On Quantum Electrodynamics of atomic resonances

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Outline of the talk

Part I

The model

The atom (1)

Assumptions

- The atom is non-relativistic
- The atom is assumed to have only finitely many excited states

Internal degrees of freedom

- Internal degrees of freedom described by an N-level system
- Hilbert space: $\mathbb{C}^N$
- Hamiltonian: $N \times N$ matrix given by

$$H_{is} := \begin{pmatrix} E_N & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & E_1 \end{pmatrix}, \quad E_N > \cdots > E_1$$

- The energy scale of transitions between internal states of the atom is measured by the quantity

$$\delta_0 := \min_{i \neq j} |E_i - E_j|$$

The atom (2)

External degrees of freedom

- Usual Hilbert space of orbital wave functions: $L^2(\mathbb{R}^3)$
- Position of the (center of mass of the) atom: $\vec{x} \in \mathbb{R}^3$
- Kinetic energy of the free center of mass motion: $-\frac{1}{2}\Delta$

Atomic Hamiltonian

- Hilbert space

$$\mathcal{H}_{at} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^N$$

- Hamiltonian:

$$H_{at} := -\frac{1}{2}\Delta + H_{is},$$

with domain $D(H_{at}) = H^2(\mathbb{R}^3) \otimes \mathbb{C}^N$

Electric dipole moment

Represented by

$$\vec{d} = (d_1, d_2, d_3),$$

where, for $j = 1, 2, 3$, $d_j \equiv \mathbb{I} \otimes d_j$ is an $N \times N$ hermitian matrix

The quantized electromagnetic field (1)

Fock space

- Wave vector of a photon: $\vec{k} \in \mathbb{R}^3$
- Helicity of a photon: $\lambda \in \{1, 2\}$
- Notation:

$$\underline{\mathbb{R}}^3 := \mathbb{R}^3 \times \{1,2\} = \{\underline{k} := (\vec{k}, \lambda) \mid \vec{k} \in \mathbb{R}^3, \lambda \in \{1,2\}\}$$

Moreover, $\underline{\mathbb{R}}^{3n} := (\underline{\mathbb{R}}^3)^{\times n}$, and, for $B \subset \mathbb{R}^3$,

$$\underline{B} := B \times \{1,2\}, \qquad \int_{\underline{B}} d\underline{k} := \sum_{\lambda=1,2} \int_B d\vec{k}$$

- Hilbert space of states of photons given by

$$\mathcal{H}_f := \mathcal{F}_+(L^2(\underline{\mathbb{R}}^3)),$$

where $\mathcal{F}_+(L^2(\underline{\mathbb{R}}^3))$ is the symmetric Fock space over the space $L^2(\underline{\mathbb{R}}^3)$ of one-photon states:

$$\mathcal{H}_f = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathrm{L}^2_s(\underline{\mathbb{R}}^{3n})$$

The quantized electromagnetic field (2)

Photon creation- and annihilation operators

Denoted by

$$a^*(\underline{k}) \equiv a^*_\lambda(\vec{k}), \quad a(\underline{k}) \equiv a_\lambda(\vec{k}), \quad \text{for all } \underline{k} = (\vec{k}, \lambda) \in \underline{\mathbb{R}}^3$$

Fock vacuum

Fock space $\mathcal{H}_f$ contains a unit vector, Ω, called "vacuum (vector)" and unique up to a phase, with the property that

$$a(\underline{k})\Omega = 0, \quad \text{for all } \underline{k}$$

Hamiltonian

Hamiltonian of the free electromagnetic field given by

$$H_f = \int_{\underline{\mathbb{R}}^3} |\vec{k}| a^*(\underline{k}) a(\underline{k}) d\underline{k}$$

Total physical system (1)

Hilbert space

Total Hilbert space:

$$\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{H}_f$$

Interaction of the atom with the quantized electromagnetic field

Interaction Hamiltonian:

$$H_I := -\vec{d} \cdot \vec{E}(\vec{x}),$$

where $\vec{E}$ denotes the quantized electric field:

$$\vec{E}(\vec{x}) := -i \int_{\underline{\mathbb{R}}^3} \Lambda(\vec{k}) |\vec{k}|^{\frac{1}{2}} \vec{\epsilon}(\underline{k}) \left(e^{i\vec{k}\cdot\vec{x}} \otimes a(\underline{k}) - e^{-i\vec{k}\cdot\vec{x}} \otimes a^*(\underline{k}) \right) d\underline{k}$$

