. 244.

§ 393. P. 254, Display (10.3.1). Write $s_X : X \to Y_X$ for the morphism in $\mathcal{N}Q$ with Y_X in $\mathcal{I}$ which exists by assumption. Then Display (10.3.1) can be written as

$$R_{\mathcal{N}}^{\mathcal{N}'}F(Q(X)) := Q'(F(Y_X)).$$

Moreover, the structural morphism $Q' \circ F \to RF \circ Q$ is given by $Q'(F(X)) \xrightarrow{F(s_X)} Q'(F(Y_X)) = RF(Q(X))$. (See §254(e).)

§ 394. P. 254, Proposition 10.3.5 (ii). Write R for $R_{\mathcal{N}}^{\mathcal{N}'}, R_{\mathcal{N}}^{\mathcal{N}''}$ and $R_{\mathcal{N}'}^{\mathcal{N}''}$; write $s_X : X \to Y_X$ for the morphism in $\mathcal{N}Q$ with Y_X in $\mathcal{I}$ which exists by assumption; write $t_{X'} : X' \to Y'_{X'}$ for the morphism in $\mathcal{N}'Q'$ with Y_X in $\mathcal{I}$ which exists by assumption; and define RF, RF' and $R(F' \circ F)$ is in §393 above. We may suppose that $t_{X'} = \mathrm{id}_{X'}$ whenever X' is already in $\mathcal{I}'$. Then we have

$$R(F' \circ F)(Q(X)) = Q''(F'(F(Y_X))),$$

$$RF'(RF(Q(X))) = RF'(Q'(F(Y_X))) = Q''(F'(Y'_{F(Y_X)})) = Q''(F'(F(Y_X))).$$

(It is crucial that the above displays are chains of equalities, as opposed to chains of isomorphisms.)

§ 395. Proof of Theorem 10.4.1 p. 257. The fact that "the canonical functor $\mathcal{T}_{X_1} \times \mathcal{T}_{X_2} \to \mathcal{T}_{X_i}$ (i=1,2) is cofinal" follows from Lemma 215 p. 136.

We rewrite the first display on page 257 of the book as

$$\begin{aligned}(\varphi^\dagger F)(X_1 \oplus X_2) = \operatorname*{colim}_{Y \in \mathcal{T}_{X_1} \oplus \mathcal{T}_{X_2}} F(Y) &\xleftarrow[\text{(a)}]{\sim} \operatorname*{colim}_{(Y_1,Y_2) \in \mathcal{T}_{X_1} \times \mathcal{T}_{X_2}} F(Y_1 \oplus Y_2) \\ &\underset{\text{(b)}}{\simeq} \operatorname*{colim}_{(Y_1,Y_2) \in \mathcal{T}_{X_1} \times \mathcal{T}_{X_2}} F(Y_1) \oplus F(Y_2) \\ &\underset{\text{(c)}}{\simeq} \left(\operatorname*{colim}_{(Y_1,Y_2) \in \mathcal{T}_{X_1} \times \mathcal{T}_{X_2}} F(Y_1)\right) \oplus \left(\operatorname*{colim}_{(Y_1,Y_2) \in \mathcal{T}_{X_1} \times \mathcal{T}_{X_2}} F(Y_2)\right) \\ &\xrightarrow[\text{(d)}]{\sim} \left(\operatorname*{colim}_{Y_1 \in \mathcal{T}_{X_1}} F(Y_1)\right) \oplus \left(\operatorname*{colim}_{Y_2 \in \mathcal{T}_{X_2}} F(Y_2)\right) = (\varphi^\dagger F)(X_1) \oplus (\varphi^\dagger F)(X_2).\end{aligned}$$

Isomorphism (a) follows from the cofinality of ξ, Isomorphism (b) follows from the additivity of F, Isomorphism (c) is straightforward, Isomorphism (d) follows from the cofinality of $\mathcal{T}_{X_1} \times \mathcal{T}_{X_2} \to \mathcal{T}_{X_i}$ for $i = 1, 2$.

For $i = 1, 2$ let

$$f : \operatorname*{colim}_{\varphi(Y) \to X_1 \oplus X_2} F(Y) \to \operatorname*{colim}_{\varphi(Y_i) \to X_i} F(Y_i)$$

be the morphism defined by the above display.

For $i = 1, 2$ define

$$g : \operatorname*{colim}_{\varphi(Y) \to X_1 \oplus X_2} F(Y) \to \operatorname*{colim}_{\varphi(Y_i) \to X_i} F(Y_i)$$

as follows: If

$$p[\varphi(Y) \to X_1 \oplus X_2] : F(Y) \to \operatorname*{colim}_{\varphi(Y) \to X_1 \oplus X_2} F(Y)$$

is the coprojection corresponding to $\varphi(Y) \to X_1 \oplus X_2$, and

$$p[\varphi(Y) \to X_1 \oplus X_2] : F(Y) \to \operatorname*{colim}_{\varphi(Y) \to X_1 \oplus X_2} F(Y)$$

is the coprojection corresponding to $\varphi(Y_i) \to X_i$, then g is defined by the commutative diagrams

$$\begin{array}{ccc}
\operatorname*{colim}\limits_{\varphi(Y) \to X_1 \oplus X_2} F(Y) & \xrightarrow{\ g\ } & \operatorname*{colim}\limits_{\varphi(Y_i) \to X_i} F(Y_i) \\
{\scriptstyle p[\varphi(Y) \to X_1 \oplus X_2]}\big\uparrow & & \big\uparrow{\scriptstyle q[\varphi(Y) \to X_1 \oplus X_2 \to X_i]} \\
F(Y) & =\!=\!=\!= & F(Y).
\end{array}$$

We must show $f = g$. It suffices to check that the diagrams

$$\begin{array}{ccc}
\operatorname*{colim}\limits_{\varphi(Y) \to X_1 \oplus X_2} F(Y) & \xrightarrow{\ f\ } & \operatorname*{colim}\limits_{\varphi(Y_i) \to X_i} F(Y_i) \\
{\scriptstyle p[\varphi(Y) \to X_1 \oplus X_2]}\big\uparrow & & \big\uparrow{\scriptstyle q[\varphi(Y) \to X_1 \oplus X_2 \to X_i]} \\
F(Y) & =\!=\!=\!= & F(Y).
\end{array}$$

commute. This verification is left to the reader.

The fact that $\mathcal{T}_X$ is cofinally small follows from Proposition 3.4.5 (iii) p. 89 of the book.

12.3 Brown's Representability Theorem

§ 396. P. 260, Remark 10.5.4, phrase "it is easy to see that (iii) implies (ii)". For the reader's convenience we spell out the argument.

Let us define $\mathcal{S}$ as in Remark 10.5.4. We have a family $(X_i \to Y_i)$ of morphisms in $\mathcal{D}$ and a morphism $C \to \oplus X_i$ in $\mathcal{D}$ with C in $\mathcal{F}$, we assume that the obvious maps $\mathrm{Hom}_{\mathcal{D}}(C, X_i) \to \mathrm{Hom}_{\mathcal{D}}(C, Y_i)$ vanish, and we must show that the composition $C \to \oplus X_i \to \oplus Y_i$ vanishes. By hypothesis there is a morphism $C \to \oplus C_i$ with C_i in $\mathcal{S}$ such that the composition $C \to \oplus C_i \to \oplus X_i$ coincides with the above morphism $C \to \oplus X_i$, so that it suffices to prove that the composition $\oplus C_i \to \oplus X_i \to \oplus Y_i$ vanishes, or even to prove that the composition $C_{i_0} \to \oplus X_i \to \oplus Y_i$ vanishes for all i_0 in I. But this follows from (ii) and the definition of $\mathcal{F}$.

§ 397. P. 260, Remark 10.5.4. We also spell out the proof of the implication (ii)$'$ $\Rightarrow$(iii).

Let $f : C \to \oplus X_i$ be given. We must find a family of morphisms $(u_i : C_i \to X_i)$ in $\mathcal{D}$ with C_i in $\mathcal{S}$ and a morphism $g : C \to \oplus C_i$ such that

$$f = (C \xrightarrow{g} \oplus C_i \to \oplus X_i).$$

Set

$$C_i := \bigoplus_{C \in \mathcal{F}} \bigoplus_{C \to X_i} C,$$

let $p_i[C, C \to X_i] : C \to C_i$ be the coprojections and let u_i be the unique morphism $C_i \to X_i$ such that $u_i \circ p_i[C, C \to X_i] = (C \to X_i)$ for all C and all $C \to X_i$. Then (ii)$'$ implies that the obvious map

$$\mathrm{Hom}_{\mathcal{D}}(C, \oplus C_i) \to \mathrm{Hom}_{\mathcal{D}}(C, \oplus X_i)$$

is surjective, and it suffices to let g be a pre-image of f.

§ 398. P. 261, proof of Lemma 10.5.6. Let us adhere to the notation of Lemma 10.5.6 and its proof.

(a) Let F be in $\mathcal{S}^{\wedge,prod}$, set

$$V_F := \bigoplus_{C \in \mathcal{S}_0} \bigoplus_{C \to F} C,$$

where $C \to F$ runs over $\mathrm{Hom}_{\mathcal{S}^{\wedge,prod}}(C, F)$, write $p[C, C \to F] : C \to X$ for the coprojections and let $e_F : V_F \to F$ be the unique morphism from V_F to F in $\mathcal{S}^{\wedge,prod}$ which satisfies $e_F \circ p[C, C \to F] = (C \to F)$ for all $C \to F$ in $\mathrm{Hom}_{\mathcal{S}^{\wedge,prod}}(C, F)$.

We claim that $e_F : V_F \to F$ is an epimorphism.

To prove the claim it suffices to let C be in $\mathcal{S}_0$ and to verify that $e_F(C) : V_F(C) \to F(C)$ is surjective. But this is clear because we have $e_F(C)(p[C, C \to F]) = (C \to F)$.

(b) Define $\widetilde{\varphi} : \mathcal{D} \to \mathcal{S}^{\wedge,prod}$ by $\widetilde{\varphi}(X)(C) := \mathrm{Hom}_{\mathcal{D}}(C, X)$ and let X be in $\mathcal{D}$. Then we have a canonical bijection $\mathrm{Hom}_{\mathcal{S}^{\wedge,prod}}(C, \widetilde{\varphi}(X)) \simeq \mathrm{Hom}_{\mathcal{D}}(C, X)$. Abusing the notation we identify these two sets. Let $e_X : V_{\widetilde{\varphi}(X)} \to X$ be the unique morphism from $V_{\widetilde{\varphi}(X)}$ to X in $\mathcal{D}$ which satisfies $e_X \circ p[C, C \to X] = (C \to X)$ for all $C \to X$ in $\mathrm{Hom}_{\mathcal{D}}(C, X)$. Then we have

$$\widetilde{\varphi}(e_X) = e_{\widetilde{\varphi}(X)}. \tag{165}$$

(c) Let H be in $\mathcal{D}^{\wedge,prod}$ and let $H_0 \in \mathcal{S}^{\wedge,prod}$ be the restriction of H to $\mathcal{S}$. Then $e_{H_0} \in H_0(V_{H_0}) = H(V_{H_0}) \simeq \mathrm{Hom}_{\mathcal{D}^{\wedge,prod}}(\widetilde{\varphi}(V_{H_0}), H)$, that is, e_{H_0} can be viewed as a morphism $e_H : V_{H_0} \to H$ in $\mathcal{D}^{\wedge,prod}$.

§ 399. P. 262, proof of Lemma 10.5.7 (i). Here is the implicit commutative diagram:

$$\begin{array}{ccccc} F(\oplus_i X_i) & & \xrightarrow{\sim} & & \prod_i H(\varphi X_i, F) \\ \downarrow & \searrow^{\sim} & & \nearrow^{\sim} & \downarrow \\ & H(\varphi \oplus_i X_i, F) & \longrightarrow & H(\oplus_i \varphi X_i, F) & \\ & \swarrow & & \searrow & \\ F(X_i) & & \longrightarrow & & H(\varphi X_i, F), \end{array}$$

where we have written H for $\mathrm{Hom}_{\mathcal{S}^{\wedge,prod}}$ to save space.

§ 400. P. 261, Lemma 10.5.7 (ii). The proof shows that the direct sum in $\mathcal{S}^{\wedge,add}$ of a small family of objects of $\mathcal{S}^{\wedge,prod}$ belongs to $\mathcal{S}^{\wedge,prod}$.

More precisely, if $\iota : \mathcal{S}^{\wedge,prod} \to \mathcal{S}^{\wedge,add}$ is the natural functor, then the proof of Lemma 10.5.7 (ii) in the book shows, in the notation used there, that $\oplus_i \, \iota F_i$ is isomorphic to $\iota \, \mathrm{Coker}(\varphi(\oplus_i X_i) \to \varphi(\oplus_i Y_i))$.

In view of Lemma 10.5.5 p. 260 of the book, this implies that the limit of a small inductive system in $\mathcal{S}^{\wedge,prod}$ exists in $\mathcal{S}^{\wedge,add}$ and lies in $\mathcal{S}^{\wedge,prod}$. Moreover, §315 p. 192 entails that small filtrant inductive limits exist and are exact in $\mathcal{S}^{\wedge,add}$ and $\mathcal{S}^{\wedge,prod}$. The exactness of small filtrant inductive limits in $\mathcal{S}^{\wedge,prod}$ will be used in §403 p. 251.

Right after the proof of Lemma 10.5.7 it is written: "Note that, for a small family $(F_i)_i$ of objects in $\mathcal{S}^{\wedge,prod}$ and $X \in \mathcal{S}$, the map $\oplus_i(F_i(X)) \to (\oplus_i F_i)(X)$ may be not bijective". I think this is not true, that is, I think that the map $\oplus_i(F_i(X)) \to (\oplus_i F_i)(X)$ is *always* bijective. (If I'm wrong on this, then the present part of this text is incorrect.)

Let us insist on the main point:

$$\boxed{\text{Small filtrant colimits exist in } \mathcal{S}^{\wedge,prod} \text{ and are exact.}}$$

§ 401. P. 263, proof of Lemma 10.5.8 (ii).

• In the notation of §398(b) p. 248 we set $Y_i := V_{\widetilde{\varphi}(X_i)}$ and $(Y_i \to X_i) := e_{X_i}$ and we get $\widetilde{\varphi}(e_{X_i}) = e_{\widetilde{\varphi}(X_i)}$ by (165) p. 248.

• The fact that $\oplus_i W_i \to \oplus_i Y_i \to \oplus_i X_i \to T(\oplus_i W_i)$ is a d.t. follows from Proposition 10.1.19 p. 247 of the book.

• Last sentence.

Variant 1: Consider the commutative diagram

$$\begin{array}{ccccccc} \oplus_i \varphi(Z_i) & \longrightarrow & \oplus_i \varphi(Y_i) & \longrightarrow & \oplus_i \widetilde{\varphi}(X_i) & \longrightarrow & 0 \\ \| & & \| & & \downarrow & & \\ \oplus_i \varphi(Z_i) & \longrightarrow & \oplus_i \varphi(Y_i) & \longrightarrow & \widetilde{\varphi}(\oplus_i X_i) & \longrightarrow & 0, \end{array}$$

whose rows are complexes. We already know that the bottom row is exact. The exactness of the top row follows (as in the proof of Lemma 10.5.7 (ii) p. 261 of the book) from the isomorphisms

$$\operatorname{Coker}(\oplus_i \varphi(Z_i) \to \oplus_i \varphi(Y_i)) \simeq \oplus_i \operatorname{Coker}(\varphi(Z_i) \to \varphi(Y_i)) \simeq \oplus_i \widetilde{\varphi}(X_i).$$

Variant 2: Apply the Five Lemma to the commutative diagram

$$\begin{array}{ccccccc} \oplus_i\varphi(Z_i) & \longrightarrow & \oplus_i\varphi(Y_i) & \longrightarrow & \oplus_i\widetilde{\varphi}(X_i) & \longrightarrow & 0 \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \\ \varphi(\oplus_i Z_i) & \longrightarrow & \varphi(\oplus_i Y_i) & \longrightarrow & \widetilde{\varphi}(\oplus_i X_i) & \longrightarrow & 0. \end{array}$$

§ 402. P. 263, proof of Lemma 10.5.9.

By §398(c) p. 248) the morphism $X_0 \to H_0$ in $\mathcal{S}^{\wedge,prod}$ extends to a morphism $X_0 \to H$ in $\mathcal{D}^{\wedge,prod}$.

Vanishing of $Z_n \to X_n \to H$: Let $0 \to F \to X_n \to H$ be exact in $\mathcal{D}^{\wedge,add}$, in the notation of §398(a) p. 248 set $Z_n := V_F$ and

$$(Z_n \longrightarrow X_n) := (Z_n \xrightarrow{e_F} F \longrightarrow X_n).$$

The vanishing of $Z_n \to X_n \to H$ is then clear.

Before the sentence "Since Z_n and X_n belong to $\mathcal{K}$, X_{n+1} also belongs to $\mathcal{K}$", one could add "We may, and do, assume that $\mathcal{K}$ is saturated".

Recall the Yoneda isomorphisms

$$\mathrm{Hom}_{\mathcal{S}^{\wedge,\mathrm{prod}}}(\varphi(X), H_0) \simeq H(X) \simeq \mathrm{Hom}_{\mathcal{D}^\wedge}(X, H)$$

for X in $\mathcal{S}$.

Note that Convention 264 p. 167 can be applied.

§ 403. P. 264, proof of Lemma 10.5.11. As observed in §400 p. 249, small filtrant inductive limits exist and are exact in $\mathcal{S}^{\wedge,prod}$.

12.4 Exercise 10.3 p. 265

It suffices to prove:

If $L : \mathcal{D} \to \mathcal{D}'$ is a triangulated functor of triangulated categories and $R : \mathcal{D}' \to \mathcal{D}$ is right adjoint to L, then R is triangulated.

The following argument is taken from Tag 0A8D in the Stacks Project:

http://stacks.math.columbia.edu/tag/0A8D

Proof. Let X be an object of $\mathcal{D}$ and X' an object of $\mathcal{D}'$. Since L is triangulated we have isomorphisms functorial in X and X'

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(X, R(T(X')) &\simeq \mathrm{Hom}_{\mathcal{D}'}(L(X), T(X'))\\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(T^{-1}(L(X)), X')\\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(T^{-1}(L(X)), X')\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(T^{-1}(X), R(X'))\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X, T(R(X'))).\end{aligned}$$

By Yoneda's lemma we obtain an isomorphism $T(R(X')) \simeq T(R(X'))$ functorial in X'. Let

$$X' \to Y' \to Z' \to T(X')$$

be a distinguished triangle in $\mathcal{D}'$. Choose a distinguished triangle

$$R(X') \to R(Y') \to Z \to T(R(X'))$$

in $\mathcal{D}$. Then

$$L(R(X')) \to L(R(Y')) \to L(Z) \to T(L(R(X')))$$

is a distinguished triangle in $\mathcal{D}'$. By TR4 we can choose a morphism of distinguished triangles

$$\begin{array}{ccccccc} L(R(X')) & \longrightarrow & L(R(Y')) & \longrightarrow & L(Z) & \longrightarrow & T(L(R(X'))) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

Since R is right adjoint to L the morphism $L(Z) \to Z'$ determines a morphism $Z \to R(Z')$ such that the diagram

$$\begin{array}{ccccccc} R(X') & \longrightarrow & R(Y') & \longrightarrow & Z & \longrightarrow & T(R(X')) \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ R(X') & \longrightarrow & R(Y') & \longrightarrow & R(Z') & \longrightarrow & T(R(X')) \end{array}$$

commutes. Applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(W, \ )$ for an object W of $\mathcal{D}$, we get a commutative diagram of abelian groups of the form

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & U_0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & \cdots \end{array}$$

The top row is an exact complex for obvious reasons, whereas the bottom row is an exact complex because of the isomorphism $\mathrm{Hom}_{\mathcal{D}}(W, R(\ )) \simeq \mathrm{Hom}_{\mathcal{D}'}(L(W), \ )$. We deduce from the 5 lemma that $\mathrm{Hom}_{\mathcal{D}}(W, Z) \to \mathrm{Hom}_{\mathcal{D}}(W, R(Z'))$ is bijective, and using the Yoneda lemma once more we conclude that $Z \to R(Z')$ is an isomorphism. Hence we conclude that $R(X') \to R(Y') \to R(Z') \to T(R(X'))$ is a distinguished triangle, which is what we wanted to show. $\square$

12.5 Exercise 10.11 p. 266

Recall the statement:

(i) Let $\mathcal{N}$ be a null system in a triangulated category $\mathcal{D}$, let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the localization functor, and let $f : X \to Y$ be a morphism in $\mathcal{D}$ satisfying $Q(f) = 0$. Then f factors through some object of $\mathcal{N}$.

(ii) The following conditions on X in $\mathcal{D}$ are equivalent:

(a) $Q(X) \simeq 0$, (b) $X \oplus Y \in \mathcal{N}$ for some $Y \in \mathcal{D}$, (c) $X \oplus TX \in \mathcal{N}$.

Proof.

(i) The definition of $\mathcal{D}/\mathcal{N}$ and the assumption $Q(f) = 0$ imply the existence of a morphism $s : Y \to Z$ in $\mathcal{N}Q$ such that $s \circ f = 0$ (see (7.1.5) p. 155 of the book), and thus, in view of the definition of $\mathcal{N}Q$ (see (10.2.1) p. 249 of the book), the existence of a d.t. $W \to Y \to Z \to TW$ with W in $\mathcal{N}$, and the conclusion follows from the fact that $\mathrm{Hom}_{\mathcal{D}}(X,\ )$ is cohomological (see Proposition 10.1.13 p. 245 of the book).

(ii)

(a)⇒(b): As $Q(\mathrm{id}_X) = 0$, the first part of the exercise implies that id_X factors as $X \xrightarrow{f} Z \xrightarrow{g} X$ with Z in $\mathcal{N}$. By TR2 there is a d.t.

$$X \xrightarrow{f} Z \xrightarrow{h} Y \xrightarrow{k} TX.$$

Since $g \circ f = \mathrm{id}_X$, the morphism f is a monomorphism, and so is Tf. As $Tf \circ k = 0$ by Proposition 10.1.11 p. 245 of the book, this implies $k = 0$. Hence we have a morphism of d.t.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Z & \xrightarrow{h} & Y & \xrightarrow{0} & TX \\ \| & & \downarrow{\scriptstyle (g,h)} & & \| & & \| \\ X & \longrightarrow & X \oplus Y & \longrightarrow & Y & \longrightarrow & TX \end{array}$$

(the bottom is a d.t. by Corollary 10.1.20 (ii) p. 248 of the book) and Proposition 10.1.15 p. 246 of the book implies that (g, h) is an isomorphism.

(b)⇒(c): Let $\Delta_1, \ldots, \Delta_5$ be the triangles

$$X \longrightarrow 0 \longrightarrow TX \xrightarrow{\text{id}} TX$$

$$Y \xrightarrow{\text{id}} Y \longrightarrow 0 \longrightarrow TY$$

$$X \oplus Y \longrightarrow Y \longrightarrow TX \longrightarrow TX \oplus TY$$

$$0 \longrightarrow X \xrightarrow{\text{id}} X \longrightarrow 0$$

$$X \oplus Y \longrightarrow X \oplus Y \longrightarrow X \oplus TX \longrightarrow TX \oplus TY,$$

with $\Delta_3 := \Delta_1 \oplus \Delta_2$ and $\Delta_5 := \Delta_3 \oplus \Delta_4$. It is easy to see that Δ_1, Δ_2 and Δ_4 are distinguished. Then Δ_3 and Δ_5 are distinguished by Proposition 10.1.19 p. 247 of the book, and, as $X \oplus Y$ is in $\mathcal{N}$, Condition N'3 of Lemma 10.2.1 (b) p. 249 of the book implies that $X \oplus TX$ is in $\mathcal{N}$.

(c)⇒(a): Follows from Theorem 10.2.3 (iv) p. 249 of the book. □

13 About Chapter 11

13.1 Brief comments

§ 404. P. 270. Recall that $(\mathcal{A}, T)$ is an additive category with translation. Let

$$(d_{X,i} : X_i \to TX_i)_{i \in I} \tag{166}$$

be an inductive system in $\mathcal{A}_d$. Assume that $X := \operatorname{colim}_i X_i$ exists in $\mathcal{A}$. Then the natural morphism $\operatorname{colim}_i d_{X,i} : X \to TX$ is an inductive limit of (166) in $\mathcal{A}_d$. There are analogous statements with "projective" instead of "inductive" and $\mathcal{A}_c$ instead of $\mathcal{A}_d$.

§ 405. P. 270, Definition 11.1.3. Here is a "picture" of the mapping cone of $f : X \to$

Y:

$$\begin{array}{ccc} TX & \xrightarrow{-T(d_X)} & T^2X \\ & \searrow^{T(f)} & \\ \oplus & & \oplus \\ Y & \xrightarrow[d_Y]{} & TY. \end{array}$$

§ 406. P. 271, Remark 11.1.5. We have:

$$d_{\mathrm{Mc}(T(f))} = \begin{pmatrix} d_{T^2X} & 0 \\ T^2(f) & d_{TY} \end{pmatrix}, \quad d_{T(\mathrm{Mc}(f))} = \begin{pmatrix} d_{T^2X} & 0 \\ -T^2(f) & d_{TY} \end{pmatrix},$$

and

$$T(\mathrm{Mc}(f)) = T^2X \oplus TY \xrightarrow{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} T^2X \oplus TY = \mathrm{Mc}(T(f))$$

is a differential isomorphism.

13.2 Theorem 11.2.6 p 273

Here is a minor comment about the verification of Axiom TR5 in the proof of Theorem 11.2.6. We stated Axiom TR5 right after Corollary 384 p. 240 above. For the reader's convenience we restate it.

