

---

# RANDOM WEIGHTED SOBOLEV INEQUALITIES ON $\mathbb{R}^d$ AND APPLICATION TO HERMITE FUNCTIONS

by

Aurélien Poiret, Didier Robert & Laurent Thomann

---

**Abstract.** — We extend a randomisation method, introduced by Shiffman-Zelditch and developed by Burq-Lebeau on compact manifolds for the Laplace operator, to the case of  $\mathbb{R}^d$  with the harmonic oscillator. We construct measures, thanks to probability laws which satisfy the concentration of measure property, on the support of which we prove optimal weighted Sobolev estimates on  $\mathbb{R}^d$ . This construction relies on accurate estimates on the spectral function in a non-compact configuration space. As an application, we show that there exists a basis of Hermite functions with good decay properties in  $L^\infty(\mathbb{R}^d)$ , when  $d \geq 2$ .

## 1. Introduction and results

**1.1. Introduction.** — During the last years, several papers have shown that some basic results concerning P.D.E. and Sobolev spaces can be strikingly improved using randomization techniques. In particular Burq-Lebeau developed in [2] a randomisation method based on the Laplace operator on a compact Riemannian manifold, and showed that almost surely, a function enjoys better Sobolev estimates than expected, using ideas of Shiffman-Zelditch [18]. This approach depends heavily on spectral properties of the operator one considers. In this paper we are interested in estimates in Sobolev spaces based on the harmonic oscillator in  $L^2(\mathbb{R}^d)$

$$H = -\Delta + |x|^2 = \sum_{j=1}^d (-\partial_j^2 + x_j^2).$$

We get optimal stochastic weighted Sobolev estimates on  $\mathbb{R}^d$  using the Burq-Lebeau method. Indeed we show that there is a unified setting for these results, including the case of compact manifolds. We also make the following extension: In [2], the construction of the measures relied on Gaussian random variables, while in our work we consider general random variable which satisfy concentration

---

**2000 Mathematics Subject Classification.** — 35R60 ; 35P05 ; 35J10 ; 33D45.

**Key words and phrases.** — Harmonic oscillator, spectral analysis, concentration of measure, Hermite functions.

D. R. was partly supported by the grant “NOSEVOL” ANR-2011-BS01019 01.

L.T. was partly supported by the grant “HANDDY” ANR-10-JCJC 0109.

of measure estimates (including discrete random variables, see Section 2). However, we obtain the optimal estimates only in the case of the Gaussians.

We will see that the extension from a compact manifold to an operator on  $\mathbb{R}^d$  with discrete spectrum is not trivial because of the complex behaviour of the spectral function on a non-compact configuration space.

In our forthcoming paper [15], we will give some applications to the well-posedness of nonlinear Schrödinger equations with Sobolev regularity below the optimal deterministic index.

Most of the results stated here can be extended to more general Schrödinger Hamiltonians  $-\Delta + V(x)$  with confining potentials  $V$ . This will be detailed in [17].

Let  $d \geq 2$ . We want to define probability measures on finite dimensional subspaces  $\mathcal{E}_h \subset L^2(\mathbb{R}^d)$ , based on spectral projections with respect to  $H$ . We denote by  $\{\varphi_j, j \geq 1\}$  an orthonormal basis of  $L^2(\mathbb{R}^d)$  of eigenvectors of  $H$  (the Hermite functions), and we denote by  $\{\lambda_j, j \geq 1\}$  the non decreasing sequence of eigenvalues (each is repeated according to its multiplicity):  $H\varphi_j = \lambda_j\varphi_j$ .

For  $h > 0$ , we define the interval  $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$  and we assume that  $a_h$  and  $b_h$  satisfy, for some  $a, b, D > 0, \delta \in [0, 1]$ ,

$$(1.1) \quad \lim_{h \rightarrow 0} a_h = a, \quad \lim_{h \rightarrow 0} b_h = b, \quad 0 < a \leq b \quad \text{and} \quad b_h - a_h \geq Dh^\delta,$$

with any  $D > 0$  if  $\delta < 1$  and  $D \geq 2$  in the case  $\delta = 1$ . This condition ensures that  $N_h$ , the number (with multiplicities) of eigenvalues of  $H$  in  $I_h$  tends to infinity when  $h \rightarrow 0$ . Indeed, we can check that  $N_h \sim ch^{-d}(b_h - a_h)$ , in particular  $\lim_{h \rightarrow 0} N_h = +\infty$ , since  $d \geq 2$ . In the sequel, we write  $\Lambda_h = \{j \geq 1, \lambda_j \in I_h\}$  and  $\mathcal{E}_h = \text{span}\{\varphi_j, j \in \Lambda_h\}$ , so that  $N_h = \#\Lambda_h = \dim \mathcal{E}_h$ . Finally, we denote by  $\mathbf{S}_h = \{u \in \mathcal{E}_h : \|u\|_{L^2(\mathbb{R}^d)} = 1\}$  the unit sphere of  $\mathcal{E}_h$ .

In the sequel, we will consider sequences  $(\gamma_n)_{n \in \mathbb{N}}$  so that there exists  $K_0 > 0$

$$(1.2) \quad |\gamma_n|^2 \leq \frac{K_0}{N_h} \sum_{j \in \Lambda_h} |\gamma_j|^2, \quad \forall n \in \Lambda_h, \quad \forall h \in ]0, 1].$$

This condition means that on each level of energy  $\lambda_n, n \in \Lambda_h$ , one coefficient  $|\gamma_k|$  cannot be much larger than the others. Sometimes, in order to prove lower bound estimates, we will need the stronger condition ( $K_1 > 0$ )

$$(1.3) \quad \frac{K_1}{N_h} \sum_{j \in \Lambda_h} |\gamma_j|^2 \leq |\gamma_n|^2 \leq \frac{K_0}{N_h} \sum_{j \in \Lambda_h} |\gamma_j|^2, \quad \forall n \in \Lambda_h, \quad \forall h \in ]0, 1].$$

This so-called ‘‘squeezing’’ condition means that on each level of energy  $\lambda_n, n \in \Lambda_h$ , the coefficients  $|\gamma_k|$  have almost the same size. For instance (1.2) or (1.3) hold if there exists  $(d_h)_{h \in ]0, 1]}$  so that  $\gamma_n = d_h$  for all  $n \in \Lambda_h$ .

Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\{X_n, n \geq 1\}$  be independent standard complex Gaussians  $\mathcal{N}_{\mathbb{C}}(0, 1)$ . In fact, in our work we will consider more general probability laws, which satisfy concentration of measure estimates (see Assumption 1), but for sake of clarity, we first state the results in this particular case. If  $(\gamma_n)_{n \in \mathbb{N}}$  satisfies (1.2), we define the random vector in  $\mathcal{E}_h$

$$v_\gamma(\omega) := v_{\gamma,h}(\omega) = \sum_{j \in \Lambda_h} \gamma_j X_j(\omega) \varphi_j.$$

We define a probability measure  $\mathbf{P}_{\gamma,h}$  on  $\mathbf{S}_h$  by: for all measurable and bounded function  $f : \mathbf{S}_h \rightarrow \mathbb{R}$

$$\int_{\mathbf{S}_h} f(u) d\mathbf{P}_{\gamma,h}(u) = \int_{\Omega} f \left( \frac{v_\gamma(\omega)}{\|v_\gamma(\omega)\|_{L^2(\mathbb{R}^d)}} \right) d\mathbb{P}(\omega).$$

We can check that in the isotropic case ( $\gamma_j = \frac{1}{\sqrt{N_h}}$  for all  $j \in \Lambda_h$ ),  $\mathbf{P}_{\gamma,h}$  is the uniform probability on  $\mathbf{S}_h$  (see Appendix C).

Finally, let us recall the definition of harmonic Sobolev spaces for  $s \geq 0, p \geq 1$ .

$$(1.4) \quad \begin{aligned} \mathcal{W}^{s,p} &= \mathcal{W}^{s,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d), H^{s/2}u \in L^p(\mathbb{R}^d)\}, \\ \mathcal{H}^s &= \mathcal{H}^s(\mathbb{R}^d) = \mathcal{W}^{s,2}. \end{aligned}$$

The natural norms are denoted by  $\|u\|_{\mathcal{W}^{s,p}}$  and up to equivalence of norms we have (see [23, Lemma 2.4]) for  $1 < p < +\infty$

$$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x \rangle^s u\|_{L^p}.$$

## 1.2. Main results of the paper. —

**1.2.1. Estimates for frequency localised functions.** — Our first result gives properties of the elements on the support of  $\mathbf{P}_{\gamma,h}$ , which are high frequency localised functions. Namely

**Theorem 1.1.** — *Let  $d \geq 2$ . Assume that  $0 \leq \delta < 2/3$  in (1.1) and that condition (1.3) holds. Then there exist  $0 < C_0 < C_1, c_1 > 0$  and  $h_0 > 0$  such that for all  $h \in ]0, h_0]$ .*

$$\mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : C_0 |\log h|^{1/2} \leq \|u\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \leq C_1 |\log h|^{1/2} \right] \geq 1 - h^{c_1}.$$

Moreover the estimate from above is satisfied for any  $\delta \leq 1$  with  $D$  large enough.

It is clear that under condition (1.3), there exist  $0 < C_2 < C_3$ , so that for all  $u \in \mathbf{S}_h$ , and  $s \geq 0$

$$C_2 h^{-s/2} \leq \|u\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq C_3 h^{-s/2},$$

since all elements of  $\mathbf{S}_h$  oscillate with frequency  $h^{-1/2}$ . Thus Theorem 1.1 shows a gain of  $d/2$  derivatives in  $L^\infty$ , and this induces a gain of  $d$  derivatives compared to the usual deterministic Sobolev embeddings. This can be compared with the results of [2] where the authors obtain a gain of  $d/2$  derivatives on compact manifolds: this comes from different behaviours of the spectral function, see Section 3. Notice that the bounds in Theorem 1.1 (and in the results of [2] as well) do not depend on the length of the interval of the frequency localisation  $I_h$  (see (1.1)), but only on the size of the frequencies. This is a consequence of the randomisation, and from the bound (3.15).

We will see in Theorem 4.1 that the upper bound in Theorem 1.1 holds for any  $0 \leq \delta \leq 1$  and for more general random variables  $X$  which satisfy the concentration of measure property. However, to prove the lower bound (see Corollary 4.8), we have to restrict to the case of Gaussians: in the general case, under Assumption 1, we do not reach the factor  $|\ln h|^{1/2}$ . Following the approach of [18, 2], we first prove estimates of  $\|u\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)}$  with large  $r$  and uniform constants (see Theorem 4.12), and which are essentially optimal for general random variables (see Theorem 4.13).

The condition  $\delta < 2/3$  is needed to prove the lower bound, thanks to a reasonable functional calculus based on the harmonic oscillator (see Appendix B).

Finally we point out that in a very recent paper [6], Feng and Zelditch prove similar estimates for the mean and median for the  $L^\infty$ -norm of random holomorphic fields.

**1.2.2. Global Sobolev estimates.** — Using a dyadic Littlewood-Paley decomposition, we now give general estimates in Sobolev spaces; we refer to Subsection 4.1 for more details. For  $s \in \mathbb{R}$  and  $p, q \in [1, +\infty]$ , we define the harmonic Besov space by

$$(1.5) \quad \mathcal{B}_{p,q}^s(\mathbb{R}^d) = \left\{ u = \sum_{n \geq 0} u_n : \sum_{n \geq 0} 2^{nqs/2} \|u_n\|_{L^p(\mathbb{R}^d)}^q < +\infty \right\},$$

where the  $u_n$  have frequencies of size  $\sim 2^n$ . The space  $\mathcal{B}_{p,q}^s(\mathbb{R}^d)$  is a Banach space with the norm in  $\ell^q(\mathbb{N})$  of  $\{2^{ns/2} \|u_n\|_{L^p(\mathbb{R}^d)}\}_{n \geq 0}$ .

We assume that  $\gamma$  satisfies (1.2) and

$$\sum_{n \geq 0} |\gamma|_{\Lambda_n} < +\infty \quad \text{where} \quad |\gamma|_{\Lambda_n}^2 := \sum_{k: \lambda_k \in [2^n, 2^{n+1}[} |\gamma_k|^2.$$

Then we set

$$v_\gamma(\omega) = \sum_{j=0}^{+\infty} \gamma_j X_j(\omega) \varphi_j,$$

so that almost surely  $v_\gamma \in \mathcal{B}_{2,1}^0(\mathbb{R}^d)$  and its probability law defines a measure  $\mu_\gamma$  in  $\mathcal{B}_{2,1}^0(\mathbb{R}^d)$ . Notice that we have

$$\mathcal{H}^s(\mathbb{R}^d) \subset \mathcal{B}_{2,1}^0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d), \quad \forall s > 0.$$

We have the following result

**Theorem 1.2.** — *For every  $(s, r) \in \mathbb{R}^2$  such that  $r \geq 2$  and  $s = d(\frac{1}{2} - \frac{1}{r})$  there exists  $c_0 > 0$  such that for all  $K > 0$  we have*

$$(1.6) \quad \mu_\gamma \left[ u \in \mathcal{B}_{2,1}^0(\mathbb{R}^d) : \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \geq K \|u\|_{\mathcal{B}_{2,1}^0(\mathbb{R}^d)} \right] \leq e^{-c_0 K^2}.$$

*In particular  $\mu_\gamma$ -almost all functions in  $\mathcal{B}_{2,1}^0(\mathbb{R}^d)$  are in  $\mathcal{W}^{s,r}(\mathbb{R}^d)$ .*

If  $\gamma$  satisfies (1.2) and the (weaker) condition  $\sum_{n \geq 0} |\gamma|_{\lambda_n}^2 < +\infty$ , then  $\mu_\gamma$  defines a probability measure on  $L^2(\mathbb{R}^d)$  and we can prove the estimate

$$(1.7) \quad \mu_\gamma \left[ u \in L^2(\mathbb{R}^d) : \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \geq K \|u\|_{L^2(\mathbb{R}^d)} \right] \leq e^{-c_0 K^2},$$

with  $s = d(\frac{1}{2} - \frac{1}{r})$  when  $r < +\infty$  and  $s < d/2$  in the case  $r = +\infty$ . From this result it is easy to deduce space-time estimates (Strichartz) for the linear flow  $e^{-itH}u$ , which can be used to study the nonlinear problem. This will be pursued in [15].

**1.2.3. An application to Hermite functions.** — Similarly to [2], the previous results give some information on Hilbertian bases. We prove that there exists a basis of Hermite functions with good decay properties.

**Theorem 1.3.** — *Let  $d \geq 2$ . Then there exists a Hilbertian basis of  $L^2(\mathbb{R}^d)$  of eigenfunctions of the harmonic oscillator  $H$  denoted by  $(\varphi_n)_{n \geq 1}$  such that  $\|\varphi_n\|_{L^2(\mathbb{R}^d)} = 1$  and so that for some  $M > 0$  and all  $n \geq 1$ ,*

$$(1.8) \quad \|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq M \lambda_n^{-\frac{d}{4}} (1 + \log \lambda_n)^{1/2}.$$

We refer to Theorem 5.1 for a more quantitative result, and where we prove that for a natural probability measure, almost all Hermite basis satisfies the property of Theorem 1.3 (see also Corollary 4.14). For the proof of this result, we need the finest randomisation with  $\delta = 1$  and  $D = 2$  in (1.1), so that  $\mathbf{P}_{\gamma,h}$  is a probability measure on each eigenspace.

The result of Theorem 1.3 does not hold true in dimension  $d = 1$ . Indeed, in this case one can prove the optimal bound (see [11])

$$(1.9) \quad \|\varphi_n\|_{L^\infty(\mathbb{R})} \leq C n^{-1/12}.$$

Let us compare (1.10) with the general known bounds on Hermite functions. We have  $H\varphi_n = \lambda_n \varphi_n$ , with  $\lambda_n \sim cn^{1/d}$ , therefore (1.10) can be rewritten

$$(1.10) \quad \|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq M n^{-1/4} (1 + \log n)^{1/2}.$$

For a general basis with  $d \geq 2$ , Koch and Tataru [11] (see also [12]) prove that

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq C \lambda_n^{\frac{d}{4} - \frac{1}{2}},$$

which shows that (1.8) induces a gain of  $d - 1$  derivatives compared to the general case. We stress that we don't now any explicit example of  $(\varphi_n)_{n \geq 1}$  which satisfy the conclusion of the Theorem. For instance, the basis  $(\varphi_n^\top)_{n \geq 1}$  obtained by tensorisation of the 1D basis does not realise (1.10) because of (1.9) which implies the optimal bound

$$\|\varphi_n^\top\|_{L^\infty(\mathbb{R}^d)} \leq C \lambda_n^{-1/12}.$$

Observe also that the basis of radial Hermite functions does not satisfy (1.10) in dimension  $d \geq 2$ . As in [2, Théorème 8], it is likely that the log term in (1.10) can not be avoided.

