Supplement to "Rate of convergence to equilibrium of fractional driven stochastic differential equations with rough multiplicative noise"

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We have gathered here the proofs of a few selected results from [**?**].

1. A Lyapunov property for rough differential equations

We consider the general rough equation

$$dy_t = b(y_t)\,dt + \sigma(y_t)\,d\mathbf{x}_t \quad , \quad t \in [0,1] \quad , \quad y_0 = a \in \mathbb{R}^d , \tag{1.1}$$

where $\mathbf{x}$ is a given (deterministic) γ-rough path on $[0,1]$, in the sense of [**?**, Definition 2.1], for some fixed parameter $\gamma \in (\frac{1}{3}, \frac{1}{2})$. In what follows, we will write $\|\mathbf{x}\|_\gamma$ for $\|\mathbf{x}\|_{\gamma;[0,1]}$. Let us also recall the two assumptions at the core of our study:

Hypothesis (H1): $b : \mathbb{R}^d \to \mathbb{R}^d$, resp. $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$, is a $\mathcal{C}^3$, resp. $\mathcal{C}^4$, vector field such that

$$\begin{aligned} &\sup_{v\in\mathbb{R}^d} \|(D^{(\ell)}b)(v)\| \ < \ \infty \text{ for } \ell \in \{1,2,3\} , \\ \text{resp. } &\sup_{v\in\mathbb{R}^d} \|(D^{(\ell)}\sigma)(v)\| \ < \ \infty \text{ for } \ell \in \{0,\ldots,4\} . \end{aligned} \tag{1.2}$$

Hypothesis (H2): There exist $C_1, C_2 > 0$ such that for every $v \in \mathbb{R}^d$, one has

$$\langle v, b(v)\rangle \leqslant C_1 - C_2\|v\|^2 \quad . \tag{1.3}$$

This section is devoted to the proof of the following statement:

Theorem 1.1. *Under Hypothesis* (**H1**) *and for every initial condition* $y_0 \in \mathbb{R}^d$*, Equation (1.1) admits a unique solution* y *on* $[0,1]$*, in the sense of [?, Definition 2.3]. Besides, if we assume in addition that Hypothesis* (**H2**) *holds true, then there exists a constant* C *(which depends on* $b, \sigma, \gamma, C_1, C_2$*, but not on* $\mathbf{x}$*) such that*

$$\|y_1\|^2 \leqslant e^{-C_2/2}\|y_0\|^2 + C\big\{1 + \|\mathbf{x}\|_\gamma^\mu\big\} \quad , \quad \text{with } \mu := \frac{8}{3\gamma - 1} \quad . \tag{1.4}$$

Under Hypothesis (**H1**), the fact that there exists at most one solution to (1.1) (in other words, the uniqueness part of our statement) is a standard result, which can for instance be found in [**?**, Theorem 3.3]. On the opposite, due to the unboundedness of b, it seems that the proof of existence of a global solution on $[0,1]$ cannot be found as such in the literature, and we shall therefore provide a few details below.

In brief, our strategy towards Theorem 1.1 is based on a careful analysis of the natural discrete numerical scheme associated with (1.1), in the same spirit as in [**?**]. Let us thus introduce the sequence of dyadic partitions $\mathcal{P}_n := \{t_i = t_i^n := \frac{i}{2^n}\, ;\ i = 0, \ldots, 2^n\}$ of $[0,1]$, and consider the discrete path y^n defined on $\mathcal{P}_n$ along the iterative formula

$$y_0^n := a \quad , \quad \delta y^n_{t_i t_{i+1}} = b(y^n_{t_i})\, \delta \mathcal{T}_{t_i t_{i+1}} + \sigma(y^n_{t_i})\, \delta x_{t_i t_{i+1}} + (D\sigma \cdot \sigma)(y^n_{t_i})\, \mathbf{x}^{\mathbf{2}}_{t_i t_{i+1}} \ , \tag{1.5}$$

where we recall that $\delta \mathcal{T}_{st} = t - s$. We shall also be led to handle the following quantities associated with y^n: for $s, t \in \mathcal{P}_n$,

$$\begin{aligned} L^{y,n}_{st} &:= \delta y^n_{st} - \sigma(y^n_s)\, \delta x_{st} - (D\sigma \cdot \sigma)(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} \\ R^{y,n}_{st} &:= \delta y^n_{st} - b(y^n_s)\, \delta \mathcal{T}_{st} - \sigma(y^n_s)\, \delta x_{st} - (D\sigma \cdot \sigma)(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} \\ Q^{y,n}_{st} &:= \delta y^n_{st} - \sigma(y^n_s)\, \delta x_{st} \ . \end{aligned}$$

For every $s < t \in [0,1]$, we will write $[\![s,t]\!] = [\![s,t]\!]_n := [s,t] \cap \mathcal{P}_n$, and set

$$\mathcal{N}[f; \mathcal{C}_2^\mu([\![\ell_1, \ell_2]\!])] := \sup_{s<t \in [\![\ell_1,\ell_2]\!]} \frac{\|f_{st}\|}{|t-s|^\mu} \ ,$$

$$\mathcal{N}[f; \mathcal{C}_1^\mu([\![\ell_1, \ell_2]\!])] := \mathcal{N}[\delta f; \mathcal{C}_2^\mu([\![\ell_1, \ell_2]\!])] \ .$$

The starting point of our analysis is the following local estimate for $R^{y,n}$, which can be obtained as a straightforward application of the forthcoming Proposition 2.7:

Proposition 1.2. *Fix* $\kappa := \frac{1}{2}\big(\frac{1}{3} + \gamma\big)$*. Then, under Hypothesis* (**H1**)*, there exists a constant* c_0 *(which depends only on* b, σ, γ*) such that if we set*

$$T_0 = T_0(\|\mathbf{x}\|) := \min\Big(1, \big(c_0\{1 + \|\mathbf{x}\|_\gamma\}\big)^{-1/(\gamma-\kappa)}\Big) \ ,$$

one has, for every $\tau \in \mathcal{P}_n$ *satisfying* $0 < \tau \leqslant T_0$ *and every* $k \leqslant 1/\tau$*,*

$$\mathcal{N}[R^{y,n}; \mathcal{C}_2^{3\kappa}([\![k\tau, (k+1)\tau \wedge 1]\!])] \leqslant c_0\big\{1 + \|y^n_{k\tau}\|\big\} \ . \tag{1.6}$$

Corollary 1.3. *In the setting of Proposition 1.2, there exists a constant c_1 (which depends only on b, σ, γ) such that for every $\tau \in \mathcal{P}_n$ satisfying $0 < \tau \leqslant T_0$ and every $k \leqslant 1/\tau$, one has*

$$\mathcal{N}[y^n; \mathcal{C}_1^0([\![k\tau, (k+1)\tau \wedge 1]\!])] \leqslant c_1\{1 + \|y^n_{k\tau}\|\} \ , \tag{1.7}$$

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma([\![k\tau, (k+1)\tau \wedge 1]\!])] \leqslant c_1\{1 + \|y^n_{k\tau}\|\}\{1 + \|\mathbf{x}\|_\gamma\} \tag{1.8}$$

and

$$\mathcal{N}[Q^{y,n}; \mathcal{C}_2^{2\gamma}([\![k\tau, (k+1)\tau \wedge 1]\!])] \leqslant c_1\{1 + \|y^n_{k\tau}\|\}\{1 + \|\mathbf{x}\|_\gamma\} \ . \tag{1.9}$$

Proof. For every $t \in [\![k\tau, (k+1)\tau \wedge 1]\!]$, write

$$y^n_t = y^n_{k\tau} + b(y^n_{k\tau})\, \delta\mathcal{T}_{k\tau,t} + \sigma(y^n_{k\tau})\, \delta x_{k\tau,t} + (D\sigma \cdot \sigma)(y^n_{k\tau})\, \mathbf{x}^{\mathbf{2}}_{k\tau,t} + R^{y,n}_{k\tau,t} \ ,$$

so that using (1.6), we get $\|y^n_t\| \lesssim 1 + \|y^n_{k\tau}\| + \|\mathbf{x}\|_\gamma T_0^\gamma$, and (1.7) now follows from the fact that $\|\mathbf{x}\|_\gamma T_0^\gamma \leqslant \|\mathbf{x}\|_\gamma T_0^{\gamma-\kappa} \lesssim 1$.

Then, in a more general way, we have for every $s < t \in [\![k\tau, (k+1)\tau \wedge 1]\!]$

$$\delta y^n_{st} = b(y^n_s)\, \delta\mathcal{T}_{st} + \sigma(y^n_s)\, \delta x_{st} + (D\sigma \cdot \sigma)(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} + R^{y,n}_{st}$$

and

$$Q^{y,n}_{st} = b(y^n_s)\, \delta\mathcal{T}_{st} + (D\sigma \cdot \sigma)(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} + R^{y,n}_{st} \ .$$

Injecting (1.6) and (1.7) into these expressions easily yields (1.8) and (1.9). □

Corollary 1.4. *Under Hypothesis* (**H1**)*, Equation (1.1) admits a unique global solution y on $[0,1]$. Besides, with the previous notations, there exists a subsequence of (y^n), that we still denote by (y^n), such that*

$$\max_{i=0,\ldots,2^n} \|y_{t_i} - y^n_{t_i}\| \stackrel{n\to\infty}{\longrightarrow} 0 \ . \tag{1.10}$$

Proof. Although the two local estimates (1.7)-(1.8) are not uniform as such (that is, the right-hand side still depends on y^n), they easily give rise, via an obvious iterative procedure, to a uniform estimate for

$$\mathcal{N}[y^n; \mathcal{C}_1^{0,\gamma}([\![0,1]\!])] := \mathcal{N}[y^n; \mathcal{C}_1^0([\![0,1]\!])] + \mathcal{N}[y^n; \mathcal{C}_1^\gamma([\![0,1]\!])] \ .$$

Still denoting by y^n the continuous path obtained through the linear interpolation of $(y^n_{t_i})_{i=0,\ldots,2^n}$, we thus get a uniform estimate for $\mathcal{N}[y^n; \mathcal{C}_1^{0,\gamma}([0,1])]$, which, by a standard compactness argument, allows us to conclude about the existence of a path $y \in \mathcal{C}_1^\gamma([0,1])$, as well as a subsequence of y^n (that we still denote by y^n), such that $y^n \to y$ in $\mathcal{C}_1^{0,\gamma'}([0,1])$ for every $0 < \gamma' < \gamma$.