If the diagram

$$\begin{array}{ccccccccc} U & \longrightarrow & V & \longrightarrow & W' & \longrightarrow & TU & & \\ \| & & \| & & & & & & \\ & & V & \longrightarrow & W & \longrightarrow & U' & \longrightarrow & TV \\ \| & & & & \| & & & & \\ U & \longrightarrow & \longrightarrow & \longrightarrow & W & \longrightarrow & V' & \longrightarrow & TU \end{array}$$

commutes, and if the rows are distinguished triangles, then there is a distinguished triangle $W' \to V' \to U' \to TW'$ such that the diagram below commutes:

$$
\begin{array}{ccccccc}
U & \longrightarrow & V & \longrightarrow & W' & \longrightarrow & TU \\
\| & & \downarrow & & \downarrow & & \| \\
U & \longrightarrow & W & \longrightarrow & V' & \longrightarrow & TU \\
\downarrow & & \| & & \downarrow & & \downarrow \\
V & \longrightarrow & W & \longrightarrow & U' & \longrightarrow & TV \\
\downarrow & & \downarrow & & \| & & \downarrow \\
W' & \longrightarrow & V' & \longrightarrow & U' & \longrightarrow & TW'.
\end{array}
$$

Going back to the proof of TR5 on p. 275, we consider the commutative diagram

$$
\begin{array}{ccccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\alpha(f)} & TX \oplus Y & \xrightarrow{\beta(f)} & TX & & & & \\
\| & & \| & & & & & & & & \\
\| & & Y & \xrightarrow{g} & Z & \xrightarrow{\alpha(g)} & TY \oplus Z & \xrightarrow{\beta(g)} & TY & & \\
\| & & & & \| & & & & & & \\
X & & \xrightarrow[g \circ f]{} & & Z & & \xrightarrow[\alpha(g \circ f)]{} & & TX \oplus Z & \xrightarrow[\beta(g \circ f)]{} & TX.
\end{array}
$$

The goal of the proof is then to construct a commutative diagram

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\alpha(f)} & TX \oplus Y & \xrightarrow{\beta(f)} & TX \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \| \\
X & \xrightarrow{g \circ f} & Z & \xrightarrow{\alpha(g \circ f)} & TX \oplus Z & \xrightarrow{\beta(g \circ f)} & TX \\
{\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle T(f)} \\
Y & \xrightarrow{g} & Z & \xrightarrow{\alpha(g)} & TY \oplus Z & \xrightarrow{\beta(g)} & TY \\
{\scriptstyle \alpha(f)}\downarrow & & {\scriptstyle \alpha(g \circ f)}\downarrow & & \| & & \downarrow{\scriptstyle T(\alpha(f))} \\
TX \oplus Y & \xrightarrow[u]{} & TX \oplus Z & \xrightarrow[v]{} & TY \oplus Z & \xrightarrow[w]{} & T^2X \oplus TY.
\end{array}
$$

13.3 Brief comments

§ 407. P. 280, Example 11.3.5 (i). It is written "Let $f : X \to Y$ be a morphism in $\mathcal{C}$. We identify f with a morphism in $\mathrm{C}(\mathcal{C})$. Then $\mathrm{Mc}(f)$ is the complex

$$\cdots \to 0 \to X \xrightarrow{f} Y \to 0 \to \cdots$$

where Y stands in degree 0."

I would have said something slightly different, namely:

Let $U^\bullet$ (resp. $V^\bullet$) be the complex having X (resp. Y) in degree 0 and 0 in the other degrees, let $g : U^\bullet \to V^\bullet$ the morphism whose zeroth component is f, and let $W^\bullet$ be the complex $\mathrm{Mc}(g)$. Then we have $W^\bullet = U^{1+\bullet} \oplus V^\bullet$, that is, $W^\bullet$ has X in degree -1, Y in degree 0, and 0 elsewhere, and the differentials of $W^\bullet$ are all equal to 0. The shifted object $W^{1+\bullet}$ has X in degree -2, Y in degree -1, and 0 elsewhere, and the differentials of $W^{1+\bullet}$ are all equal to 0. The -1 component of $d_{W^\bullet} : W^\bullet \to W^{1+\bullet}$ is f.

$*$ P. 282, Definition 11.3.12: see §15 p. 18.

§ 408. P. 286, Notation 11.5.1. Here is a minor variant: Define the functor F_I : $\mathrm{C}^2(\mathcal{C}) \to \mathrm{C}(\mathrm{C}(\mathcal{C}))$ by the formulas

$$(F_I(X)^n)^m := X^{n,m}, \quad d^m_{F_I(X)^n} := d''^{n,m}_X, \quad (d^n_{F_I(X)})^m := d'^{n,m}_X.$$

§ 409. P. 289, beginning of Section 11.6. The key formula in the definition of $\mathrm{C}(F) : \mathrm{C}(\mathcal{C}) \times \mathrm{C}(\mathcal{C}') \to \mathrm{C}(\mathcal{C}'')$ is

$$d''^{n,m}_{F(X,Y)} := (-1)^m \, F(X^n, d^m_Y).$$

§ 410. P. 290, Example 11.6.2 (i) (see §16 p. 19). Writing $F(U,V)$ for $\mathrm{Hom}_{\mathcal{C}}(U,V)$, the differential $d_{F(X,Y)}$ is given, in the notation right before Proposition 11.5.5 p. 287 of the book, by the diagram

$$\begin{array}{ccc} & & F(X^{-m},Y^{n-1}) \\ & & \downarrow{\scriptstyle F(X^{-m},d^{n-1}_Y)} \\ F(X^{1-m},Y^n) & \xrightarrow[(-1)^{m+n}F(d^{-m}_X,Y^n)]{} & F(X^{-m},Y^n). \end{array}$$

Here is another way of writing the same formula:

$$(d^{m+n-1}_{F(X,Y)}((f_{i,j})))_{n,-m} = d^{n-1}_Y \circ f_{n-1,-m} + (-1)^{m+n} f_{n,1-m} \circ d^{-m}_X.$$

§ 411. P. 290, Example 11.6.2 (i). (As already stated, there is a typo; see §16 p. 19.) Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be additive categories with translation. If $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ is a bifunctor of additive categories with translation and if $\mathcal{C}''$ admits countable direct

sums, then, as explained in the book, F induces a bifunctor of additive categories with translation
$$F_\oplus : \mathrm{C}(\mathcal{C}) \times \mathrm{C}(\mathcal{C}') \to \mathrm{C}(\mathcal{C}'').$$
If $\mathcal{C}''$ admits countable products instead of direct sums, then F induces a bifunctor of additive categories with translation

$$F_\pi : \mathrm{C}(\mathcal{C}) \times \mathrm{C}(\mathcal{C}') \to \mathrm{C}(\mathcal{C}'').$$

The precise formulas are given in the book. If $F : \mathcal{C} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{C}''$ is a bifunctor of additive categories with translation and if $\mathcal{C}''$ admits countable products, then F induces again a bifunctor of additive categories with translation

$$F_\pi : \mathrm{C}(\mathcal{C}) \times \mathrm{C}(\mathcal{C}')^{\mathrm{op}} \to \mathrm{C}(\mathcal{C}'').$$

The formulas defining F_π in this setting are almost the same as in the previous setting, and we give them without further comments:

$$F_\pi(Y,X)^{n,m} = F(Y^n, X^{-m}),$$

$$d'^{n,m} = F(d_Y^n, X^{-m}),$$
$$d''^{n,m} = (-1)^{m+1} F(Y^n, d_X^{-m-1}),$$
$$\theta_{Y,X} : F_\pi(TY,X) \to TF_\pi(Y,X),$$
$$\theta'_{Y,X} : F_\pi(Y,T^{-1}X) \to TF_\pi(Y,X),$$
$$\theta_{Y,X}^{i+j} : F_\pi(TY,X)^{i+j} \to (TF_\pi(Y,X))^{i+j},$$
$$\theta_{Y,X}^{i,j} : F_\pi((TY)^i, X^{-j}) = F(Y^{i+1}, X^{-j}) \to F_\pi(Y,X)^{i+j+1} = (TF_\pi(Y,X))^{i+j},$$
$$\theta'^{i+j}_{Y,X} : F_\pi(Y,T^{-1}X)^{i+j} \to (TF_\pi(Y,X))^{i+j},$$
$$\theta'^{i,j}_{Y,X} : F_\pi(Y^i, (T^{-1}X)^{-j}) = F(Y^i, X^{-j-1}) \to F_\pi(Y,X)^{i+j+1} = (TF_\pi(Y,X))^{i+j},$$
the morphism $\theta'^{i,j}_{Y,X}$ being $(-1)^i$ times the canonical embedding.

§ 412. P. 296, Exercise 11.12, partial solution. Let $f : X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$, where $\mathcal{C}$ is an additive category. One of the sub-exercises asks for a proof of the existence of a distinguished triangle

$$\mathrm{Mc}(\sigma^{>a} f) \to \mathrm{Mc}(f) \to \mathrm{Mc}(\sigma^{\le a} f) \to \mathrm{Mc}(\sigma^{>a} f)[1]$$

in $\mathrm{K}(\mathcal{C})$, which is equivalent to the existence of a distinguished triangle

$$\mathrm{Mc}(\sigma^{\le a} f)[-1] \to \mathrm{Mc}(\sigma^{>a} f) \to \mathrm{Mc}(f) \to \mathrm{Mc}(\sigma^{\le a} f) \tag{167}$$

in $\mathrm{K}(\mathcal{C})$. We claim that there is a morphism $g : \mathrm{Mc}(\sigma^{\le a} f)[-1] \to \mathrm{Mc}(\sigma^{>a} f)$ in $\mathrm{C}(\mathcal{C})$ and an isomorphism $\mathrm{Mc}(g) \simeq \mathrm{Mc}(f)$ in $\mathrm{C}(\mathcal{C})$. Clearly, the claim implies the existence of a distinguished triangle (167). We shall define the morphism g and leave the verification of the claim to the reader.

We define $g : \mathrm{Mc}(\sigma^{\le a} f)[-1] \to \mathrm{Mc}(\sigma^{>a} f)$ as follows:

Firstly we define $g_a : \mathrm{Mc}(\sigma^{\le a} f)[-1]^a \to \mathrm{Mc}(\sigma^{>a} f)^a$ as follows: We identify $\mathrm{Mc}(\sigma^{\le a} f)[-1]^a$ to $X^a \oplus Y^{a-1}$ and $\mathrm{Mc}(\sigma^{>a} f)^a$ to X^{a+1}, and decree that $g_a : X^a \oplus Y^{a-1} \to X^{a+1}$ is the obvious morphism induced by d_X^a.

Secondly we define $g_{a+1} : \mathrm{Mc}(\sigma^{\le a} f)[-1]^{a+1} \to \mathrm{Mc}(\sigma^{>a} f)^{a+1}$ as follows: We identify $\mathrm{Mc}(\sigma^{\le a} f)[-1]^{a+1}$ to Y^a and $\mathrm{Mc}(\sigma^{>a} f)^a$ to $X^{a+2} \oplus Y^{a+1}$, and decree that $g_{a+1} : Y^a \to X^{a+2} \oplus Y^{a+1}$ is the obvious morphism induced by d_Y^a.

Thirdly we set $g_n := 0$ for $n \ne a, a+1$.

We leave it to the reader to check g is indeed a morphism of complexes, and that we have $\mathrm{Mc}(g) \simeq \mathrm{Mc}(f)$ in $\mathrm{C}(\mathcal{C})$.

14 About Chapter 12

14.1 Avoiding the Snake Lemma p. 297

This is about Sections 12.1 and 12.2 of the book. I think the Snake Lemma can be avoided as follows:

Let $\mathcal{A}$ be an abelian category.

Lemma 413. *If*

$$\begin{array}{ccccccccc}
 & & X_1 & \xrightarrow{f} & X_2 & \xrightarrow{g} & X_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_3} & & \\
0 & \longrightarrow & Y_1 & \xrightarrow[f']{} & Y_2 & \xrightarrow[g']{} & Y_3 & &
\end{array}$$

is a commutative diagram in $\mathcal{A}$ with exact rows, then the sequence

$$\operatorname{Ker} u_1 \to \operatorname{Ker} u_2 \to \operatorname{Ker} u_3 \xrightarrow{0} \operatorname{Coker} u_1 \to \operatorname{Coker} u_2 \to \operatorname{Coker} u_3$$

is exact at $\operatorname{Ker} u_2$ *and* $\operatorname{Coker} u_2$*. If in addition u_3 is a monomorphism or u_1 is an epimorphism, then the whole sequence is exact.*

The proof of the above lemma is slightly easier than that of the Snake Lemma. Of course one can argue that the Snake Lemma has an intrinsic interest, and, as it can be proved with only a modest additional effort, it is worth proving it. The limited purpose of this section is to describe an alternative, not to claim that this alternative is better.

Proof. The exactness at $\operatorname{Ker} u_2$ and $\operatorname{Coker} u_2$ is straightforward. Assume u_1 is an epimorphism (the case when u_3 is a monomorphism being). It suffices to show that the morphism g_0 defined by the commutative square

$$\begin{array}{ccc} \operatorname{Ker} u_2 & \xrightarrow{g_0} & \operatorname{Ker} u_3 \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a_3} \\ X_2 & \xrightarrow[g]{} & X_3, \end{array}$$

where a_2 and a_3 are the natural morphisms, is an epimorphism. Let $b_3 : Z_3 \to \operatorname{Ker} u_3$ be a morphism. It suffices to complete the commutative square

$$\begin{array}{ccc} Z_2 & \overset{b}{\dashrightarrow\!\!\!\!\rightarrow} & Z_3 \\ {\scriptstyle b_2}\vdots & & \downarrow{\scriptstyle b_3} \\ \operatorname{Ker} u_2 & \xrightarrow[g_0]{} & \operatorname{Ker} u_3. \end{array}$$

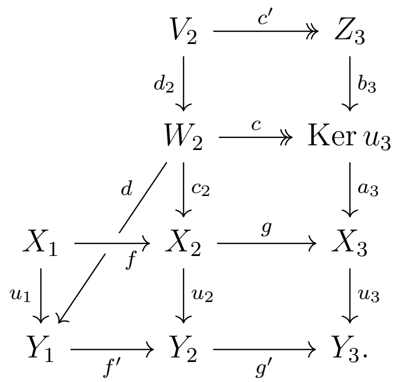
Completing successively the commutative squares

$$\begin{array}{ccc} W_2 & \overset{c}{\dashrightarrow\!\!\!\!\rightarrow} & \operatorname{Ker} u_3 \\ {\scriptstyle c_2}\vdots & & \downarrow{\scriptstyle a_3} \\ X_2 & \underset{g}{\twoheadrightarrow} & X_3 \end{array}$$

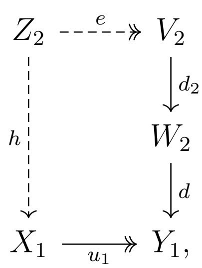
and

$$\begin{array}{ccc} V_2 & \overset{c'}{\dashrightarrow\!\!\!\!\rightarrow} & Z_3 \\ {\scriptstyle d_2}\vdots & & \downarrow{\scriptstyle b_3} \\ W_2 & \underset{c}{\twoheadrightarrow} & \operatorname{Ker} u_3, \end{array}$$

we get the commutative diagram



We complete the commutative square



and define $b : Z_2 \to Z_3$ by

$$b := c'e.$$

(In this proof we write xy for $x \circ y$.)

It remains to define $b_2 : Z_2 \to \operatorname{Ker} u_2$ and to prove $b_3 b = g_0 b_2$.

To define b_2 it suffices to define a morphism $b_2' : Z_2 \to X_2$ such that $u_2 b_2' = 0$. This will give us a morphism b_2 satisfying $a_2 b_2 = b_2'$.

We set $b_2' := c_2 d_2 e - fh$. We have

$$u_2 c_2 d_2 e = f' d d_2 e = f' u_1 h = u_2 f h,$$

and thus $u_2 b_2' = 0$. We also have

$$a_3 b_3 b = a_3 b_3 c' e = g c_2 d_2 e = g c_2 d_2 e - g f h = g b_2' = g a_2 b_2 = a_3 g_0 b_2,$$

and thus $b_3 b = g_0 b_2$ since a_3 is a monomorphism. □

Let $(\mathcal{A}, T)$ be an abelian category with translation.

Lemma 414 (see Theorem 12.2.4 p. 301)**.** *If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in $\mathcal{A}_c$, then the sequence $H(X) \to H(Y) \to H(Z)$ is exact. If, in addition, $H(T^n X) \simeq 0$ (respectively $H(T^n Z) \simeq 0$) for all n, then $T^n Y \to T^n Z$ (respectively $T^n X \to T^n Y$) is a qis for all n.*

Proof. Taking into account Display (12.2.1) p. 300 of the book, apply Lemma 413 to the commutative diagram

$$\begin{array}{ccccccc} & & \operatorname{Coker} T^{-1}d_X & \xrightarrow{f} & \operatorname{Coker} T^{-1}d_Y & \xrightarrow{g} & \operatorname{Coker} T^{-1}d_Z & \longrightarrow 0 \\ & & \downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} & & \downarrow{\scriptstyle d_Z} & \\ 0 & \longrightarrow & \operatorname{Ker} Td_X & \xrightarrow[f]{} & \operatorname{Ker} Td_Y & \xrightarrow[g]{} & \operatorname{Ker} Td_Z. & \end{array}$$

□

Proposition 415 (Corollary 12.2.5 p. 301)**.** *The functor*

$$H : \mathrm{K}_c(\mathcal{A}) \to \mathcal{A}$$

is cohomological.

Proof. Let $X \to Y \to Z \to TX$ be a d.t. in $K_c(\mathcal{A})$. It is isomorphic to

$$V \xrightarrow{\alpha(u)} \mathrm{Mc}(u) \xrightarrow{\beta(u)} TU \to TV$$

for some morphism $u : U \to V$. Since the sequence

$$0 \to V \to \mathrm{Mc}(u) \to TU \to 0$$

in $\mathcal{A}_c$ is exact, it follows from Lemma 414 that the sequence

$$H(V) \to H(\mathrm{Mc}(u)) \to H(TU)$$

is exact. □

Proposition 416 (Corollary 12.2.6 p. 302)**.** *To each short an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}_c$ is attached in a natural way a d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ in $\mathrm{K}_c(\mathcal{A})$. More precisely, let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathcal{A}_c$ and* *define* $\varphi : \mathrm{Mc}(f) \to Z$ *by* $\varphi := (0, g)$. *Then* φ *is a morphism in* $\mathcal{A}_c$, *this morphism is a qis, and it satisfies* $\varphi \circ \alpha(f) = g$. *In particular, there are natural morphisms* $H(T^n Z) \to H(T^{n+1} X)$ *such that the sequence*

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(TX) \to \cdots$$

is exact.

Proof. The commutative diagram in $\mathcal{A}_c$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0 \end{array}$$

yields the exact sequence

$$0 \to \mathrm{Mc}(\mathrm{id}_X) \to \mathrm{Mc}(f) \xrightarrow{\varphi} \mathrm{Mc}(0 \to Z) \to 0$$

in $\mathcal{A}_c$. As $H(\mathrm{Mc}(\mathrm{id}_X)) \simeq 0$, a homotopy $\mathrm{Mc}(\mathrm{id}_X) \to T^{-1}\,\mathrm{Mc}(\mathrm{id}_X)$ being given by the matrix

$$\begin{pmatrix} 0 & \mathrm{id}_X \\ 0 & 0 \end{pmatrix},$$

the morphism φ is a qis by Lemma 414. □

14.2 Brief comments

§ 417. Pp. 300-301, the cohomology functor. In the paragraph just before Lemma 12.2.2 p. 300 it is written:

"We have obtained an additive functor: $H : \mathcal{A}_c \to \mathcal{A}$",

and in the paragraph just before Definition 12.2.3 p. 301 it is written:

"Hence the functor H defines a functor (denoted by the same symbol) $H : \mathrm{K}_c(\mathcal{A}) \to \mathcal{A}$".

In fact we have additive functors

$$H : (\mathcal{A}_c, T) \to (\mathcal{A}, T) \quad \text{and} \quad H : (\mathrm{K}_c(\mathcal{A}), T) \to (\mathcal{A}, T).$$

In Corollary 12.2.5 p. 301 the display $H : \mathrm{K}_c(\mathcal{A}) \to \mathcal{A}$ can also be replaced with $H : (\mathrm{K}_c(\mathcal{A}), T) \to (\mathcal{A}, T)$, and the display

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(TX) \to \cdots$$

can be written

$$\cdots \to H(X) \to H(Y) \to H(Z) \to TH(X) \to \cdots$$

§ 418. P. 303, Display (12.3.3). The morphisms $\tau^{\le n}X \to \widetilde{\tau}^{\le n}X \to X$ induce isomorphisms in the degree $\le n$ cohomology. The morphisms $X \to \widetilde{\tau}^{\ge n}X \to \tau^{\ge n}X$ induce isomorphisms in the degree $\ge n$ cohomology.

§ 419. P. 313, beginning of Section 12.5. Defining $H_I : \mathrm{C}^2(\mathcal{C}) \to \mathrm{C}^2(\mathcal{C})$ by $H_I := F_I^{-1} \circ H \circ F_I$, we get

$$H_I(X)^{n,m} = H^n(X^{\bullet,m}, d_X'^{\bullet,m}), \quad d'_{H_I(X)} = 0, \quad d''^{n,m}_{H_I(X)} = H^n(X^{\bullet,m}, d_X''^{\bullet,m}).$$

The functor $H_I^n : \mathrm{C}^2(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$ is defined by $H_I^n := H^n \circ F_I$.

§ 420. P. 314, proof of Lemma 12.5.2. Define $Y \in \mathrm{C}(\mathrm{C}(\mathcal{C}))$ by

$$Y := \operatorname{Im} d^q_{F_I(X)}[-q-1]$$

(see §408 p. 257), that is:

$$(Y^n)^m = \begin{cases} \operatorname{Im} d_X'^{q,m} & \text{if } n = q+1 \\ 0 & \text{if } n \ne q+1, \end{cases}$$

$d_Y = 0$ and $d^m_{Y^{q+1}}$ is induced by $d_X''^{q+1,m}$. Then we have the exact sequence

$$0 \to \tau^{\le q}F_I(X) \to \widetilde{\tau}^{\le q}F_I(X) \to \mathrm{Mc}(\mathrm{id}_Y) \to 0 \tag{168}$$

in $\mathrm{C}(\mathrm{C}(\mathcal{C}))$. Define $Z \in \mathrm{C}(\mathcal{C})$ by $Z^m := (Y^{q+1})^m$ and $d_Z^m := d^m_{Y^{q+1}}$. Applying $\mathrm{tot} \circ F_I^{-1}$ to (168) we get the exact sequence

$$0 \to \mathrm{tot}\, \tau_I^{\le q}F_I(X) \to \mathrm{tot}\, \widetilde{\tau}_I^{\le q}F_I(X) \to \mathrm{Mc}(\mathrm{id}_Z) \to 0$$

in $\mathrm{C}^2(\mathcal{C})$. As observed at the end of the proof of Proposition 416 p. 262, the complex $\mathrm{Mc}(\mathrm{id}_Z)$ is exact.

§ 421. P. 315, Corollary 12.5.5 (ii).

Statement. *If the columns $X^{\bullet,j}$ of X are are exact for all $j \ne p$, then* $\mathrm{tot}(X)$ *is qis to* $X^{\bullet,p}[-p]$.

Proof. Apply Theorem 12.5.4 to the morphisms $\sigma_{II}^{\le p}\sigma_{II}^{\ge p}X \leftarrow \sigma_{II}^{\ge p}X \to X$.

15 About Chapter 13

15.1 Brief comments

§ 422. Beginning of Section 13.1 p. 319. Recall that $\mathcal{N}Q$ is defined in (10.2.1) p. 249 of the book. Then §417 p. 263 implies that, in the context of the beginning of Section 13.1, we have $f \in \mathcal{N}Q \Leftrightarrow f$ is a qis.

§ 423. P. 319, Display (13.1.1). This display reads: $H : \mathrm{K}_c(\mathcal{A}) \to \mathcal{A}$. As observed in §417 p. 263 we have in fact $H : (\mathrm{K}_c(\mathcal{A}), T) \to (\mathcal{A}, T)$. (See also §422.)

§ 424. P. 319. Sentence "One shall be aware that the category $\mathrm{D}_c(\mathcal{A})$ may be a big category" after Definition 13.1.1. See §427 p. 265 and §458 p. 274 below.

§ 425. P. 320, Display (13.1.2). It might be worth stating explicitly the equality

$$N^{ub}(\mathcal{C})Q = \mathrm{Qis} \tag{169}$$

(see §422 p. 265).

§ 426. P. 320, Parts (i) and (ii) of Proposition 13.1.5. Part (i): the argument showing that the cohomology functor $H : \mathrm{K}(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$ factors through $Q : \mathrm{K}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ is implicit in §417, §422 and (169). Part (ii): the statement that, in the notation of the proposition, f is an isomorphism if $H(f)$ is, also follows from the argument implicit in §417, §422 and (169). To be slightly more explicit, one considers the long exact sequence

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(TX) \to \cdots$$

attached to a d.t. $X \to Y \to Z \to TX$ in $\mathrm{D}(\mathcal{C})$. The fact that H commutes with T implies $H(Z) \simeq 0$, Remark 13.1.4 (i) implies $Z \simeq 0$, and Exercise 10.1 p. 265 implies that f is an isomorphism.

§ 427. P. 322, Notation 13.1.9, sentence "Remark that the set $\mathrm{Ext}^k_{\mathcal{C}}(X, Y)$ is not necessarily $\mathcal{U}$-small". See §424 p. 265 above and §458 p. 274 below.

