Holomorphe Vektorbündel auf nichtalgebraischen Flächen

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Holomorphic vector bundles on non-algebraic surfaces

To the memory of Constantin Bănică

Contents

Zusammenfassung

Diese Arbeit behandelt Existenzprobleme von holomorphen Vektorbündeln auf nichtalgebraischen Flächen.

Zuerst wird in diesem Zusammenhang die Frage untersucht, welche topologischen komplexen Vektorbündel auf einer vorgegebenen nichtalgebraischen Fläche X holomorphe Strukturen besitzen.

Die topologischen komplexen Vektorbündel auf einer kompakten komplexen Fläche X werden vollständig durch ihren Rang r und ihre Chernklassen $(c_1, c_2) \in H^2(X, \mathbf{Z}) \times H^4(X, \mathbf{Z})$ bestimmt. Also beschränkt sich das Problem darauf holomorphe Vektorbündel mit gegebenen topologischen Invarianten (r, c_1, c_2) zu konstruieren.

Für algebraische Flächen gab Schwarzenberger eine vollständige Antwort (vgl. I.2. Theorem 1). In diesem Fall können alle holomorphen Vektorbündel als Extensionen geeigneter kohärenter Garben von niedrigerem Rang konstruiert werden.

Dies stimmt jedoch nicht mehr für Vektorbündel über einer nichtalgebraischen Basis. Zuerst führen wir einige Definitionen ein. Ein holomorphes Vektorbündel E auf einer kompakten komplexen Fläche X heißt **reduzibel**, wenn eine kohärente Untergarbe $\mathcal{F} \subset E$ existiert mit $0 < \operatorname{Rang} \mathcal{F} < \operatorname{Rang} E$, andernfalls **irreduzibel**. Weiter heißt E **filtrierbar** wenn es eine Filtration $0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_r = E$ durch kohärente Untergarben $\mathcal{F}_i$ mit $\operatorname{Rang} \mathcal{F}_i = i$ gibt, sonst heißt E **nichtfiltrierbar**. Im Gegensatz zum algebraischen Fall gibt es auf nichtalgebraischen Flächen nichtfiltrierbare und sogar irreduzible Vektorbündel. Dieses Phänomen wurde zuerst von G. Elencwajg und O. Forster [EF] untersucht und weiter von C. Bănică und J. Le Potier studiert [BL]. Tatsächlich können ihre Existenzsätze für irreduzible Vektorbündel [BL; §5] im Fall $\operatorname{Rang} E = 2$ auf alle nichtalgebraische Flächen ausgedehnt werden (vgl. II.1.1.).

Durch die Anwendung von Extensionen gaben C. Bănică und J. Le Potier eine vollständige Beschreibung von topologischen Vektorbündeln auf nicht-algebraischen Flächen, die filtrierbare holomorphe Strukturen ermöglichen (vgl. I.2., Theorem 3). Für manche Klassen nicht-algebraischer Flächen wird damit das gestellte Problem gelöst. Im allgemeinen jedoch gibt es topologische Vektorbündel, die holomorphe Strukturen besitzen aber keine filtrierbare Struktur, nicht einmal eine reduzible. In Kapitel II §1 zeigen wir dies für den Fall, daß die Basis ein nicht-algebraischer 2-dimensionaler Torus ist.

Eine weitere Frage ist die Existenz von holomorphen Strukturen mit zusätzlichen analytischen Eigenschaften auf einem topologischen komplexen Vektorbündel über eine Fläche.

In Kapitel II.§2. betrachten wir das Problem der Existenz von **stark irreduziblen**

Vektorbündeln auf eine Fläche X. Dies sind holomorphe Vektorbündel E, die irreduzibel bleiben nach einem Basiswechsel $X' \xrightarrow{f} X$, wobei X' wieder eine kompakte komplexe Fläche ist und f eine surjektive holomorphe Abbildung. Wenn E vom Rang 2 ist, passiert das genau dann, wenn das zugehörige projektive Bündel $\mathbf{P}(E)$ keine "horizontale" Divisoren besitzt. Wir zeigen die Existenz solcher Bündel in der versellen Deformation eines irreduziblen Bündels, wenn X ein 2-Torus ohne Kurven oder eine K3-Fläche ohne Kurven ist.

In Kapitel III betrachten wir schließlich einfache reduzible Vektorbündel von Rang 2 und nennen notwendige und hinreichende Bedingungen an die topologischen Invarianten (r, c_1, c_2) für die Existenz solcher Vektorbündel auf Flächen der algebraischen Dimension null. Dabei heißt ein holomorphes Vektorbündel E **einfach**, wenn jeder von Null verschiedene Endomorphismus von E ein Automorphismus ist. Die Bedeutung dieser Vektorbündel liegt darin, daß sie Modulräume besitzen. Im letzten Abschnitt werden einige Bemerkungen über die Trennungseigenschaften dieser Modulräume gemacht.

Introduction

This paper is about existence problems for holomorphic vector bundles on non-algebraic surfaces.

A first such problem to be considered is determining what topological complex vector bundles on a given non-algebraic surface X admit holomorphic structures.

The topological complex vector bundles on a compact surface X are completely characterized by their rank r and their Chern classes $(c_1, c_2) \in H^2(X, \mathbf{Z}) \times H^4(X, \mathbf{Z})$. So the problem comes to constructing holomorphic vector bundles with given topological invariants (r, c_1, c_2).

When X is algebraic one has a complete answer due to Schwarzenberger (cf. I.2. Theorem 1). In this case all holomorphic vector bundles can be constructed as extensions of appropriate coherent sheaves of inferior ranks.

This however is no longer true over a non-algebraic base. Consider first the following definitions. A holomorphic vector bundle E on a compact complex surface X is called reducible if there exists a coherent subsheaf $\mathcal{F} \subset E$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} E$, (and irreducible otherwise). E is called filtrable if it admits a filtration $0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_r = E$ by coherent subsheaves $\mathcal{F}_i$, with $\operatorname{rank} \mathcal{F}_i = i$ (and non-filtrable otherwise). In contrast to the algebraic case there exist non-filtrable and even irreducible vector bundles on non-algebraic surfaces. This phenomenon was brought forward by G. Elencwajg and O. Forster [EF], and further studied by C. Bănică and J. Le Potier [BL]. In fact their theorems of existence for irreducible vector bundles, [BL;§5], can be extended in order to cover all non-algebraic surfaces (for $\operatorname{rank} E = 2$) as in II.1.1..

Using the method of extensions C. Bănică and J. Le Potier gave a complete characterization of topological vector bundles on non-algebraic surfaces admitting filtrable holomorphic structures (see I.2. Theorem 3). For some classes of non-algebraic surfaces this solves the given problem. In general, however, there exist topological vector bundles admitting holomorphic structures but no filtrable structure, or even no reducible structure. We show this when the base is a non-algebraic 2-dimensional torus, in Chapter II §1.

One can formulate further problems concerning the existence of holomorphic structures together with some extra analytic properties for a topological vector bundle on a surface.

In Chapter II §2 we consider the problem of existence of strongly irreducible vector bundles on a surface X. These are holomorphic vector bundles E which remain irreducible after any base change $X' \xrightarrow{f} X$, where X' is again a compact complex surface and f is a surjective morphism. When rank $E = 2$, this happens precisely when the associated projective bundle $\mathbf{P}(E)$ has no " horizontal" divisors. We show the existence of such bundles in the versal deformation of reducible bundles when X is a 2-torus without curves or a K3 surface without curves.

Finally in Chapter III we consider simple reducible rank 2 vector bundles and give necessary and sufficient conditions, in terms of topological invariants, for their existence on surfaces of algebraic dimension zero. Recall that a holomorphic vector bundle is called simple if every non-zero endomorphism of it is an automorphism. Their importance stems from the fact that they admit moduli spaces. Some remarks on non-separation phenomena in these moduli spaces are made in the last paragraph.

This work wouldn't have been possible without the constant guidance of late Professor Constantin Bănică . I dedicate it as a small tribute to his memory. I also wish to thank Professor M. Schneider for supervising the last stages of this work. These last stages also profited from the kind hospitality of Bayreuth University and from the financial support of its Graduiertenkolleg "Komplexe Mannigfaltigkeiten".

Chapter I

Preliminaries

§1. Non-algebraic surfaces

1.1.

A 2-dimensional compact complex connected manifold will be simply called a surface, and any 1-dimensional compact complex space - a curve. For general facts on surfaces we refer to [BPV]. For X a surface, we denote its Picard group by $\mathrm{Pic}(X) := H^1(X, \mathcal{O}^*)$, its Néron-Severi group by $\mathrm{NS}(X) := c_1(\mathrm{Pic}(X)) \subset H^2(X, \mathbf{Z})$, while $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ will be the kernel of

$$c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbf{Z}).$$

The rank of $\mathrm{NS}(X)$ is called the Picard number of X and is denoted by $\rho(X)$. The free part of $\mathrm{NS}(X)$, i.e. its image in $H^2(X, \mathbf{C})$ through the natural morphism $j : H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C})$, is described by

Theorem 1 (Lefschetz' theorem on (1,1)-classes)

$$j(\mathrm{NS}(X)) = j(H^2(X, \mathbf{Z})) \cap H^{1,1}(X)$$

where $H^{1,1}(X) \subset H^2(X, \mathbf{C})$ *is the set of classes of closed differential forms of type* $(1,1)$ *on* X*. In other words, an element of* $H^2(X, \mathbf{C})$ *is in the image of* $\mathrm{NS}(X)$ *if and only if it is integral and can be represented by a real closed* $(1,1)$*-form on* X*.*

1.2.

The algebraic dimension of a surface X is the transcendence degree of its meromorphic function field $\mathcal{M}(X)$ over $\mathbf{C}$. This is finite and does not exceed $\dim X$. We denote it by $a(X)$.

A surface X is called algebraic if it admits an algebraic variety structure compatible with its complex-analytic one. Otherwise X is called non-algebraic.

In the special case of dimension 2 we have the following characterization:

Theorem 2 (Kodaira) *For a surface X the following statements are equivalent:*

i) X is projective

ii) X is algebraic

iii) $a(X)=2$

iv) there exists an element $L \in \mathrm{Pic}(X)$ with $L^2 > 0$.

Here L^2 denotes self intersection $L^2 = c_1(L)^2 \in H^4(X, \mathbf{Z}) \cong \mathbf{Z}$.

Corollary 1 *If X is non-algebraic the intersection form on $\mathrm{NS}(X)$ is negative semidefinite. In particular if $a, b \in \mathrm{NS}(X)$ then $a^2 = 0 \Rightarrow a \cdot b = 0$.* (Just look at $(na+b)^2 \leq 0$ for $n \in \mathbf{Z}$.)

Definition A connected holomorphic map $f : X \longrightarrow S$ from a surface X onto a non-singular curve S is called an **elliptic fibration** if the general fiber X_s, $s \in S$, is a smooth elliptic curve. A surface X is called **elliptic** if it admits some elliptic fibration.

Corollary 2 *Any connected holomorphic map from a non-algebraic surface X onto a smooth curve S is an elliptic fibration. Moreover any irreducible curve on X is contained in some fiber and thus the fibration is unique.*

Proof. The general fibre C is non-singular and $C^2 = 0$. Corollary 1 now gives $K_X \cdot C = 0$ where K_X denotes the class of the canonical bundle on X. By the adjunction formula

$$2g(C) - 2 = K_X \cdot C + C^2$$

we get $g(C) = 1$, i.e. C is elliptic. Let now D be an irreducible curve contained in no fiber. Then D intersects the generic fiber C in $D \cdot C > 0$ points. This would contradict Corollary 1. □

The algebraic dimension of a surface X can be expressed in terms of "number of curves" on X. When $a(X) = 2$, X is projective hence through each point we have an infinity of curves of X. By Corollary 2 this is no longer true when $a(X) \leq 1$. The following two theorems settle the cases $a(X) = 1$ and $a(X) = 0$ from this point of view.

Theorem 3 *For X non-algebraic we have $a(X) = 1$ if and only if X is elliptic.*

Proof. If $a(X) = 1$ there exists a non-constant meromorphic function f on X. If f had indeterminacy points we would have $D^2 > 0$ where $D = (f)$ is the associated divisor of f. But this cannot be, hence f defines a morphim $f : X \to \mathbf{P}_1$. Its Stein factorization gives a nosingular curve S and a connected morphism $X \longrightarrow S$ which is an elliptic fibration by Corollary 2.

Conversely if X admits an elliptic fibration over some curve S then there exist non-constant meromorphic functions on X lifted from S. It follows that $a(X) = 1$, since X was assumed non-algebraic. □

Theorem 4 *If X is a surface having $a(X) = 0$ then the number of irreducible curves on X is finite, not exceeding $h^{1,1}(X) + 2$.*

Proof. Let $\mathcal{M}^1$, Ω^1 be the sheaves of germs of meromorphic, resp. holomorphic, 1-forms on X and $\mathcal{L} := \mathcal{M}^1/\Omega^1$. A curve on X of local equation $f = 0$ defines an element $\frac{df}{f}$ in $H^0(X, \mathcal{L})$. One can see that different irreducible curves give linearly independent elements in $H^0(X, \mathcal{L})$. One considers then the long cohomology sequence

$$\ldots \to H^0(X, \mathcal{M}^1) \to H^0(X, \mathcal{L}) \to H^1(X, \Omega^1) \to \ldots$$

and we only need to prove $h^0(X, \mathcal{M}^1) \leq 2$. Let $\omega_1, \omega_2 \in H^0(X, \mathcal{M}^1)$ linearly independent over $\mathbf{C}$. Locally on $U \subset X$ they are linearly independent over $\mathcal{M}(U)$, otherwise $\omega_1 \wedge \omega_2 = 0$ on U, hence on X and $\frac{\omega_1}{\omega_2} \in \mathcal{M}(X) = \mathbf{C}$. Thus locally on U every third form $\omega \in \mathcal{M}^1(X)$ can be uniquely written as a linear combination in ω_1, ω_2 over $\mathcal{M}(U)$ which globalizes to give $\omega = f_1\omega_1 + f_2\omega_2$ with

$$f_1, f_2 \in \mathcal{M}(X) = \mathbf{C}. \qquad \square$$

For the following Proposition we need a

Definition A coherent sheaf $\mathcal{E}$ on a complex, (connected) manifold X is called **reducible** if it admits a coherent subsheaf $\mathcal{F}$ such that

$$0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E},$$

and **irreducible** otherwise.

Recall that any coherent sheaf is locally free on some open Zariski subset of X and its rank can be defined as being the rank of the corresponding holomorphic vector bundle on this open subset.

Proposition *Let X be a surface having $a(X) = 0$ and $\mathcal{E}$ a coherent sheaf without torsion on X. Then*

$$h^0(X, \mathcal{E}) \leq \operatorname{rank} \mathcal{E}$$

Proof. We use induction on $r = \operatorname{rank} \mathcal{E}$.

For $r = 1$, we can restrict ourselves to the locally free case by considering the inclusion of $\mathcal{E}$ into its double dual $\mathcal{E}^{\vee\vee}$ which is invertible. Now if $\mathcal{E}$ had 2 linearly independent global sections their quotient would give a non-constant meromorphic map on X contradicting the hypothesis.

Let now $r \geq 2$. When $\mathcal{E}$ is irreducible we have $h^0(X, \mathcal{E}) = 0$, since non-zero global sections in $\mathcal{E}$ would generate rank 1 coherent subsheaves. For $\mathcal{E}$ reducible one can choose $\mathcal{F}$ as in the definition above such that moreover $\mathcal{E}/\mathcal{F}$ has no torsion. In order to see this take an arbitrary $\mathcal{F}$, $p : \mathcal{E} \to \mathcal{E}/\mathcal{F}$, and replace it by $\mathcal{F}' = p^{-1}$ (Tors $\mathcal{E}/\mathcal{F}$). Apply now the induction hypothesis to the exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}/\mathcal{F} \to 0. \qquad \square$$

1.3. Classification

Theorem 5 (Kodaira) *Every non-algebraic surface has a unique minimal model in one of the classes of the following table*

Nr	*class of X*	kod(X)	$b_1(X)$	*order of* K_X *in* Pic(X)	$a(X)$
1)	*minimal surfaces in class VII*	$-\infty$	1		0, 1
2)	*non-algebraic tori*	0	4	1	0, 1
3)	*non-algebraic K3-surfaces*	0	0	1	0, 1
4)	*primary Kodaira surfaces*	0	3	1	1
5)	*secondary Kodaira surfaces*	0	1	2, 3, 4, 6	1
6)	*properly elliptic minimal non-algebraic surfaces*	1			1

Definitions and examples

1) A surface X is in **class VII** when kod$(X) = -\infty$ and $b_1(X) = 1$ (we follow the convention in [BPV]).