The fact that y actually defines a solution of (1.1) is then an easy consequence of the bound (1.6). The details of the procedure can for instance be found at the end of [**?**, Section 3.3]. Finally, and as we have already evoked it in the beginning of the section, the uniqueness of this solution is a standard result from the rough-path literature (see [**?**, Theorem 3.3]). □

Let us now turn to the proof of the second part of Theorem 1.1, that is to the proof of (1.4) under Hypotheses (**H1**) and (**H2**). To this end, we introduce, for every $n \geqslant 0$, the additional discrete path $z^n : \mathcal{P}_n \to \mathbb{R}$ defined for every $t \in \mathcal{P}_n$ as

$$z^n_t := \frac{1}{2}\|y^n_t\|^2 \ .$$

In the same vein as above, we will lean on the following quantities related to z^n: for every $s, t \in \mathcal{P}_n$,

$$\begin{array}{rcl} R^{z,n}_{st} & := & \delta z^n_{st} - \langle y^n_s, b(y^n_s)\rangle\, \delta\mathcal{T}_{st} - \langle y^n_s, \sigma(y^n_s)\rangle\, \delta x_{st} - \Sigma(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} \\ L^{z,n}_{st} & := & \delta z^n_{st} - \langle y^n_s, \sigma(y^n_s)\rangle\, \delta x_{st} - \Sigma(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} \\ Q^{z,n}_{st} & := & \delta z^n_{st} - \langle y^n_s, \sigma(y^n_s)\rangle\, \delta x_{st} \ , \end{array}$$

where we have set

$$\Sigma(y^n_s) := \langle \sigma(y^n_s), \sigma(y^n_s)\rangle + \langle y^n_s, (D\sigma\cdot\sigma)(y^n_s)\rangle \ .$$

Just to be clear, the notation for the second-order term in $R^{z,n}, L^{z,n}$ specifically refers to the sum

$$\Sigma(y^n_s)\, \mathbf{x}^{\mathbf{2}}_{st} = \big\{\langle \sigma_j(y^n_s), \sigma_k(y^n_s)\rangle + \langle y^n_s, (D\sigma_j\cdot\sigma_k)(y^n_{st})\rangle\big\}\, \mathbf{x}^{\mathbf{2},jk}_{st} \ .$$

Finally, along the same lines as in the subsequent Section 2, we set, if $s = \frac{p}{2^n}$ and $t = \frac{q}{2^n}$ and $G : [\![0,1]\!] \to \mathbb{R}^d$,

$$\mathcal{M}^\mu\big[G; [\![s,t]\!]\big] := \sup_{p \leqslant i \leqslant q} \frac{\|G_{t_i t_{i+1}}\|}{|t_{i+1}-t_i|^\mu} \ .$$

Let us start with a few estimates on $R^{z,n}$, for which Hypothesis (**H2**) is still not required:

Lemma 1.5. *Under Hypothesis* (**H1**) *and with the above notations, there exists a constant c_2 (which depends only on b, σ, γ) such that for every $s < t \in \mathcal{P}_n$, one has*

$$\mathcal{M}^{3\gamma}\big[R^{z,n}; [\![s,t]\!]\big] \leqslant c_2\{1+\|\mathbf{x}\|^2_\gamma\}\{1+\mathcal{N}[y^n; \mathcal{C}^0_1([\![s,t]\!])]^2\} \ . \qquad (1.11)$$

Proof. We have

$$\delta z^n_{t_i t_{i+1}} = \langle y^n_{t_i}, \delta y^n_{t_i t_{i+1}}\rangle + \frac{1}{2}\langle \delta y^n_{t_i t_{i+1}}, \delta y^n_{t_i t_{i+1}}\rangle \ ,$$

and so, injecting (1.5) into the first term gives

$$\begin{array}{rcl} R^{z,n}_{t_i t_{i+1}} & = & \frac{1}{2}\langle \delta y^n_{t_i t_{i+1}}, \delta y^n_{t_i t_{i+1}}\rangle - \langle \sigma(y^n_{t_i}), \sigma(y^n_{t_i})\rangle \mathbf{x}^{\mathbf{2}}_{t_i t_{i+1}} \\ & = & \langle \sigma(y^n_{t_i})\, \delta x_{t_i t_{i+1}} + \frac{1}{2} Q^{y,n}_{t_i t_{i+1}}, Q^{y,n}_{t_i t_{i+1}}\rangle \\ & & + \left\{ \frac{1}{2}\langle \sigma(y^n_{t_i})\delta x_{t_i t_{i+1}}, \sigma(y^n_{t_i})\delta x_{t_i t_{i+1}}\rangle - \langle \sigma(y^n_{t_i}), \sigma(y^n_{t_i})\rangle \mathbf{x}^{\mathbf{2}}_{t_i t_{i+1}} \right\} \\ & = & \langle \sigma(y^n_{t_i})\, \delta x_{t_i t_{i+1}} + \frac{1}{2} Q^{y,n}_{t_i t_{i+1}}, Q^{y,n}_{t_i t_{i+1}}\rangle \ , \end{array} \qquad (1.12)$$

where we have used the basic identity

$$\mathbf{x}^{\mathbf{2};ij}_{st} + \mathbf{x}^{\mathbf{2};ji}_{st} = \delta x^i_{st}\, \delta x^j_{st} \ .$$

Finally, since $Q^{y,n}_{t_i t_{i+1}} = b(y^n_{t_i})\, \delta\mathcal{T}_{t_i t_{i+1}} + (D\sigma\cdot\sigma)(y^n_{t_i})\, \mathbf{x}^{\mathbf{2}}_{t_i t_{i+1}}$, it is immediate that

$$\|Q^{y,n}_{t_i t_{i+1}}\| \lesssim |t_{i+1}-t_i|\{1+\|y^n_{t_i}\|\} + |t_{i+1}-t_i|^{2\gamma}\|\mathbf{x}\|_\gamma \ .$$

Going back to (1.12), we get the conclusion. □

Proposition 1.6. *Assume Hypothesis* (**H1**) *holds true and let* $T_0 = T_0(\|\mathbf{x}\|_\gamma)$ *be the time defined in Proposition 1.2. Then there exists a constant* c_3 *(which depends only on* b, σ, γ*) such that for every* $\tau \in \mathcal{P}_n$ *satisfying* $0 < \tau \leqslant T_0$ *and every* $k \leqslant 1/\tau$*, one has*

$$\mathcal{N}[R^{z,n}; \mathcal{C}^{3\gamma}_2([\![k\tau, (k+1)\tau \wedge 1]\!])] \leqslant c_3\{1+\|\mathbf{x}\|^3_\gamma\}\{1+z^n_{k\tau}\} \ .$$

Proof. Thanks to the forthcoming Lemma 2.4, we can rely on the estimate

$$\begin{aligned}&\mathcal{N}[R^{z,n}; \mathcal{C}^{3\gamma}_2([\![k\tau, (k+1)\tau \wedge 1]\!])]\\ &\lesssim \mathcal{M}^{3\gamma}\big[R^{z,n}; [\![k\tau, (k+1)\tau \wedge 1]\!]\big] + \mathcal{N}[\delta R^{z,n}; \mathcal{C}^{3\gamma}_3([\![k\tau, (k+1)\tau \wedge 1]\!])] \ .\end{aligned}$$

As far as the first term is concerned, combining (1.11) and (1.7) allows us to assert that

$$\mathcal{M}^{3\gamma}\big[R^{z,n}; [\![k\tau, (k+1)\tau \wedge 1]\!]\big] \lesssim \{1+\|\mathbf{x}\|^2_\gamma\}\{1+z^n_{k\tau}\} \ .$$

Then, for every $s < u < t \in [\![k\tau, (k+1)\tau \wedge 1]\!]$, decompose $\delta R^{z,n}_{sut}$ as

$$\delta R^{z,n}_{sut} = -\delta(\langle y^n, b(y^n)\rangle)_{su}\, \delta\mathcal{T}_{ut} + \delta L^{z,n}_{sut} \ .$$

On the one hand, one has, by (1.7) and (1.8),

$$\begin{aligned}\big|\delta(\langle y^n, b(y^n)\rangle)_{su}\big| &\leqslant \big|\langle \delta y^n_{su}, b(y^n_u)\rangle\big| + \big|\langle y^n_s, \delta b(y^n)_{su}\rangle\big|\\ &\lesssim |u-s|^\gamma\{1+\|\mathbf{x}\|_\gamma\}\{1+z^n_{k\tau}\} \ .\end{aligned}$$

On the other hand, combining Chen's identity with elementary Taylor expansions easily leads us to the decomposition

$$\delta L^{z,n}_{sut} = \big\{I^i_{su} + II^i_{su} + III^i_{su} + IV^i_{su}\big\}\, \delta x^i_{ut} + \delta\Sigma_{ij}(y^n)_{su}\, \mathbf{x}^{\mathbf{2},ij}_{ut} \ ,$$

with

$$\begin{aligned}\Sigma_{ij}(y^n) &:= \langle \sigma_i(y^n), \sigma_j(y^n)\rangle + \langle y^n, (D\sigma_i\cdot\sigma_j)(y^n)\rangle \ ,\\ I^i_{su} &:= \langle \delta y^n_{su}, \delta\sigma_i(y^n)_{su}\rangle \ , \quad II^i_{su} := \langle \sigma_i(y^n_s), Q^{y,n}_{su}\rangle \ ,\\ III^i_{su} &:= \int_0^1 d\xi\, \langle y^n_s, D\sigma_i(y^n_s + \xi\, \delta y^n_{su})Q^{y,n}_{su}\rangle,\end{aligned}$$

and finally

$$IV^i_{su} := \int_0^1 d\xi \, \langle y^n_s, \big[D\sigma_i(y^n_s + \xi\, \delta y^n_{su}) - D\sigma_i(y^n_s)\big](\sigma_j(y^n_s))\rangle\, \delta x^j_{su} \ .$$

With the above expressions in mind and using the three estimates (1.7), (1.8) and (1.9), it is not hard to check that

$$\big|\delta L^{z,n}_{sut}\big| \lesssim |t-s|^{3\gamma}\{1+\|\mathbf{x}\|^3_\gamma\}\{1+z^n_{k\tau}\} \ ,$$

which achieves the proof of our assertion.

□

Let us finally involve Hypothesis (**H2**) into the picture:

Corollary 1.7. *Assume Hypotheses* (**H1**) *and* (**H2**) *hold true and let* $T_0 = T_0(\|\mathbf{x}\|_\gamma)$ *be the time defined in Proposition 1.2. Then there exist constants* c_4, c_5 *(both depending only on* $b, \sigma, \gamma, C_1, C_2$*) such that if we set*

$$T_1 = T_1(\|\mathbf{x}\|_\gamma) := \min\left(T_0, \frac{2}{C_2}, \left(\frac{1}{c_4\{1+\|\mathbf{x}\|^3_\gamma\}}\right)^{1/(3\gamma-1)}\right) \ ,$$

one has, for every $\tau \in \mathcal{P}_n$ *satisfying* $0 < \tau \leqslant T_1$ *and every* $k \leqslant 1/\tau$*,*

$$z^n_{(k+1)\tau\wedge 1} \leqslant \Big(1 - \frac{C_2}{2}\tau\Big) z^n_{k\tau} + c_5\{1+\|\mathbf{x}\|^2_\gamma\}\tau^{2\gamma-1} \ , \qquad (1.13)$$

where we recall that the two parameters C_1, C_2 *have been introduced in Hypothesis (H2).*

Proof. Using Hypothesis (**H2**), we get that for every $\tau \in \mathcal{P}_n$ and every such that $k \leqslant 1/\tau$,

$$\begin{aligned} & z^n_{(k+1)\tau\wedge 1} \\ &= z^n_{k\tau} + \langle y^n_{k\tau}, b(y^n_{k\tau})\rangle\, \delta\mathcal{T}_{k\tau,(k+1)\tau\wedge 1} \\ &\quad + \langle y^n_{k\tau}, \sigma(y^n_{k\tau})\rangle\, \delta x_{k\tau,(k+1)\tau\wedge 1} + \Sigma(y^n_{k\tau})\, \mathbf{x}^{\mathbf{2}}_{k\tau,(k+1)\tau\wedge 1} + R^{z,n}_{k\tau,(k+1)\tau\wedge 1} \\ &\leqslant (1-C_2\tau) z^n_{k\tau} + C_1\tau \\ &\quad + \langle y^n_{k\tau}, \sigma(y^n_{k\tau})\rangle\, \delta x_{k\tau,(k+1)\tau\wedge 1} + \Sigma(y^n_{k\tau})\, \mathbf{x}^{\mathbf{2}}_{k\tau,(k+1)\tau\wedge 1} + R^{z,n}_{k\tau,(k+1)\tau\wedge 1} \ , \end{aligned}$$

and so, thanks to Proposition 1.6, we can conclude that for every $0 < \tau \leqslant \min\big(T_0, \frac{2}{C_2}\big)$ and every $k \leqslant 1/\tau$, one has

$$\begin{aligned} z^n_{(k+1)\tau\wedge 1} &\leqslant (1-C_2\tau) z^n_{k\tau} + C_1\tau \\ &\quad + c_4\Big[\|\mathbf{x}\|_\gamma \tau^\gamma\big\{1+(z^n_{k\tau})^{\frac12}\big\} + \frac{C_2}{4}\tau^{3\gamma}\{1+\|\mathbf{x}\|^3_\gamma\}\{1+z^n_{k\tau}\}\Big] \ , \end{aligned} \qquad (1.14)$$

for some constant $c_4 = c_4(b, \sigma, \gamma, C_2)$. Now, by the very definition of T_1, we know that if $0 < \tau \leqslant T_1$, then

$$c_4\tau^{3\gamma}\{1+\|\mathbf{x}\|^3_\gamma\} \leqslant \tau\big(c_4\tau^{3\gamma-1}\{1+\|\mathbf{x}\|^3_\gamma\}\big) \leqslant \tau \ ,$$

and thus we can recast relation (1.14) into:

$$z^n_{(k+1)\tau\wedge 1} \leqslant \left(1 - \frac{3C_2}{4}\tau\right) z^n_{k\tau} + C_1\tau + c_4\|\mathbf{x}\|_\gamma \tau^\gamma \{1 + (z^n_{k\tau})^{\frac{1}{2}}\}$$

