On the separation cut-off phenomenon for Brownian motions on high dimensional rotationally symmetric compact manifolds

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Abstract

Given a family of rotationally symmetric compact manifolds indexed by the dimension and a weight function, the goal of this paper is to investigate the cut-off phenomenon for the Brownian motions on this family. We provide a class of compact manifolds with non-negative Ricci curvatures for which the cut-off in separation with windows occurs, in high dimension, with different explicit mixing times. We also produce counter-examples, still with non-negative Ricci curvatures, where there are no cut-off in separation. In fact we show a phase transition for the cut-off phenomenon concerning the Brownian motions on a rotationally symmetric compact manifolds. Our proof is based on a previous construction of a sharp strong stationary times by the authors, and some quantitative estimates on the two first moments of the covering time of the dual process. The concentration of measure phenomenon for the above family of manifolds appears to be relevant for the study of the corresponding cut-off.

Keywords: Rotationally symmetric Brownian motions, strong stationary times, separation discrepancy, hitting times.

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1 Introduction

1.1 Overview

The main purpose of the present paper is to investigate the cut-off phenomenon in separation for Brownian motion on high dimensional compact manifolds, especially for model space that are the rotationally symmetric compact manifolds. In the context of card shuffling, the cut-off phenomenon was discovered by Diaconis and Shahshahani [6] and Aldous and Diaconis [1]. Cut-off phenomenon is an abrupt transition from out of equilibrium to equilibrium, which occurs for certain Markov processes, when the size of the state space become large. Afterward, the cut-off phenomenon has been proven for a large variety of finite Markov chains, see e.g. Diaconis [5], Diaconis and Fill [7], Levin, Peres and Wilmer [11] and Ding, Lubetzky and Peres [9]. Nevertheless the literature on the cut-off phenomenon for Markov processes on a continuous state space is rather sparse. For example Saloff-Coste [14] has proven the cut-off phenomenon in total variation distance for the Brownian motions on the spheres $\mathbb{S}^n$ for high dimensions n, with a mixing time of order $\ln(n)/(2n)$, see also Méliot [13] for extensions to classical symmetric spaces of compact type. Their approach are based on complete knowledge of the spectral decomposition. It is shown in Hermon, Lacoin and Peres [10] that total variation and separation cut-off are not equivalent and neither one implies the other. In a precedent paper [3] we have shown that the cut-off in separation also occurs for the Brownian motion on the sphere of high dimensions n with a mixing time of order $\ln(n)/n$. In the present paper we generalize such a result for a large class of manifold, and as example we strengthen this result on spheres with a cut-off in separation with windows. Note that controlling the separation discrepancy is essentially (but not exactly) a L^∞ control while the control of the total variation is a L^1 control. Heuristically, the difference in the mixing times in total variation and separation comes from the fact that L^1 estimates only require the dual process to see a big part of the volume, and by concentration of measure phenomenon it is sufficient to see the "equator", while to get L^∞ estimates we have wait for the dual process to cover all the sphere, namely to reach the opposite pole, and this takes twice as long.

Our goal here is to check that there is a cut-off phenomenon in separation with windows for a large class of family of rotationally symmetric manifolds with non-negative Ricci curvature Theorems 12 and 21, including the case of spheres Corollary 13. We also give examples of rotationally symmetric manifolds with non-negative Ricci curvature where there is no cut-off in separation Theorems 18 and 23. In fact we show a phase transition for the cut-off phenomenon concerning the Brownian motions see Theorem 4. Our results are connected with those of Salez, concerning sequences of irreducible Markov chains with symmetric support and non-negative coarse Ricci curvature that exhibit cut-off in total variation when an additional product condition hypothesis is satisfied, see [15] for the precise statement.

Our proof is based on two ingredients, the resort to the strong stationary times for X_n presented in [2] and the detailed quantitative estimates on the cover time of dual process (see [4]) that appear to be an one-dimensional diffusion processes in the case of rotationnaly symmetric manifolds. The concentration of volume phenomenon plays a crucial role to detect the scale on which the cut-off phenomenon occurs. This alternative point of view differs from the traditional approach based on spectral analysis and could be extended to other situations where spectral information is less available.

1.2 Geometric framework

For $n \geqslant 2$, let M_f^n be the product manifold $[0, L] \times \mathbb{S}^{n-1}/\sim$, where $(r_1, \theta_1) \sim (r_2, \theta_2)$ if $(r_1, \theta_1) = (r_2, \theta_2)$ or $r_1 = r_2 = 0$ or $r_1 = r_2 = L$, endowed with the warping product metric

$$ds^2 = dr \otimes dr + f^2(r) d\theta \otimes d\theta,$$

where $\mathbb{S}^{n-1}$ is the usual sphere of dimension $n-1$ and radius 1, $d\theta \otimes d\theta$ is the standard metric on the sphere and f is a regular real function that satisfies the following assumption:

$$\left\{ \begin{array}{ccc} f : [0, L] & \to & \mathbb{R}_+, \\ f(s) \sim_0 s & , & f(L-s) \sim_0 s \\ f^{(2k)}(0) & = & f^{(2k)}(L) = 0, k \in \mathbb{Z}_+ \end{array} \right. \tag{1}$$

We will call such function a **weight** function, we will assume all along the paper that f is a weight function. Later, further conditions will be required to ensure the regularity of the metric at $\widetilde{0} \sim (0, .)$ and $\widetilde{L} \sim (L, .)$. The volume of the geodesic ball $B(\widetilde{0}, r)$ in M_f^n centered at $\widetilde{0}$ of radius $r \in [0, L]$ is given by

$$\mathrm{Vol}_n(B(\widetilde{0}, r)) = c_n \int_0^r f^{n-1}(s) ds,$$

where $c_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the volume of $\mathbb{S}^{n-1}$. The area of the geodesic sphere $\partial B(\widetilde{0}, r)$ is $c_n f^{n-1}(r)$ and the mean curvature of any point in $\partial B(\widetilde{0}, r)$ is given by $(n-1)\frac{f'(r)}{f(r)}$. We have $\mathrm{Ric}(v) = \left((n-2)\frac{1-f'(r)^2}{f^2(r)} - \frac{f''(r)}{f(r)}\right) v$ if $v \in T\mathbb{S}^{n-1}$ and $\mathrm{Ric}(\partial_r) = \left(-(n-1)\frac{f''(r)}{f(r)}\right) \partial_r$, where Ric denote the Ricci tensor. For a good introduction to warped products, see Chapter 3 in Petersen [12].

Here is our main object of interest.

Definition 1 For any $n \in \mathbb{N}\backslash\{1\}$, $X_n \coloneqq (X_n(t))_{t \geqslant 0}$ stands for the Brownian motion on M_f^n started at $\widetilde{0}$ and time-accelerated by a factor 2, i.e. the Δ-diffusion in M_f^n. So the generator of X_n is the Laplacian Δ and not the Laplacian divided by 2 as it is sometimes more usual in Probability Theory.

□

It was seen in [4] that X_n can be intertwined with the dual process $D \coloneqq (D(t))_{t \geqslant 0}$ taking values in the closed balls of M_f^n centered at $\widetilde{0}$, starting at $\{\widetilde{0}\}$ and absorbed in finite time τ_n in the whole set M_f^n. In [2], several couplings of X_n and D were constructed, so that for any time $t \geqslant 0$, the conditional law of $X_n(t)$ knowing the trajectory $D([0, t]) \coloneqq (D(s))_{s \in [0,t]}$ is the normalized uniform law over $D(t)$, which will be denoted $\Lambda(D(t), \cdot)$ in the sequel. Furthermore, D is progressively measurable with respect to X_n, in the sense that for any $t \geqslant 0$, $D([0, t])$ depends on X_n only through $X_n([0, t])$. Due to these couplings and to general arguments from Diaconis and Fill [7], τ_n is a strong stationary time for X_n, meaning that τ_n and $X_n(\tau_n)$ are independent and $X_n(\tau_n)$ is uniformly distributed over M_f^n. As a consequence we have

$$\forall\ t \geqslant 0, \qquad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n) \quad \leqslant \quad \mathbb{P}[\tau_n > t]$$

where the l.h.s. is the separation discrepancy between the law of $X_n(t)$ and the uniform distribution $\mathcal{U}_n$ over M_f^n. Notice that $\mathcal{U}_n(B(\widetilde{0}, r)) = \frac{\int_0^r f^{n-1}(s)ds}{\int_0^L f^{n-1}(s)ds}$ for any $r \in [0, L]$.

Recall that the separation discrepancy between two probability measures μ and ν defined on the same measurable space is given by

$$\mathfrak{s}(\mu, \nu) \quad = \quad \operatorname*{ess\,sup}_{\nu} 1 - \frac{d\mu}{d\nu}$$

where $d\mu/d\nu$ is the Radon-Nikodym density of μ with respect to ν. Note that $\|\mu - \nu\|_{\mathrm{tv}} \leqslant \mathfrak{s}(\mu, \nu)$, where $\|\cdot\|_{\mathrm{tv}}$ stands for the total variation.

Remark 2 Note that for any $t \in [0, \tau_n)$, the "opposite pole" $\widetilde{L}$ does not belong to the support of $\Lambda(D(t), \cdot)$. It follows from an extension of Remark 2.39 of Diaconis and Fill [7] that τ_n is even a sharp strong stationary time for X_n, meaning that

$$\forall\ t \geqslant 0, \qquad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n) \quad = \quad \mathbb{P}[\tau_n > t]$$

Thus the understanding of the convergence in separation of X_n toward $\mathcal{U}_n$ amounts to understanding the distribution of τ_n.

□

1.3 Cut-off phenomenon

For fixed n, the Brownian motion X_n in M_f^n converges in law to $\mathcal{U}_n$, namely

$$X_n(t) \xrightarrow{\mathcal{L}}_{t\to+\infty} \mathcal{U}_n.$$

Quantifying this convergence to equilibrium is relevant when the dimension n becomes large. This speed of convergence or mixing time, depends one the way the difference between the time marginal and the uniform distribution is measured. A cut-off phenomenon in separation at time a_n is a kind of phase transition, namely the separation discrepancy between X_n and the equilibrium abruptly drops from the largest value 1 to the smallest one 0 on a small interval around a_n. More precisely, we say that the family of diffusion processes $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has a cut-off in separation with mixing times $(a_n)_{n\in\mathbb{N}\setminus\{1\}}$ and windows $(b_n)_{n\in\mathbb{N}\setminus\{1\}}$ when

$$\forall n \geqslant 1, \quad 0 < b_n \leqslant a_n,$$

$$\begin{aligned}
\forall\, r > 0, \qquad & \lim_{n\to\infty} \mathfrak{s}(\mathcal{L}(X_n(a_n + rb_n)), \mathcal{U}_n) = \lim_{n\to\infty} \mathbb{P}\left[\tau_n > a_n + rb_n\right] &= 0 \\
\forall\, r \in (0,1), \qquad & \lim_{n\to\infty} \mathfrak{s}(\mathcal{L}(X_n(a_n - rb_n)), \mathcal{U}_n) = 1 - \lim_{n\to\infty} \mathbb{P}\left[\tau_n \leqslant a_n - rb_n\right] &= 1.
\end{aligned}$$

When $\forall n \geqslant 1$, $b_n = a_n$, we simply say that the family of diffusion processes $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has a cut-off in separation with mixing times $(a_n)_{n\in\mathbb{N}\setminus\{1\}}$.

1.4 Intertwining relations

Writing $B(\widetilde{0}, R(t)) \coloneqq D(t)$ for $t \in [0, \tau_n]$, it has been seen in [4] that $R \coloneqq (R(t))_{t\in[0,\tau_n]}$ is solution to the stochastic differential equation

$$\forall\, t \in (0, \tau_n), \qquad dR(t) \;=\; \sqrt{2}dB(t) + b_n(R(t))dt \tag{2}$$

and

$$\tau_n \;=\; \inf\{t \geqslant 0 \,:\, R(t) = L\} \tag{3}$$

where $(B(t))_{t\geqslant 0}$ is a standard Brownian motion in $\mathbb{R}$ and the mapping b_n is given by

$$\forall\, r \in (0, L), \qquad b_n(r) \;\coloneqq\; 2\frac{f^{n-1}(r)}{\int_0^r f^{n-1}(u)\,du} - (n-1)\frac{f'(r)}{f(r)} \tag{4}$$

It is not difficult to check that as r goes to 0_+

$$b_n(r) \;\sim\; \frac{n+1}{r}$$

and this is sufficient to insure that 0 is an entrance boundary for R, so that starting from 0, it will never return to 0 at positive times.

In the following corollary we explicit two intertwining relations, which were constructed in [2] Theorems 3.5 and 4.1, enabling to deduce τ_n from the Brownian motion X_n (and independent randomness for the second construction):

Corollary 3 *Consider the Brownian motion $X_n \coloneqq (X_n(t))_{t\geqslant 0}$ in M_f^n described in Definition 1. For $x \in M_f^n\setminus\{\widetilde{0}, \widetilde{L}\}$, denote by $N(x)$ the unit vector at x normal to the sphere centred at $\widetilde{0}$ with radius $\rho(\widetilde{0}, x)$ where ρ is the distance in M_f^n, pointing towards $\widetilde{0}$: $N(x) = -\nabla\rho(\widetilde{0}, \cdot)(x)$.*

(1) ***Full coupling***. *Let $D_1(t)$ be the ball in M_f^n centred at $\widetilde{0}$ with radius $R_1(t)$ solution started at* 0 *to the Itô equation*

$$dR_1(t) = -\sqrt{2}\langle N(X_n(t)), dX_n(t)\rangle + n\left[2\frac{f'}{f}(\rho(\widetilde{0}, X_n(t))) - \frac{f'}{f}(R_1(t))\right] dt$$

This evolution equation is considered up to the hitting time $\tau_n^{(1)}$ of L by $R_1(t)$.

(2) ***Full decoupling, reflection of** D **on** X_n*. *Let $D_2(t)$ be the ball in M_f^n centered at $\widetilde{0}$ with radius $R_2(t)$ solution started at* 0 *to the Itô equation*

$$dR_2(t) = -\sqrt{2}dW_t + 2dL_t^{R_2}[\rho(\widetilde{0}, X_n)] - n\frac{f'}{f}(R_2(t))\, dt$$

where $(W_t)_{t\geqslant 0}$ is a real-valued Brownian motion independent of $(X_n(t))_{t\geqslant 0}$ and $\left(L_t^{R_2}[\rho(\widetilde{0}, X_n)]\right)_{t\in[0,\tau_n^{(2)}]}$ is the local time at 0 *of the process $R_2 - \rho(\widetilde{0}, X_n)$. These considerations are valid up to the hitting time $\tau_n^{(2)}$ of L by $R_2(t)$.*

Let $D(t)$ be the ball in M_f^n centered at $\widetilde{0}$ with radius $R(t)$, defined in (2)*, and let τ_n be the stopping time defined in* (3)*.*

Then we have:

(1) for $i = 1, 2$ $X_n(\tau_n^{(i)})$ is uniformly distributed in M_f^n,

(2) the pairs $(\tau_n^{(1)}, (D_1(t))_{t\in[0,\tau_n^{(1)}]})$, $(\tau_n^{(2)}, (D_2(t))_{t\in[0,\tau_n^{(2)}]})$ and $(\tau_n, (D(t))_{t\in[0,\tau_n]})$ have the same law. In particular $\tau_n^{(1)}$ and $\tau_n^{(2)}$ satisfy Proposition 7 and Theorem 8 below.

1.5 Outline of the paper and main result

The paper is organized as follow in Section 2 we compute the Green functional of the one-dimensional diffusion associated to the radius of the dual process, and we give a tractable formulation of all moments of τ_n, the covering time of the dual process. In Section 3, we compute the mixing time for several rotationnaly symmetric manifold and we show that depending on the shape of the weight function f, the cut-off occurs in separation for the Brownian motion on M_f^n for high dimensions n, see Theorem 12 and 21, or there is no cut-off in separation, see Theorem 18 and 23.

