Continued fractions and numeration in the Fibonacci base

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Abstract

Let ϕ be the golden ratio. We define and study a continued ϕ-fraction algorithm, inspired by Euclid's algorithm. We show that any non-negative element of $\mathbb{Q}(\phi)$ has a finite continued ϕ-fraction.

Keywords Continued fractions, Fibonacci numeration, Fibonacci substitution, β-numeration, β-integers

AMS classification 11A55, 11J70, 11Y65, 68R01

Introduction

The β-numeration, introduced by Rényi [26] and Parry [23], is a numeration system in a non-integral base. Let $\beta > 1$. In the same way as in the case of an integral base, one may expand any $x \in [0,1]$ as $x = \sum_{k\in\mathbb{N}^*} v_k\beta^{-k}$, where the sequence $(v_k)_{k\in\mathbb{N}^*}$, which takes values in $\mathcal{A}_\beta = [\![0,\ldots,[\beta]]\!]$, is called *expansion of x in base β*. Among the expansions of x in base β, the greatest sequence for the lexicographical order is called *β-expansion of x*, and is denoted by $d_\beta(x)$. The β-expansion of x is constructed by the *greedy algorithm*, that is, $d_\beta(x) = 0.\varepsilon_1\varepsilon_2\ldots$, where the elements of the sequence $(\varepsilon_k)_{k\in\mathbb{N}^*}$ are defined, using the map $T_\beta : [0,1] \to [0,1], x \longmapsto \{\beta x\}$, by $\varepsilon_k = [T_\beta^k(x)]$ for all $k \in \mathbb{N}^*$. Note that the map d_β is increasing if $\mathcal{A}_\beta^{\mathbb{N}^*}$ is endowed with the lexicographical order. When $d_\beta(x) = 0.v_1\ldots$ contains only finitely many non-zero elements, one may remove the ending consecutive occurences of 0's, that is, $d_\beta(x) = 0.v_1\ldots v_n$. In the particular case where $d_\beta(1)$ is either finite or ultimately periodic, β is said to be a *Parry number*, respectively *simple* or *non-simple*.

Parry showed in [23] that a sequence $v = (v_k)_{k\in\mathbb{N}^*}$ is the β-expansion of a real number $x \in [0,1[$ if and only if the following condition, called the *Parry condition*, holds:

$$\text{for all } i \in \mathbb{N}, S^i(v) <_{lex} (\varepsilon_k)_{k\in\mathbb{N}^*}, \tag{1}$$

where S denotes the shift map, that is, $S((v_k)_{k\in\mathbb{N}}) = (v_{k+1})_{k\in\mathbb{N}}$, and where $(\varepsilon_k)_{k\in\mathbb{N}^*}$ is the greatest sequence for the lexicographical order among the expansions of 1 in base β that are not finite, denoted by $d_\beta^*(1)$. A word or a sequence which satisfies (1) is said to be *admissible*. The set of admissible words is a language denoted by $\mathcal{L}_\beta$.

The notion of β-expansion is naturally extended to non-negative real numbers by applying the greedy algorithm. Note however that we do not use the expansion 1.0^∞ for the real number 1; this expansion seems more natural and does not depend on the base β, however it does not allow us to define the condition of admissibility that appears in the Parry condition (1). Any $x > 1$ may be uniquely expanded as $x = \sum_{k=0}^{n} v_{-k}\beta^k + \sum_{k\in\mathbb{N}^*} v_k\beta^{-k}$, where $(v_{k-n})_{k\in\mathbb{N}}$ is an admissible sequence. The sum which consists of non-negative powers of β is called *β-integer part of* x, and is denoted by $[x]_\beta$. The sum which consists of negative powers of β is called *β-fractionary part of* x, and is denoted by $\{x\}_\beta$.

In the framework of β-numeration, the elements which play the role of non-negative integers are the non-negative real numbers x such that $x = [x]_\beta$, which are called non-negative *β-integers*. The set of non-negative β-integers is denoted by $\mathbb{Z}_\beta^+$. Since $\mathbb{Z}_\beta^+$ is a discrete set for any $\beta > 1$, one may define the *β-successor* of $x \in \mathbb{Z}_\beta^+$ as $s_\beta(p) = \min\{q \in \mathbb{Z}_\beta^+, p < q\}$. The set $\{s_\beta(x) - x, x \in \mathbb{Z}_\beta^+\}$ is finite if and only if β is a Parry number, see for instance [8, 28].

It is natural to ask whether usual properties in the framework of classical numeration systems are preserved in a non-integral base. In this article, we are interested in studying a continued fraction algorithm introduced by Enomoto, where the sequence of partial quotients consists of β-integers instead of integers. This study is performed with the golden mean, that is, $\beta = \frac{1+\sqrt{5}}{2}$, that we denote by ϕ. This choice is accounted for the following properties:

1. ϕ is quadratic over $\mathbb{Q}$,
2. ϕ is a Pisot number,
3. $\phi < 2$, hence expansions in base ϕ are defined on the alphabet $\mathcal{A} = \{0, 1\}$.

The aim of this article is to prove the following result, conjectured by Akiyama [29].

Theorem 5.3 The positive real numbers whose continued ϕ-fraction is finite are the positive elements of $\mathbb{Q}(\phi)$.

This article is structured in the following way. Section 1 gathers all elementary definitions, notation and preliminary results. We introduce the notion of ϕ-fractions, which are fractions whose numerators and denominators are ϕ-integers, and also the notion of length on ϕ-integers. We use in Section 2 the Dumont-Thomas algorithm (see [13]), which allows us to expand the prefixes of a fixed point of a primitive substitution in a canonical way. Thus, there is an explicit one-to-one map between $\mathbb{Z}_\phi^+$ and the set of prefixes of ω, the fixed point of the Fibonacci substitution σ defined by $\sigma(a) = ab$ and $\sigma(b) = a$ (Propositions 2.3 and 2.7).

In Section 3, we introduce intervals $\mathcal{I}_W$, defined for any admissible word W. We prove that $\mathcal{I}_W$ contains the images under the Galois map τ of ϕ-integers whose ϕ-expansion admit W as a suffix. Furthermore, the bounds of $\mathcal{I}_W$ are determined by W (Lemmas 3.1 and 3.5). This provides a geometrical characterization of elements in $\mathbb{Z}^2$ that are abelianizations of prefixes of the fixed point of the Fibonacci substitution, as follows: they need to belong to a particular semi-window $\mathcal{B}_\phi$ (Theorem 3.8, Corollary 3.9, and see Figure 3). The semi-window $\mathcal{B}_\phi$ is in fact defined by a cut-and-project scheme, which admits a Rauzy fractal as a window of acceptance; this Rauzy fractal, which is $]-1, \phi[$ in the Fibonacci case, allows us to define a self-similar tiling of $\mathbb{R}$. Thanks to this characterization, it is possible to determine whether any real number constructed by adding, subtracting or multiplying ϕ-integers is a ϕ-integer.

Section 4 deals with continued ϕ-fractions. These are continued fractions, constructed according to a generalization of Euclid's algorithm, where the sequence of partial quotients consists of ϕ-integers. First, we study the construction of continued ϕ-fractions (Proposition 4.5). Then, we try to extend to positive elements of $\mathbb{Q}(\phi)$ the following classical result: the continued fraction of any positive rational number is finite. Having this prospect in mind, we go back to the approach used in classical continued fractions, in order to apply it to continued ϕ-fractions. Since the set of ϕ-fractions is $\mathbb{Q}(\phi)^+$ (Proposition 4.6), we define an algorithm A

on pairs of ϕ-integers which represents, when it is defined, the action of the map $[0,1] \to [0,1]$, $x \longmapsto \{\frac{1}{x}\}_\phi$. Hence, starting from a pair (p_0, q_0) of ϕ-integers such that $x = \frac{p_0}{q_0}$, the algorithm A constructs by iteration a sequence of pairs of ϕ-integers $(p_i, q_i)_{i\in\mathbb{N}}$, such that the sequence of partial quotients of x is $([\frac{p_i}{q_i}]_\phi)_{i\in\mathbb{N}}$. Then, using a notion of length on pairs of ϕ-integers, denoted by t, we compute an upper bound for the quantity $t(p,q) - t(A(p,q))$ (Lemma 4.8). Studying more closely several cases which depend on $[\frac{p}{q}]_\phi$, we obtain a more accurate upper bound for $t(A(p,q)) - t(p,q)$ (Propositions 4.11, 4.14, 4.15).

We prove in Section 5 that the sequence $(t(p_i, q_i))_i$ of lengths of pairs of ϕ-integers that are produced when iterating the algorithm A is bounded. This implies that the continued ϕ-fraction of any $x \in \mathbb{Q}(\phi)^+$ is either finite or eventually periodic. Finally, we prove by contradiction that elements having an ultimately periodic continued ϕ-fraction are not in $\mathbb{Q}(\phi)$, which proves Theorem 5.3.

The definitions introduced in this article may easily be extended to the class of Parry numbers, and several results obtained in Sections 2, 3, 4 may hold for other numbers than the golden ratio. However, we do not known for which numbers one can generalize the result provided by Theorem 5.3.

1 Definitions and notation

1.1 Generalities

For convenience, we define for any set $E \subset \mathbb{R}$ the sets $E^* = E \setminus \{0\}$ and $E^+ = E \cap \mathbb{R}_+$.

Let $\mathcal{A}$ be a finite set, called *alphabet*. Endowed with the concatenation, $\mathcal{A}$ generates a monoid $\mathcal{A}^*$. For any $v \in \mathcal{A}^*$, we denote by $|v|$ the number of letters of v, and by $|v|_{a_i}$ the number of occurrences of the letter a_i in v. The empty word is denoted by ε.

Let $(\vec{e_i})_{i\in[\![1,\dots,d]\!]}$ be the canonical basis of $\mathbb{Z}^d$. Let $f : \mathcal{A} \to \mathbb{Z}^d$ be the morphism of monoid called *abelianization morphism* or *Parikh map*, defined by $f(a_i) = \vec{e_i}$ for all $i \in [\![1,\dots,d]\!]$ (for more details, see [24]).

A substitution is a map from $\mathcal{A}$ to $\mathcal{A}^*$ which naturally extends to a morphism on $\mathcal{A}^*$. Let σ be a substitution defined on $\mathcal{A} = \{a_1, \dots, a_d\}$. The *incidence matrix* of σ is the square matrix M_σ of size d, whose coefficients are defined by $M_\sigma[i,j] = |\sigma(a_j)|_{a_i}$ for all $(i,j) \in [\![1,\dots,d]\!]^2$.

When $d_\beta(1)$ is either finite or ultimately periodic, β is said to be a Parry number. Let us recall that any Pisot number is a Parry number ([7, 27]). When β is a Parry number, one can define a substitution σ associated to β called *β-substitution*. The eigenvalues of the incidence matrix M_σ of σ are the roots of the polynomial whose coefficients are defined by $d_\beta(1)$. In particular, β and its Galois conjugates are eigenvalues of M_σ. See [28, 15] for more details on β-substitutions. The notion of admissibility introduced in (1), which depends on $d^*_\beta(1)$, can be defined using the associated β-substitution when β is a Parry number. In this case, the set of admissible words is the set of words that are recognized by a finite automaton associated to β called the *prefix-suffix automaton*. One may refer to [11, 12] for more details.

1.2 The Fibonacci numeration system

We denote by ϕ the golden mean $\frac{1+\sqrt{5}}{2}$, which is the positive root of the polynomial $X^2 - X - 1$. Since the Galois conjugate of ϕ is $-\phi^{-1}$, whose modulus is less than 1, ϕ is a Pisot number. We denote by τ the field morphism defined on $\mathbb{Q}(\phi)$ by $\tau(\phi) = -\phi^{-1}$.

Since $T_\phi(1) = \phi^{-1}$ and $T_\phi(\phi^{-1}) = 0$, the ϕ-expansion of 1 is $d_\phi(1) = 0.110^\infty = 0.11$, which means that ϕ is a *simple Parry number*. Moreover, $d^*_\phi(1) = 0.(10)^\infty$, which implies that any word is admissible if and only if it is defined on the alphabet $\mathcal{A}_\phi = \{0, 1\}$ and it does not admit the word 11 as a factor. More details about Parry numbers can be found in [7, 14, 22, 27].

Let $x > 0$. When there are only finitely many non-zero elements in $d_\phi(x)$, we say that x has a *finite ϕ-expansion*. In this case, we omit the ending of consecutive zeros. The set of real numbers having a finite ϕ-expansion is denoted by $\mathrm{Fin}(\phi)$. Note that ϕ satisfies the finiteness property $(\mathcal{F})$, that is, $\mathrm{Fin}(\phi) = \mathbb{Z}(\phi^{-1})$. See [17, 1, 2] for more details on the finiteness property.

The set of non-negative ϕ-integers is the set of real numbers that can be expanded as $x = \sum_{k=0}^{n} v_k \phi^k$, where $v_k \in \{0, 1\}$ for all $k \in [\![0, \dots, n]\!]$. Note that $\mathbb{Z}^+_\phi$ is a subset of $\mathbb{Z}[\phi] \simeq \mathbb{Z}[X]/(X^2 - X - 1)$.

Remark 1.1. *Since ϕ is a confluent Parry number ([16]), we may obtain the set of ϕ-integers without using the admissibility condition.*

Definition 1.2. *Let $d_\phi(x) = v_N v_{N-1} \dots v_1 v_0 . v_{-1} \dots v_{-N'}$, where $x \in Fin(\phi)$, $x \neq 0$. We call* ϕ-integer length of x *the quantity $|d_\phi([x]_\phi)| = N + 1$, that we denote by $t_+(x)$, and* ϕ-fractional length of x *the quantity $|d_\phi(\{x\}_\phi)| = N'$, that we denote by $t_-(x)$. We call* global length of x *the quantity $|d_\phi(x)| = N + N' + 1$, that we denote by $t(x)$. We set $t_+(0) = t_-(0) = t(0) = -\infty$.*

Remark 1.3. *The positive real number x belongs to $\mathbb{Z}^+_\phi$ if and only if $t_+(x) = t(x)$.*

Definition 1.4. *Let $p, q \in \mathbb{Z}^+_\phi$ with $q > 0$. Then $\frac{p}{q} \in \mathbb{Q}(\phi)$ is called* ϕ-fraction. *The pair (p, q) is called* ϕ-fractionary expansion *of x. The set of ϕ-fractions is denoted by $\mathbb{Q}^+_\phi$.*

Definition 1.5. *Let $p, q \in \mathbb{Z}^+_\phi$. We define the* length *of (p, q) as $t(p, q) = t(p) + t(q) - 1$. We define the* length *of $x \in \mathbb{Q}^+_\phi$ as $\min_{p,q \in \mathbb{Z}^+_\phi} \{t(p) + t(q) - 1, x = \frac{p}{q}\}$.*

Let $x \in \mathbb{Q}^+_\phi$. When (p, q) is a ϕ-fractionary expansion of x such that $t(x) = t(p) + t(q) - 1$, (p, q) is called reduced ϕ-fractionary expansion *of x.*

Example 1.6. *A ϕ-fractionary expansion of 2 is $(\phi^3 + 1, \phi^2)$. One checks that $(\phi^3 + 1, \phi^2)$ is in fact the unique reduced ϕ-fractionary expansion of 2.*

Remark 1.7. *Any ϕ-fraction has a unique reduced ϕ-fractionary expansion. Since we do not need this property for our study, we do not include its proof. See [6] for more details.*

The ϕ-substitution associated to ϕ is defined by $\sigma(a) = ab$ and $\sigma(b) = a$. This substitution is called the *Fibonacci substitution*; the eigenvalues of the incidence matrix M_σ of σ are exactly the roots of $X^2 - X - 1$, namely ϕ and $\tau(\phi) = -\phi^{-1}$. We denote by ω the unique fixed point of σ. For all $k \in \mathbb{N}^*$, we denote by ω_k the prefix of ω such that $|\omega_k| = k$. The following proposition is a particular case of Theorem 1.5 in [13].