To achieve the proof, it now suffices to use the basic inequality

$$c_4\|\mathbf{x}\|_\gamma \tau^\gamma (z^n_{k\tau})^{\frac{1}{2}} \leqslant \frac{C_2}{4}\tau z^n_{k\tau} + \frac{c_4^2}{C_2}\|\mathbf{x}\|^2_\gamma \tau^{2\gamma-1} \ .$$

□

At this point, we are very close to (1.4). With the notations of Corollary 1.7, consider n large enough such that we can exhibit $\tau_0 \in \mathcal{P}_n$ satisfying $\frac{1}{2}T_1 \leqslant \tau_0 \leqslant T_1$, and then let K be the integer such that $(K-1)\tau_0 \leqslant 1 < K\tau_0$. Iterating the bound (1.13) with $\tau = \tau_0$ yields that

$$\begin{aligned} z^n_1 &\leqslant \left(1 - \frac{C_2}{2}\tau_0\right)^K z^n_0 + c_5\, K\{1 + \|\mathbf{x}\|^2_\gamma\}\tau_0^{2\gamma-1} \\ &\leqslant \left(1 - \frac{C_2}{2}\tau_0\right)^{\frac{1}{\tau_0}} z^n_0 + c_5\, K\{1 + \|\mathbf{x}\|^2_\gamma\}\tau_0^{2\gamma-1} \\ &\leqslant e^{-C_2/2} z^n_0 + c_5\, K\{1 + \|\mathbf{x}\|^2_\gamma\}\tau_0^{2\gamma-1} \ . \end{aligned}$$

Thanks to (1.10), the conclusion is now immediate, by noting that $K\tau_0^{2\gamma-1} \lesssim T_1^{2\gamma-2}$ and then using the explicit description of T_1, T_0 in terms of $\|\mathbf{x}\|_\gamma$.

2. Singular rough equations

We fix two parameters for the whole section: $\gamma \in (\frac{1}{3}, \frac{1}{2})$ (for the general Hölder roughness) and $\beta \in [\gamma, 1]$ (encoding the singularity at time 0).

2.1. Singular rough solutions and well-posedness results

We define singular extensions of the usual Hölder spaces through the following seminorms: given a Banach space V, an interval $I \subset [0,1]$ and two parameters $\alpha \in (0,1], \mu \geqslant \alpha$, set, for any map $f : I^2 \to V$, resp. $f : I^3 \to V$,

$$\mathcal{N}[f; \mathcal{C}^{\alpha,\mu}_{2;\beta}(I;V)] := \max\left(\sup_{s<t\in I} \frac{\|f_{st}\|_V}{|t-s|^\alpha}, \sup_{0<s<t\in I} \frac{\|f_{st}\|_V}{|t-s|^\mu\, s^{\beta-1}} \right) , \qquad (2.1)$$

resp.

$$\mathcal{N}[f; \mathcal{C}^{\alpha,\mu}_{3;\beta}(I;V)] := \max\left(\sup_{s<u<t\in I} \frac{\|f_{sut}\|_V}{|t-s|^\alpha}, \sup_{0<s<u<t\in I} \frac{\|f_{sut}\|_V}{|t-s|^\mu\, s^{\beta-1}} \right) , \qquad (2.2)$$

and then

$$\mathcal{C}^{\alpha,\mu}_{1;\beta}(I;V) := \left\{ f \in \mathcal{C}_1(I;V) : \ \delta f \in \mathcal{C}^{\alpha,\mu}_{2;\beta}(I;V) \right\} . \qquad (2.3)$$

Of course, it holds that $\mathcal{C}^{\alpha,\mu}_{i;\beta}(I;V) \subset \mathcal{C}^{\alpha}_{i}(I;V)$ and $\mathcal{C}^{\alpha,\mu}_{i;1}(I;V) = \mathcal{C}^{\mu}_{i}(I;V)$.

Let us now introduce the related notion of a singular rough solution. In the sequel, given two Banach spaces V, W and a smooth map $F : V \to W$, we will denote by $D^{(\ell)}F : V \to \mathcal{L}(V^{\otimes \ell}; W)$ the ℓ-th derivative of F, understood in the usual Fréchet sense.

Definition 2.1. *Consider a path $h \in \mathcal{C}^{\gamma,1}_{1;\beta}([0,1];\mathbb{R}^m)$ and a γ-rough path $\mathbf{z} = (z, \mathbf{z}^{\mathbf{2}})$, in the sense of [?, Definition 2.1]. Then, for any fixed Banach space V, any interval $I = [t_0, t_1] \subset [0,1]$, any $v_0 \in V$ and any continuous, resp. differentiable, vector field*

$$B : V \to \mathcal{L}(\mathbb{R}^m; V) \quad , \ \textit{resp. } \Sigma : V \to \mathcal{L}(\mathbb{R}^n; V) \ ,$$

we call $y \in \mathcal{C}^{\gamma}_{1}(I;V)$ a solution (on I) of the equation

$$dy_t = B(y_t)\, dh_t + \Sigma(y_t)\, d\mathbf{z}_t \quad , \quad y_{t_0} = v_0 \ , \tag{2.4}$$

if the two-parameter path R^y defined as

$$R^y_{st} := (\delta y)_{st} - B_i(y_s)\, (\delta h^i)_{st} - \Sigma_j(y_s)\, (\delta z^j)_{st} - (D\Sigma_j \cdot \Sigma_k)(y_s)\, \mathbf{z}^{\mathbf{2},jk}_{st}$$

belongs to $\mathcal{C}^{\gamma,\mu}_{2;\beta}(I;V)$, for some parameter $\mu > 1$. Here, the notation $D\Sigma_j \cdot \Sigma_k$ stands for

$$(D\Sigma_j \cdot \Sigma_k)(v) := (D\Sigma_j)(v)(\Sigma_k(v)) \ , \ \textit{for every } v \in V \ .$$

Let us now turn to the presentation of our main results about existence and uniqueness of a solution for the rough singular equation (2.4). We will either be concerned with the classical situation of bounded vector fields (Hypothesis (VF1)) or the more general possibility of linear growth (Hypothesis (VF2)).

Hypothesis (VF1). The vector field B, resp. Σ, is $\mathcal{C}^2$, resp. $\mathcal{C}^3$, and

$$\begin{aligned} \sup_{v \in V} \|D^{(\ell)} B(v)\| \ &< \ \infty \quad \text{for } \ell \in \{1,2\} \\ \text{resp. } \sup_{v \in V} \|D^{(\ell)} \Sigma(v)\| \ &< \ \infty \quad \text{for } \ell \in \{0,\dots,3\} \ . \end{aligned}$$

Hypothesis (VF2). The vector field B, resp. Σ, is $\mathcal{C}^2$, resp. $\mathcal{C}^3$, and the following bounds hold true: for all $\ell \in \{0,1,2\}, m \in \{0,\dots,3\}$,

$$\|(D^{(\ell)}B)(v)\| \lesssim 1 + \|v\| \quad , \quad \|(D^{(m)}\Sigma)(v)\| \lesssim 1 + \|v\| \ , \tag{2.5}$$

and also, for every $v, w \in V$,

$$\|(D\Sigma \cdot \Sigma)(v)\| \lesssim 1 + \|v\| \quad , \quad \|(D\Sigma \cdot \Sigma)(v) - (D\Sigma \cdot \Sigma)(w)\| \lesssim \|v - w\|\{1 + \|v\|\} \ . \tag{2.6}$$

Theorem 2.2 ((VF1)-situation)**.** *Under Hypothesis (VF1), and for any $v_0 \in V$, Equation (2.4) admits a unique solution on $[0,1]$ with initial condition v_0, in the sense of Definition 2.1.*

Theorem 2.3 ((VF2)-situation)**.** *Under Hypothesis (VF2), the following assertions hold true:*

(i) For any $v_0 \in V$, Equation (2.4) admits at most one solution on $[0,1]$ with initial condition v_0, in the sense of Definition 2.1.

(ii) For every $K \geqslant 1$, there exists $M_K > 0$ such that if

$$\|v_0\| \leqslant K \ , \ \mathcal{N}[\delta h; \mathcal{C}_{2;\beta}^{\gamma,1}([0,1])] \leqslant K \text{ and } \|\mathbf{z}\|_{\gamma;[0,1]} \leqslant M_K \ ,$$

then Equation (2.4) admits a unique solution y on $[0,1]$ with initial condition v_0, in the sense of Definition 2.1. Besides,

$$\mathcal{N}[y; \mathcal{C}_1^0([0,1];V)] + \mathcal{N}[y; \mathcal{C}_1^\gamma([0,1];V)] \leqslant C(K) \ , \tag{2.7}$$

for some growing function $C : \mathbb{R}^+ \to \mathbb{R}^+$.

Just as in the above Section 1, and in the same spirit as in [**?**], our proof for both Theorem 2.2 and Theorem 2.3 relies on the examination of the discrete scheme associated with the equation. Set $t_i = t_i^n := \frac{i}{2^n}$, $\mathcal{P}_n := \{t_i : i = 0, \ldots, 2^n\}$ and define y^n along the iterative formula: $y_0^n = v_0$ and

$$(\delta y^n)_{t_i t_{i+1}} = B(y_{t_i}^n)\,(\delta h)_{t_i t_{i+1}} + \Sigma(y_{t_i}^n)\,(\delta z)_{t_i t_{i+1}} + (D\Sigma \cdot \Sigma)(y_{t_i}^n)\,\mathbf{z}^{\mathbf{2}}_{t_i t_{i+1}} \ .$$

Then, for every $s, t \in \mathcal{P}_n$, set

$$R_{st}^n := (\delta y^n)_{st} - B(y_s^n)\,(\delta h)_{st} - \Sigma(y_s^n)\,(\delta z)_{st} - (D\Sigma \cdot \Sigma)(y_s^n)\,\mathbf{z}^{\mathbf{2}}_{st} \ ,$$

noting in particular that $R^n_{t_i t_{i+1}} = 0$. We will also consider the paths

$$L_{st}^n := (\delta y^n)_{st} - \Sigma(y_s^n)\,(\delta z)_{st} - \ (D\Sigma \cdot \Sigma)(y_s^n)\,\mathbf{z}^{\mathbf{2}}_{st} \quad \big(= R_{st}^n + B(y_s^n)(\delta h)_{st} \ \big)$$

and

$$Q_{st}^n := (\delta y^n)_{st} - \Sigma(y_s^n)\,(\delta z)_{st} \ .$$

Finally, for every $s < t \in [0,1]$, we will write $[\![s,t]\!] = [\![s,t]\!]_n := [s,t] \cap \mathcal{P}_n$, and we extend (or rather restrict) the norms (2.1)-(2.3) to discrete paths as

$$\mathcal{N}[f; \mathcal{C}_{2;\beta}^{\alpha,\mu}([\![s,t]\!];V)] := \max\left(\sup_{u<v\in[\![s,t]\!]} \frac{\|f_{uv}\|_V}{|v-u|^\alpha}, \sup_{0<u<v\in[\![s,t]\!]} \frac{||f_{uv}\|_V}{|v-u|^\mu\, u^{\beta-1}} \right) ,$$

with a similar definition for $\mathcal{N}[f; \mathcal{C}_{i;\beta}^{\alpha,\mu}([\![s,t]\!];V)]$, $i \in \{1,3\}$.