- $\underline{k} \mapsto \vec{\epsilon}(\underline{k}) \in \mathbb{R}^3$ represents the polarization vector:

$$|\vec{\epsilon}(\underline{k})| = 1, \quad \vec{\epsilon}(\underline{k}) \cdot \vec{k} = 0, \quad \vec{\epsilon}((r\vec{k}, \lambda)) = \vec{\epsilon}((\vec{k}, \lambda)), \ \forall r > 0, \ \forall \underline{k} \in \underline{\mathbb{R}}^3$$

- $\Lambda : \mathbb{R}^3 \mapsto \mathbb{R}$ is an ultraviolet cut-off:

$$\Lambda(\vec{k}) = e^{-|\vec{k}|^2/(2\sigma_\Lambda^2)}, \quad \sigma_\Lambda \geq 1$$

Total physical system (2)

Total Hamiltonian

Total Hamiltonian of the system:

$$\mathbf{H} := H_{at} + H_f + \lambda_0 H_I, \qquad \lambda_0 \in \mathbb{R}$$

Translation invariance

- Photon momentum operator:

$$\vec{P}_f := \int_{\underline{\mathbb{R}}^3} \vec{k} a^*(\underline{k}) a(\underline{k}) d\underline{k}$$

- Total momentum operator:

$$\vec{P}_{tot} := -i\vec{\nabla} + \vec{P}_f$$

-

$$[\mathbf{H}, \vec{P}_{tot,j}] = 0, \quad j = 1, 2, 3$$

The fibre Hamiltonian

Direct integrals

- Isomorphism

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^N \otimes \mathcal{H}_f \cong L^2(\mathbb{R}^3; \mathbb{C}^N \otimes \mathcal{H}_f)$$

- Direct integral decomposition

$$\mathcal{H} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_{\vec{p}} d\vec{p}, \quad H = \int_{\mathbb{R}^3}^{\oplus} H(\vec{p}) d\vec{p},$$

where the fibre space is

$$\mathcal{H}_{\vec{p}} := \mathbb{C}^N \otimes \mathcal{H}_f,$$

and the fibre Hamiltonian is

$$H(\vec{p}) := H_{is} + \frac{1}{2}(\vec{p} - \vec{P}_f)^2 + H_f + \lambda_0 H_{I,0},$$

where

$$H_{I,0} := i \int_{\underline{\mathbb{R}}^3} \Lambda(\vec{k})|\vec{k}|^{\frac{1}{2}} \left(\vec{\epsilon}(\underline{k}) \cdot \vec{d} \otimes a(\underline{k}) - \vec{\epsilon}(\underline{k}) \cdot \vec{d} \otimes a^*(\underline{k}) \right) d\underline{k}$$

Spectrum of $H_0(P)$

Simplification

Subtracting the trivial term $\vec{p}^2/2$, we obtain the Hamiltonian

$$H(\vec{p}) := H_{is} + \frac{1}{2}\vec{P}_f^2 - \vec{p}\cdot\vec{P}_f + H_f + \lambda_0 H_{I,0}$$

Non-interacting Hamiltonian

$$H_0(\vec{p}) := H_{is} + \frac{1}{2}\vec{P}_f^2 - \vec{p}\cdot\vec{P}_f + H_f$$

Spectrum

- $$\sigma(H_0(\vec{p})) = \begin{cases} [E_1, \infty) & \text{if } |\vec{p}| \leq 1, \\ [E_1 + |\vec{p}| - \frac{1}{2} - \frac{\vec{p}^2}{2}, \infty) & \text{if } |\vec{p}| \geq 1. \end{cases}$$
- Pure point spectrum

$$\sigma_{\rm pp}(H_0(\vec{p})) = \{E_1, E_2, \dots E_N\} \text{ for all } \vec{p} \in \mathbb{R}^3$$