### 1.3. Notations and plan of the paper. —

**Notations.** — In this paper  $c, C > 0$  denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters. We denote by  $H = -\Delta + |x|^2 = \sum_{j=1}^d (-\partial_j^2 + x_j^2)$  the harmonic oscillator on  $\mathbb{R}^d$ , and for  $s \geq 0$  we define the Sobolev space  $\mathcal{H}^s$  by the norm  $\|u\|_{\mathcal{H}^s} = \|H^{s/2}u\|_{L^2(\mathbb{R}^d)} \approx \|u\|_{H^s(\mathbb{R}^d)} + \|\langle x \rangle^s u\|_{L^2(\mathbb{R}^d)}$ . More generally, we define the spaces  $\mathcal{W}^{s,p}$  by the norm  $\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p(\mathbb{R}^d)}$ . We write  $L^{r,s}(\mathbb{R}^d) = L^r(\mathbb{R}^d, \langle x \rangle^s dx)$ , and its norm  $\|u\|_{r,s}$ .

The rest of the paper is organised as follows. In Section 2 we describe the general probabilistic setting and we prove large deviation estimates on Hilbert spaces. In Section 3 we state crucial estimates on the spectral function of the harmonic oscillator. Section 4 is devoted to the proof of weighted Sobolev estimates and of the mains results. In Section 5 we prove Theorem 1.3.

**Acknowledgements.** — The authors thank Nicolas Burq for discussions on this subject and for his suggestion to introduce conditions (1.2)-(1.3).

## 2. A general setting for probabilistic smoothing estimates

Our aim in this section is to unify several probabilistic approaches to improve smoothing estimates established for dispersive equations. This setting is inspired by papers of Burq-Lebeau [2], Burq-Tzvetkov [3, 4] and their collaborators.

### 2.1. The concentration of measure property. —

**Definition 2.1.** — We say that a family of Borelian probability measures  $(\nu_N, \mathbb{R}^N)_{N \geq 1}$  satisfies the concentration of measure property if there exist constants  $c, C > 0$  independent of  $N \in \mathbb{N}$  such that for all Lipschitz and convex function  $F : \mathbb{R}^N \rightarrow \mathbb{R}$

$$(2.1) \quad \nu_N \left[ X \in \mathbb{R}^N : |F(X) - \mathbb{E}(F(X))| \geq r \right] \leq c e^{-\frac{Cr^2}{\|F\|_{Lip}^2}}, \quad \forall r > 0,$$

where  $\|F\|_{Lip}$  is the best constant so that  $|F(X) - F(Y)| \leq \|F\|_{Lip} \|X - Y\|_{\ell^2}$ .

For a comprehensive study of these phenomena, we refer to the book of Ledoux [13]. Notice that one of the main features of (2.1) is that the bound is independent of the dimension of space, which enables to take  $N$  large.

Typically, in our applications,  $F$  will be a norm in  $\mathbb{R}^N$ .

Let us give some significative examples of such measures.

- If  $(\nu_N, \mathbb{R}^N)_{N \geq 1}$  is a family of probability measures which satisfies a Log-Sobolev estimate with constant  $C^* > 0$ , then (2.1) is satisfied for all Lipschitz function  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  (see [1, Théorème 7.4.1,

page 123]). Recall that a probability measure  $\nu_N$  on  $\mathbb{R}^N$  satisfies a Log-Sobolev estimate if there exists  $C > 0$  independent of  $N \geq 1$  so that for all  $f \in \mathcal{C}_b(\mathbb{R}^N)$

$$(2.2) \quad \int_{\mathbb{R}^N} f^2 \ln \left( \frac{f^2}{\mathbb{E}(f^2)} \right) d\nu_N(x) \leq C \int_{\mathbb{R}^N} |\nabla f|^2 d\nu_N(x), \quad \mathbb{E}(f^2) = \int_{\mathbb{R}^N} f^2 d\nu_N(x).$$

Such a property is usually difficult to check. See [1] for more details. Notice that the convexity of  $F$  is not needed.

- A probability measure of the form  $d\nu_N(x) = c_{\alpha,N} \exp(-\sum_{j=1}^N |x_j|^\alpha) dx$ ,  $x \in \mathbb{R}^N$ , satisfies (2.1) if and only if  $\alpha \geq 2$  (see [1, page 109]).

- Assume that  $\nu$  is a measure on  $\mathbb{R}$  with bounded support, then  $\nu_N = \nu^{\otimes N}$  satisfies the concentration of measure property. This is the Talagrand theorem [20] (see also [21] for an introduction to the topic).

**Assumption 1.** — Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\{X_n, n \geq 1\}$  be a sequence of independent, identically distributed, real or complex random values. In the sequel we can assume that they are real with the identification  $\mathbb{C} \approx \mathbb{R}^2$ . Moreover, we assume that for all  $n \geq 1$ ,

- (i) Denote by  $\nu$  law of the  $X_n$ . We assume that the family  $(\nu^{\otimes N}, \mathbb{R}^N)_{N \geq 1}$  satisfies the concentration of measure property in the sense of Definition 2.1.
- (ii) The r.v.  $X_n$  is centred:  $\mathbb{E}(X_n) = 0$ .
- (iii) The r.v.  $X_n$  is normalized:  $\mathbb{E}(X_n^2) = 1$ .

Under Assumption 1, for all  $n \geq 1$ , and  $\varepsilon > 0$  small enough

$$(2.3) \quad \mathbb{E}(e^{\varepsilon X_n^2}) < +\infty.$$

Indeed, by Definition 2.1 with  $F(X) = X_n$

$$\mathbb{E}(e^{\varepsilon X_n^2}) = \int_0^{+\infty} \nu(e^{C\varepsilon X_n^2} > \lambda) d\lambda = 1 + \int_1^{+\infty} \nu(|X_n| > \sqrt{\frac{\ln \lambda}{\varepsilon}}) d\lambda \leq 1 + 2 \int_1^{+\infty} \lambda^{-\frac{1}{2\varepsilon C}} d\lambda < +\infty.$$

Next, with the inequality  $s|x| \leq \varepsilon x^2/2 + s^2/(2\varepsilon)$ , we obtain that for all  $s \in \mathbb{R}$ ,  $\mathbb{E}(e^{sX_n}) \leq C e^{Cs^2}$  which in turn implies (see [14, Proposition 46]) that there exists  $C > 0$  so that for all  $s \in \mathbb{R}$

$$(2.4) \quad \mathbb{E}(e^{sX_n}) \leq e^{Cs^2}.$$

**Remark 2.2.** — Condition (2.4) is weaker than (2.2): a family of independent centred r.v.  $\{X_n, n \geq 1\}$  which satisfies (2.4) does not necessarily satisfy (2.1) for all Lipschitz function  $F$ . Indeed, using Kolmogorov estimate, one can prove (see [13]) that condition (2.1) is equivalent to

$$(2.5) \quad \int_{\mathbb{R}^d} e^{sF} d\nu \leq e^{Cs^2 \|F\|_{Lip}^2}, \quad \forall s \in \mathbb{R},$$

for all Lipschitz function  $F$  with  $\nu$ -mean 0.

We conclude with the elementary property

**Lemma 2.3.** — Assume that  $\{X_n\}$  satisfies (2.4) and that  $\{\alpha_j, 1 \leq j \leq N\}$  are real numbers such that  $\sum_{1 \leq j \leq N} \alpha_j^2 \leq 1$ . Then  $X := \sum_{1 \leq j \leq N} \alpha_j X_j$  satisfies (2.4) with the same constant  $C$ .

*Proof.* — It is a direct application of (2.5) with  $F(X) = \sum_{j=1}^N \alpha_j X_j$ . □

**2.2. Probabilities on Hilbert spaces.** — In this sub-section  $\mathcal{K}$  is a separable complex Hilbert space and  $K$  is a self-adjoint, positive operator on  $\mathcal{K}$  with a compact resolvent. We denote by  $\{\varphi_j, j \geq 1\}$  an orthonormal basis of eigenvectors of  $K$ ,  $K\varphi_j = \lambda_j\varphi_j$ , and  $\{\lambda_j, j \geq 1\}$  is the non decreasing sequence of eigenvalues of  $K$  (each is repeated according to its multiplicity). Then we get a natural scale of Sobolev spaces associated with  $K$  defined for  $s \geq 0$  by  $\mathcal{K}^s = \text{Dom}(K^{s/2})$ .

Now we want to introduce probability measures on these spaces and on some finite dimensional spaces of  $\mathcal{K}$ .

Let us describe in our setting the randomization technique deeply used by Burq-Tzvetkov in [3]. Let  $\gamma = \{\gamma_j\}_{j \geq 1}$  a sequence of complex numbers such that  $\sum_{j \geq 1} \lambda_j^s |\gamma_j|^2 < +\infty$ .

Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\{X_n, n \geq 1\}$  be independent, identically distributed random variables which satisfy Assumption 1.

We denote by  $v_\gamma^0 = \sum_{j \geq 1} \gamma_j \varphi_j \in \mathcal{K}^s$ , and we define the random vector  $v_\gamma(\omega) = \sum_{j \geq 1} \gamma_j X_j(\omega) \varphi_j$ . We have  $\mathbb{E}(\|v_\gamma\|_{\mathcal{K}}^2) < +\infty$ , therefore  $v_\gamma \in \mathcal{K}^s$ , a.s. We define the measure  $\mu_\gamma$  on  $\mathcal{K}^s$  as the law of the random vector  $v_\gamma$ .

**2.2.1. The Kakutani theorem.** — The following proposition gives some properties of the measures  $\mu_\gamma$  (see [5] for more details).

**Proposition 2.4.** — Assume that all random variables  $X_j$  have the same law  $\nu$ .

- (i) If the support of  $\nu$  is  $\mathbb{R}$  and if  $\gamma_j \neq 0$  for all  $j \geq 1$  then the support of  $\mu_\gamma$  is  $\mathcal{K}^s$ .
- (ii) If for some  $\varepsilon > 0$  we have  $v_\gamma^0 \notin \mathcal{K}^{s+\varepsilon}$  then  $\mu_\gamma(\mathcal{K}^{s+\varepsilon}) = 0$ .
- (iii) Assume that we are in the particular case where  $d\nu(x) = c_\alpha e^{-|x|^\alpha} dx$  with  $\alpha \geq 2$ . Let  $\gamma = \{\gamma_j\}$  and  $\beta = \{\beta_j\}$  be two complex sequences and assume that

$$(2.6) \quad \sum_{j \geq 1} \left( \left| \frac{\gamma_j}{\beta_j} \right|^{\alpha/2} - 1 \right)^2 = +\infty.$$

Then the measures  $\mu_\gamma$  and  $\mu_\beta$  are mutually singular, i.e there exists a measurable set  $A \subset \mathcal{H}^s$  such that  $\mu_\gamma(A) = 1$  and  $\mu_\beta(A) = 0$ .

We give the proof of (iii) in Appendix A.

We shall see now that condition (1.2) (resp. (1.3)) can be perturbed so that Proposition 2.4 gives us an infinite number of mutually singular measures on  $\mathcal{K}^s$ .



**Lemma 2.5.** — *Let  $\gamma$  satisfying (1.2) (resp. (1.3)) and  $\delta = \{\delta_n\}_{n \geq 1}$  such that  $|\delta_n| \leq \varepsilon |\gamma_n|$  for every  $n \geq n_0$ . Then for every  $\varepsilon \in [0, \sqrt{2} - 1[$ , the sequence  $\gamma + \delta$  satisfies (1.2) (resp. (1.3)) (with new constants).*

We do not give the details of the proof. From this Lemma and Proposition 2.4 we get an infinite number of measures  $\mu_\gamma$  with  $\gamma$  satisfying (1.2) (resp. (1.3)). Let  $\varepsilon_j$  be any sequence such that  $\sum_{j \geq 1} \varepsilon_j^2 = +\infty$  and  $\limsup \varepsilon_j < \sqrt{2} - 1$  and denote by  $\varepsilon \otimes \gamma$  the sequence  $\varepsilon_j \gamma_j$ . Then  $\mu_\gamma$  and  $\mu_{\gamma + \varepsilon \cdot \gamma}$  are mutually singular.

**2.2.2. Measures on the sphere  $\mathbf{S}_h$ .** — Now we consider finite dimensional subspaces  $\mathcal{E}_h$  of  $\mathcal{K}$  defined by spectral localizations depending on a small parameter  $0 < h \leq 1$  ( $h^{-1}$  is a measure of energy for the quantum Hamiltonian  $K$ ). In the sequel, we use the notations  $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$ ,  $N_h$ ,  $\Lambda_h$  and  $\mathcal{E}_h$  introduced in Section 1.1, and we assume that (1.1) is satisfied. Observe that  $\mathcal{E}_h$  is the spectral subspace of  $K$  in the interval  $I_h$ :  $\mathcal{E}_h = \Pi_h \mathcal{K}$  where  $\Pi_h$  is the orthogonal projection on  $\mathcal{K}$ . For simplicity, we sometimes denote by  $N = N_h$ ,  $\Lambda = \Lambda_h$ ,  $\dots$ , with implicit dependence in  $h$ . Our goal is to find uniform estimates in  $h \in ]0, h_0[$  for a small constant  $h_0 > 0$ .

Let us consider the random vector in  $\mathcal{E}_h$

$$(2.7) \quad v_\gamma(\omega) := v_{\gamma,h}(\omega) = \sum_{j \in \Lambda} \gamma_j X_j(\omega) \varphi_j,$$

and assume that (1.3) is satisfied. In the sequel we denote by  $|\gamma|_\Lambda^2 = \sum_{n \in \Lambda} \gamma_n^2$ .

Now we consider probabilities on the unit sphere  $\mathbf{S}_h$  of the subspaces  $\mathcal{E}_h$ . The random vector  $v_\gamma$  in (2.7) defines a probability measure  $\nu_{\gamma,h}$  on  $\mathcal{E}_h$ . Then we can define a probability measure  $\mathbf{P}_{\gamma,h}$  on  $\mathbf{S}_h$  as the image of by  $v \mapsto \frac{v}{\|v\|}$ . Namely, we have for every Borel and bounded function  $f$  on  $\mathbf{S}_h$ ,

$$(2.8) \quad \int_{\mathbf{S}_h} f(w) \mathbf{P}_{\gamma,h}(dw) = \int_{\mathcal{E}_h} f\left(\frac{v}{\|v\|_{\mathcal{K}}}\right) \nu_{\gamma,h}(dv) = \int_{\Omega} f\left(\frac{v_\gamma(\omega)}{\|v_\gamma(\omega)\|_{\mathcal{K}}}\right) \mathbb{P}(d\omega).$$

Remark that we have

$$\|v_\gamma(\omega)\|_{\mathcal{K}}^2 = \sum_{j \in \Lambda} |\gamma_j|^2 |X_j(\omega)|^2$$

and

$$\mathbb{E}(\|v_\gamma\|_{\mathcal{K}}^2) = \sum_{j \in \Lambda} |\gamma_j|^2 = |\gamma|_\Lambda^2.$$

Let us detail two particular cases of interest:

- If  $|\gamma_n| = \frac{1}{\sqrt{N}}$  for all  $j \in \Lambda$  and if  $X_n$  follows the complex normal law  $\mathcal{N}_{\mathbb{C}}(0, 1)$  then  $\mathbf{P}_{\gamma,h}$  is the uniform probability on  $\mathbf{S}_h$  considered in [2]. This follows from (2.8) and property of Gaussian laws.

• Assume that for all  $n \in \mathbb{N}$ ,  $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$ , then  $\mathbf{P}_{\gamma,h}$  is a convex sum of  $2^N$  Dirac measures. Indeed we have  $\|v_\gamma(\omega)\|_{\mathcal{K}}^2 = \sum_{j \in \Lambda} |\gamma_j|^2 = |\gamma|_\Lambda^2$ . Denote by  $(\varepsilon^{(k)})_{1 \leq k \leq 2^N}$  all the sequences so that  $\varepsilon_j^{(k)} = \pm 1$  for all  $1 \leq j \leq N$ , and set

$$\Phi_k = \frac{1}{|\gamma|} \sum_{j \in \Lambda} \gamma_j \varepsilon_j^{(k)} \varphi_j, \quad 1 \leq k \leq 2^N.$$

Then

$$\mathbf{P}_{\gamma,h} = \frac{1}{2^N} \sum_{k=1}^{2^N} \delta_{\Phi_k}.$$

To get an optimal lower bound for  $L^\infty$  estimates we shall need a stronger normal concentration estimate than estimate given in (2.1). Hence we make the following assumptions:

**Assumption 2.** — *We assume that*

- (i) *The random variables  $X_j$  are standard independent Gaussians  $\mathcal{N}_{\mathbb{C}}(0, 1)$ .*
- (ii) *The sequence  $\gamma$  satisfies (1.3).*

Let  $L$  be a linear form on  $\mathcal{E}_h$ , and denote by  $e_L = \sum_{j \in \Lambda_h} |L(\varphi_j)|^2$ . The main result of this section is the following

**Theorem 2.6.** — *Let  $L$  be a linear form on  $\mathcal{E}_h$ . Suppose that (1.2) holds and that Assumption 1 is satisfied. Then there exist  $C_2, c_2 > 0$  so that*

$$(2.9) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right] \leq C_2 e^{-c_2 \frac{N}{e_L} t^2}, \quad \forall t \geq 0, \quad \forall h \in ]0, h_0],$$

Moreover, if (1.3) holds, there exist  $C_1, c_1 > 0$  and  $\varepsilon_0, h_0 > 0$  so that

$$(2.10) \quad C_1 e^{-c_1 \frac{N}{e_L} t^2} \leq \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right], \quad \forall t \in \left[ 0, \varepsilon_0 \frac{\sqrt{e_L}}{\sqrt{N}} \right], \quad \forall h \in ]0, h_0].$$

Furthermore, if Assumption 2 is satisfied, there exist  $C_1, C_2, c_1, c_2, \varepsilon_0, h_0 > 0$  so that

$$(2.11) \quad C_1 e^{-c_1 \frac{N}{e_L} t^2} \leq \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right] \leq C_2 e^{-c_2 \frac{N}{e_L} t^2}, \quad \forall t \in [0, \varepsilon_0 \sqrt{e_L}], \quad \forall h \in ]0, h_0].$$

Since  $\mathbf{P}_{\gamma,h}$  is supported by  $\mathbf{S}_h$ , the bounds in the previous result don't depend on  $|\gamma|_\Lambda$ . The restriction on  $t \geq 0$  in (2.11) is natural, because by the Cauchy-Schwarz inequality we have

$$|L(u)| \leq \sqrt{e_L}, \quad \forall u \in \mathbf{S}_h.$$

In the applications we give, there is some embedding  $\mathcal{K}^s \rightarrow C(M)$ , for  $s > 0$  large enough, where  $M$  is a metric space. We have  $\mathcal{E} \subseteq \bigcap_{s \in \mathbb{R}} \mathcal{K}^s$ , thus we can consider the Dirac evaluation linear form  $\delta_x(v) = v(x)$ . In this case we have  $e_L = \sum_{j \in \Lambda} |\varphi_j(x)|^2 = e_x$ , which is usually called the spectral function of  $K$  in the interval  $I$ .