These results are essentially summarized by the following theorem, showing a phase transition (with respect to the parameter $\alpha \in (-1, +\infty)$ introduced below) for the cut-off phenomenon concerning the Brownian motions on the model M_f^n for high dimensions n, depending on the shape of the function f at $L/2$. Let us first introduce, in order to simplify the exposition, another set of assumptions on f:

$$\begin{cases} \forall\ s \in [0, L], & f(L - s) = f(s), \\ \forall\ s \in [0, L/2), & f'(s) > 0, \\ \forall\ s \in [0, L]\backslash\{L/2\}, & f''(s) \leqslant 0, \end{cases} \tag{5}$$

Theorem 4 *Consider a C^2 function f on $[0, L]\backslash\{L/2\}$ and C^1 in $[0, L]$, satisfying Assumptions* (1) *and* (5)*. Assume there exist $\alpha \in (-1, +\infty)$ and $C > 0$ such that for $h \neq 0$ small enough,*

$$f''(L/2 - h) = -C|h|^\alpha + o(|h|^\alpha) \tag{6}$$

Let $X_n := (X_n(t))_{t\geqslant 0}$ be the Brownian motion described in Definition 1.

- *if $\alpha \in (-1, 0)$ then $(X_n)_{n\in\mathbb{N}\backslash\{1\}}$ has a cut-off in separation at time C_1/n, with*

$$C_1 = 2\int_0^{L/2} \frac{f(s)}{f'(s)},$$

- *if* $\alpha = 0$ *then* $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ *has a cut-off in separation at time* $C_2 \ln(n)/n$*, with*

$$C_2 = \frac{f(L/2)}{C},$$

- *if* $\alpha > 0$ *then* $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ *has no cut-off in separation,*

An instance where (6) is satisfied is when there exist $\alpha \in (-1, +\infty)$, $C > 0$ and $\epsilon \in (0, L/2)$ such that

$$\forall\ h \in [-\epsilon, \epsilon], \qquad f(L/2 + h) \quad = \quad f(L/2) - C|h|^{2+\alpha}.$$

Note the additional factor $\ln(n)$ at the critical case $\alpha = 0$ for the phase transition. We conjecture that in the supercritical cases $\alpha > 0$, $\tau_n/\mathbb{E}[\tau_n]$ converges in distribution for large n toward a particular law depending on α, that would reflect the fact that the larger $\alpha > 0$, the more difficult is the mixing. To go toward this result, we should investigate more moments of the strong stationary times τ_n than just the two first ones, as we will do below.

2 Preliminary results

Define for any $r \in [0, L]$,

$$\begin{aligned} I_n(r) &\coloneqq \int_0^r f^{n-1}(s)ds \\ b_n(r) &\coloneqq \frac{d}{dr} \ln\left(\frac{I_n^2(r)}{f^{n-1}(r)}\right) \end{aligned}$$

Let $L_n := \partial_r^2 + b_n(r)\partial_r$ be the generator of R defined in (2). Here is our first preliminary result:

Proposition 5 *Given* $g \in C_b([0, L])$*, the bounded solution* ϕ_n *of the Poisson equation*

$$\begin{cases} L_n\phi_n &= -g \\ \phi_n(L) &= 0 \end{cases}$$

is given by:

$$\forall\ r \in [0, L], \qquad \phi_n(r) \quad = \quad \int_r^L \frac{f^{n-1}(t)}{I_n^2(t)} \left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} g(s)ds\right) dt. \tag{7}$$

So the Green operator G_n *associated to* L_n *is given by*

$$\forall\ g \in C_b([0, L]), \forall\ r \in [0, L], \qquad G_n[g](r) = \int_r^L \frac{f^{n-1}(t)}{I_n^2(t)} \left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} g(s)ds\right) dt.$$

Proof

Use the Remark 6 below to justify integrability of

$$[0, L] \ni t \mapsto \frac{f^{n-1}(t)}{I_n^2(t)} \int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} g(s)ds$$

at 0 and L. For the function defined in (7), we clearly have, $\phi_n(L) = 0$, and for any $r \in [0, L]$,

$$\phi_n'(r) = -\frac{f^{n-1}(r)}{I_n^2(r)} \int_0^r \frac{I_n^2(s)}{f^{n-1}(s)} g(s)ds$$

$$\phi_n''(r) = -\left(\frac{f^{n-1}}{I_n^2}\right)'(r)\int_0^r \frac{I_n^2(s)}{f^{n-1}(s)}g(s)ds - g(r).$$

It follows that

$$\begin{aligned}
L_n\phi_n(r) &= -g(r) - \left(\frac{f^{n-1}}{I_n^2}\right)'(r)\int_0^r \frac{I_n^2(s)}{f^{n-1}(s)}g(s)ds \\
&\quad + \left(\ln\frac{I_n^2}{f^{n-1}}\right)'(r)\phi_n'(r) \\
&= -g(r) - \left(\frac{f^{n-1}}{I_n^2}\right)'(r)\int_0^r \frac{I_n^2(s)}{f^{n-1}(s)}g(s)ds \\
&\quad - \left(\ln\frac{f^{n-1}}{I_n^2}\right)'(r)\left(-\frac{f^{n-1}(r)}{I_n^2(r)}\int_0^r \frac{I_n^2(s)}{f^{n-1}(s)}g(s)ds\right) \\
&= -g(r).
\end{aligned}$$

Remark 6

Let us show that the integral $\int_0^L \frac{f^{n-1}(t)}{I_n^2(t)}\left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}ds\right)dt$ is finite. Since $f(s) \sim_{s\to 0_+} s$ we have $I_n(s) \sim_{s\to 0_+} \frac{s^n}{n}$, so $\frac{I_n^2(s)}{f^{n-1}(s)} \sim_{s\to 0_+} \frac{s^{n+1}}{n^2}$ hence $\frac{f^{n-1}(t)}{I_n^2(t)}\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}ds$ is integrable at 0.

Concerning the integrability at L, since $I_n(L)$ is positive and finite, it is sufficient to see that for $\varepsilon \in (0,L)$, $\int_\varepsilon^L f^{n-1}(t)\left(\int_\varepsilon^t \frac{1}{f^{n-1}(s)}ds\right)dt$ is finite and this is indeed true since $f(L-s) \sim_{s\to 0_+} s$.

The above considerations further enable us to see that

$$\lim_{t\to 0_+} \frac{1}{I_n(t)}\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}ds \quad = \quad 0$$

justifying the following integration by parts:

$$\begin{aligned}
\int_0^L \frac{f^{n-1}(t)}{I_n^2(t)}\left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}ds\right)dt &= \int_0^L \left(-\frac{1}{I_n(t)}\right)'\left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}ds\right)dt \\
&= -\frac{1}{I_n(L)}\int_0^L \frac{I_n^2(s)}{f^{n-1}(s)}ds + \int_0^L \frac{I_n(s)}{f^{n-1}(s)}ds \\
&= \int_0^L \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)ds \\
&= \frac{1}{\mathrm{Vol}_n(M)}\int_0^L \frac{\mathrm{Vol}_n(B(\tilde{0},s))\mathrm{Vol}_n(B^c(\tilde{0},s))}{\mathrm{Vol}_{n-1}(\partial B(\tilde{0},s))}ds
\end{aligned}$$

where Vol_{n-1} is the $(n-1)$-dimensional Hausdorff measure. The last r.h.s. and the following Proposition 7 show that $\mathbb{E}[\tau_n] \leqslant \frac{L}{h_n(M_f^n)}$, with the Cheeger constant $h_n(M) := \inf_{D\subseteq M, \mathrm{Vol}_n(D)\leqslant \mathrm{Vol}_n(M)/2} \frac{\mathrm{Vol}_{n-1}(\partial D)}{\mathrm{Vol}_n(D)}$. □

Let $u_{n,0} := \mathbb{1}$, the constant function taking the value 1 on $[0,L]$, and consider the following sequence $(u_{n,k})_{k\in\mathbb{N}}$, defined inductively by bounded solution of

$$\forall\ k \in \mathbb{N}, \qquad \begin{cases} L_n u_{n,k} &= -k u_{n,k-1} \\ u_{n,k}(L) &= 0. \end{cases} \tag{8}$$

We have for all $n, k \in \mathbb{Z}^+$,

$$\frac{u_{n,k}}{k!} = G_n^{\circ k}[\mathbb{1}] := G_n[G_n[\cdots[G_n[\mathbb{1}]]\cdots]].$$

Proposition 7 *We have for all $n \geqslant 2$:*

$$\begin{aligned}\mathbb{E}[\tau_n] &= u_{n,1}(0) = \int_0^L \frac{f^{n-1}(t)}{I_n^2(t)} \left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} ds \right) dt \\ &= \int_0^L \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)} \right) ds \end{aligned} \tag{9}$$

$$\mathbb{E}[\tau_n^2] = u_{n,2}(0) = 2\int_0^L \frac{f^{n-1}(t)}{I_n^2(t)} \left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} u_{n,1}(s) ds \right) dt,$$

and more generally, for any $k \in \mathbb{Z}_+$,

$$\mathbb{E}[\tau_n^k] = u_{n,k}(0) = k! G_n^{\circ k}[\mathbb{1}](0)$$

Proof

Suppose by induction that $u_{n,k}(x) = \mathbb{E}_x[\tau_n^k]$. This is clearly satisfied for $k = 0$. Using Itô's formula, we have for all $0 \leqslant t \leqslant \tau_n$, for the process R defined in (2) and starting with $R(0) = x \in [0, L]$,

$$u_{n,k+1}(R(t)) - u_{n,k+1}(x) = -(k+1) \int_0^t u_{n,k}(R(s)) ds + M_t$$

where $(M_t)_{t \in [0,\tau_n]}$ is a martingale. Consider this equality with $t = \tau_n$, take expectation and use the Markov property to get

$$\begin{aligned} u_{n,k+1}(x) &= (k+1)\mathbb{E}_x \left[\int_0^{\tau_n} u_{n,k}(R(s)) ds \right] \\ &= (k+1)\mathbb{E}_x \left[\int_0^{\tau_n} \mathbb{E}_{R(s)}[\tau_n^k] ds \right] \\ &= (k+1)\mathbb{E}_x \left[\int_0^{\tau_n} (\tau_n - s)^k ds \right] = \mathbb{E}_x[\tau_n^{k+1}]. \end{aligned}$$

From Remark 6, $G_n(g)(r)$ is defined and bounded if g is bounded. Moreover $G_n(g)(0) \geqslant G_n(g)(r)$ if $g \geqslant 0$. This implies that if $u_{n,k}(0) < \infty$, then $u_{n,k+1}$ is defined and bounded.

■

The following characterisation of the cut-off phenomenon holds in general and in particular for the Brownian motion in M_f^n with initial value $\widetilde{0}$. The underlying idea of comparing the variance and the square of the expectation of sharp strong stationary times was also used by Diaconis and Saloff-Coste [8].

As usual, we say that $f_n = o(g_n)$ when $\frac{f_n}{g_n} \to 0$ as n goes to infinity, and $f_n = O(g_n)$ when there exists a constant c such that $f_n \leqslant c g_n$.
Let $a_n \sim_{n\to\infty} \mathbb{E}[\tau_n] = \int_0^L \frac{f^{n-1}(t)}{I_n^2(t)} \left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} ds \right) dt$.

Theorem 8 *Suppose that for some sequence $(b_n)_{n\geqslant 1}$ we have $\forall n \geqslant 1$, $0 < b_n \leqslant a_n$,*

$$a_n - \mathbb{E}[\tau_n] = o(b_n) \quad \textit{and} \quad \mathrm{Var}(\tau_n) = o(b_n^2),$$

then the family of diffusion processes $(X_n)_{n \in \mathbb{N}\setminus\{1\}}$ has a cut-off in separation with mixing times $(a_n)_{n \in \mathbb{N}\setminus\{1\}}$ and windows $(b_n)_{n \in \mathbb{N}\setminus\{1\}}$ in the sense of Section 1.3.

Proof

Since τ_n is a sharp strong stationary time for X_n, we have

$$\forall\, t \geqslant 0, \qquad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n) \;=\; \mathbb{P}\left[\tau_n > t\right].$$

Using Bienaymé-Tchebychev inequality we have for any $r > 0$,

$$\begin{aligned}
\mathbb{P}\left[\tau_n > a_n + r b_n\right] &= \mathbb{P}\left[\tau_n - \mathbb{E}[\tau_n] > r b_n + a_n - \mathbb{E}[\tau_n]\right] \\
&\leqslant \frac{\mathrm{Var}(\tau_n)}{(r b_n + a_n - \mathbb{E}[\tau_n])^2} \\
&= \frac{o(b_n)^2}{(r b_n + o(b_n))^2} \;=\; o(1)
\end{aligned}$$

For the behavior before a_n, write for $r \in (0,1)$,

$$\mathbb{P}[\tau_n \leqslant a_n - r b_n] \;=\; \mathbb{P}[\tau_n - \mathbb{E}[\tau_n] \leqslant a_n - r b_n - \mathbb{E}[\tau_n]]$$

and note that $a_n - r b_n - \mathbb{E}[\tau_n] = -r(1 + o(1)) b_n < 0$ for n large. Thus we get

$$\mathbb{P}[\tau_n \leqslant a_n - r_n b] \;\leqslant\; \frac{\mathrm{Var}(\tau_n)}{(r b_n + o(b_n))^2} \;=\; o(1)$$

■

Proposition 7 could be used to compute the variance of τ_n, but it is not well-adapted to compute an equivalent, so let us give an alternative computation of the variance.

Proposition 9 *The variance of τ_n is given by*

$$\mathrm{Var}(\tau_n) = 2 \int_0^L \frac{f^{n-1}(t)}{I_n^2(t)} \int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} (u'_{n,1}(s))^2 ds$$

Proof

Recall that $u_{n,1}$ is the solution of

$$\begin{cases} L_n u_{n,1} &= -1 \\ u_{n,1}(L) &= 0 \end{cases}$$

Using (2) and Itô formula, we have:

$$u_{n,1}(R(\tau_n)) - u_{n,1}(0) = -\tau_n + \sqrt{2} \int_0^{\tau_n} u'_{n,1}(R(s)) dB_s,$$

and since $u_{n,1}(0) = \mathbb{E}[\tau_n]$, we have

$$\mathrm{Var}(\tau_n) = 2\mathbb{E}\left[\int_0^{\tau_n} (u'_{n,1})^2(R(s)) ds\right].$$

Let ϕ_n be the solution of

$$\begin{cases} L_n \phi_n &= -2(u'_{n,1})^2 \\ \phi_n(L) &= 0 \end{cases}$$

Again by Itô formula, we get

$$\phi_n(R(\tau_n)) - \phi_n(0) = -2 \int_0^{\tau_n} (u'_{n,1})^2 (R(s)) ds + M_{\tau_n},$$

where $(M_t)_{t \in [0, \tau_n]}$ is a martingale. After taking the expectation in the above formula, we get from Proposition 2:

$$\mathrm{Var}(\tau_n) = \phi_n(0) = 2 \int_0^L \frac{f^{n-1}(t)}{I_n^2(t)} \left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} (u'_{n,1}(s))^2 ds \right) dt$$

■

3 Application to cut-off for rotationnaly symmetric

In this section we derive the cut-off in separation phenomenon for a class of rotationally symmetric manifolds that contains spheres.

From now on, all constants will be denoted c, their exact values can change from one line to another. When these constants depend on a parameter, such as A, we will rather write $c(A)$.

Proposition 10 *Let f be a C^3 function on $[0, L]$ satisfying Assumptions* (1) *and* (5). *Assume that $f''(L/2) < 0$. Denote for $n \geqslant 1$ $a_n := \frac{f(L/2)}{|f''(L/2)|}\frac{\ln(n)}{n}$, and let $(b_n)_{n\geqslant 1}$ satisfy $\frac{\sqrt{\ln n}}{n} = o(b_n)$ and $b_n \leqslant a_n$. Then*

$$\mathbb{E}[\tau_n] - a_n = o(b_n). \tag{10}$$

Proof

From Proposition 7 and Remark 6 we get: $\mathbb{E}[\tau_n] = u_{n,1}(0)$. Let us write, for $A_n := e^{\sqrt{\ln n}}$:

$$\begin{aligned} u_{n,1}(0) &= \int_0^L \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds = 2\int_0^{L/2} \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds \\ &= 2\left(\alpha_n + \gamma_n + \beta_n\right) \end{aligned} \tag{11}$$

with

$$\alpha_n := \int_0^{L/2 - A_n/\sqrt{n}} \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds, \tag{12}$$

$$\gamma_n := \int_{L/2 - A_n/\sqrt{n}}^{L/2 - 1/\sqrt{n}} \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds \tag{13}$$

and

$$\beta_n := \int_{L/2 - 1/\sqrt{n}}^{L/2} \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds. \tag{14}$$

We will prove that

$$a_n - 2\alpha_n = o(b_n), \quad \gamma_n = o(b_n) \quad \text{and} \quad \beta_n = o(b_n) \tag{15}$$

and this will establish (10).