The whole key towards the desired estimates lies in the following "singular sewing lemma":

Lemma 2.4. *Let $0 < \alpha \leqslant \lambda \leqslant 1$, $\mu_1 \geqslant 1$ and $\mu_2 > 1$. Then there exists a constant $c_{\alpha,\lambda,\mu_1,\mu_2} > 0$ such that for every path $G : \mathcal{P}_n^2 \to V$ and all $s \leqslant t \in \mathcal{P}_n$, one has*

$$\mathcal{N}[G; \mathcal{C}_{2;\lambda}^{\alpha,\mu_1 \wedge \mu_2}([\![s,t]\!];V)] \leqslant c_{\alpha,\lambda,\mu_1,\mu_2}\big\{\mathcal{M}_\lambda^{\alpha,\mu_1}\big[G;[\![s,t]\!]\big] + \mathcal{N}[\delta G; \mathcal{C}_{3;\lambda}^{\alpha,\mu_2}([\![s,t]\!];V)]\big\} \ ,$$

where we have set, if $s = \frac{p}{2^n}$ and $t = \frac{q}{2^n}$,

$$\mathcal{M}_\lambda^{\alpha,\mu_1}\big[G;[\![s,t]\!]\big] := \max\bigg(\sup_{p\leqslant i\leqslant q} \frac{\|G_{t_i t_{i+1}}\|}{|t_{i+1}-t_i|^\alpha} \ , \ \sup_{p+1\leqslant i\leqslant q} \frac{\|G_{t_i t_{i+1}}\|}{|t_{i+1}-t_i|^{\mu_1} t_i^{\lambda-1}}\bigg) \ .$$

Proof. See Appendix 4. ☐

2.2. Existence of a solution in the (VF2)-situation

Proposition 2.5. *Let Hypothesis (VF2) prevail and assume additionally that*

$$\mathcal{N}[\delta h; \mathcal{C}_{2;\beta}^{\gamma,1}([0,1];V)] \leqslant K \ , \ \textit{for some } K \geqslant 1 \ .$$

Then there exists a constant c_0 (which depends only on B, Σ, γ and β) such that if we set $T_0 = T_0(K) := \min\big(1, (c_0 K)^{-6/(3\gamma-1)}\big)$, the following assertion holds true for every $k \leqslant 1/T_0$: if $\|\mathbf{z}\|_{\gamma;[0,1]} \leqslant \big(1 + \|y^n_{kT_0}\|\big)^{-1}$, then

$$\mathcal{N}[L^n; \mathcal{C}_{2;\beta}^{\gamma,1}([\![kT_0,(k+1)T_0 \wedge 1]\!];V)] \leqslant c_0\, K\big\{1 + \|y^n_{kT_0}\|\big\} \ . \tag{2.8}$$

Proof. The strategy consists in an iteration procedure over the points of the partition. So, assume that (2.8) holds true on an interval $[\![0,t_q]\!]$, with $t_q \leqslant T_0$ (for some time T_0 to be determined along the proof). In other words, assume that

$$\mathcal{N}[L^n; \mathcal{C}_{2;\beta}^{\gamma,1}([\![0,t_q]\!];V)] \leqslant c_K\big\{1 + \|y^n_0\|\big\} \ , \tag{2.9}$$

where we denote from now on $c_K := c_0 K$ (for some constant c_0 to be fixed later on). Due to (2.5) and (2.6), it is then easy to check that the following bounds hold true as well:

$$\mathcal{N}[y^n; \mathcal{C}_1^0([\![0,t_q]\!])] \lesssim \{1 + \|y^n_0\|\}\{1 + c_K T_0^\gamma\} \ , \tag{2.10}$$

and

$$\max\big(\mathcal{N}[y^n; \mathcal{C}_1^\gamma([\![0,t_q]\!])], \mathcal{N}[Q^n; \mathcal{C}_{2;\beta}^{\gamma,2\gamma}([\![0,t_q]\!])]\big) \lesssim \{1 + \|y^n_0\|\}\{1 + c_K\} \ . \tag{2.11}$$

Now, in order to extend (2.9) on $[\![0,t_{q+1}]\!]$ (assuming that $t_{q+1} \leqslant T_0$), let us first apply Lemma 2.4 to L^n and assert that

$$\mathcal{N}[L^n; \mathcal{C}_{2;\beta}^{\gamma,1}([\![0,t_{q+1}]\!])] \lesssim \mathcal{M}_\beta^{\gamma,1}\big[L^n;[\![0,t_{q+1}]\!])] + \mathcal{N}[\delta L^n; \mathcal{C}_{3;\beta}^{\gamma,3\kappa}([\![0,t_{q+1}]\!])] \ , \tag{2.12}$$

where we set from now on $\kappa := \frac{1}{2}\big(\frac{1}{3} + \gamma\big)$, so that $1 < 3\kappa < 3\gamma$.

As far as the first term is concerned, we can use the fact that $R^n_{t_i t_{i+1}} = 0$, and then (2.5) and (2.10), to deduce that

$$\begin{aligned}\mathcal{M}^{\gamma,1}_{\beta}\big[L^n; [\![0,t_{q+1}]\!])\big] &= \mathcal{N}[B(y^n)\,\delta h; \mathcal{C}^{\gamma,1}_{2;\beta}([\![0,t_{q+1}]\!])] \\ &\leqslant K\cdot\mathcal{N}[B(y^n); \mathcal{C}^0_1([\![0,t_q]\!]] \lesssim K\{1+\|y^n_0\|\}\{1+c_K T_0^{\gamma}\} .\end{aligned}$$

In order to estimate $\mathcal{N}[\delta L^n; \mathcal{C}^{\gamma,3\kappa}_{3;\beta}([\![0,t_{q+1}]\!])]$, let us first rely on Chen relation and decompose the increments of L^n as $\delta L^n = I^i\,\delta z^i + II^i\,\delta z^i + III^{ij}\,\mathbf{z}^{\mathbf{2},ij}$, where we have set

$$I^i_{st} := \int_0^1 d\lambda\,(D\Sigma_i)(y^n_s + \lambda(\delta y^n)_{st})\,Q^n_{st} , \tag{2.13}$$

$$II^i_{st} := \int_0^1 d\lambda\,\big[(D\Sigma_i)(y^n_s+\lambda(\delta y^n)_{st}) - (D\Sigma_i)(y^n_s)\big]\,\Sigma_j(y^n_s)\,(\delta z^j)_{st} , \tag{2.14}$$

$$III^{ij}_{st} := \delta(D\Sigma_i\cdot\Sigma_j)(y^n)_{st} . \tag{2.15}$$

For $I^i\,\delta z^i$, we can combine (2.5), (2.10) and (2.11) to get that

$$\begin{aligned}&\mathcal{N}[I^i\,\delta z^i; \mathcal{C}^{\gamma,3\kappa}_{3;\beta}([\![0,t_{q+1}]\!])] \\ &\lesssim T_0^{3(\gamma-\kappa)}\big(\{1+\|y^n_0\|\}\{1+c_K\}\big)^2\,\|\mathbf{z}\|_{\gamma;[0,1]} \\ &\lesssim \{1+\|y^n_0\|\}\{1+T_0^{3(\gamma-\kappa)}c_K^2\}\cdot\Big(\{1+\|y^n_0\|\}\|\mathbf{z}\|_{\gamma;[0,1]}\Big) \\ &\lesssim \{1+\|y^n_0\|\}\{1+T_0^{3(\gamma-\kappa)}c_K^2\} ,\end{aligned}$$

where we have used the assumption $\{1+\|y^n_0\|\}\|\mathbf{z}\|_{\gamma;[0,1]}\leqslant 1$ to derive the third inequality.

With similar arguments, we can show that

$$\begin{aligned}&\mathcal{N}[II^i\,\delta z^i; \mathcal{C}^{\gamma,3\kappa}_{3;\beta}([\![0,t_{q+1}]\!])] \\ &\lesssim \{1+\|y^n_0\|\}\{1+T_0^{3(\gamma-\kappa)}c_K^3\}\cdot\Big(\{1+\|y^n_0\|\}^2\|\mathbf{z}\|^2_{\gamma;[0,1]}\Big) \\ &\lesssim \{1+\|y^n_0\|\}\{1+T_0^{3(\gamma-\kappa)}c_K^3\} .\end{aligned}$$

Finally, thanks to the second estimate in (2.6), we obtain that

$$\mathcal{N}[III^{ij}\,\mathbf{z}^{\mathbf{2},ij}; \mathcal{C}^{\gamma,3\kappa}_{3;\beta}([\![0,t_{q+1}]\!])] \lesssim \{1+\|y^n_0\|\}\{1+T_0^{3(\gamma-\kappa)}c_K^2\} .$$

Going back to (2.12), we have shown that, for some constant c_1 depending only on B, Σ and (γ,κ,β),

$$\mathcal{N}[L^n; \mathcal{C}^{\gamma,1}_{2;\beta}([\![0,t_{q+1}]\!])] \leqslant \{1+\|y^n_0\|\}\cdot\Big(c_1 K\{1+T_0^{3(\gamma-\kappa)}c_K^3\}\Big) .$$

Let us now set $c_0 := 2c_1$, $c_K := c_0 K$ and $T_0 := \min(1,(2c_1K)^{-1/(\gamma-\kappa)})$, in such a way that

$$c_1 K\{1+T_0^{3(\gamma-\kappa)}c_K^3\} \leqslant c_K ,$$

and accordingly $\mathcal{N}[L^n;\mathcal{C}_{2;\beta}^{\gamma,1}([\![0,t_{q+1}]\!])] \leqslant c_K\{1+\|y_0^n\|\}$ as desired.

This iteration procedure allows us to extend the bound (2.9) over the interval $[\![0,T_0]\!]$. Then it is easy to see that the very same arguments can be used for any interval $[\![kT_0,(k+1)T_0]\!]$, which completes the proof.

□

Corollary 2.6. *Let Hypothesis (VF2) prevail and assume additionally that*

$$\mathcal{N}[\delta h;\mathcal{C}_{2;\beta}^{\gamma,1}([0,1];V)] \leqslant K \quad \text{and} \quad \|v_0\| \leqslant K \ , \ \text{for some } K \geqslant 1 \ .$$

Then there exists $M_K>0$ such that if $\|\mathbf{z}\|_{\gamma;[0,1]} \leqslant M_K$, one has

$$\sup_{n\geqslant 0} \max\Big(\mathcal{N}[y^n;\mathcal{C}_1^0([\![0,1]\!])],\mathcal{N}[y^n;\mathcal{C}_1^\gamma([\![0,1]\!])], \\ \mathcal{N}[Q^n;\mathcal{C}_{2;\beta}^{\gamma,2\gamma}([\![0,1]\!])],\mathcal{N}[L^n;\mathcal{C}_{2;\beta}^{\gamma,1}([\![0,1]\!])]\Big) \leqslant C(K) \ , \tag{2.16}$$

for some growing function $C:\mathbb{R}^+\to\mathbb{R}^+$. As a result, under the same assumptions and if $\|\mathbf{z}\|_{\gamma;[0,1]} \leqslant M_K$, it holds that

$$\sup_{n\geqslant 0} \mathcal{N}[R^n;\mathcal{C}_{2;\beta}^{\gamma,3\gamma}([\![0,1]\!])] \leqslant D(K) \ , \tag{2.17}$$

for some growing function $D:\mathbb{R}^+\to\mathbb{R}^+$.