For example, one can consider the Laplace-Beltrami operator on compact Riemannian manifolds, namely  $K = -\Delta$  and  $\mathcal{K}^s = \mathcal{H}^s(M)$  are the usual Sobolev spaces: this is the framework of [2]. In Section 3 we will apply the result of Theorem 2.6 to the Harmonic oscillator  $K = -\Delta + |x|^2$  on  $\mathbb{R}^d$ . In this latter case  $\mathcal{K}^s$  is the weighted Sobolev space

$$\mathcal{K}^s = \{ u \in \mathcal{H}^s(\mathbb{R}^d), |x|^s u \in L^2(\mathbb{R}^d) \}, s \geq 0.$$

**Remark 2.7.** — In the particular case where  $\mathbf{P}_{\gamma,h}$  is the uniform probability on  $\mathbf{S}_h$ , we have the explicit computation

$$\mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right] = \Phi \left( \frac{t}{\sqrt{e_L}} \right),$$

where

$$(2.12) \quad \Phi(t) = \mathbb{I}_{[0,1]}(t)(1-t^2)^{N-1},$$

and (2.11) follows directly. For a proof of (2.12), see [2] or in Appendix C of this paper for an alternative argument.

For the proof of Theorem 2.6 we will need the following result.

**Proposition 2.8.** — Assume that  $\gamma$  satisfies (1.2). Let  $L$  be a linear form on  $\mathcal{E}_h$ . Then we have the large deviation estimate

$$\mathbb{P} \left[ \omega \in \Omega : |L(v_\gamma)| \geq t \right] \leq 4e^{-\kappa_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2},$$

where  $\kappa_1 = \frac{\kappa_0}{4K_1}$ . As a consequence, if  $\nu_{\gamma,h}$  denotes the probability law of  $v_\gamma$ , then

$$\nu_{\gamma,h} \left[ w \in \mathcal{E}_h : |L(w)| \geq t \right] \leq 4e^{-\kappa_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2}.$$

*Proof.* — We have

$$L(v_\gamma) = \sum_{j \in \Lambda} \gamma_j X_n(\omega) L(\varphi_j).$$

It is enough to assume that  $L(v_\gamma)$  is real and to estimate  $\mathbb{P}[\omega \in \Omega : L(v_\gamma) \geq t]$ . Using the Markov inequality, we have for all  $s > 0$

$$\mathbb{P}[L(v_\gamma) \geq t] \leq e^{-st} \mathbb{E}(e^{sL(v_\gamma)}),$$

and thanks to (1.2) we have

$$\sum_{j \in \Lambda} |\gamma_j L(\varphi_j)|^2 \leq K_1 \frac{|\gamma|_\Lambda^2}{N} \sum_{j \in \Lambda} |L(\varphi_j)|^2.$$

Using Lemma 2.3 we get

$$\mathbb{P}[L(v_\gamma) \geq t] \leq e^{-st} e^{\kappa_0 K_1 \frac{e_L}{N} |\gamma|_\Lambda^2 s^2},$$

and with the choice  $s = \frac{\kappa_0}{2} \frac{tN}{K_1 e_L |\gamma|_\Lambda^2}$  we obtain  $\mathbb{P}[L(v_\gamma) \geq t] \leq e^{-\kappa_1 \frac{N}{|\gamma|_\Lambda^2 e_L} t^2}$ .  $\square$

It will be useful to show that  $\|v_\gamma(\omega)\|_{\mathcal{K}}^2$  is close to its expectation for large  $N$ .

**Lemma 2.9.** — *Let  $\gamma$  satisfying the squeezing condition (1.3). Then there exists  $c_0 > 0$  (depending only on  $K_0$  and  $K_1$ ) such that for every  $\varepsilon > 0$*

$$\mathbb{P}\left[\omega \in \Omega : \left| \|v_\gamma(\omega)\|_{\mathcal{K}}^2 - |\gamma|_\Lambda^2 \right| > \varepsilon \right] \leq 2e^{-\frac{\varepsilon c_0 N}{|\gamma|_\Lambda^2}}.$$

*Proof.* — It is enough to consider the real case, so we assume that  $\gamma_n$  and  $X_n$  are real and  $\{X_n, n \geq 1\}$  have a common law  $\nu$ . We also assume that  $|\gamma|_\Lambda^2 = 1$ .

We have

$$\|v_\gamma(\omega)\|_{\mathcal{K}}^2 = \sum_{j \in \Lambda} |\gamma_j|^2 X_j^2(\omega) := M_N(\omega).$$

From large number law,  $\|v_\gamma(\omega)\|_{\mathcal{K}}^2$  converges to 1 a.s. To estimate the tail we use the Cramer-Chernoff large deviation principle (see e.g. [19, § 5, Chapter IV]). This applies because from (2.3) we know that  $f(s) := \mathbb{E}(e^{sX_1^2})$  is  $\mathcal{C}^2$  in  $]-\infty, s_0[$  for some  $s_0 > 0$ .

We reproduce here a well known computation in large deviation theory. Define the cumulant function  $g(s) = \log(f(s))$  which is well defined for  $s < s_0$ . Now, since the  $X_j$  are i.i.d., for  $t, s \geq 0$  we have

$$\begin{aligned} \mathbb{P}[M_N > t] &= \mathbb{P}[e^{sNM_N} > e^{sNt}] \leq \mathbb{E}(e^{sNM_N})e^{-sNt} \\ &= \prod_{j \in \Lambda} e^{-(Ns|\gamma_j|^2 t - g(Ns|\gamma_j|^2))}. \end{aligned}$$

Next, apply the Taylor formula to  $g$  at 0:  $g(0) = 0$ ,  $g'(0) = 1$  so  $t\tau - g(\tau) = (t-1)\tau + \mathcal{O}(\tau^2)$ , hence there exists  $s_1 > 0$  such that for  $0 \leq \tau \leq s_1$ ,  $t\tau - g(\tau) \geq (t-1)\frac{\tau}{2}$ . Then, with  $t = 1 + \varepsilon$ , and since  $N|\gamma_j|^2 \leq K_0$  we get

$$\mathbb{P}[M_N > 1 + \varepsilon] \leq \prod_{j \in \Lambda} e^{-\varepsilon N s |\gamma_j|^2 / 2} = e^{-\varepsilon s N / 2},$$

provided  $s > 0$  is small enough, but independent of  $\varepsilon > 0$  and  $N \geq 1$ . The same computation applied to  $-M_N$  gives as well  $\mathbb{P}[M_N < 1 - \varepsilon] \leq e^{-\varepsilon c_0 N}$ .  $\square$

*Proof of (2.9).* — By homogeneity, we can assume that  $|\gamma|_\Lambda = 1$ . Denote by

$$(2.13) \quad A = \left\{ \omega \in \Omega : \left| \|v_\gamma(\omega)\|_{\mathcal{K}}^2 - 1 \right| \leq 1/2 \right\}.$$

By the Cauchy-Schwarz inequality, for all  $u \in \mathbf{S}_h$ , we obtain  $|L(u)| \leq e_L^{1/2}$ . Thus in the sequel we can assume that  $t \leq e_L^{1/2}$ . Then, from Proposition 2.8 and Lemma 2.9 we have

$$(2.14) \quad \begin{aligned} \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right] &= \mathbb{P} \left[ \omega \in \Omega : |L(v(\omega))| \geq t \|v(\omega)\|_{L^2} \right] \\ &= \mathbb{P} \left[ (|L(v(\omega))| \geq t \|v(\omega)\|_{L^2}) \cap A \right] + \mathbb{P} \left[ (|L(v(\omega))| \geq t \|v(\omega)\|_{L^2}) \cap A^c \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right] &\leq \mathbb{P} \left[ |L(v(\omega))| \geq t/2 \right] + \mathbb{P}(A^c) \\ &\leq C_1 e^{-c_1 \frac{N}{e_L} t^2} + 2e^{-c_2 N} \leq C e^{-c \frac{N}{e_L} t^2}, \end{aligned}$$

which implies (2.9).  $\square$

We now turn to the proof of (2.10). We will need the following result

**Lemma 2.10.** — *We suppose that  $\gamma$  satisfies (1.3) and that Assumption 1 is satisfied. Then there exist  $C_1 > 0, c_1 > 0, h_0 > 0, \varepsilon_0 > 0$  such that*

$$\mathbb{P}\left[\omega \in \Omega : |L(v_\gamma(\omega))| \geq t\right] \geq C_1 e^{-c_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2}, \quad \forall t \in [0, \varepsilon_0 \frac{\sqrt{e_L} |\gamma|_\Lambda}{\sqrt{N}}], \quad \forall h \in ]0, h_0].$$

*Proof.* — Let us first recall the Paley-Zygmund inequality<sup>(1)</sup>: Let  $Z \in L^2(\Omega)$  be a r.v such that  $Z \geq 0$ , then for all  $0 < \lambda < 1$ ,

$$(2.15) \quad \mathbb{P}(Z > \lambda \|Z\|_1) \geq \left((1 - \lambda) \frac{\|Z\|_1}{\|Z\|_2}\right)^2.$$

We apply (2.15) to the random variable  $Z = |Y_N|^2$ , with

$$Y_N = \frac{\sqrt{N}}{\sqrt{e_L} |\gamma|_\Lambda} L(v_\gamma) = \frac{\sqrt{N}}{\sqrt{e_L} |\gamma|_\Lambda} \sum_{j \in \Lambda} \gamma_j X_j L(\varphi_j),$$

and  $\lambda = 1/2$ . By (1.3), we have  $c_0 \leq \|Y_N\|_2 \leq C_0$  uniformly in  $N \geq 1$ . Next, recall the Khinchin inequality (see e.g. [3, Lemma 4.2] for a proof) : there exists  $C > 0$  such that for all real  $k \geq 2$  and  $(a_n) \in \ell^2(\mathbb{N})$

$$\left\| \sum_{n \in \Lambda} X_n(\omega) a_n \right\|_{L^k_{\mathbb{P}}} \leq C \sqrt{k} \left( \sum_{n \in \Lambda} |a_n|^2 \right)^{\frac{1}{2}}.$$

Therefore, there exists  $C_1 > 0$  such that  $\|Y_N\|_4 \leq C_1$ . As a result, there exist  $\eta > 0$  and  $\varepsilon > 0$  so that for all  $N \geq 1$ ,  $\mathbb{P}(|Y_N| > \eta) > \varepsilon$ , which implies the result.  $\square$

*Proof of (2.10).* — We assume that  $|\gamma|_\Lambda = 1$ , and consider the set  $A$  defined in (2.13). Then by (2.14) and the inequality  $\mathbb{P}(B \cap A) \geq \mathbb{P}(B) - \mathbb{P}(A^c)$  we get

$$\begin{aligned} \mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : |L(u)| \geq t \right] &\geq \mathbb{P} \left[ (|L(v(\omega))| \geq t \|v(\omega)\|_{L^2}) \cap A \right] \\ &\geq \mathbb{P} \left[ |L(v(\omega))| \geq 3t/2 \right] - \mathbb{P}(A^c) \\ &\geq C_1 e^{-c_1 \frac{N}{e_L} t^2} - 2e^{-c_2 N}, \end{aligned}$$

where in the last line we used Lemma 2.10 and Lemma 2.9. This yields the result if  $t \leq \varepsilon_0 \frac{\sqrt{e_L}}{\sqrt{N}}$  with  $\varepsilon_0 > 0$  small enough.  $\square$

We now prove (2.11). To begin with, we can state

**Lemma 2.11.** — *We suppose that Assumption 2 is satisfied. Then there exist  $C_1 > 0, c_1 > 0, h_0 > 0, \varepsilon_0 > 0$  such that*

$$\mathbb{P}\left[\omega \in \Omega : |L(v_\gamma(\omega))| \geq t\right] \geq C_1 e^{-c_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2}, \quad \forall t \geq 0, \forall h \in ]0, h_0].$$

<sup>(1)</sup>We thank Philippe Sosoe for this suggestion.

*Proof.* — Denote by  $\gamma \otimes L(\varphi)$  the vector  $(\gamma \otimes L(\varphi))_j = \gamma_j L(\varphi_j)$ . Observe that, thanks to (1.3),

$$K_1 \frac{|\gamma|_\Lambda^2 e_L}{N} \leq |\gamma \otimes L(\varphi)|^2 = \sum_{j \in \Lambda_h} \gamma_j^2 |L(\varphi_j)|^2 \leq K_0 \frac{|\gamma|_\Lambda^2 e_L}{N}.$$

Then, using the rotation invariance of the Gaussian law and the previous line, we get

$$\begin{aligned} \mathbb{P} \left[ \omega \in \Omega : |L(v_\gamma(\omega))| \geq t \right] &= \mathbb{P} \left[ \left| \left\langle \frac{\gamma \otimes L(\varphi)}{|\gamma \otimes L(\varphi)|}, X \right\rangle \right| \geq \frac{t}{|\gamma \otimes L(\varphi)|} \right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{|s| \geq \frac{t}{|\gamma \otimes L(\varphi)|}} e^{-s^2/2} ds \\ &\geq C e^{-\frac{e_N}{e_L |\gamma|_\Lambda^2} t^2}. \end{aligned}$$

□

The estimate (2.11) then follows from Lemma 2.11 and with the same argument as for Lemma 2.10.

**2.2.3. Concentration phenomenon.** — We now state a concentration property for  $\mathbf{P}_{\gamma,h}$ , inherited from Assumption 1 and condition (1.3). See [13] for more details on this topic.

**Proposition 2.12.** — *Suppose that the i.i.d. random variables  $X_j$  satisfy Assumption 1 and suppose that condition (1.3) is satisfied. Then there exist constants  $K > 0$ ,  $\kappa > 0$  (depending only on  $C^*$ ) such that for every Lipschitz function  $F : \mathbf{S}_h \rightarrow \mathbb{R}$  satisfying*

$$|F(u) - F(v)| \leq \|F\|_{Lip} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h,$$

we have

$$(2.16) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |F - \mathcal{M}_F| > r \right] \leq K e^{-\frac{\kappa N r^2}{\|F\|_{Lip}^2}}, \quad \forall r > 0, h \in ]0, 1],$$

where  $\mathcal{M}_F$  is a median for  $F$ .

Recall that a median  $\mathcal{M}_F$  for  $F$  is defined by

$$\mathbf{P}_{\gamma,h} [u \in \mathbf{S}_h : F \geq \mathcal{M}_F] \geq \frac{1}{2}, \quad \mathbf{P}_{\gamma,h} [u \in \mathbf{S}_h : F \leq \mathcal{M}_F] \geq \frac{1}{2}.$$

In Proposition 2.12, the distance in  $L^2$  can be replaced with the geodesic distance  $d_S$  on  $\mathbf{S}_h$ , since we can check that

$$\|u - v\|_{L^2(\mathbb{R}^d)} \leq d_S(u, v) = 2 \arcsin \left( \frac{\|u - v\|_{L^2(\mathbb{R}^d)}}{2} \right) \leq \frac{\pi}{2} \|u - v\|_{L^2(\mathbb{R}^d)}.$$

When  $\mathbf{P}_{\gamma,h}$  is the uniform probability on  $\mathbf{S}_h$ , Proposition 2.12 is proved in [13, Proposition 2.10], and the proof can be adapted in the general case (see Appendix D). The factor  $N$  in the exponential of r.h.s of (2.16) will be crucial in our application.

### 3. Some spectral estimates for the harmonic oscillator

Our goal here is to apply the general setting of Section 2 to the harmonic oscillator in  $\mathbb{R}^d$ . This way we shall get probabilistic estimates analogous to results proved in [2] for the Laplace operator in a compact Riemannian manifold.