15.2 Lemma 13.2.1 p. 325

§ 428. P. 325. Lemma 13.2.1 will be used to prove Proposition 13.2.2 p. 326, Proposition 13.2.6 p. 327, Theorem 13.2.8 p. 329, Proposition 13.3.5 p. 330 and Lemma 14.4.1 p. 358 of the book.

§ 429. P. 325. Just before the last display it is written "There is a monomorphism $Z^p \rightarrowtail W^p$." This results from the following fact, whose proof is left to the reader:

Let

$$\begin{array}{ccccc} Z & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & V & \longrightarrow & U \end{array}$$

be a commutative diagram in $\mathcal{C}$. If $ZXUW$ is cocartesian, then so is $YXUV$.

We apply this to the commutative diagram

$$\begin{array}{ccccc} \operatorname{Coker} d_X^{p-1} & \longrightarrow & \operatorname{Ker} d_X^{p+1} & \longrightarrow & X^{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Coker} d_Y^{p-1} & \longrightarrow & Z^p & \longrightarrow & W^p \end{array}$$

and we use Lemma 290 (b) (ii) p. 178.

§ 430. P. 325, exactness of

$$0 \to H^p(X) \to \operatorname{Coker} d_X^{p-1} \to \operatorname{Ker} d_X^{p+1} \to H^{p+1}(X) \to 0 :$$

- Exactness at $H^p(X)$: apply the Five Lemma (Theorem 297 p. 180) to

$$\begin{array}{ccccccc} X^{p-1} & \xrightarrow{d_X^{p-1}} & \operatorname{Ker} d_X^p & \longrightarrow & H^p(X) & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \downarrow \\ X^{p-1} & \xrightarrow[d_X^{p-1}]{} & X^p & \longrightarrow & \operatorname{Coker} d_X^{p-1} & \longrightarrow & 0. \end{array}$$

- Exactness at $\operatorname{Coker} d_X^{p-1}$: It suffices to complete the commutative diagram

$$\begin{tikzcd} W \arrow[r, dashed, two heads, "h"] \arrow[d, dashed, "i"'] & Z \arrow[d, "g"] \arrow[dr, "0"] & \\ H^p(X) \arrow[r, "a"'] & \operatorname{Coker} d_X^{p-1} \arrow[r, "b"'] & \operatorname{Ker} d^{p+1}. \end{tikzcd}$$

Consider the commutative diagrams

$$\begin{array}{ccccc}
X^{p-1} & \xrightarrow{d_X^{p-1}} & \operatorname{Ker} d_X^p & \xrightarrow{c} & H^p(X) \\
\| & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle a} \\
X^{p-1} & \xrightarrow{d_X^{p-1}} & X^p & \xrightarrow{f} & \operatorname{Coker} d_X^{p-1} \\
 & & & \searrow{\scriptstyle d_X^p} & \downarrow{\scriptstyle b} \\
 & & & & \operatorname{Ker} d_X^{p+1}
\end{array}$$

and

$$\begin{array}{ccc}
W & \xrightarrow{h} & Z \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g} \\
X^p & \xrightarrow[f]{} & \operatorname{Coker} d_X^{p-1}.
\end{array}$$

The equalities $d_X^p j = bfj = bgh = 0$ yield commutative diagram

$$\begin{array}{ccc}
 & & W \\
 & \swarrow{\scriptstyle k} & \downarrow{\scriptstyle j} \\
\operatorname{Ker} d_X^p & \xrightarrow[e]{} & X^p.
\end{array}$$

(In this Section we write uv for $u \circ v$.) Setting $i := ck$ we get $ai = ack = fek = fj = gh$.

- Exactness at $\operatorname{Ker} d_X^{p+1}$: It suffices to complete the commutative diagram

$$\begin{array}{ccccc}
W & \overset{e}{\dashrightarrow} & Z & & \\
{\scriptstyle f}\,\vdots & & \downarrow{\scriptstyle c} & \searrow{\scriptstyle 0} & \\
\operatorname{Coker} d_X^{p-1} & \xrightarrow[a]{} & \operatorname{Ker} d^{p+1} & \xrightarrow[b]{} & H^{p+1}(X).
\end{array}$$

Forming the commutative diagram

$$\begin{array}{ccccc}
W & \xrightarrow{e} & Z & & \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle c} & \searrow{\scriptstyle 0} & \\
X^p & \xrightarrow[d_X^p]{} & \operatorname{Ker} d^{p+1} & \xrightarrow[b]{} & H^{p+1}(X)
\end{array}$$

and setting $f := hg$, where $h : X^p \to \operatorname{Coker} d_X^{p-1}$ is the natural morphism, yields $af = ahg = d_X^p g = ce$.

• Exactness at $H^{p+1}(X)$: Obvious.

§ 431. P. 325, exactness of

$$0 \to H^p(X) \to \operatorname{Coker} d_Y^{p-1} \to Z^p \to H^{p+1}(X) \to 0 :$$

• Exactness at $H^p(X)$: As mentioned in the book, the exactness at $H^p(X)$ holds by assumption.

• Exactness at $\operatorname{Coker} d_Y^{p-1}$: This is an immediate consequence of Lemma 295 (b) p. 180.

• Exactness at $H^{p+1}(X)$: The morphism $Z^p \to H^{p+1}(X)$ is an epimorphism because the composition $\operatorname{Ker} d_X^{p+1} \to Z^p \to H^{p+1}(X)$ is an epimorphism.

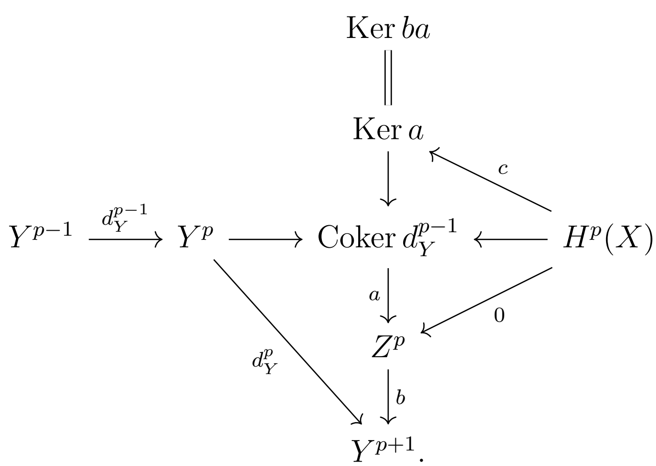
§ 432. P. 325, claim that $Z^p \in \mathcal{J}$. We have

$$\operatorname{Ker}(Z^p \to H^{p+1}(X)) \simeq \operatorname{Coker}(H^p(X) \to \operatorname{Coker} d_Y^{p-1}).$$

§ 433. P. 325, morphism $f^{p+1} : X^{p+1} \to Y^{p+1}$. This morphism is defined as the composition $X^{p+1} \to W^p \to Y^{p+1}$.

§ 434. P. 325, morphism $d_Y^p : Y^p \to Y^{p+1}$. This morphism is defined as the composition $Y^p \to \operatorname{Coker} d_Y^{p-1} \to Z^p \to Y^{p+1}$.

§ 435. P. 326, first display. Consider the commutative diagram



The precise claim is that the morphism c is an isomorphism. The key point is the fact that b is a monomorphism.

§ 436. P. 326, second display. The morphism $H^{p+1}(X) \to H^{p+1}(Y_{\leq p+1})$ in the commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Coker} d_X^{p-1} & \longrightarrow & \operatorname{Ker} d_X^{p+1} & \longrightarrow & H^{p+1}(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
\operatorname{Coker} d_Y^{p-1} & \longrightarrow & Z^p & \longrightarrow & H^{p+1}(X) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\operatorname{Coker} d_Y^{p-1} & \longrightarrow & Y^{p+1} & \longrightarrow & H^{p+1}(Y_{\leq p+1}) & \longrightarrow & 0
\end{array}$$

is a monomorphism by the Five Lemma (Theorem 297 p. 180).

§ 437. P. 326. The first isomorphism in the third display follows from the fact that $d_Y^p : Y^p \to Y^{p+1}$ factors as $Y^p \twoheadrightarrow \operatorname{Coker} d_Y^{p-1} \to Z^p \rightarrowtail Y^{p+1}$, by definition of d_Y^p.

15.3 Brief comments

§ 438. P. 326, proof of Proposition 13.2.2 (ii). See §418 p. 264.

§ 439. P. 328, proof of Proposition 13.2.6, Step (a). We write fg for $f \circ g$.

• $H^{n-1}(X) \xrightarrow{\sim} H^{n-1}(Z)$. It suffices to show

$$\operatorname{Ker} d_X^{n-1} \xrightarrow{\sim} \operatorname{Ker} f^n d_X^{n-1}.$$

Let

$$a : \operatorname{Ker} f^n d_X^{n-1} \rightarrowtail X^{n-1}, \quad b : \operatorname{Ker} d_X^n \rightarrowtail X^n, \quad c : X^{n-1} \to \operatorname{Ker} d_X^n$$

be the natural morphisms. It suffices to show that the composition

$$\operatorname{Ker} f^n d_X^{n-1} \overset{a}{\rightarrowtail} X^{n-1} \xrightarrow{d_X^{n-1}} X^n$$

vanishes. By assumption the composition

$$\operatorname{Ker} d_X^n \overset{b}{\rightarrowtail} X^n \xrightarrow{f^n} Y^n$$

is a monomorphism. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Ker} f^n d_X^{n-1} & & \\ {\scriptstyle a}\downarrow & & \\ X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\ {\scriptstyle c}\downarrow & \overset{b}{\nearrow} & \downarrow{\scriptstyle f^n} \\ \operatorname{Ker} d_X^n & \rightarrowtail & Y^n. \end{array}$$

The equalities $f^n bca = f^n d_X^{n-1} a = 0$ imply $ca = 0$ and thus $0 = bca = d_X^{n-1} a$.

• $H^n(X) \rightarrowtail H^n(Z)$. Let

$$\begin{array}{ccc} V & \xrightarrow{b} & W \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle a} \\ X^{n-1} & \xrightarrow{(d_X^{n-1})'} & \operatorname{Ker} d_X^n \\ \| & & \downarrow{\scriptstyle (f^n)'} \\ X^{n-1} & \xrightarrow[(f^n d_X^{n-1})']{} & \operatorname{Ker} d_Y^n \end{array}$$

be a diagram, where $(g)'$ denotes the morphism induced by g. The bottom square commutes, and it suffices to show that the commutativity of the big rectangle implies that if the top square, which is clear.

• $H^n(X) \twoheadrightarrow H^n(Z)$. Even if this is unsatisfactory, let us use the Freyd-Mitchell Theorem (Theorem 9.6.10. p. 238 of the book). In other words we may assume that our abelian category $\mathcal{C}$ is $\mathrm{Mod}(R)$ for some ring R. We omit the $\circ$ symbol and most of the parenthesis. Let $y^n \in \operatorname{Ker} d_Y^n$. It suffices to show that there is an x^n in $\operatorname{Ker} d_X^n$ and an x^{n-1} in X^{n-1} such that $y^n = f^n x^n + f^n d_X^{n-1} x^{n-1}$. By assumption there is an x^n in $\operatorname{Ker} d_X^n$ and a y^{n-1} in Y^{n-1} such that $y^n = f^n x^n + d_Y^{n-1} y^{n-1}$. We can replace y^n with $y^n - d_Y^{n-1} y^{n-1}$ and take $x^{n-1} := 0$.

§ 440. P. 327, Lemma 13.2.4. As noticed $\mathcal{C}^+(\mathcal{I}_\mathcal{C})$ should be $\mathrm{C}^+(\mathcal{I}_\mathcal{C})$. Using Definition 14.1.4 (i) p. 348 of the book, one can say that any X in $\mathrm{C}^+(\mathcal{I}_\mathcal{C})$ is homotopically injective.

§ 441. P. 327, Proposition 13.2.46. As noticed $\mathcal{N}$ should be $\mathrm{N}(\mathcal{C})$. Here is a corollary:

If $\mathcal{C}$ has enough injectives and $\mathcal{J} = \mathcal{I}_\mathcal{C}$ (and (13.1.2) holds), then $\mathrm{K}(\mathcal{J}) \xrightarrow{\sim} \mathrm{D}(\mathcal{C})$.

Proof. It suffices to show that any exact X in $\mathrm{K}(\mathcal{J})$ is homotopic to zero, that is, it suffices to show that $\operatorname{Ker} d_X^n$ is injective for all n. But this follows from the exact sequence

$$X^{n-d} \to X^{n-d+1} \to \cdots \to X^{n-1} \to \operatorname{Ker} d_X^n \to 0.$$

§ 442. P. 328, proof of Proposition 13.2.6, Step (b), Isomorphism $\operatorname{Coker} d_M^{i-2} \simeq X^i \oplus_{X^{i-1}} \operatorname{Coker} d_Y^{i-2}$. Let

$$a : Y^{i-1} \to \operatorname{Coker} d_Y^{i-2}, \quad b : X^i \oplus Y^{i-1} \to \operatorname{Coker} d_M^{i-2},$$

$$c : X^i \oplus \operatorname{Coker} d_Y^{i-2} \to X^i \oplus_{X^{i-1}} \operatorname{Coker} d_Y^{i-2}$$

be the canonical morphisms. One checks that there is a unique morphism

$$f : \operatorname{Coker} d_M^{i-2} \to X^i \oplus_{X^{i-1}} \operatorname{Coker} d_Y^{i-2}$$

such that $f \circ b = c \circ (\mathrm{id} \oplus a)$, and a unique morphism

$$g : X^i \oplus_{X^{i-1}} \operatorname{Coker} d_Y^{i-2} \to \operatorname{Coker} d_M^{i-2}$$

such that $g \circ c \circ (\mathrm{id} \oplus a) = b$, and that f and g are mutual inverses.

§ 443. P. 328, proof of Proposition 13.2.6, Step (b), Isomorphism

$$\operatorname{Ker} d_M^i \simeq \operatorname{Ker} d_X^{i+1} \times_{Y^{i+1}} Y^i.$$

Let

$$a : \operatorname{Ker} d_X^{i+1} \to X^{i+1}, \quad b : \operatorname{Ker} d_M^i \to X^{i+1} \times Y^i,$$
$$c : \operatorname{Ker} d_X^{i+1} \times_{Y^{i+1}} Y^i \to \operatorname{Ker} d_X^{i+1} \times Y^i$$

be the canonical morphisms. One checks that there is a unique morphism

$$f : \operatorname{Ker} d_M^i \to \operatorname{Ker} d_X^{i+1} \times_{Y^{i+1}} Y^i$$

such that $(\mathrm{id} \times a) \circ c \circ f = b$, and a unique morphism

$$g : \operatorname{Ker} d_X^{i+1} \times_{Y^{i+1}} Y^i \to \operatorname{Ker} d_M^i$$

such that $b \circ g = (\mathrm{id} \times a) \circ c$, and that f and g are mutual inverses.

§ 444. P. 328, proof of Proposition 13.2.6, Step (c), proof of $H^i(X) \xrightarrow{\sim} H^i(Z)$ for $i = a+1, a, a-1$.

- $i = a+1$: the isomorphism $\operatorname{Im} d_Z^a \simeq \operatorname{Im} d_X^a$ is induced by $X \to Z$,
- $i = a$: the isomorphism $\operatorname{Ker} d_Z^a \simeq \operatorname{Ker} d_Y^a$ is induced by $Z \to Y$,
- $i = a-1$: the isomorphism $\operatorname{Ker} d_Z^{a-1} \simeq \operatorname{Ker} d_Y^{a-1}$ is induced by $X \to Z$.

§ 445. P. 330. Right after Definition 13.3.1 it is written:

By the definition, the functor F admits a right derived functor on $\mathrm{K}^*(\mathcal{C})$ [by the way I think the authors meant $\mathrm{D}^*(\mathcal{C})$] if

$$\underset{(X\to X')\in\mathrm{Qis},X\in\mathrm{K}^*(C)}{\text{“colim”}} Q'\circ\mathrm{K}(F)(X')$$

exists in $\mathrm{D}^*(\mathcal{C}')$ for all $X\in\mathrm{K}^*(\mathcal{C})$. In such a case, this object is isomorphic to $R^*F(X)$.

It is implicitly assumed that the underlying universe $\mathcal{U}$ has been chosen so that $\mathcal{C}$ is $\mathcal{U}$-small (Definition 5 p. 10). This is justified by Theorem 95 p. 68 (the “Universal Kan Extension Theorem”).

§ 446. P. 330, phrase “R^*F is a triangulated functor from $\mathrm{D}^*(\mathcal{C})$ to $\mathrm{D}^*(\mathcal{C}')$ if it exists” just before Notation 13.3.2. Here is a proof:

Consider the (non-commutative) diagram

$$\begin{array}{ccc} \mathrm{K}(\mathcal{C}) & \xrightarrow{\mathrm{K}(F)} & \mathrm{K}(\mathcal{C}') \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q'} \\ \mathrm{D}(\mathcal{C}) & \xrightarrow[RF]{} & \mathrm{D}(\mathcal{C}'). \end{array}$$

We have

$$RF(X)\simeq\underset{Y\to X}{\mathrm{colim}}\ \mathrm{K}(F)(Y),$$

where $Y\to X$ runs over the morphisms from Y to X in $\mathrm{D}(\mathcal{C})$. Using this expression for $RF(X)$ it is easy to see that RF commutes with finite products. By Proposition 8.2.15 p. 173 of the book, this implies that RF is additive.

As RF commutes with finite products, it commutes with mapping cones, and is thus triangulated.

(We have implicitly used the fact that $\mathrm{K}(\mathcal{C})$ and $\mathrm{D}(\mathcal{C})$ have the same set of objects, that Q acts on this set as the identity, and that the same holds for $\mathcal{C}'$.)

§ 447. P. 330, Corollary 13.3.3. More generally: Let X be in $\mathrm{D}^+(\mathcal{C})$. Then $RF(X)$ exists if and only if $R^+F(X)$ exists.

§ 448. P. 330. It is observed in the book just before the statement of Proposition 13.3.5:

A full additive subcategory $\mathcal{J}$ of $\mathcal{C}$ is F-injective if and only if

(A) For all X in $\mathrm{K}^+(\mathcal{C})$ there is a qis $X \to Y$ with Y in $\mathrm{K}^+(\mathcal{J})$,

(B) $F(Y)$ is exact whenever Y is an exact complex in $\mathrm{K}^+(\mathcal{J})$.

Part "(a) $\Rightarrow$ (b) (2)" of the proof of Proposition 13.3.5 p. 331 shows that (A) above implies that $\mathcal{J}$ is cogenerating in $\mathcal{C}$.

§ 449. P. 331, Part "(a) $\Rightarrow$ (b) (2)" of the proof of Proposition 13.3.5. We have

$$\begin{aligned} \operatorname{Im}(F(X) \to F(X'')) &\xrightarrow{\sim} \operatorname{Im}(F(X) \to F(Z^0)) \\ &\xrightarrow{\sim} \operatorname{Ker}(F(Z^0) \to F(Z^1)) \xleftarrow{\sim} F(X''), \end{aligned}$$

the first isomorphism following from the fact that $F(X'') \to F(Z^0)$ is a monomorphism.

§ 450. P. 331, Remark 13.3.6 (i). See §448 p. 272. (I think that, in view of the context, the implicit assumptions are that F is an additive functor and that there is an F-injective subcategory $\mathcal{J}$ of $\mathcal{C}$.)

§ 451. P. 331, Remark 13.3.6 (iii). Lemma 13.2.1 p. 325 of the book is also used.

§ 452. P. 332. One can derive Corollary 13.3.8 from Theorem 13.8.7 as follows. Let $\mathcal{J}$ be a full additive cogenerating subcategory of $\mathcal{C}$, and consider the condition

(C) If $0 \to X' \to X \to X'' \to 0$ is exact in $\mathcal{C}$, and if $X', X \in \mathcal{J}$, then $X'' \in \mathcal{J}$ and $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

We assume that (C) holds, and we must show that $\mathcal{J}$ is F-injective. Let $Y' \rightarrowtail X$ be a monomorphism in $\mathcal{C}$ with Y' in $\mathcal{J}$, let $X \rightarrowtail Y$ be a monomorphism in $\mathcal{C}$ with Y in $\mathcal{J}$, and let $0 \to Y' \to Y \to Y'' \to 0$ be an exact sequence in $\mathcal{C}$. Then (C) implies that Y'' is in $\mathcal{J}$ and that $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact. Now Theorem 13.8.7 entails that $\mathcal{J}$ is F-injective.

§ 453. P. 332, first two sentences of the proof of Lemma 13.3.10. We get the commutative diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0. \end{array}$$

The square (Z, X'', Y, Y'') is cartesian, and the top and bottom rows are exact. The middle row is exact by Lemma 294 p. 179, and $X \to Z$ is an isomorphism by the Five Lemma.

§ 454. P. 334, Lemma 13.3.12. We add the assumption that $\mathcal{J}$ is cogenerating. (See also §484 p. 284 below. The notion of cogenerating full subcategory is defined on p. 184 of the book, and the notion of F-injective full subcategory is defined on pages 253 and 330 of the book.)

§ 455. P. 334, proof of Lemma 13.3.12. The existence of an exact sequence $R^jF(X) \to R^jF(X'') \to R^{j+1}F(X')$ attached to a given exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ follows from Proposition 416 p. 262.

Unsolved Problem 456. P. 334, phrase "even if F is right derivable, there may not exist an F-injective subcategory". I failed to prove this.

§ 457. P. 335, Display (13.3.4). The existence of $\pi_{\mathcal{C}}$ follows from Proposition 6.3.1 p. 139 of the book.

§ 458. P. 337, Theorem 13.4.1. (See §17 p. 20.) Here is a corollary. Consider the following "pathological conditions" on an abelian $\mathcal{U}$-category $\mathcal{C}$:

(a) there is an n in $\mathbb{Z}$ and there are X and Y in $\mathcal{C}$ such that $\mathrm{Ext}^n(X, Y)$ is *not* equipotent to any $\mathcal{U}$-set,

(b) $\mathrm{D}^*(\mathcal{C})$ is *not* equivalent to a $\mathcal{U}$-category,

(c) $R^* \operatorname{Hom}_{\mathcal{C}}$ does *not* exist.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). (See §424 p. 265 and §427 p. 265 above.)

Here is an abelian $\mathcal{U}$-category $\mathcal{C}$ satisfying (a). Let $\mathcal{U}$ be a universe, let k be a field belonging to $\mathcal{U}$, let V be a k-vector space whose dimension is larger than the cardinal of $\mathcal{U}$, let A be the tensor algebra of V, let $\mathcal{C}$ be the category of A-modules belonging to $\mathcal{U}$, and denote again by k the field k regarded as an A-module on which the vectors of V act by zero. Then we have $\mathrm{Ext}^1_{\mathcal{C}}(k, k) \simeq V^* \notin \mathcal{U}$.

§ 459. P. 337, Theorem 13.4.1. (See §17 p. 20.) The natural morphism

$$\operatorname*{colim}_{(X' \to X), (Y \to Y') \in \mathrm{Qis}} \operatorname{Hom}_{\mathrm{K}(\mathcal{C})}(X', Y') \to \operatorname{Hom}_{\mathrm{D}(\mathcal{C})}(X, Y)$$

an isomorphism by Remark 7.1.18 (ii) p. 156 of the book. See also Theorem 10.2.3 (i) p. 249.

§ 460. P. 337. (See §17 p. 20.) In view of §261 p. 164 above and Theorem 13.4.1 p. 337 of the book, the functors

$$\mathrm{Hom}^\bullet_{\mathcal{C}} : \mathrm{K}(\mathcal{C}) \times \mathrm{K}(\mathcal{C})^{\mathrm{op}} \to \mathrm{K}(\mathrm{Mod}(\mathbb{Z}))$$

and

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C})} : \mathrm{K}(\mathcal{C}) \times \mathrm{K}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$$

give rise to the commutative diagram

$$\begin{array}{ccccc} R^0\,\mathrm{H}^\bullet_{\mathcal{C}}(X,\ \ )(Y) & \longrightarrow & R^0\,\mathrm{H}^\bullet_{\mathcal{C}}(X,Y) & \longleftarrow & R^0\,\mathrm{H}^\bullet_{\mathcal{C}}(\ \ ,Y)(X) \\ \updownarrow & & \updownarrow & & \updownarrow \\ R\,\mathrm{H}_{\mathrm{K}(\mathcal{C})}(X,\ \ )(Y) & \longrightarrow & R\,\mathrm{H}_{\mathrm{K}(\mathcal{C})}(X,Y) & \longleftarrow & R\,\mathrm{H}_{\mathrm{K}(\mathcal{C})}(\ \ ,Y)(X) \\ & \searrow\!\!\!\nwarrow & \updownarrow & \swarrow\!\!\!\nearrow & \\ & & \mathrm{H}_{\mathrm{D}(\mathcal{C})}(X,Y), & & \end{array} \tag{170}$$

where we have written H for Hom to save space, and where the horizontal arrows are the natural maps and the other arrows are the natural bijections, and where

$$R(\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X,\ \ ))(Y), \quad R\,\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X,Y), \quad R(\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(\ \ ,Y))(X)$$

are defined by Notation 10.3.8 p. 255 of the book. Then (170) commutes, and all its arrows are bijective. This implies

$$R\,\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y) \simeq R(\mathrm{Hom}^\bullet_{\mathcal{C}}(X,\ \ ))(Y) \simeq R(\mathrm{Hom}^\bullet_{\mathcal{C}}(\ \ ,Y))(X).$$

15.4 Exercise 13.15 p. 342

Here is a partial solution. Let $\mathcal{C}$ be an abelian category, let X and Y be in $\mathcal{C}$, let E be the set of short exact sequences

$$0 \to Y \to Z \to X \to 0,$$

and let $\sim$ be the following equivalence relation on E: the exact sequences

$$0 \to Y \to Z \to X \to 0$$

and

$$0 \to Y \to W \to X \to 0$$

are equivalent if and only if there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

(This is easily seen to be indeed an equivalence relation.) To the element

$$0 \to Y \to Z \to X \to 0$$

of E we attach the morphism in

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, Y[1]) = \mathrm{Ext}^1_{\mathcal{C}}(X, Y)$$

suggested by the diagram

$$\begin{array}{ccc}
 & & X \\
 & & \uparrow \\
Y & \longrightarrow & Z \\
\| & & \\
Y, & &
\end{array}$$

where each row is a complex (viewed as an object of $\mathrm{D}(\mathcal{C})$), with the convention that only the possibly nonzero terms are indicated (the top morphism being a qis).