An important class of examples here consists of the **Hopf surfaces**. These are by definition those surfaces admitting $\mathbf{C}^2\setminus\{0\}$ as universal covering space. They divide into **primary** when $\pi_1(X) = \mathbf{Z}$ and **secondary** if they appear as quotients of primary Hopf surfaces through cyclic transformation groups. In the first situation the infinite cyclic group acting on $\mathbf{C}^2\setminus\{0\}$ is generated by a contraction of the form

$$f(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$$

where (z_1, z_2) are suitable coordinates, $m \in \mathbf{Z}$, $m \geq 1$; $\lambda, \alpha_1, \alpha_2 \in \mathbf{C}$, $0 < |\alpha_1| \leq |\alpha_2| < 1$ and $(\alpha_1 - \alpha_2^m)\lambda = 0$, whereas in the second case one has to consider also the action of a cyclic group of order l generated by

$$e(z_1, z_2) = (\epsilon_1 z_1, \epsilon_2 z_2)$$

where ϵ_1, ϵ_2 are primitive l-roots of the unity and $(\epsilon_1 - \epsilon_2^m)\lambda = 0$ (cf. [K]). Since primary Hopf surfaces are diffeomorphic to $S^1 \times S^3$, all Hopf surfaces have $b_1 = 1$, $b_2 = 0$. In particular they are minimal. Moreover they admit a global meromorphic 2-form (descending from $\mathbf{C}^2\setminus\{0\}$) of the form:

$$\begin{cases} \frac{1}{z_1 \cdot z_2}\, dz_1 \wedge dz_2 & \text{for } \lambda = 0 \\[2ex] \frac{1}{z_2^{m+1}}\, dz_1 \wedge dz_2 & \text{for } \lambda \neq 0 \end{cases}$$

which shows that

$$K_X = \begin{cases} \mathcal{O}(-C_1 - C_2) & \text{when } \lambda = 0 \\ \mathcal{O}(-(m+1)C_2) & \text{when } \lambda \neq 0 \end{cases}$$

hence kod$(X) = -\infty$, where the curves C_i are the projections of the coordinate axes (when $\lambda \neq 0$ only the axis $\{z_2 = 0\}$ is invariated by the covering group). Hence any Hopf surface admits at least one irreducible curve.

Theorem 6 (Kodaira)

a) Any minimal surface X with $a(X) = 1$ and $\mathrm{kod}(X) = -\infty$ is a Hopf surface.

b) Any surface X with $a(X) = 0$, $b_1(X) = 1$, $b_2(X) = 0$ and admitting at least one curve is a Hopf surface.

For the fact that both values 0 and 1 are attained by the algebraic dimension of Hopf surfaces we refer to [BPV,V.18].

There exist examples of surfaces in class VII with $b_2 = 0$ and no curves, or with $b_2 > 0$. For $b_2 = 0$ their classification is complete (cf.[I], [LYZ]).

2) We consider 2-dimensional complex tori $X = \mathbf{C}^2/\Gamma$, where Γ is a rank 4 lattice in $\mathbf{C}^2$. According to Lefshetz' theorem $\mathrm{NS}(X) = H^{1,1}_{\mathbf{R}}(X) \cap H^2(X, \mathbf{Z})$. Now $H^{1,1}_{\mathbf{R}}(X)$ can be seen as the space of harmonic real $(1,1)$-forms, hence of the form

$$\eta = \frac{i}{2} \sum_{\alpha,\beta=1}^{2} h_{\alpha\overline{\beta}}\, dz_\alpha \wedge d\overline{z}_\beta\,,$$

where (z_1, z_2) are complex coordinates for $\mathbf{C}^2$, $h_{\alpha\overline{\beta}} \in \mathbf{C}$ and $h_{\alpha\overline{\beta}} = \overline{h}_{\beta\overline{\alpha}}$, $\alpha, \beta \in \{1, 2\}$. These give integral elements if and only if they get integral coeficients when expressed in terms of a real basis of $\mathbf{C}^2$ consisting of generators for Γ (cf. [W;ch.VI]). Thus we obtain a natural isomorphism

$$\mathrm{NS}(X) \cong \Big\{A \mid A \text{ hermitian } 2 \times 2\text{-matrix with } \mathrm{Im}({}^t\Pi A \overline{\Pi}) \in M_4(\mathbf{Z})\Big\} \tag{1}$$

where $\Pi := (\gamma_1, \cdots, \gamma_4)$ is the period matrix of X (i.e. the column vectors γ_i of Π form a set of generators for Γ), while $M_n(\mathbf{Z})$ denotes the set of integer valued $n \times n$ matrices.

Using this, one computes the intersection form on $\mathrm{NS}(X)$ (through exterior product of forms) as follows. For a in $\mathrm{NS}(X)$ consider its associated hermitian 2×2-matrix A given by (1) and the matrices $A_1, A_2, A_3 \in M_2(\mathbf{Z})$ giving the decomposition

$$\begin{pmatrix} A_1 & A_2 \\ -{}^tA_2 & A_3 \end{pmatrix} = \mathrm{Im}({}^t\Pi A \overline{\Pi})\,.$$

Since $\mathrm{Im}({}^t\Pi A \overline{\Pi})$ is skew-symmetric we get the above form and A_1, A_3 skew-symmetric. Let

$$A_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}.$$

Now a straightforward computation shows

$$a^2 = 2(\alpha\delta - \beta\gamma - \theta\tau) \quad . \tag{2}$$

The algebraic dimension of X can be expressed in the following way (cf.[W; VI, 8]):

$$\begin{aligned} a(X) \;=\; & \max\{\mathrm{rank}\, A \mid A \text{ positive semi-definite hermitian} \\ & \qquad 2 \times 2\text{-matrix with } \mathrm{Im}({}^t\Pi A \overline{\Pi}) \in M_4(\mathbf{Z})\}\,. \end{aligned} \tag{3}$$

For examples of 2-tori admitting various algebraic dimensions and Picard numbers we refer to [EF; Appendix].

We give here the following two examples which will be used later:

A. For every positive integer n there exist complex 2-tori X having $\mathrm{NS}(X)$ cyclic generated by a with $a^2 = -2n$ (they are automatically non-algebraic with $a(X) = 0$). Let $\Pi = (\Pi_1, \Pi_2)$ with

$$\Pi_1, \Pi_2 \in M_2(\mathbf{C}), \quad \Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Pi_2 = iP, \quad P \in M_2(\mathbf{R}),$$

$$\det P = \frac{1}{n}, \quad P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \mathrm{rank}_{\mathbf{Q}}(p,q,r,s) = 4$$

and let X be the 2-torus having period matrix Π. Then using (1) one gets that $\mathrm{NS}(X)$ is generated by an element a having the associated hermitian matrix

$$A = \begin{pmatrix} 0 & in \\ -in & 0 \end{pmatrix}$$

This and (2) gives $a^2 = -2n$. By (3), $a(X) = 0$.

B. For every positive integer n there exist complex 2-tori X such that $\mathrm{NS}(X)$ is free with two generators a, b such that $a^2 = -2n$, $b^2 = 0$. (In this case $a(X) = 1$).

Take Π as before but this time

$$P = \begin{pmatrix} \frac{\sqrt[3]{2}}{n} & -1 \\ 0 & \sqrt[3]{2} \end{pmatrix}.$$

Then an element in $\mathrm{NS}(X)$ has an associated hermitian matrix of the form

$$A(x,y) = \begin{pmatrix} 0 & \frac{n}{\sqrt[3]{2}}x \\ \frac{n}{\sqrt[3]{2}}x & \frac{n}{\sqrt[3]{2}}x + \frac{1}{\sqrt[3]{2}}y \end{pmatrix}$$

for $x, y \in \mathbf{Z}$.

Taking a, b in $\mathrm{NS}(X)$ corresponding to $A(1,0)$ and $A(0,1)$ respectively, one gets the desired statement by direct computation as in example A. For a general criterion giving $a(X)$ in these examples see the end of this section.

3) X is by definition a **K3 surface** if $q(X) = 0$ and the canonical bundle is trivial ($K_X \cong \mathcal{O}_X$), where $q(X) := h^{0,1}(X) := h^1(\mathcal{O}_X)$.

Examples of such surfaces are provided by the Kummer surfaces. They are constructed as follows. One starts with a complex 2-torus T and considers its involution $i : T \to T$, $i(x) = -x$, an origin having been fixed. This has 16 fixed points giving 16 ordinary double points for the quotient $T/\{id, i\}$. Resolving these sixteen singularities one obtains a surface X, **the Kummer surface associated to** T. One obtains the same thing by first blowing up the 16 fixed points of i on T and then taking the quotient through the induced involution. X is easily seen to be a K3 surface [BPV; V 16] and $a(X) = a(T)$. The whole range for the algebraic dimension is thus attained.

For K3 surfaces $H^1(X, \mathcal{O}) = 0$, hence $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \hookrightarrow H^2(X, \mathbf{Z})$. $H^2(X, \mathbf{Z})$ is shown to be free and its intersection form to be even.

Remarks. Let X be a K3 surface with $a(X) = 0$.

i) If $L \in \mathrm{Pic}(X)$ is non-trivial then $L^2 \leq -2$ and equality holds if and only if $L \cong \mathcal{O}(\pm D)$ for some effective divisor D. (By abuse of notation we do not distinguish between a vector bundle and its associated sheaf of holomorphic sections). To see this we use the Riemann-Roch formula for a line bundle L on a surface X:

$$\chi(L) := h^0(L) - h^1(L) + h^2(L) = \frac{1}{2}(L^2 - L \cdot K_X) + \chi(\mathcal{O}_X). \qquad (4)$$

Now since $a(X) = 0$ and L non-trivial we have

$$h^0(L) + h^0(L^\vee) \leq 1$$

by 1.2. (Proposition) and the fact that in this case $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ implies $D_1 = D_2$ for any two divisors D_1,D_2 on X.

The conclusion follows now immediately from (4) and Serre duality.

ii) Any irreducible curve C on X is smooth rational with $C^2 = -2$. (Such curves are called (-2)-curves). This is a direct consequence of i) and the adjunction formula for C. (cf. [BPV; II 11]):

$$2g(C) - 2 = C^2 + C \cdot K_X.$$

iii) Any connected reduced curve C on X is an A-D-E curve (i.e. the intersection form restricted to the group generated by the irreducible components of C gives a root-lattice of type A, D or E; (cf. [BPV; p.74]).

This is a consequence of i) and ii) and of the classification of lattices of this type.

For later use we mention here the existence for all $g \in \mathbf{Z}$ of **special K3 surfaces of type** g**,** X, i.e. such that $\mathrm{NS}(X)$ is freely generated by one element L with $L^2 = 2g - 2$, (see [L]), and also the existence of K3 surfaces X with $\mathrm{NS}(X) = 0$.

4 and 5) A **primary Kodaira surface** is a surface with $b_1 = 3$ and admitting a locally trivial elliptic fibration over an elliptic curve.

A surface which is not primary Kodaira but admits as an unramified cover a primary Kodaira surface is called a **secondary Kodaira surface**. See [BPV; V 5] for examples.

6) A **properly elliptic surface** is an elliptic surface having Kodaira dimension 1. A product of an elliptic curve and a curve of higher genus is properly elliptic. Non-algebraic surfaces of this type can be obtained by applying logarithmic transformations to a product of an elliptic curve with a rational curve; (cf.[BPV; V 13]).

One corollary of the classification is the following

Theorem 7 *For a surface X with $a(X)=0$ the intersection form on* $\mathrm{NS}(X)$ *is negative definite modulo torsion.*

This is a direct consequence of the Signature theorem on $H^{1,1}_{\mathbf{R}}(X)$ (cf. [BPV; IV 2.13]) in the case of class VII surfaces, it is part of Remark i) for K3-surfaces, and was proven in [BF1] for tori using the description (1) of $\mathrm{NS}(X)$.

§2. Existence problem for holomorphic vector bundles on surfaces

Let X be a compact complex surface. Consider the following question: which topological complex vector bundles E on X admit holomorphic structures? When $r := \operatorname{rank} E = 1$ the answer is given by the exponential sequences. Indeed considering the exponential sequence for the sheaf $\mathcal{C}$ of germs of continuous complex functions

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{C} \xrightarrow{\mathbf{e}} \mathcal{C}^* \longrightarrow 0\,,$$

$\mathbf{e}(f) := \exp(2\pi i f)$, we get since $\mathcal{C}$ is a fine sheaf

$$H^1(X,\mathcal{C}^*) \xrightarrow[\sim]{c_1} H^2(X,\mathbf{Z})\,,$$

i.e. the isomorphism classes of complex topological line bundles on X are parametrized by $H^2(X,\mathbf{Z})$,by taking the Chern class c_1. Now using the long cohomology sequence of the exponential sequence for $\mathcal{O}$

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \xrightarrow{\mathbf{e}} \mathcal{O}^* \longrightarrow 0\,,$$

we get a commutative square

$$\begin{array}{ccc} H^1(X,\mathcal{C}^*) & \overset{c_1}{\underset{\sim}{\longrightarrow}} & H^2(X,\mathbf{Z}) \\ \uparrow & & \| \\ H^1(X,\mathcal{O}^*) & \overset{c_1}{\longrightarrow} & H^2(X,\mathbf{Z}) \end{array}$$

showing that a topological line bundle E admits a holomorphic structure if and only if $c_1(E) \in \mathrm{NS}(X)$. This remains a necessary condition also for $r > 1$, since if E has a holomorphic structure then $\det E := \wedge^r E$ also admits one, hence $c_1(E) = c_1(\det E) \in \mathrm{NS}(X)$.In the surface case the topological classification of vector bundles is known ([Wu]): for every rank $r > 1$ and every pair $(c_1, c_2) \in H^2(X,\mathbf{Z}) \times H^4(X,\mathbf{Z})$ there exists a unique (up to isomorphism) topological complex vector bundle E of rank r with $c_1(E) = c_1$, $c_2(E) = c_2$. Thus the problem comes to constructing holomorphic vector bundles having prescribed rank and prescribed Chern classes.

A method of construction is the method of extensions. We present it for $r = 2$. Let E be a reducible holomorphic rank 2 vector bundle (see 1.2). Then E admits a coherent rank 1 subsheaf $\mathcal{F}$ such that $E/\mathcal{F}$ is torsion free. One deduces that $\mathcal{F}$ is invertible. Indeed the bidual of the inclusion morphism is a monomorphism

$$\mathcal{F}^{\vee\vee} \longrightarrow E$$

hence $\mathcal{F}^{\vee\vee}/\mathcal{F} \hookrightarrow E/\mathcal{F}$. Since $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is a torsion sheaf we get $\mathcal{F}^{\vee\vee} \cong \mathcal{F}$. This and $\operatorname{rank}\mathcal{F} = 1$ imply $\mathcal{F}$ is invertible.

$E/\mathcal{F}$ can be described as follows. The zero locus of the section in $E \otimes \mathcal{F}^\vee$ corresponding to the inclusion morphism is a locally complete intersection 2-codimensional analytic subspace Y of X.Hence $E/\mathcal{F} \mid_{X\setminus Y}$ is an invertible sheaf L extending to an invertible sheaf on X (in fact $L = (E/\mathcal{F})^{\vee\vee}$) and one has an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow L \otimes \mathcal{J}_Y \longrightarrow 0$$

to be named a **devissage** of E. $\mathcal{J}_Y$ denotes the ideal sheaf of Y.Thus E appears as an extension of $L \otimes \mathcal{J}_Y$ through $\mathcal{F}$.

Conversely, starting with L_1, L_2 in $\operatorname{Pic}(X)$ and Y a locally complete intersection 2-codimensional analytic subspace of X one may ask when there exist extensions of $L_2 \otimes \mathcal{J}_Y$ through L_1

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \otimes \mathcal{J}_Y \longrightarrow 0 \tag{5}$$

such that E be locally free. By Serre (cf.[OSS; I.5]) the central term of (5) is locally free if and only if the image of the element $\theta \in \operatorname{Ext}^1(X; L_2 \otimes \mathcal{J}_Y, L_1)$ associated to the extension (5), through the canonical mapping

$$\operatorname{Ext}^1(X; L_2 \otimes \mathcal{J}_Y, L_1) \longrightarrow H^0(X, \mathcal{E}\mathrm{xt}^1(\mathcal{J}_Y \otimes L_2, L_1))$$

generates the sheaf $\mathcal{E}\mathrm{xt}^1(L_2 \otimes \mathcal{J}_Y, L_1)$.

Let $L := L_2^\vee \otimes L_1$. Since in our case $\mathcal{H}om(L_2 \otimes \mathcal{J}_Y, L_1) \cong L_2^\vee \otimes L_1 = L$ the exact sequence of the first terms of the Ext spectral sequence becomes:

$$0 \longrightarrow H^1(X, L) \longrightarrow \operatorname{Ext}^1(\mathcal{J}_Y, L) \longrightarrow H^0(X; \mathcal{E}\mathrm{xt}^1(\mathcal{J}_Y, L)) \longrightarrow H^2(X, L)$$

Let $l(Y)$ be the length of $\mathcal{O}_Y$ over $\mathbf{C}$. Out of the preceeding facts we deduce the

Remarks

1. Serre's condition is fulfilled for some θ in $\operatorname{Ext}^1(X; L_2 \otimes \mathcal{J}_Y, L_1)$ in the following cases:

   i) Y locally complete intersection (hence $\mathcal{E}\mathrm{xt}^1(\mathcal{J}_Y; L) \cong \mathcal{O}_Y$) and $H^2(X; L) = 0$.

   ii) $a(X) = 0$, Y locally complete intersection, $L^\vee \otimes K \cong \mathcal{O}(D)$ with D an effective divisor, and $Y \subset D$ as analytic spaces.(By duality we get that the map $H^0(X, \mathcal{E}\mathrm{xt}^1(\mathcal{J}_Y, L)) \to H^2(X, L)$ is zero).

   iii) $a(X) = 0$, Y consists of simple points,$l(Y) > 1$, $H^2(X, L) \neq 0$, $L^\vee \otimes K = \mathcal{O}(D)$ and $Y \cap \operatorname{supp} D = \emptyset$ (one can find sections in $H^0(X, \mathcal{E}\mathrm{xt}^1(\mathcal{J}_Y, L)) \cong H^0(Y, \mathcal{O}_Y)$ mapped to zero in $H^2(X, L) = H^0(X, \mathcal{O}(D))^\vee$ having non-zero components in each fiber $\mathcal{O}_{Y,y}, y \in Y$).