In the following, we consider the Hamiltonian  $H = -\Delta + V(x)$  with  $V(x) = |x|^2$ ,  $x \in \mathbb{R}^d$  for  $d \geq 2$ . For this model, all the necessary spectral estimates are already known. More general confining potentials  $V$  shall be considered in the forthcoming paper [17].

A first and basic ingredient in probabilistic approaches of weighted Sobolev spaces is a good knowledge concerning the asymptotic behavior of eigenvalues and eigenfunctions of  $H$ . The eigenvalues of this operator are the  $\{2(j_1 + \dots + j_d) + d, j \in \mathbb{N}^d\}$ , and we can order them in a non decreasing sequence  $\{\lambda_j, j \in \mathbb{N}\}$ , repeated according to their multiplicities. We denote by  $\{\varphi_j, j \in \mathbb{N}\}$  an orthonormal basis in  $L^2(\mathbb{R}^d)$  of eigenfunctions (the Hermite functions), so that  $H\varphi_j = \lambda_j\varphi_j$ . The spectral function is then defined as  $\pi_H(\lambda; x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)}$  (recall that this definition does not depend on the choice of  $\{\varphi_j, j \in \mathbb{N}\}$ ). When the energy  $\lambda$  is localized in  $I \subseteq \mathbb{R}^+$  we denote by  $\Pi_H(I)$  the spectral projector of  $H$  on  $I$ . The range  $\mathcal{E}_H(I)$  of  $\Pi_H(I)$  is spanned by  $\{\varphi_j; \lambda_j \in I\}$  and  $\Pi_H(I)$  has an integral kernel given by

$$\pi_H(I; x, y) = \sum_{[j: \lambda_j \in I]} \varphi_j(x)\overline{\varphi_j(y)}.$$

We will also use the notation  $\mathcal{E}_H(\lambda) = \mathcal{E}_H([0, \lambda])$ ,  $N_H(\lambda) = \dim[\mathcal{E}_H(\lambda)]$ .

**3.1. Interpolation inequalities.** — We begin with some general interpolation results which will be needed in the sequel. In  $\mathbb{R}^d$ , the spectral function  $\pi_H(\lambda; x, x)$  is fast decreasing for  $|x| \rightarrow +\infty$  so it is natural to work with weighted  $L^p$  norms. We denote by  $\langle x \rangle^s = (1 + |x|^2)^{s/2}$  and introduce the following Lebesgue space with weight

$$L^{p,s}(\mathbb{R}^d) = \left\{ u, \text{ Lebesgue measurable} : \int |u(x)|^p \langle x \rangle^s dx < +\infty \right\} = L^p(\mathbb{R}^d, \langle x \rangle^s dx),$$

endowed with its natural norm, which we denote by  $\|u\|_{p,s}$ . For  $p = \infty$ , we set  $\|u\|_{\infty,s} = \sup_{x \in \mathbb{R}^d} \langle x \rangle^s |u(x)|$ .

The following interpolation inequalities hold true. Let  $1 \leq p_1 \leq p \leq p_0 \leq +\infty$  and  $\kappa \in ]0, 1[$  such that  $\frac{1}{p} = \frac{\kappa}{p_1} + \frac{1-\kappa}{p_0}$ . Then for  $p_0 < +\infty$  we have

$$(3.1) \quad \|u\|_{L^{p,s}(\mathbb{R}^d)} \leq (\|u\|_{L^{p_0,s_0}(\mathbb{R}^d)})^{1-\kappa} (\|u\|_{L^{p_1,s_1}(\mathbb{R}^d)})^\kappa, \quad \text{with } s = \frac{p_1 - p}{p_1 - p_0} s_0 + \frac{p_0 - p}{p_0 - p_1} s_1.$$

In the case  $p_0 = +\infty$ , we have

$$(3.2) \quad \|u\|_{L^{p,s}(\mathbb{R}^d)} \leq (\sup_{\mathbb{R}^d} \langle x \rangle^{s_0} |u(x)|)^{1-p_1/p} (\|u\|_{L^{p_1,s_1}(\mathbb{R}^d)})^{p_1/p}, \quad \text{with } s = (p - p_1)s_0 + s_1.$$

**3.2. Rough estimates of the harmonic oscillator.** — We recall here some more or less standard properties stated in [10]. To begin with, we state a "soft" Sobolev inequality.

**Lemma 3.1.** — For all  $u \in \mathcal{E}_H(I)$

$$(3.3) \quad |u(x)| \leq (\pi_H(I; x, x))^{1/2} \|u\|_{L^2(\mathbb{R}^d)}.$$

*Proof.* — We have

$$u(x) = \Pi u(x) = \int_{\mathbb{R}^d} \pi_H(I; x, y) u(y) dy.$$

Using the Cauchy-Schwarz inequality

$$(3.4) \quad |u(x)| \leq \left( \int_{\mathbb{R}^d} |\pi_H(I; x, y)|^2 dy \right)^{1/2} \|u\|_{L^2(\mathbb{R}^d)}.$$

Now we use that  $\Pi_H(I)$  is an orthonormal projector.

$$(3.5) \quad \pi_H(I; x, y) = \int_{\mathbb{R}^d} \pi_H(I; x, z) \pi_H(I; z, y) dz \quad \text{and} \quad \pi_H(I; x, y) = \overline{\pi_H(I; y, x)}.$$

Finally, from (3.4) and (3.5) with  $y = x$  we get (3.3). □

The next result gives a bound on  $\pi_H$ .

**Lemma 3.2.** — The following bound holds true

$$(3.6) \quad \pi_H(\lambda; x, x) \leq C \lambda^{d/2} \exp\left(-c \frac{|x|^2}{\lambda}\right), \quad \forall x \in \mathbb{R}^d, \lambda \geq 1.$$

*Proof.* — Let  $K(t; x, y)$  be the heat kernel of  $e^{-tH}$ . It is given by the following Mehler formula <sup>(2)</sup>

$$(3.7) \quad K(t; x, y) = (2\pi \sinh 2t)^{-d/2} \exp\left(-\frac{\tanh t}{4} |x + y|^2 - \frac{|x - y|^2}{4 \tanh t}\right).$$

So we have

$$(3.8) \quad K(t; x, x) = \int_{\mathbb{R}} e^{-t\mu} d\pi_H(\mu; x, x) = (2\pi \sinh 2t)^{-d/2} \exp(-|x|^2 \tanh t).$$

We set  $t = \lambda^{-1}$ , integrate in  $\mu$  on  $[0, \lambda]$  and get

$$\pi_H(\lambda; x, x) \leq e K(\lambda^{-1}; x, x).$$

Assuming  $\lambda \geq \lambda_0$ ,  $\lambda_0$  large enough, we easily see that (3.6) is a consequence of (3.8). □

Let  $u \in \mathcal{E}_H(\lambda)$ . From (3.3) and (3.6) we get

$$|u(x)| \leq C \lambda^{d/4} \exp\left(-c \frac{|x|^2}{2\lambda}\right) \|u\|_{L^2(\mathbb{R}^d)},$$

where  $c, C > 0$  do not depend on  $x \in \mathbb{R}^d$  nor  $\lambda \geq 1$ .

<sup>(2)</sup>The Mehler formula can also be obtained from the Fourier transform computation of the Weyl symbol of  $e^{-tH}$  (see [16, Exercise IV-2]).



**Remark 3.3.** — From (3.6), we can deduce that for every  $\theta > 0$  there exists  $C_\theta > 0$  such that

$$\pi_H(\lambda; x, x) \leq C_\theta \lambda^{(d+\theta)/2} \langle x \rangle^{-\theta},$$

which by (3.3) implies with the semiclassical parameter  $h = \lambda^{-1}$

$$\langle x \rangle^{\theta/2} h^{(d+\theta)/4} |u(x)| \leq C_\theta \|u\|_{L^2(\mathbb{R}^d)}, \quad \forall u \in \mathcal{E}_H(h^{-1}).$$

We can easily see that this uniform estimate is true for  $u \in \mathcal{E}(I_h)$  where  $I_h = [\frac{a}{h}, \frac{b}{h}]$  with  $a < b$ . For smaller energy intervals we can get much better estimates, as we will see in Lemma 3.5.

**Remark 3.4.** — Let us compare the previous results with the case of a compact Riemannian manifold  $M$ , and when  $H = -\Delta$  is the Laplace operator. We have the uniform Hörmander estimate [7]:

$$(3.9) \quad \pi_H(\lambda; x, x) = c_d(x) \lambda^{d/2} + \mathcal{O}(\lambda^{(d-1)/2}),$$

where  $0 < c_d(x)$  is a continuous function on  $M$ . Thus from (3.4) and (3.9) we get for some constant  $C_S > 0$ ,

$$\|u\|_{L^\infty(M)} \leq C_S \lambda^{d/4} \|u\|_{L^2(M)}, \quad \forall u \in \mathcal{E}(\lambda).$$

Let us emphasize here that it results from the uniform Weyl law (3.9) that  $\pi_H(\lambda; x, x)$  has an upper bound and a lower bound of order  $\lambda^{d/2}$ . For confining potentials like  $V$  the behavior of  $\pi_H(\lambda; x, x)$  is much more complicated because of the turning points:  $\{|x|^2 = \lambda\}$ . This behavior was analyzed in [10].

**3.3. More refined estimates for the spectral function.** — From the Weyl law for the harmonic oscillator we have

$$N_H(\lambda) = c_d \lambda^d + \mathcal{O}(\lambda^{d-1}), \quad c_d > 0,$$

we deduce that if (1.1) is satisfied with  $\delta = 1$  then we have

$$(3.10) \quad \alpha h^{-d}(b_h - a_h) \leq N_h \leq \beta h^{-d}(b_h - a_h), \quad \alpha > 0, \beta > 0.$$

The main result of this section is the following lemma. It is a consequence of the work of Thangavelu [22, Lemma 3.2.2, p. 70] on Hermite functions. This was proved later Karadzhov [10] with a different method. It could also be deduced from much more general results by Koch, Tataru and Zworski [11, 12] and it is also related, after rescaling, with results obtained by Ivrii [8, Theorem 4.5.4].

**Lemma 3.5.** — Let  $d \geq 2$  and assume that  $|\mu| \leq c_0$ ,  $1 \leq p \leq +\infty$  and  $\theta \geq 0$ . Then there exists  $C > 0$  so that for all  $\lambda \geq 1$

$$\|\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)\|_{L^{p, (p-1)\theta}(\mathbb{R}^d)} \leq C \lambda^\alpha,$$

with  $\alpha = \frac{d}{2}(1 + \frac{1}{p}) - 1 + \frac{\theta}{2}(1 - \frac{1}{p})$ .

*Proof.* — Recall the following estimates proved in [10, Theorem 4]: For  $d \geq 2$  and  $x \in \mathbb{R}$

$$(3.11) \quad |\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C\lambda^{d/2-1}, \quad \lambda \geq 1, |\mu| \leq 1.$$

and for every  $\varepsilon_0 > 0$  and every  $N \geq 1$  there exists  $C_{\varepsilon_0, N}$  such that

$$(3.12) \quad \pi_H(\lambda; x, x) \leq C_{\varepsilon_0, N}|x|^{-N}, \quad \text{for } |x|^2 \geq (1 + \varepsilon_0)\lambda.$$

From (3.11) we get that for every  $C_0 > 0$  there exists  $C > 0$  such that

$$(3.13) \quad |\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C(1 + |\mu|)\lambda^{d/2-1}, \quad \lambda \geq 1, |\mu| \leq C_0\lambda.$$

Then from (3.13) and (3.12) we get that for every  $\theta \geq 0$  there exists  $C$  such that

$$(3.14) \quad |\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C(1 + |\mu|)\lambda^{d/2-1+\theta/2}\langle x \rangle^{-\theta}, \quad \lambda \geq 1, |\mu| \leq C_0\lambda.$$

Therefore, by (3.12), to get the result of Lemma 3.5, it is enough to integrate the previous inequality on  $|x| \leq c_0\lambda^{1/2}$ .  $\square$

From (3.14), we easily get an accurate estimate for the spectral function

$$e_x = \pi_H\left(\frac{b_h}{h}; x, x\right) - \pi_H\left(\frac{a_h}{h}; x, x\right).$$

**Lemma 3.6.** — Assume that (1.1) is satisfied with  $0 < \delta \leq 1$ . For any  $\theta \geq 0$  there exists  $C > 0$  such that

$$(3.15) \quad \langle x \rangle^\theta e_x \leq CN_h h^{(d-\theta)/2}.$$

Using (3.3) and interpolation inequalities we get Sobolev type inequalities for  $u \in \mathcal{E}_h$ ,  $\theta \geq 0$ ,  $p \geq 2$ .

$$(3.16) \quad \|u\|_{L^\infty, \theta/2(\mathbb{R}^d)} \leq C \left( N_h h^{(d-\theta)/2} \right)^{1/2} \|u\|_{L^2(\mathbb{R}^d)},$$

which in turn implies, by (3.1)

$$(3.17) \quad \|u\|_{L^{p, \theta(p/2-1)}(\mathbb{R}^d)} \leq C \left( N_h h^{(d-\theta)/2} \right)^{\frac{1}{2} - \frac{1}{p}} \|u\|_{L^2(\mathbb{R}^d)}.$$

By (3.10), the previous inequality can be written as

$$\|u\|_{L^{p, \theta(p/2-1)}(\mathbb{R}^d)} \leq C(b_h - a_h)^{\frac{1}{2} - \frac{1}{p}} h^{-\frac{(d+\theta)}{2}(\frac{1}{2} - \frac{1}{p})} \|u\|_{L^2(\mathbb{R}^d)}, \quad \forall p \in [2, +\infty], \forall \theta \in [0, d].$$

**Remark 3.7.** — For similar bounds for eigenfunctions or quasimodes, we refer to [12].

#### 4. Probabilistic weighted Sobolev estimates

We apply here the general probabilistic setting of Section 2 when  $K = H$  is the harmonic oscillator,  $\mathcal{K} = L^2(\mathbb{R}^d)$  and  $\{\varphi_j, j \in \mathbb{N}\}$  an orthonormal basis of Hermite functions. Recall that  $\mathbf{S}_h$  is the unit sphere of the complex Hilbert space  $\mathcal{E}_h$ , identified with  $\mathbb{C}^N$  or  $\mathbb{R}^{2N}$ , and that  $\mathbf{P}_{\gamma, h}$  is the probability on  $\mathbf{S}_h$  defined as in Section 2.

We divide this section in two parts: in the first part, under Assumption 1, we establish upper bounds and in the second part we obtain lower bounds, but only in the case of Gaussian random variables (Assumption 2), and under the condition  $0 \leq \delta < 2/3$ .

**4.1. Upper bounds.** — We suppose here that Assumption 1, (1.2) and (1.1) with  $0 \leq \delta \leq 1$  are satisfied. Our result is the following

**Theorem 4.1.** — *There exist  $h_0 \in ]0, 1]$ ,  $c_2 > 0$  and  $C > 0$  such that if  $c_1 = d(1 + d/4)$ , we have*

$$(4.1) \quad \mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : h^{-\frac{d-\theta}{4}} \|u\|_{L^\infty, \theta/2}(\mathbb{R}^d) > \Lambda \right] \leq Ch^{-c_1} e^{-c_2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in ]0, h_0].$$

*Proof.* — We adapt here the argument of [2]. To begin with, by (3.15) and (2.9), there exists  $c_2 > 0$  such that for every  $\theta \in [0, d]$ , every  $x \in \mathbb{R}^d$ , and every  $\Lambda > 0$  we have

$$(4.2) \quad \mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : \langle x \rangle^{\frac{\theta}{2}} h^{-\frac{d-\theta}{4}} |u(x)| > \Lambda \right] \leq e^{-c_2 \Lambda^2}.$$

Now, we will need a covering argument. Our configuration space is not compact but using (3.12) we have, for every  $u \in \mathbf{S}_h$ ,

$$|u(x)| \leq C_N |x|^{-N}, \quad \text{for } |x| \geq (1 + \varepsilon_0)h^{-1/2}.$$

So choosing  $R > 0$  large enough it is sufficient to estimate  $u$  inside the box  $B_{R_h} = \{x \in \mathbb{R}^d, |x|_\infty \leq Rh^{-1/2}\}$ . We divide  $B_{R_h}$  in small boxes of side with length  $\tau$  small enough. We use the gradient estimate

$$|\nabla_x u(x)| \leq Ch^{-1/2-d/4}, \quad \forall u \in \mathbf{S}_h,$$

and (4.2) at the center of each small box to get the result.