Part II

Results

Complex dilatations in Fock space

Dilatation operator in the 1-photon space

(Unitary) dilatation operator: for $\theta \in \mathbb{R}$,

$$\gamma_\theta(\phi)(\vec{k}, \lambda) := e^{-3\theta/2} \phi(e^{-\theta} \vec{k}, \lambda), \quad \text{for } \phi \in L^2(\mathbb{R}^3)$$

Second quantization

Second quantization of γ_θ: $\Gamma_\theta := \Gamma(\gamma_\theta)$ operator on $\mathcal{H}_f$ defined by:

$$\Gamma_\theta(\Phi)(\underline{k}_1, \dots, \underline{k}_n) := e^{-3n\theta/2} \Phi(e^{-\theta} \vec{k}_1, \lambda_1, \dots, e^{-\theta} \vec{k}_n, \lambda_n)$$

Dilated Hamiltonian

$$H_\theta(\vec{p}) := \Gamma_\theta H(\vec{p}) \Gamma_\theta^* = H_{is} + \frac{1}{2} e^{-2\theta} \vec{P}_f^2 - e^{-\theta} \vec{p} \cdot \vec{P}_f + e^{-\theta} H_f + \lambda_0 H_{I,\theta},$$

where

$$H_{I,\theta} := i e^{-2\theta} \int_{\underline{\mathbb{R}}^3} \Lambda(e^{-\theta} \vec{k}) |\vec{k}|^{\frac{1}{2}} \left(\vec{\epsilon}(\underline{k}) \cdot \vec{d} \otimes a(\underline{k}) - \vec{\epsilon}(\underline{k}) \cdot \vec{d} \otimes a^*(\underline{k}) \right) d\underline{k}.$$

Analytically extended to $D(0, \pi/4) := \{\theta \in \mathbb{C} \, : \, |\theta| < \pi/4\}$.

Spectrum of the non-interacting dilated Hamiltonian

Non-interacting dilated Hamiltonian

$$H_{\theta,0}(\vec{p}) := H_{is} + e^{-2\theta}\frac{\vec{P}_f^2}{2} - e^{-\theta}\vec{p}\cdot\vec{P}_f + e^{-\theta}H_f$$

Spectrum

For $\delta_0 > 0$, $E_1, \dots, E_N$ are simple eigenvalues of $H_{\theta,0}(\vec{p})$. For $|\vec{p}| < 1$ and $\theta = i\vartheta$, $\vartheta \in \mathbb{R}$, the spectrum of $H_{\theta,0}(\vec{p})$ is included in a region of the following form:

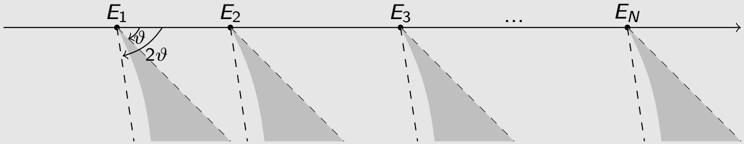

Figure: Shape of the spectrum of $H_{\theta,0}(\vec{p})$ for $\vec{p} \in \mathbb{R}^3$, $|\vec{p}| < 1$.

Main results

Theorem (Ballesteros, F, Fröhlich, Schubnel)

Let $0 < \nu < 1$. There exists $\lambda_c(\nu) > 0$ such that, for all $|\lambda_0| < \lambda_c(\nu)$ and $\vec{p} \in \mathbb{R}^3$, $|\vec{p}| < \nu$, the following properties are satisfied:

a) $E(\vec{p}) := \inf \sigma(H(\vec{p}))$ is a non-degenerate eigenvalue of $H(\vec{p})$,

b) For all $i_0 \in \{1, \cdots, N\}$ and $\theta \in \mathbb{C}$ with $0 < \mathrm{Im}(\theta) < \pi/4$ large enough, $H_\theta(\vec{p})$ has an eigenvalue, $z^{(\infty)}(\vec{p})$, such that $z^{(\infty)}(\vec{p}) \to E_{i_0}$ as $\lambda_0 \to 0$. For $i_0 = 1$, $z^{(\infty)}(\vec{p}) = E(\vec{p})$.