2. Serre's condition cannot be fulfilled when $a(X) = 0$, $l(Y) = 1$ (hence Y is a simple point on X), $L^\vee \otimes K = \mathcal{O}(D)$, D effective and Y is not on supp D.

   To see this note that the morphism $H^0(X, \mathcal{E}\mathrm{xt}^1(\mathcal{J}_Y, L)) \to H^2(X, L)$ is the dual of the restriction
$$H^0(X, L^\vee \otimes K) \longrightarrow H^0(Y, L^\vee \otimes K\mid_Y).$$

Using Remark 1 i) one can answer completely the question of existence of holomorphic structures for topological vector bundles over algebraic surfaces. We have:

Theorem 1 (Schwarzenberger) *A complex topological vector bundle E over an algebraic surface X admits a holomorphic structure if and only if $c_1(E) \in \mathrm{NS}(X)$.*

Proof. To prove the theorem, we have to construct for given $r > 1$ and $(c_1, c_2) \in \mathrm{NS}(X) \times H^4(X, \mathbf{Z})$, holomorphic vector bundles with $\operatorname{rank} E = r$, $c_1(E) = c_1$, $c_2(E) = c_2$. In fact it is enough to do this for $r = 2$ because then one can add a trivial vector bundle of suitable rank.

One uses extensions. For E as in (5) we get:

$$\begin{aligned}
c_1(E) &= c_1(\det E) = c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \\
c_2(E) &= c_2(E \otimes L_1^{-1} \otimes L_1) = c_2(E \otimes L_1^{-1}) + c_1(E \otimes L_1^{-1}) \cdot c_1(L_1) + c_1(L_1)^2 = \\
&= l(Y) + c_1(L_1) \cdot c_1(L_2) = \\
&= l(Y) + L_1 \cdot L_2 \in \mathbf{Z} \cong H^4(X, \mathbf{Z})
\end{aligned}$$

Thus
$$\begin{cases} c_1(E) = c_1(L_1) + c_1(L_2) \\ c_2(E) = l(Y) + L_1 \cdot L_2 \end{cases} \tag{6}$$

Choose now an ample line bundle H on X ,L in $\mathrm{Pic}(X)$ such that $c_1(L) = c_1(E)$, $L_1 = H^{\otimes n}$, $L_2 = L \otimes H^{-n}$, and n big enough so that:

$$H^2(X, L_1 \otimes L_2^\vee) = 0, \text{ and}$$

$$L_1 \cdot L_2 = nL \cdot H - n^2 H^2 \leq c_2.$$

Then for Y a set of $c_2 - L_1 \cdot L_2$ simple points on X, an extension (5) gives a holomorphic vector bundle with the wanted invariants.

The conclusion of Theorem 1 is no longer true when X is non-algebraic. Indeed, letting

$$\Delta(E) := \Delta := \frac{1}{r}(c_2 - \frac{r-1}{2r} c_1^2)$$

be the **discriminant** of a vector bundle E over X having invariants $(c_1(E), c_2(E)) = (c_1, c_2) \in H^2(X, \mathbf{Z}) \times \mathbf{Z}$ and $r = \operatorname{rank} E$, Bănică and Le Potier obtained the following

Theorem 2 ([BL]) *Holomorphic vector bundles on non-algebraic surfaces have non-negative discriminants ($\Delta \geq 0$).*

Before stating their existence theorem for holomorphic structures we give a

Definition A rank r holomorphic vector bundle E on a surface X is called **filtrable** if its sheaf of sections (again denoted by E as before) admits a filtration

$$0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = E$$

by coherent subsheaves $\mathcal{F}_i$ such that

$$\operatorname{rank} \mathcal{F}_i = i.$$

Remark 3 If X is algebraic all holomorphic vector bundles on it are filtrable. (A non-zero section in $E \otimes H^{\otimes n}$, where H is an ample line bundle on X and $n >> 0$, yields a rank 1 coherent subsheaf of E. Take the quotient and repeat the procedure ...).

For a non-algebraic surface X, $a \in \mathrm{NS}(X)$ and r a positive integer consider the following rational number

$$m(r,a) = -\frac{1}{2r} \max \left\{ \sum_{i=1}^{r} \left(\frac{a}{r} - \mu_i \right)^2 \mid \mu_1, \ldots, \mu_r \in \mathrm{NS}(X) \text{ with } \sum_{i=1}^{r} \mu_i = a \right\}.$$

From Kodaira's theorem (§1.thm 2) we have

$$m(r,a) \geq 0.$$

Theorem 3 ([BL]) *A rank r topological complex vector bundle E on a non-algebraic surface X admits a filtrable holomorphic structure if and only if*

$$\begin{cases} c_1(E) \in \mathrm{NS}(X) \text{ and} \\ \Delta(E) \geq m(r, c_1(E)) \end{cases}$$

except when X is K3, $a(X) = 0$, $c_1(E) \in r\mathrm{NS}(X)$ and $\Delta(E) = \frac{1}{r}$. In this excepted case E admits no holomorphic structure.

The idea of the proof is to construct like before holomorphic vector bundles with given invariants as extensions, seeing that Serre's condition is fulfilled for the different cases given by the surface classification. In this way only filtrable bundles are obtained.

As a corollary, since $c_1(E) = 0$ implies $m(r, c_1(E)) = 0$, the existence problem is completely settled for the case $c_1(E) = 0$.

Chapter II

Irreducible and strongly irreducible vector bundles

§1. Irreducible vector bundles; a class of examples on two-dimensional tori

1.1.

A holomorphic vector bundle E on a surface X is called **irreducible** if its associated sheaf of sections is irreducible (cf.I.1.2). If E has rank 2, then this is equivalent to $h^0(E \otimes L) = 0$ for every L in $\mathrm{Pic}(X)$, the notion thus coinciding with non-filtrability; cf. the proof of the Proposition in I.1.2. If $\mathrm{rank}\, E = 3$, irreducibility amounts to $h^0(E \otimes L) = h^0(E^\vee \otimes L) = 0$ for every L in $\mathrm{Pic}(X)$.

On an algebraic surface every holomorphic vector bundle is filtrable (Remark 3 in I.2) and a fortiori reducible. This is no longer true in the non-algebraic case and a first example is provided by the holomorphic tangent bundle T_X of a $K3$ surface X with $\mathrm{NS}(X) = 0$, as one can easily check. Moreover in this case its symmetric power S^2T_X is a rank 3 irreducible vector bundle as was remarked by I. Coandā. This follows out of the fact that in this situation T_X is **simple** i.e. any non-zero endomorphism of it is an automorphism.

More generally we have:

Remark 1. Irreducible vector bundles are simple.

To check it consider any non-zero endomorphism $f : E \to E$. Since E is irreducible, $\ker f = 0$ and $\det f : \det E \to \det E$ must be an isomorphism, hence also f.

In [EF] Elencwajg and Forster showed the existence of irreducible rank 2 vector bundles on two-dimensional tori X with $\mathrm{NS}(X) = 0$.

They did this by comparing the versal deformation of a filtrable rank 2 vector bundle with the space parametrizing extensions producing filtrable vector bundles. In this way they proved that in general the versal deformation is richer hence it must contain also irreducible vector bundles.

Using the relative Douady space of quotients associated to the versal deformation of a filtrable vector bundle, Bănică and Le Potier showed the existence of irreducible vector bundles E in any rank on surfaces X such that $a(X) = 0$, covering a wide range of Chern classes $c_1(E)$, $c_2(E)$; cf.[BL]. They also showed the existence of irreducible rank 2 vector bundles on surfaces X with $a(X) = 1$ and trivial canonical bundles. Their proof can actually be extended to any surface X with $a(X) = 1$, making use in [BL; (8)] of the following simple

Proposition *Let X be a minimal non-algebraic surface. Then there is a constant c depending only on X, such that for any rank 1 torsion-free coherent sheaf $\mathcal{F}$ on X one has:*

$$h^0(\mathcal{F} \otimes K_X) - h^0(\mathcal{F}) \leq c.$$

Proof. For $a(X) = 0$ we can take $c = 1$ by I.1.2. Proposition. Let now $a(X) = 1$, $\pi : X \to S$, the unique elliptic fibration of X and $X_{s_1} = m_1F_1, \ldots, X_{s_k} = m_kF_k$ its multiple fibers. Then the canonical bundle formula gives ([BPV; V 12.3]):

$$K_X = \pi^*(L) \otimes \mathcal{O}_X(\sum_{i=1}^{k}(m_i - 1)F_i)$$

where L is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_S)$ on S. It follows that there exixts an effective divisor D on S of degree d depending only on X such that

$$K_X \hookrightarrow \pi^*\mathcal{O}_S(D).$$

Then

$$h^0(\mathcal{F} \otimes K_X) - h^0(\mathcal{F}) \leq h^0(\mathcal{F} \otimes \pi^*\mathcal{O}_S(D)) - h^0(\mathcal{F}) = h^0(S\ ;\ \pi_*\mathcal{F}(D)) - h^0(S\ ;\ \pi_*\mathcal{F})$$

Now $\pi_*\mathcal{F}$ is torsion free on S hence locally free. Its rank equals $h^0(\mathcal{F}|_F)$ where F is the general fibre of π. But $\mathcal{F}|_F$ is locally free for F general. Since $F^2 = 0$ we get by Corollary 1 in I 1.2.:

$$\deg(\mathcal{F}|_F) = \deg(\mathcal{F}^{\vee\vee}|_F) = c_1(\mathcal{F}) \cdot F = 0.$$

Hence

$$\operatorname{rank}(\pi_*\mathcal{F}) = h^0(\mathcal{F}|_F) \leq 1.$$

For $\operatorname{rank}(\pi_*\mathcal{F}) = 0$ any positive c will do. Let now $\operatorname{rank}(\pi_*\mathcal{F}) = 1$. Out of the exact sequence:

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{F}(D) \to \pi_*\mathcal{F}(D)|_D \to 0$$

we get

$$h^0(\pi_*\mathcal{F}(D)) - h^0(\pi_*\mathcal{F}) \leq \deg D = d.$$

Hence

$$h^0(\mathcal{F} \otimes K_X) - h^0(\mathcal{F}) \leq d. \ \square$$

Thus there exist irreducible vector bundles(of rank 2) on any non-algebraic surface.

One reason for investigating the existence of irreducible vector bundles is filling the gap left in the range of Chern classes of holomorphic vector bundles on a non-algebraic surface X. Indeed by the theorems 2 and 3 in I.2, for fixed rank r and first Chern class $c_1 \in \mathrm{NS}(X)$ the only unknown situations are for

$$\Delta \in [0, m(r, c_1)) \tag{7}$$

When $m(r, c_1) \neq 0$ this interval is non-empty and the canditates to "filling it" are non-filtrable vector bundles by theorem 3 in I.2. But all the known examples of irreducible or non-filtrable vector bundles admit also filtrable structures on their underlying topological types.

The main result of this paragraph will show in particular that this is not necessarily the case and values for discriminants of holomorphic vector bundles can appear also in the interval (7).

Theorem *Every complex topological vector bundle E on a two-dimensional complex torus X having $c_1(E) \in \mathrm{NS}(X)$ and $\Delta(E) = 0$ admits some holomorphic structure.*

1.2. Proof of the theorem

First a

Lemma *Let X be a two-dimensional complex torus, $a \in \mathrm{NS}(X)$ and p a prime number with $p \mid \frac{1}{2}a^2$. Then there exists an unramified covering $q : X' \to X$ of degree p and $a' \in \mathrm{NS}(X')$ such that*

$$pa' = q^*(a).$$

Proof. Let Γ be a lattice generated by $\gamma_1, \ldots, \gamma_4$ in $\mathbf{C}^2$ with $X \cong \mathbf{C}^2/\Gamma$. We use the notations of I.1.3. Recall the natural isomorphism (1):

$NS(X) \cong \{A \mid A \text{ hermitian } 2 \times 2 \text{ matrix with } \mathrm{Im}^t\Pi A\overline{\Pi} \in M_4(\mathbf{Z})\}$

where $\Pi := (\gamma_1, \ldots, \gamma_4)$ is the period matrix of X.

Consider as before for a 2×2 hermitian matrix A in $NS(X)$ the decomposition

$$\begin{pmatrix} A_1 & A_2 \\ -{}^tA_2 & A_3 \end{pmatrix} = \mathrm{Im}({}^t\Pi A\overline{\Pi})$$

and let

$$A_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}.$$

Then for the selfintersection of an element a in $\mathrm{NS}(X)$ represented by A as above we had the formula (2):

$$a^2 = 2(\alpha\delta - \beta\gamma - \theta\tau).$$

The hypothesis of the Lemma becomes:

$$p | (\alpha\delta - \beta\gamma - \theta\tau).$$

We shall consider tori X' appearing by factorizing $\mathbf{C}^2$ through lattices obtained by multiplying by p one of Γ's generators γ_i and preserving the others. The projection $q : X' \to X$ will be an unramified covering of degree p. If $\widetilde{\Pi}$ is the period matrix thus obtained for X' we will need to get:

$$\operatorname{Im}({}^t\widetilde{\Pi}A\overline{\widetilde{\Pi}}) \in M_4(p\mathbf{Z})$$

The element $\frac{1}{p}A \in \operatorname{NS}(X')$ would be the looked for a'. We denote

$$\begin{pmatrix} \widetilde{A_1} & \widetilde{A_2} \\ -{}^t\widetilde{A_2} & \widetilde{A_3} \end{pmatrix} = \operatorname{Im}({}^t\widetilde{\Pi}A\overline{\widetilde{\Pi}})\,.$$

Notice that if $\widetilde{\Pi}$ is obtained from Π multiplying by p column 1 or 2 (resp.3 or 4) then $\widetilde{A_1} \in M_2(p\mathbf{Z})$ (resp.$\widetilde{A_3} \in M_2(p\mathbf{Z})$) and line (resp.column) 1 or 2 of $\widetilde{A_2}$ will take values in $p\mathbf{Z}$. In order to reach our purpose (i.e. that $A_i \in M_2(p\mathbf{Z})$ for all $i \in \{1,2,3\}$) we will make a suitable base change for Γ. Another base of Γ, $(\gamma_i')_i = \overline{1,4}$, is related to the previous one by a matrix $M \in M_4(\mathbf{Z})$ with $\det M = \pm 1$:

$$\gamma_i' = \sum_j m_{ji}\gamma_j$$

giving the corresponding period matrix

$$\Pi' = \Pi M$$

hence

$$\begin{pmatrix} A_1' & A_2' \\ -{}^tA_2' & A_3' \end{pmatrix} = \operatorname{Im}({}^t\Pi' A\overline{\Pi}') = \operatorname{Im}({}^tM{}^t\Pi A\overline{\Pi}M) = {}^tM \begin{pmatrix} A_1 & A_2 \\ {}^t-A_2 & A_3 \end{pmatrix} M.$$

Writing $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d, \in M_2(\mathbf{Z})$ we get

$$A_1' = {}^taA_1a - {}^tc{}^tA_2a + {}^taA_2c + {}^tcA_3c$$

$$A_2' = {}^taA_1b - {}^tc{}^tA_2b + {}^taA_2d + {}^tcA_3d$$

$$A_3' = {}^tbA_1b - {}^td{}^tA_2b + {}^tbA_2d + {}^tdA_3d$$

From now on we reduce all computations modulo p, all equalities taking place in $\mathbf{Z}_p$. By assumption $\det A_2 - \tau\theta = 0$. We distinguish two cases:

i) $\det A_2 \neq 0$

ii) $\det A_2 = 0$

i) $\det A_2 \neq 0$
Then $\theta \neq 0$, $\tau \neq 0$. It will be enough, considered the form of $\tilde{A}$, to find

M giving one null line (say the first) of A_2' and $A_3' = 0$. Choose $c = 0$, $a = d = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} A_2' &= A_1 b + A_2, && (8) \\ A_3' &= {}^t b A_1 b - {}^t A_2 b + {}^t b A_2 + A_3, && (9) \\ A_1 b &= \theta \begin{pmatrix} b_3 & b_4 \\ -b_1 & -b_2 \end{pmatrix}, \text{ where } b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}. \end{aligned}$$

From (8) and the requirement that the first line of A_2' should be null we find:

$$b_3 = -\theta^{-1}\alpha, \quad b_4 = -\theta^{-1}\beta. \tag{10}$$

If $A_3' = \begin{pmatrix} 0 & \tau' \\ -\tau' & 0 \end{pmatrix}$, (9), (10) and the hypothesis imply

$$\tau' = \theta(b_1 b_4 - b_2 b_3) - \alpha b_2 - \gamma b_4 + \beta b_1 + \delta b_3 + \tau = 0$$

Hence b_1, b_2 can be arbitrarily chosen.

ii) $\det A_2 = 0$
Then $\theta = 0$ or $\tau = 0$. Assume $\theta = 0$ (the case $\tau = 0$ is similar) Choose $b = c = 0$, $a = I$. Then $A_1' = A_1 = 0$ and $A_2' = A_2 d$. It is enough to find d such that A_2' has a null column, the first for instance. If $d = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ this comes to $\begin{cases} \alpha d_1 + \beta d_3 = 0 \\ \gamma d_1 + \delta d_3 = 0 \end{cases}$.
Since $\det A_2 = 0$ this system admits non-trivial solutions (d_1, d_3). Moreover, one can find a solution with one coordinate equal to 1, say $d_1 = 1$. Then we choose $d_2 = 0$ and $d_4 = 1$ and get

$$d = \begin{pmatrix} 1 & 0 \\ d_3 & 1 \end{pmatrix}.$$

Returning to integer values we can choose in both cases (i and ii) representatives $M \in M_4(\mathbf{Z})$ with $\det M = \pm 1$ for the classes in $M_4(\mathbf{Z}_p)$ found above. One can take, for example all integer representatives from $\{0, 1, \ldots, p-1\}$. The Lemma is proved.