Proposition 1.8. *Let $k \in \mathbb{N}^*$. Then ω_k can be uniquely expanded as $\omega_k = \sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$, with:*

1. *$\varepsilon_n = a$;*
2. *for all $i \in [\![0, \dots, n]\!]$, $\varepsilon_i \in \{\varepsilon, a\}$;*

3. for all $i \in [\![0, \dots, n]\!]$, $\varepsilon_i \varepsilon_{i+1} \neq aa$.

The expansion $\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$ is called the Dumont-Thomas expansion of ω_k. We denote by $\sigma^0(\varepsilon)$ the Dumont-Thomas expansion of $\omega_0 = \varepsilon$.

Let $(F_n)_{n\in\mathbb{N}}$ be the Fibonacci sequence. This sequence is defined by the following linear recurrence, which may be extended to $\mathbb{Z}$: for all $i \in \mathbb{N}^*$, $F_{i+1} = F_i + F_{i-1}$, with the initial conditions $F_0 = 1$ and $F_1 = 2$.

Remark 1.9. *We use later the following relations: for all $n \in \mathbb{N}$, $|\sigma^n(a)| = F_n$, $|\sigma^n(a)|_a = F_{n-1}$ and $|\sigma^n(a)|_b = F_{n-2}$.*

2 Link between expansions, ϕ-integers and the prefixes of ω

The aim of this section is to find connections between $\mathbb{Z}_\phi^+$ and the set of prefixes of ω, using the abelianization map f and the projections defined by the eigenvectors of the matrix M_σ. This study allows us to define and construct the algorithm of expansion in continued ϕ-fraction. We will use several results proved in [5] as well.

2.1 Abelianization of the prefixes of ω

The sequence $(f(\omega_k))_{k\in\mathbb{N}}$ defines a path in $\mathbb{Z}^2$, where the k-th vertex is $(|\omega_k|_a, |\omega_k|_b)$. This path is depicted in Figure 1. Let $||.||$ denote the euclidean norm on $\mathbb{Z}^2$.

Proposition 2.1. *The vector $\lim\limits_{k\to\infty} \frac{f(\omega_k)}{||f(\omega_k)||}$ is an eigenvector of M_σ, whose eigenvalue is ϕ.*

This property is a direct consequence of the fact that the substitution σ is of Pisot type, that is, the dominant eigenvalue β of M_σ, the incidence matrix of σ, is such that, for any other eigenvalue λ of M_σ, one has $0 < \lambda < 1 < \beta$ (see [5] for more details). As a consequence, we define $(\vec{f_1}, \vec{f_2})$, a new basis of $\mathbb{R}^2$, where $\vec{f_1}$ and $\vec{f_2}$ are eigenvectors whose associated eigenvalues are respectively ϕ and $-\phi^{-1}$, such that $\vec{e_1} = \vec{f_1} + \vec{f_2}$. Since M_σ is symmetric, this new basis is orthogonal. We denote by Δ_1 and Δ_2 the subspaces respectively generated by $\vec{f_1}$ and $\vec{f_2}$.

Definition 2.2. *We denote by $\pi_1(X)$ and $\pi_2(X)$ the coordinates of X in the basis $(\vec{f_1}, \vec{f_2})$.*

Proposition 2.3. *Let $\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$ be the Dumont-Thomas expansion of ω_k. The following relations hold:*

1. $\pi_1(f(\omega_k)) = \sum\limits_{i=0}^{n} |\varepsilon_i|(-\phi)^{-i}$,
2. $\pi_2(f(\omega_k)) = \sum\limits_{i=0}^{n} |\varepsilon_i|\phi^i$.

Proof. The vectors of the basis $(\vec{f_1}, \vec{f_2})$ are eigenvectors of the matrix M_σ. We additionally check the equality $f \circ \sigma = M_\sigma \circ f$ on $\mathcal{A}^*$; since f and σ are morphisms, we only have to check this relation on $\mathcal{A}$. As $f(\sigma(a)) = \vec{e_1} + \vec{e_2} = M_\sigma f(a)$ and $f(\sigma(b)) = \vec{e_1} = M_\sigma f(b)$, the equality holds. Since we have also $f(a) = \vec{f_1} + \vec{f_2}$, we deduce $f(\sigma^n(a)) = (-\phi)^{-n}\vec{f_1} + \phi^n \vec{f_2}$. Hence $f(\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)) = (\sum\limits_{i=0}^{n} |\varepsilon_i|(-\phi)^{-i}, \sum\limits_{i=0}^{n} |\varepsilon_i|\phi^i)$. □

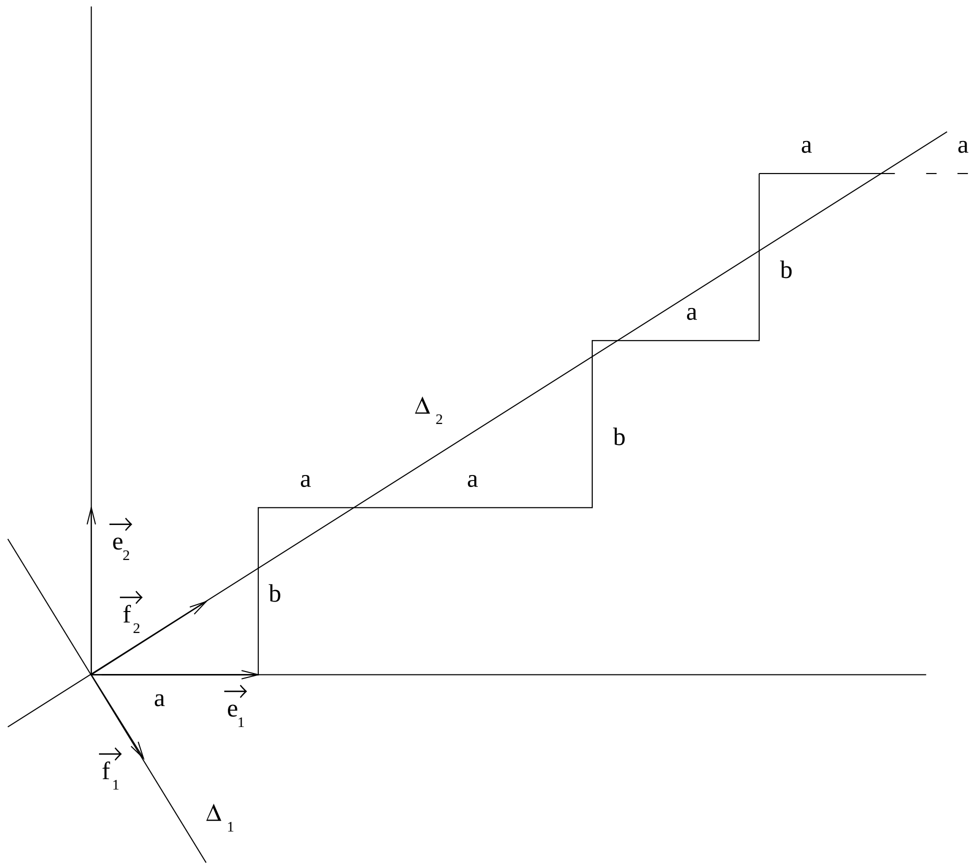

Figure 1: Abelianizations of ω and the basis $(\vec{f_1}, \vec{f_2})$

2.2 Relation between expansions of ϕ-integers and prefixes of ω

Let Γ be the map defined by

$$\Gamma : \mathcal{L}_\phi \longrightarrow \{\omega_k, k \in \mathbb{N}^*\}, v_n \dots v_0 \longmapsto \omega_k,$$

where $\omega_k = \sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$ is such that for all $i \in [\![0, \dots, n]\!]$, $|\varepsilon_i| = v_i$. Note that Γ is defined on expansions in base ϕ of ϕ-integers, and that $\Gamma(W') = \Gamma(d_\phi(x))$ for any expansion W' in base ϕ of x. If we restrict Γ to the set of admissible words which admit 1 as a prefix, we obtain an invertible map, with $\Gamma^{-1}(\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)) = |\varepsilon_n| \dots |\varepsilon_0|$. Since the coordinates of $\vec{e_1}$ and $\vec{e_2}$ in the basis $(\vec{f_1}, \vec{f_2})$ are rationally independent, the projections $\vec{\pi_1} : \mathbb{Z}^2 \to \Delta_1$ and $\vec{\pi_2} : \mathbb{Z}^2 \to \Delta_2$ are one-to-one. Hence the following maps π_1 and π_2 are bijections:

$$\pi_1 : \mathbb{Z}^2 \longrightarrow \mathbb{Z}[\phi], (p, q) \longmapsto p - \phi q,$$

$$\pi_2 : \mathbb{Z}^2 \longrightarrow \mathbb{Z}[\phi], (p, q) \longmapsto p + \phi^{-1} q = p - q + \phi q.$$

Remark 2.4. *These projections are not exactly those usually defined in the associated cut-and-project scheme, see for example [19]. The basis $(\vec{f_1}, \vec{f_2})$ may be seen as the image of the canonical basis under the action of a dilatation and a rotation. This explains why we do not retrieve exactly the usual notation for this scheme.*

Let $x \in \mathbb{Z}_\phi^+$, $d_\phi(x) = v_n \dots v_0$. Then $\pi_2^{-1}(x) = (\sum_{i=0}^{n} v_i F_{i-1}, \sum_{i=0}^{n} v_i F_{i-2})$.

Notation 2.5. *The map $\pi_1 \circ \pi_2^{-1}$ coincides with τ on $\mathbb{Z}[\phi]$.*

2.3 Basic properties of $\mathbb{Z}_\phi^+$

The set $\mathbb{Z}_\phi^+$ is not stable under addition and multiplication. For instance, one checks that 1 and $\phi^2 + 1 \in \mathbb{Z}_\phi^+$; however $1 + 1 = 2 = \phi + \phi^{-2} \notin \mathbb{Z}_\phi^+$ and $(\phi^2 + 1)^2 = \phi^5 + \phi + \phi^{-2} \notin \mathbb{Z}_\phi^+$. However, it is proved in [17] that the finiteness property $(\mathcal{F})$ holds in the case of the Fibonacci numeration system, that is, $\mathrm{Fin}(\phi) = \mathbb{Z}[\phi^{-1}]$. Hence the sum and the product of two ϕ-integers may be expanded as a finite sum of powers of ϕ whose coefficients satisfy the admissibility condition.

One may define two laws $\oplus$ and $\otimes$ on $\mathbb{Z}_\phi^+$, such that $\mathbb{Z}_\phi^+$ is stable under $\oplus$ and $\otimes$. This point of view is developed for instance in [4, 9, 10]. We do not use such a point of view, because we need to work with usual laws.

Proposition 2.6. *Let $x \in \mathbb{Z}_\phi^+$. Then:*

1. *$s_\phi(x) = x + 1$ if and only if $d_\phi(x)$ admits 0 as a suffix,*
2. *$s_\phi(x) = x + \phi^{-1}$ if and only if $d_\phi(x)$ admits 1 as a suffix.*

One can easily deduce this particular result from [8]. Note that the successor function s_ϕ has been extensively studied, see for instance [18].

Proposition 2.7. *One has $\mathbb{Z}_\phi^+ = \{\pi_2(f(\omega_k)), k \in \mathbb{N}\}$.*

Proof. Let x be a ϕ-integer. Let $d_\phi(x) = v_n \dots v_0$. Then, using Remark 1.9, one has:

$$\begin{aligned} x &= \sum_{i=0}^{n} v_i (F_{i-1} + F_{i-2}\phi^{-1}) = \pi_2(\sum_{i=0}^{n} v_i F_{i-1}, \sum_{i=0}^{n} v_i F_{i-2}) \\ &= \pi_2 \circ f(\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)) \text{ with for all } i \in [\![0, \dots, n]\!], |\varepsilon_i| = v_i. \end{aligned}$$

Since $v_i v_{i+1} \neq 11$ implies $\varepsilon_i \varepsilon_{i+1} \neq aa$, there exists $k \in \mathbb{N}$ such that $\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$ is the Dumont-Thomas expansion of ω_k. Conversely, if the Dumont-Thomas expansion of ω_k is $\sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$, then $x = \pi_2(f(\omega_k))$ can be expanded as $\sum_{i=0}^{n} |\varepsilon_i| \phi^i \in \mathbb{Z}_\phi^+$. $\square$

Hence $\pi_2 \circ f$ defines a bijection between $\mathbb{Z}_\phi^+$ and the set of prefixes of ω.

3 Algebraic and geometric characterization of $\mathbb{Z}_\phi^+$

The aim of this section is to establish relations between ϕ-integers and their images under the Galois map τ. We prove below that, if $x \in \mathrm{Fin}(\phi)^+$, then x is a ϕ-integer if and only if $\tau(x) \in]-1, \phi[$. In this case, $f(\Gamma(d_\phi(x)))$ fulfills a geometrical condition, that is, $f(\Gamma(d_\phi(x)))$ belongs to an open semi-band $\mathcal{B}_\phi^+$ that we define in Section 3.3.

3.1 Repartition of the image under $\pi_2 \circ f$ of admissible words on Δ_1

Due to Proposition 2.3, the images under $\pi_1 \circ f$ of the prefixes of ω belong to the interval $]\min\{\sum_{k\in\mathbb{N}} v_k(-\phi)^{-k}, (v_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\}, \max\{\sum_{k\in\mathbb{N}} v_k(-\phi)^{-k}, (v_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\}[=]-1,\phi[$. More generally, given $S \in \mathcal{L}_\phi$, we define an interval $\mathcal{I}_S$ which satisfies the following property: if $x \in \mathbb{Z}_\phi^+$ is such that $d_\phi(x)$ admits S as a suffix, then $\tau(x) = \sum_{i=0}^{n} v_i(-\phi)^{-i}$ belongs to $\mathcal{I}_S$.

Lemma 3.1. *Let $W = w_n \ldots w_0 \in \mathcal{L}_\phi$. Let U be an expansion in base ϕ which admits W as a suffix. Then: $-\phi^{-2[\frac{n+1}{2}]} < \pi_1(f(\Gamma(U))) - \pi_1(f(\Gamma(W))) < \phi^{-2[\frac{n}{2}]-1}$.*

Proof. Let $W = w_n \ldots w_0 \in \mathcal{L}_\phi$. Let $U = w_{n'} \ldots w_0$ be an expansion in base ϕ which admits W as a suffix. Then: $\pi_1(f(\Gamma(U))) = \sum_{i=0}^{n'} w_i(-\phi)^{-i} = \sum_{i=0}^{n} w_i(-\phi)^{-i} + \sum_{i=n+1}^{n'} w_i(-\phi)^{-i} = \pi_1(f(\Gamma(W))) + \sum_{i=n+1}^{n'} w_i(-\phi)^{-i}$.