Proof. Using (2.8) as well as its spin-offs (2.10) and (2.11), it is not hard to exhibit a growing sequence (c_k) that depends only on (B,Σ,γ,β) (and not on K) such that the following property holds true: for every $k\geqslant 0$, if $\|\mathbf{z}\|_{\gamma;[0,1]} \leqslant (1+c_k\{1+K\})^{-1}$, then one has both

$$\mathcal{N}[y^n;\mathcal{C}_1^0(I_k)] \leqslant c_{k+1}\{1+K\} \tag{2.18}$$

and

$$\max\big(\mathcal{N}[y^n;\mathcal{C}_1^\gamma(I_k)],\mathcal{N}[Q^n;\mathcal{C}_{2;\beta}^{\gamma,2\gamma}(I_k)],\mathcal{N}[L^n;\mathcal{C}_{2;\beta}^{\gamma,1}(I_k)]\big) \leqslant c_{k+1}\{1+K^2\} \ , \tag{2.19}$$

where we have set $I_k := [\![kT_0,(k+1)T_0]\!]$. As a result, if we denote by N_K the smallest integer such that $T_0N_K \geqslant 1$ and assume that $\|\mathbf{z}\|_{\gamma;[0,1]} \leqslant M_K := (1+c_{N_K}(1+K))^{-1}$, then both bounds (2.18) and (2.19) hold true for $k=0,\ldots,N_K-1$. The extension of these local bounds into global ones (that is, on the interval $[\![0,1]\!]$) is then a matter of standard arguments, which achieves the proof of (2.16).

As far as (2.17) is concerned, apply first Lemma 2.4 to the path R^n, which, since $R^n_{t_it_{i+1}}=0$, entails that

$$\mathcal{N}[R^n;\mathcal{C}_{2;\beta}^{\gamma,3\gamma}([\![0,1]\!];V)] \lesssim \mathcal{N}[\delta R^n;\mathcal{C}_{3;\beta}^{\gamma,3\gamma}([\![0,1]\!];V)]\} \ .$$

Then, just as in the proof of Proposition 2.5, observe that we can decompose the increments of R^n as

$$(\delta R^n)_{sut} = \delta B(y^n)_{su}(\delta h)_{ut} \\ + (\delta L^n)_{sut} = \delta B(y^n)_{su}(\delta h)_{ut} + I^i_{su}\delta z^i_{ut} + II^i_{su}\delta z^i_{ut} + III^{ij}_{su}\mathbf{z}^{\mathbf{2},ij}_{ut} \ , \tag{2.20}$$

where the paths (I, II, III) have been defined through (2.13)-(2.15). The conclusion is now easy to derive from the bound (2.16).

□

Proof of Theorem 2.3, point (ii). Consider the sequence (still denoted by y^n) of continuous paths on $[0,1]$ defined through the linear interpolation of the points of the previous (discrete) sequence y^n. Define M_K as in Corollary 2.6 and assume that $\|\mathbf{z}\|_{\gamma;[0,1]} \leqslant M_K$. Then it is readily checked that (2.16) gives rise to a uniform bound for $\mathcal{N}[y^n; \mathcal{C}_1^\gamma([0,1];V)]$, and we can therefore conclude about the existence of a path $y \in \mathcal{C}_1^\gamma([0,1];V)$, as well as a subsequence of y^n (that we still denote by y^n), such that $y^n \to y$ in $\mathcal{C}_1^\kappa([0,1];V)$ for every $0 < \kappa < \gamma$.

The fact that y actually defines a solution of (2.4) is essentially obtained by passing to the limit in the uniform estimate (2.17). The details of this (easy) procedure can for instance be found at the end of [**?**, Section 3.3]. As for the bound (2.7), it is a straightforward consequence of (2.16). □

2.3. Existence of a solution in the (VF1)-situation

Under Hypothesis (VF1), the exhibition of a uniform bound for

$$\mathcal{N}[R^n; \mathcal{C}^{\gamma,\mu}_{2;\beta}([\![0,1]\!])] \quad (\text{with } \mu > 1)$$

essentially follows the same general procedure as in the classical ('non-singular') situation treated in [**?**] or [**?**]. As we here consider slightly more specific topologies, let us briefly review the result at the core of this procedure.

Proposition 2.7. *Let Hypothesis (VF1) prevail and assume additionally that*

$$\mathcal{N}[\delta h; \mathcal{C}^{\gamma,1}_{2;\beta}([0,1];\mathbb{R}^m)] \leqslant K \ , \ \textit{for some } K \geqslant 1 \ .$$

Also, fix a parameter κ such that $1 < 3\kappa < 3\gamma$. Then there exists a a constant c_0 (which depends only on B, Σ, γ, β and κ) such that if we set

$$T_0 = T_0(\|\mathbf{z}\|_\gamma, K) := \min\Big(1, \Big(c_0\big\{1 + \|\mathbf{z}\|_\gamma\big\}K\Big)^{-1/(\gamma-\kappa)}\Big) \ ,$$

the following property holds true: for every $0 < T_1 < T_0$ and every $k \leqslant 1/T_1$,

$$\mathcal{N}[R^n; \mathcal{C}^{\gamma,3\kappa}_{2;\beta}([\![kT_1, (k+1)T_1 \wedge 1]\!])] \leqslant c_0\, K\big\{1 + \|y^n_{kT_1}\|\big\} \ . \tag{2.21}$$

Proof. Just as in the proof of Proposition 2.5, the strategy consists in an iteration procedure over the points of $\mathcal{P}_n$. The argument actually relies on the following two readily-checked assertions: (i) If $\mathcal{N}[R^n;\mathcal{C}_{2;\beta}^{\gamma,3\kappa}([\![s,t]\!])] \leqslant c_0\, K\big\{1+\|y_s^n\|\big\}$, then one has

$$\max\big(\mathcal{N}[y^n;\mathcal{C}_1^{\gamma}([\![s,t]\!])],\mathcal{N}[Q^n;\mathcal{C}_{2;\beta}^{\gamma,2\gamma}([\![s,t]\!])]\big)$$
$$\leqslant c_1\, K\big[\|\mathbf{z}\|_\gamma+\big\{1+\|y_s^n\|\big\}\big\{1+c_0\big\}\big\{1+K|t-s|^\gamma\big\}\big]$$

for some constant c_1 that depends only on (B,Σ); (ii) With decomposition (2.20) in mind, one has

$$\mathcal{N}[\delta R^n;\mathcal{C}_{3;\beta}^{\gamma,3\kappa}([\![s,t]\!])]$$
$$\leqslant c_2\,|t-s|^{3(\gamma-\kappa)}\big[\mathcal{N}[y^n;\mathcal{C}_1^{\gamma}([\![s,t]\!])]\{1+K+\|\mathbf{z}\|_\gamma^2\}+\mathcal{N}[Q^n;\mathcal{C}_{2;\beta}^{\gamma,2\gamma}([\![s,t]\!])]\|\mathbf{z}\|_\gamma\big]$$

for some constant c_2 that depends only on (B,Σ).

It is now easy to inject (i) and (ii) into the iteration scheme exhibited in the previous section for L^n (note that we can additionally use the fact that $R^n_{t_it_{i+1}}=0$ here). The details of the procedure are therefore left to the reader. □

Proof of the existence statement in Theorem 2.2. Starting from (2.21) and using the same steps as in the proof of Corollary 2.6, one easily gets uniform estimates for both $\mathcal{N}[y^n;\mathcal{C}_1^{\gamma}([\![0,1]\!])]$ and $\mathcal{N}[R^n;\mathcal{C}_{2;\beta}^{\gamma,3\kappa}([\![0,1]\!])]$. The derivation of a solution then follows from the same convergence argument as in the above proof of Theorem 2.3, point (ii). □

2.4. Uniqueness of the solution

It is a well-known fact that uniqueness statements are usually less demanding than existence statements as far as global boundedness of the vector fields is concerned. Accordingly, in opposition with the previous existence proof (where specific sharp estimates had to be displayed), the strategy towards uniqueness essentially follows the same lines as in the standard situation. We briefly review the transposition of the main arguments in this singular setting.

Assume here that either Hypothesis (VF1) or Hypothesis (VF2) prevails and consider two solutions $U,\widetilde{U}$ of (2.4) with identical initial conditions. Then set

$$R_{st}=R(y)_{st}:=(\delta y)_{st}-B_i(y_s)\,(\delta h^i)_{st}-\Sigma_j(y_s)\,(\delta z^j)_{st}-(D\Sigma_j\cdot\Sigma_k)(y_s)\,\mathbf{z}^{\mathbf{2},jk}_{st}\ ,$$

$$Q_{st}=Q(y)_{st}:=(\delta y)_{st}-\Sigma_j(y_s)\,(\delta z^j)_{st}\ ,$$

and similarly $\widetilde{R}:=R(\widetilde{y})$, $\widetilde{Q}:=Q(\widetilde{y})$. Also, fix μ, resp. $\tilde{\mu}>1$ such that $\mathcal{N}[R;\mathcal{C}_{2;\beta}^{\gamma,\mu}([0,1])]<\infty$, resp. $\mathcal{N}[\widetilde{R};\mathcal{C}_{2;\beta}^{\gamma,\mu}([0,1])]<\infty$, as well as a parameter κ satisfying both $\frac{1}{3}<\kappa<\gamma$ and $3\kappa<\mu\wedge\tilde{\mu}$.

Lemma 2.8. *There exists a finite constant $c_{R,\tilde{R}} > 0$ such that for every $s < t \in \mathcal{P}_n$, one has*

$$\mathcal{N}[R - \tilde{R}; \mathcal{C}_{2;\beta}^{\kappa,3\kappa}([\![s,t]\!])] \leqslant c_{R,\tilde{R}} \cdot \{2^{-n\varepsilon} + \mathcal{N}[\delta(R - \tilde{R}); \mathcal{C}_{3;\beta}^{\kappa,3\kappa}([\![s,t]\!])]\} \ , \quad (2.22)$$

where $\varepsilon := \inf(\gamma - \kappa, (\mu \wedge \tilde{\mu}) - 3\kappa) > 0$.

Proof. It is a mere application of Lemma 2.4. Observe indeed that

$$\begin{aligned} \mathcal{M}_\beta^{\kappa,3\kappa}\big[R - \tilde{R}; [\![s,t]\!]\big] &\leqslant \mathcal{M}_\beta^{\kappa,3\kappa}\big[R; [\![s,t]\!]\big] + \mathcal{M}_\beta^{\kappa,3\kappa}\big[\tilde{R}; [\![s,t]\!]\big] \\ &\leqslant c_{R,\tilde{R}} \{2^{-n(\gamma-\kappa)} + 2^{-n(\mu_1 - 3\kappa)} + 2^{-n(\mu_2 - 3\kappa)}\} \ . \end{aligned}$$

□

Lemma 2.9. *There exists a finite constant $C_{y,\tilde{y}} > 0$ such that for every $s < t \in \mathcal{P}_n$, one has*

$$\mathcal{N}[\delta(R - \tilde{R}); \mathcal{C}_{3;\beta}^{\kappa,3\kappa}([\![s,t]\!])] \leqslant C_{y,\tilde{y}} \, |t-s|^{\gamma-\kappa} \, \mathcal{N}_\beta^{\kappa,2\gamma}\big[(y,\tilde{y}); [\![s,t]\!]\big] \ , \quad (2.23)$$

where we have set

$$\begin{aligned} &\mathcal{N}_\beta^{\kappa,2\gamma}\big[(y,\tilde{y}); [\![s,t]\!]\big] := \\ &\mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0([\![s,t]\!])] + \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\kappa([\![s,t]\!])] + \mathcal{N}[Q - \tilde{Q}; \mathcal{C}_{2;\beta}^{\kappa,2\gamma}([\![s,t]\!])] \ . \end{aligned} \quad (2.24)$$

Proof. First, note that the increments of R (or $\tilde{R}$) can be decomposed just as the increments of R^n in the proof of Corollary 2.6 (see (2.20)), which allows us to write

$$\begin{aligned} \delta(R - \tilde{R})_{sut} &= \delta(B(y) - B(\tilde{y}))_{su} \, \delta h_{ut} \\ &\quad + \big[I_{su}^i - \tilde{I}_{su}^i\big] \, \delta z_{ut}^i + \big[II_{su}^i - \widetilde{II}_{su}^i\big] \, \delta z_{ut}^i + \big[III_{su}^{ij} - \widetilde{III}_{su}^{ij}\big] \, \mathbf{z}_{ut}^{\mathbf{2},ij} \ , \end{aligned}$$

where the paths I, II, III, resp. $\tilde{I}, \widetilde{II}, \widetilde{III}$, are defined along (2.13)-(2.15) (replace (y^n, Q^n) with (y, Q), resp. $(\tilde{y}, \tilde{Q})$). The bound (2.23) is then obtained through standard differential-calculus arguments based on relations (2.5) and (2.6).