Now we prove the theorem using induction on the number n of prime factors of $r = \operatorname{rank} E$:

$$r = \prod_{i=1}^{k} p_i^{n_i}, \quad n = \sum_{i=1}^{k} n_i.$$

For $n = 0$ we have $r = 1$ and the statement is true since $c_1(E) \in \mathrm{NS}(X)$ by assumption. Assume now the theorem is true for n. We shall prove it for $n+1$. Let p be a prime factor of r.

$$0 = \Delta(E) = \frac{1}{r}(c_2(E) - \frac{r-1}{2r} c_1(E)^2) \quad \text{implies}$$

$p|\frac{1}{2}c_1(E)^2$ (the intersection form on NS(X) is even). By the Lemma there exists some unramified covering of degree p, $q : X' \to X$ and $a' \in$ NS(X') such that $pa' = q^*(c_1(E))$. X' is again a torus. Consider on X' the topological vector bundle F having

$$\mathrm{rank}(F) = \frac{r}{p}, \quad c_1(F) = a', \quad c_2(F) = \frac{\frac{r}{p} - 1}{2\frac{r}{p}} a'^2 = \frac{\frac{r}{p} - 1}{2r} c_1(E)^2 \in \mathbf{Z}.$$

($q^* : H^4(X, \mathbf{Z}) \cong \mathbf{Z} \to H^4(X', \mathbf{Z}) \cong \mathbf{Z}$ is the multiplication by p). Then $\Delta(F) = 0$ and F admits holomorphic structures by the induction hypothesis.

Let $G = \{1, \tau, \cdots, \tau^{p-1}\}$ be the deck-transformation group of X'/X (these are translations of X') and $E' = F \bigoplus \tau^*(F) \bigoplus \cdots \bigoplus (\tau^{p-1})^*(F)$.

Then $c_1(E') = pc_1(F) = pa' = q^*(c_1(E))$ $\quad (c_i(\tau^*(F)) = c_i(F)$ since τ is homotopous to 1), $c_2(E') = \frac{p(p-1)}{2} \quad c_1(F)^2 + pc_2(F) = p^2 \frac{r-1}{r} a'^2, \quad \Delta(E') = 0$.

One has canonical isomorphisms $E' \to (\tau^m)^*(E')$ compatible with the action of G on X' hence E' induces a holomorphic vector bundle E'' on X such that $q^*E'' = E'$. It follows that $\Delta(E'') = 0$ and $c_1(E'') = c_1(E)$, hence the underlying topological vector bundle of E'' is E, which closes the proof of the theorem.

1.3. Corollaries and Remarks

Let X be a compact complex surface, $a \in NS(X)$ and r a positive interger. We make the following notations:

$$s(r, a) := -\frac{1}{2} \sup_{\mu \in NS(X)} (\frac{a}{r} - \mu)^2,$$

$$t(r, a) := \inf \left\{ \frac{1}{k(r-k)} s(r, ka) \Big| \;\; k = \overline{1, r-1} \right\}$$

When X is non-algebraic these numbers are non-negative.

Remark 2. For X non-algebraic and E a filtrable bundle of rank r on it:

$$\Delta(E) \geq s(r, c_1(E)).$$

(This follows from Theorem 3 of I.2 and the inequality $s(r, a) \leq m(r, a)$.)

Remark 3. For X non-algebraic and E a reducible bundle of rank r on it:

$$\Delta(E) \geq t(r, c_1(E)).$$

Proof. Let $0 \to E_1 \to E \to E_2 \to 0$ be an exact sequence with E_i coherent sheaves without torsion of ranks r_i and having $c_1(E_i) = a_i$, $(i = 1, 2)$. Let $a = c_1(E)$. Then $a_1 + a_2 = a$, $r_1 + r_2 = r$ and by Riemann-Roch's formula for E, E_1 and E_2 we find

$$\Delta(E) = \frac{1}{2r} \left(\frac{a^2}{r} - \frac{a_1^2}{r_1} - \frac{a_2^2}{r_2} \right) + \frac{r_1}{r} \Delta(E_1) + \frac{r_2}{r} \Delta(E_2).$$

Since $\Delta(E_i) \geq 0$ (see[BL]) we have

$$\Delta(E) \geq \frac{1}{2r} \left(\frac{a^2}{r} - \frac{a_1^2}{r_1} - \frac{a_2^2}{r_2} \right) = -\frac{1}{2r_1 r_2} \left(\frac{r_2 a}{r} - a_2 \right)^2 \geq \frac{1}{r_2(r - r_2)} \; s(r, r_2 a) \geq t(r, c_1(E)).$$

Corollary 1 *Let X be a non-algebraic 2-torus, r a positive integer $a \in NS(X)$ such that $r | \frac{1}{2}a^2$ and $r^2 \not| \frac{1}{2}a^2$. Then there exists a topological vector bundle E on X having rank r, $c_1(E) = a$ and $\Delta(E) = 0$ admitting holomorphic structures but not filtrable holomorphic structures.*

Proof. We choose the topological vector bundle E having $c_1(E) = a$, $c_2(E) = \frac{r-1}{2r}a^2$. Hence $\Delta(E) = 0$ and E admits holomorphic structures by the theorem. Using Remark 2 it will be enough to prove that

$$s(r,a) > 0. \text{ If this were not so we'd have}$$

$$s(r,a) = 0 \text{ i.e.}$$

$$\sup_{\mu \in \mathrm{NS}(X)} (\frac{a}{r} - \mu)^2 = 0, \text{ hence } (\frac{a}{r} - \mu)^2 = 0$$

for some μ in $NS(X)$. This implies $a = r\mu + c$ with $c \in NS(X)$ and $c^2 = 0$. Then c is orthogonal on $NS(X)$ since X is non-algebraic (Corollary 1 to Theorem 2 in I.1). It follows that $a^2 = r^2\mu^2$ and $2r^2 \mid a^2$, a contradiction!

Corollary 2 *If X is a complex 2-torus and n a positive integer as in the examples A and B in I.1.3.,i.e. having $NS(X)$ freely generated by a with $a^2 = -2n$, or by a and b with $a^2 = -2n$ and $b^2 = 0$ respectively, then the topological vector bundle E on X of rank n having $c_1(E) = a$ and $\Delta(E) = 0$ admits holomorphic structures but not reducible structures.*

Proof. In this case $t(r,a) > 0$.

Remark 4. The vector bundles E contructed in the proof of the theorem are of the form $E = \pi_* L$ where $\pi : X' \to X$ is an unramified covering of degree r and $L \in \mathrm{Pic}(X')$. In these terms we get the following criterion of reducibility:

Remark 5. Let X be a complex 2-torus with $a(X) = 0$, $X' \xrightarrow{\pi} X$ a covering of degree r, G the covering transformation group, $L \in \mathrm{Pic}(X')$, $G' = \{g \in G \mid g^* L \cong L\}$ and $E = \pi_* L$. Then E is reducible if and only if $G' \neq id$. If r is prime:

$$E \text{ reducible } \Leftrightarrow E \text{ filtrable.}$$

Proof. If $G' \neq \{id\}$ one considers $\pi' : X' \to X'/G'$, $\pi'' : X'/G' \to X$.

Since $\pi_*(L) = \pi''_* \pi'_* L$ it is enough to check that $\pi'_* L$ is reducible. We may assume that G' is cyclic. Then $\pi'_* L = L' \otimes \pi'_* \mathcal{O}_{X'}$, where $L' \in \mathrm{Pic}(X'/G')$ is such that $\pi^*(L') = L$, hence the assertion.

Conversely, let $G' = \{id\}$ and assume E reducible. Then take a torsion free quotient C of E such that $0 < \mathrm{rank}\, C < r$, and the exact sequences

$$0 \to K \to E \xrightarrow{p} C \to 0$$

$$0 \to \pi^* K \to \pi^* E \to \pi^* C \to 0.$$

$$\pi^*E = \bigoplus_{g\in G} g^*L.$$

Let $i_g : g^*L \to \pi^*E$, $g \in G$, denote the cononical indusion. Then $u_g := \pi^*p \circ i_g$ are simultaneously (for $g \in G$) zero or non-zero, and they cannot be all zero since $0 \neq \pi^*p = \sum_{g\in G} u_g$. Take

$$H := \bigoplus_{\substack{g\in G\\ g\neq id}} g^*L, \quad v := \sum_{g\neq id} u_g : H \to \pi^*C, \quad u_{id} : L \to \pi^*C$$

and

$$p_1 : \pi^*E \to L, \quad p_2 : \pi^*E \to H,$$

the projections. We get in the same way as before $p_1(\pi^*K) \neq 0$. Then $u_{id} \circ p_1(\pi^*K) = v \circ p_2(\pi^*K)$ as subsheaves in π^*C, hence $u_{id}(p_1(\pi^*K)) \subset v(H)$.

Since $a(X) = 0$, this induces an inclusion $L \otimes \mathcal{J}_Z \hookrightarrow v(H)$ where Z is a 2-codimensional subspace of X. We get thus on $X \setminus Z$ a non-zero morphism $L \to H$ which extends to the whole X and hence a non-trivial morphism $L \to g^*L$ for some $g \neq id$. This is a contradiction.

I thank V. Brînzănescu for pointing out a mistake in the initial version of the above proof.

§2. Strongly irreducible vector bundles and associated threefolds without divisors

2.1.

Let X be a surface without curves and E a rank 2 holomorphic vector bundle on it. Then the projective bundle $\mathbf{P}(E)$ is a threefold whose only curves are the vertical lines of the fibering $\mathbf{P}(E) \to X$. But when does $\mathbf{P}(E)$ have no surface? It will turn out that this happens if and only if E is "strongly irreducible".

Definition We call a vector bundle E on a surface X **strongly irreducible** if for every "base change" $X' \xrightarrow{f} X$, meaning by this a proper holomorphic surjective map between surfaces, f^*E is irreducible.

In this paragraph X, X' will always denote compact complex surfaces while E will be a holomorphic vector bundle of rank 2 on X.

Lemma 1 *Let $X' \xrightarrow{f} X$, be a bimeromorphic mapping. Then E is irreducible (resp. strongly irreducible) on X if and only if f^*E is irreducible (resp. strongly irreducible) on X'.*

Proof. If $L \hookrightarrow E$ for L in $\mathrm{Pic}(X)$, then $f^*L \to f^*E$ is injective on a Zariski open set hence the image of this morphism is a coherent subsheaf or rank 1 in f^*E.

Conversely, let $L \hookrightarrow f^*E$, with L in $\mathrm{Pic}(X')$. Applying f_* we get an injection

$$f_*L \hookrightarrow f_*f^*E$$

where rank $f_*L = 1$. The natural morphism

$$E \to f_*f^*E$$

is an isomorphism on a Zariski open set so the inverse image of f_*L through it is a coherent subsheaf of rank 1 in E.

Coming now to the strong irreducibility, let $X'' \xrightarrow{g} X$ be a base change with g^*E reducible and $Y \to X'' \times_X X'$ a resolution of singularities. In the commutative diagram

$$\begin{array}{cccc}
Y & \sigma & & \\
 & & & \\
 & X'' \times_X X' & \longrightarrow & X' \\
\tau & \downarrow & & \downarrow f \\
 & X'' & \xrightarrow{g} & X
\end{array}$$

τ is bimeromorphic, hence $\tau^*g^*E = \sigma^*f^*E$ is reducible, and so f^*E is not strongly irreducible. The converse is obvious.

Consequently, the bimeromorphic mappings do not change the irreducibility of bundles. But this is no longer true for arbitrary base changes. Indeed, the irreducible vector bundles obtained in §1 (Corollary 2) are not strongly irreducible since they are direct images of line bundles through finite covering maps and hence their pullbacks through these maps will be decomposable.

The compact complex threefolds we study are projective bundles $\mathbf{P}(E)$ associated to holomorphic vector bundles E of rank 2 on X. We denote by $\pi : \mathbf{P}(E) \to X$ the natural projection and by $\mathcal{O}_{\mathbf{P}(E)}(-1)$ the tautological line subbundle in π^*E. In the sequel we use the standard notation $\mathcal{O}_{\mathbf{P}(E)}(n)$, $n \in \mathbf{Z}$, for its tensor powers. One has the following exact sequence on $\mathbf{P}(E)$:

$$0 \to \mathcal{O}_{\mathbf{P}(E)} \to \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \to T_{\mathbf{P}(E)} \to \pi^*T_X \to 0 \tag{11}$$

where the first morphism is induced by the inclusion $\mathcal{O}_{\mathbf{P}(E)}(-1) \hookrightarrow \pi^*E$. One also has $\mathrm{Pic}(\mathbf{P}(E)) \cong \mathrm{Pic}(X) \bigoplus \mathbf{Z}$, any invertible sheaf on $\mathbf{P}(E)$ being of the form $\pi^*L \otimes \mathcal{O}_{\mathbf{P}(E)}(n)$ for some L in $\mathrm{Pic}(X)$ and n in $\mathbf{Z}$. For $n > 0$ and $\mathcal{F} \in \mathrm{Coh}(X)$ the following isomorphisms are well known

$$\pi_*\mathcal{O}_{\mathbf{P}(E)}(n) \cong S^nE^\vee$$

$$\pi_*(\pi^*\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}(E)}(n)) \cong \mathcal{F} \otimes S^nE^\vee$$

where S^nE are the symmetric powers of E.

Definition A **horizontal divisor** of $\mathbf{P}(E)$ is an effective divisor in $\mathbf{P}(E)$ such that the restriction of π to its support covers X.

Proposition *For a non-singular compact complex surface X and a holomorphic vector bundle E of rank 2 on X the following statements are equivalent:*

1. *E is strongly irreducible.*

2. *$\mathbf{P}(E)$ does not admit horizontal divisors.*

3. *$h^0(L \otimes S^n E) = 0$ for all L in $\mathrm{Pic}(X)$ and all positive integers n.*

In the proof we shall use the

Lemma 2 *E is reducible if and only if $\mathbf{P}(E)$ admits a divisor whose projection on X is bimeromorphic.*

Proof. Let D be an irreducible divisor as above and $v : \overline{D} \to D$ a resolution of singularities. One has the diagram

$$\begin{array}{ccccc} \overline{D} & \xrightarrow{v} & D & \overset{i}{\hookrightarrow} & \mathbf{P}(E) \\ & & p = \pi|D & \searrow & \downarrow \pi \\ & & & & X \end{array}$$

Having $\mathcal{O}_{\mathbf{P}(E)}(-1) \hookrightarrow \pi^* E$ and applying $(i \circ v)^*$ we get an injective bundle morphism on $\overline{D}$

$$(i \circ v)^* \mathcal{O}_{\mathbf{P}(E)}(-1) \hookrightarrow (i \circ v)^* \circ \pi^* E$$

Hence $(p \circ v)^* E = (\pi \circ i \circ v)^* E$ is reducible and so E must be reducible according to Lemma 1.

Conversely, if E is reducible there is an element L of $\mathrm{Pic}\, X$ and a non-zero section $\mathcal{O} \hookrightarrow E \otimes L^{-1}$, having Z as zero divisor. Then

$$\mathcal{O} \overset{s}{\hookrightarrow} E \otimes L^{-1} \otimes \mathcal{O}(-Z)$$

is a section which vanishes on a finite or empty set $A \subset X$. On $X \setminus A$, $L \otimes \mathcal{O}(Z)$ is a line subbundle of E, hence it induces a section

$$X \setminus A \longrightarrow \mathbf{P}(E) \setminus \pi^{-1}(A).$$

The closure in $\mathbf{P}(E)$ of the image of this section is an analytic set (cf.[GR, Prop. 10.6.3.]) which defines a horizontal divisor whose projection on X, is bimeromorphic.

Proof of the Proposition "$1 \Longrightarrow 2$" Suppose D is a horizontal divisor of $\mathbf{P}(E)$. Then as in the proof of Lemma 2 one gets that $(p \circ v)^* E$ is reducible hence E cannot be strongly irreducible.

"$2 \Longrightarrow 1$" Suppose now there is base change $X' \xrightarrow{f} X$ such that $f^* E$ is reducible. One has the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(E) \times_X X' & \cong & \mathbf{P}(f^* E) & \xrightarrow{\overline{f}} & \mathbf{P}(E) \\ & & \pi' \downarrow & & \downarrow \pi \\ & & X' & \xrightarrow{f} & X \end{array}$$

where $\overline{f}$ is induced by the projection. By Lemma 2 there is a horizontal divisor D' in $\mathbf{P}(f^*E)$. Since $f \circ \pi'(D') = X$ it follows by commutativity, that $\pi_{|}\overline{f}(D') : \overline{f}(D') \to X$ is surjective hence $\overline{f}(D')$ is a horizontal divisor of $\mathbf{P}(E)$.

"1 $\Leftrightarrow$ 3" E is not strongly irreducible $\Leftrightarrow$ $E^\vee$ is not strongly irreducible $\Leftrightarrow$ $\mathbf{P}(E^\vee)$ admits horizontal divisors.