Moreover, for $|\vec{p}| < \nu$, $|\lambda_0|$ small enough and $0 < \mathrm{Im}(\theta) < \pi/4$ large enough, the ground state energy, $E(\vec{p})$, its associated eigenprojection, $\pi(\vec{p})$, and resonances energies, $z^{(\infty)}(\vec{p})$, are analytic in $\vec{p}$, λ_0 and θ. In particular, they are independent of θ

Renormalized mass

Renormalized mass

- Rotation symmetry: $E(\vec{p}) = E(|\vec{p}|)$
- The renormalized mass of the atom can be defined by

$$m_{\mathrm{ren}} = \frac{1}{(\partial^2_{|\vec{p}|} E)(0) + 1} \quad \text{where} \quad \partial_{|\vec{p}|} = \frac{\vec{p}}{|\vec{p}|} \cdot \nabla_{\vec{p}}$$

Cerenkov radiation

Conjecture

- For $|\vec{p}| > 1$, $E(\vec{p})$ is not an eigenvalue
- Preliminary results: [De Roeck,Fröhlich,Pizzo '13]
- In what follows, we always assume that $|\vec{p}| < 1$

Ground states of related (translation invariant) models

Free electron

- Nelson model
  - [Fröhlich '73], [Pizzo '03]: $E(\vec{p})$ is not an eigenvalue (unless an infrared regularization is imposed)
  - [Abdesselam,Hasler '13]: $E(\vec{p})$ analytic in $\vec{p}$ and λ_0
- Pauli-Fierz model
  - [Chen,Fröhlich '07], [Chen '08], [Hasler,Herbst '08] [Chen,Fröhlich,Pizzo '09]

    $$E(\vec{p}) \text{ is an eigenvalue} \Leftrightarrow \nabla E(\vec{p}) = 0 \Leftrightarrow \vec{p} = \vec{0}.$$

    For $\vec{p} \neq \vec{0}$, a ground state exists in a "non-Fock representation"
  - [Bach,Chen,Fröhlich,Sigal '07], [Chen '08], [Chen,Fröhlich,Pizzo '09], [Fröhlich,Pizzo '10]: $\vec{p} \mapsto E(\vec{p})$ is twice differentiable near 0

Atoms and ions

[Amour,Grébert,Guillot '06], [Loss,Miyao,Spohn '07], [Fröhlich,Griesemer,Schlein '07], [Hasler,Herbst '08]: (for Pauli-Fierz models)

$$E(\vec{p}) \text{ is an eigenvalue} \Leftrightarrow \ (\text{Total charge vanishes}) \text{ or } (\vec{p} = \vec{0})$$

Analyticity in the coupling constant

Models with static nuclei

[Griesemer,Hasler '09], [Hasler,Herbst '11]: For different models related to non-relativistic QED, analyticity in the coupling constant, proven using spectral renormalization group

Resonances

Models with static nuclei

[Bach,Fröhlich,Sigal '98], [Abou Salem,F,Fröhlich,Sigal '09], [Sigal '09], [Bach,Ballesteros,Fröhlich '13]: For different models related to non-relativistic QED, existence of resonances, proven using spectral renormalization group or iterative perturbation theory

Moving Hydrogen atom (but center of mass confined)

[F '08] Existence of resonances proven using spectral renormalization group

Main results (2)

Theorem (Ballesteros, F, Fröhlich, Schubnel)

Let $0 < \nu < 1$. There exists $\lambda_c(\nu) > 0$ such that, for all $|\lambda_0| < \lambda_c(\nu)$ and $\vec{p} \in \mathbb{R}^3$, $|\vec{p}| < \nu$, the following properties are satisfied:

a) $E(\vec{p}) := \inf \sigma(H(\vec{p}))$ is a non-degenerate eigenvalue of $H(\vec{p})$,

b) For all $i_0 \in \{1, \cdots, N\}$ and $\theta \in \mathbb{C}$ with $0 < \mathrm{Im}(\theta) < \pi/4$ large enough, $H_\theta(\vec{p})$ has an eigenvalue, $z^{(\infty)}(\vec{p})$, such that $z^{(\infty)}(\vec{p}) \to E_{i_0}$ as $\lambda_0 \to 0$. For $i_0 = 1$, $z^{(\infty)}(\vec{p}) = E(\vec{p})$.