Let us prove that $a_n - 2\alpha_n = o(b_n)$.

We introduce

$$J_n(s) \quad := \quad \frac{I_n(s)}{f^{n-1}(s)}\frac{I_n(L) - I_n(s)}{I_n(L)} = \frac{I_n(s)}{f^{n-1}(s)}\frac{I_n(L-s)}{I_n(L)}. \tag{16}$$

We have

$$\alpha_n := \int_0^{L/2 - A_n/\sqrt{n}} J_n(s) ds \tag{17}$$

We will prove that

$$\forall\ s \in \left[0, L/2 - A_n/\sqrt{n}\right], \quad \left|\frac{f(s)}{nf'(s)} - J_n(s)\right| = O\left(\frac{f(s)}{nA_n^2 f'(s)}\right) \tag{18}$$

uniformly in s, and then that

$$\frac{1}{n}\int_0^{L/2 - A_n/\sqrt{n}} \frac{f(s)}{f'(s)} ds - \frac{a_n}{2} + \frac{\sqrt{\ln n}}{n} = O(1/n). \tag{19}$$

From estimates (18) and (19) we will get $a_n - 2\alpha_n = o(b_n)$.

For $s = \frac{L}{2}$, since f is increasing in $[0, L/2]$ and $f'(L/2) = 0$ we get using Laplace's method:

$$\begin{aligned} I_n\left(\frac{L}{2}\right) &= \int_0^{\frac{L}{2}} f^{n-1}(t)dt = \int_0^{\frac{L}{2}} \exp((n-1)\ln(f)(t))dt \\ &\sim_{n\to\infty} \sqrt{\frac{\pi}{2|\ln(f(s))''|_{s=L/2}|}} \frac{f^{n-1}(L/2)}{\sqrt{n-1}} \\ &\sim_{n\to\infty} \sqrt{\frac{\pi f(L/2)}{2n|f''(L/2)|}} f^{n-1}(L/2). \end{aligned} \tag{20}$$

If $s < \frac{L}{2} - \frac{A_n}{\sqrt{n}}$ then we have the following expansion, since $f' > 0$ on the interval $[0, L/2)$:

$$\begin{aligned} I_n(s) &= \int_0^s f^{n-1}(t)dt = \int_0^s \exp((n-1)\ln(f(t)))dt \\ &\leqslant \frac{L}{2}\exp\left((n-1)\ln\left(f\left(\frac{L}{2} - \frac{A_n}{\sqrt{n}}\right)\right)\right) \\ &= \frac{L}{2} f\left(\frac{L}{2}\right)^{n-1} \exp\left[(n-1)\ln\left(1 + \frac{A_n^2 f''(L/2)}{2nf(L/2)} + O\left(\frac{A_n^3}{n^{3/2}}\right)\right)\right] \\ &= \frac{L}{2} f\left(\frac{L}{2}\right)^{n-1} \exp\left[\frac{A_n^2 f''(L/2)}{2f(L/2)} + O\left(\frac{A_n^3}{n^{1/2}}\right)\right]. \end{aligned} \tag{21}$$

This implies that

$$\begin{aligned} \left|\frac{I_n(L) - I_n(s)}{I_n(L)} - 1\right| = \frac{L}{2}\frac{I_n(s)}{I_n(L)} &\leqslant \frac{L}{2}\frac{f\left(\frac{L}{2}\right)^{n-1}}{I_n(L)} \exp\left[\frac{A_n^2 f''(L/2)}{2f(L/2)} + O\left(\frac{A_n^3}{n^{1/2}}\right)\right] \\ &\sim \frac{L}{4}\sqrt{\frac{2n|f''(L/2)|}{\pi f(L/2)}} \exp\left[\frac{A_n^2 f''(L/2)}{2f(L/2)} + O\left(\frac{A_n^3}{n^{1/2}}\right)\right] \\ &\leqslant \frac{C}{n} \end{aligned}$$

for some $C > 0$ (this majoration will be enough for our purpose). We get

$$1 - \frac{C}{n} \leqslant \frac{I_n(L) - I_n(s)}{I_n(L)} \leqslant 1. \tag{22}$$

We now investigate the term $I_n(s)/f^{n-1}(s)$ of $J_n(s)$. After integration by parts, for $0 \leqslant s < L/2$, and since $f' > 0$ on $[0, L/2)$, we have

$$\begin{aligned} I_n(s) &= \int_0^s f^{n-1}(t)dt = \int_0^s \frac{f'(t)f^{n-1}(t)}{f'(t)}dt \\ &= \frac{f^n(s)}{nf'(s)} + \int_0^s \frac{f^n(t)f''(t)}{n(f'(t))^2}dt. \end{aligned} \tag{23}$$

Since f' is decreasing and positive on $[0, L/2)$, we have for $s \in [0, L/2)$,

$$\forall\ t \in [0, s], \qquad \frac{f^n(t)}{(f'(t))^2} \quad \leqslant \quad \frac{f^n(t)}{(f'(s))^2},$$

hence, with $m := \min_{[0,L/2]} f'' < 0$,

$$\frac{f^n(s)}{nf'(s)} + \frac{m}{n(f'(s))^2} I_{n+1}(s) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)}.$$

From the above equation we get

$$\frac{f^n(s)}{nf'(s)}\left(1+\frac{mf(s)}{(n+1)(f'(s))^2}\right) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)}. \tag{24}$$

Since f is increasing, f' is non-increasing in $(0, L/2)$ and $m < 0$ we deduce that for $s \in [0, L/2-A_n/\sqrt{n}]$:

$$\frac{f^n(s)}{nf'(s)}\left(1-\frac{|m|f(L/2)}{(n+1)(f'(L/2-A_n/\sqrt{n}))^2}\right) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)}. \tag{25}$$

From

$$f'\left(L/2-A_n/\sqrt{n}\right)+\frac{A_n}{\sqrt{n}}f''(L/2) = O\left(A_n^2/n\right) \tag{26}$$

we get

$$f'\left(L/2-A_n/\sqrt{n}\right)^2-\frac{A_n^2}{n}f''(L/2)^2 = O\left(\frac{A_n^3}{n^{3/2}}\right)$$

which yields

$$\frac{|m|f(L/2)}{(n+1)(f'(L/2-A_n/\sqrt{n}))^2} = O\left(\frac{1}{A_n^2}\right).$$

This estimate together with (25) give for $s \in [0, L/2-A_n/\sqrt{n}]$:

$$0 \leqslant \frac{f(s)}{nf'(s)}-\frac{I_n(s)}{f^{n-1}(s)} \leqslant \frac{C}{A_n^2}\frac{f(s)}{nf'(s)} \tag{27}$$

for some $C > 0$. Multiplying by $\dfrac{I_n(L)-I_n(s)}{I_n(L)}$ and using (22) we get for all $s \in [0, L/2-A_n/\sqrt{n}]$,

$$\left|\frac{f(s)}{nf'(s)}-J_n(s)\right| \leqslant \frac{Cf(s)}{A_n^2 nf'(s)} \tag{28}$$

for some $C > 0$. This is (18).

For proving (19) we remark that a Taylor expansion of $\dfrac{f(s)}{f'(s)}$ yields on $s \in [0, L/2)$

$$\frac{f(s)}{f'(s)} = \frac{f(L/2)}{|f''(L/2)|(L/2-s)}+g(s) \tag{29}$$

with $g(s)$ uniformly bounded in $[0, L/2)$. So

$$\begin{aligned}\frac{1}{n}\int_0^{L/2-A_n/\sqrt{n}}\frac{f(s)}{f'(s)}\,ds &= \frac{f(L/2)}{|f''(L/2)|}\left(\frac{\ln n}{2n}-\frac{\ln A_n}{n}\right)+O(1/n)\\ &= \frac{a_n}{2}-\frac{\sqrt{\ln n}}{n}+O(1/n)\end{aligned}$$

which is (19). So $a_n-2\alpha_n = o(b_n)$.

Next we prove that $\gamma_n = o(b_n)$. We already know as an immediate consequence of (24) that on $[0, L/2-1/\sqrt{n}]$, we have

$$J_n(s) \leqslant \frac{f(s)}{nf'(s)}. \tag{30}$$

On the other hand, by (29), $\dfrac{f(s)}{f'(s)}-\dfrac{f(L/2)}{|f''(L/2)|(L/2-s)}$ is bounded in $[0, L/2)$ .Consequently,

$$\begin{aligned}\frac{1}{n}\int_{L/2-A_n/\sqrt{n}}^{L/2-1/\sqrt{n}}\frac{f(s)}{f'(s)}\,ds &\sim \frac{1}{n}\int_{L/2-A_n/\sqrt{n}}^{L/2-1/\sqrt{n}}\frac{f(L/2)}{|f''(L/2)|(L/2-s)}\,ds\\ &= \frac{1}{n}\frac{f(L/2)}{|f''(L/2)|}\ln(A_n) = \frac{1}{n}\frac{f(L/2)}{|f''(L/2)|}\sqrt{\ln n} = o(b_n).\end{aligned} \tag{31}$$

This proves the second estimate in (15).

Finally we prove that $\beta_n = o(b_n)$.

If $s \in [\frac{L}{2} - \frac{1}{\sqrt{n}}, \frac{L}{2} + \frac{1}{\sqrt{n}}]$, then write $s = L/2 + a/\sqrt{n}$, with $a \in [-1, 1]$. Since $f'(L/2) = 0$ and f is C^3, we have uniformly in $a \in [-1, 1]$:

$$
\begin{aligned}
I_n(L/2 + a/\sqrt{n}) &= I_n(L/2) + \int_{L/2}^{L/2+a/\sqrt{n}} f^{n-1}(x)dx \\
&= I_n(L/2) + \frac{1}{\sqrt{n}} \int_0^a f^{n-1}\left(\frac{L}{2} + \frac{h}{\sqrt{n}}\right) dh \\
&= I_n(L/2) + \frac{1}{\sqrt{n}} \int_0^a \left(f(L/2) + \frac{f''(L/2)h^2}{2n} + O(1/n^{3/2})\right)^{n-1} dh \\
&\sim \frac{f^{n-1}(L/2)}{\sqrt{n}} \left(\sqrt{\frac{\pi f(L/2)}{2|f''(L/2)|}} + \int_0^a e^{\frac{f''(L/2)h^2}{2f(L/2)}} dh\right) \\
&= \frac{f^{n-1}(L/2)}{\sqrt{n}} \int_{-\infty}^a e^{\frac{f''(L/2)h^2}{2f(L/2)}} dh. \qquad (32)
\end{aligned}
$$

Hence, letting $h(a) = \int_{-\infty}^a e^{\frac{f''(L/2)h^2}{2f(L/2)}} dh$, we get that for β_n defined in (11)

$$
\begin{aligned}
\beta_n &:= \int_{L/2-1/\sqrt{n}}^{L/2} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds \\
&\sim \frac{1}{\sqrt{n} I_n(L)} \int_{-1}^0 \frac{h(a)h(-a)f^{2n-2}(L/2)}{n f^{n-1}(L/2) e^{\frac{f''(L/2)a^2}{2f(L/2)}}} da \\
&\sim \frac{c}{n} = o(b_n), \qquad (33)
\end{aligned}
$$

where we used the following uniform estimate in $a \in [-1, 1]$ obtained as in (21)

$$
f^n(L/2 - a/\sqrt{n}) \sim f^n(L/2) e^{\frac{f''(L/2)a^2}{2f(L/2)}}, \qquad (34)
$$

and next (20) for the last equivalent.

■

Proposition 11 *Let f be a C^3 function on $[0, L]$ satisfying Assumptions* (1) *and* (5). *Assume that $f''(L/2) < 0$, then*

$$
\mathrm{Var}(\tau_n) = O\left(\frac{\ln n}{n^2}\right).
$$

Proof

From Proposition 9 and after integration by parts, it follows that for all A large enough and for all n large enough:

$$
\begin{aligned}
\frac{\mathrm{Var}(\tau_n)}{2} &= \int_0^L \frac{f^{n-1}(t)}{I_n^2(t)} \int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} (u'_{n,1}(s))^2 ds \\
&= \left[-\frac{1}{I_n(t)} \int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} (u'_{n,1}(s))^2 ds\right]_0^L + \int_0^L \frac{I_n(s)}{f^{n-1}(s)} (u'_{n,1}(s))^2 ds \\
&= \int_0^L \frac{I_n(s)}{f^{n-1}(s)} \frac{I_n(L) - I_n(s)}{I_n(L)} (u'_{n,1}(s))^2 ds
\end{aligned}
$$

$$= \underbrace{\int_0^{L/2-A/\sqrt{n}} J_n(s)(u'_{n,1}(s))^2 ds}_{A_n} + \underbrace{\int_{L/2-A/\sqrt{n}}^{L/2+A/\sqrt{n}} J_n(s)(u'_{n,1}(s))^2 ds}_{B_n} \tag{35}$$

$$+ \underbrace{\int_{L/2+A/\sqrt{n}}^{L} J_n(s)(u'_{n,1}(s))^2 ds}_{C_n} \tag{36}$$

We will analyze the three last terms separately. Recall that by Proposition 5 and from the fact that $u_{n,1} = G_n[\mathbb{1}]$, we have

$$(u'_{n,1}(t))^2 \quad = \quad \left(\frac{f^{n-1}(t)}{I_n^2(t)} \int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} ds \right)^2 \tag{37}$$

- Let us start by the term A_n, using computations (23) and (24), since f is increasing, f' is non-increasing in $(0, L/2)$ and $m < 0$, it follows that for A big enough and for all n sufficiently large, and for all $s \in [0, L/2 - A/\sqrt{n}]$,

$$\frac{f^{n+1}(s)}{n^2(f'(s))^2} \left(1 - \frac{1}{A}\right)^2 \leqslant \frac{I_n^2(s)}{f^{n-1}(s)} \leqslant \frac{f^{n+1}(s)}{n^2(f'(s))^2}. \tag{38}$$

Let $W_n(t) = \int_0^t \frac{f^{n+1}(s)}{(f'(s))^2} ds$, for $0 \leqslant t < L/2$, we have after integration by parts:

$$\begin{aligned} W_n(t) &= \int_0^t \frac{f^{n+1}(s) f'(s)}{(f'(s))^3} ds \\ &= \frac{f^{n+2}(t)}{(n+2)(f'(t))^3} + \frac{3}{n+2} \int_0^t \frac{f^{n+2}(s) f''(s)}{(f'(s))^4} ds. \end{aligned}$$

Since $f'' \leqslant 0$, we deduce, using (38),

$$\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)} ds \leqslant \frac{f^{n+2}(t)}{n^3(f'(t))^3} \tag{39}$$

and that for A big enough and for all n sufficiently large:

$$\begin{aligned} (u'_{n,1}(t))^2 \quad &\leqslant \quad \left(\frac{n^2(f'(t))^2 f^{n+2}(t)}{f^{n+1}(t)(1-\frac{1}{A})^2 n^3 (f'(t))^3} \right)^2 \\ &= \quad \left(\frac{f(t)}{n f'(t)(1-\frac{1}{A})^2} \right)^2. \end{aligned}$$

Also by (30),

$$J_n(s) := \frac{I_n(s)}{f^{n-1}(s)} \frac{I_n(L-s)}{I_n(L)} \leqslant \frac{f(s)}{n f'(s)}.$$

Hence for A big enough, for all n sufficiently large, and for A_n defined in (36), by (38), we have

$$\begin{aligned} A_n &\leqslant \frac{1}{n^3(1-\frac{1}{A})^4} \int_0^{L/2-A/\sqrt{n}} \frac{f^3(s)}{(f'(s))^3} ds \\ &\leqslant \frac{f(L/2)^3}{n^3(1-\frac{1}{A})^4} \int_0^{L/2-A/\sqrt{n}} \frac{1}{(f'(s))^3} ds \end{aligned}$$

$$\sim \frac{c}{A^2 n^2},$$

where in the last equivalent we used $\frac{1}{f'(s)} \sim_{s \to L/2-} \frac{1}{|f''(L/2)|(L/2-s)}$.
Hence for A big enough,

$$A_n = O\left(\frac{\ln(n)}{n^2}\right). \tag{40}$$

- For the term B_n in (36): for A big enough, for all n large enough and for $a \in [-A, A]$, we have

$$\left|u'_{n,1}\left(\frac{L}{2} + \frac{a}{\sqrt{n}}\right)\right| = \frac{f^{n-1}\left(\frac{L}{2} + \frac{a}{\sqrt{n}}\right)}{I_n^2\left(\frac{L}{2} + \frac{a}{\sqrt{n}}\right)}\left(\int_0^{L/2-A/\sqrt{n}} \frac{I_n^2(s)}{f^{n-1}(s)} ds + \int_{L/2-A/\sqrt{n}}^{L/2+\frac{a}{\sqrt{n}}} \frac{I_n^2(s)}{f^{n-1}(s)} ds\right).$$