For a horizontal divisor D in $\mathbf{P}(E^\vee)$

$$\mathcal{O}(D) \cong \pi^* L \otimes \mathcal{O}_{\mathbf{P}}(E^\vee)(n)$$

with L in $\mathrm{Pic}(X)$ and $n > 0$. On the other side

$$\begin{aligned} H^0(\mathbf{P}(E^\vee), \pi^* L \otimes \mathcal{O}_{\mathbf{P}}(E^\vee)(n)) &\cong H^0(X, \pi_*(\pi^* L \otimes \mathcal{O}_{\mathbf{P}}(E^\vee)(n))) \\ &\cong H^0(X, L \otimes S^n E^{\vee\vee}) \\ &\cong H^0(X, L \otimes S^n E) \end{aligned}$$

and the wanted equivalence follows.

Remark. E strongly irreducible $\Rightarrow$ S^2E strongly irreducible.

Proof. Since $f^*(S^2E) \cong S^2(f^*E)$ it will be enough to prove that the strongly irreducibility of E implies the irreducibility of S^2E. For this we have to show that $h^0(S^2E \otimes L) = h^0((S^2E)^\vee \otimes L) = 0$ for all L in $\mathrm{Pic}(X)$. But $(S^2E)^\vee \cong S^2(E^\vee)$ and the conclusion follows using the Proposition for both E and $E^\vee$.

For later use we make another

Remark. An indecomposable reducible rank 2 vector bundle E on a surface X without divisors admits exactly one devissage (up to multiplying the morphisms by constants)

$$0 \longrightarrow L_1 \xrightarrow{\alpha} E \xrightarrow{\beta} L_2 \otimes J_Y \longrightarrow 0 \qquad (*)$$

where L_1, L_2 are in $\mathrm{Pic}(X)$ and Y is a 2-codimensional locally complete intersection subspace of X.

Proof. The existence was proven in I.2. Let $L \xrightarrow{\gamma} E$ be any non-trivial morphism from a line bundle into E (it induces a devissage of E). Then $\beta \circ \gamma = 0$, otherwise we'd get $L_1 \cong L_2$ (no divisors!) and a splitting of $(*)$. There exists then a non-zero morphism $\epsilon : L \to L_1$ such that $\alpha \circ \epsilon = \gamma$.

$$\begin{array}{ccc} L_1 & \xrightarrow{\alpha} & E \\ & \epsilon \nwarrow & \uparrow \gamma \\ & & L \end{array}$$

Now ϵ must be the multiplication by some constant hence the quotients E/L_1, E/L (and the devissages) are the same.

2.2. On the existence of strongly irreducible bundles

According to the proposition the existence of the compact analytic threefolds of type $\mathbf{P}(E)$ without divisors is ensured if and only if the base X has no curves and E is strongly irreducible. The following theorem gives an answer to the problem of existence of such bundles.

Theorem 4 *Let X be a K3-surface without divisors or a 2-dimensional torus without divisors. Then there exist strongly irreducible holomorphic vector bundles of rank 2 on X. More precisely:*

- *on K3-surfaces without curves every irreducible 2-bundle is strongly irreducible and for every pair $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbf{Z}$ verifying*

$$\Delta(c_1, c_2) \geq \max\left(m(2, c_1), 2 - m(2, c_1)\right)$$

there exist such bundles E with $c_i(E) = c_i$,

- *on tori without curves there exist strongly irreducible bundles E with Chern classes $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbf{Z}$ as soon as*

$$\Delta(c_1, c_2) \geq \sup\left(m(2, c_1), 1 - m(2, c_1)\right) .$$

Proof. Any base change $X' \to X$ has a factorization $X' \xrightarrow{g} \widetilde{X} \xrightarrow{h} X$ where $\widetilde{X}$ is normal, g has connected fibers and h is finite. Since the branch locus of h on $\widetilde{X}$ is purely 1-codimensional [F,p.170], if X has no curves it follows that h is a finite unramified covering, $\widetilde{X}$ nonsingular and g bimeromorphic. By lemma 1 we can restrict ourselves to the study of base changes which are finite unramified coverings. When X is K3, hence simply connected, these are trivial and the statement of the theorem follows (for existence see [BL,§5.10]).

Let now X be a 2-torus without curves with $X = \mathbf{C}^2/\Gamma$, Γ a lattice in $\mathbf{C}^2$, $\mathbf{C}^2 \to X$ the universal covering. Every finite unramified covering $f : X' \to X$ is obtained from the universal covering factorizing through a sublattice $\Gamma' \subset \Gamma$, where $X' \cong \mathbf{C}^2/\Gamma'$. Hence X' is a complex torus without curves. The condition $\Delta \geq m(2, c_1)$ ensures the existence of an extension on X

$$0 \to L_1 \to E \to L_2 \otimes \mathcal{J}_Z \to 0$$

where $L_1, L_2 \in \mathrm{Pic}(X)$, Z is a 2-codimensional subspace in X and E a locally free sheaf of rank 2 having Chern classes c_1, c_2 by theorem 3 of I.2.

We want f^*E to be simple (i.e. $\mathrm{End}(f^*E) \cong \mathbf{C}$) for any base change $f : X' \to X$ as above. Since X' has no curves this happens if and only if in the extension

$$0 \to f^*L_1 \to f^*E \to f^*L_2 \otimes \mathcal{J}_{f^*Z} \to 0 \tag{12}$$

one has $f^*L_1 \not\cong f^*L_2$ (this is easily verified; see however Lemma 1 of Ch. III). It is necessary, therefore, to have for every sublattice $\Gamma' \subset \Gamma$:

$$f^*\left(L_2^{-1} \otimes L_1\right) \not\cong \mathcal{O}_{X'}$$

where $f : X' \to X$ is the associated covering. If this doesn't happen we modify firstly L_1 by tensorizing it with a suitable bundle L_0 in $\mathrm{Pic}(X)$. There exists such an L_0 because we can choose, for example $L_2^{-1} \otimes L_1 \otimes L_0 \in \mathrm{Pic}^0(X) \cong \mathrm{Hom}(\Gamma, \{z \in \mathbf{C} \mid |z| = 1\})$ to correspond to an injective morphism $\alpha : \Gamma \to \{z \in \mathbf{C} \mid |z| = 1\}$ (it will remain injective hence non-zero on any sublattice), cf. [M], th. Appell-Humbert. Then we remark that after modifying L_1 as shown, $H^2(X, L_2^{-1} \otimes L_1) \cong H^0(X, L_2 \otimes L_1^{-1}) = 0$ ($L_1 \not\cong L_2$ and we have no curves), and this ensures the existence of a new extension with the required property.

The base $(S, 0)$ of the universal deformation $\mathcal{E} \to S \times X$, of $E = E_0$ (simple) will be smooth (see [EF],§3.6). Moreover, shrinking if necessary S around 0, we can assume that all bundles E_s, $s \in S$, are simple. It follows by Serre duality

$$\dim \operatorname{Ext}^2(E_s, E_s) = \dim \operatorname{Ext}^0(E_s, E_s) = 1$$

for $s \in S$, and by Riemann-Roch one gets $\dim \operatorname{Ext}^1(E_s, E_s)$ constant on S, hence equal to $\dim \operatorname{Ext}^1(E_0, E_0)$. This entails that the deformation $\mathcal{E} \to S \times X$ is versal at each $s \in S$. Therefore the conditions for S required in the proof of theorem 5.1. of [BL] are fulfilled (without having to leave the center $0 \in S$).

Let $D(\mathcal{E})$ be the relative Douady space of $\mathcal{E}$, $D \subset D(\mathcal{E})$ the open subset corresponding to the torsion-free quotients of rank 1 of E_s, $s \in S$, and $\pi : D \to S$ the projection. Let $s \in S$, and E'' in D quotient of E_S through a coherent subsheaf E' (E' will be a line bundle). One has the following exact sequence

$$0 \to \operatorname{Hom}(E', E'') \to T_{E''}D \xrightarrow{T_{E''}\pi} T_sS \xrightarrow{\omega_+} \operatorname{Ext}^1(E', E'')$$

where ω_+ is the composition

$$\begin{array}{ccc} T_sS & \xrightarrow{\omega} & \operatorname{Ext}^1(E_s, E_s) \\ & \omega_+ \searrow & \downarrow \\ & & \operatorname{Ext}^1(E', E'') \end{array}$$

and ω the Kodaira-Spencer morphism (see [BL;§5.5]). Moreover, in the chosen situation for S and Δ one shows that $T_{E''}\pi$ isn't surjective for any E'', fact which entails the existence of irreducible bundles on X (see [BL;§5.1]).

We fix a covering $f : X' \to X$ and consider the deformation

$$f^*\mathcal{E} \to X' \times S\,.$$

Since f^*E_0 is simple, we can choose a neighbourhood S' of 0 in S, such that f^*E_s are simple for $s \in S'$, S' is Stein and $H^2(S', \mathbf{Z}) = 0$. These conditions will be necessary later in order to apply a result of [EF]. Let D_f be the open set corresponding to the torsion-free quotients of rank 1 in the relative Douady space associated to the restriction of $f^*\mathcal{E}$ to S' and $\pi^f : D_f \to S'$ the projection. We denote $E' = L_1$, $E'' = L_2 \otimes \mathcal{J}_Z$ and we derive from (12), as above, an exact sequence

$$T_{f^*E''}D_f \xrightarrow{T_{f^*E''}\pi^f} T_0S \xrightarrow{\omega_+^f} \operatorname{Ext}^1(f^*E', f^*E'')\,.$$

We shall show that $T_{f^*E''}\pi^f$ isn't surjective or, equivalently, that $\omega_+^f \neq 0$. Using the natural commutative diagram

$$\begin{array}{ccccc} T_0S & \xrightarrow{\omega} & H^1(X, E^\vee \otimes E) & \xrightarrow{f^*} & H^1(X', f^*(E^\vee \otimes E)) \\ & \omega_+ \searrow & \downarrow & & \downarrow \\ & & H^1(X, E'^\vee \otimes E'') & \xrightarrow{f^*} & H^1(X', f^*(E'^\vee \otimes E'')) \end{array}$$

and the definitions through the double point $(0, \mathbf{C}[\epsilon])$, one easily gets

$$\omega^f = f^* \circ \omega \quad \text{and} \quad \omega_+^f = f^* \circ \omega_+ \,.$$

Since $\omega_+ \neq 0$ it is enough to prove that

$$f^* : H^1(X, E' \otimes E'') \to H^1(X', f^*(E' \otimes E''))$$

is injective. Let $\mathcal{F} = E'^* \otimes E''$. f^* is obtained by composition in the diagram

$$\begin{array}{ccc} H^1(X, \mathcal{F}) & f^* & \\ \downarrow & \searrow & \\ H^1(X, f_* f^* \mathcal{F}) & \xrightarrow{\sim} & H^1(X', f^* \mathcal{F}) \end{array}$$

hence we must only show that the vertical arrow is injective. Since the natural mapping $\mathcal{F} \to f_* f^* \mathcal{F}$ has a section $tr : f_* f^* \mathcal{F} \to \mathcal{F}$ there exists a section at H^1-level too, hence the wanted injectivity.

$T_{f^* E''} \pi^f$ not being surjective we deduce now that the morphism $D_f \xrightarrow{\pi^f} S'$ is not surjective in the following way. Assuming its surjectivity we would have $f^* E_s$ reducible and indecomposable for all s in S'. Then there would exist L, M in $\mathrm{Pic}(X' \times S')$, Y a 2-codimensional subspace in $X' \times S'$, flat over S' and an extension

$$0 \to L \to f^* \mathcal{E} \to M \otimes \mathcal{J}_Y \to 0 \tag{13}$$

whose restriction to each fiber $X' \times \{s\}$ is the uniquely determined devissage of E_s. This follows from [EF;th.2.3] (in order to have the morphism q biholomorphic on loc. cit. one needs $\mathrm{Pic}^0 X$ to be compact, which is the case in our situation). The sheaf $M \otimes \mathcal{J}_Y$ is S'-flat hence there exists an S'-morphism $\lambda : S' \to D_f$ such that (13) is the pull-back of the universal extension. In particular

$$\pi^f \circ \lambda = \mathrm{id}_{S'} \,. \tag{14}$$

$f^* E_s$ being indecomposable, they have at most one devissage, hence π^f is injective (even bijective in the hypothesis we made) and passing in (14) to the tangent morphism in 0 we get a contradiction.

$D_f \xrightarrow{\pi^f} S'$ not being surjective, there exist elements s in S such that $f^* E_s$ is irreducible. We want to show that the set N_f of elements of S which do not have this property is a countable union of proper analytic subsets of S. Let

$$R_f = \Big\{ (\xi, s) \in \mathrm{Pic}(X') \times S | H^0(X', \mathbf{P}_\xi \otimes f^* E_s) \neq 0 \Big\}$$

where $\mathbf{P}_\xi$ is the fiber in ξ of the Poincaré bundle $\mathbf{P}$ of X'. By Grauert's semi-continuity theorem, it follows that R_f is an analytic subset in $\mathrm{Pic}(X') \times S$. Let $p : R_f \to S$ be the morphism induced by projection. We have

$$N_f = p(R_f) \,.$$

Thus p isn't surjective, by the above fact.

$\mathrm{Pic}(X')$ is a countable union of connected components each isomorphic to $\mathrm{Pic}^0(X')$ which in its turn is a 2-dimensinal complex torus and therefore compact. The restriction of p to each such compact is proper, hence its image is a closed analytic set. It follows that N_f is a countable union of proper closed analytic subsets of S.

This closes the proof of the theorem because making the union of all N_f after all finite coverings $f : X' \to X$ (which form a countable set, up to isomorphisms) we find that the complementary set consisting of those s in S for which E_s is strongly irreducible is dense in S.

2.3. Remarks

1. The Chern numbers c_1^3, c_1c_2, c_3 of $\mathbf{P}(E)$ can be computed using (11) and one finds:

$$\begin{aligned} c_1^3 &= 2\left[c_1(E)^2 - 4c_2(E) + 3c_1(X)^2\right] \\ c_1c_2 &= 2\left[c_1(X)^2 + c_2(X)\right] \\ c_3 &= 2c_2(X) \end{aligned}$$

   Letting now X be either a torus as in example A of I.1.3. with $n = 1, 2$, or a special K3-surface of type g, $g = -2, -3$ (cf. I.1.3), and letting $c_1(E)$ be in all cases a generator for $\mathrm{NS}(X)$, we obtain from the theorem the following region in the "geography" of threefolds without divisors:

	c_1^3	c_1c_2	c_3
X a torus	$-4k$, k integer ≥ 2	0	0
X K3	$-4k$, k integer ≥ 4	48	48

2. If E is as in the theorem then $h^0(S^nE) = 0$ for all $n > 0$. In particular, for X a K3-surface with $\mathrm{NS}(X) = 0$, since T_X is irreducible, hence strongly irreducible, we have
$$h^0(S^nT_X) = 0, \quad \text{for all} \quad n > 0\,.$$

3. We couldn't obtain examples of strongly irreducible bundles on any compact complex surface without curves. Indeed, the only case left, that of the surfaces of class VII, doesn't admit an analogous proof, because here $\mathrm{Pic}^0 X \cong \mathbf{C}^*$ isn't compact.

4. It is easy to get examples of strongly irreducible bundles on some surfaces having divisors (for all surfaces whose minimal model is as in the theorem, K3 or torus without curves, by lemma 1).

Chapter III

Simple reducible vector bundles

§1. Existence theorem for simple reducible rank 2 vector bundles

Simple holomorphic vector bundles admit coarse moduli spaces which are only locally Hausdorff in general; cf.[N2],[N1]. A question which arises is to decide when these moduli spaces are not empty, hence the question of existence of simple vector bundles.

Unlike irreducible vector bundles which are always simple (cf. Remark 1 in II. 1.1.), reducible ones may have many endomorphisms in general.

In this paragraph we determine the range of Chern classes c_1, c_2 of reducible simple rank 2 vector bundles on minimal surfaces of algebraic dimension zero, extending the results for surfaces without divisors of [BF2]. Apart from some precise exceptions this range will be the same as that of filtrable rank 2 vector bundles given by Theorem 3 of I.2.. Determining the above range is equivalent to determining which topological rank 2 vector bundles admit simple reducible holomorphic structures as was pointed out in I.2..

According to the classification a minimal surface X of algebraic dimension zero can only belong to one of the classes:

1. tori
2. class VII surfaces
3. K3 surfaces.

We recall the notations of I.2. and add some more. First, we shall denote $m(2,a)$ shortly by $m(a)$, for a in $NS(X)$. Note that

$$m(a) = \inf M(a)$$

where $M(a) := \{-\frac{1}{2}(\frac{a}{2} - \mu)^2 : \mu \in NS(X)\}$. Next, we set

$$m'(a) := \begin{cases} \inf(M(a) \setminus \{m(a)\}), & \text{if } M(a) \neq \{m(a)\} \\ \infty, & \text{if } M(a) = \{m(a)\} \end{cases}$$

Let now E denote a holomorphic rank 2 vector bundle on a surface X, henceforth called simply "bundle". For a bundle E one has $2(\Delta(E) - m(c_1(E))) \in \mathbf{Z}$. One can immediately see that $M(c_1) = 0$ for $c_1 \in 2NS(X)$ and $m'(0) = \infty$ when $NS(X) = 0$. (For more examples see Remark 2 in this paragraph). We can now state our result.