If n is even, then $-\phi^{-n} < \sum_{i=n+1}^{n'} w_i(-\phi)^{-i} < \phi^{-n-1}$. On the other hand, if n is odd, then $-\phi^{-n-1} < \sum_{i=n+1}^{n'} w_i(-\phi)^{-i} < \phi^{-n}$.

We deduce that $\pi_1(f(\Gamma(W))) - \phi^{-2[\frac{n+1}{2}]} \leqslant \pi_1(f(\Gamma(U))) \leqslant \pi_1(f(\Gamma(W))) + \phi^{-2[\frac{n}{2}]-1}$. $\square$

3.2 Cylinders and intervals: a tiling of $]-1,\phi[$

Due to Lemma 3.1, any $S \in \mathcal{L}_\phi$ defines an interval $\mathcal{I}_S$ such that, if any expansion in base ϕ of $x \in \mathbb{Z}_\phi^+$ admits S as a suffix, then $\tau(x) \in \mathcal{I}_S$. It is natural to ask whether we can establish a reciprocal property. Thus, if $S \in \mathcal{L}_\phi$, and if x is a ϕ-integer such that $\tau(x)$ belongs to $\mathcal{I}_S$, we want to determine whether there exists an expansion in base ϕ of x which admits S as a suffix.

Definition 3.2. *For $W \in \mathcal{L}_\phi$, we define the cylinder $\mathcal{C}_W$ as the set of expansions in base ϕ which admit W as a suffix. Let $\mathcal{P}_W = \{\pi_1(f(\Gamma(W'))), W' \in \mathcal{C}_W\}$. We define the interval $\mathcal{I}_W$ as the convex hull of $\mathcal{P}_W$.*

Remark 3.3. *One has $\mathcal{I}_0 =]-1,\phi^{-1}[$ and $\mathcal{I}_1 =]\phi^{-1},\phi[$.*

Proposition 3.4. *The following properties are fulfilled:*

1. *the set $\mathcal{P}_W$ is dense in $\mathcal{I}_W$;*
2. *we get $\mathcal{I}_W =]-\phi^{2[\frac{n+1}{2}]} + \pi_1(f(\Gamma(W))), \phi^{-2[\frac{n}{2}]-1} + \pi_1(f(\Gamma(W)))[$.*

Proof. First, we show that $\mathcal{P}_W$ is dense in $\mathcal{I}_W$ if and only if $\mathcal{P}_\varepsilon$ is dense in $]-1,\phi[$. Let $W = v_n \ldots v_0 \in \mathcal{L}_\phi$. Let $x \in]-\phi^{2[\frac{n+1}{2}]} + \pi_1(f(\Gamma(W))), \phi^{-2[\frac{n}{2}]-1} + \pi_1(f(\Gamma(W)))[$. Using Lemma 3.1, we get $\pi_1(f(\Gamma(W))) = \sum_{i=0}^{n} v_i(-\phi)^{-i}$. We note that $(v_i)_{i>N}$ is an admissible sequence such that $\sum_{i=N+1}^{\infty} v_i(-\phi)^{-i} \in]-\phi^{2([\frac{n+1}{2}])}, \phi^{-2[\frac{n}{2}]-1}[$ if and only if $(v'_i)_{i\in\mathbb{N}}$ is an admissible sequence such that $\sum_{i=0}^{\infty} v'_i(-\phi)^{-i} \in]-1,\phi[$, where the sequences $(v_i)_{i>N}$ and $(v'_i)_{i\in\mathbb{N}}$ are in relation by $v'_i = v_{i+N+1}$ for all $i \in \mathbb{N}$. Hence, in order to prove the first assertion, we prove now that $\mathcal{P}_\varepsilon$ is dense in $]-1,\phi[$.

Let $x' \in]-1,\phi[$. We define $x_0 = x'$, and $v'_0 = 0$ if $x_0 \in]-1,\phi^{-1}[$, $v'_0 = 1$ otherwise. Then, since $x_0 - v'_0 \in]-1,\phi^{-1}[$, we have $x_1 = -\phi(x_0 - v'_0) \in]-1,\phi[$. By induction, if the terms of the sequences $(v'_i)_{i\in[\![0,\ldots,n-1]\!]}$ and $(x_i)_{i\in[\![0,\ldots,n]\!]}$ are defined, we set $v'_n = 0$ if $x_n \in]-1,\phi^{-1}[$, $v'_n = 1$ otherwise, and $x_{n+1} = -\phi(x_n - v'_n)$. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence with values in $]-1,\phi[$, and $(v'_n)_{n\in\mathbb{N}}$ is an admissible sequence. Since $x - \sum_{i=0}^{n} v'_i(-\phi)^{-i} = (-\phi)^{n+1}x_{n+1}$ for all $n \in \mathbb{N}$, we have constructed an admissible sequence $(v_i)_{i\in\mathbb{N}}$ such that $x = \sum_{i=0}^{\infty} v_i(-\phi)^{-i}$, which proves the density of $\mathcal{P}_W$ in $]-\phi^{2[\frac{n+1}{2}]} + \pi_1(f(\Gamma(W))), \phi^{-2[\frac{n}{2}]-1} + \pi_1(f(\Gamma(W)))[$. We deduce that any interval which contains $\mathcal{P}_W$ contains also the interval $]-\phi^{2[\frac{n+1}{2}]} + \pi_1(f(\Gamma(W))), \phi^{-2[\frac{n}{2}]-1} + \pi_1(f(\Gamma(W)))[$, hence $]-\phi^{2[\frac{n+1}{2}]} + \pi_1(f(\Gamma(W))), \phi^{-2[\frac{n}{2}]-1} + \pi_1(f(\Gamma(W)))[\subset \mathcal{I}_W$. Since $\mathcal{I}_W$ is the intersection of all intervals that contain $\mathcal{P}_W$, $\mathcal{I}_W \subset]-\phi^{2[\frac{n+1}{2}]} + \pi_1(f(\Gamma(W))), \phi^{-2[\frac{n}{2}]-1} + \pi_1(f(\Gamma(W)))[$. □

Lemma 3.5. *Let $W \in \mathcal{L}_\phi$. Then:*

1. *for any prefix P of $W = PS$, the word $W' = P\,0^{|S|}$ is admissible and $\mathcal{I}_W - \pi_1(f(\Gamma(S))) \subset \mathcal{I}_{W'}$;*
2. *for all $k \in \mathbb{N}$, $\mathcal{I}_{W0^k} = (-\phi)^{-k}\mathcal{I}_W$;*
3. *for any suffix S of W, $\mathcal{I}_W \subset \mathcal{I}_S$.*

Proof. Let $W = PS$, and let $\sigma^n(\varepsilon_n)\ldots\sigma^{n'+1}(\varepsilon_{n'+1})\sigma^{n'}(\varepsilon_{n'})\ldots\sigma^0(\varepsilon_0)$ and $\sigma^{n'}(\varepsilon_{n'})\ldots\sigma^0(\varepsilon_0)$ be respectively the Dumont-Thomas expansions of $\Gamma(W)$ and $\Gamma(S)$. Using the linear properties of f and π_1, it follows:

$$\begin{aligned}
\pi_1(f(\sigma^n(\varepsilon_n)\ldots\sigma^0(\varepsilon_0))) - \pi_1(f(\sigma^{n'}(\varepsilon_{n'})\ldots\sigma^0(\varepsilon_0))) &= \pi_1(f(\sigma^n(\varepsilon_n)\ldots\sigma^{n'+1}(\varepsilon_{n'+1})\varepsilon)) \\
&= \pi_1 \circ f \circ \Gamma(P0^{|S|}).
\end{aligned}$$

Thus, $\pi_1(f(\Gamma(W))) - \pi_1(f(\Gamma(S)))$ belongs to $\{\pi_1(f(\Gamma(W'))), W' \in \mathcal{C}_{P0^{|S|}}\}$ for any admissible word $W = PS$, which ends the proof of the first assertion.

Let $W = w_n \ldots w_0$ and $W' = W0^k$. One has :

$$\pi_1(f(\Gamma(W'))) = \sum_{i=0}^{n} w_i(-\phi)^{-(i+k)} = (-\phi)^{-k}\sum_{i=0}^{n} w_i(-\phi)^{-i} = (-\phi)^{-k}\pi_1(f(\Gamma(W))).$$

Hence $\mathcal{I}_{W0^k} = (-\phi)^{-k}\mathcal{I}_W$, which proves the second assertion.

Finally, if S is a suffix of W, then $\mathcal{C}_W \subset \mathcal{C}_S$, hence $\mathcal{I}_W \subset \mathcal{I}_S$. □

Proposition 3.6. *If W and W' are expansions in base ϕ of ϕ-integers such that $\mathcal{I}_W \cap \mathcal{I}_{W'} \neq \varnothing$, then either $\mathcal{C}_W \subset \mathcal{C}_{W'}$ or $\mathcal{C}_{W'} \subset \mathcal{C}_W$.*

Proof. Suppose that W and W' are expansions in base ϕ of ϕ-integers such that none of them is a suffix of the other. Let S be the common suffix of W and W' which is of maximal length. Then $W = PS$, $W' = P'S$, where P and P' have a different suffix of length 1 (for instance, 1 is a suffix of P).

Suppose that $\mathcal{I}_W$ and $\mathcal{I}_{W'}$ are not disjoint. Using successively the three assertions of Lemma 3.5, this implies, first that $\mathcal{I}_{P0^{|S|}}$ and $\mathcal{I}_{P'0^{|S|}}$ are not disjoint, second that $\mathcal{I}_P$ and $\mathcal{I}_{P'}$ are not disjoint, and finally that $\mathcal{I}_1$ and $\mathcal{I}_0$ are not disjoint. This is absurd, since $\mathcal{I}_0 =]-1, \phi^{-1}[$ and $\mathcal{I}_1 =]\phi^{-1}, \phi[$, see Remark 3.3. $\square$

Thus, the image of the set of admissible expansions having W as a suffix under $\pi_1 \circ f \circ \Gamma$ is dense in $\mathcal{I}_W$. Additionally, Proposition 3.6 establishes that $\mathcal{I}_W$ and $\mathcal{I}_{W'}$ are disjoint when W and W' are distincts admissible words of the same length. Hence, for $k \in \mathbb{N}^*$, the sets $\mathcal{P}_{W_{k_i}}$ form a partition of $\mathcal{P}_\varepsilon$, where $\{W_{k_i}\}_{i \in [\![1 \ldots F_k]\!]}$ denotes the set of admissible words of length k. We deduce a subdivision of $\mathcal{I}_\varepsilon$ into F_k intervals $\mathcal{I}_{W_{k_i}}$. When $k = 2$, the associated subdivision is depicted in Figure 2.

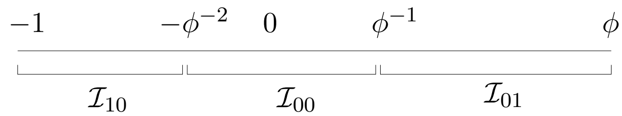

Figure 2: Tiling of $\mathcal{I}_\varepsilon =]-1, \phi[= \mathcal{I}_{10} \cup \mathcal{I}_{00} \cup \mathcal{I}_{01}$

Any admissible word W admits either 0 or 01 as a suffix, and, due to the first assertion of Lemma 3.5, the maps $\mathcal{C}_\varepsilon \to \mathcal{C}_0, v \longmapsto v0$ and $\mathcal{C}_\varepsilon \to \mathcal{C}_{01}, v \longmapsto 01$ are in one-to-one correspondance. Hence $\mathcal{I}_\varepsilon$ satisfies the relation $\mathcal{I}_\varepsilon = \mathcal{I}_1 \cup \mathcal{I}_{01} = (-\phi^{-1}\mathcal{I}_\varepsilon) \cup (1 + \phi^{-2}\mathcal{I}_\varepsilon)$. This relation provides a tiling of the self-similar set $\mathcal{I}_\varepsilon$. A closer study of such tilings is performed in [1, 3, 28]. The tiling of $\mathcal{I}_\varepsilon =]-1, \phi[$ defined by the cylinders is a self-similar tiling of $]-1, \phi[$. The dual tiling defined by $\bigcup_{k \in \mathbb{N}} \pi_2(f(\omega_k))$ is a discrete tiling of $\mathbb{R}_+$, which corresponds to the quasicrystal associated to ϕ, see [9].

As it is possible to extend the tiling of $]-1, \phi[$ to $\mathbb{R}$, we deduce that sums and products of the images under τ of ϕ-integers belong to unions of tiles of the tiling defined by π_1. These tiles can be geometrically characterized, and have a combinatorial signification. The property of defining an associated tiling remains true for any Pisot number; in particular, when β is the positive root of the polynomial $X^3 - X^2 - X - 1$ (called then the *Tribonacci number*), we get a *Rauzy fractal* $\mathcal{T}$ which is a subset of the hyperplane generated by the eigenvectors of the incidence matrix having an associated eigenvalue of modulus less than 1. One may find more details about Rauzy fractals in chapter 7 of [24] and in [25]. Note that, from a general point of view, the Rauzy fractal $\mathcal{T}$ has a fractal structure; however, when β is a quadratic unit, the Rauzy fractal $\mathcal{T}$ defined by the associated β-substitution is an interval. As a consequence, many studies, including the one performed in this article, are less complicated when we consider quadratic unit numbers.

Corollary 3.7. *Let $x \in Fin(\phi)$, $x > 0$ with $d_\phi(x) = v_N \ldots v_0.v_{-1} \ldots v_{-N'}$. Then $\tau(x) \in (-\phi)^{-N'}]\phi^{-1}, \phi[$.*

Proof. If $x = \sum_{i=-N'}^{N} v_i \phi^i$ with $v_{-N'} = 1$, then $\phi^{N'}x$ is a ϕ-integer whose expansion admits 1 as a suffix. Since $\mathcal{I}_1 =]\phi^{-1}, \phi[$, one gets $\tau(\phi^{N'}x) = \tau(\phi)^{N'}\tau(x) \in]\phi^{-1}, \phi[$, which implies

$\tau(x) \in (-\phi)^{-N'}]\phi^{-1}, \phi[$. $\square$

3.3 Characterization of ϕ-integers

Due to Corollary 3.7 and Proposition 3.4, knowing a suffix S of the ϕ-expansion of $x \in \mathrm{Fin}(\phi)$ provides an interval $\mathcal{I}$, which depends on S, such that $\tau(x) \in \mathcal{I}_S$. Conversely, we want to determine whether, knowing an interval $\mathcal{I}$ which contains $\tau(x) \in \mathbb{Q}(\phi)$, one may find $k \in \mathbb{N}$, in the case where such an integer does exist, such that $\phi^k x \in \mathbb{Z}_\phi^+$. This problem is closely related to determining the ϕ-fractional length of the ϕ-expansion of $x \in \mathrm{Fin}(\phi)$.

Let $\mathcal{B}_\phi^+$ be the semi-window of $\mathbb{R}^2$ defined by $\pi_1(X) \in]-1, \phi[$ and $\pi_2(X) \geqslant 0$, which is depicted in Figure 3. We have the following property.