Let $C(f,2) = \left(\frac{2f(L/2)}{|f''(L/2)|}\right)^{1/2}$ we have (similarly to the computation in (21) and (26)):

$$f^n(L/2 - A/n^{1/2}) \sim f^n(L/2) e^{\frac{-A^2}{C(f,2)^2}}, \tag{41}$$

and

$$f'(L/2 - A/n^{1/2}) \sim \frac{|f''(L/2)|A}{n^{1/2}}. \tag{42}$$

By the above computation and (39), we have

$$\int_0^{L/2-A/\sqrt{n}} \frac{I_n^2(s)}{f^{n-1}(s)} ds \leqslant \frac{f^{n+2}(L/2 - \frac{A}{\sqrt{n}})}{n^3 (f'(L/2 - \frac{A}{\sqrt{n}}))^3} \sim \frac{f^{n+2}(L/2) e^{-\frac{A^2}{C(f,2)^2}}}{n^{3/2} A^3 |f''(L/2)|^3}.$$

Recall that from (32), and for $h_1(a) = \int_{-\infty}^a e^{\frac{-h^2}{C(f,2)^2}} dh$, we have uniformly over $a \in [-A, A]$

$$I_n(L/2 + a/\sqrt{n}) \sim f^{n-1}(L/2) \frac{h_1(a)}{\sqrt{n}}, \tag{43}$$

hence since uniformly in $a \in [-A, A]$,

$$f^n(L/2 - a/n^{1/2}) \sim f^n(L/2) e^{\frac{-a^2}{C(f,2)^2}},$$

taking (41) into account,

$$\begin{aligned} \int_{L/2-A/\sqrt{n}}^{L/2+\frac{a}{\sqrt{n}}} \frac{I_n^2(s)}{f^{n-1}(s)} ds &= \frac{1}{\sqrt{n}} \int_{-A}^a \frac{I_n^2(L/2 + \frac{\tilde{a}}{\sqrt{n}})}{f^{n-1}(L/2 + \frac{\tilde{a}}{\sqrt{n}})} d\tilde{a} \\ &\sim \frac{f^{n-1}(L/2)}{\sqrt{n}} \int_{-A}^a \frac{h_1^2(\tilde{a}) e^{\frac{\tilde{a}^2}{C(f,2)^2}}}{n} d\tilde{a} = \frac{f^{n-1}(L/2)\theta_A(a)}{n^{3/2}}, \end{aligned} \tag{44}$$

where $\theta_A(a) := \int_{-A}^a h_1^2(\tilde{a}) e^{\frac{\tilde{a}^2}{C(f,2)^2}} d\tilde{a}$.
We have for A big enough

$$|u'_{n,1}\left(\frac{L}{2} + \frac{a}{\sqrt{n}}\right)| \leqslant c \frac{e^{-\frac{a^2}{C(f,2)^2}}}{\sqrt{n} h_1^2(a)} \left(\theta_A(a) + \frac{1}{A^3}\right).$$

Hence

$$\begin{aligned} B_n &:= \int_{L/2-A/\sqrt{n}}^{L/2+A/\sqrt{n}} J_n(s)(u'_{n,1}(s))^2 ds \\ &= \frac{1}{\sqrt{n}} \int_{-A}^{A} J_n(L/2+a/\sqrt{n})(u'_{n,1}(L/2+a/\sqrt{n}))^2 da \\ &\leqslant \frac{1}{\sqrt{n}} \int_{-A}^{A} J_n(L/2+a/\sqrt{n}) \left(c\frac{e^{-\frac{a^2}{C(f,2)^2}}}{\sqrt{n}h_1^2(a)} \left(\theta_A(a) + \frac{1}{A^3}\right)\right)^2 da \\ &\sim \frac{c(A)}{n^2}, \end{aligned}$$

where, taking (43) into account, we use for the last term that

$$J_n(L/2+a/\sqrt{n}) = \frac{I_n(L/2+a/\sqrt{n})}{f^{n-1}(L/2+a/\sqrt{n})} \frac{I_n(L/2-a/\sqrt{n})}{I_n(L)} \sim c\frac{h_1(a)h_1(-a)e^{\frac{a^2}{C(f,2)^2}}}{\sqrt{n}},$$

and $c(A)$ is a constant that depends on A. It follows that once A is fixed, for n big enough,

$$B_n = O\left(\frac{\ln(n)}{n^2}\right). \tag{45}$$

- For the last term C_n in (36), note that $J_n(s) = J_n(L-s)$ so

$$\begin{aligned} C_n &= \int_{L/2+A/\sqrt{n}}^{L} J_n(s)(u'_{n,1}(s))^2 ds \\ &= \int_{0}^{L/2-A/\sqrt{n}} J_n(s)(u'_{n,1}(L-s))^2 ds. \end{aligned}$$

Also for $s \leqslant L/2 - A/\sqrt{n}$

$$\begin{aligned} |u'_{n,1}(L-s)| &= \frac{f^{n-1}(s)}{I_n^2(L-s)} \int_0^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt \\ &= \frac{f^{n-1}(s)}{I_n^2(L-s)} \left(\int_0^{L/2-A/\sqrt{n}} \frac{I_n^2(t)}{f^{n-1}(t)} dt + \int_{L/2-A/\sqrt{n}}^{L/2+A/\sqrt{n}} \frac{I_n^2(t)}{f^{n-1}(t)} dt + \int_{L/2+A/\sqrt{n}}^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt \right). \end{aligned}$$

Using (39), (41) and (42), we have that for A big enough, for all n sufficiently large

$$\int_0^{L/2-A/\sqrt{n}} \frac{I_n^2(s)}{f^{n-1}(s)} ds \leqslant 2\frac{f^{n+2}(L/2)e^{-\frac{A^2}{C(f,2)^2}}}{n^{3/2}A^3(f''(L/2))^3}$$

and using (44),

$$\int_{L/2-A/\sqrt{n}}^{L/2+A/\sqrt{n}} \frac{I_n^2(t)}{f^{n-1}(t)} dt \leqslant 2\frac{\theta_A(A)f^{n-1}(L/2)}{n^{3/2}}.$$

For the last term since for $s \leqslant L/2 - A/\sqrt{n}$,

$$\int_{L/2+A/\sqrt{n}}^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt \leqslant I_n^2(L-s) \int_{L/2+A/\sqrt{n}}^{L-s} \frac{1}{f^{n-1}(t)} dt$$

$$
\begin{aligned}
&= I_n^2(L-s)\int_s^{L/2-A/\sqrt{n}} \frac{1}{f^{n-1}(t)}dt \\
&= I_n^2(L-s)\int_s^{L/2-A/\sqrt{n}} \frac{f'(t)}{f^{n-1}(t)f'(t)}dt \\
&\leqslant \frac{I_n^2(L-s)}{f'(L/2-A/\sqrt{n})}\left(\frac{f^{-n+2}(s)}{n-2}\right) \\
&\sim c\frac{f^{-n+2}(s)I_n^2(L-s)}{n^{1/2}A}
\end{aligned}
$$

Since $L-s \geqslant L/2$, we have for A big enough and for all n sufficiently large,

$$
\begin{aligned}
|u'_{n,1}(L-s)| &\leqslant c\frac{f^{2n-2}(L/2)}{n^{3/2}I_n^2(L/2)}\left(\frac{e^{-\frac{A^2}{C(f,2)^2}}}{A^3}+\theta_A(A)\right)+c\frac{f(s)}{n^{1/2}A} \\
&\leqslant \frac{c}{n^{1/2}}\left(\frac{e^{-\frac{A^2}{C(f,2)^2}}}{A^3}+\theta_A(A)\right)+c\frac{f(s)}{n^{1/2}A}\leqslant c(A)\frac{1}{n^{1/2}}
\end{aligned}
$$

where in the second inequality, we used (20). Since, for $s \leqslant L/2-A/\sqrt{n}$, by (24) $J_n(s) \leqslant \frac{f(s)}{nf'(s)}$, for A big enough and for all n sufficiently large,

$$
\begin{aligned}
C_n &= \int_0^{L/2-A/\sqrt{n}} J_n(s)(u'_{n,1}(L-s))^2 ds. \\
&\leqslant \frac{c(A)^2}{n}\int_0^{L/2-A/\sqrt{n}} J_n(s)ds \\
&\leqslant \frac{c(A)^2 f(L/2)}{n^2}\int_0^{L/2-A/\sqrt{n}} \frac{1}{f'(s)}ds \\
&\sim \frac{c(A)^2 f(L/2)}{n^2}\left(\frac{1}{2}\ln(n)\right) \sim c(A)^2\frac{\ln(n)}{n^2}
\end{aligned}
$$

Hence

$$
C_n = O\left(\frac{\ln(n)}{n^2}\right). \tag{46}
$$

Putting (40), (45) and (46) together, we deduce that $\mathrm{Var}(\tau_n) = O\left(\frac{\ln(n)}{n^2}\right)$.

■

We deduce the following result

Theorem 12 *Let f be a C^3 function on $[0,L]$ satisfying Assumptions (1) and (5) and $f''(L/2)<0$. For $n \in \mathbb{N}\backslash\{1\}$, consider the Brownian motion $X_n := (X_n(t))_{t\geqslant 0}$ described in Definition 1. Then the family of diffusion processes $(X_n)_{n\in\mathbb{N}\backslash\{1\}}$ has a cut-off in separation in the sense of Section 1.3, with mixing times $(a_n)_{n\in\mathbb{N}\backslash\{1\}} = \left(\frac{f(L/2)}{|f''(L/2)|}\frac{\ln(n)}{n}\right)_{n\in\mathbb{N}\backslash\{1\}}$ and windows $(b_n)_{n\in\mathbb{N}\backslash\{1\}}$ satisfying $\forall\, n \geqslant 1$, $b_n \leqslant a_n$, together with $\frac{\sqrt{\ln(n)}}{n} = o(b_n)$.*

Proof

Use Theorem 8, Proposition 10 and Proposition 11.

■

The previous arguments provide an alternative proof to the main result from [3], and strengthens the result with a cut-off in separation with windows:

Corollary 13 *Let $X_n := (X_n(t))_{t\geqslant 0}$ be the Brownian motion described in Definition 1, where M_f^n is replaced by the sphere $\mathbb{S}^n$ and where $\widetilde{0}$ now stands for any point of $\mathbb{S}^n$. Then the family of diffusion processes $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has a cut-off in separation in the sense of Section 1.3, with mixing times $(a_n)_{n\in\mathbb{N}\setminus\{1\}} = \left(\frac{\ln(n)}{n}\right)_{n\in\mathbb{N}\setminus\{1\}}$ and windows $(b_n)_{n\in\mathbb{N}\setminus\{1\}}$ satisfying $\forall\ n \geqslant 1,\ b_n \leqslant a_n$, together with $\dfrac{\sqrt{\ln(n)}}{n} = o(b_n)$.*

Proof

Use Theorem 12, with $f = \sin$ and $L = \pi$, and note that by symmetry in this case the starting point is not relevant.

■

We also deduce the following consequences.

Corollary 14 *Let f be a C^3 function on $[0, L]$ satisfying Assumptions (1) and (5) and $f''(L/2) < 0$. For $n \in \mathbb{N}\setminus\{1\}$, consider the Brownian motion $X_n := (X_n(t))_{t\geqslant 0}$ in M_f^n described in Definition 1. There exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $r > 0$, $0 < r' < 1$ and for all $n \geqslant n_0$,*

$$\begin{aligned}
\left\|\mathcal{L}\left(X_n((1+r)\frac{f(L/2)}{|f''(L/2)|}\frac{\ln(n)}{n})\right) - \mathcal{U}_n\right\|_{\mathrm{tv}} &\leqslant \frac{C}{r^2\ln(n)}\\
\forall\ y \in M_f^n, \qquad P^{(n)}_{(1+r)\frac{f(L/2)}{|f''(L/2)|}\frac{\ln(n)}{n}}(\tilde{0}, y) &\geqslant \left(1 - \frac{C}{r^2\ln(n)}\right)\frac{1}{\mathrm{Vol}(M_f^n)}\\
\inf_{y\in M_f^n} P^{(n)}_{(1-r')\frac{f(L/2)}{|f''(L/2)|}\frac{\ln(n)}{n}}(\tilde{0}, y) &\leqslant \left(\frac{C}{r'^2\ln(n)}\right)\frac{1}{\mathrm{Vol}(M_f^n)}
\end{aligned}$$

where $\|\cdot\|_{\mathrm{tv}}$ stands for the total variation norm, $\mathcal{L}(X_n(t))$ is the law of $X_n(t)$, $\mathcal{U}_n$ is the uniform measure in M_f^n, and $P_t^{(n)}(\cdot,\cdot)$ is the heat kernel density at time $t > 0$ associated to the Laplacian on M_f^n.

Proof

From the computations in the proof of Proposition 10 and Proposition 11, with $b_n = a_n$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $r > 0$ and for all $n \geqslant n_0$,

$$\mathbb{P}[\tau_n > (1+r)a_n] \leqslant \frac{\mathrm{Var}(\tau_n)}{((1+r)a_n - \mathbb{E}[\tau_n])^2} \leqslant \frac{C}{r^2\ln(n)}$$

The first conclusion follows, since

$$\|\mathcal{L}(X_n((1+r)a_n)) - \mathcal{U}_n\|_{\mathrm{tv}} \quad\leqslant\quad \mathfrak{s}(\mathcal{L}(X_n((1+r)a_n)), \mathcal{U}_n) \quad\leqslant\quad \mathbb{P}[\tau_n > (1+r)a_n]$$

The second conclusion follows by definition of the separation discrepancy, since for all $y \in M_f^n$ and $t > 0$,

$$1 - P_t^{(n)}(\tilde{0}, y)vol(M_f^n) \quad\leqslant\quad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n)$$

The last conclusion follows in the same way.

■

Proposition 15 *Let f be a C^3 function on $[0, L]$ satisfying Assumptions (1) and (5). Assume that for some $k \geqslant 2$,*

$$f(L/2 + h) = f(L/2) + \frac{f^{(2k)}(L/2)}{(2k)!} h^{2k} + o(h^{2k})$$

where $f^{(2k)}(L/2) < 0$, then

$$\mathbb{E}[\tau_n] \sim \frac{2kC(2k)C(f, 2k)^2}{n^{1/k}\Gamma\left(\frac{1}{2k}\right)},$$

where Γ is the usual Gamma functional and

$$C(f, 2k) := \left(\frac{(2k)! f(L/2)}{|f^{(2k)}(L/2)|}\right)^{1/2k} \tag{47}$$

$$C(2k) := \int_0^\infty h_{1,k}(a) h_{1,k}(-a) e^{a^{2k}} da, \tag{48}$$

and for any $x \in \mathbb{R}$, $h_{1,k}(x) := \int_{-\infty}^x e^{-a^{2k}} da$.