Theorem *If X is a minimal surface with $a(X) = 0$ there exists a simple reducible rank 2 vector bundle on X having Chern classes $c_1 \in NS(X)$, $c_2 \in \mathbf{Z}$, if and only if*

$$\Delta(c_1, c_2) \geq m(c_1),$$

excepting exactly the following cases:

1. *if X is a torus*
$$\Delta(c_1, c_2) = m(c_1) = 0,$$
2. *if X is in class VII*
$$\Delta(c_1, c_2) = m(c_1) = 0,$$
unless $b_2(X) = 0$, X without divisors and $c_1 \in 2NS(X)$,
3. *if X is a K3 surface*
$$m(c_1) = 0 \text{ and } 0 \leq \Delta(c_1, c_2) < \sup\{m'(c_1), 2\},$$
$$\Delta(c_1, c_2) = m(c_1) = \frac{1}{4},$$
$$\Delta(c_1, c_2) = m(c_1) = \frac{1}{2}$$

The proof will be given in the following paragraph. We end this one by making some remarks about the statement.

Remark. 1 Since for tori X with $a(X) = 0$ the intersection form on $NS(X)$ is negative definite (Theorem 6 in I.1.), we have $m(c_1) = 0$ if and only if $c_1 \in 2NS(X)$. Thus the only topological rank 2 vector bundles admitting reducible but not simple reducible holomorphic structures are the twistings of the trivial rank 2 bundle by elements in $NS(X)$.

Remark. 2 Using the remarks on K3 sufaces made in I.1.3., one derives the following consequences for X in this class.

If $m(c_1) = \frac{1}{4}$ then X admits divisors.

When $m(c_1) = 0$ it follows $c_1 \in 2NS(X)$ and $m'(c_1) < 2$ if and only if there are curves on X (in this case $m'(c_1) = 1$).

When X is a special K3 surface of type g (see I.1.3.), and $g < 0$, one gets

$$\sup\{m'(0), 2\} = m'(0) = 1 - g$$

and $m(c_1(L)) = \frac{(1-g)}{4}$.

One can get an even wider domain of exceptions when $NS(X) = 0$. Since in this case $m'(0) = \infty$, there are no simple reducible bundles (this can also be seen directly).

Remark. 3 Comparing our result with the range of Chern classes given by the existence theorem for irreducible bundles of [BL;§5](see also the theorem in II.2.2.), we find that there exist topological vector bundles admitting simple holomorphic structures and reducible holomorphic structures but no simple reducible holomorphic structures. This happens for instance, on special K3 -surfaces of type g with $g < -1$, for $c_1 = 0$ and $2 \leq \Delta < 1 - g$. It never happens if X is a torus with $a(X) = 0$ (use also [BL] Prop. 4.7.).

Remark. 4 On the other hand there exist topological vector bundles admitting simple reducible holomorphic structures but not irreducible structures, on tori and K3-surfaces of zero algebraic dimension.

This can be seen using Prop. 4.3. and 4.7. in [BL] and our theorem. The simplest examples one gets are on tori for $c_1 = 0$, $c_2 = 1$ and on K3-surfaces with curves for $c_1 =$ the class of a (-2)-curve and $c_2 = 1$ or 2.

§2. Proof of the theorem

2.1.

A rank 2 holomorphic vector bundle E on surface X is reducible if and only if it admits a devissage

$$0 \longrightarrow L_1 \xrightarrow{\alpha} E \xrightarrow{\beta} L_2 \otimes \mathcal{J}_Y \longrightarrow 0 \quad (*)$$

where L_1, $L_2 \in \mathrm{Pic}(X)$ and Y is a 2-codimensional subspace of X or empty, as we have seen in I.2.. To construct simple reducible bundles we will use the method of extensions described in I.2.. We want to decide when a reducible bundle given by an extension $(*)$ is simple. A first criterion is

Lemma 1 *i) If E is simple then $H^0(L_2^\vee \otimes L_1) = 0$.*

ii) If $H^0(L_2^\vee \otimes L_1) = H^0(L_1^\vee \otimes L_2 \otimes \mathcal{J}_Y) = 0$ and $(*)$ doesn't split then E is simple. In particular if X has no divisors E is simple if and only if $L_1 \not\cong L_2$ and $(*)$ doesn't split.

Proof.

i) For a non-zero section φ in $H^0(L_2^\vee \otimes L_1)$ the composition

$$E \xrightarrow{\beta} L_2 \otimes \mathcal{J}_Y \hookrightarrow L_2 \xrightarrow{\varphi} L_1 \xrightarrow{\alpha} E$$

would give a non-constant endomorphism of E.

ii) It is enough to show that there are no non-zero non-invertible elements in $\mathrm{End}(E)$.

Assume now $\epsilon \in \mathrm{End}(E)$ is such an element. Since $\beta\circ\epsilon\circ\alpha = 0$ we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & L_1 & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & L_2 \otimes \mathcal{J}_Y & \longrightarrow 0 \\ & \gamma \downarrow & & \downarrow \epsilon & & \downarrow \delta & \\ 0 \longrightarrow & L_1 & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & L_2 \otimes \mathcal{J}_Y & \longrightarrow 0 \end{array}$$

where γ and δ are homotheties or zero. Using the Ker-Coker Lemma one finds that γ and δ cannot be simultaneously isomorphisms nor simultaneously zero.

If $\delta = 0$, $\gamma \neq 0$ there exists $\psi : E \to L_1$ such that $\alpha\circ\psi = \epsilon$. Hence $\alpha\circ\gamma = \epsilon\circ\alpha = \alpha\circ\psi\circ\alpha$ which gives $\gamma = \psi \circ \alpha$ and $(*)$ splits. In the same way we exclude the case $\gamma = 0$, $\delta \neq 0$.

Remark. The tensor product of a devissage of E with a line bundle L gives a devissage of $E \otimes L = E'$ and one has

$$\Delta(E') = \Delta(E)$$
$$c_1(E') = c_1(E) + 2c_1(L)$$
$$m(c_1(E')) = m(c_1(E))$$
$$m'(c_1(E')) = m'(c_1(E))$$

so we have to consider only the classes $c_1 + 2NS(X)$ of c_1 modulo $2NS(X)$. In particular it will be enough for our problem to consider devissages with L_2 trivial:

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0 \qquad (15)$$

More precisely, a topological rank 2-vector bundle characterized by (c_1, Δ) admits simple reducible holomorphic structures if and only if there exists some simple holomorphic bundle given by an extension of type (15) having the same discriminant Δ, and first Chern class congruent modulo $2NS(X)$ with c_1.

From (15) and (6) one derives

$$c_1(E) = c_1(L)$$
$$c_2(E) = l(Y).$$

Finally note that if E is given by (15) and $\Delta(E) = m(c_1(E))$, then $Y = \emptyset$ and $L^2 = -8m(c_1)$. Indeed $\Delta(E) = \frac{1}{2}(l(Y) - \frac{1}{4}L^2) \geq -\frac{1}{8}L^2 \geq m(c_1(E))$.

2.2. Proof of the case: X a torus

Let X be a 2-torus of algebraic dimension zero. Then X has no divisors by theorem 3 of I.1. and the fact that X is homogeneous.

Consider first $c_1 \in NS(X)$, $c_2 \in \mathbf{Z}$, such that

$$\Delta(c_1, c_2) \geq m(c_1) > 0.$$

Let L in $Pic(X)$ be such that

$$c_1(L) \in c_1 + 2NS(X), \text{ and}$$

$$m(c_1) = -\frac{1}{2}\left(\frac{1}{2}c_1(L)\right)^2.$$

Since L is non-trivial and X has no divisors we have $H^2(L) \cong H^2(L^{-1}) = 0$ and there exist extensions (15) with E locally free having the needed Chern classes if Y is a union of $2(\Delta - m(c_1))$ distinct simple points (cf. Remark 1 in I.2.). Moreover, by Riemann-Roch,(4)

$$h^1(L) = -\frac{1}{2}L^2 = 4m(c_1) > 0,$$

hence also in the case $Y = \emptyset$ one has non-trivial extensions (15).

By Lemma 1 in these situations we get the middle term E simple.

By Theorems 2 and 3 of I.2. all that is left is to show that for $c_1 \in NS(X)$, $c_2 \in \mathbf{Z}$ such that

$$\Delta(c_1, c_2) = m(c_1) = 0$$

there exist no simple reducible bundles with these Chern classes.

But now Remark 1 from §1 shows that $c_1 \in 2NS(X)$ and by the Remark in 2.1. we need only to verify that if E is given by an extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

with L in $\mathrm{Pic}^0(X)$, then E is not simple. This is obvious when L is trivial, whereas in the other case all such extensions split since $h^1(L) = 0$ by Riemann-Roch.

2.3. Proof of the case X in class VII

Let now X be in class VII, $c_1 \in NS(X)$, $c_2 \in \mathbf{Z}$. Let $\Delta = \Delta(c_1, c_2)$.

a) If $\Delta > m(c_1)$ choose L_1 in $\mathrm{Pic}(X)$ such that $c_1(L_1) \in c_1 + 2NS(X)$ and $m(c_1) = -\frac{1}{2}(\frac{1}{2}c_1(L_1))^2$. Twist L_1 by L_0 in $\mathrm{Pic}_0(X)$ in order to have for $L = L_1 \otimes L_0$:

$$H^0(L^\vee \otimes K) = H^0(L) = H^0(L^\vee) = 0.$$

This is possible since $\mathrm{Pic}_0(X) \cong \mathbf{C}\setminus\{0\}$ and the elements in $\mathrm{Pic}(X)$ admitting non-trivial sections form a countable subset. As we have said, we can assume $c_1 = c_1(L)$. The assumption $\Delta > m(c_1)$ implies $c_2 > 0$. The condition $H^0(L^\vee \otimes K) = 0$ ensures the existence of an extension (15) with E locally free having the wanted Chern classes if Y is the union of c_2 distinct simple points. The other two vanishings ensure the simplicity of E by Lemma 1.

b) If $\Delta = m(c_1) > 0$ take L as before, and $Y = \emptyset$. One has $\Delta = -\frac{1}{8}c_1(L)^2 = m(c_1)$. Then, by Riemann-Roch's formula

$$h^1(L) = 4\Delta + \frac{1}{2}L \cdot K$$

$$h^1(L^\vee) = 4\Delta - \frac{1}{2}L \cdot K + h^0(L \otimes K)$$

and at least one of these numbers is positive. Consider a corresponding non-trivial extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0 \quad \text{or}$$
$$0 \longrightarrow L^{\vee} \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

which will give a bundle E of the wanted type.

c) Let now $\Delta = m(c_1) = 0$.

First we shall show that if E is a simple reducible vector bundle of rank 2 having these Chern classes, then necessarily $b_2(X) = 0$, X has no divisors and $c_1 \in 2NS(X)$. We can assume that E has a devissage of type (15). The hypothesis implies $Y = \emptyset$ and $c_1(L)^2 = 0$. Thus E is the middle term of a non-trivial extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0 \tag{16}$$

with $L^2 = 0$,$h^0(L) = 0$ (as E is simple) and $h^0(L^{\vee} \otimes K) = 1$. For this last fact, first deduce $L \cdot K = 0$ from $L^2 = 0$ (examine $(K + mL)^2, n \in \mathbf{Z}$), then apply Riemann-Roch for L.

Thus $K = L(\sum_{i=i}^{k} r_i C_i)$ with r_i non-negative integers and C_i irreducible curves on X. If $K \cdot C_i < 0$ then $C_i^2 < 0$ (examine again $(K + nC_i)^2$) hence C_i is exceptional which is absurd. It follows that $K^2 = K \cdot L + \sum_{i=1}^{k} r_i K \cdot C_i \geq 0$. But using Noether's formula, $\mathcal{X}(\mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X))$ and the fact that $b_1(X) = 1$ implies $h^1(\mathcal{O}_X) = 1$,([BPV; IV.2.6.]), we get for our surfaces $b_2(X) = -K^2$, hence $b_2(X) = 0$.

If X has no curves it must be an Inoue surface ([LYZ]), and one can see in this case using the explicit universal coverings that $c_1(X) = 0$. Now $K = L$ (since $h^0(K^{\vee} \otimes L) = 1$!), hence $c_1 = c_1(X) = 0$ and the statement is proved.

Assume now X admits divisors. We'll show that this leads to a contradiction.

Since $a(X) = 0$, $b_1(X) = 1$, $b_2(X) = 0$, X must be a Hopf surface by theorem 5 in I.1.3.. Hence one has the following form for the canonical bundle:

$$K = \begin{cases} \mathcal{O}(-C_1 - C_2), \text{if } X \text{ admits two irreducible curves } C_1, C_2, \\ \mathcal{O}(-(m+1)C), \text{if } X \text{ admits only one irreducible curve } C \end{cases}$$

where m is a positive integer depending on the transformation group on $\mathbf{C}^2 \setminus \{0\}$ giving X. Denote $D_1 = \sum_{i=1}^{k} r_i C_i \geq 0$, so that $L = K(-D_1)$. There are divisors $D_2 > 0$, $D_3 > 0$ such that $K = \mathcal{O}(-D_2 - D_3)$. We have a commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{O} \\ \searrow & & \nearrow \\ & \mathcal{O}(-D_2) & \end{array}$$

where all the arrows are natural inclusions. Passing to cohomology in dimension one , since $H^1(\mathcal{O}(-D_2)) = 0$ by Riemann-Roch (recall $b_2(X) = 0$), one finds that the natural map

$$H^1(L) \longrightarrow H^1(\mathcal{O})$$

is zero.

Since (16) is non-trivial the connecting homomorphism in

$$0 \longrightarrow H^0(L) \longrightarrow H^0(E) \longrightarrow H^0(\mathcal{O}) \xrightarrow{\delta} H^1(L) \longrightarrow H^1(E) \quad (17)$$

is non-trivial and $H^0(E) = H^0(L) = 0$. Twisting (16) by $L^\vee$ one finds

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & E^\vee & \longrightarrow & L^\vee & \longrightarrow & 0 \end{array}$$

with vertical arrows given by functoriality by the natural inclusion $\mathcal{O} \hookrightarrow L^\vee = \mathcal{O}(D_1 + D_2 + D_3)$. Hence

$$\begin{array}{ccccccccc} & & & & & & H^0(L^{\mathcal{O}}) & \overset{\delta}{\underset{\sim}{\to}} & H^1(L) \\ & & & & & & \downarrow \wr & & \downarrow \\ 0 & \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & H^0(E^\vee) & \longrightarrow & H^0(L^\vee) & \longrightarrow & H^1(\mathcal{O}) \end{array}$$

is commutative. (Counting dimensions shows that δ and the first vertical map are isomorphisms). Since the second vertical map is zero we obtain

$$h^0(E^\vee) = h^0(\mathcal{O}) + h^0(L^\vee) = 2$$

Any element in $H^0(E^\vee)$ induces twisting (16) by $E^\vee$ a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & E \otimes E^\vee & \longrightarrow & E^\vee & \longrightarrow & 0 \end{array}$$

Hence

$$\begin{array}{ccccccc} & & & & H^0(\mathcal{O}) & \xrightarrow{\delta} & H^1(L) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(E \otimes E^\vee) & \longrightarrow & H^0(E^\vee) & \longrightarrow & H^1(E) \end{array} \quad (18)$$

is commutative. We want to prove that the arrow $H^0(E^\vee) \to H^1(E)$ in (18) is zero thus finding $h^0(E \otimes E^\vee) = 2$, a contradiction.

Let α_1, α_2 linearly independent in $H^0(E^\vee)$ with α_1 induced by the morphism $L \to E$ of (16). By (17) the morphism $H^1(L) \xrightarrow{H^1(\alpha_1)} H^1(E)$ is zero. Denote by β_1 the morphism $E \to \mathcal{O}$ appearing in (16). Then $\beta_1 \circ \alpha_2 \neq 0$ hence$\beta_1 \circ \alpha_2$ must be proportional to the natural inclusion $L \to \mathcal{O}$. We denoted also by α_2 the morphism $L \to E$ inducing $\alpha_2 \in H^0(E^\vee)$. Thus $H^1(\beta_1 \circ \alpha_2) = 0$ as we have already seen. Since by (17), $H^1(\beta_1)$ is injective we find that $H^1(L) \xrightarrow{H^1(\alpha_2)} H^1(E)$ is zero.

As α_1, α_2 generate $H^0(E^\vee)$, it follows from (18) that the arrow $H^0(E^\vee) \to H^1(E)$ must be null and E is not be simple: a contradiction.

d) When $\Delta = m(c_1) = 0$, X in class VII without divisors, $b_2(X) = 0$ and $c_1 \in 2NS(X)$ one derives the existence of simple reducible holomorphic structures out of a non-trivial extension

$$0 \longrightarrow K \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

(use $h^1(K) = 1$, Lemma 1 and the fact we already mentioned that for these surfaces $c_1(X) = 0$).

2.4. Proof of the case: X a K3 surface

For X a K3 surface with $a(X) = 0$ we shall use the properties already mentioned in the Remarks of I.1.3. We also use the fact that in this case ($a(X) = 0$) an isomorphism $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ implies $D_1 = D_2$ for two divisors D_1,D_2.

Let $c_1 \in NS(X)$, $c_2 \in \mathbf{Z}$ and $\Delta = \Delta(c_1, c_2)$. One sees that $4m(c_1)$, 4Δ and $2(\Delta - m(c_1))$ are integers. We shall subdivide the proof into cases depending on the values of $m(c_1)$ and Δ. These cases cover the interesting range in the following way:

	0	$\frac{1}{4}$	$\frac{1}{2}$	$> \frac{1}{2}$
0	c	c	c	c
$\frac{1}{2}$	e	d	b	b
1 and $\frac{3}{2}$	f and g	a	a or b	a or b
≥ 2	h	a	a or b	a or b

a) $m(c_1) > 0$, $\Delta \geq m(c_1) + 1$.