We claim:

(a) this process induces a map from $E/\!\sim$ to $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$,

(b) this map (a) is bijective.

Claim (a) is left to the reader. To prove (b) we construct the inverse map. To this end, we start with a complex $W^\bullet$, a qis $f : W^\bullet \to X$, and a morphism $g : W^\bullet \to Y[1]$ representing our given element of $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$. The natural morphism $\tau^{\le 0} W^\bullet \to W^\bullet$ being a qis, we can replace $W^\bullet$ with $\tau^{\le 0} W^\bullet$, or, in other words, we may, and will, assume $W^n \simeq 0$ for $n > 0$. We have the commutative diagram

$$\begin{array}{ccccccccc}
W^{-2} & \xrightarrow{d_W^{-2}} & W^{-1} & \xrightarrow{d_W^{-1}} & W^0 & \xrightarrow{f} & X & \longrightarrow & 0 \\
 & \underset{0}{\searrow} & \downarrow{\scriptstyle g} & & & & & & \\
 & & Y & & & & & &
\end{array}$$

whose top row is an exact complex. It gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & W^0 & \xrightarrow{f} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0, \end{array}$$

where the top row is an exact complex, where g' is induced by g and where the square $(\operatorname{Ker} f, W^0, Y, Z)$ is cocartesian. Then Lemma 294 p. 179 implies that the bottom row is an exact complex. It is easy to see that this process defines a map from $\operatorname{Ext}^1_{\mathcal{C}}(X, Y)$ to $E/\sim$, and that this map is inverse to the map constructed before. q.e.d.

16 About Chapter 14

16.1 Brief comments

§ 461. P. 348, paragraph before Lemma 14.1.2. Define $u : M(X) \to T^{-1}M(X)$ by

$$u = \begin{pmatrix} 0 & \mathrm{id}_{T^{-1}X} \\ 0 & 0 \end{pmatrix} : X \oplus T^{-1}X \to T^{-1}X \oplus T^{-2}X.$$

Then u is a homotopy between 0 and $\mathrm{id}_{M(X)}$. In particular any morphism $M(X) \to M(Y)$ is a qis.

§ 462. P. 348, proof of Lemma 14.1.5. We define a map

$$\operatorname{Hom}_{\mathrm{D}_c(\mathcal{A})}(X, I) := \operatorname*{colim}_{(X' \to X) \in \mathrm{Qis}} \operatorname{Hom}_{\mathrm{K}_c(\mathcal{A})}(X', I) \to \operatorname{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) \tag{171}$$

by inverting the bijection $\operatorname{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) \to \operatorname{Hom}_{\mathrm{K}_c(\mathcal{A})}(X', I)$ described in the proof given in the book. It is easy to see that (171) is the inverse of the natural map $\operatorname{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) \to \operatorname{Hom}_{\mathrm{D}_c(\mathcal{A})}(X, I)$.

§ 463. P. 348, Lemma 14.1.5. The following corollary to Lemma 14.1.5 is almost obvious, but its importance might warrant an explicit statement and an explicit proof.

Corollary 464. *Let $(\mathcal{A}, T)$ be an abelian category with translation such that for all X in $\mathcal{A}_c$ there is a qis $X \to I$ with I in $\mathrm{K}_{c,hi}(\mathcal{A})$, let $\mathcal{D}$ be a triangulated category and $F : \mathrm{K}_c(\mathcal{A}) \to \mathcal{D}$ a triangulated functor. Then $RF : \mathrm{D}_c(\mathcal{A}) \to \mathcal{D}$ exists and satisfies $RF(X) \simeq F(I)$ in the above notation.*

Proof. This follows from Lemma 14.1.5 and Proposition 7.3.2 p. 160 of the book. □

§ 465. P. 349, Display (14.1.2), definition of QM. To be consistent with Remark 372 p. 236 we define QM as the full subcategory of $\mathrm{Mor}(\mathcal{A}_c)$ whose objects are the morphisms which are qis and monomorphisms.

§ 466. P. 349, proof of Proposition 14.1.6, Step (i) (b), claim "$M(v)$ belongs to QM": see §461 p. 277.

16.2 Proposition 14.1.6 p. 349

Here are some additional details about Step (ii) of the proof of Proposition 14.1.6.

The fact that Z is qis to 0 follows from Proposition 416 p. 262.

We refer the reader to the book for a precise description of the setting. The following facts can be easily verified:

We have morphisms

$$f : X \to Y, \quad \varphi : X \to I, \quad \psi : Y \to I, \quad h : T^{-1}Y \to I$$

in $\mathcal{A}$ which satisfy

$$\varphi = \psi \circ f, \tag{172}$$

$$h = T^{-1}d_I \circ T^{-1}\psi - \psi \circ T^{-1}d_Y, \tag{173}$$

$$h = T^{-1}d_I \circ T^{-1}\psi + \psi \circ d_{T^{-1}Y}, \tag{174}$$

$$f \text{ and } \varphi \text{ are in fact morphisms in } \mathcal{A}_c. \tag{175}$$

It is straightforward to check that h is also a morphism in $\mathcal{A}_c$. To prove $h \circ T^{-1}f = 0$, we note:

$$\begin{aligned} h \circ T^{-1}f &= T^{-1}d_I \circ T^{-1}\psi \circ T^{-1}f + \psi \circ d_{T^{-1}Y} \circ T^{-1}f && \text{by } (174) \\ &= T^{-1}d_I \circ T^{-1}\varphi + \psi \circ d_{T^{-1}Y} \circ T^{-1}f && \text{by } (172) \\ &= T^{-1}d_I \circ T^{-1}\varphi + \psi \circ f \circ d_{T^{-1}X} && \text{by } (175) \\ &= T^{-1}d_I \circ T^{-1}\varphi + \varphi \circ d_{T^{-1}X} && \text{by } (172) \\ &= 0 && \text{by } (175). \end{aligned}$$

We also have morphisms

$$g : Y \to Z, \quad \widetilde{h} : T^{-1}Z \to I, \quad \xi : Z \to I \quad \widetilde{\psi} : Y \to I$$

in $\mathcal{A}$ with

$$g \text{ is in fact a morphism in } \mathcal{A}_c, \tag{176}$$

$$h = \widetilde{h} \circ T^{-1}g, \tag{177}$$

$$\widetilde{h} = T^{-1}d_I \circ T^{-1}\xi - \xi \circ T^{-1}d_Z, \tag{178}$$

$$\widetilde{\psi} = \psi - \xi \circ g. \tag{179}$$

To prove that $\widetilde{\psi}$ is a morphism in $\mathcal{A}_c$, we note:

$$\begin{aligned} d_I \circ \widetilde{\psi} - T\widetilde{\psi} \circ d_Y &= d_I \circ \psi - d_I \circ \xi \circ g - T\psi \circ d_Y + T\xi \circ Tg \circ d_Y && \text{by (179)} \\ &= (d_I \circ \psi - T\psi \circ d_Y) - (d_I \circ \xi \circ g - T\xi \circ Tg \circ d_Y) \\ &= Th - (d_I \circ \xi \circ g - T\xi \circ Tg \circ d_Y) && \text{by (173)} \\ &= Th - (d_I \circ \xi \circ g - T\xi \circ d_Z \circ g) && \text{by (176)} \\ &= Th - T\widetilde{h} \circ g && \text{by (178)} \\ &= 0 && \text{by (177).} \end{aligned}$$

16.3 Brief comments

§ 467. P. 350, last paragraph. In view of the comments made about Corollary 347 p. 215 and Theorem 376 p. 237, one could replace “there exists an essentially small full subcategory $\mathcal{S}$ of $\mathcal{A}_c$ such that ...” with “there exists an infinite cardinal π such that $(\mathcal{A}_c)_\pi$ is essentially small and satisfies ...”, and replace $\mathcal{S}$ with $(\mathcal{A}_c)_\pi$ in (14.1.4) p. 351 of the book.

§ 468. As pointed out to me by Olaf Schnürer, one should also assume in (14.1.4) that the category $\mathcal{S}$ is closed under translation. It is used in the proof of Lemma 14.1.9 p. 351.

§ 469. P. 351, proof of (14.1.4) (iii). Given

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X' & \twoheadrightarrow & X \end{array}$$

with Y in $\mathcal{S}$ we get

$$\begin{array}{ccc} Z & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \twoheadrightarrow & X \end{array}$$

by Lemma 290 (a) (ii) p. 178, and then

$$\begin{array}{ccc} Y' & & \\ \downarrow & & \\ Z & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \twoheadrightarrow & X \end{array}$$

with Y' in $\mathcal{S}$ and $Y' \rightarrowtail Z \twoheadrightarrow Y$ an epimorphism by Theorem 376 (vi) p. 237

§ 470. P. 351, definition of $\mathcal{F}'$ and $\mathcal{F}$. To be consistent with Remark 372 p. 236 we define $\mathcal{F}'$ and $\mathcal{F}$ as follows: $\mathcal{F}'$ is the full subcategory of QM whose objects are are the objects $u : X \to Y$ of QM such that X and Y are in $\mathcal{S}$, and $\mathcal{F}$ is the full subcategory of $\mathcal{F}'$ obtained collecting a representative of each isomorphism class of objects of $\mathcal{F}'$.

§ 471. P. 352, proof of Lemma 14.1.10. I think

$$\mathrm{Ker}(H(V_{n-1}) \to H(X)) \to \mathrm{Ker}(H(V_n) \to H(X))$$

should be

$$\mathrm{Ker}(H(V_{n-1}) \to H(X)) \to H(V_n).$$

By taking the colimit over n we see that

$$\mathrm{Ker}(H(V') \to H(X)) \to H(V')$$

vanishes, which means that $H(V') \to H(X)$ is indeed a monomorphism.

§ 472. P. 352, Corollary 14.1.12. Part (i) implies:

Let $\mathcal{U}_0 \subset \mathcal{U}$ be universes, let $(\mathcal{A}, T)$ be a Grothendieck $\mathcal{U}$-category with translation, and let $\mathcal{A}_0 \subset \mathcal{A}$ be a fully abelian subcategory with translation. Assume that $\mathcal{A}_0$ is a Grothendieck $\mathcal{U}_0$-category. Then the natural functor $\mathrm{D}_c(\mathcal{A}_0) \to \mathrm{D}_c(\mathcal{A})$ is fully faithful.

Part (iii). I would change

"the functor $Q : \mathrm{K}_c(\mathcal{A}) \to \mathrm{D}_c(\mathcal{A})$ admits a right adjoint $R_q : \mathrm{D}_c(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A}), Q \circ R_q \simeq \mathrm{id}_{\mathrm{D}_c(\mathcal{A})}$, and R_q is the composition of $\iota : \mathrm{K}_{c,hi}(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$ and a quasi-inverse of $Q \circ \iota$"

to

"the functor $R_q : \mathrm{D}_c(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$ defined by $R_q := \iota \circ (Q \circ \iota)^{-1}$, where $\iota : \mathrm{K}_{c,hi}(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$ is the natural functor and $(Q \circ \iota)^{-1}$ is a quasi-inverse of $Q \circ \iota$, is a right localization of the identity functor $\mathrm{K}_c(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$ and satisfies $Q \circ R_q \simeq \mathrm{id}_{\mathrm{D}_c(\mathcal{A})}$".

This follows from Proposition 7.3.2 p. 160 of the book, together with its proof.

§ 473. P. 355, Theorem 14.3.1:

(i) follows from Lemma 14.1.5 p. 348 of the book,

(ii) follows from Corollary 14.1.8 p. 350 of the book,

(iii) follows from Corollary 14.1.12 (i) p. 352 of the book,

(iv) follows from Corollary 14.1.12 (ii) p. 352 of the book,

(v) follows (with the change suggested in §472 p. 280) from Corollary 14.1.12 (iii) p. 352 of the book,

(vi) follows from Corollary 14.1.12 (vi) p. 352 of the book,

(vii) follows from Theorem 14.2.1 p. 353 of the book,

(viii) follows from Corollary 14.2.2 p. 353 of the book,

(ix) follows from Corollary 14.2.3 p. 353 of the book.

§ 474. Corollary 14.3.2 p. 356. Let us add one sentence to the statement:

Corollary 475. *Let k be a commutative ring and let $\mathcal{C}$ be a Grothendieck k-abelian category. Then* $(\mathrm{K}_{\mathrm{hi}}(\mathcal{C}), \mathrm{K}(\mathcal{C})^{\mathrm{op}})$ *is* $\mathrm{Hom}_{\mathcal{C}}$*-injective, and the functor* $\mathrm{Hom}_{\mathcal{C}}$ *admits a right derived functor*

$$\mathrm{RHom}_{\mathcal{C}} : \mathrm{D}(\mathcal{C}) \times \mathrm{D}(\mathcal{C})^{\mathrm{op}} \to \mathrm{D}(k).$$

If X and Y are in $\mathrm{K}(\mathcal{C})$*, then for any qis $Y \to I$ with I in* $\mathrm{K}_{\mathrm{hi}}(\mathcal{C})$ *(such exist) we have*

$$\mathrm{RHom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, I).$$

Moreover, $H^0(\mathrm{RHom}_{\mathcal{C}}(X, Y)) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, Y)$ *for X, Y in* $\mathrm{D}(\mathcal{C})$*.*

§ 476. P. 358. Lemma 14.4.1 will be used to prove Theorems 14.4.3 and 14.4.5 p. 359 of the book.

§ 477. P. 358, proof of Lemma 14.4.1.

• We have a commutative diagram

$$\begin{array}{ccccccc} Q_n & \longrightarrow & P_n & \longrightarrow & \tau^{\leq n}X & & \\ & \searrow & & & & \searrow & \\ & & P_{n+1} & \longrightarrow & \tau^{\leq n+1}X & \longrightarrow & X \end{array}$$

in $\mathrm{K}(\mathcal{C})$.

• We have the following chain of isomorphisms in $\mathrm{D}(\mathcal{C})$:

$$Q_n \xrightarrow{\sim} \mathrm{Mc}(P_n \oplus P_{n+1} \to \tau^{\leq n+1}X)[-1] \xrightarrow{\sim} \mathrm{Mc}(P_{n+1} \to P_n \oplus P_{n+1})$$

$$\xrightarrow{\sim} \mathrm{Mc}(0 \to P_n) \oplus \mathrm{Mc}(P_{n+1} \to P_{n+1}) \xrightarrow{\sim} P_n \oplus 0 \xrightarrow{\sim} P_n.$$

• Here is a rewriting of the last four lines:

Hence, φ_i and φ_{i+1} are monomorphisms by Exercise 8.37. Note that $\mathrm{id} - \mathrm{sh}$ in Exercise 8.37 corresponds to φ_i and $X_0 \to X_1 \to \cdots$ corresponds to $H^i(X) \xrightarrow{\mathrm{id}} H^i(X) \xrightarrow{\mathrm{id}} \cdots$. It is straightforward to check that the obvious diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^i(Q) & \xrightarrow{\varphi_i} & H^i(P) & \longrightarrow & H^i(R) & \longrightarrow & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \Big\downarrow & & \\ 0 & \longrightarrow & \bigoplus_{n \geq i} H^i(Q_n) & \longrightarrow & \bigoplus_{n \geq i} H^i(P_n) & & \Big\downarrow & & \\ & & \downarrow\wr & & \downarrow\wr & & \Big\downarrow & & \\ 0 & \longrightarrow & \bigoplus_{n \geq i} H^i(X) & \xrightarrow{\mathrm{id} - \mathrm{sh}} & \bigoplus_{n \geq i} H^i(X) & \longrightarrow & H^i(X) & \longrightarrow & 0 \end{array}$$

commutes. This implies that $H^i(R) \to H^i(X)$ is an isomorphism.

§ 478. P. 359, proof of Lemma 14.4.2. The fact that the full subcategory $\mathrm{K}_{hp}(\mathcal{C})$ of $\mathrm{K}(\mathcal{C})$ consisting of homotopically projective objects contains $\mathrm{K}^-(\mathcal{P})$ follows from Lemma 13.2.4 p. 327 of the book.

§ 479. P. 359. In the setting of Theorem 14.4.3, the functor $\mathrm{RHom}_{\mathcal{C}}$ exists.

§ 480. P. 359, Theorem 14.4.5.

We add the assumption that $\mathrm{RHom}_{\mathcal{C}}$ exists.

Theorem 14.3.1 (vi) p. 355 implies that RF exists. This fact is implicit in the statement and the proof of Theorem 14.4.5.

Let us denote by (b1) the statement "the left derived functor $LG : \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ exists", and by (b2) the statement "(LG, RF) is a pair of adjoint functors".

The proof proves successively (a), (b1), (c), (b2). More precisely:

• The second sentence of Step (v) of the proof is "Hence $\widetilde{\mathcal{P}}$ is $\mathrm{K}(G)$-projective and LG exists". Thus, (a) and (b1) have been proved at this point.

• The penultimate sentence of the proof is "Hence we obtain (c)".

• The last sentence of the proof is "By taking the cohomologies, we obtain (b)", but what is really meant is "By taking the cohomologies, we obtain (b2)".

§ 481. P. 360, Step (ii) of the proof of Theorem 14.4.5: the fact that $\mathrm{Qis}_X \cap \mathrm{K}^-(\mathcal{P})$ is co-cofinal to Qis_X follows from Proposition 3.2.4 p. 79 of the book.

§ 482. P. 360, Theorem 14.4.5 (c). Here is the implicit underlying lemma:

Lemma 483. *In the setting described by the diagram*

$$\mathcal{A} \xrightarrow{E} \mathcal{B} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C},$$

where E is an equivalence, the map

$$\mathrm{Hom}_{\mathrm{Fct}(\mathcal{B},\mathcal{C})}(F, G) \to \mathrm{Hom}_{\mathrm{Fct}(\mathcal{A},\mathcal{C})}(F \circ E, G \circ E), \quad \theta \mapsto \theta \star E$$

(see Definition 35 p. 32) is bijective.

(Note that the above lemma is a particular case of Lemma 7.1.3 p. 150 of the book.)

The statement of Theorem 14.4.5 (c) is

We have an isomorphism in $\mathrm{D}(k)$, functorial with respect to $X \in \mathrm{D}(\mathcal{C})$ and $Y \in \mathrm{D}(\mathcal{C}')$:

$$\mathrm{RHom}_{\mathcal{C}}(X, RF(Y)) \simeq \mathrm{RHom}_{\mathcal{C}'}(LG(X), Y),$$

and the above lemma enables us to assume that X is in $\widetilde{\mathcal{P}}$ and Y in $\mathrm{K}_{\mathrm{hi}}(\mathcal{C}')$.

§ 484. P. 361, Corollary 14.4.6. We add the assumption that $\mathcal{P}$ is generating. (See also §454 p. 274 above. The notion of generating full subcategory is defined on p. 184 of the book, and the notion of F-projective full subcategory is defined on pages 253 and 330 of the book.)

§ 485. P. 361, proof of Corollary 14.4.6. The last display can be written

$$\operatorname{colim} H^n(LG(\alpha)) = \operatorname{colim} H^n(G(p_\bullet)) \xrightarrow{\sim} H^n(G(\operatorname{colim} p_\bullet)) = H^n(LG(\operatorname{colim} \alpha)).$$

§ 486. P. 361, Theorem 14.4.8. We add the assumption that $\mathrm{RHom}_{\mathcal{C}_1}$ and $\mathrm{RHom}_{\mathcal{C}_2}$ exist.

§ 487. P. 362, Line 8: as already indicated $\mathrm{K}(G)$-projective should be G-projective (see Definition 13.4.2 p. 338 of the book).

§ 488. P. 364, proof of Theorem 14.4.8. I don't understand Step (f). Here is another argument. We must prove that there is a functorial isomorphism

$$\mathrm{RHom}_{\mathcal{C}_3}(LG(X_1, X_2), X_3) \simeq \mathrm{RHom}_{\mathcal{C}_1}(X_1, RF(X_2, X_3))$$

for $X_1 \in \mathrm{K}(\mathcal{C}_1), X_2 \in \mathrm{K}(\mathcal{C}_2), X_3 \in \mathrm{K}(\mathcal{C}_3)$. We can assume that $X_1 \in \widetilde{\mathcal{P}}_1, X_2 \in \widetilde{\mathcal{P}}_2, X_3 \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3)$ (see §482 p. 283). We have

$$\mathrm{RHom}_{\mathcal{C}_3}(LG(X_1, X_2), X_3) \simeq \mathrm{Hom}^\bullet_{\mathcal{C}_3}(\mathrm{K}(G)(X_1, X_2), X_3)$$

$$\simeq \mathrm{Hom}^\bullet_{\mathcal{C}_1}(X_1, \mathrm{K}(F_1)(X_2, X_3)) \simeq \mathrm{RHom}_{\mathcal{C}_1}(X_1, RF(X_2, X_3)).$$

17 About Chapter 16

17.1 Sieves and local epimorphisms

This section is about the beginning of Section 16.1 p. 389 of the book. Let $\mathcal{C}$ be a category whose hom-sets are disjoint, let M be the set of morphisms in $\mathcal{C}$, and for each U in $\mathcal{C}$ let $M_U \subset M$ be the set of morphisms whose target is U. A subset S of M_U is a **sieve** over U if it is a right ideal of M, in the sense that S contains all morphism of the form $s \circ f$ with s in S. If S is a sieve over U and $f : V \to U$ is a morphism, we put

$$S \times_U V := \{W \to V \mid (W \to V \to U) \in S\}. \tag{180}$$

One easily checks that this is a sieve over V.

To a sieve S over U we attach the subobject A_S of U in $\mathcal{C}^\wedge$ by the formula

$$A_S(V) := S \cap \mathrm{Hom}_{\mathcal{C}}(V, U).$$

Conversely, to an object $A \to U$ of $(\mathcal{C}^\wedge)_U$ we attach the sieve $S_{A \to U}$ over U by putting

$$S_{A \to U} := \{V \to A \to U\}.$$

In particular we get a natural bijection between the sieves over U and the subobjects of U in $\mathcal{C}^\wedge$, and this bijection is compatible with (180) p. 284. We may sometimes tacitly identify these two sets, so that, given a sieve S over U, the datum of a morphism $(V \to U) \in S$ is equivalent to that of a morphism $V \to S$ in $\mathcal{C}^\wedge$. (We say that $A \in \mathcal{C}^\wedge$ is a subobject of $B \in \mathcal{C}^\wedge$ if $A(U) \subset B(U)$ for all U.)

Let Σ_U be the set of sieves over U. Let $(J(U))_{U \in \mathcal{C}}$ be a subfamily of the family $(\Sigma_U)_{U \in \mathcal{C}}$ and consider the following conditions:

Condition 489.

GT1: for all U in $\mathcal{C}$ we have: $M_U \in J(U)$,

GT2: for all U in $\mathcal{C}$ we have: $J(U) \ni S \subset S' \in \Sigma_U \implies S' \in J(U)$,

GT3: for all U in $\mathcal{C}$ we have: $S \in J(U),\ (V \to U) \in M \implies S \times_U V \in J(V)$,

GT4: for all U in $\mathcal{C}$ we have:

$$S \in J(U),\ S' \in \Sigma_U,\ S' \times_U V \in J(V)\ \forall\ (V \to U) \in S \implies S' \in J(U).$$

The membres of $J(U)$ are called **covering sieves of** U.

Proposition 490. *Axiom GT2 follows from GT4.*

Proof. In the notation of GT2, if $(V \to U)$ be in S, then we have $S' \times_U V = S \times_U V$. □

Consider the following conditions on a set $\mathcal{E}$ of morphisms in $\mathcal{C}^\wedge$:

LE1: id_U is in $\mathcal{E}$ for all U in $\mathcal{C}$,

LE2: if the composition of two elements of $\mathcal{E}$ exists, it belongs to $\mathcal{E}$, LE3: if the composition $v \circ u$ of two morphisms of $\mathcal{C}^\wedge$ exists and is in $\mathcal{E}$, then v is in $\mathcal{E}$,

LE4: a morphism $A \to B$ in $\mathcal{C}^\wedge$ is in $\mathcal{E}$ if and only if, for any morphism $U \to B$ in $\mathcal{C}^\wedge$ with U in $\mathcal{C}$, the projection $A \times_B U \to U$ is in $\mathcal{E}$.

As proved in the book,

$$\mathcal{E} \text{ contains all epimorphisms.} \tag{181}$$

For the reader's convenience we paste the proof of (181) (see p. 391 in the book):

Assume that $u : A \to B$ is an epimorphism in $\mathcal{C}^\wedge$. If $w : U \to B$ is a morphism with U in $\mathcal{C}$, there exists $v : U \to A$ such that $w = u \circ v$ by Proposition 161 p. 105, and Exercise 3.4 (i) p. 90 of the book, stated above as Proposition 165 p. 106. Hence, $\mathrm{id}_U : U \to U$ factors as $U \to A \times_B U \to U$. Therefore $A \times_B U \to U$ is in $\mathcal{E}$ by LE1 and LE3, and this implies that $A \to B$ is in $\mathcal{E}$ by LE4. q.e.d.

The elements of $\mathcal{E}$ are called **local epimorphisms**.