Proof

Recall from Proposition 7 that $\mathbb{E}[\tau_n] = u_{n,1}(0) = \int_0^L \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds$. Let us write, for A big enough and n sufficiently large:

$$u_{n,1}(0) = \int_0^L \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds = 2\int_0^{L/2} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds$$

$$= 2 \left(\underbrace{\int_0^{L/2 - A/n^{1/2k}} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds}_{A_n} \right. \tag{49}$$

$$\left. + \underbrace{\int_{L/2 - A/n^{1/2k}}^{L/2} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L) - I_n(s)}{I_n(L)}\right) ds}_{B_n} \right) \tag{50}$$

- Volume of M_f^n: Since f is increasing in $[0, L/2]$, using Laplace method we get:

$$\begin{aligned} I_n\left(\frac{L}{2}\right) &= \int_0^{\frac{L}{2}} f^{n-1}(t) dt = \int_0^{\frac{L}{2}} \exp((n-1)\ln(f)(t)) dt \\ &\sim_{n\to\infty} \frac{\Gamma\left(\frac{1}{2k}\right)}{2k} \left(\frac{(2k)!}{|(\ln f)^{(2k)}(L/2)|}\right)^{1/2k} \frac{f^{n-1}(L/2)}{(n-1)^{1/2k}} \\ &\sim_{n\to\infty} \frac{1}{2k}\Gamma\left(\frac{1}{2k}\right) \left(\frac{(2k)! f(L/2)}{n|f^{(2k)}(L/2)|}\right)^{1/2k} f^{n-1}(L/2) \\ &\sim_{n\to\infty} \frac{1}{2k}\Gamma\left(\frac{1}{2k}\right) C(f, 2k) \frac{f^{n-1}(L/2)}{n^{1/2k}} \end{aligned} \tag{51}$$

Since f'' is non-positive, $f'(0) = 1$, $f^{(i)}(L/2) = 0$ for $i \in \{1, ..., 2k-1\}$, and $f^{(2k)}(L/2) < 0$, there exists $M > 0$ such that, for all $u \in [0, L]$

$$\left| \frac{f''(u)}{(f'(u))^{(2k-2)/(2k-1)}} \right| \leqslant M.$$

It follows that for all $s \in [0, L/2)$ and for all $t \in [0, s]$

$$\left|\frac{f^n(t)f''(t)}{n(f'(t))^2}\right| \leqslant \frac{f^n(t)M}{n(f'(t))^{2k/(2k-1)}} \leqslant \frac{f^n(t)M}{n(f'(s))^{2k/(2k-1)}}.$$

Hence (23) gives

$$\frac{f^n(s)}{nf'(s)} - \frac{M}{n(f'(s))^{2k/(2k-1)}} I_{n+1}(s) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)}, \tag{52}$$

it follows that

$$\frac{f^n(s)}{nf'(s)}\left(1 - \frac{Mf(s)}{(n+1)(f'(s))^{2k/(2k-1)}}\right) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)},$$

and

$$1 - \frac{f^n(s)}{nf'(s)I_n(L)} \leqslant \frac{I_n(L) - I_n(s)}{I_n(L)} \leqslant 1. \tag{53}$$

- For $s \in [0, L/2 - A/n^{1/2k}]$.
  The above equation gives:

$$1 - \frac{f^n(L/2 - A/n^{1/2k})}{nf'(L/2 - A/n^{1/2k})I_n(L)} \leqslant \frac{I_n(L) - I_n(s)}{I_n(L)} \leqslant 1.$$

Since

$$f^n(L/2 - A/n^{1/2k}) \quad \sim \quad f^n(L/2)e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}, \tag{54}$$

and

$$f'(L/2 - A/n^{1/2k}) \quad \sim \quad \frac{|f^{(2k)}(L/2)|A^{2k-1}}{(2k-1)!n^{(2k-1)/2k}}, \tag{55}$$

using (51), we get for A big enough and n large enough

$$\frac{f^n(L/2 - A/n^{1/2k})}{nf'(L/2 - A/n^{1/2k})I_n(L)} \sim \frac{f(L/2)e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{\frac{2}{(2k)!}\Gamma\left(\frac{1}{2k}\right)C(f,2k)|f^{(2k)}(L/2)|A^{2k-1}} \leqslant e^{\frac{-A^{2k}}{C(f,2k)^{2k}}},$$

and so using the above equations, we get that for A big enough uniformly over $s \in [0, L/2 - A/n^{1/2k}]$

$$(1 - e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}) \leqslant \frac{I_n(L) - I_n(s)}{I_n(L)} \leqslant 1. \tag{56}$$

Since

$$\frac{Mf(s)}{(n+1)(f'(s))^{2k/(2k-1)}} \leqslant \frac{Mf(L/2)}{(n+1)(f'(L/2 - A/n^{1/2k}))^{2k/(2k-1)}} \sim \frac{c}{A^{2k}}$$

it follows with (55) that for A big enough and n sufficiently large

$$\frac{f^n(s)}{nf'(s)}\left(1 - \frac{1}{A^{2k-1}}\right) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)}. \tag{57}$$

Hence using (56) we get that for A big enough and for all n sufficiently large, uniformly in $s \in [0, L/2 - A/n^{1/2k}]$:

$$\frac{f(s)}{nf'(s)}\left(1 - \frac{1}{A^{2k-1}}\right)(1 - e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}) \leqslant J_n(s) \leqslant \frac{f(s)}{nf'(s)} \tag{58}$$

where J_n is defined in (16).
Since $\frac{f(s)}{f'(s)} \sim_{s\to L/2-} \frac{(2k-1)!f(L/2)}{|f^{(2k)}(L/2)|(L/2-s)^{(2k-1)}}$, we have

$$\frac{1}{n}\int_0^{L/2-A/n^{1/2k}} \frac{f(s)}{f'(s)}ds \sim \frac{1}{n}\int_0^{L/2-A/n^{1/2k}} \frac{(2k-1)!f(L/2)}{|f^{(2k)}(L/2)|(L/2-s)^{(2k-1)}}ds \sim \frac{c}{n^{1/k}A^{2k-2}}.$$

We get, for all A big enough, and for A_n defined in (49):

$$\limsup_{n\to\infty} \frac{A_n}{\frac{c}{n^{1/k}}} = \limsup_{n\to\infty} \frac{\int_0^{L/2-A/n^{1/2k}} J_n(s)ds}{\frac{c}{n^{1/k}}} \leqslant \frac{1}{A^{2k-2}} \tag{59}$$

and

$$\liminf_{n\to\infty} \frac{A_n}{\frac{c}{n^{1/k}}} = \liminf_{n\to\infty} \frac{\int_0^{L/2-A/n^{1/2k}} J_n(s)ds}{\frac{c}{n^{1/k}}} \geqslant \frac{1}{A^{2k-2}}\left(1-\frac{1}{A^{2k-1}}\right)(1-e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}) \tag{60}$$

- If $s \in [\frac{L}{2}-\frac{A}{n^{1/2k}}, \frac{L}{2}+\frac{A}{n^{1/2k}}]$, then write $s = L/2 + a/n^{1/2k}$, with $a \in [-A, A]$. Since $f^{(i)}(L/2) = 0$ for $i \in \{1, ..., 2k-1\}$, $f^{(2k)}(L/2) < 0$ and $f \in C^{2k+1}$, we deduce that uniformly in $a \in [-A, A]$:

$$\begin{aligned}
I_n(L/2+a/n^{1/2k}) &= I_n(L/2) + \int_{L/2}^{L/2+a/n^{1/2k}} f^{n-1}(x)dx \\
&= I_n(L/2) + \frac{1}{n^{1/2k}}\int_0^a f^{n-1}\left(\frac{L}{2}+\frac{h}{n^{1/2k}}\right)dh \\
&= I_n(L/2) + \frac{1}{n^{1/2k}}\int_0^a \left(f(L/2) - \frac{|f^{(2k)}(L/2)|}{(2k)!n}h^{2k} + O(\frac{1}{n^{(2k+1)/2k}})\right)^{n-1} dh \\
&\sim \frac{f^{n-1}(L/2)}{n^{1/2k}}\left(\frac{1}{2k}\Gamma\left(\frac{1}{2k}\right)C(f,2k) + \int_0^a e^{\frac{-h^{2k}}{C(f,2k)^{2k}}}dh\right) \\
&= \frac{f^{n-1}(L/2)}{n^{1/2k}}\int_{-\infty}^a e^{\frac{-h^{2k}}{C(f,2k)^{2k}}}dh.
\end{aligned} \tag{61}$$

Let $h_k(a) = \int_{-\infty}^a e^{\frac{-h^{2k}}{C(f,2k)^{2k}}}dh$. Since $I_n(L) - I_n(s) = I_n(L-s)$ we have uniformly in $a \in [-A, A]$

$$I_n(L/2+a/n^{1/2k})(I_n(L) - I_n(L/2+a/n^{1/2k})) \sim \frac{f^{2n-2}(L/2)}{n^{1/k}}h_k(a)h_k(-a).$$

Since

$$\begin{aligned}
&\int_{L/2-A/n^{1/2k}}^{L/2} \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)ds \\
&\qquad = \frac{1}{n^{1/2k}I_n(L)}\int_{-A}^0 \frac{I_n(L/2+a/n^{1/2k})(I_n(L)-I_n(L/2+a/n^{1/2k}))}{f^{n-1}(L/2+a/n^{1/2k})}da
\end{aligned}$$

we get for all A big enough, and for B_n defined in (49)

$$\begin{aligned}
B_n &\sim \frac{f^{n-1}(L/2)}{n^{3/2k}I_n(L)}\int_{-A}^0 h_k(a)h_k(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}}da \\
&\sim \frac{1}{n^{1/k}\frac{1}{k}\Gamma\left(\frac{1}{2k}\right)C(f,2k)}\int_{-A}^0 h_k(a)h_k(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}}da
\end{aligned} \tag{62}$$

Also

$$\int_0^{\infty} h_k(a)h_k(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}}\,da = C(f,2k)^3 C(2k)$$

where $C(2k) = \int_0^{\infty} h_{1,k}(a)h_{1,k}(-a)e^{a^{2k}}da$ and $h_{1,k}(x) = \int_{-\infty}^{x} e^{-a^{2k}}da$ (note that $C(2k)$ is finite since $k \geqslant 2$). We have for all A large enough

$$\begin{aligned}\int_{-A}^{0} h(a)h(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}}\,da &+ \frac{c}{A^{2k-2}}(1-\frac{1}{A^{2k-1}})(1-e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}) \\ &\leqslant \liminf_{n\to\infty} \frac{\mathbb{E}[\tau_n]}{\frac{2k}{n^{1/k}\Gamma(\frac{1}{2k})C(f,2k)}} \leqslant \limsup_{n\to\infty} \frac{\mathbb{E}[\tau_n]}{\frac{2k}{n^{1/k}\Gamma(\frac{1}{2k})C(f,2k)}} \\ &\leqslant \frac{c}{A^{2k-2}} + \int_{-A}^{0} h(a)h(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}}\,da,\end{aligned}$$

and so letting A go to infinity, we get

$$\mathbb{E}[\tau_n] \sim \frac{2kC(2k)C^2(f,2k)}{n^{1/k}\Gamma\left(\frac{1}{2k}\right)} = \frac{2kC(2k)}{n^{1/k}\Gamma\left(\frac{1}{2k}\right)}\left(\frac{(2k!f(L/2)}{|f^{(2k)}(L/2)|}\right)^{1/k}.$$

■

Remark 16 Note that the dominant term of $\mathbb{E}[\tau_n]$ comes from $\int_0^{L/2-A/\sqrt{n}} J_n(s)\,ds$ when $k=1$, and comes from $\int_{L/2-A/n^{1/2k}}^{L/2} J_n(s)ds$ when $k \geqslant 2$, this essentially leads to two different proofs, despite the apparent similarity. We will find this feature again in the sequel.

□

Proposition 17 *Let f be a C^{2k+1} function on $[0,L]$ satisfying Assumption (1) and (5). Assume that for some $k \geqslant 2$,*

$$f(L/2+h) = f(L/2) - \frac{|f^{(2k)}(L/2)|}{(2k)!}h^{2k} + o(h^{2k})$$

where $f^{(2k)}(L/2) < 0$ then

$$\lim_{n\to\infty} \frac{\mathbb{E}[\tau_n^2]}{\mathbb{E}[\tau_n]^2} = 1 + \frac{2}{c_k}\int_{-\infty}^{\infty} h_k^2(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}}\int_{-\infty}^{a} h_k^2(\tilde{a})e^{\frac{\tilde{a}^{2k}}{C(f,2k)^{2k}}}\,d\tilde{a}da,$$

where $c_k = C(2k)^2C(f,2k)^{3/k}$, and $C(2k)$ and $C(f,2k)$ are defined in (48) and (47). In particular

$$\mathrm{Var}(\tau_n) = O(\mathbb{E}[\tau_n]^2), \quad \mathrm{Var}(\tau_n) \neq o(\mathbb{E}[\tau_n]^2).$$

Proof

Using Proposition 7 and integration by parts we get

$$\begin{aligned}\frac{1}{2}\mathbb{E}[\tau_n^2] &= \int_0^L \frac{f^{n-1}(t)}{I_n^2(t)}\left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}u_{n,1}(s)ds\right)dt. \\ &= \int_0^L \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)u_{n,1}(s)ds.\end{aligned}$$

Also by Proposition 5 and integration by parts we have

$$u_{n,1}(r) = \int_r^L \frac{f^{n-1}(t)}{I_n^2(t)}\left(\int_0^t \frac{I_n^2(s)}{f^{n-1}(s)}ds\right)dt$$

$$
\begin{aligned}
&= \frac{1}{I_n(r)}\int_0^r \frac{I_n^2(s)}{f^{n-1}(s)}ds - \frac{1}{I_n(L)}\int_0^L \frac{I_n^2(s)}{f^{n-1}(s)}ds + \int_r^L \frac{I_n(s)}{f^{n-1}(s)}ds \\
&= \left(\frac{I_n(L)-I_n(r)}{I_n(L)I_n(r)}\right)\int_0^r \frac{I_n^2(s)}{f^{n-1}(s)}ds + \int_r^L \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}[\tau_n^2] &= \int_0^L \frac{1}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds \\
&+ \int_0^L \frac{I_n(s)}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)\int_s^L \frac{I_n(t)}{f^{n-1}(t)}\left(\frac{I_n(L)-I_n(t)}{I_n(L)}\right)dtds. \\
&= \int_0^L \frac{1}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds \\
&+ \frac{1}{2}\left(\int_0^L \frac{I_n(t)}{f^{n-1}(t)}\left(\frac{I_n(L)-I_n(t)}{I_n(L)}\right)dt\right)^2 \\
&= \int_0^L \frac{1}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds + \frac{1}{2}\mathbb{E}[\tau_n]^2,
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{\mathbb{E}[\tau_n^2]}{\mathbb{E}[\tau_n]^2} &= 1 + \frac{2\int_0^L \frac{1}{f^{n-1}(s)}\left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}{\mathbb{E}[\tau_n]^2} \\
&= 1 + \frac{2\int_0^L \frac{1}{f^{n-1}(s)}\left(I_n(L)-I_n(s)\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}{\mathbb{E}[\tau_n]^2 I_n(L)^2}
\end{aligned}
$$

Note that the numerator in the right hand side of the above first equation is the variance of τ_n. Let us write for all A large enough and for all n large enough:

$$
\begin{aligned}
\int_0^L \frac{(I_n(L)-I_n(s))^2}{f^{n-1}(s)}\int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds &= \underbrace{\int_0^{L/2-A/n^{1/2k}} \frac{(I_n(L-s))^2}{f^{n-1}(s)}\int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}_{A_n} \\
+ \underbrace{\int_{L/2-A/n^{1/2k}}^{L/2+A/n^{1/2k}} \frac{(I_n(L-s))^2}{f^{n-1}(s)}\int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}_{B_n} &+ \underbrace{\int_{L/2+A/n^{1/2k}}^{L} \frac{(I_n(L-s))^2}{f^{n-1}(s)}\int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}_{C_n}
\end{aligned}
\tag{63}
$$

- Let us start by the term A_n in (63), using (57), for A big enough and for all n sufficiently large we have, for all $s \in [0, L/2 - A/n^{1/2k}]$,

$$
\frac{f^{n+1}(s)}{n^2(f'(s))^2}(1-\frac{1}{A^{2k-1}})^2 \leqslant \frac{I_n^2(s)}{f^{n-1}(s)} \leqslant \frac{f^{n+1}(s)}{n^2(f'(s))^2}. \tag{64}
$$

  Let $W_{n+1}(t) = \int_0^t \frac{f^{n+1}(s)}{(f'(s))^2}ds$, after integration by parts, we have:

$$
W_{n+1}(t) = \int_0^t \frac{f^{n+1}(s)f'(s)}{(f'(s))^3}ds
$$

$$= \frac{f^{n+2}(t)}{(n+2)(f'(t))^3} + 3\int_0^t \frac{f^{n+2}(s)f''(s)}{(n+2)(f'(s))^4}ds$$

Since f'' is negative,

$$W_{n+1}(t) \leqslant \frac{f^{n+2}(t)}{(n+2)(f'(t))^3}.$$

It follows that:

$$\int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dt \leqslant \frac{f^{n+2}(s)}{n^3(f'(s))^3}. \tag{65}$$