Moreover, for $|\vec{p}| < \nu$, $|\lambda_0|$ small enough and $0 < \mathrm{Im}(\theta) < \pi/4$ large enough, the ground state energy, $E(\vec{p})$, its associated eigenprojection, $\pi(\vec{p})$, and resonances energies, $z^{(\infty)}(\vec{p})$, are analytic in $\vec{p}$, λ_0 and θ. In particular, they are independent of θ

Main contributions

- Existence of resonances for translation invariant models
- Analyticity of resonances energies in $\vec{p}$ and λ_0
- Proof: Inductive construction ("replacing" the spectral renormalization group analysis and) involving a sequence of 'smooth Feshbach-Schur maps', which yields an algorithm for the calculation of the resonances energies that converges super-exponentially fast

Fermi Golden Rule

Proposition (Ballesteros, F, Fröhlich, Schubnel)

Let $i_0 > 1$ and $\vec{p} \in \mathbb{R}^3$, $|\vec{p}| < 1$. Suppose that

$$\sum_{j<i_0} \int_{\underline{\mathbb{R}}^3} \Big| \sum_{s\in\{1,2,3\}} (d_s)_{N-j+1,N-i_0+1} \epsilon_s(\underline{k}) \Big|^2 |\vec{k}| |\Lambda(\vec{k})|^2$$

$$\delta\big(E_j - E_{i_0} + |\vec{k}| - \vec{p}\cdot\vec{k} + \frac{\vec{k}^2}{2}\big) d\underline{k} > 0,$$

Then, under the conditions of our main theorem and for $|\lambda_0|$ small enough, the imaginary part of $z^{(\infty)}(\vec{p})$ is strictly negative

Part III

Ingredients of the proof

Feshbach-Schur map (1)

Definition (Feshbach-Schur Pairs)

Let P be an operator on a separable Hilbert space $\mathcal{V}$, $0 \leq P \leq 1$. Assume that P and $\overline{P} := \sqrt{1 - P^2}$ are both non-zero. Let H and T be two closed operators on $\mathcal{V}$ with identical domains. Assume that P and $\overline{P}$ commute with T. We set $W := H - T$ and assume that $\overline{P}WP$ and $PW\overline{P}$ are bounded operators. We define

$$H_P := T + PWP, \quad H_{\overline{P}} := T + \overline{P}W\overline{P}.$$

The pair (H, T) is called a Feshbach-Schur pair associated with P iff

(i) $H_{\overline{P}}$ and T are bounded invertible on $\overline{P}[\mathcal{V}]$

(ii) $H_{\overline{P}}^{-1}\overline{P}WP$ can be extended to a bounded operator on $\mathcal{V}$

For an arbitrary Feshbach-Schur pair (H, T) associated with P, we define the (smooth) Feshbach-Schur map by

$$F_P(\cdot, T) : H \mapsto F_P(H, T) := T + PWP - PW\overline{P}H_{\overline{P}}^{-1}\overline{P}WP$$

Feshbach-Schur map (2)

Theorem ([Bach,Chen,Fröhlich,Sigal '03], [Griesemer,Hasler '08])

Let $0 \le P \le 1$, and let (H, T) be a Feshbach-Schur pair associated with P (i.e., satisfying properties (i) and (ii) of the previous definition). Define

$$Q_P(H,T) := P - \overline{P} H_{\overline{P}}^{-1} \overline{P} W P.$$

Then the following hold true:

(i) H is bounded invertible on $\mathcal{V}$ if and only if $F_P(H,T)$ is bounded invertible on $P[\mathcal{V}]$.