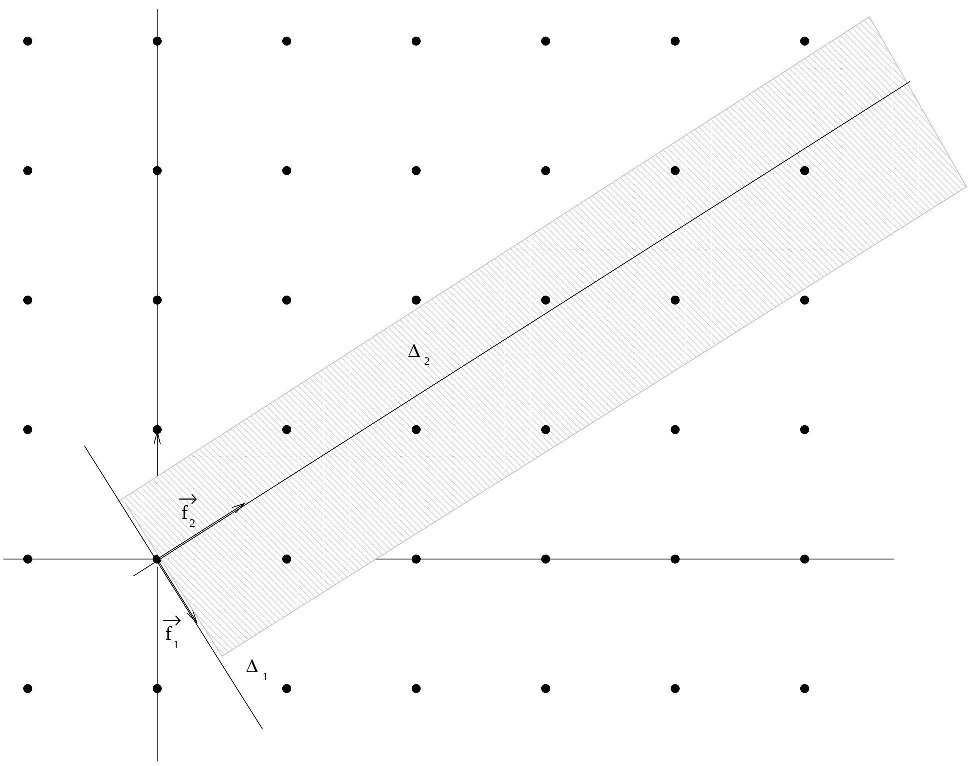

Figure 3: Geometrical representation of $\mathcal{B}_\phi^+$

Proposition 3.8. *Let $X \in \mathbb{Z}^2$. Then $\pi_2(X) \in \mathbb{Z}_\phi^+$ if and only if $X \in \mathcal{B}_\phi^+$.*

One can find the proof of this proposition in [5], and in [6] as well.

Corollary 3.9. *Let $X \in \mathbb{Z}^2$ such that $\pi_2(X) > 0$. There exists $N \in \mathbb{Z}$ such that $\pi_1(X)$ belongs to $(-\phi)^N]\phi^{-1}, \phi[$. Moreover, if $N \in \mathbb{N}^*$, then $d_\phi(\{\pi_2(X)\}_\phi) = 0.x_{-1} \dots x_N$.*

Proof. One has $\bigcup_{N\in\mathbb{Z}} (-\phi)^{N'}]\phi^{-1}, \phi[= \mathbb{R} \setminus \bigcup_{Z\in\mathbb{N}} -(-\phi)^N$. Suppose that there exists $N \in \mathbb{Z}$ such that $\pi_1(X) = -(-\phi)^N$. Then $\pi_2(X) = \tau(\pi_1(X)) = -\phi^N$, hence $\pi_2(X) < 0$. We deduce that, if $\pi_2(X) > 0$, there exists $N \in \mathbb{Z}$ such that $\pi_1(X) \in (-\phi)^N]\phi^{-1}, \phi[$.

If $\pi_1(X) \in (-\phi)^{-N'}]\phi^{-1}, \phi[$, then $\pi_1(X)(-\phi)^{N'} \in]\phi^{-1}, \phi[$ and $\pi_1(X)\tau(\phi^{-N'}) \in]\phi^{-1}, \phi[$. Let $x = \pi_2(X)$. Since $\tau(x)\tau(\phi^{-N'}) \in]\phi^{-1}, \phi[$, $x' = x\phi^{-N'}$ fulfills the relation $\tau(x') \in]\phi^{-1}, \phi[$. Additionally, since M_σ is invertible, there exists $X' \in \mathbb{Z}^2$ such that $\pi_2^{-1}(x') = M_\sigma^{-N'} X = X'$. Since $\pi_2(X') \geqslant 0$, we may use Proposition 3.8; there exists $\omega_k = \sigma^n(\varepsilon_n) \dots \sigma^0(\varepsilon_0)$ such that $X' = f(\omega_k)$, hence $\pi_2(X') = \sum_{i=0}^{n} |\varepsilon_i| \phi^i$. Thus, $x = \sum_{i=0}^{n} |\varepsilon_i| \phi^{i-N'}$, hence $d_\phi(\pi_2(X)) = |\varepsilon_n| \dots |\varepsilon_0| . |\varepsilon_{-1}| \dots |\varepsilon_{-N'}|$. ☐

Corollary 3.10. *Let $x, y \in \mathbb{Z}_\phi^+$. Then $\phi^2(x+y)$, $\phi^2(x-y)$ and $\phi^2 xy \in \mathbb{Z}_\phi^+$.*

Proof. Since $\phi^2(x+y)$, $\phi^2(x-y)$ and $\phi^2 xy$ are positive, we can use Corollary 3.9. Then:

1. $\tau(x+y) \in]-2, 2\phi[=]-\phi-\phi^{-2}, \phi^2+\phi^{-1}[$, hence $\tau(\phi^2(x+y)) \in]-1, \phi[$.
2. $\tau(x-y) \in]-1-\phi, 1+\phi[=]-\phi^2, \phi^2[$; this implies $\tau(\phi^2(x-y)) \in]-1, \phi[$.
3. $\tau(xy) \in]-\phi, \phi^2[$, hence $\tau(\phi^2 xy) \in]-\phi^{-1}, 1[\subset]-1, \phi[$.

☐

Remark 3.11. *These results were first proved in the framework of quasicrystals in [9, 10].*

Note that images under τ of the sum, the subtraction or the product of two non-negative ϕ-integers belong in fact to an interval which is strictly included in $]-\phi^2, \phi^3[$. This provides additional information about the suffixes of the ϕ-integers $\phi^2(x+y)$, $\phi^2(x-y)$ and $\phi^2 xy$, when $x, y \in \mathbb{Z}_\phi^+$. For instance, since $\tau(\phi^2(x-y)) \in]-1, 1[$, then, as a consequence of Lemma 3.1, 101 is not a suffix of $d_\phi(\phi^2(x-y))$.

4 Continued ϕ-fraction algorithm

Notation 4.1. *Let $(p_i)_{i \in \mathbb{N}}$ be a sequence which consists of positive real numbers. We denote by $[p_0; p_1 \dots, p_{n-1}, p_n]$ the finite continued fraction $p_0 + \frac{1}{p_1 + \dots + \frac{1}{p_n}}$.*

4.1 Definition of the generalized Euclid's algorithm

We explain here how to generalize Euclid's algorithm which generates the expansion in continued fraction of a positive real number. This study is very similar to the classical one with usual continued fractions, that can be found for instance in [20] or in [21].

We define the representation of $x \in \mathbb{R}_+$ by a continued ϕ-fraction using the following algorithm. Let $x_0 = x$. Since $x_0 - [x_0]_\phi \in [0, 1[$ according to Proposition 2.6, we define $x_1 = \frac{1}{x_0 - [x_0]_\phi}$ if $x_0 - [x_0]_\phi > 0$, otherwise the algorithm ends. More generally, at step $i \in \mathbb{N}$, we define $x_{i+1} = \frac{1}{x_i - [x_i]_\phi}$ if x_i is not a ϕ-integer, otherwise the algorithm ends. The constructed sequence $(x_i)_i$ is generated using the function T defined as follows.

$$T :]0, 1[\longrightarrow]0, 1[, x \longmapsto \{\frac{1}{x}\}_\phi.$$

Hence, while $x_i \notin \mathbb{Z}_\phi^+$, that is, while $T^i(x_0) > 0$, x_{i+1} is defined by $\frac{1}{x_{i+1}} = T(\frac{1}{x_i}) = T^{i+1}(\frac{1}{x_0})$. If the sequence $(x_i)_i$ is finite, then the representation of x is $[x_0]_\phi + \frac{1}{\ldots + \frac{1}{x_N}}$. Otherwise, we get:

$$\begin{aligned} x_0 &= [x_0]_\phi + T(\frac{1}{x_0}) = [x_0]_\phi + \frac{1}{x_1} = [x_0]_\phi + \frac{1}{[x_1]_\phi + T(\frac{1}{x_1})} \\ &= [x_0]_\phi + \frac{1}{[x_1]_\phi + \frac{1}{\ldots + \frac{1}{[x_n]_\phi + \ldots}}} = [[x_0]_\phi; [x_1]_\phi, \ldots]. \end{aligned}$$

Definition 4.2. *We define the n-th partial ϕ-quotient of x as the ϕ-integer $a_n = [x_n]_\phi$, and the n-th ϕ-convergent of x as $c_n = [a_0; a_1 \ldots, a_n]$. The expansion $a_0 + \frac{1}{a_1 + \ldots \frac{1}{a_n + \ldots}}$ is called the* continued ϕ-fraction of x.

Lemma 4.3. *Let $(a_i)_i$ be the sequence of partial quotients of $x \in \mathbb{R}_+$. Let $k \in \mathbb{N}$ be such that a_k and a_{k+1} are defined. If $d_\phi(a_k)$ admits 1 as a suffix, then $a_{k+1} \geqslant \phi$.*

Proof. Let $(a_i)_i$ be the sequence of partial quotients of $x \in \mathbb{R}_+$. Suppose that a_k and a_{k+1} are defined, and that $d_\phi(a_k)$ admits 1 as a suffix. Then, due to Proposition 2.6, $x_k - a_k$ belongs to $[0, \phi^{-1}[$. Since a_{k+1} is defined, $x_k \neq a_k$, and $\frac{1}{x_i - a_i} > \phi$. Hence $a_{k+1} \geqslant \phi$. □

Remark 4.4. *We deduce that there exist sequences of ϕ-integers that are not sequences of partial quotients. For instance, $[1; 1^\infty]$, which is the classical continued fraction of ϕ, is not a continued ϕ-fraction. Since $\phi \in \mathbb{Z}_\phi^+$, the continued ϕ-fraction of ϕ is $[\phi; 0^\infty]$.*

Proposition 4.5. *The sequence of ϕ-convergents of x tends to x.*

Proof. Let $m, n \in \mathbb{N}$ with $m > n$. Then:

$$\begin{aligned} |[a_0; \ldots, a_n] - [a_0; \ldots, a_m]| &= |[0; a_1, \ldots a_n] - [0; a_1, \ldots, a_m]| = \left| \frac{1}{[a_1; \ldots a_n]} - \frac{1}{[a_1; \ldots a_m]} \right| \\ &= \left| \frac{[a_1; \ldots a_n] - [a_1; \ldots a_m]}{[a_1; \ldots a_n][a_1; \ldots a_m]} \right| \leqslant \frac{1}{a_1^2} |[a_1; \ldots a_n] - [a_1; \ldots a_m]|. \end{aligned}$$

Due to Lemma 4.3, $a_i = 1$ implies $a_{i+1} \geqslant \phi$. Hence the inequality:

$$\begin{aligned} |[a_0; \ldots, a_n] - [a_0; \ldots, a_m]| &\leqslant (\prod_{i=1}^{n} \frac{1}{a_i^2}) \times |a_n - [a_n; a_{n+1}, \ldots, a_m]| \leqslant \frac{1}{a_1} \prod_{i=1}^{n-1} \frac{1}{(a_i a_{i+1})} \\ &\leqslant \phi^{1-n}. \end{aligned}$$

Thus, the sequence of ϕ-convergents of x is a Cauchy sequence. Since $|[a_0; \ldots a_n] - x| \leqslant \phi^{1-n}$ holds as well, the sequence $([a_0; \ldots a_n])_{n \in \mathbb{N}}$ tends to x. □

It is clear that a finite continued ϕ-fraction represents a positive element of $\mathbb{Q}(\phi)$, and it is natural to ask whether the reciprocal property holds. Thus, we will prove the following result, conjectured by Akiyama [29]: any positive element of $\mathbb{Q}(\phi)$ can be represented by a finite continued ϕ-fraction.

We remark that we need first to define a canonical way to expand elements of $\mathbb{Q}(\phi)^+$. The following proposition allows us to expand any positive element of $\mathbb{Q}(\phi)$ as a quotient of positive ϕ-integers.

Proposition 4.6. *One has* $\mathbb{Q}_\phi^+ = \mathbb{Q}(\phi)^+$.

Proof. Let $x \in \mathbb{Q}(\phi)^+$. There exist a and $b \in \mathbb{Z}[\phi]$, both positive real numbers such that $x = \frac{a}{b}$. There exist k and $k' \in \mathbb{N}$ such that the quantities $\tau(\phi^k a)$ and $\tau(\phi^{k'} b)$ both belong to $]-1, \phi[$. Due to Proposition 3.8, it implies that $\phi^k a$ and $\phi^{k'} b$ are ϕ-integers. Let $l = \max\{k, k'\}$. Then, $x = \frac{\phi^l a}{\phi^l b}$, so $x \in \mathbb{Q}_\phi^+$. Since $\mathbb{Q}_\phi^+$ is a subset of $\mathbb{Q}(\phi)^+$, the required equality is proved. $\square$

Remark 4.7. *The result provided by Proposition 4.6 may easily be extended to the class of numbers such that the finiteness property* $(\mathcal{F})$ *holds. Indeed, since* $\mathbb{Q}(\beta)^+ = \mathbb{Q}(\beta^{-1})^+$, *any element* $x \in \mathbb{Q}(\beta)$ *can be expanded as* $\frac{p}{q}$, *where* $p, q \in \mathbb{Z}[\beta^{-1}]$. *If the finiteness property* $(\mathcal{F})$ *holds,* p *and* q *have a finite* β*-expansion. Let* $l = \max\{|d_\beta(\{p\}_\beta)|, |d_\beta(\{q\}_\beta)|\}$. *Then* $p' = p\beta^l$ *and* $q' = q\beta^l$ *are* β*-integers which satisfy* $x = \frac{p}{q}$.

4.2 An algorithm applied on ϕ-fractions

We are interested in studying the sequence of partial ϕ-quotients when we apply the continued ϕ-fraction algorithm on $x \in \mathbb{Q}(\phi)^+$. Since $[0,1] \setminus \mathbb{Q}(\phi)^+$ is stable under T, and due to Proposition 4.6, it is possible to expand the elements of the sequence $(x_i)_i$ as ϕ-fractionary expansions (p_i, q_i). Thus, we define an algorithm A that constructs a sequence of ϕ-fractionary expansions $(p_i, q_i)_i$, such that for all i, $x_i = \frac{p_i}{q_i}$. Then, we establish connections between $t(p_i, q_i)$ and $t(p_{i+1}, q_{i+1})$.

Lemma 4.8. *Let* $p, q \in \mathbb{Z}_\phi^+$ *with* $q \neq 0$. *Then* $\phi^3(p - [\frac{p}{q}]_\phi q) \in \mathbb{Z}_\phi^+$.