Let $J = (J(U))_{U \in \mathcal{C}}$ be a subfamily of the family $(\Sigma_U)_{U \in \mathcal{C}}$ satisfying GT1-GT4, let $\mathcal{U}$ be a universe such that $\mathcal{C}$ is $\mathcal{U}$-small (Definition 5 p. 10), and let

$$\mathcal{E} = \mathcal{E}(J,\mathcal{U}) \tag{182}$$

be the set of those morphisms $A \to B$ in $\mathcal{C}^\wedge$ such that, for any morphism $U \to B$ in $\mathcal{C}^\wedge$ with U in $\mathcal{C}$, the sieve $S_{A \times_B U \to U}$ is in $J(U)$.

Lemma 491. *A morphism $A \to U$ in $\mathcal{C}^\wedge$ with U in $\mathcal{C}$ is in $\mathcal{E}$ if and only if $S_{A \to U}$ is in $J(U)$.*

Proof. Observe first that, in the setting

$$A \to U \leftarrow V$$

(obvious notation), we have

$$S_{A \times_U V \to V} = S_{A \to U} \times_U V. \tag{183}$$

Let $A \to U$ be a morphism in $\mathcal{C}^\wedge$ with U in $\mathcal{C}$. Consider the conditions (with obvious notation)

$$(A \to U) \in \mathcal{E}, \tag{184}$$

$$S_{A\times_U V\to V}\in J(V)\quad\forall\quad V\to U, \tag{185}$$

$$S_{A\to U}\times_U V\in J(V)V\quad\forall\quad V\to U, \tag{186}$$

$$S_{A\to U}\in J(U). \tag{187}$$

We have (184) $\Leftrightarrow$ (185) by definition of $\mathcal{E}$, and (185) $\Leftrightarrow$ (186) by Lemma 491, (186) $\Rightarrow$ (187) by GT1 and GT4, and (187) $\Rightarrow$ (186) by GT3. □

Let us check that $\mathcal{E}$ satisfies LE1-LE4:

LE1 follows immediately from GT1.

LE2: Let $A\to B\to C$ be a diagram in $\mathcal{C}^\wedge$, and assume that the two arrows are in $\mathcal{E}$. Consider the diagram of solid arrows with cartesian squares

$$\begin{array}{ccccc} F & \dashrightarrow & V & & \\ \downarrow & & \downarrow & & \\ D & \longrightarrow & E & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

in $\mathcal{C}^\wedge$ (with U in $\mathcal{C}$). We have that $S_{E\to U}$ is in $J(U)$ (because $B\to C$ is in $\mathcal{E}$) and we must prove that $S_{D\to U}$ is in $J(U)$. Let $V\to U$ be in $S_{E\to U}$, and let us complete the diagram with cartesian squares as indicated. By GT4 it suffices to check that $S_{F\to V}$ is in $J(V)$. But this follows from the assumption that $A\to B$ is in $\mathcal{E}$ (together with a transitivity property of cartesian squares which has already been tacitly used).

LE3 follows immediately from GT2.

LE4 follows immediately from Lemma 491.

Conversely, given an object U of $\mathcal{C}$ and a set $\mathcal{E}$ of morphisms in $\mathcal{C}^\wedge$ satisfying LE1-LE4, put

$$J(U):=\{S\in\Sigma_U\mid(A_S\to U)\in\mathcal{E}\}.$$

Recall that we have attached to a sieve S over U the subobject A_S of U in $\mathcal{C}^\wedge$ defined by

$$A_S(V):=S\cap\operatorname{Hom}_{\mathcal{C}}(V,U).$$

Let us check that $J_{\mathcal{E}}:=(J(U))_{U\in\mathcal{C}}$ satisfies GT1-GT4.

GT1 follows from LE1 and the equality $(A_{M_U}\to U)=\mathrm{id}_U$.

GT2 follows from GT4 (Proposition 490 p. 285).

To prove GT3, note that if S is a sieve over U and $V \to U$ is a morphism in $\mathcal{C}$, then we have the equality

$$(A_{S' \times_U V} \to V) = (A_{S'} \times_U V \to V), \tag{188}$$

in $(\mathcal{C}^\wedge)_V$. In view of LE4, this implies GT3.

The lemma below will helps us verify GT4.

Lemma 492. *For any sieve S over U we have*

$$S_{A_S \to U} = S.$$

For any morphism $A \to U$ (obvious notation) there is a canonical isomorphism

$$A_{S_{A \to U}} \simeq \operatorname{Im}(A \to U).$$

Proof. The proof of the second sentence is straightforward and left to the reader. To prove the first sentence let $s : V \to U$ be a morphism in $\mathcal{C}$ and S a sieve over U. It suffices to show that s is in S if and only if s factors through the natural morphism $i : A_S \to U$.

By the Yoneda Lemma (Lemma 36 p. 33), there is a bijection

$$S \cap \operatorname{Hom}_{\mathcal{C}}(V, U) \xrightarrow{\varphi} \operatorname{Hom}_{\mathcal{C}^\wedge}(V, A_S)$$

such that $\varphi(s)_W = s\circ$ for all W in $\mathcal{C}$.

Assume that s is in S and let us show that there is a morphism $v : V \to A_S$ satisfying $i \circ v = s$. It suffices to prove $i \circ \varphi(s) = s$ and to put $v := \varphi(s)$. We have for all W in $\mathcal{C}$

$$(i \circ \varphi(s))_W = i_W \circ \varphi(s)_W = i_W \circ (s\circ) = s\circ = s_W.$$

Conversely, assuming that v is in $\operatorname{Hom}_{\mathcal{C}^\wedge}(V, A_S)$, it suffices to prove that $i \circ v$ is in S. We have

$$i \circ v = (i \circ v) \circ \operatorname{id}_V = (i \circ v)_V(\operatorname{id}_V) = i_V(v_V(\operatorname{id}_V)) = v_V(\operatorname{id}_V) \in A_S(V) \subset S.$$

□

Lemma 493. *Condition GT4 holds.*

Proof. Let us assume

$$S \in J(U),\ S' \in \Sigma_U,\ S' \times_U V \in J(V)\ \forall\ (V \to U) \in S. \tag{189}$$

It suffices to check $S' \in J(U)$, or, equivalently,

$$(A_{S'} \to U) \in \mathcal{E}. \tag{190}$$

Form the cartesian square

$$\begin{array}{ccc} B & \longrightarrow & A_S \\ \downarrow & & \downarrow \\ A_{S'} & \longrightarrow & U. \end{array}$$

As $A_S \to U$ is in $\mathcal{E}$ by assumption, it suffices, by LE2 and LE3, to check

$$(B \to A_S) \in \mathcal{E}. \tag{191}$$

Let $V \to A_S$ be a morphism in $\mathcal{C}^\wedge$ with V in $\mathcal{C}$, and let

$$\begin{array}{ccc} C & \longrightarrow & V \\ \downarrow & & \downarrow \\ B & \longrightarrow & A_S \end{array}$$

be a cartesian square. By LE4 it is enough to verify

$$(C \to V) \in \mathcal{E}. \tag{192}$$

The morphism $V \to U$ being in S by the first sentence of Lemma 492, the sieve $S' \times_U V$ is in $J(V)$ by (189), and $A_{S' \times_U V} \to V$ is in $\mathcal{E}$ by definition of $J(V)$. We have

$$\mathcal{E} \ni (A_{S' \times_U V} \to V) = (A_{S'} \times_U V \to V) \simeq (C \to V).$$

Indeed, the equality holds by (188) p. 288, and the isomorphism holds because the rectangle

$$\begin{array}{ccc} C & \longrightarrow & V \\ \downarrow & & \downarrow \\ B & \longrightarrow & A_S \\ \downarrow & & \downarrow \\ A_{S'} & \longrightarrow & U \end{array}$$

is cartesian. This proves successively (192), (191), (190) and the lemma. □

We have proved that $(J(U))_{U\in\mathcal{C}}$ satisfies GT1-GT4.

It is now easy, thanks to Lemma 492 p. 288, to prove

Theorem 494. *If $\mathcal{C}$ is a $\mathcal{U}$-small category (Definition 5 p. 10), if J is a subfamily of $(\Sigma_U)_{U\in\mathcal{C}}$ satisfying GT1-GT4, and if $\mathcal{E}$ is a set of morphisms in $\mathcal{C}^\wedge_\mathcal{U}$ satisfying LE1-LE4, then the equalities $\mathcal{E}=\mathcal{E}(J,\mathcal{U})$ and $J=J_\mathcal{E}$ are equivalent.*

Corollary 495. *Let $\mathcal{U}\subset\mathcal{U}'$ be universes, let $\mathcal{C}$ be a $\mathcal{U}$-small category, let J be a subfamily of $(\Sigma_U)_{U\in\mathcal{C}}$ satisfying GT1-GT4 (see Conditions 489 p. 285), let $u:A\to B$ be a morphism in $\mathcal{C}^\wedge_\mathcal{U}$, and let $u':A'\to B'$ be the corresponding morphism in $\mathcal{C}^\wedge_{\mathcal{U}'}$. Then u is in $\mathcal{E}(J,\mathcal{U})$ (see* (182) *p. 286) if and only if u' is in $\mathcal{E}(J,\mathcal{U}')$.*

17.2 Brief comments

§ 496. P. 390, Display (16.1.1). Note that A_S is *not* a subobject of U in $\mathcal{C}^\wedge$ in the sense of Definition 1.2.18 (i) p. 18 of the book. See §27 p. 27.

§ 497. P. 390, Axioms LE1-LE4. The set of local epimorphisms attached to the natural Grothendieck topology associated with a small topological space X can be described as follows.

Let $f:A\to B$ be a morphism in $\mathcal{C}^\wedge$, where $\mathcal{C}$ is the category of open subsets of X. For each pair (U,b) with U in $\mathcal{C}$ and b in $B(U)$ let $\Sigma(U,b)$ be the set of those V in $\mathcal{C}_U$ such that there is an a in $A(V)$ satisfying $f_V(a)=b_V$, where b_V is the restriction of b to V. Then f is a local epimorphism if and only if

$$U=\bigcup_{V\in\Sigma(U,b)}V$$

for all (U,b) as above.

Moreover, a morphism $u:A\to U$ in $(\mathrm{Op}_X)^\wedge$ with U in Op_X is a local epimorphism if and only if for all x in U there is a V in Op_X such that $x\in V$ and $A(V)\neq\varnothing$.

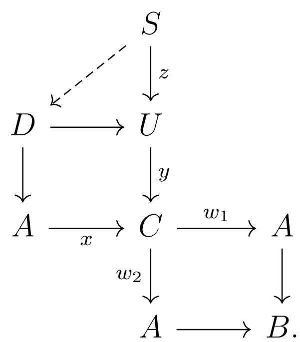
§ 498. For any universe $\mathcal{U}$, any $\mathcal{U}$-small category (Definition 5 p. 10) $\mathcal{C}$ and any subfamily J of $(\Sigma_U)_{U\in\mathcal{C}}$ satisfying GT1-GT4 (see Conditions 489 p. 285), let $\mathcal{M}(J,\mathcal{U})$ and $\mathcal{I}(J,\mathcal{U})$ denote respectively the set of local monomorphisms and local isomorphisms attached to $\mathcal{E}(J,\mathcal{U})$ (see (182) p. 286). Corollary 495 p. 290 implies:

Let $\mathcal{U} \subset \mathcal{U}'$ be universes, let $\mathcal{C}$ be a $\mathcal{U}$-small category, let J be a subfamily of $(\Sigma_U)_{U\in\mathcal{C}}$ satisfying GT1-GT4, let $u : A \to B$ be a morphism in $\mathcal{C}^\wedge_{\mathcal{U}}$, and let $u' : A' \to B'$ be the corresponding morphism in $\mathcal{C}^\wedge_{\mathcal{U}'}$. Then u is in $\mathcal{M}(J,\mathcal{U})$ (resp. in $\mathcal{I}(J,\mathcal{U})$) if and only if u' is in $\mathcal{M}(J,\mathcal{U}')$ (resp. in $\mathcal{I}(J,\mathcal{U}')$).

§ 499. P. 395, proof of Lemma 16.2.3 (iii), sentence:

"Notice first that a morphism $U \to A \times_B A$ is nothing but a diagram $U \rightrightarrows A \to B$ such that the two compositions coincide, and then any diagram $S \to U \rightrightarrows A$ such that the two compositions coincide factorizes as $S \to A \underset{A\times_B A}{\times} U \to U$."

Let us prove the factorization statement, even if it is straightforward. Consider the commutative diagram with cartesian squares



We have

$$w_1 x = \mathrm{id}_A = w_2 x \tag{193}$$

and

$$w_1 y z = w_2 y z \tag{194}$$

by assumption (we omit the composition symbol $\circ$), and it suffices to show $x w_1 y z = y z$, that is

$$w_i x w_1 y z = w_i y z \qquad (i = 1, 2). \tag{195}$$

But (195) follows immediately from (193) and (194).

§ 500. P. 395, Lemma 16.2.3 (iii). Consider the conditions

(b) for any diagram $C \rightrightarrows A \to B$ such that C is in $\mathcal{C}$ and the two compositions coincide, there exists a local epimorphism $D \to C$ such that the two compositions $D \to C \rightrightarrows A$ coincide,

(c) for any diagram $C \rightrightarrows A \to B$ such that C is in $\mathcal{C}^\wedge$ and the two compositions coincide, there exists a local epimorphism $D \to C$ such that the two compositions $D \to C \rightrightarrows A$ coincide,

(d) for any diagram $C \rightrightarrows A \to B$ such that C is in $\mathcal{C}$ and the two compositions coincide, there exists a local isomorphism $D \to C$ such that the two compositions $D \to C \rightrightarrows A$ coincide,

(e) for any diagram $C \rightrightarrows A \to B$ such that C is in $\mathcal{C}^\wedge$ and the two compositions coincide, there exists a local isomorphism $D \to C$ such that the two compositions $D \to C \rightrightarrows A$ coincide.

Recall that (a) is the condition that $A \to B$ is a local monomorphism. Lemma 16.2.3 p. 395 of the book implies

$$\text{Conditions (a), (b), (c), (d), (e) are equivalent.} \tag{196}$$

Indeed, Part (iii) of the lemma says that (a), (b) and (c) are equivalent. Clearly (e) implies (c) and (d), and (d) implies (b). It suffices to check that (c) implies (e). Let $C \rightrightarrows A \to B$ be as in the assumption (c), let $D \to C$ be the local epimorphism furnished by (c), and let I be its image. The two compositions $I \to C \rightrightarrows A$ coincide because $D \to I$ is an epimorphism, and $I \to C$ is a local isomorphism by Part (ii) of the lemma. q.e.d.

§ 501. P. 395, Lemma 16.2.4 (i). The statement says that local monomorphisms are stable by base change. The last sentence of Step (a) in the proof follows from the fact that local epimorphisms are stable by base change (Proposition 16.1.11 (i) p. 394 of the book).

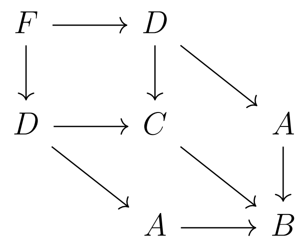
§ 502. P. 396, Step (a) in the proof of Lemma 16.2.4 (i). Starting with the cartesian square

$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B, \end{array}$$

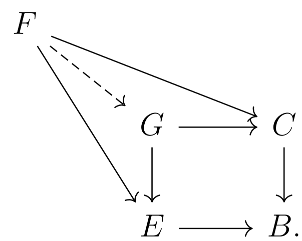
we form the cartesian squares

$$\begin{array}{ccc} E & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} F & \longrightarrow & D \\ \downarrow & & \downarrow \\ D & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} G & \longrightarrow & C \\ \downarrow & & \downarrow \\ E & \longrightarrow & B, \end{array}$$

we define a morphism $F \to E$ as suggested by the commutative diagram



and we check that there is an obvious morphism $F \to G$ suggested by the commutative diagram



We claim that $F \to G$ is an isomorphism.

To prove this, we can assume that the above commutative diagrams take place in **Set**. We leave it to the reader to verify that the formula

$$((a_1, a_2), c) \mapsto ((a_1, c), (a_2, c))$$

defines a (unique) map $G \to F$, and that this map is inverse to $F \to G$.

§ 503. P. 396, proof of Lemma 16.2.4 (ii). The fact that h is a local epimorphism follows from Proposition 16.1.11 (i) p. 394 of the book.

§ 504. P. 397, Notation 16.2.5 (ii). The fact that

$$\text{such a } w \text{ is necessarily a local isomorphism} \tag{197}$$

follows from Lemma 16.2.4 (vii) p. 396.

§ 505. P. 398, proof of Lemma 16.2.7: see §86 p. 64.

§ 506. Right after Display (16.3.1) p. 399 of the book, in view of the natural isomorphism

$$A^a(U) \simeq \mathrm{Hom}_{(\mathcal{C}^\wedge)_{\mathcal{LI}}}(Q(U), Q(A)),$$

the map $A^a(U') \to A^a(U)$ induced by a morphism $U \to U'$ can also be described by the diagram

$$Q(U) \to Q(U') \to Q(A).$$

Similarly, the map $A(U) \to A^a(U)$ at the top of p. 400 of the book can also be described by the diagram

$$A(U) \simeq \mathrm{Hom}_{\mathcal{C}^\wedge}(U, A) \to \mathrm{Hom}_{(\mathcal{C}^\wedge)_{\mathcal{LI}}}(Q(U), Q(A)) \simeq A^a(U).$$

Then Lemma 16.3.1 can be stated as follows.

If

$$U \xleftarrow{s} B \xrightarrow{u} A$$

is a diagram in $\mathcal{C}^\wedge$ with U in $\mathcal{C}$ and s a local isomorphism, and if

$$v = Q(u) \circ Q(s)^{-1} \in A^a(U) \simeq \mathrm{Hom}_{(\mathcal{C}^\wedge)_{\mathcal{LI}}}(Q(U), Q(A)),$$

then

$$v \circ s = \varepsilon(A) \circ u. \tag{198}$$

Indeed, (198) is equivalent to $v \circ Q(s) = Q(u)$.

§ 507. P. 400, Step (ii) in the proof of Lemma 16.3.2 (additional details):

We want to prove that $A \to A^a$ is a local monomorphism. In view of (196) p. 292 it suffices to check that Condition (b) of §500 p. 291 holds.

Recall that the functor

$$\alpha : (\mathcal{LI}_U)^{\mathrm{op}} \to \mathbf{Set}, \quad (B \xrightarrow{s} U) \mapsto \mathrm{Hom}_{\mathcal{C}^\wedge}(B, A)$$

satisfies $\operatorname{colim} \alpha \simeq A^a(U)$ (see (16.3.1) p. 399 of the book). Let $i(s) : \alpha(s) \to A^a(U)$ be the coprojection, and let $f_1, f_2 : U \rightrightarrows A$ be two morphisms such that the compositions $U \rightrightarrows A \to A^a$ coincide. By definition of the natural morphism $A \to A^a$, we have

$$i(\mathrm{id}_U)(f_1) = i(\mathrm{id}_U)(f_2).$$

By the fact that $\mathcal{LI}_U$ is cofiltrant, and by Proposition 3.1.3 p. 73 of the book, there is a morphism

$$\varphi : (B \xrightarrow{s} U) \to (U \xrightarrow{\mathrm{id}_U} U)$$

in $\mathcal{LI}_U$ such that $\alpha(\varphi)(f_1) = \alpha(\varphi)(f_2)$. This means that the compositions $B \to U \rightrightarrows A$ coincide. q.e.d.

§ 508. P. 401, Step (i) of the proof of Proposition 16.3.3. See (196) p. 292 and (197) p. 293. (As already mentioned, $B'' \to B$ should be $B'' \to B'$.)

18 About Chapter 17

18.1 Brief comments

§ 509. P. 405, Chapter 17. It seems to me it would be more convenient to denote by f^t the functor from $(\mathcal{C}_Y)^{\mathrm{op}}$ to $(\mathcal{C}_X)^{\mathrm{op}}$ (and *not* the functor from $\mathcal{C}_Y$ to $\mathcal{C}_X$) which defines f. To avoid confusion, we shall adopt here the following convention:

If $f : X \to Y$ is a morphism of presites, then we keep the notation f^t for the functor from $\mathcal{C}_Y$ to $\mathcal{C}_X$, and we designate by f^τ the functor from $(\mathcal{C}_Y)^{\mathrm{op}}$ to $(\mathcal{C}_X)^{\mathrm{op}}$:

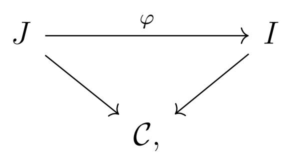
$$f^t : \mathcal{C}_Y \to \mathcal{C}_X, \quad f^\tau : (\mathcal{C}_Y)^{\mathrm{op}} \to (\mathcal{C}_X)^{\mathrm{op}}. \tag{199}$$

In other words, we set

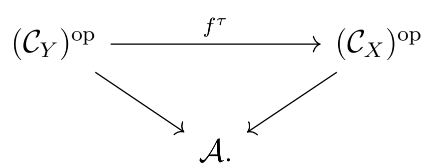
$$\boxed{f^\tau := (f^t)^{\mathrm{op}}}$$

We keep the same definition of left exactness (based on f^t) of $f : X \to Y$ as in the book.

The motivation for introducing the functor f^τ can be described as follows: The diagram



representing the general setting of Section 2.3 p. 50 of the book, is now replaced by the commutative diagram



(See also §510 p. 295 and §512 p. 296.)

§ 510. P. 406. Recall that, in the first line of the second display, $(\mathcal{C}_Y)^\wedge$ should be $\mathcal{C}_Y$ (twice). In notation (199), Formula (64) p. 91 gives, for B in $\mathcal{C}_Y^\wedge$ and U in $\mathcal{C}_X$,

$$(f^t)\hat{\ }(B)(U) \simeq \operatorname*{colim}_{(V \to B) \in (\mathcal{C}_Y)_B} \mathrm{Hom}_{(\mathcal{C}_X)}(U, f^t(V)) \simeq \operatorname*{colim}_{(U \to f^t(V)) \in (\mathcal{C}_Y)^U} B(V). \tag{200}$$

For the sake of emphasis, we state:

Proposition 511. *The functor* $(f^t)^\frown$ *commutes with small inductive limits (Proposition 2.7.1 p. 62 of the book, Remark 117 p. 90 above). Moreover, if* f *is left exact, then* $(f^t)^\frown$ *is exact (Corollary 3.3.19 p. 87 of the book).*

If $f : X \to Y$ is a continuous map of small topological spaces, if B is in $(\mathrm{Op}_Y)^\wedge$ and U in Op_X, then (200) gives

$$(f^t)^\frown(B)(U) \simeq \operatorname*{colim}_{f^{-1}(V)\supset U} B(V). \tag{201}$$

§ 512. P. 407. Let $f : X \to Y$ be a morphism of presites and let $\mathcal{A}$ be a category admitting small inductive and projective limits. In the notation of (199) p. 295, we set

$$\boxed{f_* := (f^\tau)_*} \quad , \quad \boxed{f^\dagger := (f^\tau)^\dagger} \quad , \quad \boxed{f^\ddagger := (f^\tau)^\ddagger} \quad ,$$

yielding

$$f^\dagger, f^\ddagger : \mathrm{PSh}(X\mathcal{A}) \to \mathrm{PSh}(Y, \mathcal{A}).$$

Then (17.1.3) and (17.1.4) follow respectively from (2.3.6) and (2.3.7) p. 52 of the book. For the sake of completeness, let us rewrite (17.1.3) and (17.1.4) (in the notation of (199)):

$$f^\dagger(G)(U) = \operatorname*{colim}_{(f^\tau(V)\to U)\in((\mathcal{C}_Y)^{\mathrm{op}})_U} G(V), \tag{202}$$

with G in $\mathrm{PSh}(Y, \mathcal{A}), U$ in $\mathcal{C}_X$, $f^\tau(V) \to U$ being a morphism in $(\mathcal{C}_X)^{\mathrm{op}}$ (corresponding to a morphism $U \to f^t(V)$ in $\mathcal{C}_X$),

$$f^\ddagger(G)(U) = \lim_{(U\to f^\tau(V))\in((\mathcal{C}_Y)^{\mathrm{op}})^U} G(V), \tag{203}$$

with G in $\mathrm{PSh}(Y, \mathcal{A}), U$ in $\mathcal{C}_X$, $U \to f^\tau(V)$ being a morphism in $(\mathcal{C}_X)^{\mathrm{op}}$ (corresponding to a morphism $f^t(V) \to U$ in $\mathcal{C}_X$).

§ 513. P. 408, comment preceding Convention 17.1.6. Let us recall the comment:

We extend presheaves over X to presheaves over $\widehat{X}$ using the functor $\mathrm{h}_X^\ddagger$ associated with the Yoneda embedding $\mathrm{h}_X^t = \mathrm{h}_{\mathcal{C}_X}$. Hence, for F in $\mathrm{PSh}(X, \mathcal{A})$ and A in $\mathcal{C}_X^\wedge$, we have

$$(\mathrm{h}_X^\ddagger F)(A) = \lim_{(U\to A)\in(\mathcal{C}_X)_A} F(U).$$

By Corollary 2.7.4 p. 63 of the book, the functor

$$\mathrm{h}_X^\ddagger : \mathrm{PSh}(X, \mathcal{A}) \to \mathrm{PSh}(\widehat{X}, \mathcal{A})$$

induces an equivalence of categories between $\mathrm{PSh}(X, \mathcal{A})$ and the full subcategory of $\mathrm{PSh}(\widehat{X}, \mathcal{A})$ whose objects are the $\mathcal{A}$-valued presheaves over $\widehat{X}$ which commute with small projective limits.