Hence since $\frac{1}{f'(s)} \sim_{s\to L/2-} \frac{(2k-1)!}{|f^{(2k)}(L/2)|(L/2-s)^{2k-1}}$, we have

$$\begin{aligned} A_n &\leqslant \frac{I_n(L)^2}{n^3}\int_0^{L/2-A/n^{1/2k}} \frac{f^3(s)}{(f'(s))^3}ds \\ &\leqslant \frac{I_n(L)^2 f^3(L/2)}{n^3}\int_0^{L/2-A/n^{1/2k}} \frac{1}{(f'(s))^3}ds \\ &\sim c\frac{I_n(L)^2}{n^{2/k}A^{6k-4}}. \end{aligned}$$

Since by Proposition 15, $\mathbb{E}[\tau_n] \sim \frac{c}{n^{1/k}}$, it follows that for all A big enough,

$$\limsup_{n\to\infty} \frac{A_n}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \leqslant \frac{c}{A^{6k-4}} \tag{66}$$

- for the term B_n in (63): for A big enough, for all n sufficiently large we write

$$\begin{aligned} B_n &= \int_{L/2-A/n^{1/2k}}^{L/2+A/n^{1/2k}} \frac{(I_n(L-s))^2}{f^{n-1}(s)}\int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds \\ &= \underbrace{\int_{L/2-A/n^{1/2k}}^{L/2+A/n^{1/2k}} \frac{(I_n(L-s))^2}{f^{n-1}(s)}\int_0^{L/2-A/n^{1/2k}} \frac{I_n^2(t)}{f^{n-1}(t)}dtds}_{B_n^{(1)}} \\ &+ \underbrace{\int_{L/2-A/n^{1/2k}}^{L/2+A/n^{1/2k}} \frac{(I_n(L-s))^2}{f^{n-1}(s)}\int_{L/2-A/n^{1/2k}}^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}_{B_n^{(2)}}. \end{aligned}$$

  - Let us estimate the term $B_n^{(1)}$. Using (54) and (65) we have,

$$\begin{aligned} \int_0^{L/2-A/n^{1/2k}} \frac{I_n^2(t)}{f^{n-1}(t)}dt &\leqslant \frac{f^{n+2}(L/2-A/n^{1/2k})}{n^3(f'(L/2-A/n^{1/2k}))^3} \\ &\sim c\frac{f^{n+2}(L/2)e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^3(\frac{A^{2k-1}}{n^{(2k-1)/2k}})^3} \\ &\sim c\frac{f^{n+2}(L/2)e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^{3/2k}A^{6k-3}}. \end{aligned}$$

So using (61), we have uniformly in $a \in [-A, A]$

$$I_n(L/2 + a/n^{1/2k}) \sim \frac{f^{n-1}(L/2)}{n^{1/2k}} h_k(a). \tag{67}$$

Since uniformly in $a \in [-A, A]$ we have, recall (54),

$$f^n(L/2 - a/n^{1/2k}) \sim f^n(L/2) e^{\frac{-a^{2k}}{C(f,2k)^{2k}}}, \tag{68}$$

we deduce that

$$\begin{aligned}
B_n^{(1)} &\leqslant c\frac{1}{n^{1/2k}} \int_{-A}^{A} \frac{I_n^2(L/2 - a/n^{1/2k})}{f^{n-1}(L/2 + a/n^{1/2k})} \frac{f^{n+2}(L/2) e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^{3/2k} A^{6k-3}} da \\
&\sim c\frac{f^{n+2}(L/2) e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^{2/k} A^{6k-3}} \int_{-A}^{A} \frac{I_n^2(L/2 - a/n^{1/2k})}{f^{n-1}(L/2 + a/n^{1/2k})} da \\
&\sim c\frac{f^{n+2}(L/2) e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^{2/k} A^{6k-3}} \int_{-A}^{A} \left(\frac{f^{n-1}(L/2) h_k(-a)}{n^{1/2k}}\right)^2 \frac{1}{f^{n-1}(L/2) e^{\frac{-a^{2k}}{C(f,2k)^{2k}}}} da \\
&\sim c\frac{f^{2n+1}(L/2) e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^{3/k} A^{6k-3}} \int_{-A}^{A} (h_k(-a))^2 e^{\frac{a^{2k}}{C(f,2k)^{2k}}} da \\
&\leqslant c\frac{f^{2n+1}(L/2)}{n^{3/k} A^{6k-3}} \int_{-A}^{A} (h_k(-a))^2 da \\
&\leqslant 2c \parallel h_k \parallel_\infty^2 \frac{f^{2n+1}(L/2)}{n^{3/k} A^{6k-4}}.
\end{aligned}$$

Also by (51) and Proposition 15 we have

$$\mathbb{E}[\tau_n]^2 I_n(L)^2 \sim \frac{c_k f^{2n-2}(L/2)}{n^{3/k}}.$$

Note that the constant $c_k = C(2k)^2 C(f,2k)^{3/k}$, where $C(2k)$ and $C(f,2k)$ are defined in (48) and (47). It follows that for all A big enough,

$$\limsup_{n\to\infty} \frac{B_n^{(1)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \leqslant \frac{c}{A^{6k-4}} \tag{69}$$

– Let us estimate the term $B_n^{(2)}$. Using (68) and (67) we have

$$\begin{aligned}
B_n^{(2)} &= \int_{L/2-A/n^{1/2k}}^{L/2+A/n^{1/2k}} \frac{(I_n(L-s))^2}{f^{n-1}(s)} \int_{L/2-A/n^{1/2k}}^{s} \frac{I_n^2(t)}{f^{n-1}(t)} dtds \\
&= \frac{1}{n^{1/2k}} \int_{-A}^{A} \frac{\left(I_n(L/2 - a/n^{1/2k})\right)^2}{f^{n-1}(L/2 + a/n^{1/2k})} \int_{L/2-A/n^{1/2k}}^{L/2+a/n^{1/2k}} \frac{I_n^2(t)}{f^{n-1}(t)} dtda \\
&= \frac{1}{n^{1/k}} \int_{-A}^{A} \frac{\left(I_n(L/2 - a/n^{1/2k})\right)^2}{f^{n-1}(L/2 + a/n^{1/2k})} \int_{-A}^{a} \frac{I_n^2(L/2 + \tilde{a}/n^{1/2k})}{f^{n-1}(L/2 + \tilde{a}/n^{1/2k})} d\tilde{a}da \\
&\sim \frac{1}{n^{1/k}} \int_{-A}^{A} \frac{\left(\frac{f^{n-1}(L/2) h_k(-a)}{n^{1/2k}}\right)^2}{f^{n-1}(L/2) e^{\frac{-a^{2k}}{C(f,2k)^{2k}}}} \int_{-A}^{a} \frac{\left(\frac{f^{n-1}(L/2) h_k(\tilde{a})}{n^{1/2k}}\right)^2}{f^{n-1}(L/2) e^{\frac{-\tilde{a}^{2k}}{C(f,2k)^{2k}}}} d\tilde{a}da
\end{aligned}$$

$$\sim \quad \frac{f^{2n-2}(L/2)}{n^{3/k}} \int_{-A}^{A} h_k^2(-a) e^{\frac{a^{2k}}{C(f,2k)^{2k}}} \int_{-A}^{a} h_k^2(\tilde{a}) e^{\frac{\tilde{a}^{2k}}{C(f,2k)^{2k}}} d\tilde{a} da$$

It follows that for all A big enough

$$\lim_{n\to\infty} \frac{B_n^{(2)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} = \frac{C(A)}{c_k}, \tag{70}$$

where $C(A) = \int_{-A}^{A} h_k^2(-a) e^{\frac{a^{2k}}{C(f,2k)^{2k}}} \int_{-A}^{a} h_k^2(\tilde{a}) e^{\frac{\tilde{a}^{2k}}{C(f,2k)^{2k}}} d\tilde{a} da$

- for the term C_n in (63): for A big enough, and for all n sufficiently large we write

$$\begin{aligned}
C_n &= \int_{L/2+A/n^{1/2k}}^{L} \frac{(I_n(L-s))^2}{f^{n-1}(s)} \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)} dt ds \\
&= \underbrace{\int_{L/2+A/n^{1/2k}}^{L} \frac{(I_n(L-s))^2}{f^{n-1}(s)} \int_0^{L/2-A/n^{1/2k}} \frac{I_n^2(t)}{f^{n-1}(t)} dt ds}_{C_n^{(1)}} \\
&+ \underbrace{\int_{L/2+A/n^{1/2k}}^{L} \frac{(I_n(L-s))^2}{f^{n-1}(s)} \int_{L/2-A/n^{1/2k}}^{L/2+A/n^{1/2k}} \frac{I_n^2(t)}{f^{n-1}(t)} dt ds}_{C_n^{(2)}} \\
&+ \underbrace{\int_{L/2+A/n^{1/2k}}^{L} \frac{(I_n(L-s))^2}{f^{n-1}(s)} \int_{L/2+A/n^{1/2k}}^{s} \frac{I_n^2(t)}{f^{n-1}(t)} dt ds}_{C_n^{(3)}}.
\end{aligned}$$

  – For $C_n^{(1)}$, using (65), we have

$$C_n^{(1)} = \left(\int_0^{L/2-A/n^{1/2k}} \frac{I_n^2(t)}{f^{n-1}(t)} dt \right)^2 \leqslant \left(c \frac{f^{n+2}(L/2) e^{\frac{-A^{2k}}{C(f,2k)^{2k}}}}{n^{3/2k} A^{6k-3}} \right)^2,$$

  and so

$$\limsup_{n\to\infty} \frac{C_n^{(1)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \leqslant \frac{c}{A^{12k-6}} \tag{71}$$

  – For $C_n^{(2)}$, note that after a change of variable it is equal to $B_n^{(1)}$ hence

$$\limsup_{n\to\infty} \frac{C_n^{(2)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \leqslant \frac{c}{A^{6k-4}} \tag{72}$$

  – For $C_n^{(3)}$, since $f'' \leqslant 0$ and by (64) we have

$$\begin{aligned}
C_n^{(3)} &= \int_{L/2+A/n^{1/2k}}^{L} \frac{(I_n(L-s))^2}{f^{n-1}(s)} \int_{L/2+A/n^{1/2k}}^{s} \frac{I_n^2(t)}{f^{n-1}(t)} dt ds \\
&= \int_0^{L/2-A/n^{1/2k}} \frac{(I_n(s))^2}{f^{n-1}(s)} \int_{L/2+A/n^{1/2k}}^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt ds \\
&= \int_0^{L/2-A/n^{1/2k}} \frac{(I_n(s))^2}{f^{n-1}(s)} \int_s^{L/2-A/n^{1/2k}} \frac{I_n^2(L-t)}{f^{n-1}(t)} dt ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant I_n^2(L)\int_0^{L/2-A/n^{1/2k}} \frac{(I_n(s))^2}{f^{n-1}(s)} \int_s^{L/2-A/n^{1/2k}} \frac{f'(t)}{f^{n-1}(t)f'(t)}dtds \\
&\leqslant \frac{I_n^2(L)}{f'(L/2-A/n^{1/2k})}\int_0^{L/2-A/n^{1/2k}} \frac{(I_n(s))^2}{f^{n-1}(s)} \int_s^{L/2-A/n^{1/2k}} \frac{f'(t)}{f^{n-1}(t)}dtds \\
&\leqslant \frac{I_n^2(L)}{(n-2)f'(L/2-A/n^{1/2k})}\int_0^{L/2-A/n^{1/2k}} \frac{(I_n(s))^2}{f^{n-1}(s)} \frac{1}{f^{n-2}(s)}ds \\
&\leqslant \frac{I_n^2(L)}{(n-2)f'(L/2-A/n^{1/2k})}\int_0^{L/2-A/n^{1/2k}} \left(\frac{I_n(s)}{f^{n-1}(s)}\right)^2 f(s)ds \\
&\leqslant \frac{I_n^2(L)}{(n-2)f'(L/2-A/n^{1/2k})}\int_0^{L/2-A/n^{1/2k}} \frac{f^3(s)}{n^2(f'(s))^2}ds \\
&\leqslant \frac{I_n^2(L)f^3(L/2)}{n^2(n-2)f'(L/2-A/n^{1/2k})}\int_0^{L/2-A/n^{1/2k}} \frac{1}{(f'(s))^2}ds \\
&\sim c\frac{I_n^2(L)f^3(L/2)}{n^3(A/n^{1/2k}))^{2k-1}}\frac{n^{2-3/2k}}{A^{4k-3}} \\
&\sim c\frac{I_n^2(L)f^3(L/2)}{n^{2/k}A^{6k-4}}.
\end{aligned}
$$

It follows that

$$
\limsup_{n\to\infty} \frac{C_n^{(3)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \leqslant \frac{c}{A^{6k-4}} \tag{73}
$$

Using equations (66),(69),(70), (71),(72) ,(73) we get, for all A big enough

$$
\begin{aligned}
\frac{2C(A)}{c_k} = \liminf_{n\to\infty} \quad & 2\frac{B_n^{(1)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \\
\leqslant \quad & \liminf_{n\to\infty} \frac{2\int_0^L \frac{1}{f^{n-1}(s)}\left(I_n(L)-I_n(s)\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \\
\leqslant \quad & \limsup_{n\to\infty} \frac{2\int_0^L \frac{1}{f^{n-1}(s)}\left(I_n(L)-I_n(s)\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \\
\leqslant \quad & 2\limsup_{n\to\infty} \frac{A_n + B_n^{(2)} + B_n^{(1)} + C_n^{(1)} + C_n^{(2)} + C_n^{(3)}}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \\
\leqslant \quad & \frac{c}{A^{6k-4}} + \frac{2C(A)}{c_k}
\end{aligned}
$$

Passing to the limit when A goes to infinity, and using dominated convergence theorem , we get

$$
\begin{aligned}
&\lim_{n\to\infty} 2\frac{\int_0^L \frac{1}{f^{n-1}(s)}\left(I_n(L)-I_n(s)\right)^2 \int_0^s \frac{I_n^2(t)}{f^{n-1}(t)}dtds}{\mathbb{E}[\tau_n]^2 I_n(L)^2} \\
&\quad = \frac{2}{c_k}\int_{-\infty}^{\infty} h_k^2(-a)e^{\frac{a^{2k}}{C(f,2k)^{2k}}} \int_{-\infty}^{a} h_k^2(\tilde{a})e^{\frac{\tilde{a}^{2k}}{C(f,2k)^{2k}}} d\tilde{a}da,
\end{aligned}
$$

and

$$\lim_{n\to\infty} \frac{\mathbb{E}[\tau_n^2]}{\mathbb{E}[\tau_n]^2} = 1 + \frac{2}{c_k} \int_{-\infty}^{\infty} h_k^2(-a) e^{\frac{a^{2k}}{C(f,2k)^{2k}}} \int_{-\infty}^{a} h_k^2(\tilde{a}) e^{\frac{\tilde{a}^{2k}}{C(f,2k)^{2k}}} d\tilde{a} da,$$

It follows that

$$\mathrm{Var}(\tau_n) \quad \sim \quad \mathbb{E}[\tau_n]^2 \Big(\frac{2}{c_k} \int_{-\infty}^{\infty} h_k^2(-a) e^{\frac{a^{2k}}{C(f,2k)^{2k}}} \int_{-\infty}^{a} h_k^2(\tilde{a}) e^{\frac{\tilde{a}^{2k}}{C(f,2k)^{2k}}} \Big)$$

and $\mathrm{Var}(\tau_n) \neq o(\mathbb{E}[\tau_n]^2)$.

■

Theorem 18 *Let f be a C^{2k+1} function on $[0, L]$ satisfying Assumptions (1) and (5). Assume that for some $k \geqslant 2$,*

$$f(L/2 + h) \quad = \quad f(L/2) + \frac{f^{(2k)}(L/2)}{(2k)!} h^{2k} + o(h^{2k})$$

where $f^{(2k)}(L/2) < 0$. For any $n \in \mathbb{N} \backslash \{1\}$, let $X_n \coloneqq (X_n(t))_{t \geqslant 0}$ be the Brownian motion described in Definition 1. Then the family of diffusion processes $(X_n)_{n \in \mathbb{N} \backslash \{1\}}$ has no cut-off in separation.