Proof. Since p, $[\frac{p}{q}]_\phi$ and q are ϕ-integers, their images under τ belong to $]-1, \phi[$. Hence $\tau(p - [\frac{p}{q}]_\phi q) \in]-\phi^2 - 1, \phi^2 + \phi^{-1}[\subset (-\phi)^3] - 1, \phi[$. Using Corollary 3.9, we get $\phi^3(p - [\frac{p}{q}]_\phi q) \in \mathbb{Z}_\phi^+$. $\square$

Due to the previous properties, we define an algorithm on the set of pairs of ϕ-integers which performs the following operations.

1. It subtracts from the first element of the pair (p, q) the quantity $[\frac{p}{q}]_\phi q$.
2. It multiplies each element of the pair $(p - [\frac{p}{q}]_\phi q, q)$ by ϕ^M, choosing M minimal among the integers $k \in \mathbb{Z}$ such that $\phi^k(p - [\frac{p}{q}]_\phi q) \in \mathbb{Z}_\phi^+$.
3. It exchanges the elements of the pair $(\phi^M(p - [\frac{p}{q}]_\phi q), \phi^M q)$.

Remark 4.9. *As a consequence of Lemma 4.8, the value of* M *defined at step 2. of the algorithm* A *satisfies* $M \leqslant 3$. *Moreover, by definition of* M, *0 cannot be a common suffix of* $d_\phi(\phi^M(p - [\frac{p}{q}]_\phi q))$ *and* $d_\phi(\phi^M q)$.

Example 4.10.
Let $p = \phi^3 + 1$ *and* $q = \phi^2 + 1$. *Then* $p = q \times 1 + \phi$, *hence* $M = 0$, $p' = q = \phi^2 + 1$ *and* $q' = 1$.
Let $p = \phi^3$ *and* $q = \phi^2 + 1$. *Then* $p = q \times 1 + \phi^{-1}$, *hence* $M = 1$, $p' = \phi q = \phi^3 + \phi$ *and* $q' = 1$.
Let $p = \phi^4$ *and* $q = \phi^2 + 1$. *Then* $p = q \times \phi + \phi + \phi^{-2}$, *hence* $M = 2$, $p' = \phi^2 q = \phi^4 + \phi^2$ *and* $q' = \phi^3 + 1$.

Let $p = \phi^7 + \phi^5 + \phi$ and $q = \phi^4 + \phi^2 + 1$. Then $p = q \times (\phi^2 + 1) + \phi^2 + 1 + \phi^{-3}$, hence $M = 3$, $p' = \phi^3 q = \phi^7 + \phi^5 + \phi^3$ and $q' = \phi^5 + \phi^3 + 1$.

We recall that Definition 1.2 introduces the notion of positive length and global length, respectively denoted by t_+ and t, which are defined for elements that belong to $\mathrm{Fin}(\phi)^+$. By construction, if (p, q) is a pair of ϕ-integers, then $A(p, q) = (p', q')$ is a pair of ϕ-integers such that $\frac{q'}{p'} = T(\frac{q}{p})$. Thus, the sequence of partial ϕ-quotients of $x \in \mathbb{Q}(\phi)^+$ is finite if and only if $(t(p_i, q_i))_i = (t(p_i) + t(q_i) - 1)_i$, the sequence of the lengths of the pairs of ϕ-integers constructed by iteration of the algorithm A, that is, such that $(p_{i+1}, q_{i+1}) = A(p_i, q_i)$ for all $i \in \mathbb{N}$, is decreasing.

In the rational case, when we iterate Euclid's algorithm on $\frac{p}{q}$, we get a fraction $\frac{q}{p-[\frac{p}{q}]q}$. The sequence of fractions constructed by iteration of Euclid's algorithm is such that the sequence of the numerators, or of the denominators, is decreasing. This proves that, for any $x \in \mathbb{Q}^+$, the continued fraction of x is finite.

There is an additional difficulty in comparison with the classical rational case. By definition of the algorithm A, the operations performed at steps 1. and step 3. do not increase the sum of the positive lengths of the studied elements. However, since we have to multiply at step 2. each element of the pair $(p - [\frac{p}{q}]_\phi q, q)$ by ϕ^M, the sum of the positive lengths of the studied elements may increase by $2M$. Since M belongs to $\{0, 1, 2, 3\}$, we deduce the inequality:

$$t(A(p, q)) \leqslant t(p, q) + 6. \tag{2}$$

Hence $(t(p_i, q_i))_i$ may be a sequence which does not decrease.

There exist examples for which $t(A(p, q)) \leqslant t(p, q)$ does not hold. For instance, $t(A(p, q)) = t(p, q) + 1$ for the third and the fourth cases of Example 4.10. Hence, contrarily to the classical rational case, we cannot directly prove that the sequence of the numerators $(p_i)_i$ produced when we iterate the algorithm A decreases. Instead, we study the sequence of the sum of lengths $(t(p_i)+t(q_i))_i$ when we iterate the algorithm A starting from a ϕ-fractionary expansion (p_0, q_0).

We see in Section 4.3 that, starting a closer study of t which depends on $[\frac{p}{q}]_\phi$, we may improve (2). More precisely, $t(p', q') > t(p, q)$ may hold in a small number of particular cases. As a consequence, the sequence of the lengths of the ϕ-fractionary expansions that are produced by the generalized Euclid's algorithm A is almost decreasing. By studying in Section 5.1 the particular cases for which t does not decrease, we prove that the sequence $(t(p_i) + t(q_i) - 1)_i$ of the lengths of the ϕ-fractionary expansions produced by iteration of A is bounded. Finally, a closer study performed in Section 5.2 allows us to prove that $(t(p_i) + t(q_i) - 1)_i$ tends to 0, hence $(t(p_i) + t(q_i) - 1)_i$ is finite.

4.3 Study of the sequence $(t(p_i, q_i))_{i\in\mathbb{N}}$

In this section, we show that the way $t(A(p, q)) - t(p, q)$ may decrease depends closely on $[\frac{p}{q}]_\phi$. More precisely, we give a better upper bound for $t(A(p, q)) - t(p, q)$ than 6, which depends, first on $t([\frac{p}{q}]_\phi)$, and second on the suffixes of $d_\phi([\frac{p}{q}]_\phi)$. Let us recall that, when $x \in \mathrm{Fin}(\phi)^+$ with $d_\phi(x) = v_N v_{N-1} \dots v_1 v_0 . v_{-1} \dots v_{-N'}$, then, according to Definition 1.2, $t_+(x)$ and $t(x)$ denote respectively the length of the ϕ-integer part of $d_\phi(x)$ and the length of $d_\phi(x)$, that is, $t_+(x) = N + 1$ and $t(x) = N + N' + 1$.

Proposition 4.11. *Let $p, q \in \mathbb{Z}_\phi$ with $p \geqslant q > 0$. Let $\lambda = [\frac{p}{q}]_\phi$.*

1. *One has* $t(\lambda) = t(p) - t(q) + 1$ *or* $t(p) - t(q)$.

2. *If* $d_\phi(\lambda)$ *admits 0 as a suffix, then* $t_+(p - \lambda q) \leqslant t(q)$.

3. *If* $d_\phi(\lambda)$ *admits 1 as a suffix, then* $t_+(p - \lambda q) \leqslant t(q) - 1$.

Proof. For p and $q \in \mathbb{Z}_\phi^+$, the relations $\phi^{t(p)-1} \leqslant p < \phi^{t(p)}$ and $\phi^{t(q)-1} \leqslant q < \phi^{t(q)}$ hold. Thus, $\phi^{t(p)-t(q)-1} < \frac{p}{q} < \phi^{t(p)-t(q)+1}$, hence $t(p) - t(q) \leqslant t(\lambda) \leqslant t(p) - t(q) + 1$, which proves the first assertion.

If 0 is suffix of $d_\phi(\lambda)$, then, due to Proposition 2.6, one has $s_\phi(\lambda) = \lambda + 1$. Since $0 \leqslant \frac{p-\lambda q}{q} < 1$, we get $t_+(p - \lambda q) \leqslant t(q)$.

Suppose now that 1 is a suffix of $d_\phi(\lambda)$. Due to Proposition 2.6, $s_\phi(\lambda) = \lambda + \phi^{-1}$. Thus, $\lambda \leqslant \frac{p}{q} < \lambda + \phi^{-1}$. Since $0 \leqslant \frac{p-\lambda q}{q} < \phi^{-1}$, we get $t_+(p - \lambda q) \leqslant t_+(q\phi^{-1})$. As we have also $t_+(q\phi^{-1}) = t(q) - 1$, then $t_+(p - \lambda q) \leqslant t(q) - 1$. □

Let (p, q) be a ϕ-fractionary expansion of x. Then, due to Proposition 4.11, one gets the following relation, where M is set at step 2 of the algorithm A:

$$\begin{aligned}
t(A(p,q)) &\leqslant 2M - 1 + |d_\phi(q)| + |d_\phi([p - [\frac{p}{q}]_\phi q]_\phi)| \\
&\leqslant 2M - 1 + 2|d_\phi(q)| \\
&\leqslant 2M - 1 + |d_\phi(p)| + |d_\phi(q)| - (|d_\phi(p)| - |d_\phi(q)|) \\
&\leqslant 2M + t(p,q) - |d_\phi([\frac{p}{q}]_\phi)|.
\end{aligned}$$

Hence $t(A(p,q)) - t(p,q)$ depends on $[\frac{p}{q}]_\phi$; more precisely, the quantity $t(A(p,q)) - t(p,q)$ may be non-negative in only a small number of cases. The following proposition starts the study in a more precise way.

Remark 4.12. *Starting from now on, we use some specific properties of* ϕ. *If we replace* ϕ *by any number which satisfies the finiteness property* $(\mathcal{F})$, *it is still possible to define the algorithm* A, *introduced in Section 4.2. In this case, (2) becomes* $t(A(p,q)) \leqslant t(p,q) + 2(L_\oplus + L_\otimes)$, *where* $L_\oplus$ *and* $L_\otimes$ *respectively denote the maximal possible length for the* β*-fractional part of the sum, or of the product, of two* β*-integers. There is still a finite number of cases for which the quantity* $t(A(p,q)) - t(p,q)$ *may be positive, but the study of this set of possibilities is more complicated than the present study performed in the Fibonacci case. In particular, we do not know for which numbers* β *the result provided by Theorem 5.3 holds, or even for which numbers the weaker result that, for any* $p, q \in \mathbb{Z}_\beta^+$ *with* $q > 0$, *the continued* β*-fraction of* $\frac{p}{q}$ *is either finite or ultimately periodic, holds.*

Remark 4.13. *Note that, when* β *satisfies* $d_\beta(1) = 0.41$, *then* $\beta = \phi^3$. *Since* $\phi = \phi^3 - \phi - 1$, *one has* $\phi = \frac{\beta-1}{2}$, *hence* $\mathbb{Q}(\beta) = \mathbb{Q}(\phi)$. *We check that the continued* β*-fraction of* ϕ *corresponds in this case to the classical continued fraction of* ϕ, *that is,* $\phi = [1; 1^\infty]$. *Hence Theorem 5.3 does not hold for the numeration system defined by* $d_\beta(1) = 0.41$.

Proposition 4.14. *Let* $p, q \in \mathbb{Z}_\phi^+$, *with* $p \geqslant q > 0$. *Let* $\lambda = [\frac{p}{q}]_\phi$.

1. *If* $d_\phi(\lambda) = (10)^k$ *with* $k \in \mathbb{N}^*$, *then* $t_+(p - \lambda q) \leqslant t(p) - t(\lambda)$.

2. *If* $d_\phi(\lambda) = (10)^k 1$ *with* $k \in \mathbb{N}$, *then* $t_+(p - \lambda q) \leqslant t(p) - t(\lambda) - 1$.

Proof. Assume that $d_\phi(\lambda) = (10)^k$. Since $t(\lambda) = 2k$, $t(p) - t(q) = 2k$ or $2k - 1$ according to Proposition 4.11. If $t(p) - t(q) = 2k$, then, since $t_+(p - \lambda q) \leqslant t(q)$, $t_+(p - \lambda q) \leqslant t(p) - 2k$ holds, and we get the required inequality. Otherwise, suppose that $t(p) - t(q) = 2k - 1$. Since $t_+(p - \lambda q) > t(p) - 2k$, we get $t_+(p - \lambda q) = t(q) = t(p) - 2k + 1$. Moreover, $\lambda q = \sum_{i=1}^{k} \phi^{2i-1} q = q(\phi^{2k} - \phi^{-1})$. Let $t(q) = n$. Then $q \geqslant \phi^{n-1}$ by definition of t. Hence $\lambda q \geqslant \phi^{n+2k-1} - \phi^{n-2}$. Since $t(p - \lambda q) = t(q) = n$, $p - \lambda q \geqslant \phi^{n-1}$ and $p \geqslant \phi^{n+2k-1}$, which implies $t(p) \geqslant n + 2k$. This contradicts the relation $t(p) - t(q) = 2k - 1$.

The second assertion can be proved in the same way. If $d_\phi(\lambda) = (10)^k 1$, then $t(\lambda) = 2k+1$, thus $t(p) - t(q) = 2k + 1$ or $2k$, using the first point of Proposition 4.11. Since 1 is a suffix of $d_\phi(\lambda) = (10)^k 1$, we get $t_+(p - \lambda q) \leqslant t(q) - 1$, using the second point of Proposition 4.11. Thus, we need $t(p) - t(q) = 2k$ and $t_+(p - \lambda q) = t(q) - 1$ to fulfill the relation $t_+(p - \lambda q) > t(p) - 2(k + 1)$. However we prove, as in the first assertion, that $t(p) = t_+(p - \lambda q) + 2k + 2$, which contradicts $t(p) = t_+(p - \lambda q) + 2k + 1$. Hence $t_+(p - \lambda q) \leqslant t(p) - 2(k + 1)$. $\square$

Proposition 4.15. *Let $p, q \in \mathbb{Z}_\phi^+$, with $p \geqslant q > 0$. Let $\lambda = [\frac{p}{q}]_\phi$. Let $r = p - \lambda q$.*

1. *If $d_\phi(\lambda) = 1$,10 or 100, then $t_+(r) \leqslant t(p) - 2$.*
2. *If $d_\phi(\lambda) = 1000$, then $t_+(r) \leqslant t(p) - 3$.*
3. *In all other cases, $t_+(r) \leqslant t(p) - 4$.*

Proof. Since $t(q) \geqslant t_+(r)$, we assume that $t(p) - t(q) \leqslant 3$, otherwise $t_+(r) \leqslant t(p) - 4$ holds. Using the first point of Proposition 4.11, we get $t(\lambda) \leqslant 4$. Hence the only possible values for λ are 1, ϕ, ϕ^2, ϕ^2+1, ϕ^3, ϕ^3+1 and $\phi^3+\phi$. The case where $d_\phi(\lambda)$ belongs to $\{1, 10, 101, 1010\}$ is a particular case of Proposition 4.14. We get the inequalities $t_+(r) \leqslant t(p)-2$ for $d_\phi(\lambda) \in \{1, 10\}$, and $t_+(r) \leqslant t(p) - 4$ for $d_\phi(\lambda) \in \{101, 1010\}$. Assertions 1 and 2 of Proposition 4.11 provide the inequalities $t_+(r) \leqslant t(p) - 2$ when $d_\phi(\lambda) = 100$, and $t_+(r) \leqslant t(p) - 3$ when $d_\phi(\lambda) = 1000$. Finally, if $d_\phi(\lambda) = 1001$, then $t(q) \leqslant t(p) - 3$ according to the first assertion of Proposition 4.11. Using the second assertion of this proposition, we deduce $t_+(r) \leqslant t(p) - 4$. $\square$

Corollary 4.16. *Let $\frac{p}{q} \in \mathbb{Q}_\phi^+$, $\lambda = [\frac{p}{q}]_\phi$ and $(p', q') = A(p, q)$. Then $t(p', q') > t(p, q)$ can only hold in the following cases:*

1. *$d_\phi(\lambda) \in \{1, 10, 100, 1000\}$ and $\tau(p - \lambda q) \in]\phi, \phi^2 + \phi^{-1}[$,*
2. *$\tau(p - \lambda q) \in] - \phi^2 - 1, -\phi^2[$, and, either $t(\lambda) \leqslant 5$, or $t(\lambda) = 6$ with 0 suffix of $d_\phi(\lambda)$.*

Proof. If $\tau(p - \lambda q) \in]-\phi^2, \phi[$, then $t(p', q') \leqslant 2 + t(p - \lambda q, q) \leqslant 2 + t_+(p - \lambda q) + t(q) \leqslant t(p) + t(q)$, where the last inequality follows from Proposition 4.15. We deduce that $t(p', q') > t(p, q)$ only holds when $\tau(p - \lambda q) \notin] - \phi^2, \phi[$. This is possible when either $\tau(p - \lambda q) \in] - \phi^2 - 1, -\phi^2[$ or $\tau(p - \lambda q) \in]\phi, \phi^2 + \phi^{-1}[$.