We can choose $L \in \operatorname{Pic}(X)$ such that $c_1(L) \in c_1 + 2NS(X)$ and $L^2 = -8m(c_1)$. Then L is not trivial hence at least one of $H^0(L)$, $H^0(L^\vee)$ must be zero. Assume $H^0(L) = 0$, otherwise replace L by $L^\vee$. Take Y a set of $2(\Delta - m(c_1))$ simple points on X. If $H^0(L^\vee) \neq 0$ one has $L^\vee = \mathcal{O}(D)$ for some divisor $D > 0$ and we can assume the points of Y do not belong to supp$D$. When $H^0(L^\vee) \neq 0$ we have no more restrictions on the choice of Y.

In both cases an extension of type (15) produces a locally free sheaf E by Remark 1 in I.2. (note that $H^2(L) \cong H^0(L^\vee)$ by duality).

Moreover E has the wanted Chern classes and it is simple by Lemma 1.

b) $\Delta > m(c_1) > \frac{1}{4}$

We choose L in $\operatorname{Pic}(X)$ such that $c_1(L) \in c_1 + 2NS(X)$ and $L^2 = -8m(c_1)$. We shall show that this choice can be made such that $H^0(L) = H^0(L^\vee) = 0$; this is clearly true when L is not divisorial. Then taking Y the union of $2(\Delta - m(c_1))$ simple points on X, the extension (15) produces a simple locally free sheaf E with the requested Chern classes by Remark 1 in I.2. and Lemma 1.

The existence of an element L in $\operatorname{Pic}(X)$ with the properties we need is a consequence of the following facts:

Lemma 2 *Let R be a reduced irreducible root system in the $\mathbf{R}$-vector space V, of type A, D or E, let $B = (\alpha_i)_i$ be a basis for R and $Q(R)$ the subgroup of V generated by the vectors in R. Then for any $x \in Q(R)$ there is some $y \in x + 2Q(R)$ such that if $y = \sum_i y_i \alpha_i$ with $y_i \in \mathbf{Z}$ the following conditions are satisfied:*

i) $< y, y > \leq < x, x >$

ii) y is a root or one of its coordinates y_i is zero.

Corollary *If D is a divisor on X with $D^2 = -8m(D) < -2$ then there exists some divisor C on X with $C \in D + 2NS(X)$, $C^2 = D^2$ and $H^0(\mathcal{O}(C)) = H^0(\mathcal{O}(-C)) = 0$ (i.e. C is neither positive nor negative).*

Proof of the corollary One can assume supp$D$ connected and D positive otherwise it's easy. For example if $D = D_1 + D_2$ with $D_1 \cdot D_2 = 0$, $D_1 > 0$, $D_2 > 0$ we can choose $C = D - 2D_2 = D_1 - D_2$.

Since the intersection form on $\mathrm{Div}(X)$ is negative definite supp$D$ is an A-D-E curve (cf. Remark iii in I.1.3). Let R_D be the root system of the type given by supp$D$, the roots corresponding to the irreducible components of D forming a basis B, with the opposite of the intersection form as scalar product.

Now Lemma 2 produces some divisor $D' \in D + 2NS(X)$ with $D'^2 \geq D^2$, such that D' is a root or suppD' is strictly contained in supp$D$. The equality $D^2 = -8m(D)$ and the definition of $m(D)$ imply $D'^2 = D^2$. It follows that D' is not a root (in the A-D-E case all roots have the same length), hence suppD' is strictly contained in supp$D$.

If suppD' is not connected we are over. If it is , one repeats the same argument starting with D' and so on. Finally we get a divisor with a non-connected support. (Otherwise we get a divisor of the form kC with C a (-2)-curve which implies $m(D) = 0$ or $\frac{1}{4}$).

Proof of Lemma 2 First we can assume that x is minimal in the following sense: for every $x' \in x + 2Q(R)$, $< x', x' > \geq < x, x >$.

Let $x = \sum x_i \alpha_i$. Assume also that all x_i are non-zero and x is not a root, otherwise one can take $y = x$.

In particular supp$x$ is connected (supp$x$ has the meaning which can be adapted to divisors as above) and x or $-x$ is positive (i.e. all its coordinates are positive). Indeed, if $x = x_+ - x_-$, where x_+ (resp x_-) are its positive (resp. negative) part, then

$$< x_+ + x_-, x_+ + x_- > = < x_+ - x_-, x_+ - x_- > + 4 < x_+, x_- >$$

But $< x_+, x_- > \leq 0$ and we have even equality by the minimality of x. Then x_+ or x_- must be null by connectedness.

We shall use the following modulo 2 reduction:

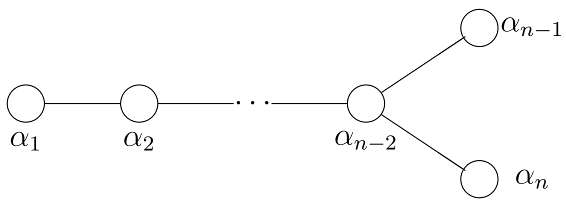
(R): replace the coordinates of x by 0 or 1 according to their parity.

If R is of type A_n the reduction (R) solves our problem. Indeed, in this case

$$< x, x >= x_1^2 + (x_1 - x_2)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2$$

and one immediately sees that by (R) we obtain at most $\frac{1}{2} < x, x >$ connected components, each such component being a root.

Let now R be of type D_n:



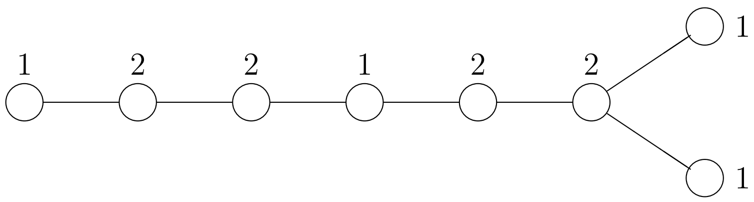
$$\begin{aligned} < x, x > &= x_1^2 + (x_1 - x_2)^2 + \cdots + (x_{n-3} - x_{n-2})^2 + \\ &+ \quad (x_{n-1} + x_n - x_{n-2})^2 + (x_{n-1} - x_n)^2. \end{aligned}$$

If x_{n-2} is odd or one of x_{n-1}, x_n are even one applies (R) and reasons as above.

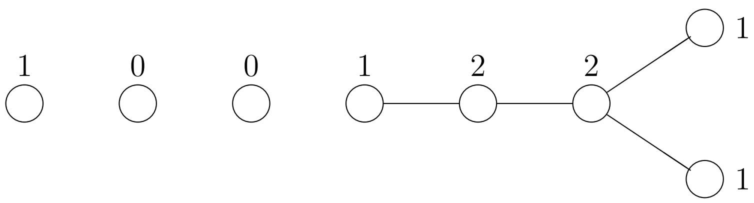
If x_{n-2} is even and x_{n-1}, x_n are odd apply the following reduction:

(R'): replace all odd x_i by 1 and all even ones by zero excepting x_{n-2} and all chains of even coordinates connected to x_{n-2} which are replaced by 2.

For example

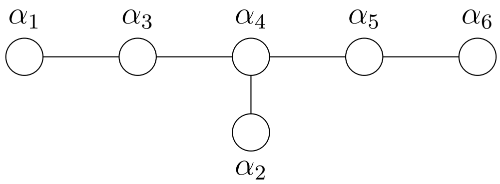


is reduced in this way to



Looking at the quadratic form $< x, x >$ one sees that (R') solves the problem in this case since it doesn't increase $< x, x >$ and all the connected components obtained will be roots.

The E_6-case



Since after (R) one obtains at most 3 connected components the only difficult case is $< x, x >= 4$. Noting that

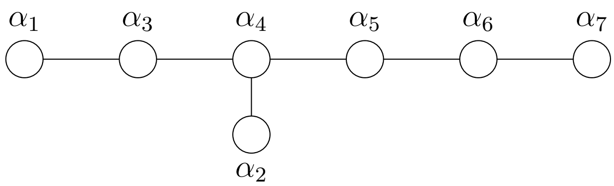
$$\begin{aligned} < x, x > &= (\tfrac{1}{2}x_1 + x_2 - x_3)^2 + (-\tfrac{1}{2}x_1 + x_2 + x_3 - x_4)^2 + \\ &+(-\tfrac{1}{2}x_1 + x_4 - x_5)^2 + (-\tfrac{1}{2}x_1 + x_5 - x_6)^2 + \\ &+(-\tfrac{1}{2}x_1 + x_6)^2 + \tfrac{3}{4}x_1^2 \end{aligned}$$

one can verify that (R) solves the problem in this case too. We omit the computations. In order to make them shorter one can also assume that

$$S_i \in [2x_i - 1, 2x_i + 2]$$

where S_i is the sum of the neighbouring coordinates of x_i on the Dynkin diagram of R, otherwise one can contradict the minimality replacing x_i by some x_i' of the same parity and leaving all the other coordinates unchanged.

The E_7-case:



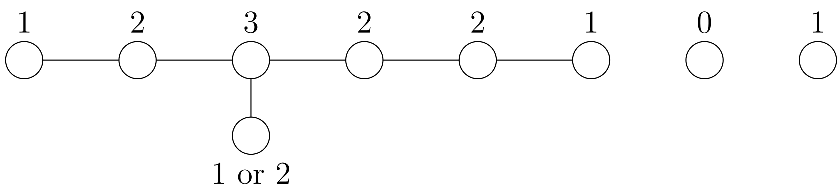
This time the difficult cases are $< x, x >\in \{4, 6\}$.

$$\begin{aligned} < x, x > &= (\frac{1}{2}x_1 + x_2 - x_3)^2 + (-\frac{1}{2}x_1 + x_2 + x_3 - x_4)^2 + \\ &+ (-\frac{1}{2}x_1 + x_4 - x_5)^2 + (\frac{1}{2}x_1 + x_5 - x_6)^2 + \\ &+ (-\frac{1}{2}x_1 + x_6 - x_7)^2 + (-\frac{1}{2}x_1 + x_7)^2 + \frac{1}{2}x_1^2 \end{aligned}$$

If $x_6 = x_7$ one gets the same quadratic form as for E_6 and in this case we know the Lemma is true. Similarly one can reduce the cases $x_5 = x_6$ and $x_5 = x_4$. The remaining situations can be solved by (R) or (R') relative to x_4 instead of x_{n-2}.

The E_8-case

As before we can restrict ourselves to the cases $< x, x >\in 4, 6$ and $x_8 \neq x_7 \neq x_6 \neq x_5 \neq x_4$. This time beside the situations solved by (R) or (R') we obtain the following solutions



c) $\Delta = m(c_1)$ In this case any extension (15) giving a bundle E with the corresponding Chern classes, must have $Y = \emptyset$ and $L^2 = -8m(c_1)$.

When $m(c_1) > \frac{1}{2}$ we can choose as in case b) an element $L \in \mathrm{Pic}(X)$ such that $L^2 = -8m(c_1)$, $H^0(L) = H^0(L^\vee) = 0$ and $c_1(L) \in c_1 + 2NS(X)$.

Hence $L^2 \leq -6$ and by Riemann-Roch's formula $h^1(L) \geq -2 - \frac{1}{2}L^2 \geq 1$. Thus there exist non-trivial extensions (16):

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0.$$

Moreover the middle term E is simple by Lemma 1.

When $m(c_1) \leq \frac{1}{2}$ we show that all extensions (16) with $L^2 = -8m(c_1)$ give a non simple bundle E. Indeed, from Riemann-Roch's formula for $\mathcal{E}nd(E)$

$$\chi(X, \mathcal{E}nd(E)) = 4\chi(X, \mathcal{O}_X) - 8\Delta(E)$$

we get

$$2h^0(\mathcal{E}nd(E)) \geq 8 - 4 = 4, \text{and } E \text{ cannot be simple.}$$

d) $m(c_1) = \frac{1}{4}$, $\Delta = \frac{3}{4}$

First we state:

Lemma 3 *If*

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0$$

is an extension with $L = \mathcal{O}(-D)$, D an effective divisor, Y a simple point belonging to $\mathrm{supp}D$ *and E locally free, then:*

i) $Y \in \mathrm{Reg}D \Rightarrow E$ *is simple*

ii) $D^2 = -8$ *and* $D \in 2NS(X) \Rightarrow E$ *is not simple.*

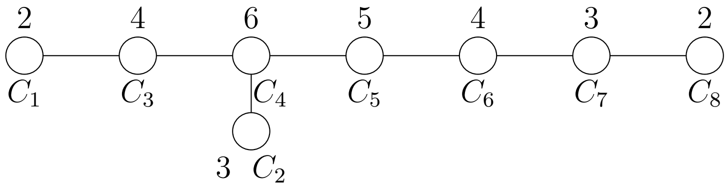
Before the proof we'll show the existence of simple reducible bundles with corresponding Chern classes.

Choose $L \in c_1 + 2NS(X)$ such that $L^2 = -2$.

If some irreducible component of D appears with multiplicity 1 in D then $\mathrm{Reg}D \neq \emptyset$. Taking Y a simple point on $\mathrm{Reg}D$ and using Remark1 in I.2. and Lemma 3, one obtains the simple reducible bundles we're after out of extensions of the type

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0.$$

Since D can be seen as a root in the A, D, E root system corresponding to $\mathrm{supp}D$ one easily sees that D admits an irreducible component of multiplicity 1 in the cases A_n,D_n, E_6, E_7, $n \geq 1$ (just look at the highest roots! ; [H] p. 66). This is also true in the E_8-case unless D corresponds to the highest root:



(it is the only root having both C_1 and C_8 of multiplicity 2). Here C_i denote the irreducible components of D.

In this case (D the highest root in E_8) we consider $D' = C_2 - C_5 - C_7 \in D + 2NS(X)$. $D'^2 = -6$ and by Lemma 1, a non-trivial extension

$$0 \longrightarrow \mathcal{O}(D') \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

provides a simple holomorphic bundle E admitting corresponding Chern classes.

Proof of Lemma 3

i) The long cohomology sequence of

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

gives $h^1(\mathcal{J}_Y) = 0$, $h^2(\mathcal{J}_Y) = 1$.
Using this in the long cohomology sequence of

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0$$

one finds $h^0(E) = 0$, $h^1(E) = h^1(L)$, $h^2(E) = 2$.
Passing to sections in

$$0 \longrightarrow E \longrightarrow E \otimes E^\vee \longrightarrow E^\vee \otimes \mathcal{J}_Y \longrightarrow 0$$

we obtain

$$h^0(E \otimes E^\vee) \leq h^0(E^\vee \otimes \mathcal{J}_Y).$$

We want to show that the subspace of $h^0(E^\vee)$ of sections vanishing on Y is one-dimensional; in fact we'll show that this subspace is the image of ι in the exact sequence (19) below. Let $Y = \{p\}$.
From

$$0 \longrightarrow \mathcal{O} \longrightarrow E^\vee \longrightarrow \mathcal{J}_Y(D) \longrightarrow 0$$

one derives the exact sequences

$$0 \longrightarrow H^0(\mathcal{O}) \xrightarrow{\iota} H^0(E^\vee) \longrightarrow H^0(\mathcal{J}_Y(D)) \longrightarrow 0 \tag{19}$$

$$0 \longrightarrow \mathcal{O}_p \xrightarrow{{}^t(-z_1,z_2)} \mathcal{O}_p^2 \xrightarrow{(z_2,z_1)} \mathcal{J}_{Y,p} \longrightarrow 0 \tag{20}$$

A non-zero section α in $H^0(\mathcal{J}_Y(D))$ can be seen as a section in $H^0(\mathcal{O}(D))$ since we have the natural inclusions $\mathcal{O} \hookrightarrow \mathcal{J}_Y(D) \hookrightarrow \mathcal{O}(D)$. Thus α vanishes of order exactly one on $\mathrm{Reg}(D)$, hence a lifting of it to $H^0(E^\vee)$ cannot vanish in p, otherwise from (20) we'd get that α vanishes twice in $p \in \mathrm{Reg}(D)$.

ii) $D > 0$, $D^2 = -8$, $D \in 2NS(X)$ imply there exists some divisor C such that $D = 2C$. Then taking sections in

$$0 \longrightarrow \mathcal{O}(-C) \longrightarrow E(C) \longrightarrow \mathcal{J}_Y(C) \longrightarrow 0$$

one finds that $h^0(E(C)) = 1$.
A non-zero morphism $\mathcal{O}(-C) \to E$ induces a devissage

$$0 \longrightarrow \mathcal{O}(-C + C') \longrightarrow E \longrightarrow \mathcal{J}_Z(-C' - C) \longrightarrow 0$$

with C' effective and Lemma 1 tells us that E is not simple.

e) $m(c_1) = 0$, $\Delta = \frac{1}{2}$

In this case not even holomorphic bundles can be found with these Chern classes by Theorem 3 in I.2..

f) $m(c_1) = 0$, $\Delta = 1$.

$m(c_1) = 0$ implies $c_1 \in 2NS(X)$. Then $m'(c_1)$ is an integer ≥ 1 with equality exactly when X admits divisors. Two kinds of extensions (15) can appear: either $L = \mathcal{O}$ and $l(Y) = 2$ or $L^2 = -8$ and $l(Y) = 0$. As usual L is taken in $2NS(X)$. Lemma 1 and an argument parallel to case ii) of Lemma 3 show that both cases give only non-simple bundles.

g) $m(c_1) = 0$, $\Delta = \frac{3}{2}$.