(ii) H is not injective if and only if $F_P(H,T)$ is not injective as an operator on $P[\mathcal{V}]$:

$$H\psi = 0,\ \psi \neq 0 \Longrightarrow F_P(H,T)P\psi = 0,\ P\psi \neq 0,$$

$$F_P(H,T)\phi = 0,\ \phi \neq 0 \Longrightarrow HQ_P(H,T)\phi = 0,\ Q_P(H,T)\phi \neq 0.$$

Wick monomials (1)

Kernels

We denote by

$$\underline{w} := \{w_{m,n}\}_{m,n\in\mathbb{N}_0}$$

a sequence of bounded measurable functions,

$$\forall m,n \ : \ \ w_{m,n} : \mathbb{R}\times\mathbb{R}^3\times\underline{\mathbb{R}}^{3m}\times\underline{\mathbb{R}}^{3n}\to\mathbb{C},$$

that are continuously differentiable in the variables, $r\in\sigma(H_f)\subset\mathbb{R}$, $\vec{l}\in\sigma(\vec{P}_f)=\mathbb{R}^3$, respectively, appearing in the first and the second argument, and symmetric in the m variables in $\underline{\mathbb{R}}^{3m}$ and the n variables in $\underline{\mathbb{R}}^{3n}$. We suppose furthermore that

$$w_{0,0}(0,\vec{0})=0$$

Wick monomials (2)

Generalized Wick monomials

With a sequence, $\underline{w}$, of functions, we associate a bounded operator

$$W_{m,n}(\underline{w}) := \mathbb{1}_{H_f \le 1} \int_{\mathbb{R}^{3m} \times \mathbb{R}^{3n}} a^*(\underline{k}_1) \cdots a^*(\underline{k}_m) \, w_{m,n}(H_f; \vec{P}_f; \underline{k}_1, \cdots, \underline{k}_m; \underline{\tilde{k}}_1, \cdots, \underline{\tilde{k}}_n) \, a(\underline{\tilde{k}}_1) \cdots a(\underline{\tilde{k}}_n) \prod_{i=1}^{m} d\underline{k}_i \prod_{j=1}^{n} d\underline{\tilde{k}}_j \mathbb{1}_{H_f \le 1}$$

Effective Hamiltonians

For every sequence of functions $\underline{w}$ and every $\mathcal{E} \in \mathbb{C}$ we define

$$H[\underline{w}, \mathcal{E}] = \sum_{m+n \ge 0} W_{m,n}(\underline{w}) + \mathcal{E}, \quad W_{\ge 1}(\underline{w}) := \sum_{m+n \ge 1} W_{m,n}(\underline{w})$$

Analyticity in the total momentum

Complexification of the total momentum

Let $\vec{p}^* \in \mathbb{R}^3$, $|\vec{p}^*| < 1$ and $\theta = i\vartheta$, $0 < \vartheta < \pi/4$. We set

$$\mu = \frac{1 - |\vec{p}^*|}{2}$$

and

$$U_\theta[\vec{p}^*] := \{\vec{p} \in \mathbb{C}^3 \mid |\vec{p} - \vec{p}^*| < \mu\} \cap \{\vec{p} \in \mathbb{C}^3 \mid |\mathrm{Im}(\vec{p})| < \frac{\mu}{2} \tan(\vartheta)\}.$$

For $\vec{p} \in U_\theta[\vec{p}^*]$, we consider the operator

$$H_\theta(\vec{p}) := H_{is} + e^{-2\theta} \frac{\vec{P}_f^2}{2} - e^{-\theta} \vec{p} \cdot \vec{P}_f + e^{-\theta} H_f + \lambda_0 H_{I,\theta}$$

The First Decimation Step of Spectral Renormalization (1)

The first spectral "projection"

- Let ψ_{i_0} denote a normalized eigenvector of H_{is} associated to the eigenvalue E_{i_0} and

$$P_{i_0} := |\psi_{i_0}\rangle\langle\psi_{i_0}|$$

- Let $\chi \in C^\infty(\mathbb{R})$ a decreasing function satisfying

$$\chi(r) := \begin{cases} 1, & \text{if } r \leq 3/4, \\ 0 & \text{if } r > 1, \end{cases}$$

and strictly decreasing on $(3/4, 1)$. For $\rho_0 \in (0,1)$, let

$$\chi_{\rho_0}(r) := \chi(r/\rho_0), \quad \overline{\chi}_{\rho_0}(r) := \sqrt{1 - \chi^2_{\rho_0}(r)}$$