□

Proof of Theorem 2.3, point (i), and uniqueness property of Theorem 2.2. Consider the above setting and notations. First, going back to the very definitions of (K, R) and $(\tilde{K}, \tilde{R})$, it is not hard to check that for every $s < t \in \mathcal{P}_n$, one has, with the notation (2.24),

$$\begin{aligned} &\mathcal{N}_\beta^{\kappa,2\gamma}\big[(y,\tilde{y}); [\![s,t]\!]\big] \\ &\leqslant c_{y,\tilde{y}} \Big\{ \|y_s - \tilde{y}_s\| + |t-s|^\kappa \, \mathcal{N}_\beta^{\kappa,2\gamma}\big[(y;\tilde{y}); [\![s,t]\!]\big] + \mathcal{N}\big[R - \tilde{R}; \mathcal{C}_{2;\beta}^{\kappa,3\kappa}([\![s,t]\!]\big] \Big\} \ , \end{aligned}$$

where the constant $c_{y,\tilde{y}}$ does not depend on n. We can then combine (2.22)-(2.23) and assert that for every $s < t \in \mathcal{P}_n$,

$$\mathcal{N}_\beta^{\kappa,2\gamma}\big[(y,\tilde{y}); [\![s,t]\!]\big] \leqslant c_{y,\tilde{y}} \Big\{ \|y_s - \tilde{y}_s\| + |t-s|^{\gamma-\kappa} \, \mathcal{N}_\beta^{\kappa,2\gamma}\big[(y;\tilde{y}); [\![s,t]\!]\big] + 2^{-n\varepsilon} \Big\} \ .$$

The uniqueness result is now immediate. Indeed, for $T_0 > 0$ such that $c_{y,\tilde{y}}\, T_0^{\gamma-\kappa} \leqslant \frac{1}{2}$, and since $y_0 = \tilde{y}_0$, we first get that

$$\mathcal{N}\big[y-\tilde{y};\mathcal{C}_1^0([\![0,T_0]\!])\big] \leqslant \mathcal{N}_\beta^{\kappa,2\gamma}\big[(y,\tilde{y});[\![0,T_0]\!]\big] \leqslant C_{y,\tilde{y}}\cdot 2^{-n\varepsilon}\ ,$$

and accordingly $y_t = \tilde{y}_t$ for every $t \in [0,T_0]$. The argument can then be repeated on $[T_0, 2T_0]$, $[2T_0, 3T_0]$, and so on. □

3. Singular paths and canonical lift

Let us recall that the space $\mathcal{E}_\gamma^2([0,1];\mathbb{R}^d)$, as well as the notation $|\!|\!| f |\!|\!|_{1;\gamma}$, have been introduced in [**?**, Section 4.1]. Besides, let us denote by $\mathcal{C}^1([0,1];\mathbb{R}^d)$ the space of differentiable $\mathbb{R}^d$-valued paths on $[0,1]$ with continuous derivative.

Proposition 3.1. *Let $z \in \mathcal{C}_1^\gamma([0,1];\mathbb{R}^d)$ be a path that can be canonically lifted into a rough path $\mathfrak{L}(z)$, in the sense of [**?**, Definition 2.2], and let $g \in \mathcal{E}_\gamma^2([0,1];\mathbb{R}^d)$, resp. $g \in \mathcal{C}^1([0,1];\mathbb{R}^d)$. Then $z+g$ can be canonically lifted into a rough path $\mathfrak{L}(z+g)$ and it holds that*

$$\begin{aligned}&\mathcal{N}[\mathfrak{L}(z+g)^{\mathbf{2}} - \mathfrak{L}(z)^{\mathbf{2}};\mathcal{C}_{2,\gamma}^{2\gamma,1+\gamma}([0,1];\mathbb{R}^{d,d})]\\ &\leqslant c_\gamma\big\{1 + |\!|\!| g|\!|\!|_{1;\gamma}^2 + \mathcal{N}[z;\mathcal{C}_1^\gamma([0,1];\mathbb{R}^d)]^2\big\}\ ,\end{aligned} \tag{3.1}$$

resp.

$$\begin{aligned}&\mathcal{N}[\mathfrak{L}(z+g)^{\mathbf{2}} - \mathfrak{L}(z)^{\mathbf{2}};\mathcal{C}_2^{1+\gamma}([0,1];\mathbb{R}^{d,d})]\\ &\leqslant c_\gamma\big\{1 + \mathcal{N}[g;\mathcal{C}^1([0,1];\mathbb{R}^d)]^2 + \mathcal{N}[z;\mathcal{C}_1^\gamma([0,1];\mathbb{R}^d)]^2\big\}\ ,\end{aligned} \tag{3.2}$$

for some constant c_γ that depends only on γ.

The two following results, which are extensively used in our analysis, are immediate consequences of (3.1) and (3.2).

Corollary 3.2. *Let $z \in \mathcal{C}_1^\gamma([0,1];\mathbb{R}^d)$ be a path that can be canonically lifted into a rough path $\mathfrak{L}(z)$ and let $g \in \mathcal{E}_\gamma^2([0,1];\mathbb{R}^d)$, resp. $g \in \mathcal{C}^1([0,1];\mathbb{R}^d)$. Then, in the setting of Definition 2.1 (with $\beta := \gamma$, resp. $\beta = 1$), a path $y : [0,1] \to V$ is a solution of*

$$dy_t = B(y_t)\, dh_t + \Sigma(y_t)\, d\mathfrak{L}(z+g)_t \quad , \quad y_0 = v_0\ ,$$

if and only if y is a solution of

$$dy_t = \big[B(y_t)\, dh_t + \Sigma(y_t)\, dg_t\big] + \Sigma(y_t)\, d\mathfrak{L}(z)_t \quad , \quad y_0 = v_0\ .$$

Corollary 3.3. *Let $z \in \mathcal{C}_1^\gamma([0,1];\mathbb{R}^d)$ be a path that can be canonically lifted into a rough path $\mathfrak{L}(z)$ and let $g \in \mathcal{E}_\gamma^2([0,1];\mathbb{R}^d)$. Then it holds that*

$$\|\mathfrak{L}(z+g)\|_{\gamma;[0,1]} \leqslant c_\gamma\big\{1 + \|\mathfrak{L}(z)\|_{\gamma;[0,1]}^2 + |\!|\!| g|\!|\!|_{1;\gamma}^2\big\}\ , \tag{3.3}$$

for some constant c_γ that depends only on γ.

We will only prove Proposition 3.1 in the situation where $g \in \mathcal{E}^2_\gamma([0,1];\mathbb{R}^d)$, but the proof when $g \in \mathcal{C}^1([0,1];\mathbb{R}^d)$ could be derived from the very same arguments.

Lemma 3.4. *Let $g \in \mathcal{E}^2_\gamma([0,1];\mathbb{R}^d)$ and denote by g^n the linear interpolation of g along the dyadic partition $\mathcal{P}_n$ of $[0,1]$. Then it holds that*

$$\sup_n \sup_{t\in(0,1]\setminus\mathcal{P}_n} t^{1-\gamma}|(g^n)'_t| \lesssim \|\!|g|\!\|_{1;\gamma} \tag{3.4}$$

and for every $0<\gamma'<\gamma$,

$$\sup_{t\in(0,1]\setminus\mathcal{P}_n} t^{1-\gamma'}|(g^n-g)'_t| \lesssim \|\!|g|\!\|_{2;\gamma} 2^{-n(\gamma-\gamma')} . \tag{3.5}$$

Proof. Pick $t\in(t^n_i,t^n_{i+1})$, for some $i=0,\dots,2^n$. One has

$$t^{1-\gamma}|(g^n)'_t| = \frac{t^{1-\gamma}}{t^n_{i+1}-t^n_i}|g_{t^n_{i+1}}-g_{t^n_i}| \leqslant \|\!|g|\!\|_{1;\gamma}\, t^{1-\gamma}\int_0^1 \frac{dr}{(t^n_i+r(t^n_{i+1}-t^n_i))^{1-\gamma}} .$$

If $i=0$, then $t\leqslant t^n_{i+1}-t^n_i$ and so $t^{1-\gamma}|(g^n)'_t| \leqslant \|\!|g|\!\|_{1;\gamma}\int_0^1 \frac{dr}{r^{1-\gamma}}$. If $i\geqslant 1$, then $\frac{t}{2}\leqslant \frac{i+1}{2^{n+1}} \leqslant \frac{i}{2^n} = t^n_i$, and so $t^{1-\gamma}|(g^n)'_t| \leqslant \|\!|g|\!\|_{1;\gamma}(t/t^n_i)^{1-\gamma} \leqslant \|\!|g|\!\|_{1;\gamma}2^{1-\gamma}$, which completes the proof of (3.4).

For (3.5), note first that if $i=0$, then $t\leqslant 2^{-n}$ and so by (3.4) we get in this case

$$t^{1-\gamma'}|(g^n-g)'_t| \leqslant 2^{-n(\gamma-\gamma')}\big\{t^{1-\gamma}|(g^n)'_t| + t^{1-\gamma}|g'_t|\big\} \lesssim \|\!|g|\!\|_{1;\gamma}2^{-n(\gamma-\gamma')} .$$

If $i\geqslant 1$, then

$$\begin{aligned} t^{1-\gamma'}|(g^n-g)'_t| &= \frac{t^{1-\gamma'}}{t^n_{i+1}-t^n_i}\left|\int_{t^n_i}^{t^n_{i+1}}\{g'_r-g'_t\}\,dr\right| \\ &\lesssim \|\!|g|\!\|_{2;\gamma}\frac{t^{1-\gamma'}}{t^n_{i+1}-t^n_i}\int_{t^n_i}^{t^n_{i+1}}\frac{|t-u|}{(t^n_i)^{2-\gamma}}\,du \\ &\lesssim \|\!|g|\!\|_{2;\gamma}\left(\frac{t}{t^n_i}\right)^{1-\gamma'}\left(\frac{t^n_{i+1}-t^n_i}{t^n_i}\right)^{1-(\gamma-\gamma')}2^{-n(\gamma-\gamma')} . \end{aligned}$$

As above, we can conclude by using the fact that in this case, one has

$$\max\big(\frac{t}{2}, t^n_{i+1}-t^n_i\big) \leqslant t^n_i .$$

□

Proof of Proposition 3.1. Denote by z^n, resp. g^n, the linear interpolation of z, resp. g, along the dyadic partition $\mathcal{P}_n$. By (3.5), the convergence of g^n to g (and accordingly the convergence of z^n+g^n to $z+g$) in $\mathcal{C}_1^{\gamma'}([0,1];\mathbb{R}^d)$ is immediate, since

$$\begin{aligned}|\delta(g^n-g)_{st}| \leqslant \int_s^t |(g^n-g)'_u|\,du &\lesssim 2^{-n(\gamma-\gamma')}(t-s)\int_0^1 \frac{dr}{(s+r(t-s))^{1-\gamma'}}\\ &\lesssim 2^{-n(\gamma-\gamma')}(t-s)^{\gamma'}\,.\end{aligned}$$

Then, by setting $x := z+g$, we have the following readily-checked decomposition

$$\mathbf{x}^{\mathbf{2},n}_{st}-\mathbf{z}^{\mathbf{2},n}_{st} = \int_s^t (\delta z^n)_{su}\otimes dg^n_u + \left(\int_s^t (\delta z^n)_{ut}\otimes dg^n_u\right)^* + \int_s^t (\delta g^n)_{su}\otimes dg^n_u\,. \quad (3.6)$$