First, note that $p - \lambda q > 0$ by definition of λ. Since $\tau(x) = -\phi^2$ or $\tau(x) = \phi$ implies respectively $x = -\phi^{-2}$ or $x = -\phi^{-1}$, then $\tau(p - \lambda q)$ cannot be equal to $-\phi^2$ or ϕ.

If $\tau(p - \lambda q) \in]\phi, \phi^2 + \phi^{-1}[$, then, using Corollary 3.9, $\phi^2(p - \lambda q) \in \mathbb{Z}_\phi^+$, hence $\frac{p'}{q'} = \frac{\phi^2 q}{\phi^2(p - \lambda q)}$, and $t(p', q') - t(p, q) \leqslant 4 + t_+(p - [\frac{p}{q}]_\phi q) - t(p)$. If $d_\phi(\lambda) \notin \{1, 10, 100, 1000\}$, then, using the third point of Proposition 4.15, we get the relation $4 + t_+(p - \lambda q) - t(p) \leqslant 0$. In this case, $t(p', q') > t(p, q)$ can only occur when $d_\phi(\lambda) \in \{1, 10, 100, 1000\}$.

If $\tau(p-\lambda q)\in]-\phi^2-1,-\phi^2[$, we get $t(p',q')-t(p,q)\leqslant 6+t_+(p-\lambda q)-t(p)$. Using Propositions 4.11, 4.14 and 4.15, this proves that $t(p',q')-t(p,q)>0$ can only occur when either $t(\lambda)<6$, or $t(\lambda)=6$ with 0 suffix of $d_\phi(\lambda)$. $\square$

5 Proof of Theorem 5.3

The proof of Theorem 5.3 consists of two steps. First, we prove that the continued ϕ-fraction of any $x\in\mathbb{Q}(\phi)^+$ is either ultimately periodic or finite. Since an ultimately periodic continued ϕ-fraction occurs only if the algorithm A produces a sequence of ϕ-fractions of bounded length, this means that there exist cycles in the automaton which represent the action of the algorithm A. Then, we compute these cycles, and we check that they correspond to quadratic numbers over $\mathbb{Q}(\phi)$ which do not belong to $\mathbb{Q}(\phi)$ itself.

5.1 Ultimately periodicity of the continued ϕ-fraction of $x\in\mathbb{Q}(\phi)^+$

When p and q are ϕ-integers, any pair $(\tau(p),\tau(q))$ belongs to $]-1,\phi[\times]-1,\phi[$. We define a subdivision of $]-1,\phi[\times]-1,\phi[$ into three parts E_1, E_2 et E_3 in the following way:

$$E_1=]-1,\phi^{-1}[\times]-1,\phi^{-1}[,$$

$$E_2=]-\phi^{-1},\phi^{-1}[\times]\phi^{-1},1[,$$

$$E_3 \text{ is the complement of } E_1\cup E_2 \text{ in }]-1,\phi[\times]-1,\phi[.$$

Let us note that, using Proposition 3.4, it is possible to give a symbolic definition of the sets of pairs (p,q) of non-negative ϕ-integers such that $(\tau(p),\tau(q))\in E_1$, E_2 or E_3. We do not give this definition, since we do not need it in the following.

Remark 5.1. *The study of $t(A(p,q))-t(p,q)$ needs to define an appropriate partition of $\mathcal{T}\times\mathcal{T}$ in the general case of a number β which satisfies the finiteness property $(\mathcal{F})$. Let us remind that the Rauzy fractal $\mathcal{T}$ is particularly easy to describe in the case of the Fibonacci numeration system, since $\mathcal{T}$ is then the interval $[-1,\phi]$. The partition (E_1,E_2,E_3) is particularly well fitted for the computations performed in this section; however we do not know whether it is possible, given β which satisfies the finiteness property $(\mathcal{F})$, to construct a canonical partition of $\mathcal{T}\times\mathcal{T}$ suited for the study of $t(A(p,q))-t(p,q)$.*

Proposition 5.2. *Let $\frac{p}{q}$ and $\frac{p'}{q'}$ be two ϕ-fractions such that $(p',q')=A(p,q)$. Then:*

1. *$(\tau(p'),\tau(q'))\notin E_1$;*
2. *if $(\tau(p),\tau(q))\in E_3$ and $(\tau(p'),\tau(q'))\in E_2$, then $t(p',q')\leqslant t(p,q)+2$;*
3. *if $(\tau(p),\tau(q))\in E_2$ and $(\tau(p'),\tau(q'))\in E_2$, then $t(p',q')\leqslant t(p,q)$;*
4. *if $(\tau(p),\tau(q))\in E_3$ and $(\tau(p'),\tau(q'))\in E_3$, then $t(p',q')\leqslant t(p,q)$;*
5. *if $(\tau(p),\tau(q))\in E_2$ and $(\tau(p'),\tau(q'))\in E_3$, then $t(p',q')\leqslant t(p,q)-2$.*

Proof. The first assertion is a consequence of the definition of M at step 2 of the algorithm A, and of Remark 4.9.

Let (p, q) be a pair of ϕ-integers. Let $(p', q') = A(p, q)$. We prove now, first that $t(p', q') \leqslant t(p, q) + 2$, second that $t(p', q') > t(p, q)$ implies $(\tau(p), \tau(q)) \in E_3$ and $(\tau(p'), \tau(q')) \in E_2$.

Let $\lambda = [\frac{p}{q}]_\phi$. Using Corollary 4.16, $t(p', q') > t(p, q)$ may hold only in one of the two following cases:

1. when $\tau(p - \lambda q) \in (-\phi)^2]\phi^{-1}, \phi[$ and $t(\lambda) \leqslant 4$;

2. when $\tau(p - \lambda q) \in (-\phi)^3]\phi^{-1}, \phi[$ and $t(\lambda) \leqslant 6$.

1. The first case can only occur when $d_\phi(\lambda) \in \{1, 10, 100, 1000\}$. Then, $\tau(\lambda) \in [-\phi^{-1}, 1]$, and it follows that $\tau(p - \lambda q) < \phi^2$. Hence $\tau(p') \in] - \phi^{-2}, \phi^{-1}[$, and $\tau(q') \in]\phi^{-1}, 1[$, that is, $(\tau(p'), \tau(q')) \in E_2$. We remark that $\tau(-\lambda q) < 1$ and $\tau(p - \lambda q) > \phi$ imply $\tau(p) > \phi^{-1}$, hence $(\tau(p), \tau(q)) \in E_3$.

   Since $\tau(p - \lambda q) \in (-\phi)^2]\phi^{-1}, \phi[$, step 2. of the algorithm A sets $M = 2$. Hence we get the relation $t(p', q') - t(p, q) = t(p') + t(q') - t(p) - t(q) = 4 + t_+(p - \lambda q) - t(p) \leqslant 2$.

2. In the second case, step 2. of the algorithm A sets $M = 3$. Then, we deduce $\tau(p') \in] -\phi^{-2}, \phi^{-3}[$ and $\tau(q') \in]\phi^{-1}, \phi^{-1} + \phi^{-3}[$, hence $(\tau(p'), \tau(q')) \in E_2$. Moreover, $\tau(p - \lambda q) < -\phi^2$ with $\tau(p) > 1$ implies $\tau(-\lambda q) < -\phi$, hence $\tau(q) > 1$ and $(\tau(p), \tau(q)) \in E_3$.

   Since $\tau(p - \lambda q) \in] - \phi^2 - 1, -\phi^2[$ implies $\tau(-\lambda q) < -\phi$, we deduce $\tau(\lambda) > 1$ and $\lambda \notin \{1, \phi, \phi^2, \phi^3\}$. Hence $t_+(p - \lambda q) \leqslant t(p) - 4$, using the third point of Proposition 4.15.

Thus, if $t(p', q') - t(p, q) > 0$, then $(\tau(p), \tau(q)) \in E_3$ and $(\tau(p'), \tau(q')) \in E_2$. Hence $t(p', q') - t(p, q) \leqslant 2$, which proves the second, the third and the fourth assertion of the theorem.

We prove now the last point. Suppose that $(\tau(p), \tau(q)) \in E_2$. We distinguish the two following cases: $d_\phi(\lambda) \in \{1, 10, 100, 1000\}$ and $d_\phi(\lambda) \notin \{1, 10, 100, 1000\}$.

1. If $d_\phi(\lambda) \in \{1, 10, 100, 1000\}$, then $\tau(\lambda) \in [-\phi^{-1}, 1]$, so $\tau(p - \lambda q) \in] - \phi, 1 + \phi^{-3}[$. Since $t_+(r) \leqslant t(p) - 2$ always holds, the relation $t(p', q') = t(p, q)$ only holds when the value M computed at step 2. in the algorithm A satisfies $M \leqslant 1$. In the case $M = 1$ and $\tau(p - \lambda q) \in] - \phi, -1[$, one has $\tau(q') \in]\phi^{-1}, 1[$ and $\tau(p') \in] - \phi^{-1}, -\phi^{-2}[$, hence $(\tau(p'), \tau(q')) \in E_2$. Thus, when $d_\phi(\lambda) \in \{1, 10, 100, 1000\}$, then, either $p - \lambda q \in \mathbb{Z}_\phi^+$, and $t(p', q') - t(p, q) = t_+(r) - t(p) \leqslant -2$, or $M = 1$ and $(\tau(p'), \tau(q')) \in E_2$. We have proven that $(\tau(p'), \tau(q')) \in E_3$ can only occur when $p - \lambda q \in \mathbb{Z}_\phi^+$, with $t(p', q') - t(p, q) \leqslant -2$.

2. If $d_\phi(\lambda) \notin \{1, 10, 100, 1000\}$, then, since $\tau(p) \in] - \phi^{-1}, \phi^{-1}[$ and $\tau(q) \in]\phi^{-1}, 1[$, we get the relation $\tau(p - \lambda q) \in] - \phi - \phi^{-1}, \phi[$. This means that $\phi(p - \lambda q) \in \mathbb{Z}_\phi^+$, and we obtain $t(p') + t(q') \leqslant 2 + t_+(r) + t(q)$. Using the third point of Proposition 4.15, we deduce $t(p', q') - t(p, q) \leqslant 2 + t_+(r) - t(p) \leqslant -2$.

We have proven that, when $(\tau(p), \tau(q)) \in E_2$ and $(\tau(p'), \tau(q')) \in E_3$, then $t(p', q') - t(p, q) \leqslant -2$ holds, which proves the fifth assertion of the theorem. $\square$

It is interesting to give a representation of these computations using a graph $\mathcal{G}$. The vertices of $\mathcal{G}$ are the subsets E_i, and the set of edges of $\mathcal{G}$ is defined as follows: the edge (E_j, E_k), indexed by $i \in \mathbb{Z}$, belongs to $\mathcal{G}$ if, for any pair of ϕ-integers (p, q) such that $(p', q') = A(p, q)$,

$(\tau(p), \tau(q)) \in E_j$ and $(\tau(p'), \tau(q')) \in E_k$, the relation $t(p', q') - t(p, q) \leqslant i$ holds. The graph $\mathcal{G}$ is depicted in Figure 4, δ_i denoting the index of the associated edge, that is, the upper bound for the quantity $t(p_{i+1}, q_{i+1}) - t(p_i, q_i)$.

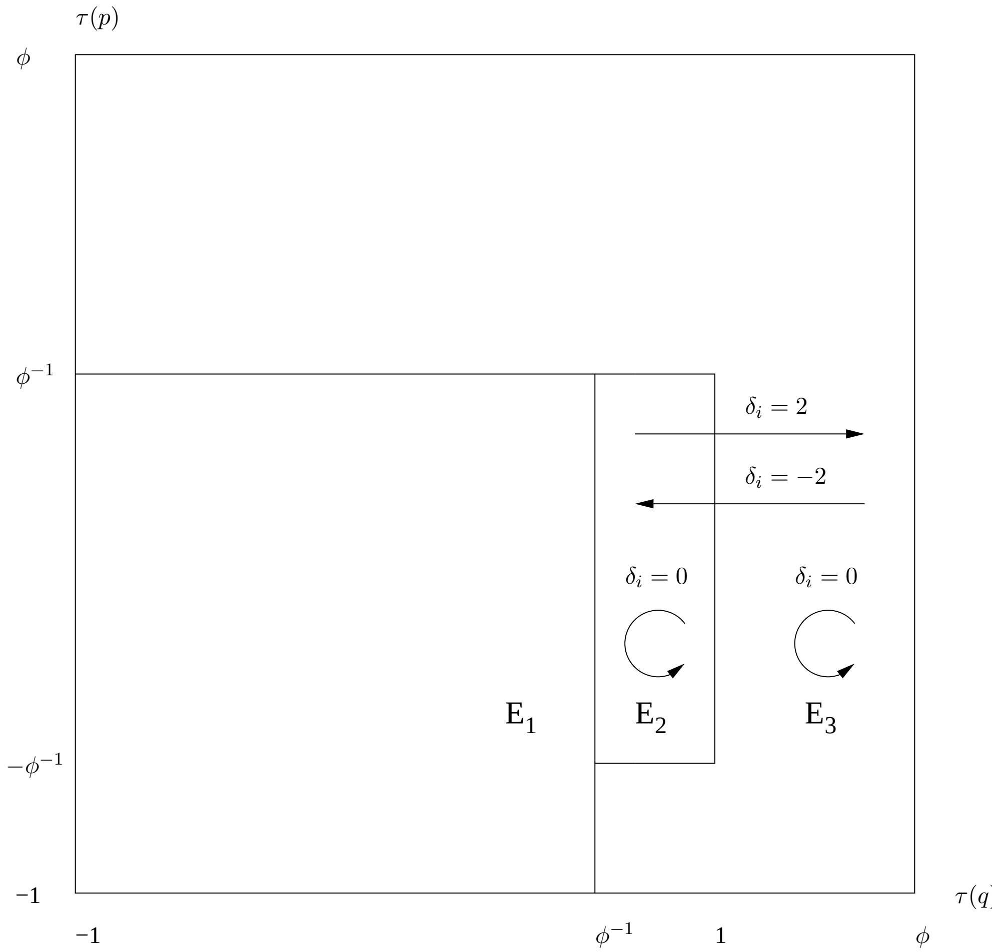

Figure 4: Partition of $\mathcal{I}_\varepsilon^2$ and upper bound for $t(A(p, q)) - t(p, q)$

We deduce from Proposition 5.2 that, starting from a ϕ-fractionary expansion (p_0, q_0) of $x \in \mathbb{Q}(\phi)^+$, the algorithm A produces by iteration a sequence $(p_i, q_i)_{i \in \mathbb{N}}$ of pairs of ϕ-integers which satisfy for all $i \in \mathbb{N}$, $t(p_i) + t(q_i) \leqslant t(p_0) + t(q_0) + 2$. This implies that, for all $i \in \mathbb{N}$, p_i and q_i are ϕ-integers less than $\phi^{t(p_0)+t(q_0)+2}$. Hence there exist $m, i \in \mathbb{N}$ such that $A^m(p_i, q_i) = (p_i, q_i)$. This proves that any $x \in \mathbb{Q}(\phi)^+$ can be represented by a continued ϕ-fraction that is either eventually periodic or finite. We prove in the next paragraph that the eventually periodic case is not possible.