One can add that a quasi-inverse is given by

$$\mathrm{h}_{X*} : \mathrm{PSh}(\widehat{X}, \mathcal{A}) \to \mathrm{PSh}(X, \mathcal{A}).$$

§ 514. P. 408, Convention 17.1.6. Recall the convention: If F is an $\mathcal{A}$-valued presheaf over X and A is a presheaf of sets over X, then we put

$$F(A) := (\mathrm{h}_X^{\ddagger} F)(A) = \lim_{(U \to A) \in (\mathcal{C}_X)_A} F(U). \tag{204}$$

(Note that the same comment is made at the beginning of Section 17.3 p. 414.) This convention of extending each presheaf F over X to a presheaf, still denoted by F, over $\widehat{X}$ which commutes with small projective limits implies that we have, for A, B in $\mathcal{C}^\wedge$,

$$B(A) \simeq \mathrm{Hom}_{\mathcal{C}_X^\wedge}(A, B).$$

In the notation of §119 p. 90, Convention 17.1.6 can be described as follows:

If X is a site, if $\mathcal{C}$ is the corresponding category, if $h : \mathcal{C} \to \mathcal{C}^\wedge$ is the Yoneda embedding, if F is an $\mathcal{A}$ valued sheaf over X, and if A is an object of $\mathcal{C}^\wedge$, then Convention 17.1.6 consists in putting

$$F(A) := (h^{\mathrm{op}})^{\ddagger}(F)(A).$$

§ 515. P. 409, Proposition 17.1.9 follows immediately from (58) p. 85, (60) p. 86 and (61) p. 88.

§ 516. P. 410, Display (17.1.15): As already indicated in §18, Display (17.1.15) p. 410 should read

$$\mathrm{Hom}_{\mathrm{PSh}(X,\mathcal{A})}(F, G) \simeq \lim_{U \in \mathcal{C}_X} \mathcal{H}om_{\mathrm{PSh}(X,\mathcal{A})}(F, G)(U).$$

§ 517. P. 412, proof of Lemma 17.2.2 (ii), (b)⇒(a), Step (1): $(f^t)^\frown$ is right exact by Proposition 161 p. 105 and Proposition 511 p. 296.

§ 518. P. 412, proof of Lemma 17.2.2 (ii), (b)⇒(a), Step (3). See §19 p. 21. This is essentially a copy and paste of the book.

Claim: if a local isomorphism $u : A \to B$ in $\mathcal{C}_Y^\wedge$ is either a monomorphism or an epimorphism, then $(f^t)^\frown(u)$ is a local isomorphism in $\mathcal{C}_X^\wedge$.

Proof of the claim: Let $V \to B$ be a morphism in $\mathcal{C}_Y^\wedge$ with V in $\mathcal{C}_Y$. Then $u_V : A \times_B V \to V$ is either a monomorphism or an epimorphism by Proposition 160 p. 105 and Proposition 165 p. 106. Let us show that $(f^t)^\frown(u_V)$ is a local isomorphism.

If u_V is a monomorphism, $(f^t)^\frown(u_V)$ is a local isomorphism by assumption.

If u_V is an epimorphism, then u_V has a section $s : V \to A \times_B V$. Since u_V is a local isomorphism by Lemma 16.2.4 (i) p. 395 of the book, s is a local isomorphism. Since

$$(f^t)^\frown(u_V) \circ (f^t)^\frown(s) \simeq \mathrm{id}_{f^t(V)}$$

is a local monomorphism, and $(f^t)^\frown(s)$ is a local epimorphism by Step (2), Lemma 16.2.4 (vi) p. 396 of the book implies that $(f^t)^\frown(u_V)$ is a local monomorphism. Since $(f^t)^\frown(u_V)$ is an epimorphism by Step (2), we see that $(f^t)^\frown(u_V)$ is a local isomorphism. This proves the claim.

Taking the inductive limit with respect to $V \in (\mathcal{C}_Y)_B$, we conclude by Proposition 16.3.4 p. 401 of the book that $(f^t)^\frown(u)$ is a local isomorphism.

§ 519. P. 413, Definition 17.2.4 (ii): see Remark 118 p. 90.

§ 520. P. 413. Lemma 17.2.5 (ii) and Exercise 2.12 (ii) p. 66 of the book imply: If $f : X \to Y$ is weakly left exact, then $(f^t)^\frown : \mathcal{C}_Y^\wedge \to \mathcal{C}_X^\wedge$ commutes with projective limits indexed by small connected categories (Definition 5 p. 10).

§ 521. P. 413, Lemma 17.2.5 (ii). Here is a corollary:

Let $f : X \to Y$ be a weekly left exact morphism of sites such that $(f^t)^\frown(u)$ is a local epimorphism if and only if u is a local epimorphism. Then $(f^t)^\frown(u)$ is a local monomorphism if and only if u is a local monomorphism, and $(f^t)^\frown(u)$ is a local isomorphism if and only if u is a local isomorphism.

§ 522. P. 413, Example 17.2.7 (i). Recall that $f : X \to Y$ is a continuous map of small topological spaces. As explained in the book, to see that f is a morphism of sites, it suffices to check that, if $u : B \to V$ is a local epimorphism in $(\mathrm{Op}_Y)^\wedge$ with V in Op_Y, then $(f^t)^\frown(B) \to f^{-1}(V)$ is a local epimorphism in $(\mathrm{Op}_X)^\wedge$. This follows immediately from §497 p. 290 and (201) p. 296.

§ 523. P. 414, Definition 17.2.8 (minor variant):

Definition 524 (Definition 17.2.8 p. 414, Grothendieck topology)**.** *Let X be a small presite. We assume, as we may, that the hom-sets of $\mathcal{C}_X$ are disjoint. A* Grothendieck topology *on X is a set $\mathcal{T}$ of morphisms of $\mathcal{C}_X$ which satisfies Axioms LE1-LE4 p. 390. Let $\mathcal{T}'$ and $\mathcal{T}$ be Grothendieck topologies. We say that $\mathcal{T}$ is* stronger than *$\mathcal{T}'$, or that $\mathcal{T}'$ is* weaker than *$\mathcal{T}$, if $\mathcal{T}' \subset \mathcal{T}$.*

Let $(\mathcal{T}_i)$ be a family of Grothendieck topologies. We observe that $\bigcap \mathcal{T}_i$ is a Grothendieck topology, and we denote by $\bigvee \mathcal{T}_i$ the intersection of all Grothendieck topologies containing $\bigcup \mathcal{T}_i$.

18.2 Definition of a sheaf (page 414)

Here is Definition 17.3.1(ii) of the book:

Definition 525. *A presheaf $F \in \mathrm{PSh}(X, \mathcal{A})$ is a sheaf if for any local isomorphism $A \to U$ such that $U \in \mathcal{C}_X$ and $A \in (\mathcal{C}_X)^\wedge$, the morphism $F(U) \to F(A)$ is an isomorphism.*

(Here $\mathcal{A}$ is a category admitting small projective limits.)

To simplify we assume $\mathcal{A} = \mathbf{Set}$.

In SGA4.II.2.1 Verdier defines a sheaf of sets by

Definition 526. *A presheaf of sets $F \in \mathrm{PSh}(X)$ is a sheaf if for any local isomorphism $A \to U$ in $\mathcal{C}_X$ such that $A \to U$ is a monomorphism and U is in $\mathcal{C}_X$, the morphism $F(U) \to F(A)$ is an isomorphism.*

See http://www.normalesup.org/∼forgogozo/SGA4/02/02.pdf

A "KS-sheaf" is obviously a "Verdier sheaf". By implication "(i) $\implies$ (ii bis)" in Proposition 5.3 of Verdier's SGA4.II Exposé linked to above, the converse is also true.

In https://mathoverflow.net/a/283271/461 Dylan Wilson proved this fact using only the beginning of Verdier's Exposé, up to Proposition 4.2. In the next section (Section 18.3 p. 301) we prove these results. In the sequel of this section we describe Dylan Wilson's argument. Our purpose was that this text, together with *Categories and Sheaves*, offer a self-contained proof of the equivalence of the two definitions.

Proposition 527. *A "Verdier sheaf" is a "KS-sheaf".*

Proof. For the duration of the proof, $\mathrm{Sh}(X)$ will denote the category of "*Verdier*-sheaves". By Theorem 3.4 in SGA4.II (Theorem 531 p. 307), the inclusion $\mathrm{Sh}(X) \subset \mathrm{PSh}(X)$ admits an exact left adjoint $\mathbf{a}$. We will also use tacitly the fact that $\mathrm{Sh}(X)$ admits finite limits and colimits (Theorem 4.1 in SGA4.II, Theorem 534 p. 308).

Claim 1: If $A \to U$, with U in $\mathcal{C}_X$, is a monomorphism and a local isomorphism in $\mathrm{PSh}(X)$, then $\mathbf{a}A \to \mathbf{a}U$ is an isomorphism in $\mathrm{Sh}(X)$.

Proof of Claim 1: Let F be in $\mathrm{Sh}(X)$. We must show that the map

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{a}U, F) \to \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{a}A, F)$$

is bijective, that is, that the map

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(U, F) \to \mathrm{Hom}_{\mathrm{PSh}(X)}(A, F)$$

is bijective, which is clear.

Claim 2: If $A \to B$ is a monomorphism and a local isomorphism in $\mathrm{PSh}(X)$, then $\mathbf{a}A \to \mathbf{a}B$ is an isomorphism in $\mathrm{Sh}(X)$.

Proof of Claim 2: Let $f : A \to B$ be a monomorphism and a local isomorphism in $\mathrm{PSh}(X)$. By Proposition 4.2 in SGA4.II (Proposition 536 p. 309) and (the dual of) Proposition 165 p. 106 above, it suffices to show that $\mathbf{a}f$ is an epimorphism in $\mathrm{Sh}(X)$. Let $x, y : B \rightrightarrows F$ be two morphisms in $\mathrm{PSh}(X)$ with F in $\mathrm{Sh}(X)$ and $x \neq y$. It suffices to prove $x \circ f \neq y \circ f$. As we have

$$\operatorname*{colim}_{U \to B} U \xrightarrow{\sim} B$$

(the colimit being taken in $\mathrm{PSh}(X)$) by (44) p. 81, there is a morphism $u : U \to B$ in $\mathrm{PSh}(X)$ such that $x \circ u \neq y \circ u$. Consider the commutative diagram

$$\begin{array}{ccccc} A_U & \xrightarrow{f_U} & U & & \\ \downarrow & & \downarrow{\scriptstyle u} & & \\ A & \xrightarrow[f]{} & B & \overset{x}{\underset{y}{\rightrightarrows}} & F \end{array}$$

in $\mathrm{PSh}(X)$, where the square is cartesian. As f_U is a monomorphism by Exercise 2.22 p. 68 of the book, and a local isomorphism by Lemma 16.2.4 (i) p. 395 of the book, we have $x \circ u \circ f_U \neq y \circ u \circ f_U$ by Claim 1. This implies $x \circ f \neq y \circ f$, as desired.

Claim 3: **a** maps local epimorphisms to epimorphisms.

Proof of Claim 3: Let $A \to B$ be a local epimorphism in $\mathrm{PSh}(X)$. Then we see that

$$\mathrm{Im}(A \to B) \to B$$

is a local epimorphism and a monomorphism (Lemma 16.2.3 (ii) p. 395 of the book), that

$$\mathbf{a}\,\mathrm{Im}(A \to B) \to \mathbf{a}B$$

is an isomorphism by Claim 2, that

$$\mathrm{Im}(\mathbf{a}A \to \mathbf{a}B) \to \mathbf{a}B$$

is an isomorphism by exactness of **a**, and that $\mathbf{a}A \to \mathbf{a}B$ is an epimorphism by Proposition 5.1.2 (iv) p. 114 of the book. This proves Claim 3.

Claim 4: **a** maps local isomorphisms to isomorphisms.

Proof of Claim 4: Claim 4 follows from Claim 3 and the exactness of **a**.

Let us show that a "Verdier sheaf" is a "KS-sheaf". Let F be a "Verdier sheaf" and let $A \to U$ be a local isomorphism in $\mathrm{PSh}(X)$ such that U is in $\mathcal{C}_X$. It suffices to prove that the map $FU \to FA$ is bijective, that is, that the map

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(U, F) \to \mathrm{Hom}_{\mathrm{PSh}(X)}(A, F)$$

is bijective, that is, that the map

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{a}U, F) \to \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{a}A, F)$$

is bijective. But this follows from Claim 4. □

18.3 Proof of some results of SGA4II

In this section we prove the results contained in the beginning of Verdier's Exposé

http://www.normalesup.org/∼forgogozo/SGA4/02/02.pdf

up to Proposition 4.2, results used in Section 18.2 p. 299 above.

Let $\mathcal{C}$ be a site and $\mathcal{U}$ a universe such that $\mathcal{C} \in \mathcal{U}$, and write $\mathcal{C}^\wedge$ for $\mathcal{C}^\wedge_{\mathcal{U}}$.

Lemma 528 (1.1.1)**.** *Let S and S' be covering sieves of some object U of $\mathcal{C}$. Then the intersection $S \cap S'$ is again a covering sieve of U. In particular, the ordered set $J(U)$ is cofiltrant.*

Proof. Let $V \to S'$ be a morphism in $\mathcal{C}^\wedge$. It suffices to show that $S \times_U V$ is a covering sieve of V. There is a commutative diagram

$$\begin{array}{ccccc} S \times_U V & \hookrightarrow & V & = & V \\ \downarrow & & \downarrow & \swarrow & \downarrow \\ S \cap S' & \hookrightarrow & S' & \hookrightarrow & U \end{array}$$

where the two small squares are cartesian. This implies that $S \times_U V$ is a covering sieve of V, as desired. $\square$

18.3.1 Sheaf associated to a presheaf

Recall that $\mathcal{C}$ is a site and $\mathcal{U}$ a universe such that $\mathcal{C} \in \mathcal{U}$, and that we write $\mathcal{C}^\wedge$ for $\mathcal{C}^\wedge_{\mathcal{U}}$. The set $J(U)$ of covering sieves of U, ordered by inclusion, is cofiltrant by Lemma 528.

For all $\mathcal{U}$-presheaf F, the "canonical" colimit

$$\operatorname*{colim}_{S \in J(U)} \operatorname{Hom}_{\mathcal{C}^\wedge}(S, F)$$

is a member $\mathcal{U}$. (By "canonical" colimit we mean the set given by Proposition 2.4.1 p. 54 of *Categories and Sheaves*.) Let $g : V \to U$ be a morphism in $\mathcal{C}$. The base change functor $g^* : J(U) \to J(V)$ defines a map

$$LF(g) : LF(U) \to LF(V),$$

making $U \mapsto LF(U)$ is a presheaf over $\mathcal{C}$.

The morphism $\mathrm{id}_U : U \to U$ being a member of $J(U)$, we have, for all object U of $\mathcal{C}$, a map

$$\ell(F)(U) : F(U) \to LF(U),$$

defining a morphism of functors

$$\ell(F) : F \to LF.$$

It is clear moreover that $F \mapsto LF$ is a functor in F, and that the morphisms $\ell(F)$ define a morphism

$$\ell : \mathrm{id} \to L.$$

Finally let $S \hookrightarrow U$ be a covering sieve of U, and let

$$Z_S : \mathrm{Hom}_{\mathcal{C}^\wedge}(S, F) \to \mathrm{Hom}_{\mathcal{C}^\wedge}(U, LF)$$

be the coprojection. For all morphism $V \xrightarrow{g} U$ in $\mathcal{C}$, the definition of the functor LF shows that the diagram

$$(*)\qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}^\wedge}(S, F) & \xrightarrow{Z_S} & \mathrm{Hom}_{\mathcal{C}^\wedge}(U, LF) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}^\wedge}(S \times_U V, F) & \xrightarrow[Z_{S \times_U V}]{} & \mathrm{Hom}_{\mathcal{C}^\wedge}(V, LF) \end{array}$$

commutes. (The vertical arrows are the obvious ones.)

Lemma 529 (Lemme 3.1)**.**

1. *For all covering sieve $i_S : S \hookrightarrow U$ and all $a : S \to F$, the diagram*

$$(**)\qquad \begin{array}{ccc} F & \xrightarrow{\ell(F)} & LF \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle Z_S(a)} \\ S & \xrightarrow[i_S]{} & U \end{array}$$

   commutes.

2. *For all morphism $b : U \to LF$, there is a covering sieve S of U and a morphism $a : S \to F$ such that $Z_S(a) = b$.*

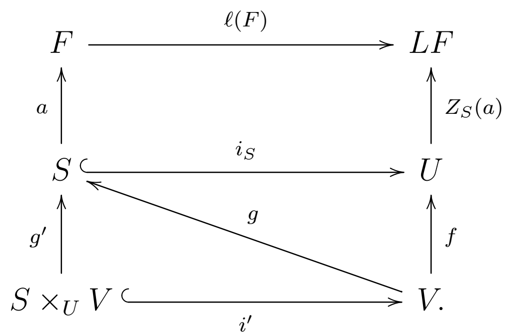
3. *Let U be an object of $\mathcal{C}$ and $a, b : V \rightrightarrows F$ two morphisms such that $\ell(F) \circ a = \ell(F) \circ b$. Then the kernel of the couple (a, b) is a covering sieve of U.*

4. *Let S and S' be two covering sieves of U, and let $a : S \to F$ and $a' : S' \to F$ be two morphisms. Then we have $Z_S(a) = Z_{S'}(a')$ if and only if a and a' coincide on a covering sieve $S'' \hookrightarrow S \times_U S'$.*

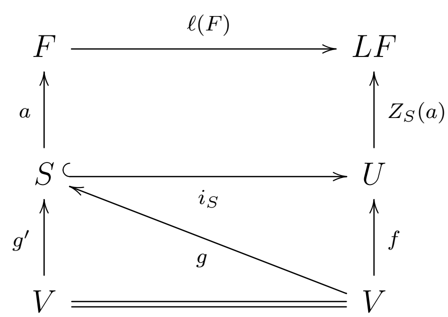
Proof. The only nontrivial assertion is Assertion 1. We must check that

$$Z_S(a) \circ i_S = \ell(F) \circ a.$$

It suffices to show that the compositions of these morphisms with a morphism $g : V \to S$ (V object of $\mathcal{C}$) are equal. Consider the morphism $f := i_S \circ g$ and the fiber product $S \times_U V$:



As the inclusion i' is admits a section, we have $S \times_U V = V$ and i' is the identity of V, so that we get the commutative diagram



with $g' = g$. We have

$$\ell(F) \circ a \circ g = Z_V(a \circ g') = Z_S(a) \circ i_S \circ g,$$

the equalities following respectively from the by definition of the $\ell(F)$ and the commutativity of the diagram (*). □

Proposition 530 (Proposition 3.2)**.**

1. *The functor L is left exact.*

2. *For all presheaf F, LF is a separated presheaf.*

3. *The presheaf F is separated if and only if the morphism $\ell(F) : F \to LF$ is a monomorphism. The presheaf LF is then a sheaf.*

4. *The following properties are equivalent:*

   (a) *$\ell(F) : F \to LF$ is an isomorphism.*

   (b) *F is a sheaf.*

Proof. Part 1. It suffices to show (Proposition 161 p. 105) that for all object U of $\mathcal{C}$, the functor $F \mapsto LF(U)$ commutes with finite limits. But, by definition of the limit, for $S \in J(U)$, the functor $F \mapsto \mathrm{Hom}_{\mathcal{C}^\wedge}(S, F)$ commutes with limits, and the colimit $\mathrm{colim}_{J(U)}$ commutes with finite limits because $J(U)$ is a cofiltrant ordered set (Proposition 125 p. 94).

Part 2. Let U be an object of $\mathcal{C}$ and $f, g : U \rightrightarrows LF$ two morphisms which coincide on a covering sieve $S \hookrightarrow U$ of U. By Part 2 of Lemma 529 p. 303, there is a covering sieve $S' \hookrightarrow U$, which we can assume to be contained in S, and there are two morphisms $a, b : S' \rightrightarrows F$ such that $Z_{S'}(a) = f$ and $Z_{S'}(b) = g$. By Part 1 of Lemma 529 p. 303 we then have $\ell(F) \circ a = \ell(F) \circ b$. Thus (Part 4 of Lemma 529 p. 303) a and b coincide on a covering sieve $S'' \hookrightarrow S'$. Letting w be the restriction of a to S'' yields

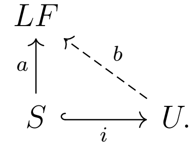
$$f = Z_{S'}(a) = Z_{S''}(w) = Z_{S'}(b) = g,$$

and thus $f = g$. Hence the presheaf LF is separated.

Part 3. Assume that F is separated and let us show that $\ell(F)$ is a monomorphism. Let U be an object of $\mathcal{C}$ and let $a, b : U \rightrightarrows F$ satisfy $\ell(F) \circ a = \ell(F) \circ b$, that is $Z_U(a) = Z_U(b)$. It suffices to prove $a = b$. By Part 4 of Lemma 529 p. 303 a and b coincide in some covering sieve of U. Since F is separated, this implies $a = b$.

If $\ell(F)$ is a monomorphism, the presheaf F, being a sub-presheaf of a separated presheaf, is separated.

To show that LF is then a sheaf, let $i : S \hookrightarrow U$ be a covering sieve of an object U of $\mathcal{C}$, and $a : S \to LF$ a morphism. It suffices to show that a factors through U:



(205)

We may, and will, view F as a subobject of LF. Form the cartesian square

$$\begin{array}{ccc} F & \xhookrightarrow{\ell(F)} & LF \\ {\scriptstyle a'}\uparrow & & \uparrow{\scriptstyle a} \\ S_1 & \xhookrightarrow{j} & S. \end{array}$$

Claim 1: For all morphism $V \to LF$, where V be an object of $\mathcal{C}$, the sieve $F \times_{LF} V$ is a covering sieve of V.

Proof of Claim 1: By Part 1 of Lemma 529 p. 303 there is a commutative diagram

$$\begin{array}{cccc} & F & \xhookrightarrow{\ell(F)} & LF \\ & \uparrow & & \uparrow \\ & S_2 & \hookrightarrow & V \\ \nearrow & & & \\ S_3 & & & \end{array}$$

where the small square is cartesian and S_3 is a covering sieve of V. We conclude that S_2 is also a covering sieve of V. This proves Claim 1.

Claim 2: S_1 is a covering sieve of U.

Proof of Claim 2: Let W be an object of $\mathcal{C}$ and $W \to S$ a morphism. There is a commutative diagram

$$\begin{array}{ccccc} F & \xhookrightarrow{\ell(F)} & LF & & \\ {\scriptstyle a'}\uparrow & & \uparrow{\scriptstyle a} & & \\ S_1 & \xhookrightarrow{j} & S & \xhookrightarrow{i} & U \\ \uparrow & & \uparrow & & \\ S_4 & \hookrightarrow & W & & \end{array}$$

where the squares (S_1, S, W, S_4) and (S_1, U, W, S_4) are cartesian. It suffices to check that S_4 is a covering sieve of W. As (F, LF, S, S_1) is cartesian too, so is (F, LF, W, S_4), and S_4 is a covering sieve of W by Claim 1. This proves Claim 2.

By Claim 2, we may define b in (205) by $b := Z_{S_1}(a')$. We must show $b \circ i = a$.

Let again W be an object of $\mathcal{C}$ and $W \to S$ a morphism, and let

$$\begin{array}{ccccc} F & \xhookrightarrow{\ell(F)} & LF & & \\ {\scriptstyle a'}\uparrow & & {\scriptstyle a}\uparrow & \overset{b}{\nwarrow} & \\ S_1 & \xhookrightarrow{j} & S & \xhookrightarrow[i]{} & U \\ {\scriptstyle c'}\uparrow & & \uparrow{\scriptstyle c} & & \\ S_4 & \xhookrightarrow[k]{} & W & & \end{array}$$

be the commutative diagram obtained in the obvious way from the previous. Proving $b \circ i = a$ reduces to proving $b \circ i \circ c = a \circ c$, which, LF being separated, reduces in turn to proving $b \circ i \circ c \circ k = a \circ c \circ k$, that is $b \circ i \circ j \circ c' = a \circ j \circ c'$. But we have

$$b \circ i \circ j = Z_{S_1}(a') \circ i \circ j = \ell(F) \circ a' = a \circ j$$

by Part 1 of Lemma 529 p. 303.

Part 4. Clear. □

Theorem 531 (Théorème 3.4)**.** *Let $\mathcal{C} \in \mathcal{U}$ be a site. The inclusion functor $i : \mathcal{C}^\sim \hookrightarrow \mathcal{C}^\wedge$ of the sheaves into the presheaves admits a left exact left adjoint functor $\mathbf{a}$ (Propositon 164 p. 106):*

$$\mathcal{C}^\sim \; \underset{i}{\overset{\mathbf{a}}{\leftrightarrows}} \; \mathcal{C}^\wedge \qquad .$$

The functor $i \circ \mathbf{a}$ is canonically isomorphic to the functor $L \circ L$. For all presheaf F the adjonction morphism $F \to i \circ \mathbf{a}(F)$ is obtained, via the previous isomorphism, from the morphism $\ell(LF) \circ \ell(F) : F \to (L \circ L)(F)$.

Definition 532 (Définition 3.5)**.** *The sheaf $\mathbf{a}F$ is called the sheaf associated to the presheaf F.*

Theorem 531 results immediately from Proposition 530 p. 304.