For  $x, x' \in \mathbb{R}^d$  we have

$$|\langle x \rangle^{\theta/2} u(x) - \langle x' \rangle^{\theta/2} u(x')| \leq C(\langle x \rangle^{\theta/2} |u(x) - u(x')| + \langle x \rangle^{\theta/2} |x - x'| |u(x')|).$$

Let  $\{Q_\tau\}_{\tau \in A}$  be a covering of  $B_{R_h}$  with small boxes  $Q_\tau$  with center  $x_\tau$  and side length  $\tau$  small enough. Then for every  $x \in Q_\tau$  we have

$$(4.3) \quad h^{(\theta-d)/4} |\langle x \rangle^{\theta/2} u(x) - \langle x_\tau \rangle^{\theta/2} u(x_\tau)| \leq C\tau h^{-1/2-d/4}.$$

We choose

$$(4.4) \quad \tau \approx \frac{\varepsilon \Lambda}{2C} h^{1/2+d/4}$$

and  $h_\varepsilon > 0$  such that

$$(4.5) \quad |x|_\infty > Rh^{-1/2} \Rightarrow h^{(\theta-d)/4} \langle x \rangle^{\theta/2} |u(x)| \leq \frac{\varepsilon \Lambda}{2}, \quad \forall h \in ]0, h_\varepsilon].$$

Then using (4.2), (4.3), (4.4) and (4.5) we get

$$(4.6) \quad \mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : h^{-\frac{d-\theta}{4}} \|u\|_{L^\infty, \theta/2}(\mathbb{R}^d) > \Lambda \right] \leq \#A e^{-c_2(1-\varepsilon)^2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in ]0, h_\varepsilon].$$

Using now that  $\#A \approx Ch^{-c_1}$  with  $c_1 = d(1 + d/4)$  we get (4.1) from (4.6).  $\square$

We can deduce probabilistic estimates for the derivatives as well. Recall that the Sobolev spaces  $\mathcal{W}^{s,p}(\mathbb{R}^d)$  are defined in (1.4).

**Corollary 4.2.** — For any multi index  $\alpha, \beta \in \mathbb{N}^d$  there exists  $\tilde{c}_2$  such that

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : h^{\frac{|\alpha|+|\beta|}{2} - \frac{d}{4}} \|x^\alpha \partial_x^\beta u\|_{L^\infty(\mathbb{R}^d)} > \Lambda \right] \leq Ch^{-c_1} e^{-\tilde{c}_2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in ]0, h_0].$$

In particular we have, for every  $s > 0$ ,

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : h^{\frac{s}{2} - \frac{d}{4}} \|u\|_{\mathcal{W}^{s, \infty}(\mathbb{R}^d)} > \Lambda \right] \leq Ch^{-c_1} e^{-\tilde{c}_2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in ]0, h_0].$$

*Proof.* — We apply (4.1) using that from the spectral localization of  $u \in \mathcal{E}_h$  we have

$$\|x^\alpha \partial_x^\beta u\|_{L^2(\mathbb{R}^d)} \leq Ch^{-\frac{|\alpha|+|\beta|}{2}} \|u\|_{L^2(\mathbb{R}^d)},$$

$$\|H^s u\|_{L^2(\mathbb{R}^d)} \leq Ch^{-s/2} \|u\|_{L^2(\mathbb{R}^d)}.$$

□

The following corollary shows that we get a probabilistic Sobolev estimate improving the deterministic one (3.16) with probability close to one as  $h \rightarrow 0$ . The improvement is "almost" of order  $N_h^{1/2} \approx ((b_h - a_h)h^{-d})^{1/2}$ . Choosing  $\Lambda = \sqrt{-K \log h}$  for  $K > 0$  we get

**Corollary 4.3.** — Let  $c_1, c_2 > 0$  be the constants given by Theorem 4.1. Then for every  $K > \frac{c_1}{c_2}$  we have

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : \|u\|_{L^\infty, \theta/2(\mathbb{R}^d)} > Kh^{\frac{d-\theta}{4}} |\log h|^{1/2} \right] \leq h^{Kc_2 - c_1}, \quad \forall h \in ]0, h_0], \forall \theta \in [0, d].$$

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : \|u\|_{\mathcal{W}^{s, \infty}(\mathbb{R}^d)} > Kh^{\frac{d}{4} - \frac{s}{2}} |\log h|^{1/2} \right] \leq h^{Kc_2 - c_1}, \quad \forall h \in ]0, h_0], \forall s \geq 0.$$

Let us give now an application to a probabilistic Sobolev embedding for the Harmonic oscillator. We shall use a Littlewood-Paley decomposition with  $h_j = 2^{-j}$ . Let  $\theta$  a  $C^\infty$  real function on  $\mathbb{R}$  such that  $\theta(t) = 0$  for  $t \leq a$ ,  $\theta(t) = 1$  for  $t \leq b/2$  with  $0 < a < b/2$ . Define  $\psi_{-1}(t) = 1 - \theta(t)$ ,  $\psi_j(t) = \theta(h_j t) - \theta(h_{j+1} t)$  for  $j \in \mathbb{N}$ . Notice that the support of  $\psi_j$  is in  $[\frac{a}{h_j}, \frac{b}{h_j}]$ . For every distribution  $u \in \mathcal{S}'(\mathbb{R}^d)$  we have the Littlewood-Paley decomposition

$$u = \sum_{j \geq -1} u_j, \quad \text{with} \quad u_j = \sum_{k \in \mathbb{N}} \psi_j(\lambda_k) \langle u, \varphi_k \rangle \varphi_k$$

and we have  $u_j \in \mathcal{E}_{h_j}$ .

The Besov spaces for the Harmonic are naturally defined as follows: if  $p, r \in [1, \infty]$  and  $s \in \mathbb{R}$ ,  $u \in \mathcal{B}_{p, r}^s$  if and only if

$$\|u\|_{\mathcal{B}_{p, r}^s} := \left( \sum_{j \geq -1} 2^{jsr/2} \|u_j\|_{L^p(\mathbb{R}^d)}^r \right)^{1/r} < +\infty.$$

We shall use here the spaces  $\mathcal{B}_{2, \infty}^s$ . For every  $s > 0$  we have

$$\mathcal{B}_{2, \infty}^s \subseteq L^2(\mathbb{R}^d) \subseteq \mathcal{B}_{2, \infty}^0.$$

Another scale of spaces is defined as

$$\mathcal{G}^m = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \sum_{j \geq 1} j^m \|u_j\|_{L^2(\mathbb{R}^d)} < +\infty \right\}, \quad m \geq 0.$$

Then for every  $s > 0$ ,  $m \geq 0$  we have  $\mathcal{B}_{2,\infty}^s \subseteq \mathcal{G}^m \subseteq L^2(\mathbb{R}^d)$ .

It is not difficult to see that  $\mathcal{G}^m$  can be compared with the domain in  $L^2(\mathbb{R}^d)$  of the operator  $\log^s H$ . This domain is denoted by  $\mathcal{H}_{\log}^s$ , the norm being the graph norm. For every  $s > 1/2$  we have

$$\mathcal{H}_{\log}^{m+s} \subset \mathcal{G}^m \subset \mathcal{H}_{\log}^m.$$

Notice that we do not need that the energy localizations  $\psi_j$  are smooth and we can define the same spaces with  $\psi(t) = \mathbb{I}_{[1,2]}(t)$  so that the energy intervals  $[2^j, 2^{j+1}]$  are disjoint.

Let us now define probabilities on  $\mathcal{G}^m$  as we did for Sobolev spaces  $\mathcal{H}^s$ . Let  $\gamma_j$  be a sequence of complex numbers satisfying (1.2) and such that

$$(4.7) \quad \sum_{j \geq 0} j^m |\gamma_j|_{\Lambda_j} < +\infty,$$

where  $\Lambda_j = \Lambda_{h_j}$  and

$$v_\gamma^0 = \sum_{j \geq 0} \gamma_j \varphi_j, \quad v_\gamma(\omega) = \sum_{j \geq 0} \gamma_j X_j(\omega) \varphi_j,$$

so that  $v_\gamma$  is a.s. in  $\mathcal{G}^m$  and its probability law defines a measure  $\mu_\gamma^m$  in  $\mathcal{G}^m$ . This measure satisfies also the following properties as in Proposition 2.4.

- (i) If the support of  $\nu$  is  $\mathbb{R}$  and if  $\gamma_j \neq 0$  for all  $j \geq 1$  then the support of  $\mu_\gamma^m$  is  $\mathcal{G}^m$ .
- (ii) If  $u_\gamma^0 \in \mathcal{G}^m$  and  $v_\gamma^0 \notin \mathcal{G}^s$  where  $s > m$  then  $\mu_\gamma^m(\mathcal{G}^s) = 0$ . In particular  $\mu_\gamma^m(\mathcal{H}^s) = 0$  for every  $s > 0$ .
- (iii) Under the assumptions (iii) in Proposition 2.4 we can construct singular measures  $\mu_\gamma^m$  and  $\mu_\beta^m$ .

Now we can state the following corollary of Theorem 4.1.

**Corollary 4.4.** — *Suppose that  $\gamma$  satisfies (1.2) with  $a < b$  and (4.7) with  $m = 1/2$ . Then for the measure  $\mu_\gamma^{1/2}$  almost all functions in the space  $\mathcal{G}^{1/2}$  are in the space  $\mathcal{C}_H^{[d/2]}$  where*

$$\mathcal{C}_H^\ell(\mathbb{R}^d) = \left\{ u \in \mathcal{C}^\ell(\mathbb{R}^d) : \|x^\alpha \partial_x^\beta u\|_{L^\infty(\mathbb{R}^d)} < +\infty, \quad \forall |\alpha| + |\beta| \leq \ell \right\}.$$

*In particular if  $v_\gamma^0 \in \mathcal{H}^{s_0}$ ,  $s_0 > 0$  and if  $v_\gamma^0 \notin \mathcal{H}^s$ ,  $s > s_0$ , then we have  $\mu_\gamma^{1/2}(\mathcal{B}_{2,\infty}^\sigma) = 1$  for every  $\sigma > 0$  and we have an a.s. embedding of the Besov space  $\mathcal{B}_{2,\infty}^\sigma$  in  $\mathcal{C}_H^{[d/2]}$ .*

*Proof.* — Let  $u = \sum_{n \geq -1} u_n \in \mathcal{G}^{1/2}$  with  $u_n \in \mathcal{E}_{h_n}$ . For  $\kappa > 0$  (chosen large enough) denote by

$$B_n^\kappa = \left\{ v \in \mathcal{E}_{h_n} : \|x^\alpha \partial_x^\beta v\|_{L^\infty(\mathbb{R}^d)} \leq \kappa \sqrt{h_n} \|v\|_{L^2(\mathbb{R}^d)}, \quad \forall |\alpha| + |\beta| \leq [d/2] \right\}.$$

We have, using Corollary 4.2

$$\nu_{\gamma,n}(B_n^\kappa) \geq 1 - e^{-n(c_2\kappa^2 - c_1)}.$$

So if  $B^\kappa = \{u \in \mathcal{G}^{1/2} : u_0 \in \mathcal{E}_{h_0}, u_n \in B_n^\kappa, \forall n \geq 1\}$ , then we have

$$\mu_\gamma^{1/2}(B^\kappa) \geq \prod_{n \geq 1} (1 - e^{-n(c_2\kappa^2 - c_1)}) \geq 1 - \varepsilon(\kappa)$$

with  $\lim_{\kappa \rightarrow +\infty} \varepsilon(\kappa) = 0$ . More precisely we have  $\varepsilon(\kappa) \approx e^{-c\kappa^2}$  for some  $c > 0$ .

Now if  $u \in B^\kappa$  we have

$$\|x^\alpha \partial_x^\beta u\|_{L^\infty(\mathbb{R}^d)} \leq \sum_{n \geq -1} \|x^\alpha \partial_x^\beta u_n\|_{L^\infty(\mathbb{R}^d)} \leq \kappa \sum_{n \geq -1} \sqrt{n} \|u_n\|_{L^2(\mathbb{R}^d)} := \kappa \|u\|_{\mathcal{G}^{1/2}}.$$

So the corollary is proved.  $\square$

**Remark 4.5.** — In the last corollary, for every  $s > 0$  we can choose  $\gamma$  such that  $\mu_\gamma^{1/2}(\mathcal{H}^s) = 0$ . So the smoothing property is a probabilistic effect similar to the Khinchin inequality.

From the proof we get a more quantitative statement. There exists  $c > 0$  such that

$$\mu_\gamma^{1/2} \left[ \|u\|_{\mathcal{W}^{d/2,\infty}} \geq \kappa \|u\|_{\mathcal{G}^{1/2}} \right] \leq e^{-c\kappa^2}.$$

**Remark 4.6.** — The proof of the corollary depends on the squeezing assumption (1.2) on  $\gamma$ . For example if (1.2) is satisfied for  $b_h - a_h \approx h$  then we can consider the energy decomposition in intervals  $[2n, 2(n+1)[$  instead of the dyadic decomposition. So when applying Theorem 4.1 with  $h$  of order  $\frac{1}{n}$  we get  $h^{-c_1} e^{-c_2 \Lambda^2} = e^{c_1 \log n - c_2 \Lambda^2}$ .

Then taking  $\Lambda = \kappa \sqrt{\log n}$  with  $\kappa$  large enough, in the construction of  $B_n^\kappa$  we have to replace  $\sqrt{n}$  by  $\sqrt{\log n}$ . In the conclusion the space  $\mathcal{G}^{1/2}$  is replaced by  $\tilde{\mathcal{G}}^{1/2}$  where

$$\tilde{\mathcal{G}}^m = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \sum_{j \geq 1} \log^m j \|u_j\|_{L^2(\mathbb{R}^d)} < +\infty \right\}, \quad u_j := \sum_{2j \leq \lambda_n < 2(j+1)} \langle u, \varphi_j \rangle \varphi_j.$$

**4.2. Lower bounds in the case of Gaussian random variables.** — Here we suppose that the stronger Assumption 2 and (1.1) with  $\delta < 2/3$  are satisfied. We are interested to get a lower bound for  $\|u\|_{L^\infty, \theta/2(\mathbb{R}^d)}$ .

The spectral condition  $\delta < 2/3$  is needed here because it seems difficult to estimate from below the variations of the spectral function of the harmonic oscillator in intervals of length  $\leq h^{-1/3}$ .

A first step is to get two sides weighted  $L^r$  estimates for large  $r$  which is a probabilistic improvement of (3.17). Denote by

$$(4.8) \quad \beta_{r,\theta} = \frac{d-\theta}{2} \left(1 - \frac{2}{r}\right).$$

**Theorem 4.7.** — Assume that  $\theta \in [0, d]$ , and denote by  $\mathcal{M}_r$  a median of  $\|u\|_{L^{r,\theta}(r/2-1)}$ . Then there exist  $0 < C_0 < C_1$ ,  $K > 0$ ,  $c_1 > 0$ ,  $h_0 > 0$  such that for all  $r \in [2, K|\log h|]$  and  $h \in ]0, h_0]$  such that

$$(4.9) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \left| \|u\|_{L^{r,\theta}(r/2-1)} - \mathcal{M}_r \right| > \Lambda \right] \leq 2 \exp \left( -c_2 N_h^{2/r} h^{-\beta_{r,\theta}} \Lambda^2 \right).$$

and where

$$C_0\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \leq \mathcal{M}_r \leq C_1\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})}, \quad \forall r \in [2, K \log N].$$

This result shows that  $\|u\|_{L^{r,\theta(r/2-1)}}$  has a Gaussian concentration around its median.

From (4.9) we deduce that for every  $\kappa \in ]0, 1[$ ,  $K > 0$ , there exist  $0 < C_0 < C_1$ ,  $c_1 > 0$ ,  $h_0 > 0$  such that for all  $r \in [2, K|\log h|^\kappa]$ ,  $h \in ]0, h_0]$  and  $\Lambda > 0$  we have

$$\mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : C_0\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \leq \|u\|_{L^{r,\theta(r/2-1)}} \leq C_1\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \right] \geq 1 - e^{-c_1|\log h|^{1-\kappa}}.$$

As a consequence of Theorem 4.7, for every  $\theta \in [0, d]$  we get a two sides weighted  $L^\infty$  estimate showing that Theorem 4.1 and its corollary are sharp.

**Corollary 4.8.** — *After a slight modification of the constants in Theorem 4.7, if necessary, we get that for all  $\theta \in [0, d]$  and  $h \in ]0, h_0]$*

$$(4.10) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : C_0|\log h|^{1/2}h^{(d-\theta)/4} \leq \|u\|_{L^\infty,\theta/2} \leq C_1|\log h|^{1/2}h^{(d-\theta)/4} \right] \geq 1 - h^{c_1}.$$

To prove these results we have to adapt to the unbounded configuration space  $\mathbb{R}^d$  the proofs of [2, Theorems 4 and 5] which hold for compact manifolds. The concentration result stated in Proposition 2.12 will prove useful.