- Operator $\boldsymbol{\chi}_{i_0}$ is defined by

$$\boldsymbol{\chi}_{i_0} := P_{i_0} \otimes \chi_{\rho_0}(H_f)$$

The First Decimation Step of Spectral Renormalization (2)

The first Feshbach-Schur map

- For $|z - E_{i_0}| \leq r_0 \ll \rho_0 \mu \sin(\vartheta)$, $(H_\theta(\vec{p}) - z, H_{\theta,0}(\vec{p}) - z)$ is a Feshbach-Schur pair associated to χ_{i_0}

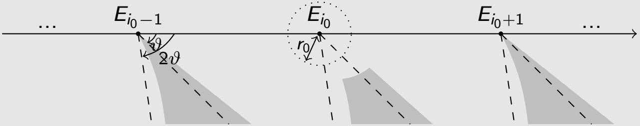

Figure: Spectrum of $H_{\theta,0}(\vec{p})$ restricted to the range of $\bar{\chi}_{i_0} = \sqrt{1 - \chi_{i_0}^2}$. The spectral parameter z is located inside $D(E_{i_0}, r_0)$

- Expanding the resolvent into a Neumann series, and using Wick ordering, one verifies that there is a sequence of functions $\underline{w}^{(0)}(\vec{p}, z)$ and $\mathcal{E}^{(0)}(\vec{p}, z) \in \mathbb{C}$ such that

$$F_{\chi_{i_0}}(H_\theta(\vec{p}) - z, H_{\theta,0}(\vec{p}) - z)_{|\mathrm{Ran}(\chi_{i_0})} = \left(P_{i_0} \otimes H[\underline{w}^{(0)}(\vec{p}, z), \mathcal{E}^{(0)}(\vec{p}, z)]\right)_{|\mathrm{Ran}(\chi_{i_0})}$$

Inductive Construction of Effective Hamiltonians (1)

Scale parameters

Let $(\rho_j)_{j\in\mathbb{N}_0}$, $(r_j)_{j\in\mathbb{N}_0}$ be defined by

$$\rho_j = \rho_0^{(2-\varepsilon)^j}, \text{ with } \varepsilon \in (0,1), \quad r_j := \frac{\mu \sin(\vartheta)}{32}\rho_j$$

Hilbert spaces

A filtration of Hilbert spaces $(\mathcal{H}^{(j)})_{j\in\mathbb{N}_0}$ is given by setting

$$\mathcal{H}^{(j)} = \mathbb{1}_{H_f \le \rho_j}[\mathcal{H}_f]$$

Inductive Construction of Effective Hamiltonians (2)

Effective Hamiltonians

We construct inductively a sequence of complex numbers $\{z^{(j-1)}(\vec{p})\}_{j\in\mathbb{N}_0}$, $z^{(-1)}(\vec{p}) := E_{i_0}$, and, for every $z \in D(z^{(j-1)}(\vec{p}), r_j)$, a sequence of functions $\underline{w}^{(j)}(\vec{p}, z)$ and a complex number $\mathcal{E}^{(j)}(\vec{p}, z)$:

(a) Let

$$W^{(j)}_{m,n}(\vec{p}, z) := W_{m,n}(\underline{w}^{(j)}(\vec{p}, z)), \quad H^{(j)}(\vec{p}, z) := H[\underline{w}^{(j)}(\vec{p}, z), \mathcal{E}^{(j)}(\vec{p}, z)],$$

acting on $\mathcal{H}^{(j)}$, (with $m, n \in \mathbb{N}_0$). Then

$$H^{(j+1)}(\vec{p}, z) = F_{\chi_{\rho_{j+1}}(H_f)}[H^{(j)}(\vec{p}, z), W^{(j)}_{0,0}(\vec{p}, z) + \mathcal{E}^{(j)}(\vec{p}, z)]|\mathbb{1}_{H_f \leq \rho_{j+1}}$$

is well defined.