5.2 Finiteness of the continued ϕ-fraction of $x \in \mathbb{Q}(\phi)^+$

According to the last remark, if $x \in \mathbb{Q}(\phi)^+$, then the algorithm A constructs by iteration a sequence of pairs of ϕ-integers $(p_i, q_i)_i$, either finite or eventually periodic. It is clear that the sequence of partial ϕ-quotients is also respectively finite or eventually periodic. Assume that this sequence is infinite, and let $(p_i, q_i)_{i\in\mathbb{N}}$ be the sequence of the pairs of ϕ-integers constructed by A. Since the lengths of the ϕ-fractions constructed are integers, the sequence $(x_i)_{i\in\mathbb{N}}$ can only be infinite when the inequalities of Proposition 5.2 become equalities from a certain index on. Then, there are four possible cases:

1. $(\tau(p_i), \tau(q_i)) \in E_2$ and $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_2$ with $t(p_i, q_i) = t(p_{i+1}, q_{i+1})$,
2. $(\tau(p_i), \tau(q_i)) \in E_3$ and $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_2$ with $t(p_i, q_i) = t(p_{i+1}, q_{i+1}) + 2$,
3. $(\tau(p_i), \tau(q_i)) \in E_3$ and $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_3$ with $t(p_i, q_i) = t(p_{i+1}, q_{i+1})$,
4. $(\tau(p_i), \tau(q_i)) \in E_2$ and $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_3$ with $t(p_i, q_i) = t(p_{i+1}, q_{i+1}) - 2$.

We see below that the study of such possibilities can be represented by a graph $\mathcal{G}'$. The vertices of $\mathcal{G}'$ define a partition of $]-1, \phi[\times]-1, \phi[$. The set of edges of $\mathcal{G}'$ consists of the edges (E_j, E_k) that are indexed by $i \in \mathbb{Z}$ such that, if (p, q) is a pair of ϕ-integers such that $(\tau(p), \tau(q)) \in E_j$ and $(\tau(p'), \tau(q')) \in E_k$, where $(p', q') = A(p, q)$, then $t(p', q') - t(p, q) \leqslant i$.

We show that there is no infinite path in $\mathcal{G}'$ that uses the allowed edges defined by the relations 1., 2., 3., 4., which proves the following result:

Theorem 5.3. *The continued ϕ-fraction of x is finite if and only if $x \in \mathbb{Q}(\phi)^+$.*

In the following computations, we use for convenience the notation $\lambda_i = [\frac{p_i}{q_i}]_\phi$. We associate graphs to the computations, where the vertices are subsets of $]-1, \phi[\times]-1, \phi[$, and the edges (E_j, E_k) are now indexed by the possible values for λ_i such that $A(p_i, q_i) = (p_{i+1}, q_{i+1})$, $(\tau(p_i), \tau(q_i)) \in E_j$ and $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_k$.

Proof. The proof is based on a closer study of the four cases 1., 2., 3., 4. that are defined above.

1. Suppose that $(\tau(p_i), \tau(q_i)) \in E_2$, $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_2$ and $t(p_i, q_i) = t(p_{i+1}, q_{i+1})$.

   Since $\tau(p_i) \in]-\phi^{-1}, \phi^{-1}[$ and $\tau(q_i) \in]\phi^{-1}, 1[$, we get $\tau(p_i - \lambda_i q_i) \in]-\phi-\phi^{-1}, \phi[$. If $\tau(p_i - \lambda_i q_i) \in]-1, \phi[$, then, since $(p_i - \lambda_i q_i, q_i) \in (\mathbb{Z}_\phi^+)^2$, $t(p_{i+1}, q_{i+1}) = t(p_i, q_i)$ is not possible, and $t_+(r) \leqslant t(p) - 2$ implies $t(p_{i+1}, q_{i+1}) - t(p_i, q_i) = t_+(r) - t(p) \leqslant -2$.

   Thus, $\tau(p_i - \lambda_i q_i) \in]-\phi-\phi^{-1}, -1[$. In this case, the step 2 of the algorithm A sets $M = 1$. Hence $t_+(r) = t(p) - 2$, which implies, due to Proposition 4.15, $\lambda_i \in \{1, \phi, \phi^2\}$. However, $\tau(\lambda_i) \leqslant \phi^{-2}$ implies $\tau(\lambda_i q_i) < \phi^{-2}$ and $\tau(p_i - \lambda_i q_i) > -1$. This contradicts $\tau(p_i - \lambda_i q_i) \in]-\phi-\phi^{-1}, -1[$. We deduce that $\tau(\lambda_i) > \phi^{-2}$, hence $\lambda_i = 1$.

   We have shown that Case 1. can only occur when $\lambda_i = 1$.

   We additionally remark that, since $\tau(q_i) \in]\phi^{-1}, 1[$, then $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$. Moreover, if $\tau(p_i - q_i) \in]-\phi, -1[$, then $\tau(p_i) < 0$. The graph associated to Case 1. is depicted in Figure 5.

2. Suppose that $(\tau(p_i), \tau(q_i)) \in E_3$, $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_2$ and $t(p_i, q_i) = t(p_{i+1}, q_{i+1}) + 2$.

   Due to Corollary 4.15, one of the two following possibilities occurs:

   (a) $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2 + \phi^{-1}[$ with $\lambda_i \in \{1, \phi, \phi^2\}$;
   (b) $\tau(p_i - \lambda_i q_i) < -\phi^2$, with $\lambda_i \in \{1, \phi, \phi^2, \phi^2+1, \phi^3, \phi^3+1, \phi^3+\phi, \phi^4, \phi^4+\phi, \phi^4+\phi^2\}$.

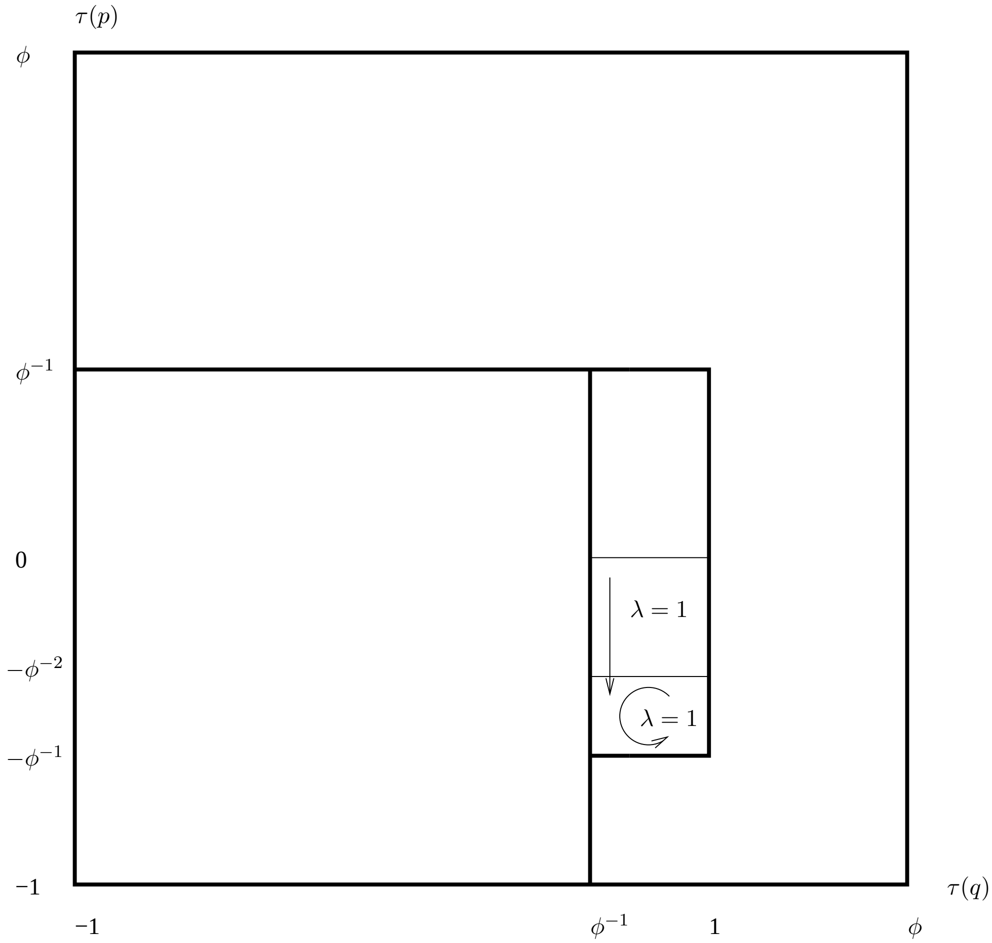

Figure 5: Graph associated to Case 1.

(a) Suppose that $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2 + \phi^{-1}[$ and $\lambda_i \in \{1, \phi, \phi^2\}$. Since $\tau(q_{i+1}) \in]\phi^{-1}, 1[$, $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2 + \phi^{-1}[$ belongs in fact to $\phi^2] - \phi^{-1}, 1[=]\phi, \phi^2[$. Let us consider the three possibilities $\lambda_i = 1$, $\lambda_i = \phi$ and $\lambda_i = \phi^2$.

i. If $\lambda_i = 1$, then $\tau(p_i - q_i) > \phi$ only occurs when $\tau(q_i) \in] - 1, 0[$ and $\tau(p_i) > \phi^{-1}$. This implies $\tau(p_{i+1}) \in] - \phi^{-2}, 0[$.

ii. If $\lambda_i = \phi$, then $\tau(-\lambda_i q_i) \in] - \phi^{-1}, 1[$. Thus, $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2[$, which implies $\tau(p_i) > \phi^{-1}$ and $\tau(-\lambda_i q_i) \in]0, 1[$. We thus have $\tau(q_i) > 0$ and $\tau(p_{i+1}) \in]0, \phi^{-1}[$.

iii. If $\lambda_i = \phi^2$, and if $\tau(p_i) \leqslant 1$ holds, then, since $\tau(-\lambda_i q_i) \in] - \phi^{-1}, \phi^{-2}[$, this implies $\tau(p_i - \lambda_i q_i) < \phi$, which contradicts $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2 + \phi^{-1}[$. Thus, $\tau(p_i) > 1$. As in the case 2(a)i., we deduce from $\tau(p_i - \lambda_i q_i) > \phi$ the relations $\tau(q_i) \in] - 1, 0[$, $\tau(p_i) > \phi^{-1}$ and $\tau(p_{i+1}) \in] - \phi^{-2}, 0[$.

(b) Consider now that $\tau(p_i - \lambda_i q_i) \in] - \phi^2 - 1, -\phi^2[$. This implies $\tau(p_i) < 0$ and $\tau(-\lambda_i q_i) < -\phi$, hence $\tau(q_i) > 1$ and $\tau(\lambda_i) > 1$. We deduce that the only possibility for λ_i is $\lambda_i = 1 + \phi^2$. Moreover, since $\tau(-\lambda_i q_i) \in] - \phi - \phi^{-1}, 1 + \phi^2[$, we get $\tau(p_i) < -\phi^{-2}$.

The possibilities related to the cases studied in 2(a)i., 2(a)ii., 2(a)iii. and 2b. show that Case 2. can only occur when one of the following conditions holds:

(a) $\lambda_i = 1$, with $\tau(q_i) < 0$, $\tau(p_i) > \phi^{-1}$ and $\tau(p_{i+1}) \in]-\phi^{-2}, 0[$;
(b) $\lambda_i = \phi$, with $\tau(q_i) > 0$, $\tau(p_i) > \phi^{-1}$ and $\tau(p_{i+1}) \in]0, \phi^{-1}[$;
(c) $\lambda_i = \phi^2$, with $\tau(q_i) < 0$, $\tau(p_i) > 1$ and $\tau(p_{i+1}) \in]-\phi^{-2}, 0[$;
(d) $\lambda_i = \phi^2 + 1$, with $\tau(p_i) < -\phi^{-2}$, $\tau(q_i) > 1$ and $\tau(p_{i+1}) \in]-\phi^{-2}, 0[$.

We note that, if $\lambda_i = \phi$, the relations $\tau(p_i) < 1$ and $\tau(q_i) < 1$ cannot both hold, since this would contradict $\tau(p_i - \lambda_i q_i) > \phi$. The set of the four edges is depicted in Figure 6.

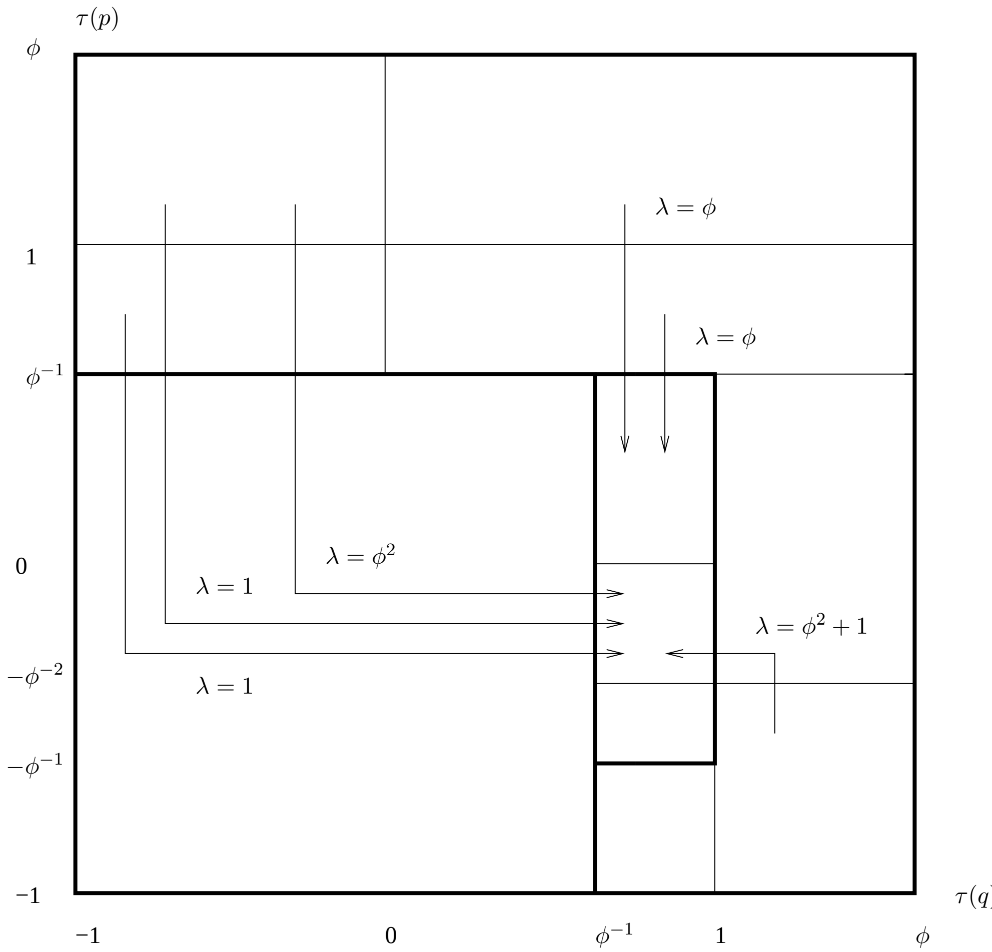

Figure 6: Graph associated to Case 2.