*Proof of Theorem 4.7.* — Denote by  $F_r(u) = \|u\|_{L^{r,\theta(r/2-1)}}$  and by  $\mathcal{M}_r$  its median. Thanks to (3.17) we have the Lipschitz estimate

$$|F_r(u) - F_r(v)| \leq C \left( N_h h^{\frac{d-\theta}{2}} \right)^{\frac{1}{2}-\frac{1}{r}} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h.$$

Therefore, by (2.16) and (4.8), we have for some  $c_2 > 0$

$$(4.11) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |F_r(u) - \mathcal{M}_r| > \Lambda \right] \leq 2 \exp \left( -c_2 N_h^{2/r} h^{-\beta_{r,\theta}} \Lambda^2 \right).$$

The next step is to estimate  $\mathcal{M}_r$ . Denote by  $\mathcal{A}_r^r = \mathbf{E}_h(F_r^r)$  the moment of order  $r$  and compute, with  $s = \theta(r/2 - 1)$ ,

$$\begin{aligned} \mathcal{A}_r^r &= \mathbf{E}_h \left( \int_{\mathbb{R}^d} \langle x \rangle^s |u(x)|^r dx \right) \\ &= r \int_{\mathbb{R}^d} \langle x \rangle^s \left( \int_0^{+\infty} s^{r-1} \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |u(x)| > s \right] ds \right) dx. \end{aligned}$$

Thus by (2.11) we get

$$C_1 r \int_{\mathbb{R}^d} \langle x \rangle^s \left( \int_0^{\varepsilon_0 \sqrt{e_x}} s^{r-1} e^{-c_1 \frac{N}{e_x} s^2} ds \right) dx \leq \mathcal{A}_r^r \leq C_2 r \int_{\mathbb{R}^d} \langle x \rangle^s \left( \int_0^{+\infty} s^{r-1} e^{-c_2 \frac{N}{e_x} s^2} ds \right) dx.$$

Performing the change of variables  $t = c_j \frac{N}{e_x} s^2$  we obtain that there exist  $C_1, C_2 > 1$  such that

$$(4.12) \quad C_1 r (c_1 N)^{-r/2} \left( \int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \int_0^{\varepsilon N} t^{r/2-1} e^{-t} dt \leq \mathcal{A}_r^r \leq C_2 r (c_2 N)^{-r/2} \left( \int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \Gamma(r/2),$$

with  $\varepsilon = c_1 \varepsilon_0^2$ . We need to estimate the term  $\int_0^{\varepsilon N} t^{r/2-1} e^{-t} dt$  from below. Using the elementary estimate

$$\int_T^{+\infty} t^{r/2-1} e^{-t} dt \leq T^{r/2} e^{1-T} \Gamma(r/2), \quad T \geq 1,$$

we get that there exists  $\varepsilon_1 > 0$  such that for  $N$  large and  $r \leq \varepsilon_1 \frac{N}{\log N}$  then we have

$$\int_0^{\varepsilon N} t^{r/2-1} e^{-t} dt \geq \frac{\Gamma(r/2)}{2}.$$

So we get the expected lower bound,  $\forall r \in [1, \varepsilon_1 \frac{N}{\log N}]$ ,

$$e^{-r/2} C^{-1} r \left( \int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) N^{-r/2} \Gamma(r/2) \leq \mathcal{A}_r \leq C_2 r N^{-r/2} \left( \int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \Gamma(r/2).$$

and where  $\Gamma(r/2)$  can be estimated thanks to the Stirling formula: there exist  $0 < C_0 < C_1$  such that

$$(C_0 r)^{r/2} \leq \Gamma(r/2) \leq (C_1 r)^{r/2}, \quad \forall r \geq 1.$$

Now we need the following lemma which will be proven in Appendix B. The upper bound can be seen as an application of Lemma 3.5 with  $\lambda = h^{-1}$  and  $\mu = (b_h - a_h) h^{-1} \sim N_h h^{d-1}$ .

**Lemma 4.9.** — *Assume that  $\theta > -d/(p-1)$ . Then there exist  $0 < C_0 < C_1$  and  $h_0 > 0$  such that*

$$C_0 N_h h^{\beta_{2p,\theta}} \leq \left( \int_{\mathbb{R}^d} \langle x \rangle^{\theta(p-1)} e_x^p dx \right)^{1/p} \leq C_1 N_h h^{\beta_{2p,\theta}},$$

for every  $p \in [1, \infty[$  and  $h \in ]0, h_0]$  where  $\beta_{r,\theta} = \frac{d-\theta}{2} (1 - \frac{2}{r})$ .

From this lemma we get

$$(4.13) \quad C_0 \sqrt{r h^{\beta_{r,\theta}}} \leq \mathcal{A}_r \leq C_1 \sqrt{r h^{\beta_{r,\theta}}}, \quad \forall r \geq 2, h \in ]0, h_0].$$

Now we have to compare  $\mathcal{A}_r$  and the median  $\mathcal{M}_r$ . We have

$$\begin{aligned} |\mathcal{A}_r - \mathcal{M}_r|^r &= \left| \|F_r\|_{L^r(\mathbf{s}_h)} - \|\mathcal{M}_r\|_{L^r(\mathbf{s}_h)} \right|^r \\ &\leq \|F_r - \mathcal{M}_r\|_{L^r(\mathbf{s}_h)}^r = r \int_0^\infty s^{r-1} \mathbf{P}_{\gamma,h} [ |F_r - \mathcal{M}_r| > s ] ds. \end{aligned}$$

Then using the large deviation estimate (4.11) we get

$$|\mathcal{A}_r - \mathcal{M}_r| \leq C N^{-1/r} \sqrt{r h^{\beta_{r,\theta}}}, \quad \forall r \geq 2.$$

Choosing  $r \leq K \log N$ , ( $K < 1$ ) and  $N$  large, from (4.13) we obtain

$$(4.14) \quad C_0 \sqrt{r h^{\beta_{r,\theta}}} \leq \mathcal{M}_r \leq C_1 \sqrt{r h^{\beta_{r,\theta}}}, \quad \forall r \in [2, K \log N]$$

and the proof of Theorem 4.7 follows using (4.14) and (4.11).  $\square$

**Remark 4.10.** — The upper-bound in Lemma 4.9 is true for  $\delta = 1$ . This is proved in Appendix B.

Now let us prove Corollary 4.8.



*Proof of Corollary 4.8.* — For simplicity we assume that  $\theta = d$ . Using (4.1) it is enough to prove that there exist  $C_0 > 0$ ,  $h_0 > 0$ ,  $c_1 > 0$  such that

$$(4.15) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \|u\|_{L^\infty,d/2} \leq C_0 |\log h|^{1/2} \right] \leq h^{c_1}, \quad \forall h \in ]0, h_0].$$

Let  $u \in \mathbf{S}_h$ , then by (3.2) we have the interpolation inequality

$$\|u\|_{L^{r,d(r/2-1)}(\mathbb{R}^d)} \leq \|u\|_{L^\infty,d/2}^{1-2/r}.$$

So we get

$$\mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \|u\|_{L^\infty,d/2} \leq C_0 |\log h|^{1/2} \right] \leq \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \|u\|_{L^{r,d(r/2-1)}} \leq \left( C_0 |\log h|^{1/2} \right)^{1-2/r} \right],$$

and choosing  $r = r_h = \varepsilon_0 |\log h|$  we obtain

$$\mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \|u\|_{L^\infty,d/2} \leq C_0 |\log h|^{1/2} \right] \leq \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \|u\|_{L^{r_h,d(r_h/2-1)}} \leq \left( \frac{C_0}{\sqrt{\varepsilon_0}} r_h^{1/2} \right)^{1-2/r_h} \right].$$

Then choosing  $h_0 > 0$ ,  $\frac{C_0}{\sqrt{\varepsilon_0}}$  small enough and  $\Lambda = c |\log h|^{1/2}$  we can conclude that (4.15) is satisfied using (4.9).  $\square$

**Remark 4.11.** — Concerning the mean  $\mathcal{M}_\infty$  of  $F_\infty(u) := \|u\|_{L^\infty,d/2}$  it results from Corollary 4.8, (4.1) and (3.16) that we have the two sides estimates

$$C_0 |\log h|^{1/2} \leq \mathcal{M}_\infty \leq C_1 |\log h|^{1/2}, \quad \forall h \in ]0, h_0].$$

It is not difficult to adapt the proof of (4.9) and (4.10) for the Sobolev norms  $\|u\|_{\mathcal{W}^{s,p}(\mathbb{R}^d)}$ . It is enough to remark that considering  $L_s u(x) = H^{s/2} u(x)$  we have

$$e_{L_s} = \sum_{j \in \Lambda} \lambda_j^s \varphi_j^2(x).$$

But for  $j \in \Lambda$ ,  $\lambda_j$  is of order  $h^{-1}$  hence there exists  $C > 0$  such that

$$C^{-1} h^{-s} e_x \leq e_{L_s} \leq C h^{-s} e_x.$$

Using this property we easily get the next result, which in particular implies Theorem 1.1. Let  $\mathcal{M}_{r,s}$  be the median of  $u \mapsto \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)}$ , and recall the definition (4.8). Then

**Theorem 4.12.** — *Let  $s \geq 0$ . There exist  $0 < C_0 < C_1$ ,  $K > 0$ ,  $c_1 > 0$ ,  $h_0 > 0$  such that for all  $r \in [2, K |\log h|]$  and  $h \in ]0, h_0]$*

$$(4.16) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : \left| \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} - \mathcal{M}_{r,s} \right| > \Lambda \right] \leq 2 \exp \left( -c_2 N_h^{2/r} h^{-\beta_{r,0+s}} \Lambda^2 \right).$$

where

$$C_0 \sqrt{r} h^{\frac{\beta_{r,0-s}}{2}} \leq \mathcal{M}_{r,s} \leq C_1 \sqrt{r} h^{\frac{\beta_{r,0-s}}{2}}, \quad \forall r \in [2, K \log N].$$

In particular, for every  $\kappa \in ]0, 1[$ ,  $K > 0$ , there exist  $C_0 > 0$ ,  $C_1 > 0$ ,  $c_1 > 0$  such that for every  $r \in [2, K|\log h|^\kappa]$  we have

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : C_0 \sqrt{r} h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \leq \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \leq C_1 \sqrt{r} h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \right] \geq 1 - e^{-c_1 |\log h|^{1-\kappa}},$$

For  $r = +\infty$  we have for all  $h \in ]0, h_0]$

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : C_0 |\log h|^{1/2} h^{\frac{d-2s}{4}} \leq \|u\|_{\mathcal{W}^{s,\infty}(\mathbb{R}^d)} \leq C_1 |\log h|^{1/2} h^{\frac{d-2s}{4}} \right] \geq 1 - h^{c_1}.$$

Namely,

$$\|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \approx h^{-s/2} \|u\|_{L^{r,0}(\mathbb{R}^d)} + \|u\|_{L^{r,s}(\mathbb{R}^d)},$$

and

$$h^{-s/2} \|u\|_{L^{r,0}(\mathbb{R}^d)} \sim h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}}, \quad \|u\|_{L^{r,s}(\mathbb{R}^d)} \sim h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2r}}.$$

**4.3. Lower bounds in the general case.** — Under Assumption 1, we prove a weaker version of Theorem 4.12.

**Theorem 4.13.** — Suppose that Assumption 1 is satisfied. Let  $s \geq 0$ ,  $\kappa \in ]0, 1[$ ,  $K > 0$ . There exist  $0 < C_0 < C_1$ ,  $K > 0$ ,  $c_1 > 0$ ,  $h_0 > 0$  such that for all  $r \in [2, K|\log h|^\kappa]$  and  $h \in ]0, h_0]$

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : C_0 h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \leq \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \leq C_1 \sqrt{r} h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \right] \geq 1 - e^{-c_1 |\log h|^{1-\kappa}},$$

For  $r = +\infty$  we have for all  $h \in ]0, h_0]$

$$\mathbf{P}_{\gamma, h} \left[ u \in \mathbf{S}_h : C_0 h^{\frac{d-2s}{4}} \leq \|u\|_{\mathcal{W}^{s,\infty}(\mathbb{R}^d)} \leq C_1 |\log h|^{1/2} h^{\frac{d-2s}{4}} \right] \geq 1 - h^{c_1}.$$

Therefore, we have optimal constants in the control of the  $\mathcal{W}^{s,r}(\mathbb{R}^d)$  norms when  $r < +\infty$  and for general random variables which satisfy the concentration property, but when  $r = +\infty$  we lose the factor  $|\log h|^{1/2}$  in the lower bound.

*Proof.* — We can follow the main lines of the proof of Theorem 4.12. Here compared to (4.12) we get

$$\begin{aligned} \mathcal{A}_r^r &\geq C r N^{-r/2} \left( \int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \int_0^\varepsilon t^{r/2-1} e^{-t} dt \\ &\geq C N^{-r/2} \left( \int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \varepsilon^{r/2}, \end{aligned}$$

and this explains the loss of the factor  $\sqrt{r}$ . □

**4.4. Global probabilistic  $L^p$ -Sobolev estimates.** — Here we extend the  $L^\infty$ - random estimates obtained before to the  $L^r$ -spaces for any real  $r \geq 2$ , and we prove Theorem 1.2. Let us recall the definition (1.5) of the Besov spaces, where we use the notations of Subsection 4.1 for the dyadic Littlewood-Paley decomposition.

*Proof of Theorem 1.2.* — Recall that for every  $\sigma > m$  we can choose  $\gamma$  such that  $\mu_\gamma(\mathcal{H}^\sigma) = 0$ . Denote by  $F_{r,s}(u) = \|u\|_{\mathcal{W}^{s,r}}$ . The Lipschitz norm of  $F_{r,s}$  satisfies

$$\|F_{r,s}\|_{Lip} \leq Ch^{-s+d(\frac{1}{2}-\frac{1}{r})} N_h^{\frac{1}{2}-\frac{1}{r}}.$$

Let us denote by  $\mathcal{M}_{r,s}$  the median of  $F_{r,s}$  on the sphere  $\mathbf{S}_h$  for the probability  $\mathbf{P}_{\gamma,h}$  and by  $\mathcal{A}_{r,s}^r$  the mean of  $F_{r,s}^r$ . From Proposition 2.12 we have, for some  $0 < c_0 < c_1$ ,

$$(4.17) \quad \mathbf{P}_{\gamma,h} \left[ u \in \mathbf{S}_h : |F_{r,s} - \mathcal{M}_{r,s}| > K \right] \leq \exp \left( -c_1 N \frac{K^2}{\|F_{r,s}\|_{Lip}^2} \right) \leq \exp \left( -c_0 N^{1/r} K^2 \right).$$

With the same computations as for (4.14) we get

$$(4.18) \quad \mathcal{A}_{r,s} \approx \sqrt{r} \quad \text{and} \quad |\mathcal{A}_{r,s} - \mathcal{M}_{r,s}| \lesssim \sqrt{r} N^{-1/r}.$$

These formulas are obtained from (2.9) applied to the linear form  $L_s u := H^s u(x)$  noticing that

$$e_{L_s} = \sum_{j \in \Lambda_h} |H^s \varphi_j(x)|^2 \approx h^{-2s} e_x.$$

Then taking  $c_0 > 0$  small enough that we have

$$(4.19) \quad \nu_{\gamma,h} \left[ v \in \mathcal{E}_h : \|v\|_{\mathcal{W}^{s,r}} \geq K \|v\|_{L^2(\mathbb{R}^d)} \right] \leq \exp \left( -c_0 N^{2/r} K^2 \right), \quad \forall K \geq 1.$$

Then from (4.19) we proceed as for the proof of Corollary 4.4. For simplicity we consider here the usual Littlewood-Paley decomposition. Then we have  $N^{2/r} \approx 2^{2nd/r}$ . So the end of the proof follows by considering

$$B_n^K = \{v \in \mathcal{E}_n : \|v\|_{\mathcal{W}^{s,r}} \leq K \|v\|_{L^2(\mathbb{R}^d)}\}.$$

So for a fixed  $r \geq 2$  we infer (1.6) from (4.17) and (4.18), taking  $c_0 > 0$  small enough, we get

$$\mu_\gamma \left( \prod_{n \geq 0} B_n^K \right) \geq 1 - e^{-c_0 K^2}.$$

□

Using the isometry  $u \mapsto H^{-m/2}u$  between  $\mathcal{B}_{2,1}^s$  and  $\mathcal{B}_{2,1}^{m+s}$  for all real  $m \geq 0$ , we can get the following corollary to Theorem 1.2.

**Corollary 4.14.** — *Let  $m \geq 0$  and assume that  $\gamma$  satisfies (1.2) and*

$$\sum_{n \geq 0} 2^{nm} |\gamma|_n < +\infty.$$

*Then for  $s = d(\frac{1}{2} - \frac{1}{r}) + m$  and  $r \geq 2$ , we have*

$$\mu_\gamma \left[ u \in \mathcal{B}_{2,1}^m : \|u\|_{\mathcal{W}^{m+s,r}} \geq K \|u\|_{\mathcal{B}_{2,1}^m} \right] \leq e^{-c_0 K^2}.$$

### 5. Application to Hermite functions

We turn to the proof of Theorem 1.3 and we can follow the main lines of [2, Section 3]. We use here the upper bounds estimates of Section 4.1 in their full strength. Firstly, we assume that for all  $j \in \Lambda_h$ ,  $\gamma_j = N_h^{-1/2}$  and that  $X_j \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ , so that  $\mathbf{P}_h := \mathbf{P}_{\gamma, h}$  is the uniform probability on  $\mathbf{S}_h$ . We set  $h_k = 1/k$  with  $k \in \mathbb{N}^*$ , and

$$a_{h_k} = 2 + dh_k, \quad b_{h_k} = 2 + (2 + d)h_k.$$

Then (1.1) is satisfied with  $\delta = 1$  and  $D = 2$ . In particular, each interval

$$I_{h_k} = \left[ \frac{a_{h_k}}{h_k}, \frac{b_{h_k}}{h_k} \right] = [2k + d, 2k + d + 2[$$

only contains the eigenvalue  $\lambda_k = 2k + d$  with multiplicity  $N_{h_k} \sim ck^{d-1}$ , and  $\mathcal{E}_{h_k}$  is the corresponding eigenspace of the harmonic oscillator  $H$ . We can identify the space of the orthonormal basis of  $\mathcal{E}_{h_k}$  with the unitary group  $U(N_{h_k})$  and we endow  $U(N_{h_k})$  with its Haar probability measure  $\rho_k$ . Then the space  $\mathcal{B}$  of the Hilbertian bases of eigenfunctions of  $H$  in  $L^2(\mathbb{R}^d)$  can be identified with

$$\mathcal{B} = \times_{k \in \mathbb{N}} U(N_{h_k}),$$

which can be endowed with the measure

$$d\rho = \otimes_{k \in \mathbb{N}} d\rho_k.$$

Denote by  $B = (\varphi_{k, \ell})_{k \in \mathbb{N}, \ell \in \llbracket 1, N_{h_k} \rrbracket} \in \mathcal{B}$  a typical orthonormal basis of  $L^2(\mathbb{R}^d)$  so that for all  $k \in \mathbb{N}$ ,  $(\varphi_{k, \ell})_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k})$  is an orthonormal basis of  $\mathcal{E}_{h_k}$ .