Proposition 533 (Proposition 3.6)**.** *Let $\mathcal{C} \in \mathcal{U}$ be a site and $\mathcal{V} \supset \mathcal{U}$ a universe. Write $\mathcal{C}^\wedge_\mathcal{U}$ and $\mathcal{C}^\wedge_\mathcal{U}$ (resp. $\mathcal{C}^\wedge_\mathcal{V}$ and $\mathcal{C}^\wedge_\mathcal{V}$) for the categories of $\mathcal{U}$-presheaves and of $\mathcal{U}$-sheaves (resp. of $\mathcal{V}$-presheaves and of $\mathcal{V}$-sheaves) and $\mathbf{a}_\mathcal{U} : \mathcal{C}^\wedge_\mathcal{U} \to \mathcal{C}^\wedge_\mathcal{U}$ (resp. $\mathbf{a}_\mathcal{V}$:*

$\mathcal{C}^\wedge_\mathcal{V} \to \mathcal{C}^\wedge_\mathcal{V}$) for the corresponding "associated sheaves" functors. The diagram

$$\begin{array}{ccc} \mathcal{C}^\wedge_\mathcal{U} & \xrightarrow{\mathbf{a}_\mathcal{U}} & \mathcal{C}^\sim_\mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{C}^\wedge_\mathcal{V} & \xrightarrow{\mathbf{a}_\mathcal{V}} & \mathcal{C}^\sim_\mathcal{V}, \end{array}$$

where the vertical functors are the canonical inclusions, commutes up to canonical isomorphism.

Proof. This follows from the construction of the functors $\mathbf{a}_\mathcal{U}$ and $\mathbf{a}_\mathcal{V}$ (Theorem 531). □

18.3.2 Exactness properties of the category of sheaves

The exactness properties of the category of sheaves follow from the exactness properties of the category of presheaves via Theorem 531. The present section spells out this philosophy by giving some of the most useful standard statements.

Theorem 534 (Théorème 4.1)**.** *Let $\mathcal{C} \in \mathcal{U}$ be a site, $\mathcal{C}^\sim$ the category of sheaves, $\mathbf{a} : \mathcal{C}^\wedge \to \mathcal{C}^\sim$ the associated sheaf functor, $i : \mathcal{C}^\sim \to \mathcal{C}^\wedge$ the inclusion functor.*

1. *The functor* $\mathbf{a}$ *commutes with colimits and is exact.*

2. *The $\mathcal{U}$-colimits in $\mathcal{C}^\sim$ are representable. For all category $I \in \mathcal{U}$ and for all functor $E : I \to \mathcal{C}^\sim$, the canonical morphism*

$$\operatorname*{colim}_I E \to \mathbf{a}\left(\operatorname*{colim}_I i \circ E\right)$$

   is an isomorphism.

3. *The $\mathcal{U}$-limits in $\mathcal{C}^\sim$ are representable. For all object U of $\mathcal{C}$, the functor $\mathcal{C}^\sim \to \mathcal{U}$-*$\mathbf{Set}$*, $F \mapsto F(U)$ commutes with limits,* i.e. *the inclusion functor $i : \mathcal{C}^\sim \to \mathcal{C}^\sim$ commutes with limits.*

Proof. These properties follow essentially from Theorem 531 and from Corollary 67 p. 56. □

So, in the category of sheaves, the products indexed by a member of $\mathcal{U}$, the fibered products, the coproducts indexed by a member of $\mathcal{U}$, the fibered coproducts, the kernels, the cokernels, the images, and the coimages are representable.

Corollary 535 (Corollaire 4.1.1)**.** *Let $\mathcal{C} \in \mathcal{U}$ be a site and F a sheaf of $\mathcal{U}$-sets over $\mathcal{C}$. The canonical morphism*

$$\underset{(U \to F) \in \mathcal{C}_F}{\text{“colim”}} \mathbf{a}(U) \to F$$

is an isomorphism.

Proof. Results from (44) p. 81 and from the fact that $\mathbf{a}$ commutes with colimits. □

Proposition 536 (Proposition 4.2)**.** *A morphism in $\mathcal{C}^\sim$, which is both an epimorphism and a monomorphism, is an isomorphism.*

Proof. Let $f : G \to H$ be a morphism in $\mathcal{C}^\sim$ which is an epimorphism and a monomorphism. First note that the morphism f is a presheaf monomorphism (Part 1 of Theorem 534 and Proposition 165 p. 106). Form the cocartesian square

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i_2} \\ H & \xrightarrow{i_1} & K. \end{array}$$

in the category of presheaves. As f is a presheaf monomorphism, the above square is cartesian (Lemma 290 (b) p. 178). Applying the “associated sheaf” functor, we hence get a cartesian and cocartesian square in the category of sheaves (Part 1 of Theorem 534):

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{a}(i_2)} \\ H & \xrightarrow{\mathbf{a}(i_1)} & \mathbf{a}(K). \end{array}$$

As f is a sheaf epimorphism, $\mathbf{a}(i_1)$ is an isomorphism, and as the above square is cartesian, f is an isomorphism. □

18.4 Brief comments

§ 537. P. 415, Isomorphism (17.3.1). Recall briefly the setting. We have

$$F \in \mathrm{PSh}(X, \mathcal{A}), \quad M \in \mathcal{A}, \quad U \in \mathcal{C}_X,$$

and we claim

$$\mathcal{H}om_{\mathrm{PSh}(X,\mathcal{A})}(M, F)(U) \simeq \mathrm{Hom}_{\mathcal{A}}(M, F(U)). \tag{206}$$

Here and in the sequel, we denote again by M the constant presheaves over X and U attached to the object M of $\mathcal{A}$. Note that, by §513 p. 296, this isomorphism can be written

$$\mathcal{H}om_{\mathcal{A}}(M, F) \simeq \mathrm{Hom}_{\mathcal{A}}(M, F(\ )).$$

To prove (206), observe that we have

$$\mathcal{H}om_{\mathrm{PSh}(X,\mathcal{A})}(M, F)(U) \simeq \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(\mathrm{j}_{U\to X*}\, M, \mathrm{j}_{U\to X*}\, F)$$

$$\simeq \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(M, \mathrm{j}_{U\to X*}\, F),$$

the two isomorphisms following respectively from the definition of $\mathcal{H}om_{\mathrm{PSh}(X,\mathcal{A})}$ given in (17.1.14) p. 410 of the book, and from the definition of the functor $\mathrm{j}_{U\to X*}$, so that we must show

$$\mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(M, \mathrm{j}_{U\to X*}\, F) \simeq \mathrm{Hom}_{\mathcal{A}}(M, F(U)).$$

We define maps

$$\mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(M, \mathrm{j}_{U\to X*}\, F) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{A}}(M, F(U))$$

as follows: If $p : M \to \mathrm{j}_{U\to X*}\, F$ is a morphism in $\mathrm{PSh}(U, \mathcal{A})$, given by morphisms $p(V \to U) : M \to F(V)$ in $\mathcal{A}$, then we put $\varphi(p) := p(U \xrightarrow{\mathrm{id}_U} U)$; if $a : M \to F(U)$ is a morphism in $\mathcal{A}$, then we put $\psi(a)(V \xrightarrow{c} U) := F(c) \circ a$; and we check that φ and ψ are mutually inverse bijections.

§ 538. P. 418, proof of Lemma 17.4.2 (minor variant): Consider the natural morphisms

$$\operatorname{colim} \alpha \xrightarrow{f} \operatorname{colim} \alpha \circ \mu_u^{\mathrm{op}} \circ \lambda_u^{\mathrm{op}} \xrightarrow{g} \operatorname{colim} \alpha \circ \mu_u^{\mathrm{op}} \xrightarrow{h} \operatorname{colim} \alpha.$$

We must show that $g \circ f$ is an isomorphism. The equality $h \circ g \circ f = \mathrm{id}_{\operatorname{colim} \alpha}$ is easily checked. Being a right adjoint, μ_u^{op} is left exact, hence cofinal by Lemma 3.3.10 p. 84 of the book, and h is an isomorphism. q.e.d.

§ 539. P. 419, proof of Proposition 17.4.4:

First sentence of the proof: see §86 p. 64.

Step (i), Line 4: The fact that K^{op} is filtrant results from Lemma 16.2.7 p. 398 and Proposition 3.2.1 (iii) p. 78 of the book.

The key ingredient to prove that K^{op} is cofinally small is Lemma 16.2.8 p. 398 which says that K^{op} is a product of cofinally small categories. To prove that K^{op} is cofinally small one must prove that a certain product $\prod P_i$ of connected categories is connected, but, as a product of connected categories is *not* connected in general, some caution is needed. Going through the proof of Lemma 16.2.8, we see that each P_i is *filtrant*. This implies that $\prod P_i$ is filtrant, and thus that it is connected.

Step (i), additional details about the chain of isomorphisms at the bottom of p. 419 of the book: The chain reads

$$\prod_i F^b(A_i) \overset{\text{(a)}}{\simeq} \prod_i \operatorname*{colim}_{(B_i\to A_i)\in\mathcal{LI}_{A_i}} F(B_i) \overset{\text{(b)}}{\simeq} \operatorname*{colim}_{(B_i\to A_i)_{i\in I}\in K} \prod_i F(B_i)$$

$$\overset{\text{(c)}}{\simeq} \operatorname*{colim}_{(B_i\to A_i)_{i\in I}\in K} F\left(\text{“}\bigsqcup_i\text{”} B_i\right) \overset{\text{(d)}}{\simeq} \operatorname*{colim}_{(B\to A)\in\mathcal{LI}_A} F(B) \overset{\text{(e)}}{\simeq} F^b(A),$$

and the isomorphisms can be justified as follows:

(a) definition of F^b,

(b) $\mathcal{A}$ satisfies IPC,

(c) F commutes with small projective limits,

(d) an inductive limit of local isomorphisms is a local isomorphism by Proposition 16.3.4 p. 401 of the book,

(e) definition of F^b.

§ 540. P. 419, proof of Proposition 17.4.4, Step (ii). More details: The morphism $\varepsilon_b(F^b)(A) : F^b(A) \to F^{bb}(A)$ is obtained as the composition

$$F^b(A) \xrightarrow{f} \operatorname*{colim}_{(B\to A)\in\mathcal{LI}_A} F^b(A) \xrightarrow{g} \operatorname*{colim}_{(B\to A)\in\mathcal{LI}_A} F^b(B).$$

Moreover, f is an isomorphism by Lemma 2.1.12 p. 41 of the book, and g is an isomorphism by Lemma 17.4.2 p. 418 of the book.

18.5 Proposition 17.4.4 p. 420

We draw a few diagrams with the hope of helping the reader visualize the argument in Step (ii) of the proof of Proposition 17.4.4.

An object of the category

$$M\big[J \to K \leftarrow M[I \to K \leftarrow K]\big]$$

can be represented by a diagram

$$\begin{array}{ccccccccc}
B' & & B' \times_B A & \xleftarrow{(u',\alpha)} & A' & \xrightarrow{(v',\alpha)} & C' \times_C A & & C' \\
\beta\downarrow & & \downarrow & & \downarrow\alpha & & \downarrow & & \downarrow\gamma \\
B & & A & = & A & = & A & & C,
\end{array}$$

and it is clear that this category is equivalent to $\mathcal{E}^{\mathrm{op}}$.

Recall that $D := B \text{“}\sqcup\text{”}_A C$, let E be one of the objects A, B, C, or D, and consider the "obvious" functors

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{p_E} & \mathcal{LI}_E \\
 & {\scriptstyle q_E}\searrow & \downarrow j_E \\
 & & \mathcal{C}^\wedge_X
\end{array}$$

(p_E is defined in the book, j_E is the forgetful functor, and q_E is the composition). We also define $r_E : \mathcal{LI}_E \to \mathcal{E}$ by mapping the object $E'' \to E$ of $\mathcal{LI}_E$ to the object

$$\begin{array}{ccccc}
B \times_E E'' & \longleftarrow & A \times_E E'' & \longrightarrow & C \times_E E'' \\
\downarrow & & \downarrow & & \downarrow \\
B & \longleftarrow & A & \longrightarrow & C
\end{array}$$

of $\mathcal{E}$. One checks that (p_E, r_E) is a pair of adjoint functors. In particular p_E is cocofinal. We have

$$F^b(D) \overset{\text{(a)}}{\simeq} \operatorname*{colim}_{y \in \mathcal{LI}_D} F(j_D(y)) \overset{\text{(b)}}{\simeq} \operatorname*{colim}_{x \in \mathcal{E}} F(q_D(x))$$

$$\overset{\text{(c)}}{\simeq} \operatorname*{colim}_{x \in \mathcal{E}} F\left(q_B(x) \text{“}\bigsqcup_{q_A(x)}\text{”} q_C(x)\right) \overset{\text{(d)}}{\simeq} \operatorname*{colim}_{x \in \mathcal{E}} (F(q_B(x)) \times_{F(q_A(x))} F(q_C(x)))$$

$$\overset{\text{(e)}}{\simeq} \left(\operatorname*{colim}_{x\in\mathcal{E}} F(q_B(x))\right) \times_{\operatorname{colim}_{x\in\mathcal{E}} F(q_A(x))} \left(\operatorname*{colim}_{x\in\mathcal{E}} F(q_C(x))\right)$$

$$\overset{\text{(f)}}{\simeq} \left(\operatorname*{colim}_{y\in\mathcal{LI}_B} F(j_B(y))\right) \times_{\operatorname{colim}_{y\in\mathcal{LI}_A} F(j_A(y))} \left(\operatorname*{colim}_{y\in\mathcal{LI}_C} F(j_C(y))\right)$$

$$\overset{\text{(g)}}{\simeq} F^b(B) \times_{F^b(A)} F^b(C).$$

Indeed, the isomorphisms can be justified as follows:

(a) definition of F^b,

(b) cocofinality of p_D,

(c) definition of p_E,

(d) left exactness of F,

(e) exactness of filtrant inductive limits in $\mathcal{A}$,

(f) cocofinality of p_D,

(g) definition of F^b.

18.6 Brief comments

§ 541. P. 421, first display:

$$F^a(U) \simeq \operatorname*{colim}_{(U\to A)\in\mathcal{LI}_U} F(A).$$

Lemma 16.2.8 p. 398 of the book and its proof, show that F^a does *not* depend on the universe such that $\mathcal{C}$ is a small category (Definition 5 p. 10) and $\mathcal{A}$ satisfies (17.4.1) p. 417 of the book.

§ 542. P. 421, proof of Lemma 17.4.6 (i): The category $\mathcal{LI}_U$ is cofiltrant by Lemma 16.2.7 p. 398 of the book, small filtrant inductive limits are exact in $\mathcal{A}$ by Display (17.4.1) p. 417 of the book, exact functors preserve monomorphisms by Proposition 165 p. 106.

§ 543. P. 422. The first sentence of the proof of Theorem 17.4.7 (iv) follows from Corollary 162 p. 105. One could add:

If $\mathcal{A}$ is abelian, then $\mathrm{PSh}(X,\mathcal{A})$ and $\mathrm{Sh}(X,\mathcal{A})$ are abelian, and $\iota : \mathrm{Sh}(X,\mathcal{A}) \to \mathrm{PSh}(X,\mathcal{A})$ and $(\ )^a : \mathrm{PSh}(X,\mathcal{A}) \to \mathrm{Sh}(X,\mathcal{A})$ are additive

§ 544. P. 423, end of the proof of Theorem 17.4.9 (iv): the functor $(\ )^a$ is exact by Theorem 17.4.7 (iv) p. 421 of the book.

§ 545. P. 424, proof of Theorem 17.5.2 (i). With the convention that a diagram of the form

$$\begin{array}{c} \mathcal{C}_1 \\ {\scriptstyle L}\big\downarrow\big\uparrow{\scriptstyle R} \\ \mathcal{C}_2 \end{array}$$

means: "(L, R) is a pair of adjoint functors", the proof of Theorem 17.5.2 (i) in the book can be visualized by the diagram

$$\begin{array}{c} \mathrm{PSh}(Y, \mathcal{A}) \\ {\scriptstyle f^\dagger}\big\downarrow\big\uparrow{\scriptstyle f_*} \\ \mathrm{PSh}(X, \mathcal{A}) \\ {\scriptstyle (\ )^a}\big\downarrow\big\uparrow{\scriptstyle \iota} \\ \mathrm{Sh}(X, \mathcal{A}). \end{array}$$

§ 546. P. 424, proof of Theorem 17.5.2 (iv). As already mentioned, there is a typo: "The functor $f^\dagger$ is left exact" should be "The functor $f^\dagger$ is exact".

§ 547. P. 424, Definition 17.6.1. By Lemma 17.1.8 p. 409 of the book, a morphism

$$\begin{array}{ccc} C & \longrightarrow & B \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

in $\mathcal{C}_A^\wedge$ is a local epimorphism if and only if $C \to B$ is a local epimorphism in $\mathcal{C}_X^\wedge$.

§ 548. P. 424, sentence following Definition 17.6.1: "It is easily checked that we obtain a Grothendieck topology on $\mathcal{C}_A$". The verification of LE1, LE2 and LE3 is straightforward. Axiom LE4 follows from Parts (iii) and (ii) of Lemma 17.2.5 p. 413 of the book.

§ 549. P. 424, Definition 17.6.1. Here is an observation which follows from §521 p. 298 and Lemma 17.2.5 (iii) p. 413 of the book:

In the setting of Definition 17.6.1, let $B \to A$ be a morphism in $\mathcal{C}_X^\wedge$, let $u : C \to B$ be a morphism in $\mathcal{C}_X^\wedge$, and let $v : (C \to A) \to (B \to A)$ be the corresponding morphism in $\mathcal{C}_A^\wedge$. Then u is a local epimorphism if and only if v is a local epimorphism, u is a local monomorphism if and only if v is a local monomorphism, and u is a local isomorphism if and only if v is a local isomorphism.

§ 550. P. 425, proof of Proposition 17.6.3:

Step (i): $\mathrm{j}_{A\to X}$ is weakly left exact by Lemma 17.2.5 (iii) p. 413 of the book, and $(\,\cdot\,)^a$ is exact by Theorem 17.4.7 (iv) p. 421 of the book.

Step (ii): "f factors as $X \xrightarrow{\mathrm{j}_{A\to X}} A \xrightarrow{g} Y$": see Definition 17.2.4 (ii) p. 413 of the book and Remark 118 p. 90. The isomorphism $f^{-1} \simeq \mathrm{j}_{A\to X}^{-1} \circ g^{-1}$ follows from Proposition 17.5.3 p. 424 of the book.

§ 551. P. 425, Display (17.6.1): Putting $j := \mathrm{j}_{A\to X}$, we have the adjunctions

$$\begin{array}{c} \mathrm{Sh}(A,\mathcal{A}) \\ {\scriptstyle j^{-1}}\big\downarrow \quad {\scriptstyle j_*}\big\uparrow \quad \big\downarrow{\scriptstyle j^\ddagger} \\ \mathrm{Sh}(X,\mathcal{A}). \end{array}$$

For the functor $j_* : \mathrm{Sh}(X,\mathcal{A}) \to \mathrm{Sh}(A,\mathcal{A})$, see Proposition 17.5.1 p. 423 of the book.

For the functor $j^{-1} : \mathrm{Sh}(A,\mathcal{A}) \to \mathrm{Sh}(X,\mathcal{A})$, see last display of p. 423 of the book.

For the functor $j^\ddagger : \mathrm{Sh}(A,\mathcal{A}) \to \mathrm{Sh}(X,\mathcal{A})$, see Proposition 17.6.2 p. 425 of the book.

§ 552. P. 426, proof of Proposition 17.6.7 (i). The isomorphism

$$(f^t)^\wedge(V \times B) \simeq f^t(V) \times (f^t)^\wedge(B) \tag{207}$$

follows from Proposition 511 p. 296, and we have

$$\begin{aligned} \mathrm{j}_{B\to Y}^\ddagger\big(f_{B*}(G)(V)\big) &\simeq f_{B*}(G)(V \times B \to B) && \text{by (17.1.12) p. 409} \\ &\simeq G\big((f^t)^\wedge(V \times B) \to (f^t)^\wedge(B)\big) && \text{by (17.1.6) p. 408} \\ &\simeq G\big(f^t(V) \times A \to A\big) && \text{by (207),} \end{aligned}$$

as well as

$$f_*\big(\mathrm{j}^{\ddagger}_{A\to X}(G)(V)\big) \simeq \mathrm{j}^{\ddagger}_{A\to X}(G)(f^t(V))$$

$$\simeq G\big(f^t(V)\times A \to A\big) \qquad \text{by (17.1.12) p. 409.}$$

§ 553. P. 427, proof of Proposition 17.6.8, Step (i). The isomorphism

$$\mathrm{j}^{\ddagger}_{A\to X}(\mathrm{j}_{A\to X*}(G)(U)) \simeq \mathrm{j}_{A\to X*}(G)(U\times A\to A)$$

follows from (17.1.12) p. 409 of the book. The fact that $p : A\times U \to U$ is a local isomorphism follows from the fact that the obvious square

$$\begin{array}{ccc} A\times U & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{pt}_X \end{array}$$

is cartesian and the bottom arrow is a local isomorphism by assumption.

§ 554. P. 427, proof of Proposition 17.6.8, Step (ii). Let $v : V \to A$ be a morphism in $\mathcal{C}_X^{\wedge}$. Here is a proof of the fact that

$$\begin{array}{ccc} V & \xrightarrow{(\mathrm{id}_V,v)} & V\times A \\ & \searrow \quad \swarrow & \\ & A & \end{array} \tag{208}$$

is a local isomorphism in $\mathcal{C}_A^{\wedge}$.

As $V\times A\to V$ is a local isomorphism in $\mathcal{C}_X^{\wedge}$ by §553, and $V\to V\times A\to V$ is the identity of V, Lemma 16.2.4 (vii) p. 396 of the book implies that $V\to V\times A$ is a local isomorphism in $\mathcal{C}_X^{\wedge}$, and thus, by §549 p. 314, that (208) is a local isomorphism in $\mathcal{C}_A^{\wedge}$.

§ 555. P. 428, just after Definition 17.6.10: $((\ )_A, \Gamma_A(\ ))$ is a pair of adjoint functors: this follows from Theorem 17.5.2 (i) p. 424 of the book.

§ 556. P. 429, top. By §88 p. 65 and Corollary 163 p. 106, the functor $\Gamma(A;\ )$ commutes with small projective limits.

§ 557. P. 430, first sentence of the proof of Proposition 17.7.1 (i). Let us make a general observation.

Let X be a site. In this §, for any A in $\mathcal{C}_X^\wedge$, we denote the corresponding site by A' instead of A. We also identify $\mathcal{C}_{A'}^\wedge$ to $(\mathcal{C}_X^\wedge)_A$ (see Lemma 17.1.8 p. 409 of the book). In particular, we get $\mathrm{pt}_{A'} \simeq (A \xrightarrow{\mathrm{id}_A} A) \in \mathcal{C}_{A'}^\wedge$.

Let $A \to B$ be a local isomorphism in $\mathcal{C}_X^\wedge$, and let us write ω for "the" terminal object $\mathrm{pt}_{B'} \simeq (B \xrightarrow{\mathrm{id}_B} B)$ of $\mathcal{C}_{B'}^\wedge$. We claim that

$$(A \to B) \to \omega \tag{209}$$

is a local isomorphism in $\mathcal{C}_{B'}^\wedge$.

Proof: (209) is a local epimorphism by §547 p. 314. It remains to check that

$$(A \to B) \to (A \to B) \times_\omega (A \to B) \simeq (A \times_B A \to B) \tag{210}$$

is a local epimorphism. But this follows again from §547 p. 314. □

Consider the morphism of presites $B' \to A'$ induced by $A \to B$ and note that the square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{j}_{A\to X}} & A' \\ \| & & \uparrow \\ X & \xrightarrow[\mathrm{j}_{B\to X}]{} & B'. \end{array}$$

commutes.

§ 558. P. 430, proof of Proposition 17.7.3. The third isomorphism follows, as indicated, from Proposition 17.6.7 (ii) p. 426 of the book. The fifth isomorphism follows from (17.6.2) (ii) p. 426 of the book.

§ 559. P. 431, Exercise 17.5 (i). Put $PX := \mathrm{PSh}(X, \mathcal{A})$, $SX := \mathrm{Sh}(X, \mathcal{A})$, and define PY and SY similarly. Let

$$\begin{array}{c} SY \\ {\scriptstyle a_Y}\uparrow\downarrow{\scriptstyle \iota_Y} \\ PY \\ {\scriptstyle f^\dagger}\downarrow\uparrow{\scriptstyle f_*} \\ PX \\ {\scriptstyle a_X}\downarrow\uparrow{\scriptstyle \iota_X} \\ SX \end{array}$$

be the obvious diagram of adjoint functors. We must show

$$a_X \circ f^\dagger \circ \iota_Y \circ a_Y \simeq a_X \circ f^\dagger.$$

Let F be in SX and G be in PY. We have (omitting most of the parenthesis)

$$\mathrm{Hom}_{SX}(a_X f^\dagger \iota_Y a_Y G, F) \simeq \mathrm{Hom}_{PX}(f^\dagger \iota_Y a_Y G, \iota_X F) \simeq \mathrm{Hom}_{PY}(\iota_Y a_Y G, f_* \iota_X F)$$

$$\overset{\text{(a)}}{\simeq} \mathrm{Hom}_{PY}(\iota_Y a_Y G, \iota_Y a_Y f_* \iota_X F) \simeq \mathrm{Hom}_{SY}(a_Y \iota_Y a_Y G, a_Y f_* \iota_X F)$$

$$\overset{\text{(b)}}{\simeq} \mathrm{Hom}_{SY}(a_Y G, a_Y f_* \iota_X F) \simeq \mathrm{Hom}_{PY}(G, \iota_Y a_Y f_* \iota_X F) \overset{\text{(c)}}{\simeq} \mathrm{Hom}_{PY}(G, f_* \iota_X F)$$

$$\simeq \mathrm{Hom}_{PX}(f^\dagger G, \iota_X F) \simeq \mathrm{Hom}_{SX}(a_X f^\dagger G, F)$$

where (a) and (c) follow from the fact that the presheaf $f_* \iota_X F$ is actually a sheaf (Proposition 17.5.1 p. 423 of the book), (b) follows from the isomorphism

$$a_Y \circ \iota_Y \circ a_Y \simeq a_Y,$$

which holds by Lemma 17.4.6 (ii) p. 421 of the book, and the other isomorphisms hold by adjunction.