As before two kinds of extensions can appear: either with $L = \mathcal{O}$ and $l(Y) = 3$ or with $L^2 = -8$ and $l(Y) = 1$. The first gives only non-simple bundles by Lemma 1. For the second we remark that $L \cong \mathcal{O}(\pm 2C)$ with C effective hence we must choose the situation $L = \mathcal{O}(-2C)$, otherwise E is not simple by Lemma 1. Then by Remark 2 in I.2., Y must be a simple point lying on C and case ii) of Lemma 3 shows that also in this case E is not simple.

h) $m(c_1) = 0$, $\Delta \geq 2$.

When X admits divisors one can choose an irreducible curve C, Y a union of $2(\Delta - 1)$ simple points on $X \setminus C$,and the extensions

$$0 \longrightarrow \mathcal{O}(-2C) \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0$$

which give simple holomorphic bundles E with the expected Chern numbers by Remark 1 in I.2. and Lemma 1.

When X admits no divisors one can show as before that there are no simple bundles E with $c_1(E) \in 2NS(X)$ and $\Delta(E) < m'(0)$ (in (15) we should have $L = \mathcal{O}$). When $\Delta(E) \geq m'(0)$ we can choose a holomorphic line bundle L in $2NS(X)$, with $L^2 = -8m'(0)$, Y a union of $2(\Delta - m'(0))$ simple points and extensions

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0$$

give simple holomorphic bundles admitting the expected Chern classes by Remark 1 in I.2. and Lemma 1.

The Theorem is proved.

§3. Reducible vector bundles and non-separation phenomena in the moduli space of simple vector bundles

In this paragraph we make some remarks about the non-separation of the moduli space of simple vector bundles on surfaces in connection to the notion of reducible vector bundle.

Let X be a surface and $S_r = S_r(X)$ the coarse moduli space of simple vector bundles of rank r on X, (it is a possibly non-reduced analytic space).

The following separation criterion is well known.

Remark 1. If E_1, E_2 represent distinct non-separated classes of isomorphism of simple vector bundles in S_r, then there exist two non-trivial morphisms $\epsilon_1 : E_1 \to E_2$, $\epsilon_2 : E_2 \to E_1$, such that

$$\epsilon_1 \circ \epsilon_2 = \epsilon_2 \circ \epsilon_1 = 0.$$

Proof. There is a sequence of points $s^{(n)}$ in S_r converging both to $< E_1 >$ and to $< E_2 >$, where $< E_i >$ denotes the class of E_i in S_r, $i = 1, 2$.

In the local families around $< E_i >$ we get vector bundles $E_i^{(n)}$ representing $s^{(n)}$ and isomorphisms $E_1^{(n)} \cong E_2^{(n)}$. By semi-continuity $\mathrm{Hom}(E_1, E_2) \neq 0$ and $\mathrm{Hom}(E_2, E_1) \neq 0$, hence there exist non-trivial morphisms $\epsilon_1 : E_1 \to E_2$, $\epsilon_2 : E_2 \to E_1$. Moreover since E_1, E_2 are simple and non-isomorphic we must have

$$\epsilon_1 \circ \epsilon_2 = \epsilon_2 \circ \epsilon_1 = 0.$$

Consequence If E_1, E_2 are simple vector bundles representing non-separated classes in S_r, $(r > 1)$, then they must be both reducible.

We restrict ourselves from now on to the case of simple vector bundles of rank 2 on a surface X.

Remark 2. If X is a Kähler surface with $a(X) = 0$ then the points of S_2 represented by reducible vector bundles form a closed analytic subset R_2 in S_2. If moreover X has no divisors then R_2 is separated.

Proof. Take a point s_0 in S_2. There is an open connected neighbourhood S of s_0 in S_2 and a holomorphic vector bundle E on $X \times S$ whose fibers E_s represent the isomorphism classes of vector bundles given by the points s of S.

E_s is reducible if and only if there exists an element L in $\mathrm{Pic}(X)$ such that $H^0(L \otimes E_s) \neq 0$.

Consider as in the proof of the Theorem in II.2.2.,

$$\mathcal{R} := \{(\xi, s) \in \mathrm{Pic}(X) \times S : H^0(X, \mathbf{P}_\xi \otimes E_s) \neq 0\}$$

where $\mathbf{P}_\xi$ denotes the fibre in ξ of the Poincaré bundle $\mathbf{P}$ of X. $\mathcal{R}$ is an analytic subset of $\mathrm{Pic}(X) \times S$ and its image through the projection

$$p : \mathrm{Pic}(X) \times S \longrightarrow S$$

is $R_2 \cap S$. p is proper on every connected component of $\mathrm{Pic}(X) \times S$ by hypothesis.

We shall show that the restriction of p to the trace of $\mathcal{R}$ on a finite number of such components gives the same image $p(\mathcal{R})$. We may assume $c_1(E_s)$, $c_2(E_s)$ constant on S, shrinking S if necessary.

E_s is reducible if and only if it has a devissage

$$0 \longrightarrow L_1 \longrightarrow E_s \longrightarrow L_2 \otimes \mathcal{J}_Y \longrightarrow 0 \quad (*)$$

with L_1, L_2 in $\mathrm{Pic}(X)$ and Y a 2-codimensional locally complete intersection subspace of X. If $L^\vee \cong L_1$ we say that L "gives a devissage" of E_s. We restrict p to those components of $\mathrm{Pic}(X)$ containing elements which give devissages of E_s, for $s \in S$.

Consider now a devissage $(*)$. Then by (6)

$$c_1(E_s) = c_1(L_1) + c_1(L_2)$$

$$c_2(E_s) = L_1 L_2 + l(Y).$$

Hence

$$\begin{aligned} L_1^2 = [c_1(L_1) + c_1(L_2)]^2 - L_2^2 - 2L_1 \cdot L_2 = \\ = c_1(E_s)^2 - 2c_2(E_s) - L_2^2 + 2l(Y) \geq \\ \geq c_1(E_s)^2 - 2c_2(E_s) \end{aligned}$$

Since the intersection form is negative definite on $NS(X)$ modulo torsion (I.1. Theorem 6) and $NS(X)$ is finitely generated, it follows that $c(L_1)$ can reach only a finite number of values in $NS(X)$. Thus we see that R_2 is a closed analytic subset of S_2.

Let now X have no divisors and E_1, E_2 give non-separated classes in R_2. There exists then a sequence $(E^{(n)})_n$ of reducible simple bundles whose corresponding isomorphism classes converge in R_2 both to $< E_1 >$ and to $< E_2 >$. Consider devissages for $E^{(n)}$

$$0 \longrightarrow L^{(n)} \longrightarrow E^{(n)} \longrightarrow M^{(n)} \otimes \mathcal{J}_Y(n) \longrightarrow 0$$

Since $\mathrm{Pic}_0(X)$ is compact and $c_1(L^{(n)})$ is bounded we can choose a subsequence (denoted in the same way) such than $L^{(n)}$ converge in $\mathrm{Pic}(X)$ to an element L. Then $H^0(L^{-1} \otimes E_i) \neq 0$ for $i = 1, 2$.

Now $(\det E^{(n)})_n$ converges to $\det E_i$, $i = 1, 2$, hence $\det E_1 = \det E_2$, (S_1 is separated by Remark 1) and we have unique devissages (II.2.1.) of the form

$$0 \longrightarrow L \xrightarrow{\alpha i} E_i \xrightarrow{\beta i} M \otimes \mathcal{J}_{Z_i} \longrightarrow 0 \ , \ i = 1, 2,$$

where $L \ncong M$ by §2 Lemma 1.

Considering morphisms $\epsilon_1 : E_1 \to E_2$, $\epsilon_2 : E_2 \to E_1$ with $\epsilon_1 \circ \epsilon_2 = \epsilon_2 \circ \epsilon_1 = 0$ one gets $\beta_2 \circ \epsilon_1 \circ \alpha_1 = 0$ and morphisms γ,δ closing the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\alpha_1} & E_1 & \xrightarrow{\beta_1} & M \otimes \mathcal{J}_{Z_1} & \longrightarrow & 0 \\
 & & \gamma \downarrow & & \downarrow \epsilon_1 & & \downarrow \delta & & \\
0 & \longrightarrow & L & \xrightarrow{\alpha_2} & E_2 & \xrightarrow{\beta_2} & M \otimes \mathcal{J}_{Z_2} & \longrightarrow & 0
\end{array}$$

If γ were zero we'd get a non-trivial morphism $M \otimes \mathcal{J}_{Z_1} \to E_2$ contradicting the unicity of the devissage. Thus γ must be an isomorphism and similary δ is an isomorphism. This implies ϵ_1 -an isomorphism, which is contradiction.

Remark 3 Let X be a surface admitting an effective divisor D such that $\mathcal{O}(D)$ generates $\mathrm{Pic}(X)$. Then $S_2(X)$ is Hausdorff.

Examples of surfaces fulfilling this hypothesis are: $\mathbf{P}^2$, special K3 surfaces of type g for $g \geq 0$ (they are algebraic for $g \geq 2$, elliptic when $g = 1$, and have algebraic dimension zero for $g = 0$; cf. I.1.3. and use Riemann-Roch to verify that D is effective), K3 surfaces X with $NS(X) = 0$ blown-up in one point.

Proof. Assume $S_2(X)$ is not Hausdorff and let E_1, E_2 be simple vector bundles giving non-separated elements in S_2. There exist then non-trivial morphisms

$$\epsilon_1 : E_1 \longrightarrow E_2 \ , \ \epsilon_2 : E_2 \longrightarrow E_1 \quad \text{with } \epsilon_2 \circ \epsilon_1 = \epsilon_1 \circ \epsilon_2 = 0$$

We consider devissages

$$0 \longrightarrow \mathcal{O}(n_i D) \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} \mathcal{J}_{Y_i}(m_i D) \longrightarrow 0$$

of E_i, n_i, $m_i \in \mathbf{Z}$, $i = 1, 2$, which we shall modify in order to make them compatible with the $\epsilon_i - s$, namely we want them to verify $\epsilon_i \circ \alpha_i = 0$ for $i = 1, 2$. If $\epsilon_1 \circ \alpha_1 \neq 0$ then $\mathcal{O}(n_1 D) \xrightarrow{\epsilon_1 \circ \alpha_1} E_2$ induces a devissage with kernel

$$\mathcal{O}(n_2' D) \overset{\alpha_2'}{\hookrightarrow} E_2 \quad \text{and} \quad 0 = \epsilon_2 \circ \epsilon_1 \circ \alpha_1 = \epsilon_2 \circ \alpha_2' \circ j,$$

where $j : \mathcal{O}(n_1 D) \to \mathcal{O}(n_2' D)$ is the inclusion. Hence $\epsilon_2 \circ \alpha_2' = 0$.

Leaving the primes aside we have now a new devissage for E_2 with $\epsilon_2 \circ \alpha_2 = 0$. Since $\epsilon_2 \neq 0$ there exists a non-zero morphism

$$\psi : \mathcal{J}_{Y_2}(m_2 D) \to E_1$$

such that $\epsilon_2 = \psi \circ \beta_2$. ψ induces a devissage of E_1 with kernel $\mathcal{O}(n_1' D) \xrightarrow{\alpha_1'} E_1$.

$0 = \epsilon_1 \circ \psi \circ \beta_2$ implies $0 = \epsilon_1 \circ \psi = \epsilon_1 \circ \alpha_1' \circ j$ where as before j denotes the inclusion

$\mathcal{J}_{Y_2}(m_2 D) \xrightarrow{j} \mathcal{O}(n_1' D)$

Hence $\epsilon_1 \circ \alpha_1' = 0$.

Leaving again the primes aside and considering the latter j we get $\epsilon_1 \circ \alpha_1 = 0$ and $\epsilon_2 = \alpha_1 \circ j \circ \beta_2$. Then $0 = \alpha_1 \circ j \circ \beta_2 \circ \epsilon_1$ implies $\beta_2 \circ \epsilon_1 = 0$. Thus there exists a non-trivial morphism

$$\chi : E_1 \longrightarrow \mathcal{O}(n_2 D)$$

such that $\epsilon_1 = \alpha_2 \circ \chi$.

Now $\alpha_2 \circ \chi \circ \alpha_1 = 0$ gives $\chi \circ \alpha_1 = 0$ and there is a non-zero morphism

$$\eta : \mathcal{J}_{Y_1}(m_1 D) \longrightarrow \mathcal{O}(n_2 D) \quad \text{with} \quad \epsilon_1 = \alpha_2 \circ \eta \circ \beta_1$$

Putting everything together we find the inequalities (the strict ones due to the simplicity of the $E_i - s$ and the two others to j and η)

$$m_2 \leq n_1 < m_1 \leq n_2 < m_2,$$

hence the wanted contradicton.

In general S_r will not be separated. In [N1] Norton gave a criterion of non-separation in S_r and used it to study the case when the base X is a Riemann surface. One can get examples of non-separation in higher dimensions considering, for instance, fibrations over a curve B and lifting non-separate elements in $S_r(B)$.

Using Norton's criterion one obtains in some cases examples of non-separation in $S_2(X)$ also for X a surface with $a(X) = 0$ as the one presently shown.

Example

If X is a surface with $a(X) = 0$ and C_1, C_2 two irreducible curves on it such that

$$H^0(K_X) = H^0(K_X(C_1 - C_2)) = H^0(K_X(C_2 - C_1)) = 0$$

$$H^1(\mathcal{O}(C_2 - C_1)) \neq 0 \quad \text{and}$$
$$H^1(\mathcal{O}(C_1 - C_2)) \neq 0,$$

then any two non-trivial extensions

$$0 \longrightarrow \mathcal{O}(C_1) \xrightarrow{\alpha_1} E_1 \xrightarrow{\beta_1} \mathcal{O}(C_2) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}(C_2) \xrightarrow{\alpha_2} E_2 \xrightarrow{\beta_2} \mathcal{O}(C_1) \longrightarrow 0$$

produce simple vector bundles E_1, E_2 having non-separated classes in $S_2(X)$.

Note first that there exist surfaces in class VIII satisfying the avove conditions on C_1, C_2 and K_X. Moreover also the condition $\Delta(E_i) + m(c_1(E_i) + c_1(X)) < -\frac{K_X^2}{8}$ can be fullfilled (see for instance [D; p.80, Ex.2.2.]). By [BL; Lemma 4.4.] this extracondition insures that no irreducible rank 2 vector bundle is continuosly isomorphic to the E_i-s and thus the non-separation will take place on $R_2(X)$.

Remark next that the E_i-s are simple by 2.1. Lemma 1. To show now that they have non-separated classes in $S_2(X)$ we only need to verify Norton's criterion. We refer to [N1] for the facts and notations used below.

Choosing a sufficiently fine open cover $\mathcal{U} = (U_\alpha)_\alpha$ of X and $f_\alpha^i = 0$ defining equations for C_i on U_α we get corresponding representative cocycles of the form

$$\varphi_{\alpha\beta} = \begin{pmatrix} f_\alpha^1 \setminus f_\beta^1 & * \\ 0 & f_\alpha^2 \setminus f_\beta^2 \end{pmatrix}$$

for E_1 and

$$\psi_{\alpha\beta} = \begin{pmatrix} f_\alpha^2 \setminus f_\beta^2 & * \\ 0 & f_\alpha^1 \setminus f_\beta^1 \end{pmatrix}$$

for E_2.

Then the morphisms $h := \alpha_2 \circ \beta_1$, $k := \alpha_1 \circ \beta_2$ are defined in terms of $(\varphi_{\alpha\beta})$, $(\psi_{\alpha\beta})$ by the matrices

$$h_\alpha = k_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

h and k have complementary ranks and are "compatible" i.e. there exist analytic deformations $h_\alpha(t)$, $k_\alpha(t)$, for a small complex parameter t, with $h_\alpha(0) = h_\alpha$, $k_\alpha(0) = k$ and $h_\alpha(t) = u(t) \cdot I_2$ for a scalar analytic function $u \neq 0$. In our case one can choose

$$h_\alpha(t) = k_\alpha(t) = \begin{pmatrix} -t & 1 \\ 0 & t \end{pmatrix}.$$

Let $\mathcal{S}$ be the kernel of the morphism

$$f : \mathcal{E}nd(E_1) \otimes \mathcal{E}nd(E_2) \longrightarrow \mathcal{H}om(E_1, E_2) \otimes \mathcal{H}om(E_2, E_1)$$

defined by $f(x, y) = (h \circ x - y \circ h, k \circ y - x \circ k)$

$\mathcal{S}$ is a locally free sheaf since it is a reflexive sheaf on a surface and its representative cocycle is easily verified to be of the form

$$\begin{pmatrix} f_\alpha^1 f_\beta^2 \setminus f_\beta^1 f_\alpha^2 & 0 & * & * \\ 0 & f_\beta^1 f_\alpha^2 \setminus f_\alpha^1 f_\beta^2 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence an exact sequence

$$0 \longrightarrow \mathcal{O}(C_1 - C_2) \oplus \mathcal{O}(C_2 - C_1) \longrightarrow \mathcal{S} \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow 0$$

from which one derives

$$H^2(\mathcal{S}) = H^0(\mathcal{S}^\vee \otimes K_X) = 0.$$

Thus since all the hypotheses of Theorem 1 in [N1] are fulfilled, $< E_1 >$ and $< E_2 >$ are non-separated in $S_2(X)$.

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