Then the main result of the section is the following, which implies Theorem 1.3.

**Theorem 5.1.** — *Let  $d \geq 2$ . Then, if  $M > 0$  is large enough, there exist  $c, C > 0$  so that for all  $r > 0$*

$$\rho \left[ B = (\varphi_{k, \ell})_{k \in \mathbb{N}, \ell \in \llbracket 1, N_{h_k} \rrbracket} \in \mathcal{B} : \exists k, \ell; \|\varphi_{k, \ell}\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \leq Ce^{-cr^2}.$$

We will need the following result

**Proposition 5.2.** — *Let  $d \geq 2$ . Then, if  $M > 0$  is large enough, there exist  $c, C > 0$  so that for all  $r > 0$  and  $k \geq 1$*

$$(5.1) \quad \rho_k \left[ B_k = (\psi_{\ell})_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k}) : \exists \ell \in \llbracket 1, N_{h_k} \rrbracket; \|\psi_{\ell}\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \leq Ck^{-2}e^{-cr^2}.$$

*Proof.* — The proof is similar to the proof of [2, Proposition 3.2]. We observe that for any  $\ell_0 \in \llbracket 1, N_{h_k} \rrbracket$ , the measure  $\rho_k$  is the image measure of  $\mathbf{P}_{h_k}$  under the map

$$U(N_{h_k}) \ni B_k = (\psi_{\ell})_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \longmapsto \psi_{\ell_0} \in \mathbf{S}_{h_k}.$$

Then we use that  $\mathbf{S}_{h_k} \subset \mathcal{E}_{h_k}$  is an eigenspace and by Theorem 4.1 we obtain that for all  $\ell_0 \in \llbracket 1, N_{h_k} \rrbracket$

$$\begin{aligned} \rho_k \left[ B_k = (\psi_\ell)_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k}) : \|\psi_{\ell_0}\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \\ = \mathbf{P}_{h_k} \left[ u \in \mathbf{S}_{h_k} : \|u\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \\ = \mathbf{P}_{h_k} \left[ u \in \mathbf{S}_{h_k} : k^{d/4} \|u\|_{L^{\infty, 0}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \\ \leq C k^{c_1 - M^2 c_2} e^{-c_2 r^2}, \end{aligned}$$

where  $c_1, c_2 > 0$  are given by Theorem 4.1. As a consequence, (5.1) is bounded by  $C k^c e^{-c_2 r^2}$ , with  $c = c_1 - M^2 c_2 + d - 1$  which implies the result.  $\square$

*Proof of Theorem 5.1.* — We set

$$\mathcal{F}_{k,r} = \left\{ B_k = (\psi_\ell)_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k}) : \forall \ell \in \llbracket 1, N_{h_k} \rrbracket; \|\psi_\ell\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \leq M(\log k)^{1/2} + r \right\},$$

and  $\mathcal{F}_r = \bigcap_{k \geq 1} \mathcal{F}_{k,r}$ . Then for all  $r > 0$

$$\rho(\mathcal{F}_r^c) \leq \sum_{k \geq 1} \rho_k(\mathcal{F}_{k,r}^c) \leq C \sum_{k \geq 1} k^{-2} e^{-cr^2} = C' e^{-cr^2},$$

and this completes the proof.  $\square$

We have the following consequence of the previous results.

**Corollary 5.3.** — *For  $\rho$ -almost all orthonormal basis  $(\varphi_{k,\ell})_{k \in \mathbb{N}, \ell \in \llbracket 1, N_{h_k} \rrbracket}$  of eigenfunctions of  $H$  we have*

$$\|\varphi_{k,\ell}\|_{L^\infty(\mathbb{R}^d)} \leq (M+1)k^{-\frac{d}{4}}(1 + \log k)^{1/2}, \quad \forall k \in \mathbb{N}, \forall \ell \in \llbracket 1, N_{h_k} \rrbracket.$$

*Proof.* — Apply (5.1) with  $r = (\log k)^{1/2}$  and denote, for  $k \geq 2$ ,  $\Omega_k$  the event

$$\Omega_k = \left\{ B = (\varphi_{k,\ell}), \exists \ell \in \llbracket 1, N_{h_k} \rrbracket, \|\varphi_{k,\ell}\|_{L^\infty(\mathbb{R}^d)} \geq (M+1)k^{-d/4}(\log k)^{1/2} \right\}.$$

We have  $\rho(\Omega_k) \leq \frac{C}{k^2}$ . Therefore from the Borel-Cantelli Lemma we have  $\rho[\limsup \Omega_k] = 0$  and this gives the corollary.  $\square$

## Appendix A

### Proof of Proposition 2.4 (iii)

*Proof.* — Denote by  $f_\gamma(x) = \frac{c_\alpha}{\gamma} e^{-(\frac{|x|}{\gamma})^\alpha}$ ,  $\gamma > 0$ . We have, with obvious identifications,

$$\mu_\gamma = \otimes_{j \geq 0} (f_{\gamma_j} dx).$$

Denote by

$$\pi_j = \int_{\mathbb{R}} \left( \frac{f_{\gamma_j}}{f_{\beta_j}} \right)^{1/2} f_{\beta_j} dx.$$

According to the main result of [9] the measures  $\mu_\gamma$  and  $\mu_\beta$  are mutually singular if the infinite product  $\prod_{j \geq 0} \pi_j$  is divergent. From elementary computations we get

$$\pi_j = \left( \frac{1}{2} \left( \frac{\gamma_j}{\beta_j} \right)^{\alpha/2} + \frac{1}{2} \left( \frac{\beta_j}{\gamma_j} \right)^{\alpha/2} \right)^{-1/\alpha}.$$

- If  $\pi_j$  has not 1 as limit then the product is divergent.
- If  $\pi_j$  has 1 as limit then the infinite product is divergent if  $\sum_{j \geq 0} (\pi_j^{-\alpha} - 1) = +\infty$ . So, using that

$$\frac{1}{2} \left( x + \frac{1}{x} \right) = 1 + \frac{1}{2} (1 - x)^2 + \mathcal{O}(1 - x)^3,$$

we see that the infinite product is divergent if (2.6) is satisfied.  $\square$

## Appendix B

### $L^p$ weighted spectral estimates for the Harmonic oscillator

Our goal here is to give a self-contained proof of Lemma 4.9. It could be proved using the semi-classical functional calculus for pseudo-differential operators [16], but for the harmonic oscillator it is possible to use the exact Mehler formula and elementary properties of Hermite functions to get the result.

**B.1. A functional calculus with parameter for the Harmonic oscillator.** — The starting point is the inverse Fourier transform

$$f(H) = \frac{1}{2\pi} \int e^{itH} \hat{f}(t) dt,$$

where  $f$  is in the Schwartz space  $\mathcal{S}(\mathbb{R})$ .

We want estimates for the integral kernel  $K_f(x, y)$  of  $f(H)$ . To do that it is convenient to first compute the Weyl symbol  $W_{f(H)}(x, \xi)$  of  $f(H)$  and use that

$$K_f(x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} W_{f(H)}\left(\frac{x+y}{2}, \xi\right) e^{i(x-y) \cdot \xi} d\xi.$$

For basic properties about the Weyl calculus see for example [16]. The unitary operator  $e^{itH}$  has an explicit Weyl symbol  $w(t, x, \xi)$  :

$$(B.1) \quad w(t, x, \xi) = \frac{1}{(\cos t)^d} e^{i \tan t (|x|^2 + |\xi|^2)}, \quad \text{for } |t| < \frac{\pi}{2}.$$

Formula (B.1) can be easily proved from the Mehler formula (3.7) and also directly (see [16, Exercise IV]).

Let us introduce a cutoff  $\chi \in C^\infty(\mathbb{R})$ ,  $\chi(t) = 1$  for  $|t| < \varepsilon_0$ ,  $\chi(t) = 0$  for  $|t| > 2\varepsilon_0$  with  $0 < \varepsilon_0 < \pi/4$ . Denote by

$$R_f = \frac{1}{2\pi} \int e^{itH} (1 - \chi(t)) \hat{f}(t) dt$$

and

$$\tilde{W}_f(x, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} w(t, x, \xi) \chi(t) \hat{f}(t) dt.$$

We apply these formulas to give estimates with  $f_h(s) = f(hs)$  where  $h > 0$  is a small parameter. We begin with an estimate for the remainder term for the kernel  $KR_{f_h}(x, y)$  of the operator  $R_{f_h}$ .

**Lemma B.1.** — *There exists  $M_0 > 0$  such that for every  $M \geq 1$  there exists  $C_M > 0$  such that*

$$(B.2) \quad |KR_{f_h}(x, y)| \leq C_M h^M (\langle x \rangle \langle y \rangle)^{-\frac{M}{2} - d - 1} \|f\|_{M+M_0}, \quad \forall h \in ]0, 1], \quad \forall x, y \in \mathbb{R}^d,$$

$$\text{where } \|f\|_m = \sup_{j+k \leq m, t \in \mathbb{R}} |t^j \frac{d^k}{dt^k} \hat{f}(t)|.$$

*Proof.* — Denote by  $\hat{g}_h(t) = (1 - \chi(t)) \hat{f}(\frac{t}{h})$ . So we have  $R_{f,h} = g_h(H)$  and for every  $M, M' \geq 1$ ,

$$|\mu^M g_h(\mu)| \leq \int_{|t| \geq \varepsilon_0} \left| \frac{d^M}{dt^M} \hat{g}_h(t) \right| dt \leq C_{M, M'}(f) h^{M'}.$$

So we have

$$|KR_{f_h}(x, y)| = \left| \sum_j g_h(\lambda_j) \varphi_j(x) \overline{\varphi_j(y)} \right| \leq Ch^M \left( \sum_j \lambda_j^{-M} |\varphi_j(x)|^2 \right)^{1/2} \left( \sum_j \lambda_j^{-M} |\varphi_j(y)|^2 \right)^{1/2}.$$

Recall the Sobolev estimate in the harmonic spaces: for every  $s > \frac{d}{2} + r$  there exists  $C = C_{sr}$  such that

$$\langle x \rangle^r |u(x)| \leq C \|u\|_{\mathcal{H}^s}, \quad \forall u \in \mathcal{H}^s(\mathbb{R}^d).$$

So we get, for  $s > \frac{d}{2} + r$ ,

$$|KR_{f_h}(x, y)| \leq Ch^M (\langle x \rangle \langle y \rangle)^{-r} \sum_j \lambda_j^{s-M}.$$

Using that  $\lambda_j \approx j^{1/d}$  and choosing  $r = \frac{M}{2} + d + 1$  we get (B.2).  $\square$

Our aim is to estimate the kernel of  $f\left(\frac{H-\nu\lambda}{\mu}\right)$  for large  $\lambda$ ,  $|\mu| \geq D\lambda^{1-\delta}$  where  $D > 0$  and  $\delta < 2/3$ . The parameter  $\nu$  is fixed in an interval  $[\nu_0, \nu_1]$ , where  $0 < \nu_0 < \nu_1$ . All our estimates will be uniform in  $\nu$ , so for convenience we shall take  $\nu = 1$ .

Denote by  $g_{\lambda, \mu}(s) = f\left(\frac{s-\lambda}{\mu}\right)$  so we have  $\hat{g}_{\lambda, \mu}(t) = \mu e^{-it\lambda} \hat{f}(\mu t)$ . We consider the dilated Weyl symbol:  $W_{\lambda, \mu}(x, \xi) = \tilde{W}_{g_{\lambda, \mu}}(\sqrt{\lambda}x, \sqrt{\lambda}\xi)$ . Then we have

$$(B.3) \quad W_{\lambda, \mu}(x, \xi) = \frac{\mu}{2\pi} \int_{\mathbb{R}} e^{i\lambda\Phi(t, x, \xi)} \frac{\chi(t)}{\cos(t)^d} \hat{f}(\mu t) dt,$$

with the phase  $\Phi(t, x, \xi) = \tan t(|x|^2 + |\xi|^2) - t$ .

**Lemma B.2.** — Assume that  $\delta < \frac{2}{3}$ . Then for every  $N, M \geq 0$  we have

$$(B.4) \quad W_{\lambda, \mu}(x, \xi) = \sum_{j(1-\delta)+k(2-3\delta) < N} c_{k,j} \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k f^{(3k+j)} \left( \frac{\lambda}{\mu} (|x|^2 + |\xi|^2 - 1) \right) + \mathcal{O}(\lambda^{-N} (1 + |x|^2 + |\xi|^2)^{-M}),$$

where  $c_{k,j}$  are real numbers,  $c_{0,0} = 1$ .

*Proof.* — Using that  $\partial_t \Phi(t, x, \xi) = (1 + \tan^2 t)(|x|^2 + |\xi|^2) - 1$  and integrating by parts we get that for every  $M$  there exists  $C_M > 0$  such that for  $||x|^2 + |\xi|^2 - 1| \geq 1/2$  we have

$$(B.5) \quad |W_{\lambda, \mu}(x, \xi)| \leq C_M \lambda^{-M\delta} (|x|^2 + |\xi|^2 + 1)^{-M}.$$

So it is enough to estimate  $W_{\lambda, \mu}(x, \xi)$  for  $|x|^2 + |\xi|^2 \approx 1$ .

To do that, we write down

$$e^{i\lambda\Phi(t, x, \xi)} = e^{i\lambda(\tan t - t)(|x|^2 + |\xi|^2)} e^{i\lambda t(|x|^2 + |\xi|^2 - 1)}.$$

Denote by  $E_t = (\tan t - t)(|x|^2 + |\xi|^2)$ , then we have

$$e^{i\lambda E_t} = \sum_{0 \leq k \leq N} \frac{(i\lambda E_t)^k}{k!} + r_N(i\lambda E_t)$$

where

$$\left| \frac{d^j}{ds^j} r_N(s) \right| \leq \frac{|s|^{N-j}}{(N-j)!}, \quad 0 \leq j \leq N.$$

Lastly, we end up the computation by expanding  $(\tan t - t)$  with Taylor

$$(\tan t - t)^k \frac{\chi(t)}{\cos(t)^d} = \sum_{j=0}^{+\infty} d_{k,j} t^{3k+j}.$$

Thus

$$\begin{aligned} W_{\lambda, \mu}(x, \xi) &= \frac{\mu}{2\pi} \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \int_{\mathbb{R}} e^{i\lambda t(|x|^2 + |\xi|^2 - 1)} d_{k,j} t^{3k+j} \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k \hat{f}(\mu t) dt \\ &= \frac{\mu}{2\pi} \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \int_{\mathbb{R}} e^{i\lambda t(|x|^2 + |\xi|^2 - 1)} d_{k,j} t^{3k+j} \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k \widehat{f^{(3k+j)}}(\mu t) dt \\ &= \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} c_{k,j} \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k f \left( \frac{\lambda}{\mu} (|x|^2 + |\xi|^2 - 1) \right), \end{aligned}$$

which implies the result with (B.5). □



**B.2. Proof of Lemma 4.9.** — First remark that when  $\theta \geq 0$ , the upper-bound is a direct consequence of (3.12) and (3.13) and this holds true for  $\delta = 1$ . The bound (3.13) being a rather difficult result, we shall prove by the same method the estimate from above and from below for  $\delta < 2/3$ .