Proof

The proof is by contradiction. Suppose that the family of diffusion processes $(X_n)_{n\in\mathbb{N}\backslash\{1\}}$ has a cut-off in separation with mixing times $(a_n)_{n\in\mathbb{N}\backslash\{1\}}$. Since for any $n \in \mathbb{N}\backslash\{1\}$, τ_n is a sharp strong stationary time for X_n, the following convergence in probability holds for large n,

$$\frac{\tau_n}{a_n} \overset{\mathbb{P}}{\to} 1.$$

Note that by Propositions 5 and 7, we have

$$\| G_n(\mathbb{1}) \|_\infty = u_{n,1}(0) = \mathbb{E}[\tau_n],$$

$$\| G_n \|_{\mathbb{L}^\infty, \mathbb{L}^\infty} = \mathbb{E}[\tau_n],$$

and

$$\mathbb{E}[\tau_n^k] = k! G_n^{\circ k}[\mathbb{1}] \leqslant k! \mathbb{E}[\tau_n]^k.$$

Hence $\frac{\tau_n}{\mathbb{E}[\tau_n]}$ is uniformly integrable. Up to extracting a subsequence, we can suppose that $\frac{\tau_n}{\mathbb{E}[\tau_n]}$ converges in law, says toward a random variable Y.

Using Slutsky Theorem we have that $(\frac{\tau_n}{\mathbb{E}[\tau_n]}, \frac{\tau_n}{a_n})$ converge in law to $(Y, 1)$, and so $\frac{a_n}{\mathbb{E}[\tau_n]}$ converge in law to Y, and there exist $\lambda \in \mathbb{R}^+$ such that

$$\frac{a_n}{\mathbb{E}[\tau_n]} \to \lambda = Y.$$

Hence $\frac{\tau_n}{\mathbb{E}[\tau_n]} \overset{\mathbb{P}}{\to} \lambda$, and since $\frac{\tau_n}{\mathbb{E}[\tau_n]}$ is uniformly integrable, the convergence takes place in $\mathbb{L}^1$, and $\lambda = 1$. Also $\frac{\tau_n^2}{\mathbb{E}[\tau_n]^2} \overset{\mathbb{P}}{\to} 1$, and $\frac{\tau_n^2}{\mathbb{E}[\tau_n]^2}$ is uniformly integrable, so the convergence takes place in $\mathbb{L}^1$, and

$$\mathbb{E}\left[\frac{\tau_n^2}{\mathbb{E}[\tau_n]^2}\right] \to 1,$$

and we get a contradiction with Proposition 17.

In the next propositions, we show that the cut-off phenomenon in separation occurs when $f(x)$ looks like $f(L/2) - C|L/2 - x|^{1+\alpha}$ for x near $L/2$, with $\alpha \in (0, 1)$, but with a different speed in comparison with the case $\alpha = 1$ in Theorem 12.

Proposition 19 *Let f be a C^2 function on $[0,L]\backslash\{L/2\}$ and C^1 on $[0,L]$ satisfying Assumptions* (1) *and* (5)*. Assume that locally around $L/2$ we have for some $\alpha \in (0,1)$ and $C>0$,*

$$f''(L/2-h) \quad = \quad -C|h|^{\alpha-1} + o(|h|^{\alpha-1})$$

then

$$\mathbb{E}[\tau_n] \sim \frac{2}{n}\int_0^{L/2} \frac{f(s)}{f'(s)} ds$$

Proof

We follow the same computations as in the proof of Proposition 10, and we adopt the same notations. For A big enough and n sufficiently large, let

$$\mathbb{E}[\tau_n] = 2\left(\underbrace{\int_0^{L/2-A/n^{1/(1+\alpha)}} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right) ds}_{A_n} \right. \tag{74}$$

$$\left. + \underbrace{\int_{L/2-A/n^{1/(1+\alpha)}}^{L/2} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right) ds}_{B_n} \right) \tag{75}$$

Since $f'(L/2)=0$ we have,

$$f'(L/2-h) = \frac{C}{\alpha}\text{sign}(h)|h|^{\alpha} + o(|h|^{\alpha}),$$

$$f(L/2-h) = f(L/2) - \frac{C}{\alpha(\alpha+1)}|h|^{1+\alpha} + o(|h|^{1+\alpha}).$$

Let $C_\alpha = \frac{C}{\alpha(\alpha+1)}$, since $\ln(f(L/2-h)) = \ln(f(L/2)) - \frac{C_\alpha}{f(L/2)}|h|^{1+\alpha} + o(|h|^{1+\alpha})$, using Laplace's method we get:

$$I_n(L/2) \quad = \quad \int_0^{\frac{L}{2}} f^{n-1}(t)dt \sim_{n\to\infty} \frac{f^{n-1}(L/2)}{n^{1/(1+\alpha)}} C(\alpha,f), \tag{76}$$

where $C(\alpha,f) = (\frac{f(L/2)}{C_\alpha})^{1/(1+\alpha)}\frac{1}{1+\alpha}\Gamma(\frac{1}{1+\alpha})$.

- For A big enough and for all n sufficiently large, and for $s \in [0, L/2 - A/n^{1/(1+\alpha)}]$, let us compute an equivalent of A_n in (74).

  Let $m_n = \inf_{[0,L/2-A/n^{1/(1+\alpha)}]} f''$ by hypothesis on f, we have for large n,

  $$m_n \sim -C\frac{A^{\alpha-1}}{n^{(\alpha-1)/(1+\alpha)}}.$$

  Since

  $$\frac{m_n f(s)}{(n+1)(f'(s))^2} \geqslant \frac{m_n f(L/2)}{(n+1)(f'(L/2-A/n^{1/(1+\alpha)}))^2} \sim_{n\to\infty} -\frac{c_1}{A^{1+\alpha}},$$

  we get that for A big enough and n sufficiently large

  $$\frac{m_n f(s)}{(n+1)(f'(s))^2} \geqslant -\frac{1}{A^{(1+\alpha)/2}}.$$

Also for large n,

$$\frac{f^n(s)}{nf'(s)I_n(L)} \leqslant \frac{f^n(L/2 - A/n^{1/(1+\alpha)})}{nf'(L/2 - A/n^{1/(1+\alpha)})I_n(L)} \sim c_1 \frac{e^{-\frac{C_\alpha}{f(L/2)}A^{1+\alpha}}}{A^\alpha},$$

so for A big enough and n sufficiently large

$$\frac{f^n(s)}{nf'(s)I_n(L)} \leqslant \frac{1}{A^{\alpha/2}}.$$

Using (24) and (53), it follows that for A big enough and n sufficiently large, uniformly in $s \in [0, L/2 - A/n^{1/(1+\alpha)}]$,

$$\frac{f^n(s)}{nf'(s)}\left(1 - \frac{1}{A^{(1+\alpha)/2}}\right) \leqslant I_n(s) \leqslant \frac{f^n(s)}{nf'(s)}, \tag{77}$$

and so

$$\frac{f(s)}{nf'(s)}\left(1 - \frac{1}{A^{\alpha/2}}\right)\left(1 - \frac{1}{A^{(1+\alpha)/2}}\right) \leqslant J_n(s) \leqslant \frac{f(s)}{nf'(s)}. \tag{78}$$

where J_n is defined in (16).

Since $\frac{f(s)}{f'(s)} \sim_{s\to(L/2)_-} \frac{\alpha}{C}\frac{f(L/2)}{|L/2-s|^\alpha}$, is integrable at $L/2$ we get that for all A big enough:

$$\left(1 - \frac{1}{A^{\alpha/2}}\right)\left(1 - \frac{1}{A^{(1+\alpha)/2}}\right) \leqslant \liminf_{n\to\infty} \frac{A_n}{\frac{1}{n}\int_0^{L/2}\frac{f(s)}{f'(s)}ds} \leqslant \limsup_{n\to\infty} \frac{A_n}{\frac{1}{n}\int_0^{L/2}\frac{f(s)}{f'(s)}ds} \leqslant 1 \tag{79}$$

- For A big enough and for all n sufficiently large, and for $s \in [L/2 - A/n^{1/(1+\alpha)}, L/2]$, let us compute the equivalent of B_n in (75).
  More generally when $s \in [\frac{L}{2} - \frac{A}{n^{1/(1+\alpha)}}, \frac{L}{2} + \frac{A}{n^{1/(1+\alpha)}}]$, write $s = L/2 + a/n^{1/(1+\alpha)}$, with $a \in [-A, A]$. We have uniformly in $a \in [-A, A]$:

$$\begin{aligned}
I_n(L/2 + a/n^{1/(1+\alpha)}) &= I_n(L/2) + \int_{L/2}^{L/2+a/n^{1/(1+\alpha)}} f^{n-1}(x)dx \\
&= I_n(L/2) + \frac{1}{n^{1/(1+\alpha)}}\int_0^a f^{n-1}(L/2 + \frac{h}{n^{1/(1+\alpha)}})dh \\
&= I_n(L/2) + \frac{1}{n^{1/(1+\alpha)}}\int_0^a \left(f\left(\frac{L}{2}\right) - \frac{C_\alpha |h|^{1+\alpha}}{n} + o\left(\frac{|h|^{1+\alpha}}{n}\right)\right)^{n-1} dh \\
&= I_n(L/2) + \frac{f(L/2)^{n-1}}{n^{1/(1+\alpha)}}\int_0^a e^{(n-1)\ln\left(1 - \frac{C_\alpha|h|^{1+\alpha}}{f(L/2)n} + o\left(\frac{|h|^{1+\alpha}}{n}\right)\right)} dh \\
&\sim \frac{f^{n-1}(L/2)}{n^{1/(1+\alpha)}}\left(C(\alpha, f) + \int_0^a e^{-\frac{C_\alpha|h|^{1+\alpha}}{f(L/2)}}dh\right) \\
&= \frac{f^{n-1}(L/2)}{n^{1/(1+\alpha)}}\int_{-\infty}^a e^{-\frac{C_\alpha|h|^{1+\alpha}}{f(L/2)}}dh.
\end{aligned} \tag{80}$$

Concerning the justification of the equivalent in the above computation, note the integral term $\int_0^a e^{(n-1)\ln(1-\frac{C_\alpha|h|^{1+\alpha}}{f(L/2)n} + o(|h|^{1+\alpha}/n))}dh$ converges for fixed a to $\int_0^a e^{-\frac{C_\alpha h^{1+\alpha}}{f(L/2)}}dh$, by the dominated convergence theorem (since the integrand is bounded by 1), finally by Dini's theorem this convergence is uniform in $a \in [-A, A]$. The last equality follows by a change of variable formula that shows that $C(\alpha, f) = \int_{-\infty}^0 e^{-\frac{C_\alpha|h|^{1+\alpha}}{f(L/2)}}dh$.

Define $h_\alpha(a) = \int_{-\infty}^a e^{-\frac{C_\alpha |h|^{1+\alpha}}{f(L/2)}} dh$, we get that for A big enough, and for B_n defined in (75)

$$\begin{aligned} B_n &:= \int_{L/2-A/n^{1/(1+\alpha)}}^{L/2} \frac{I_n(s)}{f^{n-1}(s)} \left(\frac{I_n(L)-I_n(s)}{I_n(L)}\right) ds \\ &\sim \frac{1}{n^{3/(1+\alpha)} I_n(L)} \int_{-A}^{0} \frac{h_\alpha(a)h_\alpha(-a) f^{2n-2}(L/2)}{f^{n-1}(L/2)} e^{\frac{C_\alpha |a|^{1+\alpha}}{f(L/2)}} da \\ &\sim \frac{c}{n^{2/(1+\alpha)}} = o\left(\frac{1}{n}\right), \end{aligned} \tag{81}$$

where we took (76) into account. Hence using (79) we have for all A large enough

$$\left(1-\frac{1}{A^{\alpha/2}}\right)\left(1-\frac{1}{A^{(1+\alpha)/2}}\right) \leqslant \liminf_{n\to\infty} \frac{\mathbb{E}[\tau_n]}{\frac{2}{n}\int_0^{L/2} \frac{f(s)}{f'(s)} ds} \leqslant \limsup_{n\to\infty} \frac{\mathbb{E}[\tau_n]}{\frac{2}{n}\int_0^{L/2} \frac{f(s)}{f'(s)} ds} \leqslant 1,$$

and so leting A tends to infinity we get

$$\mathbb{E}[\tau_n] \sim \frac{2}{n}\int_0^{L/2} \frac{f(s)}{f'(s)} ds.$$

■

Proposition 20 *Let f be a C^2 function on $[0,L]\backslash\{L/2\}$ and C^1 on $[0,L]$ satisfying Assumptions* (1) *and* (5). *Assume that for some $\alpha \in (0,1)$ and $C > 0$, we have for all $|h| > 0$ small enough,*

$$f''(L/2-h) = -C|h|^{\alpha-1} + o(|h|^{\alpha-1})$$

then

$$\mathrm{Var}(\tau_n) \quad = \quad o\left(\frac{1}{n^2}\right).$$

Proof

From Proposition 9 and after integration by parts, we have that for all A large enough and for all n large enough:

$$\frac{\mathrm{Var}(\tau_n)}{2} = \underbrace{\int_0^{L/2-A/n^{1/(1+\alpha)}} J_n(s)(u'_{n,1}(s))^2 ds}_{A_n} + \underbrace{\int_{L/2-A/n^{1/(1+\alpha)}}^{L/2+A/n^{1/(1+\alpha)}} J_n(s)(u'_{n,1}(s))^2 ds}_{B_n} \tag{82}$$

$$+ \underbrace{\int_{L/2+A/n^{1/(1+\alpha)}}^{L} J_n(s)(u'_{n,1}(s))^2 ds,}_{C_n} \tag{83}$$

where J_n is defined in (16) and $(u'_{n,1})^2$ in (37).

- Let us start by estimating the term A_n, using (77), it follows that for A big enough and for all n sufficiently large, and for all $s \in [0, L/2 - A/n^{1/(1+\alpha)}]$,

$$\frac{f^{n+1}(s)}{n^2(f'(s))^2}\left(1-\frac{1}{A^{(1+\alpha)/2}}\right)^2 \leqslant \frac{I_n^2(s)}{f^{n-1}(s)} \leqslant \frac{f^{n+1}(s)}{n^2(f'(s))^2}. \tag{84}$$

Since $f'' \leqslant 0$ in $[0, L/2[$, (39) holds for all $0 \leqslant t < L/2$, hence for A big enough and for all n sufficiently large, and for all $t \in [0, L/2 - A/n^{1/(1+\alpha)}]$:

$$(u'_{n,1}(t))^2 \leqslant \left(\frac{f(t)}{nf'(t)(1 - \frac{1}{A^{(1+\alpha)/2}})^2} \right)^2.$$

Also by (38),

$$J_n(s) := \frac{I_n(s)}{f^{n-1}(s)} \frac{I_n(L-s)}{I_n(L)} \leqslant \frac{f(s)}{nf'(s)}.$$

Hence for A big enough, for all n sufficiently large, and for A_n defined in (82), we have

$$\begin{aligned} A_n &\leqslant \frac{1}{n^3(1 - \frac{1}{A^{(1+\alpha)/2}})^4} \int_0^{L/2 - A/n^{1/(1+\alpha)}} \frac{f^3(s)}{(f'(s))^3} ds \\ &\leqslant \frac{f(L/2)^3}{n^3(1 - \frac{1}{A^{(1+\alpha)/2}})^4} \int_0^{L/2 - A/n^{1/(1+\alpha)}} \frac{1}{(f'(s))^3} ds \end{aligned}$$

Taking into account that $\frac{1}{f'(s)} \sim_{s \to L/2-} \frac{c}{|(L/2-s)|^\alpha}$, we get

$$\int_0^{L/2 - A/n^{1/(1+\alpha)}} \frac{1}{(f'(s))^3} ds \quad \sim \quad \begin{cases} c & \text{, if } 3\alpha < 1 \\ c \ln(n^{1/(1+\alpha)}/A) & \text{, if } 3\alpha = 1 \\ c\frac{n^{(3\alpha-1)/(1+\alpha)}}{A^{(3\alpha-1)}} & \text{, if } 3\alpha > 1. \end{cases}$$

Hence for A big enough, we get

$$A_n \quad \sim \quad \begin{cases} c/n^3 & \text{, if } 3\alpha < 1 \\ c\ln(n)/n^3 & \text{, if } 3\alpha = 1 \\ c/n^{\frac{4}{1+\alpha}} & \text{, if } 3\alpha > 1, \end{cases}$$

in particular, since $\alpha \in (0, 1)$,

$$A_n \quad = \quad o\left(\frac{1}{n^2}\right). \tag{85}$$

- For the term B_n in (82): for A big enough, for all n large enough and for $a \in [-A, A]$, let $x = L/2 + a/n^{1/(1+\alpha)}$ we have

$$\begin{aligned} &\left| u'_{n,1}\left(\frac{L}{2} + \frac{a}{n^{1/(1+\alpha)}}\right) \right| \\ &= \frac{f^{n-1}\left(\frac{L}{2} + \frac{a}{n^{1/(1+\alpha)}}\right)}{I_n^2\left(\frac{L}{2} + \frac{a}{n^{1/(1+\alpha)}}\right)} \left(\int_0^{L/2 - A/n^{1/(1+\alpha)}} \frac{I_n^2(s)}{f^{n-1}(s)} ds + \int_{L/2 - A/n^{1/(1+\alpha)}}^{L/2 + \frac{a}{n^{1/(1+\alpha)}}} \frac{I_n^2(s)}{f^{n-1}(s)} ds \right). \end{aligned}$$