Now consider the integral $\int_s^t (\delta z)_{su}\otimes dg_u$, which, due to the regularity of g, can be interpreted in the classical Lebesgue sense, and use (3.4)-(3.5) to assert that

$$\begin{aligned}&\left|\int_s^t (\delta z^n)_{su}\otimes dg^n_u - \int_s^t (\delta z)_{su}\otimes dg_u\right|\\ &\leqslant \int_s^t |\delta(z^n-z)_{su}|\otimes|dg^n_u| + \int_s^t |(\delta z)_{su}|\otimes|d(g^n-g)_u|\\ &\lesssim \mathcal{N}[z^n-z;\mathcal{C}_1^{\gamma'}([0,1];\mathbb{R}^d)]\int_s^t \frac{|u-s|^{\gamma'}}{u^{1-\gamma}}\,du\\ &\qquad\qquad + \mathcal{N}[z;\mathcal{C}_1^{\gamma}([0,1];\mathbb{R}^d)]2^{-n(\gamma-\gamma')}\int_s^t \frac{|u-s|^{\gamma}}{u^{1-\gamma'}}\,du\\ &\lesssim |t-s|^{\gamma+\gamma'}\{\mathcal{N}[z^n-z;\mathcal{C}_1^{\gamma'}([0,1];\mathbb{R}^d)]+2^{-n(\gamma-\gamma')}\}\int_0^1 \frac{dr}{r^{1-(\gamma+\gamma')}}\,.\end{aligned}$$

We can treat the two other summands in (3.6) along the same lines, which leads us to the desired conclusion, namely $\mathcal{N}[\mathbf{x}^{\mathbf{2},n}-\mathbf{z}^{\mathbf{2},n};\mathcal{C}_2^{2\gamma'}([0,1];\mathbb{R}^d)]\to 0$ as $n\to\infty$. We even get the explicit description

$$\mathfrak{L}(z+g)^{\mathbf{2}}_{st}-\mathfrak{L}(z)^{\mathbf{2}}_{st} = \int_s^t (\delta z)_{su}\otimes dg_u + \left(\int_s^t (\delta z)_{ut}\otimes dg_u\right)^* + \int_s^t (\delta g)_{su}\otimes dg_u\,.$$

With this decomposition in hand, it is now easy to exhibit the bound (3.1): for instance, for every $0<s<t$,

$$\begin{aligned}&\left|\int_s^t (\delta z)_{su}\otimes dg_u\right|\\ &\leqslant \|\!|g|\!\|_{1;\gamma}\mathcal{N}[z;\mathcal{C}_1^{\gamma}([0,1];\mathbb{R}^d)]\int_s^t \frac{|u-s|^{\gamma}}{u^{1-\gamma}}\,du\\ &\leqslant \|\!|g|\!\|_{1;\gamma}\mathcal{N}[z;\mathcal{C}_1^{\gamma}([0,1];\mathbb{R}^d)]\,|t-s|^{1+\gamma}\int_0^1 \frac{r^{\gamma}}{(s+r(t-s))^{1-\gamma}}\,dr\\ &\leqslant \|\!|g|\!\|_{1;\gamma}\mathcal{N}[z;\mathcal{C}_1^{\gamma}([0,1];\mathbb{R}^d)]\\ &\qquad\qquad \min\left(|t-s|^{2\gamma}\int_0^1 \frac{dr}{r^{1-2\gamma}},\, s^{\gamma-1}|t-s|^{1+\gamma}\int_0^1 r^{\gamma}\,dr\right)\,.\end{aligned}$$

□

4. Proof of Lemma 2.4

The argument relies on the algorithm introduced in [?, Section 6] and which aims at "removing the points one by one" between t_p and t_{q+1} in a tricky way. First, just as in [?, Section 3.1], and given any (not necessarily uniform) subpartition Π of $\mathcal{P}_n$, we define the path G^Π as follows: for every $s \leqslant t \in \mathcal{P}_n$,

$$G^{\Pi}_{st} := \begin{cases} 0 & \text{if } (s,t) \cap \Pi = \varnothing \\ (\delta G)_{sut} & \text{if } (s,t) \cap \Pi = u \\ G_{st} - G_{s\tilde{t}_1} - \sum_{k=1}^{\ell-1} G_{\tilde{t}_k \tilde{t}_{k+1}} - G_{\tilde{t}_\ell t} & \text{if } (s,t) \cap \Pi = \{\tilde{t}_1, ..., \tilde{t}_\ell\} \end{cases} .$$

With this notation, if $s = t_p$ and $t = t_{q+1}$, one has in particular

$$G_{st} = G^{[\![s,t]\!]}_{st} + \sum_{i=p}^{q} G_{t_i t_{i+1}} . \tag{4.1}$$

As far as the sum is concerned, we have on the one hand, since $\mu_1 \geqslant 1$,

$$s^{1-\lambda} \Big\| \sum_{i=p}^{q} G_{t_i t_{i+1}} \Big\| \leqslant \mathcal{M}^{\alpha,\mu_1}_{\lambda}\big[G; [\![s,t]\!]\big] \cdot \sum_{i=p}^{q} |t_{i+1} - t_i|^{\mu_1} \leqslant \mathcal{M}^{\alpha,\mu_1}_{\lambda}\big[G; [\![s,t]\!]\big] \cdot |t-s|^{\mu_1} ,$$

and on the other hand

$$\Big\| \sum_{i=p}^{q} G_{t_i t_{i+1}} \Big\| \leqslant \mathcal{M}^{\alpha,\mu_1}_{\lambda}\big[G; [\![s,t]\!]\big] \cdot \Big\{ |t_{p+1} - s|^{\alpha} + \sum_{i=p+1}^{q} t_i^{\lambda-1} |t_{i+1} - t_i|^{\mu_1} \Big\} ,$$

with

$$\sum_{i=p+1}^{q} t_i^{\lambda-1} |t_{i+1} - t_i|^{\mu_1} = \frac{1}{2^{n(\lambda+\mu_1-1)}} \sum_{i=p+1}^{q} \frac{1}{i^{1-\lambda}} \lesssim \frac{1}{2^{n(\lambda+\mu_1-1)}} |q+1-p|^{\lambda} \lesssim |t-s|^{\lambda} .$$

Going back to (4.1), it remains us to bound $\|G^{[\![s,t]\!]}_{st}\|$. For the sake of clarity, let us temporarily change the notation by setting, for s, t fixed as above,

$$t_k := t - \frac{k}{2^n} , \ k = 0, \ldots, N , \quad \text{where } N := 2^n(t-s) \ \ (= q+1-p) . \tag{4.2}$$

We make this (unnatural) choice to "reverse" the time, that is to consider a decreasing function $k \mapsto t_k$, in a such a way that the below notations will be consistent with those of [?, Section 6] (and especially those of [?, Proposition 6.2]). Consider indeed the algorithm described in [?, Section 6] to remove one by one the points between 0 and N, and accordingly the points of $\mathcal{P}_n$ between s and t (just use the transformation (4.2) to connect one with the other). Denote by $(\Pi^m)_{m=0,\ldots,N-1}$ the decreasing sequence of partitions of $[\![s,t]\!]$ that is associated with this algorithm. With the notations of [?, Section 6], it is readily checked that

$$G_{st}^{\Pi^m} - G_{st}^{\Pi^{m+1}} = (\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}} \quad , \quad G_{st}^{\Pi^0} = G_{st}^{[\![s,t]\!]} \quad , \quad G_{st}^{\Pi^{N-1}} = 0 \ ,$$

and so

$$G_{st}^{[\![s,t]\!]} = \sum_{m=1}^{N-1} (\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}} \ . \tag{4.3}$$

Now, still with the notations of [?, Section 6] in mind, write

$$\sum_{m=0}^{N} (\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}} = \sum_{r=1}^{M-1} \Big\{ (\delta G)_{s t_{k_{A_{r-1}+1}} t_{k^-_{A_{r-1}+1}}} + \sum_{m=A_{r-1}+2}^{A_r} (\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}} \Big\}$$

and so

$$\begin{aligned} \Big\| \sum_{m=0}^{N} (\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}} \Big\| &\leqslant \mathcal{N}[\delta G; \mathcal{C}_{3;\lambda}^{\alpha,\mu_2}([\![s,t]\!])] \cdot \\ &\sum_{r=1}^{M-1} \Big\{ |t_{k^-_{A_{r-1}+1}} - s|^{\alpha} + \sum_{m=A_{r-1}+2}^{A_r} t_{k_m^+}^{\lambda-1} |t_{k_m^-} - t_{k_m^+}|^{\mu_2} \Big\} \ . \end{aligned} \tag{4.4}$$

Observe at this point that

$$|t_{k^-_{A_{r-1}+1}} - s|^{\alpha} = |t-s|^{\alpha} \cdot \Big| 1 - \frac{k^-_{A_{r-1}+1}}{N} \Big|^{\alpha}$$

and

$$t_{k_m^+}^{\lambda-1} |t_{k_m^-} - t_{k_m^+}|^{\mu_2} \leqslant |t-s|^{\lambda+\mu_2-1} \cdot \frac{1}{N^{\mu_2}} \Big| 1 - \frac{k_m^+}{N} \Big|^{\lambda-1} |k_m^+ - k_m^-|^{\mu_2} \ .$$

Going back to (4.4), we get that

$$\Big\| \sum_{m=0}^{N} (\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}} \Big\| \leqslant |t-s|^{\alpha} \, \mathcal{N}[\delta G; \mathcal{C}_{3;\lambda}^{\alpha,\mu_2}([\![s,t]\!])] \cdot Q^N_{\alpha,\lambda,\mu_2} \ , \tag{4.5}$$

where we have set

$$Q^N_{\alpha,\lambda,\mu_2} := \sum_{r=1}^{M-1} \Big\{ \Big| 1 - \frac{k^-_{A_{r-1}+1}}{N} \Big|^{\alpha} + \frac{1}{N^{\mu_2}} \sum_{m=A_{r-1}+2}^{A_r} \Big| 1 - \frac{k_m^+}{N} \Big|^{\lambda-1} |k_m^+ - k_m^-|^{\mu_2} \Big\} \ .$$

Therefore, we are exactly in a position to apply [?, Proposition 6.2] and assert that $\sup_{N\geqslant 1} Q^N_{\alpha,\lambda,\mu_2} < \infty$. The combination of (4.3) and (4.5) then gives us the desired estimate, namely

$$\|G_{st}^{[\![s,t]\!]}\| \lesssim |t-s|^{\alpha} \, \mathcal{N}[\delta G; \mathcal{C}_{3;\lambda}^{\alpha,\mu_2}([\![s,t]\!])] \ .$$

The estimation of $s^{1-\lambda}\|G_{st}^{[\![s,t]\!]}\|$ is easier. Indeed, with decomposition (4.3) in mind, we simply use the fact that the above algorithm also satisfies

$$|k_m^+ - k_m^-| \leqslant \frac{2N}{(N-m+1)} \quad \text{for every } m = 1, \ldots, N-1 \ ,$$

and consequently

$$\begin{aligned} s^{1-\lambda}\|G_{st}^{[\![s,t]\!]}\| &\leqslant \sum_{m=1}^{N-1} t_{k_m^+}^{1-\lambda} \|(\delta G)_{t_{k_m^+} t_{k_m} t_{k_m^-}}\| \\ &\leqslant \mathcal{N}[\delta G; \mathcal{C}_{3;\lambda}^{\alpha,\mu_2}([\![s,t]\!])] \cdot \sum_{m=1}^{N-1} |t_{k_m^-} - t_{k_m^+}|^{\mu_2} \\ &\lesssim |t-s|^{\mu_2} \, \mathcal{N}[\delta G; \mathcal{C}_{3;\lambda}^{\alpha,\mu_2}([\![s,t]\!])] \ . \end{aligned}$$

5. Proof of [?, Lemma 8.8]

For the sake of clarity, and with the setting described in [?, Section 4.2] in mind, let us recall the statement of the result under consideration:

Lemma 5.1. *Let $\alpha > 0$ and assume that for some (fixed) calibration of the scheme, there exists $\eta \in (0,1)$ such that for all $k \geqslant 1$, $\ell \geqslant 0$ and $K > 0$,*

$$\mathbb{P}(\mathcal{E}_k|\mathcal{E}_{k-1}) \geqslant \eta \ , \qquad \mathbb{P}(F_{k,\ell}|\mathcal{E}_{k-1}) \leqslant 2^{-\alpha\ell} \quad \text{and} \quad \Delta\tau_k \geqslant a_k \ \text{ a.s.}, \tag{5.1}$$

where $a_k \geqslant 1$ for every $k \geqslant 1$. Then there exists a constant $C_{\eta,\alpha}^2 > 0$ and for every $p > 0$ there exists a constant $C_{\eta,\alpha,p}^1 > 0$ such that for every $k \geqslant 1$,

$$\mathbb{E}\Big[\Big(\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}_{\tau_{m-1},\tau_m}^{\tau_k}(t)|\Big)\Big|\,\mathcal{E}_k\Big] \leqslant \frac{C_{\eta,\alpha,p}^1}{a_k^{1/2-H}\eta^{(k-m)/p}} \quad , \quad m \in \{1,\ldots,k-1\} \ , \tag{5.2}$$

$$\mathbb{E}\Big[\Big(\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}_{-\infty,0}^{\tau_k}(t)|\Big)\Big|\,\mathcal{E}_k\Big] \leqslant \frac{C_{\eta,\alpha,p}^1}{a_k^{1/2-H}\eta^{k/p}} \ , \tag{5.3}$$

and

$$\max\Big(\mathbb{E}\Big[\Big(\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}_{\tau_{k-1},\tau_k}^{\tau_k}(t)|\Big)\Big|\,\mathcal{E}_k\Big], \mathbb{E}\Big[\Big(\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}_{-\infty,0}^{0}(t)|\Big)\Big]\Big) \leqslant C_{\eta,\alpha}^2 \ . \tag{5.4}$$

The proof of this result will rely on the following general estimate, which somehow allows us to "fix" the duration of Attempt m and to go back to a conditioning by $\mathcal{E}_{m-1}$.