(b) The complex number $z^{(j)}(\vec{p})$ is defined as the only zero of the function

$$D\left(z^{(j-1)}(\vec{p}), \frac{2}{3} r_j\right) \ni z \longrightarrow \mathcal{E}^{(j)}(\vec{p}, z) = \langle \Omega |\ H^{(j)}(\vec{p}, z)\Omega\rangle$$

Inductive Construction of Effective Hamiltonians (3)

Isospectrality properties

Using isospectrality of the Feshbach-Schur map, we have the following properties:

$$H_\theta(\vec{p}) - z \text{ is bounded invertible} \Longleftrightarrow H^{(j)}(\vec{p}, z) \text{ is bounded invertible.}$$

$$H_\theta(\vec{p}) - z \text{ is not injective} \Longleftrightarrow H^{(j)}(\vec{p}, z) \text{ is not injective.}$$

Inductive Construction of Effective Hamiltonians (4)

Estimates

- The following inequality holds:

$$|z^{(j)}(\vec{p}) - z^{(j-1)}(\vec{p})| < \frac{r_j}{2}$$

- $H^{(j)}(\vec{p}, z)$ is the sum of the unperturbed Hamiltonian, $T = W^{(j)}_{0,0}(\vec{p}, z) + \mathcal{E}^{(j)}(\vec{p}, z)$, and a perturbation given by $W = W^{(j)}_{\geq 1}(\vec{p}, z)$ whose norm tends to zero, as j tends to ∞, super-exponentially rapidly,

$$\|W^{(j)}_{\geq 1}(\vec{p}, z)\| \leq \mathbf{C}^j \rho_j^2,$$

for some constant $\mathbf{C}$

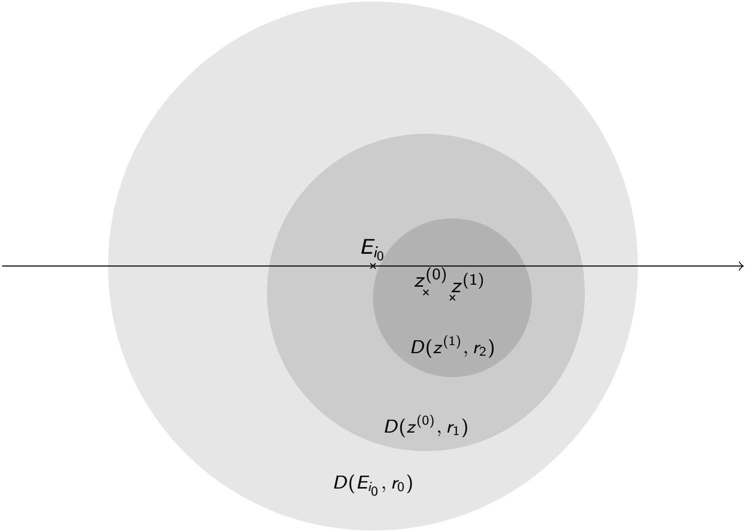

Figure: The sets $D(z^{(j)}(\vec{p}), r_{j+1})$ are shrinking super-exponentially fast with j and, for every $j \in \mathbb{N}_0$, $D(z^{(j)}(\vec{p}), r_{j+1}) \subset D(z^{(j-1)}(\vec{p}), r_j)$.

Construction of Eigenvalues and Analyticity in $\vec{p}$

Approximate resonances energies

- The sequence of approximate resonance energies $(z^{(j)}(\vec{p}))_{j\in\mathbb{N}_0}$ is a Cauchy sequence of analytic functions of $\vec{p}$. We then define

$$z^{(\infty)}(\vec{p}) := \lim_{j\to\infty} z^{(j)}(\vec{p}) = \bigcap_{j\in\mathbb{N}_0} D\big(z^{(j-1)}(\vec{p}), r_j\big),$$

  which is analytic in $\vec{p}$

- Analyticity in θ, for $\mathrm{Im}(\theta) < \frac{\pi}{4}$ large enough, and in λ_0, for $|\lambda_0|$ small enough, can be shown by very similar arguments.

Isospectrality

Using isospectrality of the Feshbach-Schur map, one verifies that $z^{(\infty)}(\vec{p})$ is an eigenvalue of $H_\theta(\vec{p})$; it is the resonance energy that we are looking for

Thank you!