By the above computation and (39), we have

$$\int_0^{L/2 - A/n^{1/(1+\alpha)}} \frac{I_n^2(s)}{f^{n-1}(s)} ds \leqslant \frac{f^{n+2}(L/2 - \frac{A}{n^{1/(1+\alpha)}})}{n^3(f'(L/2 - \frac{A}{n^{1/(1+\alpha)}}))^3} \sim c\frac{f^{n+2}(L/2)e^{-\frac{C_\alpha}{f(L/2)}A^{(1+\alpha)}}}{n^{3/(1+\alpha)}A^{3\alpha}}.$$

Recall that from (80), and for $h_\alpha(a) = \int_{-\infty}^a e^{-\frac{C_\alpha |h|^{1+\alpha}}{f(L/2)}} dh$, we have uniformly over $a \in [-A, A]$

$$I_n(L/2 + a/n^{1/(1+\alpha)}) \sim f^{n-1}(L/2)\frac{h_\alpha(a)}{n^{1/(1+\alpha)}}, \tag{86}$$

hence uniformly in $a \in [-A, A]$,

$$\begin{aligned}\int_{L/2-A/n^{1/(1+\alpha)}}^{L/2+\frac{a}{n^{1/(1+\alpha)}}} \frac{I_n^2(s)}{f^{n-1}(s)} ds &= \frac{1}{n^{1/(1+\alpha)}} \int_{-A}^{a} \frac{I_n^2(L/2+\frac{\tilde{a}}{n^{1/(1+\alpha)}})}{f^{n-1}(L/2+\frac{\tilde{a}}{n^{1/(1+\alpha)}})} d\tilde{a} \\ &\sim \frac{f^{n-1}(L/2)}{n^{3/(1+\alpha)}} \int_{-A}^{a} h_\alpha^2(\tilde{a}) e^{\frac{C_\alpha}{f(L/2)}|\tilde{a}|^{(1+\alpha)}} d\tilde{a} \\ &= \frac{f^{n-1}(L/2)\theta_{\alpha,A}(a)}{n^{3/(1+\alpha)}},\end{aligned}$$

where $\theta_{\alpha,A}(a) := \int_{-A}^{a} h_\alpha^2(\tilde{a}) e^{\frac{C_\alpha}{f(L/2)}|\tilde{a}|^{(1+\alpha)}} d\tilde{a}$.
Since uniformly in $a \in [-A, A]$,

$$f^n(L/2 - a/n^{1/(1+\alpha)}) \sim f^n(L/2) e^{-\frac{C_\alpha}{f(L/2)}|a|^{(1+\alpha)}},$$

we have for A big enough

$$\left| u'_{n,1}\left(\frac{L}{2} + \frac{a}{n^{1/(1+\alpha)}}\right)\right| \leqslant c \frac{e^{-\frac{C_\alpha}{f(L/2)}|a|^{(1+\alpha)}}}{n^{1/(1+\alpha)} h_\alpha^2(a)} \left(\theta_{\alpha,A}(a) + \frac{1}{A^{3\alpha}}\right).$$

Hence

$$\begin{aligned}B_n &:= \int_{L/2-A/n^{1/(1+\alpha)}}^{L/2+A/n^{1/(1+\alpha)}} J_n(s)(u'_{n,1}(s))^2 ds \\ &= \frac{1}{n^{1/(1+\alpha)}} \int_{-A}^{A} J_n(L/2 + a/n^{1/(1+\alpha)})(u'_{n,1}(L/2 + a/n^{1/(1+\alpha)}))^2 da \\ &\leqslant \frac{1}{n^{1/(1+\alpha)}} \int_{-A}^{A} J_n(L/2 + a/n^{1/(1+\alpha)}) \left(c \frac{e^{-\frac{C_\alpha}{f(L/2)}|a|^{(1+\alpha)}}}{n^{1/(1+\alpha)} h_\alpha^2(a)} \left(\theta_{\alpha,A}(a) + \frac{1}{A^{3\alpha}}\right)\right)^2 da \\ &\sim \frac{c(A)}{n^{4/(1+\alpha)}},\end{aligned}$$

where we use in the third line that

$$\begin{aligned}J_n(L/2 + a/n^{1/(1+\alpha)}) &= \frac{I_n(L/2 + a/n^{1/(1+\alpha)})}{f^{n-1}(L/2 + a/n^{1/(1+\alpha)})} \frac{I_n(L/2 - a/n^{1/(1+\alpha)})}{I_n(L)} \\ &\sim c \frac{h_\alpha(a) h_\alpha(-a) e^{\frac{C_\alpha}{f(L/2)}|a|^{(1+\alpha)}}}{n^{1/(1+\alpha)}},\end{aligned}$$

and $c(A)$ is a constant that depends on A. It follows that for all A large enough,

$$B_n = o\left(\frac{1}{n^2}\right). \tag{87}$$

- For the last term C_n in (83), note that $J_n(s) = J_n(L - s)$ so

$$C_n = \int_{L/2+A/n^{1/(1+\alpha)}}^{L} J_n(s)(u'_{n,1}(s))^2 ds$$

$$= \int_0^{L/2-A/n^{1/(1+\alpha)}} J_n(s)(u'_{n,1}(L-s))^2 ds.$$

Also for $s \leqslant L/2 - A/n^{1/(1+\alpha)}$

$$\begin{aligned}
|u'_{n,1}(L-s)| &= \frac{f^{n-1}(s)}{I_n^2(L-s)} \int_0^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt \\
&= \frac{f^{n-1}(s)}{I_n^2(L-s)} \left(\int_0^{L/2-A/n^{1/(1+\alpha)}} \frac{I_n^2(t)}{f^{n-1}(t)} dt \right. \\
&\quad \left. + \int_{L/2-A/n^{1/(1+\alpha)}}^{L/2+A/n^{1/(1+\alpha)}} \frac{I_n^2(t)}{f^{n-1}(t)} dt + \int_{L/2+A/n^{1/(1+\alpha)}}^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt \right).
\end{aligned}$$

The first two terms in the above bracket has been computed in the above item, for the last term since for $s \leqslant L/2 - A/n^{1/(1+\alpha)}$,

$$\begin{aligned}
\int_{L/2+A/n^{1/(1+\alpha)}}^{L-s} \frac{I_n^2(t)}{f^{n-1}(t)} dt &\leqslant I_n^2(L-s) \int_{L/2+A/n^{1/(1+\alpha)}}^{L-s} \frac{1}{f^{n-1}(t)} dt \\
&= I_n^2(L-s) \int_s^{L/2-A/n^{1/(1+\alpha)}} \frac{1}{f^{n-1}(t)} dt \\
&= I_n^2(L-s) \int_s^{L/2-A/n^{1/(1+\alpha)}} \frac{f'(t)}{f^{n-1}(t) f'(t)} dt \\
&\leqslant \frac{I_n^2(L-s)}{f'(L/2 - A/n^{1/(1+\alpha)})} \left(\frac{f^{-n+2}(s)}{n-2} \right) \\
&\sim c \frac{f^{-n+2}(s) I_n^2(L-s)}{n^{1/(1+\alpha)} A^\alpha}
\end{aligned}$$

Since $L - s \geqslant L/2$, we have for A big enough and for all n sufficiently large,

$$\begin{aligned}
|u'_{n,1}(L-s)| &\leqslant \frac{f^{2n-2}(L/2)}{n^{3/(1+\alpha)} I_n^2(L/2)} \left(\frac{1}{A^{3\alpha}} + \theta_A(A) \right) + c \frac{f(s)}{n^{1/(1+\alpha)} A^\alpha} \\
&\leqslant \frac{c}{n^{1/(1+\alpha)}} \left(\frac{1}{A^{3\alpha}} + \theta_A(A) \right) + c \frac{f(s)}{n^{1/(1+\alpha)} A^\alpha} \leqslant \frac{c(A)}{n^{1/(1+\alpha)}}
\end{aligned}$$

where in the second inequality, we used (76). Since, for $s \leqslant L/2 - A/n^{1/(1+\alpha)}$, (77) yield $J_n(s) \leqslant \frac{f(s)}{nf'(s)}$, so for A big enough and for all n sufficiently large,

$$\begin{aligned}
C_n &= \int_0^{L/2-A/n^{1/(1+\alpha)}} J_n(s)(u'_{n,1}(L-s))^2 ds. \\
&\leqslant \frac{c^2(A)}{n^{2/(1+\alpha)}} \int_0^{L/2-A/n^{1/(1+\alpha)}} J_n(s) ds \\
&\leqslant \frac{c^2(A) f(L/2)}{n^{1+2/(1+\alpha)}} \int_0^{L/2-A/n^{1/(1+\alpha)}} \frac{1}{f'(s)} ds \\
&\sim c \frac{c^2(A) f(L/2)}{n^{1+2/(1+\alpha)}} = o(\frac{1}{n^2}),
\end{aligned}$$

where for the equivalent we use that $f'(s) \sim_{(L/2)^-} c|L/2-s|^\alpha$ and the integral is convergent. The last equality follows since $\alpha \in (0,1)$. Hence

$$C_n = o\left(\frac{1}{n^2}\right). \tag{88}$$

Putting (85), (87) and (88) together, we deduce that $\mathrm{Var}(\tau_n) = o\left(\frac{1}{n^2}\right)$.

■

Theorem 21 *Let f be a C^2 function on $[0,L]\backslash\{L/2\}$ and C^1 on $[0,L]$ satisfying Assumptions (1) and (5). Assume that for some $\alpha \in (0,1)$ and $C > 0$, we have for all $|h| > 0$ small enough,*

$$f''(L/2-h) = -C|h|^{\alpha-1} + o(|h|^{\alpha-1}).$$

Let $X_n \coloneqq (X_n(t))_{t\geqslant 0}$ be the Brownian motion described in Definition 1. Then the family of diffusion processes $(X_n)_{n\in\mathbb{N}\backslash\{1\}}$ has a cut-off in separation with mixing times $(a_n)_{n\in\mathbb{N}\backslash\{1\}} = \left(\frac{2}{n}\int_0^{L/2}\frac{f(s)}{f'(s)}ds\right)_{n\in\mathbb{N}\backslash\{1\}}$, in the sense of Section 1.3.

Proof

Use Theorem 8, Proposition 19 and Proposition 20.

■

Corollary 22 *With same hypothesis as in Theorem 21, there exist $\tilde{C} > 0$ and $n_0 \in \mathbb{N}$ such that for all $r > 0$, $0 < r' < 1$ and for all $n \geqslant n_0$,*

$$\begin{aligned}
\left\|\mathcal{L}\left(X_n((1+r)\frac{2}{n}\int_0^{L/2}\frac{f(s)}{f'(s)}ds)\right) - \mathcal{U}_n\right\|_{\mathrm{tv}} &\leqslant \frac{\tilde{C}}{r^2 n^{\frac{1-\alpha}{1+\alpha}}} \\
\forall\ y \in M_f^n, \qquad P^{(n)}_{(1+r)\frac{2}{n}\int_0^{L/2}\frac{f(s)}{f'(s)}ds}(\tilde{0}, y) &\geqslant \left(1 - \frac{\tilde{C}}{r^2 n^{\frac{1-\alpha}{1+\alpha}}}\right)\frac{1}{\mathrm{Vol}(M_f^n)} \\
\inf_{y\in M_f^n} P^{(n)}_{(1-r')\frac{2}{n}\int_0^{L/2}\frac{f(s)}{f'(s)}ds}(\tilde{0}, y) &\leqslant \left(\frac{\tilde{C}}{r'^2 n^{\frac{1-\alpha}{1+\alpha}}}\right)\frac{1}{\mathrm{Vol}(M_f^n)}
\end{aligned}$$

Proof

In the proof of Proposition 20 we have in fact (since the dominant term is C_n)

$$\mathrm{Var}(\tau_n) = O\left(\frac{1}{n^{1+\frac{2}{1+\alpha}}}\right).$$

The result follows with the same proof as the proof of Corollary 14.

Theorem 23 *Let f be a C^2 function on $[0,L]\backslash\{L/2\}$ and C^1 on $[0,L]$ satisfying Assumptions (1) and (5). Assume that for some $\alpha > 1$ and $C > 0$, we have for all $|h| > 0$ small enough,*

$$f''(L/2-h) = -C|h|^{\alpha-1} + o(|h|^{\alpha-1}).$$

Let $X_n \coloneqq (X_n(t))_{t\geqslant 0}$ be the Brownian motion described in Definition 1. Then the family of diffusion processes $(X_n)_{n\in\mathbb{N}\backslash\{1\}}$ has no cut-off in separation.

Proof

Following the proof of Proposition 19, we show that

$$\mathbb{E}[\tau_n] \quad \sim \quad \frac{C_\alpha(f)}{n^{2/(2+\alpha)}}.$$

Following the proof of Proposition 17, we show that $\mathrm{Var}(\tau_n)/2$ is equivalent for n large to $\frac{B_n^{(2)}}{I_n(L)^2}$ with the same decompositions as introduced there. It follows that $\mathrm{Var}(\tau_n)/\mathbb{E}[\tau_n]^2$ converges toward a positive constant and we conclude as in Theorem 18.

■

To end the paper, let us give the

Proof of Theorem 4

The items of Theorem 4 correspond respectively to Theorem 21, Theorem 12 and Theorem 23.

■

References

[1] Aldous, David and Diaconis, Persi, Shuffling cards and stopping times, American Mathematical Monthly, 93, 333–348, 1986.

[2] Marc Arnaudon, Koléhè Coulibaly-Pasquier and Laurent Miclo. Construction of set-valued dual processes on manifolds. J. Éc. Polytech., Math., 11, 473–522, 2024.

[3] Marc Arnaudon, Koléhè Coulibaly-Pasquier and Laurent Miclo. On the separation cut-off phenomenon for Brownian motions on high dimensional spheres. Bernoulli, 30, 1007–1028, 2024.

[4] Koléhè Coulibaly-Pasquier and Laurent Miclo. On the evolution by duality of domains on manifolds. *Mém. Soc. Math. Fr., Nouv. Sér.*, 171:1–110, 2021.

[5] Persi Diaconis. The cutoff phenomenon in finite Markov chains. *Proc. Nat. Acad. Sci. U.S.A.*, 93(4):1659–1664, 1996.

[6] Diaconis, Persi and Shahshahani, Mehrdad, Generating a random permutation with random transpositions, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 57, 59–179, 1981.

[7] Persi Diaconis and James Allen Fill. Strong stationary times via a new form of duality. *Ann. Probab.*, 18(4):1483–1522, 1990.

[8] Persi Diaconis and Laurent Saloff-Coste. Separation cut-offs for birth and death chains. *Ann. Appl. Probab.*, 16(4):2098–2122, 2006.

[9] Jian Ding, Eyal Lubetzky and Yuval Peres. Total variation cutoff in birth-and-death chains. *Probab. Theory Relat. Fields*, 146(1-2):61–85, 2010.

[10] Jonathan Hermon, Hubert Lacoin and Yuval Peres. Total variation and separation cutoffs are not equivalent and neither one implies the other. *Electron. J. Probab.*, 21:36, 2016. Id/No 44.

[11] David A. Levin, Yuval Peres and Elizabeth L. Wilmer. *Markov chains and mixing times.* American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[12] Peter Petersen. *Riemannian geometry, 2nd ed..* Graduate Texts in Mathematics, 171, 2006.

[13] Pierre-Loïc Méliot. The cut-off phenomenon for Brownian motions on compact symmetric spaces. *Potential Anal.*, 40(4):427–509, 2014.

[14] L. Saloff-Coste. Precise estimates on the rate at which certain diffusions tend to equilibrium. *Math. Z.*, 217(4):641–677, 1994.

[15] Justin Salez. Cutoff for non-negatively curved Markov chains, J. Eur. Math. Soc. (2023), published online first.

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