Lemma 5.2. *In the setting of Lemma 5.1, fix $m \geqslant 1$ and, on the event $\mathcal{E}_m$, consider a generic process $(R_t)_{t>\tau_{m-1}}$. Besides, on the event $F_{m,\ell}$ ($\ell \geqslant 0$), set $\Delta(m,\ell) := \Delta\tau_m \geqslant 1$ (noticing that $\Delta\tau_m$ is deterministic in this case). Then, for every $p > 0$, it holds that*

$$\mathbb{E}[R_{\tau_m}|\mathcal{E}_k] \leqslant c_{\eta,\alpha}\eta^{(m-k)/p} \sup_{\ell\geqslant 0} \mathbb{E}[|R_{\tau_{m-1}+\Delta(m,\ell)}|^{2p}|\mathcal{E}_{m-1}]^{\frac{1}{2p}} \ , \tag{5.5}$$

where $c_{\eta,\alpha}$ depends on η and α only.

Proof. First, one can readily check that

$$\mathbb{E}[R_{\tau_m}|\mathcal{E}_m] = \sum_{\ell \geqslant 0} \mathbb{E}[R_{\tau_{m-1}+\Delta(m,\ell)}|F_{m,\ell}]\mathbb{P}(F_{m,\ell}|\mathcal{E}_m).$$

By the Cauchy-Schwarz inequality and the fact that $F_{m,\ell} \subset \mathcal{E}_{m-1}$, one has then

$$\begin{aligned}\mathbb{E}[R_{\tau_{m-1}+\Delta(m,\ell)}|F_{m,\ell}] &= \frac{\mathbb{E}[R_{\tau_{m-1}+\Delta(m,\ell)}1_{F_{m,\ell}}|\mathcal{E}_{m-1}]}{\mathbb{P}(F_{m,\ell}|\mathcal{E}_{m-1})} \\ &\leqslant \mathbb{E}[R^2_{\tau_{m-1}+\Delta(m,\ell)}|\mathcal{E}_{m-1}]^{\frac{1}{2}}\mathbb{P}(F_{m,\ell}|\mathcal{E}_{m-1})^{-\frac{1}{2}} \ .\end{aligned}$$

As $F_{m,\ell} \subset \mathcal{E}_m \subset \mathcal{E}_{m-1}$, we can obviously write

$$\mathbb{P}(F_{m,\ell}|\mathcal{E}_m) = \frac{\mathbb{P}(F_{m,\ell}|\mathcal{E}_{m-1})}{\mathbb{P}(\mathcal{E}_m|\mathcal{E}_{m-1})} \ ,$$

so that, thanks to our assumption (5.1), the following holds true:

$$\mathbb{E}[R_{\tau_m}|\mathcal{E}_m] \leqslant \eta^{-\frac{1}{2}} \sum_{\ell \geqslant 0} \mathbb{E}[R^2_{\tau_{m-1}+\Delta(m,\ell)}|\mathcal{E}_{m-1}]^{\frac{1}{2}}\mathbb{P}(F_{m,\ell}|\mathcal{E}_{m-1})^{\frac{1}{2}} \ .$$

Invoking our assumption (5.1) again, we deduce that

$$\mathbb{E}[R_{\tau_m}|\mathcal{E}_m] \leqslant \eta^{-\frac{1}{2}} \sum_{\ell \geqslant 0} 2^{-\frac{\alpha\ell}{2}} \mathbb{E}[R^2_{\tau_{m-1}+\Delta(m,\ell)}|\mathcal{E}_{m-1}]^{\frac{1}{2}}$$

which yields:

$$\mathbb{E}[R_{\tau_m}|\mathcal{E}_m] \leqslant c_{\eta,\alpha} \cdot \sup_{\ell \geqslant 0} \mathbb{E}[R^2_{\tau_{m-1}+\Delta(m,\ell)}|\mathcal{E}_{m-1}]^{\frac{1}{2}} \ . \tag{5.6}$$

Since $\mathcal{E}_k \subset \mathcal{E}_m$ for $k \geqslant m$, a similar Cauchy-Schwarz argument as above implies that for a given random variable S and for every $p > 0$,

$$|\mathbb{E}[S|\mathcal{E}_k]| \leqslant \frac{\mathbb{E}[|S|^p|\mathcal{E}_m]^{\frac{1}{p}}}{\mathbb{P}(\mathcal{E}_k|\mathcal{E}_m)^{\frac{1}{p}}} \ . \tag{5.7}$$

Using that $\mathbb{P}(\mathcal{E}_k|\mathcal{E}_m) \geqslant \eta^{k-m}$, we finally obtain the desired control:

$$\mathbb{E}[R_{\tau_m}|\mathcal{E}_k] \leqslant c_{\eta,\alpha}\eta^{(m-k)/p} \sup_{\ell \geqslant 0} \mathbb{E}[|R_{\tau_{m-1}+\Delta(m,\ell)}|^{2p}|\mathcal{E}_{m-1}]^{\frac{1}{2p}} \ , \tag{5.8}$$

where $c_{\eta,\alpha}$ depends on η and α only. □

Proof of Lemma 5.1. The reasoning is divided in three steps.

Step 1. Case $1 \leqslant m < k$. Since $\Delta\tau_k \geqslant a_k$, it is readily checked that for all $m < k$, $t \in [0,1]$ and $r \in [\tau_{m-1}, \tau_m]$, one has $t + \tau_k - r \geqslant a_k$. We then deduce from [**?**, Lemma 8.7] that

$$\sup_{t\in[0,1]} t^{1-\gamma} |\mathcal{D}^{\tau_k}_{\tau_{m-1},\tau_m}(t)| \leqslant \sup_{t\in[0,1]} |\mathcal{D}^{\tau_k}_{\tau_{m-1},\tau_m}(t)| \leqslant c_H a_k^{H-1/2} R_{\tau_m} \tag{5.9}$$

where we have set, for every $t > \tau_{m-1}$,

$$R_t := (t-\tau_{m-1})^{-1} \left|W_t - W_{\tau_{m-1}}\right| + \int_{\tau_{m-1}}^t (t+1-r)^{-2} \left|W_t - W_r\right| dr \ .$$

Since $\int_{\tau_{m-1}}^t (t+1-r)^{-3/2} dr \leqslant 2$ for every $t > \tau_{m-1}$, one can first check by Jensen's inequality that

$$\begin{aligned} &\left(\int_{\tau_{m-1}}^t (t+1-r)^{-2} \left|W_t - W_r\right| dr \right)^p \\ &\leqslant c_p \int_{\tau_{m-1}}^t (t+1-r)^{-3/2} \left((t-r)^{-\frac{1}{2}} \left|W_t - W_r\right| \right)^p dr \ . \end{aligned}$$

Then it follows from the scaling property of the Brownian motion that for every $\ell \geqslant 0$,

$$\mathbb{E}[|R_{\tau_{m-1}+\Delta(m,\ell)}|^{2p} | \mathcal{E}_{m-1}] \leqslant c_p \ ,$$

where c_p depends on p only (and the notation $\Delta(m,\ell)$ has been introduced in Lemma 5.2). Using (5.5), we get the desired bound (5.2).

Step 2. Case $m = k \geqslant 1$. Let us write here

$$\sup_{t\in(0,1]} t^{1-\gamma} |\mathcal{D}^{\tau_k}_{\tau_{k-1},\tau_k}(t)| \leqslant \sup_{t\in(0,1]} |\mathcal{D}^{\tau_k}_{\tau_{k-1},\tau_k-1}(t)| + \sup_{t\in(0,1]} t^{1-\gamma} |\mathcal{D}^{\tau_k}_{\tau_k-1,\tau_k}(t)| \ . \tag{5.10}$$

The first term in the right-hand side can then be treated along the very same arguments as above (using $p = 1/2$ in (5.5)), which gives us directly

$$\mathbb{E}\Big[\Big(\sup_{t\in(0,1]} |\mathcal{D}^{\tau_k}_{\tau_{k-1},\tau_k-1}(t)| \Big) \Big| \mathcal{E}_k \Big] \leqslant c^1_{\eta,\alpha} \ . \tag{5.11}$$

On the other hand, using the bound of [**?**, Lemma 8.7] again, we get that

$$\sup_{t\in(0,1]} t^{1-\gamma} |\mathcal{D}^{\tau_k}_{\tau_k-1,\tau_k}(t)| \leqslant c_H R_{\tau_k} \ ,$$

with, for every $t > \tau_{k-1}$,

$$R_t := |W_t - W_{t-1}| + \int_{t-1}^t |t-r|^{(H-\gamma)-3/2} |W_t - W_r| \, dr \ .$$

It is then readily checked that for every $\ell \geqslant 0$,

$$\mathbb{E}[|R_{\tau_{k-1}+\Delta(k,\ell)}||\mathcal{E}_{k-1}] \leqslant c_{H,\gamma} \ ,$$

and so we can apply (5.5) again (with $p = 1/2$) to assert that

$$\mathbb{E}\Big[\Big(\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}^{\tau_k}_{\tau_k-1,\tau_k}(t)|\Big)\Big|\,\mathcal{E}_k\Big] \leqslant c^2_{\eta,\alpha} \ . \tag{5.12}$$

The combination of (5.10), (5.11) and (5.12) provide the first part of (5.4).

Step 3. Asymptotic cases. On the one hand, we can use [?, Lemma 8.7] to obtain that for every $k \geqslant 1$,

$$\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}^{\tau_k}_{-\infty,0}(t)| \leqslant \sup_{t\in(0,1]} |\mathcal{D}^{\tau_k}_{-\infty,0}(t)| \leqslant c_H a_k^{H-1/2} \int_{-\infty}^0 |1-r|^{-2}|W_r|\, dr$$

and (5.3) then follows from the general bound (5.7) (with $m = 0$).

On the other hand, it is not hard to see that the situation where $k = 0$ can be handled with the same strategy as in Step 3, namely writing

$$\sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}^0_{-\infty,0}(t)| \leqslant \sup_{t\in(0,1]} |\mathcal{D}^0_{-\infty,-1}(t)| + \sup_{t\in(0,1]} t^{1-\gamma}|\mathcal{D}^0_{-1,0}(t)|$$

and then bounding the first, resp. second, term along the arguments of Step 2, resp. Step 3, with $p = 1/2$. This easily leads us to the second part of (5.4), and accordingly the proof of the lemma is achieved.

□

References