We use here the functional calculus with energy parameter (B.4). Let  $f$  be a non negative  $C^\infty$  function in  $] -2C_0, 2C_0[$  with a compact support, such that  $f = 1$  in  $[-C_0, C_0]$ . We choose two cutoff functions  $f_\pm$  with  $f_+$  as above and  $f_-$  such that  $\text{supp}(f_-) \subseteq ]C_1, C_0[$ ,  $f_- = 1$  in  $[2C_1, C_0/2]$  where  $C_1 < C_0/4$ . If  $K_{\pm, h}(x, y)$  is the Schwartz kernel of  $f_\pm \left( \frac{H-h^{-1}}{\mu} \right)$  ( $h = \frac{1}{\lambda}$  is now a small parameter,  $\nu \in [\nu_0, \nu_1]$ ). We have

$$K_{-, h}(x, x) \leq e_x \leq K_{+, h}(x, x).$$

So we have to prove

$$(B.6) \quad C_0 N_h h^{\beta_{2p, \theta}} \leq \left( \int_{\mathbb{R}^d} \langle x \rangle^{\theta(p-1)} K_{\pm, h}(x, x)^p dx \right)^{1/p} \leq C_1 N_h h^{\beta_{2p, \theta}}.$$

Recall that

$$K_{\pm, h}(x, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} W_{\pm, h}(x, \xi) d\xi$$

where  $W_{\pm, h}(x, \xi)$  is the Weyl symbol of the operator  $f_\pm \left( \frac{H-h^{-1}}{\mu} \right)$ . So using (B.4) it is not difficult to see that it is enough to consider only the principal term given by the following formula

$$K_{f, h}^0(x, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} f \left( \frac{|x|^2 + |\xi|^2 - h^{-1}}{\mu} \right) d\xi.$$

We shall detail now the lower-bound; the upper-bound is proved in the same way. Denote by  $K_-^0(x) = K_{f, h}^0(x, x)$  and  $s = \theta(p-1)$ . We have, with the change of variable  $x = h^{-1/2}y$ ,  $\xi = h^{-1/2}\eta$

$$\begin{aligned} \int_{\mathbb{R}^d} \langle x \rangle^s K_-^0(x)^p dx &= (2\pi)^{-dp} \int_{\mathbb{R}_\xi^d} \langle x \rangle^s \left( \int_{\mathbb{R}_\xi^d} f_- \left( \frac{|x|^2 + |\xi|^2 - h^{-1}}{\mu} \right) d\xi \right)^p dx \\ &= (2\pi)^{-dp} h^{-(1+p)d/2} \int_{\mathbb{R}_\eta^d} \langle h^{-1/2}y \rangle^s \left( \int_{\mathbb{R}_\eta^d} f_- \left( \frac{|y|^2 + |\eta|^2 - 1}{h\mu} \right) d\eta \right)^p dy \end{aligned}$$

Using the property of the support of  $f_-$  we obtain

$$\int_{\mathbb{R}_\eta^d} f_- \left( \frac{|y|^2 + |\eta|^2 - 1}{h\mu} \right) d\eta \gtrsim h\mu,$$

and that  $|y| \leq 1$  on the support of  $f_-$ . Next,

$$\int_{|y| \leq 1} \langle h^{-1/2}y \rangle^s dy = h^{d/2} \int_{|x| \leq h^{-1/2}} \langle x \rangle^s dx \sim \begin{cases} Ch^{d/2}, & \text{if } s < -d, \\ C|\ln h|h^{d/2}, & \text{if } s = -d, \\ Ch^{-s/2}, & \text{if } s > -d. \end{cases}$$

Finally, we get (B.6) using that  $\mu \approx h^{\delta-1}$  so  $\mu h \approx h^\delta \approx h^d N_h$ .

## Appendix C

### Proof of (2.12)

To begin with, we identify the complex sphere of  $\mathbb{C}^N$  with the real sphere

$$\mathbb{S}^{2N-1} = \{w \in \mathbb{R}^{2N} : w_1^2 + \dots + w_{2N}^2 = 1\} \subset \mathbb{R}^{2N}.$$

Denote by  $\mathbb{P}_N$  the uniform probability measure on  $\mathbb{S}^{2N-1}$  and by  $\mu_N$  the Gaussian measure on  $\mathbb{R}^{2N}$  of density  $d\mu = \frac{1}{(2\pi)^N} \exp\left(-\frac{1}{2} \sum_{j=1}^{2N} x_j^2\right) dx_1 \dots dx_{2N}$ . It is easy to check that  $\mathbb{P}_N$  is the image measure of  $\mu_N$  by the map

$$\begin{aligned} G : \mathbb{R}^{2N} &\longrightarrow \mathbb{S}^{2N-1} \\ (x_1, \dots, x_{2N}) &\longmapsto \frac{1}{\sqrt{\sum_{j=1}^{2N} x_j^2}} (x_1, \dots, x_{2N}). \end{aligned}$$

Indeed,  $\mu_N \circ G^{-1}$  is a probability measure on  $\mathbb{S}^{2N-1}$  which is invariant by the isometries of  $\mathbb{S}^{2N-1}$ , therefore  $\mathbb{P}_N = \mu_N \circ G^{-1}$ . For  $t \in [0, 1]$ , denote by  $\Phi(t) = \mathbb{P}_N(\sqrt{w_1^2 + w_2^2} > t)$ , then

$$\begin{aligned} \Phi(t) &= \frac{1}{(2\pi)^N} \int \mathbb{I}_{x_1^2 + x_2^2 > t^2} \sum_{j=1}^{2N} x_j^2 e^{-\frac{1}{2} \sum_{j=1}^{2N} x_j^2} dx_1 \dots dx_{2N} \\ &= \frac{1}{(2\pi)^N} \int \mathbb{I}_{x_1^2 + x_2^2 > \frac{t^2}{1-t^2} \sum_{j=3}^{2N} x_j^2} e^{-\frac{1}{2} \sum_{j=1}^{2N} x_j^2} dx_1 \dots dx_{2N}. \end{aligned}$$

We make a spherical change of variables  $(x_3, \dots, x_{2N}) \mapsto r\sigma$  and the polar change of variables  $(x_1, x_2) = (r \cos \theta, r \sin \theta)$ . Denote by  $s = t/\sqrt{1-t^2}$ , thus there exists  $C_N$  so that

$$\begin{aligned} \Phi(t) &= C_N \int r^{2N-3} e^{-\frac{1}{2}(\rho^2 + r^2)} \mathbb{I}_{\rho > sr} \rho dr d\rho \\ &= C_N \int_0^{+\infty} r^{2N-3} e^{-\frac{1}{2}(1+s^2)r^2} dr. \end{aligned}$$

Now, by the change of variables  $r' = (1+s^2)^{1/2}r$ , there exists  $C_N$  so that

$$\Phi(t) = C_N (1+s^2)^{-(N-1)} = C_N (1-t^2)^{N-1},$$

and  $C_N = \Phi(0) = 1$ .

## Appendix D

### Proof of Proposition 2.12

For simplicity we assume that the random variables, the  $\gamma_j$  and the space  $\mathcal{E}_h$  are real, and we identify  $\mathcal{E}_h$  with  $\mathbb{R}^N$ , endowed with its natural Euclidean norm  $|y|_0$ . We also consider the  $\gamma$ -dependent norm

$$|y|_\gamma^2 = \frac{1}{N} \sum_{1 \leq j \leq N} \frac{y_j^2}{\gamma_j^2}, \quad y = (y_1, \dots, y_N).$$

Condition (1.3) means that we have

$$\frac{1}{C}|y|_0^2 \leq |y|_\gamma^2 \leq C|y|_0^2.$$

We define a probability measure  $\nu_\gamma$  in  $\mathbb{R}^N$  as the pull forward of the measure  $\nu$  in  $\mathbb{R}^N$  by the mapping  $\varphi: (x_1, \dots, x_N) \mapsto \sqrt{N}(\gamma_1 x_1, \dots, \gamma_N x_N)$ . Notice that  $\nu_\gamma$  satisfies the concentration property of Definition 2.1.

Now we follow the proof of [13, Proposition 2.10]. Let  $F$  be a Lipschitz function on the sphere  $\mathbf{S}_h$ , and by homogeneity, it is enough to assume that  $F$  is 1-Lipschitz. For  $x \in \mathbb{R}^N$ ,  $G(x) = |x|_0(F(\frac{x}{|x|_0}) - F(x_0))$ , where  $x_0$  is a fixed point on  $\mathbf{S}_h$ . Thus  $G$  satisfies

$$(D.1) \quad |G(x) - G(y)|_0 \leq 2(\pi + 1)|x - y|_0,$$

and

$$(D.2) \quad F\left(\frac{x}{|x|_0}\right) - F\left(\frac{y}{|y|_0}\right) = G\left(\frac{x}{|x|_0}\right) - G\left(\frac{y}{|y|_0}\right).$$

By [13, Corollary 1.5], it is enough to prove that

$$\begin{aligned} & \mathbf{P}_{\gamma,h} \otimes \mathbf{P}_{\gamma,h} \left[ u, v \in \mathbf{S}_h : |F(u) - F(v)| \geq 3r \right] \\ &= \nu_\gamma \otimes \nu_\gamma \left[ x, y \in \mathbb{R}^N : \left| F\left(\frac{x}{|x|_0}\right) - F\left(\frac{y}{|y|_0}\right) \right| \geq 3r \right] \leq C e^{-cNr^2}. \end{aligned}$$

Denote by  $\mathcal{M}_\gamma$  the median of  $x \mapsto |x|_0$  with respect to  $\nu_\gamma$ . Then by (D.2)

$$\begin{aligned} & \nu_\gamma \otimes \nu_\gamma \left[ x, y \in \mathbb{R}^N : \left| F\left(\frac{x}{|x|_0}\right) - F\left(\frac{y}{|y|_0}\right) \right| \geq 3r \right] \\ &= \nu_\gamma \otimes \nu_\gamma \left[ x, y \in \mathbb{R}^N : \left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{y}{|y|_0}\right) \right| \geq 3r \right] \\ &\leq \nu_\gamma \otimes \nu_\gamma \left[ x, y \in \mathbb{R}^N : \left| G\left(\frac{x}{\mathcal{M}_\gamma}\right) - G\left(\frac{y}{\mathcal{M}_\gamma}\right) \right| \geq r \right] + 2\nu_\gamma \left[ x \in \mathbb{R}^N : \left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{x}{\mathcal{M}_\gamma}\right) \right| \geq r \right]. \end{aligned}$$

By (D.1) we have

$$\left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{x}{\mathcal{M}_\gamma}\right) \right| \leq 2(\pi + 1) \left| \frac{|x|_0}{\mathcal{M}_\gamma} - 1 \right|,$$

which implies from (2.1) that

$$\nu_\gamma \left[ x \in \mathbb{R}^N : \left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{x}{\mathcal{M}_\gamma}\right) \right| \geq r \right] \leq C_1 e^{-c_1 \mathcal{M}_\gamma^2 r^2}.$$

Similarly, by (D.1) and (2.1)

$$\nu_\gamma \left[ x, y \in \mathbb{R}^N : \left| G\left(\frac{x}{\mathcal{M}_\gamma}\right) - G\left(\frac{y}{\mathcal{M}_\gamma}\right) \right| \geq r \right] \leq C_2 e^{-c_2 \mathcal{M}_\gamma^2 r^2}.$$

To conclude the proof of the proposition we use the following

**Lemma D.1.** — In  $\mathbb{R}^N$ , denote by  $\mathcal{M}_\gamma(N)$  the median of  $x \mapsto |x|_0$  with respect to  $\nu_\gamma$ , and by  $\mathcal{A}_\gamma(N)$  its expectation. Then there exist  $C, C_1, C_2 > 0$  such that for all  $N \geq 1$

$$|\mathcal{M}_\gamma(N) - \mathcal{A}_\gamma(N)| \leq C \quad \text{and} \quad C_1\sqrt{N} \leq \mathcal{M}_\gamma(N) \leq C_2\sqrt{N}.$$

*Proof.* — Here we use the notation  $|x|_N := (x_1^2 + \dots + x_N^2)^{1/2}$ . By definition of  $\mathcal{M}_\gamma$ , we have for all  $t > 0$  and thanks to (2.5)

$$\frac{1}{2} = \mathbf{P}_{\gamma,h} \left[ |x|_N \geq \mathcal{M}_\gamma \right] \leq e^{-t(\mathcal{M}_\gamma - \mathcal{A}_\gamma)} \mathbf{E} \left[ e^{t(|x|_N - \mathcal{A}_\gamma)} \right] \leq e^{-t(\mathcal{M}_\gamma - \mathcal{A}_\gamma)} e^{ct^2}.$$

Then, we choose  $t = (\mathcal{M}_\gamma - \mathcal{A}_\gamma)/(2c)$  and get for some  $C > 0$ ,  $|\mathcal{M}_\gamma - \mathcal{A}_\gamma| \leq C$ . This was the first claim.

Next, by Cauchy-Schwarz, we obtain  $\mathcal{A}_\gamma^2(N) \leq \int_{\mathbb{R}^N} |x|_N^2 d\nu(x) = N$ . Now we prove that there exists  $C > 0$  so that for all  $N \geq 1$ ,  $\mathcal{A}_\gamma(N) \geq C\sqrt{N}$ . Indeed,

$$\begin{aligned} \mathcal{A}_\gamma(N+1) - \mathcal{A}_\gamma(N) &= \int_{\mathbb{R}^{N+1}} \frac{x_{N+1}^2}{|x|_N + |x|_{N+1}} d\nu(x) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^{N+1}} \frac{x_{N+1}^2}{|x|_{N+1}} d\nu(x) = \frac{\mathcal{A}_\gamma(N)}{2(N+1)}. \end{aligned}$$

This implies that for all  $N \geq 1$

$$\mathcal{A}_\gamma(N+1) \geq \left(1 - \frac{1}{2(N+1)}\right)^{-1} \mathcal{A}_\gamma(N) \geq \left(1 + \frac{1}{2(N+1)}\right) \mathcal{A}_\gamma(N),$$

and then  $\mathcal{A}_\gamma(N) \geq P_N \mathcal{A}_\gamma(1)$ , where

$$\ln P_N = \sum_{k=2}^N \ln\left(1 + \frac{1}{2k}\right) = \frac{1}{2} \ln N + \mathcal{O}(1),$$

which yields the result. □

## References

- [1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, R. Cyril, and G. Scheffer. Sur les inégalités de Sobolev logarithmiques. Panoramas et Synthèses, 10. Société Mathématique de France, Paris, 2000.
- [2] N. Burq and G. Lebeau. Injections de Sobolev probabilistes et applications. *Ann. Sci. Éc. Norm. Supér.*, to appear.
- [3] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations I: local existence theory. *Invent. Math.* 173, No. 3, (2008), 449–475.
- [4] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations II: A global existence result. *Invent. Math.* 173, No. 3 (2008), 477–496.
- [5] N. Burq and N. Tzvetkov. Probabilistic well-posedness for the cubic wave equation (2011). *Preprint* : *arXiv:1103.2222*. To appear in *JEMS*.
- [6] R. Feng and S. Zelditch. Median and mean of the Supremum of  $L^2$  normalized random holomorphic fields *Preprint* : *arXiv:1303.4096v1 [math.PR]*, (2013).
- [7] L. Hörmander. The spectral function of an elliptic operator. *Acta Mathematica*. 121, p. 193–218, (1968).

- [8] V. Ivrii. *Microlocal Analysis and Precise Spectral Asymptotics*. Springer Monographs in Mathematics (1998).
- [9] S. Kakutani. On Equivalence of Infinite Product Measures. *Annals of Math.* 49, No.1 (1948), p. 214–224.
- [10] G.E. Karadzhov. Riesz summability of multiple Hermite series in  $L^p$  spaces. *Math. Z.* 219, (1995), 107–118.
- [11] H. Koch and D. Tataru.  $L^p$  eigenfunction bounds for the Hermite operator. *Duke Math. J.* 128 (2005), no. 2, 369–392.
- [12] H. Koch, D. Tataru and M. Zworski. Semiclassical  $L^p$  Estimates. *Annales Henri Poincaré* 8 (2007), 885–916.
- [13] M. Ledoux. The concentration of measure phenomenon. *Mathematical Surveys and Monographs*, Vol.89, AMS, (2001).
- [14] A. Poiret. Solutions globales pour des équations de Schrödinger sur-critiques en toutes dimensions. *Preprint : arXiv:1207.3519*.
- [15] A. Poiret, D. Robert and L. Thomann. Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator. *Preprint arXiv:1309.0795*.
- [16] D. Robert. *Autour de l'approximation semi-classique* Birkhäuser Boston, (1987).
- [17] D. Robert and L. Thomann. Random weighted Sobolev inequalities and application to quantum ergodicity. *Preprint. hal-00919443*.
- [18] B. Shiffman and S. Zelditch. Random polynomials of high degree and Levy concentration of measure. *Asian J. Math.* 7 (2003), no. 4, 627–646.
- [19] A. N. Shiryaev. Probability. *Graduate Texts in Mathematics*, 95. Springer-Verlag, New York, 1996.
- [20] M. Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l'I.H.E.S.* 81, 73–205 (1995).
- [21] T. Tao. Blog. <http://terrytao.wordpress.com/2010/01/03/254a-notes-1-concentration-of-measure/>.
- [22] S. Thangavelu. Lectures on Hermite and Laguerre expansions. *Mathematical Notes*, 42. Princeton University Press, Princeton, NJ, 1993.
- [23] K. Yajima and G. Zhang. Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity. *J. Differential Equations* (2004), no. 1, 81–110.

---

AURÉLIEN POIRET, Laboratoire de Mathématiques, UMR 8628 du CNRS. Université Paris Sud, 91405 Orsay Cedex, France • *E-mail* : [aurelien.poiret@math.u-psud.fr](mailto:aurelien.poiret@math.u-psud.fr)

DIDIER ROBERT, Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France • *E-mail* : [didier.robert@univ-nantes.fr](mailto:didier.robert@univ-nantes.fr)

LAURENT THOMANN, Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France • *E-mail* : [laurent.thomann@univ-nantes.fr](mailto:laurent.thomann@univ-nantes.fr)