§ 560. P. 431, Exercise 17.5 (ii). By §549 p. 314 we have, for U in $\mathcal{C}_X$ and $U \to A$ in $\mathcal{C}_A$, an isomorphism

$$\mathcal{L}\mathcal{I}_{U \to A} \simeq \mathcal{L}\mathcal{I}_U.$$

Exercise 17.5 (ii) follows immediately.

19 About Chapter 18

19.1 Brief comments

§ 561. P. 437, Theorem 18.1.6 (v). If X is a site, if R a ring, if F and G are complexes of R-modules, then the complex of abelian groups $\mathrm{RHom}_R(F, G)$ (see Corollary 14.3.2 p. 356 of the book) does *not* depend on the universe chosen to define it (the universe in question being subject to the obvious conditions). This follows from §472 p. 280 and §541 p. 313.

§ 562. P. 436, Lemma 18.1.4. Note that $\mathrm{PSh}(\mathcal{R})$ is abelian, that $\mathrm{Mod}(\mathcal{R})$ is an additive subcategory of $\mathrm{PSh}(\mathcal{R})$, and that the functors

$$\mathrm{Mod}(\mathcal{R}) \underset{(\cdot)^a}{\overset{}{\rightleftarrows}} \mathrm{PSh}(\mathcal{R})$$

are additive.

§ 563. P. 437, proof of Theorem 18.1.6 (v). We prove

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}_U, F) \simeq F(U).$$

As

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}_U, F) \simeq \mathrm{Hom}_{\mathcal{R}}(\mathrm{j}^{-1}_{U\to X*}(\mathcal{R}|U), F) \simeq \mathrm{Hom}_{\mathcal{R}|U}(\mathcal{R}|U, F|U),$$

we only need to verify

$$\mathrm{Hom}_{\mathcal{R}|U}(\mathcal{R}|U, F|U) \simeq F(U).$$

We shall define maps

$$\mathrm{Hom}_{\mathcal{R}|U}(\mathcal{R}|U, F|U) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F(U)$$

and leave it to the reader to check that they are mutually inverse.

Definition of φ: Let θ be in $\mathrm{Hom}_{\mathcal{R}|U}(\mathcal{R}|U, F|U)$. In particular, for each morphism $f : V \to U$ in $\mathcal{C}_X$ we have a map $\theta(f) : \mathcal{R}(V) \to F(V)$, and we put $\varphi(\theta) := \theta(\mathrm{id}_U)(1)$.

Definition of ψ: Let x be in $F(U)$. For each morphism $f : V \to U$ in $\mathcal{C}_X$ we define $\psi(x)(f) : \mathcal{R}(V) \to F(V)$ by $\psi(x)(f)(\lambda) := \lambda\, F(f)(x)$.

§ 564. P. 438, end of Section 18.1: Γ_A is left exact by §555 p. 316. Moreover, $\Gamma(A;\ )$ commutes with small projective limits by §556 p. 316, and is thus left exact by Proposition 161 p. 105.

§ 565. P. 438, bottom: One can add that we have $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, F) \simeq F$ for all F in $\mathrm{PSh}(\mathcal{R})$.

§ 566. P. 439, after Definition 18.2.2: One can add that we have

$$\mathcal{R} \overset{\mathrm{psh}}{\otimes}_{\mathcal{R}} F \simeq F$$

for F in $\mathrm{PSh}(\mathcal{R})$ and

$$\mathcal{R} \otimes_{\mathcal{R}} F \simeq F$$

for F in $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$, as well as

$$F \overset{\mathrm{psh}}{\otimes}_{\mathcal{R}} \mathcal{R} \simeq F$$

for F in $\mathrm{PSh}(\mathcal{R}^{\mathrm{op}})$ and

$$F \otimes_{\mathcal{R}} \mathcal{R} \simeq F$$

for F in $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$.

§ 567. P. 441. The proof of Proposition 18.2.5 uses Display (17.1.11) p. 409 of the book and §559 p. 317.

§ 568. P. 441. In the notation of Remark 18.2.6 we have

$$\mathrm{Hom}_{\mathcal{R}_3}({}_3M_2 \otimes_{\mathcal{R}_2} {}_2M_1, {}_3M_4) \simeq \mathrm{Hom}_{\mathcal{R}_2}({}_2M_1, \mathcal{H}om_{\mathcal{R}_3}({}_3M_2, {}_3M_4)),$$

$$\mathcal{H}om_{\mathcal{R}_3}({}_3M_2 \otimes_{\mathcal{R}_2} {}_2M_1, {}_3M_4) \simeq \mathcal{H}om_{\mathcal{R}_2}({}_2M_1, \mathcal{H}om_{\mathcal{R}_3}({}_3M_2, {}_3M_4)),$$

$$\mathrm{Hom}_{\mathcal{R}_3^{\mathrm{op}}}({}_1M_2 \otimes_{\mathcal{R}_2} {}_2M_3, {}_4M_3) \simeq \mathrm{Hom}_{\mathcal{R}_2^{\mathrm{op}}}({}_1M_2, \mathcal{H}om_{\mathcal{R}_3^{\mathrm{op}}}({}_2M_3, {}_4M_3)),$$

$$\mathcal{H}om_{\mathcal{R}_3^{\mathrm{op}}}({}_1M_2 \otimes_{\mathcal{R}_2} {}_2M_3, {}_4M_3) \simeq \mathcal{H}om_{\mathcal{R}_2^{\mathrm{op}}}({}_1M_2, \mathcal{H}om_{\mathcal{R}_3^{\mathrm{op}}}({}_2M_3, {}_4M_3)).$$

More generally, if $\mathcal{R}, \mathcal{S}, \mathcal{T}$ are $\mathcal{O}_X$-algebras, if F is a $(\mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{R}^{\mathrm{op}})$-module, if G is an $(\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{S})$-module, and if H is an $(\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T})$-module, then we have

$$\mathrm{Hom}_{\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T}}(F \otimes_{\mathcal{R}} G, H) \simeq \mathrm{Hom}_{\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{S}}(G, \mathcal{H}om_{\mathcal{T}}(F, H)),$$

$$\mathcal{H}om_{\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T}}(F \otimes_{\mathcal{R}} G, H) \simeq \mathcal{H}om_{\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{S}}(G, \mathcal{H}om_{\mathcal{T}}(F, H)). \tag{211}$$

§ 569. P. 442, proof of Proposition 18.2.7. Here are additional details.

Proof of (18.2.12): We must show

$$F_A \simeq \mathcal{R}_A \otimes_{\mathcal{R}} F \simeq k_{XA} \otimes_{k_X} F. \tag{212}$$

We have

$$F_A \overset{\text{(a)}}{\simeq} \mathrm{j}_{A \to X}^{-1}(F|_A) \overset{\text{(b)}}{\simeq} \mathrm{j}_{A \to X}^{-1}(\mathcal{R}|_A \otimes_{\mathcal{R}|_A} F|_A) \overset{\text{(c)}}{\simeq} (\mathrm{j}_{A \to X}^{-1}(\mathcal{R}|_A)) \otimes_{\mathcal{R}} F \overset{\text{(d)}}{\simeq} \mathcal{R}_A \otimes_{\mathcal{R}} F.$$

Indeed, (a) and (d) hold by Definition 17.6.10 (i) and Display (17.6.5) p. 428 of the book, (b) follows from §566, (c) follows from (18.2.6) p. 441 of the book. The isomorphism $F_A \simeq k_{XA} \otimes_{k_X} F$ is a particular case of the isomorphism $F_A \simeq \mathcal{R}_A \otimes_{\mathcal{R}} F$ just proved.

Proof of (18.2.13): We must show

$$\Gamma_A(F) \simeq \mathcal{H}om_{\mathcal{R}}(\mathcal{R}_A, F) \simeq \mathcal{H}om_{k_X}(k_{XA}, F). \tag{213}$$

We have

$$\mathcal{H}om_{\mathcal{R}}(\mathcal{R}_A, F) \overset{\text{(a)}}{\simeq} \mathcal{H}om_{\mathcal{R}}(\mathcal{R} \otimes_{k_X} k_{XA}, F)$$
$$\overset{\text{(b)}}{\simeq} \mathcal{H}om_{k_X}(k_{XA}, \mathcal{H}om_{\mathcal{R}}(\mathcal{R}, F)) \overset{\text{(c)}}{\simeq} \mathcal{H}om_{k_X}(k_{XA}, F),$$

where (a) follows from (212), (b) follows from Display (18.2.4) p. 439 of the book (which is a particular case of (211)), and (c) follows from §565. Let us record the isomorphism

$$\mathcal{H}om_{\mathcal{R}}(\mathcal{R}_A, F) \simeq \mathcal{H}om_{k_X}(k_{XA}, F). \tag{214}$$

We also have for G in $\operatorname{Mod}(\mathcal{R})$

$$\operatorname{Hom}_{\mathcal{R}}(G, \mathcal{H}om_{k_X}(k_{XA}, F)) \overset{\text{(a)}}{\simeq} \operatorname{Hom}_{\mathcal{R}}(G \otimes_{k_X} k_{XA}, F) \overset{\text{(b)}}{\simeq} \operatorname{Hom}_{\mathcal{R}}(\mathrm{j}_{A\to X}^{-1}\, \mathrm{j}_{A\to X*}\, G, F)$$
$$\overset{\text{(c)}}{\simeq} \operatorname{Hom}_{\mathcal{R}}(G, \mathrm{j}_{A\to X}^{\ddagger}\, \mathrm{j}_{A\to X*}\, F) \overset{\text{(d)}}{\simeq} \operatorname{Hom}_{\mathcal{R}}(G, \Gamma_A(F)),$$

where (a) follows from (211) with

$$(k_X; k_X, \mathcal{R}, k_X; k_{XA}, G, F)$$

instead of

$$(\mathcal{O}_X; \mathcal{R}, \mathcal{S}, \mathcal{T}; F, G, H),$$

(b) follows from (212), Definition 17.6.10 (i) and Display (17.6.5) p. 428 of the book, (c) follows by adjunction, and (d) by Definition 17.6.10 (ii) p. 428 of the book.

19.2 Lemma 18.5.3 p. 447

We give additional details about the proof of Lemma 18.5.3 of the book (stated below as Lemma 573 p. 324) with the hope of helping the reader. We start with a technical lemma.

Lemma 570. *Let R be a ring, let A be a right R-module, let B be a left R-module, let n be a positive integer, and let*

$$(a_i)_{i=1}^n, \quad (b_i)_{i=1}^n$$

be two families of elements belonging respectively to A and B. Then Conditions (i) *and* (ii) *below are equivalent:*

(i) *We have $\sum_{i=1}^n a_i \otimes b_i = 0$ in $A \otimes_R B$.*

(ii) *There are positive integers ℓ and m with $\ell \geq n$, and there are three families*

$$(a_i)_{i=n+1}^{\ell}, \quad (\lambda_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}, \quad (b'_j)_{j=1}^{m}$$

of elements belonging respectively to A, R and B, such that, if we set $b_i = 0$ for $n < i \leq \ell$, we have:

$$\sum_{j=1}^{m} \lambda_{ij}\, b'_j = b_i \quad (\forall\ 1 \leq i \leq \ell), \tag{215}$$

$$\sum_{i=1}^{\ell} a_i\, \lambda_{ij} = 0 \quad (\forall\ 1 \leq j \leq m). \tag{216}$$

Proof. Implication (ii)⇒(i) is clear. To prove Implication (i)⇒(ii), we assume (i), and we choose a set I containing $\{1, \dots, \ell\}$, where ℓ is an integer $\geq n$ to be determined later, such that there is a family $(a_i)_{i\in I}$ which completes the family $(a_i)_{1\leq i\leq n}$ and generates A. We write C for the kernel of the epimorphism

$$f : R^{\oplus I} \twoheadrightarrow A, \quad (\mu_i) \mapsto \sum_{i\in I} a_i\, \mu_i.$$

In particular we have exact sequences

$$C \xrightarrow{g} R^{\oplus I} \xrightarrow{f} A \to 0, \qquad C \otimes_R B \xrightarrow{g'} B^{\oplus I} \xrightarrow{f'} A \otimes_R B \to 0,$$

with

$$g'\big((\mu_i) \otimes b\big) = (\mu_i\, b), \quad f'((b''_i)) = \sum_{i\in I} a_i \otimes b''_i.$$

Put $b_i := 0$ for i in $I \setminus \{1, \dots, \ell\}$. The family $(b_i)_{i\in I}$ is in $\operatorname{Ker} f'$, and thus in $\operatorname{Im} g'$. The condition $(b_i) \in \operatorname{Im} g'$ means that there is a positive integer m, a family

$$(\lambda_{ij})_{i\in I, 1\leq j\leq m}$$

of elements of R such that

$$(\lambda_{ij})_i \in C \subset R^{\oplus I}$$

for $1 \leq j \leq m$, and a family $(b'_j)_{1\leq j\leq m}$ of elements of B, such that

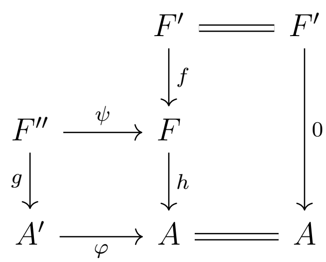
$$(b_i)_i = g'\left(\sum_{j=1}^{m} (\lambda_{ij})_i \otimes b'_j\right) = \left(\sum_{j=1}^{m} \lambda_{ij}\, b'_j\right)_i .$$

As $(\lambda_{ij})_i$ is in $R^{\oplus I}$ for all j, the set of those i in I for which there is a j such that $\lambda_{ij} \neq 0$ is finite, and we can arrange the notation so that this set is contained in $\{1, \dots, \ell\}$ with $\ell \geq n$, and we get (215). As $(\lambda_{ij})_i$ is in C for all j, we also have (216). $\square$

Here is another technical lemma:

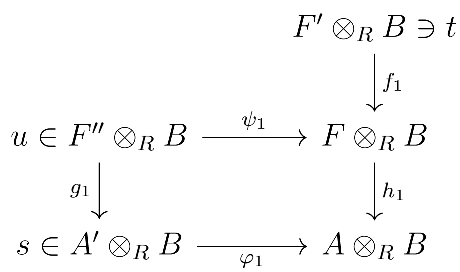
Lemma 571. *Let R be a ring, let $\varphi : A' \to A$ be a morphism of right R-modules, let B be a left R-module, and let s be an element of $\operatorname{Ker}(A' \otimes_R B \to A \otimes_R B)$. Then there exist*

- *a commutative diagram*



of right R-modules such that F, F' and F'' are free of finite rank,

- *elements $t \in F' \otimes_R B$, $u \in F'' \otimes_R B$ such that the commutative diagram*



satisfies $g_1(u) = s$ and $\psi_1(u) = f_1(t)$.

Proof. Write
$$s = \sum_{i=1}^{n} a'_i \otimes b_i$$
with a'_i in A' and b_i in B, and put $a_i := \varphi(a'_i) \in A$, so that we have
$$\sum_{i=1}^{n} a_i \otimes b_i = 0.$$

By Lemma 570 p. 321 there are positive integers ℓ, m with $\ell \geq n$, and there are three families

$$(a_i)_{i=n+1}^{\ell}, \quad (\lambda_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}, \quad (b'_j)_{j=1}^{m}$$

of elements belonging respectively to A, R and B, such that, if we set $b_i = 0$ for $n < i \leq \ell$, we get (215) and (216) p. 322. We have a commutative diagram of right R-modules

$$\begin{array}{ccc} & & R^m \\ & & \downarrow{\scriptstyle f} \\ R^n & \xrightarrow{\psi} & R^\ell \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[\varphi]{} & A \end{array} \tag{217}$$

with

$$f(x)_i = \sum_{j=1}^{m} \lambda_{ij}\, x_j, \quad g(x) = \sum_{i=1}^{n} a'_i\, x_i, \quad h(x) = \sum_{i=1}^{\ell} a_i\, x_i.$$

In particular, (216) p. 322 implies $h \circ f = 0$. □

Lemma 572. *If F and F' are two $\mathcal{R}$-modules of finite rank, then the natural map*

$$\mathrm{Hom}_{\mathcal{R}}(F, F') \to \mathrm{Hom}_{\Gamma(X,\mathcal{R})}(\Gamma(X,F), \Gamma(X,F'))$$

is bijective. (Recall that $\Gamma(X, F)$ is defined just before Proposition 17.6.14 p. 429 of the book.)

Proof. It suffices to prove the statement when $F = F' = \mathcal{R}$, which is easy. □

Let us turn to the proof of Lemma 18.5.3 p. 447. [As already pointed out, there are two typos in the proof: in (18.5.3) $M'|_U$ and $M|_U$ should be $M'(U)$ and $M(U)$, and, after the second display on p. 448, $s_1 \in ((\mathcal{R}^{\mathrm{op}})^{\oplus m} \otimes_{\mathcal{R}} P)(U)$ should be $s_1 \in ((\mathcal{R}^{\mathrm{op}})^{\oplus n} \otimes_{\mathcal{R}} P)(U)$.]

For the reader's convenience we state (in a slightly different form) Lemma 18.5.3 (see Notation 17.6.13 p. 428 of the book):

Lemma 573 (Lemma 18.5.3 p. 447)**.** *Let P be an $\mathcal{R}$-module. Assume that for all U in $\mathcal{C}_X$, all free right $\mathcal{R}$-module F', F'' of finite rank, and all $\mathcal{R}|_U$-linear morphism $u : F'|_U \to F''|_U$, the sequence*

$$0 \to \mathrm{Ker}(u) \otimes_{\mathcal{R}|_U} P|_U \to F'|_U \otimes_{\mathcal{R}|_U} P|_U \to F''|_U \otimes_{\mathcal{R}|_U} P|_U$$

is exact. Then P is a flat $\mathcal{R}$-module.

(Recall that the notation $?|_U$ is defined in Notation 17.6.13 (ii) p. 428 of the book.)

Proof. Consider a monomorphism $M' \rightarrowtail M$ of right $\mathcal{R}$-modules. It suffices to prove that the sheaf

$$K := \operatorname{Ker}(M' \otimes_{\mathcal{R}} P \to M \otimes_{\mathcal{R}} P)$$

of k_X-modules over X vanishes. Let K_0 be the presheaf of k_X-modules over X defined by

$$K_0(U) := \operatorname{Ker}\Big(M'(U) \otimes_{\mathcal{R}(U)} P(U) \to M(U) \otimes_{\mathcal{R}(U)} P(U)\Big),$$

let U be an object of $\mathcal{C}_X$, let s be an element of $K_0(U)$, and let $\overline{s}$ be the image of s in $K(U)$. We shall prove $\overline{s} = 0$. By §562 p. 319 above, Definition 18.2.2 p. 439 and Theorem 17.4.7 (iv) p. 421 of the book, K is the sheaf associated to K_0. Hence, as U and s are arbitrary, Equality $\overline{s} = 0$ will imply that the natural morphism $K_0 \to K$ vanishes. By (17.4.12) p. 421 of the book, this vanishing will entail $K \simeq 0$, and thus, the lemma. Let us record this observation:

$$\text{Equality } \overline{s} = 0 \text{ implies the lemma.} \tag{218}$$

By Lemma 571 p. 323 there exist

- a commutative diagram

$$\begin{array}{ccccc}
 & & F'(U) & = & F'(U) \\
 & & \downarrow f & & \downarrow 0 \\
F''(U) & \xrightarrow{\psi} & F(U) & & \\
g \downarrow & & \downarrow h & & \\
M'(U) & \xrightarrow[\varphi]{} & M(U) & = & M(U)
\end{array}$$

of right $\mathcal{R}(U)$-modules such that F, F' and F'' are free right $\mathcal{R}$-modules of finite rank,

- elements $t \in F'(U) \otimes_{\mathcal{R}(U)} P(U)$, $u \in F''(U) \otimes_{\mathcal{R}(U)} P(U)$ such that the commutative diagram

$$\begin{CD} @. F'(U)\otimes_{\mathcal{R}(U)}P(U)\ni t\\ @. @VV{f_1}V\\ u\in F''(U)\otimes_{\mathcal{R}(U)}P(U) @>{\psi_1}>> F(U)\otimes_{\mathcal{R}(U)}P(U)\\ @V{g_1}VV @VV{h_1}V\\ s\in M'(U)\otimes_{\mathcal{R}(U)}P(U) @>>{\varphi_1}> M(U)\otimes_{\mathcal{R}(U)}P(U) \end{CD} \tag{219}$$

satisfies $g_1(u)=s$ and $\psi_1(u)=f_1(t)$.

By Lemma 572 p. 324, the commutative diagram (217) also induces the commutative diagram

$$\Delta := \left\{ \begin{CD} N @>>> F'|_U\\ @V{k_2}VV @VV{f_2}V\\ F''|_U @>{\psi_2}>> F|_U\\ @V{g_2}VV @VV{h_2}V\\ M'|_U @>>{\varphi_2}> M|_U, \end{CD} \right.$$

the top square being cartesian. Then φ_2 is a monomorphism by Proposition 17.6.6 p. 425 and Notation 17.6.13 p. 428 of the book (recall that $M'\to M$ is a monomorphism by assumption). This implies $g_2\circ k_2=0$. Hence Δ is a commutative diagram of *complexes*. The condition that the top square is cartesian is equivalent to the exactness of

$$\Sigma := \Big(0\to N\to F'|_U\oplus F''|_U\to F|_U\Big).$$

The sequence $\Sigma\otimes_{\mathcal{R}|_U}P|_U$ being exact thanks to the assumption in Lemma 573 p. 324, we see that the commutative diagram of complexes $\Delta\otimes_{\mathcal{R}|_U}P|_U$ has a cartesian top square, and that, by left exactness of $\Gamma(U;-)$ (see §564 p. 319), the commutative diagram of complexes $\Gamma(U;\Delta\otimes_{\mathcal{R}|_U}P|_U)$, that is (see Notation 17.6.13 p. 428 of the book),

$$\begin{CD} (N\otimes_{\mathcal{R}|_U}P|_U)(U) @>>> (F'\otimes_{\mathcal{R}}P)(U)\ni t\\ @V{k_1}VV @VV{f_1}V\\ u\in(F''\otimes_{\mathcal{R}}P)(U) @>{\psi_1}>> (F\otimes_{\mathcal{R}}P)(U)\\ @V{g_3}VV @VV{h_3}V\\ \overline{s}\in(M'\otimes_{\mathcal{R}}P)(U) @>>{\varphi_1}> (M\otimes_{\mathcal{R}}P)(U) \end{CD}$$

(see (219)) has also a cartesian top square, and satisfies $g_3(u) = \overline{s}$ and

$$\psi_1(u) = f_1(t). \tag{220}$$

We have used the isomorphisms

$$\Gamma\left(U; M|_U \otimes_{\mathcal{R}|_U} P|_U\right) \simeq \Gamma\big(U; (M \otimes_{\mathcal{R}} P)\,|_U\big) \simeq (M \otimes_{\mathcal{R}} P)\,(U), \tag{221}$$

and similarly with M' instead of M. Indeed, the first isomorphism in (221) is a particular case of (18.2.5) p. 441 of the book, and the second isomorphism in (221) results from the last two displays on p. 428 of the book. In other words, we have

$$(N \otimes_{\mathcal{R}|_U} P|_U)(U) \simeq (F' \otimes_{\mathcal{R}} P)(U) \times_{(F \otimes_{\mathcal{R}} P)(U)} (F'' \otimes_{\mathcal{R}} P)(U). \tag{222}$$

Note that (220) implies

$$x := (t, u) \in (F' \otimes_{\mathcal{R}} P)(U) \times_{(F \otimes_{\mathcal{R}} P)(U)} (F'' \otimes_{\mathcal{R}} P)(U).$$

If y is the element of $(N \otimes_{\mathcal{R}|_U} P|_U)(U)$ corresponding to x under Isomorphism (222), then we get $k_1(y) = u$, and thus $\overline{s} = g_3(u) = g_3(k_1(y)) = 0$. By (218), this completes the proof. □

19.3 Brief comments

§ 574. P. 452, Part (i) (a) of the proof of Lemma 18.6.7. As already mentioned, $\mathcal{O}_U$ and $\mathcal{O}_V$ stand presumably for $\mathcal{O}_X|_U$ and $\mathcal{O}_Y|_V$ (and it would be better, in the penultimate display of the page, to write $\mathcal{O}_V$ instead of $\mathcal{O}_Y|_V$), and, a few lines before the penultimate display of the page, $f_W^{-1} : \mathcal{O}_U^{\oplus n} \xrightarrow{u} \mathcal{O}_U^{\oplus m}$ should be (I think) $f_W^{-1} : \mathcal{O}_W^{\oplus n} \to \mathcal{O}_W^{\oplus m}$.

Also, one may refer to (199) p. 295 and §512 p. 296 to describe the morphism of sites $f_W : W \to V$. More precisely, we define, in the notation (199), the functor $(f_W)^\tau : ((\mathcal{C}_Y)_V)^{\mathrm{op}} \to ((\mathcal{C}_X)_W)^{\mathrm{op}}$ by

$$(f_W)^\tau(V' \to V) := \big(f^\tau(V') \to f^\tau(V) \to W\big).$$

Finally, let us rewrite explicitly one of the key equalities (see §512 p. 296):

$$f^\dagger(\mathcal{O}_Y^{\oplus nm})(W) = \operatorname*{colim}_{(f^\tau(V) \to W) \in ((\mathcal{C}_Y)^{\mathrm{op}})_W} \mathcal{O}_Y^{\oplus nm}(V),$$

where $f^\tau(V) \to W$ is a morphism in $(\mathcal{C}_X)^{\mathrm{op}}$ (corresponding to a morphism $W \to f^t(V)$ in $\mathcal{C}_X$).

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