3. Suppose that $(\tau(p_i), \tau(q_i)) \in E_3$, $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_3$ and $t(p_i, q_i) = t(p_{i+1}, q_{i+1})$. We have to distinguish three following possibilities:

   (a) $\tau(p_i - \lambda_i q_i) \in]-\phi^2, -1[$ and $\lambda_i \in \{1, \phi, \phi^2\}$,
   (b) $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2 + \phi^{-1}[$ and $\lambda_i \in \{1, \phi, \phi^2, \phi^2+1, \phi^3, \phi^3+1, \phi^3+\phi, \phi^4, \phi^4+\phi, \phi^4+\phi^2\}$,

(c) $\tau(p_i - \lambda_i q_i) \in]-\phi^2 - 1, -\phi^2[$.

(a) Suppose that $\tau(p_i - \lambda_i q_i) \in]-\phi^2, -1[$ and $\lambda_i \in \{1, \phi, \phi^2\}$. This implies $\tau(q_{i+1}) \in]\phi^{-1}, \phi[$ and $\tau(\lambda_i) \in [-\phi^{-1}, 1]$. Additionally, since $\tau(p_i) \in]-1, \phi[$, the relation $\tau(p_i - \lambda_i q_i) < -1$ implies $\tau(-\lambda_i q_i) < 0$.
If $\lambda_i = 1$, then $\tau(q_i) > 0$ and $\tau(p_i) < \phi^{-1}$. But $(p_i, q_i) \notin E_1$ implies $\tau(q_i) > \phi^{-1}$. Thus, we have additionally the conditions $\tau(p_i) < \phi^{-1}$, $\tau(q_i) > \phi^{-1}$ and $\tau(p_{i+1}) \in]-1, -\phi^{-2}[$.
If $\lambda_i = \phi$, then $\tau(-\lambda_i q_i) \in]-\phi^{-1}, 1[$, so $\tau(p_i - \lambda_i q_i) \in]-\phi, -1[$. This implies $\tau(q_i) < 0$ and $\tau(p_i) < 0$, hence $(\tau(p_i), \tau(q_i)) \in E_1$, which is impossible. Thus, $\lambda_i \neq \phi$.
If $\lambda_i = \phi^2$, then $\tau(-\lambda_i q_i) \in]-\phi^{-1}, \phi^{-2}[$, which implies $\tau(p_i - \lambda_i q_i) \in]-\phi, -1[$. We deduce $\tau(q_{i+1}) \in]\phi^{-1}, 1[$, $\tau(p_i) < -\phi^{-2}$ and $\tau(-\lambda_i q_i) < 0$. Thus, $\tau(q_i) > 0$, and since $\tau(p_i) < \phi^{-1}$ and $(\tau(p_i), \tau(q_i)) \notin E_1$, we deduce $\tau(q_i) > \phi^{-1}$. Moreover, $\tau(q_i) \in]\phi^{-1}, \phi[$ implies $\tau(p_{i+1}) \in]-1, -\phi^{-2}[$. But $(\tau(p_{i+1}), \tau(q_{i+1})) \notin E_2$ and $\tau(q_{i+1}) \in]\phi^{-1}, 1[$ implies $\tau(p_{i+1}) \in]-1, -\phi^{-1}[$, that is, $\tau(q_i) \in]1, \phi[$.

(b) Suppose that $\tau(p_i - \lambda_i q_i) \in]\phi, \phi^2 + \phi^{-1}[$ and $\lambda_i < \phi^5$. This implies $\tau(p_i - \lambda_i q_i) > \phi$, hence $\tau(p_i) > 0$ and $\tau(-\lambda_i q_i) > 0$. Then, $\tau(p_{i+1}) \in (-\phi)^{-2}]-1, \phi[\subset]-\phi^{-1}, \phi^{-1}[$ and $\tau(q_{i+1}) \in (-\phi)^{-2}]\phi, \phi^2 + \phi^{-1}[\subset]\phi^{-1}, \phi[$. Since $(p_{i+1}, q_{i+1}) \notin E_2$, we have additionally $\tau(q_{i+1}) > 1$, which implies $\tau(p_i - \lambda_i q_i) > (-\phi)^2$. We deduce that $\tau(p_i) > 1$ and $\tau(-\lambda_i q_i) > 1$. This inequality only holds when $\tau(\lambda_i)$ and $\tau(q_i)$ are such that one of them belongs to $]-1, -\phi^{-1}[$ and the other one belongs to $]1, \phi[$. This condition gives the set of possible values for λ_i as well, that is, $\lambda_i \in \{\phi^2 + 1, \phi^3 + \phi\}$.

(c) Suppose that $\tau(p_i - \lambda_i q_i) \in]-\phi^2 - 1, -\phi^2[$. This implies $\tau(p_i - \lambda_i q_i) \in]\phi^2 - 1, -\phi^2[$ implies $\tau(p_i) < 0$, $\tau(q_i) > 1$ and $\tau(\lambda_i) > 1$. Hence $\tau(p_{i+1}) \in]-\phi^{-1}, \phi^{-1}[$ and $\tau(q_{i+1}) \in]\phi^{-1}, 1[$. This means that $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_2$, which contradicts the hypothesis $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_3$, thus this possibility do not occur.

We have shown that Case 3. can only occur when one of the following conditions is satisfied:

(a) $\lambda_i = 1$, with $\tau(p_i) < \phi^{-1}$, $\tau(q_i) > \phi^{-1}$ and $\tau(p_{i+1}) \in]-1, -\phi^{-2}[$;

(b) $\lambda_i = \phi^2$, with $\tau(p_i) < -\phi^{-2}$, $\tau(q_i) \in]1, \phi[$, $\tau(p_{i+1}) \in]-1, -\phi^{-1}[$ and $\tau(q_{i+1}) \in]\phi^{-1}, 1[$;

(c) $\lambda_i = \phi^2 + 1$, with $\tau(p_i) \in]1, \phi[$, $\tau(q_i) \in]-1, 0[$, $\tau(p_{i+1}) \in]-\phi^{-1}, \phi^{-1}[$ and $\tau(q_{i+1}) \in]1, \phi[$;

(d) $\lambda_i = \phi^3 + \phi$, with $\tau(p_i) \in]1, \phi[$, $\tau(q_i) \in]0, \phi[$, $\tau(p_{i+1}) \in]-\phi^{-1}, \phi^{-1}[$ and $\tau(q_{i+1}) \in]1, \phi[$.

The set of these conditions is depicted in Figure 7.

4. Suppose that $(\tau(p_i), \tau(q_i)) \in E_2$, $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_3$ and $t(p_i, q_i) = t(p_{i+1}, q_{i+1}) - 2$. We distinguish two possibilities:

(a) $\tau(p_i - \lambda_i q_i) \in]-1, \phi[$ and $\lambda_i \in \{1, \phi, \phi^2\}$,

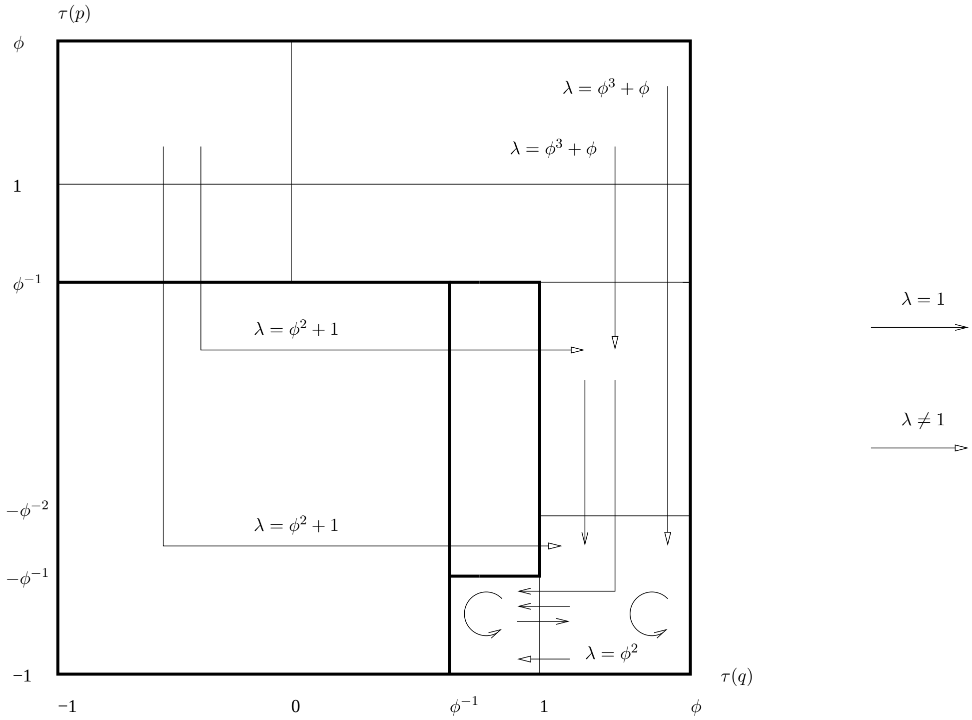

Figure 7: Graph associated to Case 3.

(b) $\tau(p_i - \lambda_i q_i) \in]-\phi^2, -1[$ and $\lambda_i \in \{1, \phi, \phi^2, \phi^2+1, \phi^3, \phi^3+1, \phi^3+\phi, \phi^4, \phi^4+\phi, \phi^4+\phi^2\}$.

(a) Suppose that $\tau(p_i - \lambda_i q_i) \in]-1, \phi[$, with $\lambda_i \in \{1, \phi, \phi^2\}$. Then $\tau(p_{i+1}) = \tau(q_i) \in]\phi^{-1}, 1[$.
If $\lambda_i = 1$, then $\tau(-\lambda_i q_i) \in]-1, -\phi^{-1}[$. Since $\tau(p_i) \in]-\phi^{-1}, \phi^{-1}[$, we have $\tau(p_i - \lambda_i q_i) \in]-1, 0[$. This implies $\tau(p_i) > -\phi^{-2}$ and $\tau(q_{i+1}) \in]-1, 0[$.
If $\lambda_i = \phi$, then $\tau(-\lambda_i q_i) \in]\phi^{-2}, \phi^{-1}[$, thus $\tau(q_{i+1}) = \tau(p_i - \lambda_i q_i) \in]-\phi^{-3}, 1+\phi^{-3}[$. Moreover, if $\tau(p_i) > -\phi^{-2}$, then $\tau(p_i - \lambda_i q_i) = \tau(q_{i+1}) > 0$.
If $\lambda_i = \phi^2$, then $\tau(-\lambda_i q_i) \in]-\phi^{-2}, -\phi^{-3}[$, hence $\tau(q_{i+1}) \in]-1, \phi^{-2}[$.

(b) Suppose that $\tau(p_i - \lambda_i q_i) \in]-\phi^2, -1[$ and $\lambda_i < \phi^5$. This implies $\tau(p_i) \in]-\phi^{-1}, \phi^{-1}[$ and $\tau(q_i) \in]\phi^{-1}, 1[$. Hence $\tau(p_i - \lambda_i q_i) \in]-\phi - \phi^{-1}, -1[$. Since $\tau(p_i) > -\phi^{-1}$, we have $\tau(-\lambda_i q_i) < -\phi^{-2}$. Thus, $\tau(\lambda_i) > \frac{\phi^{-2}}{\tau(q_i)} \geqslant \phi^{-2}$. The only possible values for λ_i that fulfill this inequality and belong to $[1, \phi^4 + \phi^2]$ are $1, \phi^2+1, \phi^3+1$ and $\phi^4+\phi^2$. Additionally, if $\tau(\lambda_i) \leqslant 1$, then $\tau(\lambda_i) \in [\phi^{-2}, 1]$, hence $\tau(-\lambda_i q_i) \in]-1, -\phi^{-3}[$. Thus, $\tau(p_i - \lambda_i q_i) \in]-\phi, -1[$, and $\tau(q_{i+1}) \in]\phi^{-1}, 1[$. Since $\tau(q_i) \in]\phi^{-1}, 1[$, $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$, we get $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_2$. This contradicts the hypothesis $(\tau(p_{i+1}), \tau(q_{i+1})) \in E_3$. Thus, $\tau(\lambda_i) > 1$, and this implies $\lambda_i = 1 + \phi^2$. Then, $\tau(-\lambda_i q_i) \in]-1-\phi^{-2}, -\phi^{-1}-\phi^{-3}[$, hence $\tau(p_i - \lambda_i q_i) \in]-\phi-\phi^{-2}, -1[$ and $\tau(q_{i+1}) \in]\phi^{-1}, 1+\phi^{-3}[$.
Moreover, $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$ and $(\tau(p_{i+1}), \tau(q_{i+1})) \notin E_2$ implies that the relation $\tau(q_{i+1}) \in]\phi^{-1}, 1[$ cannot hold. Hence $\tau(q_{i+1}) \in]1, 1+\phi^{-3}[$; this implies $\tau(p_i - \lambda_i q_i) < -\phi$ and $\tau(p_i) < -\phi^{-3}$.

We have proven that Case 4. can only occur when one of the following conditions is satisfied:

(a) $\lambda_i = 1$, with $\tau(p_i) \in]-\phi^{-2}, \phi^{-1}[$, $\tau(p_{i+1}) \in]\phi^{-1}, 1[$ and $\tau(q_{i+1}) \in]-1, 0[$,

(b) $\lambda_i = \phi$, with $\tau(p_{i+1}) \in]\phi^{-1}, 1[$, and $\tau(q_{i+1}) < 0$ implies $\tau(p_i) < -\phi^{-2}$,

(c) $\lambda_i = \phi^2$, with $\tau(p_{i+1}) \in]\phi^{-1}, 1[$,

(d) $\lambda_i = \phi^2 + 1$, with $\tau(p_i) < -\phi^{-3}$, $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$ and $\tau(q_{i+1}) \in]1, 1+\phi^{-3}[$.

The set of these conditions is depicted in Figure 8.

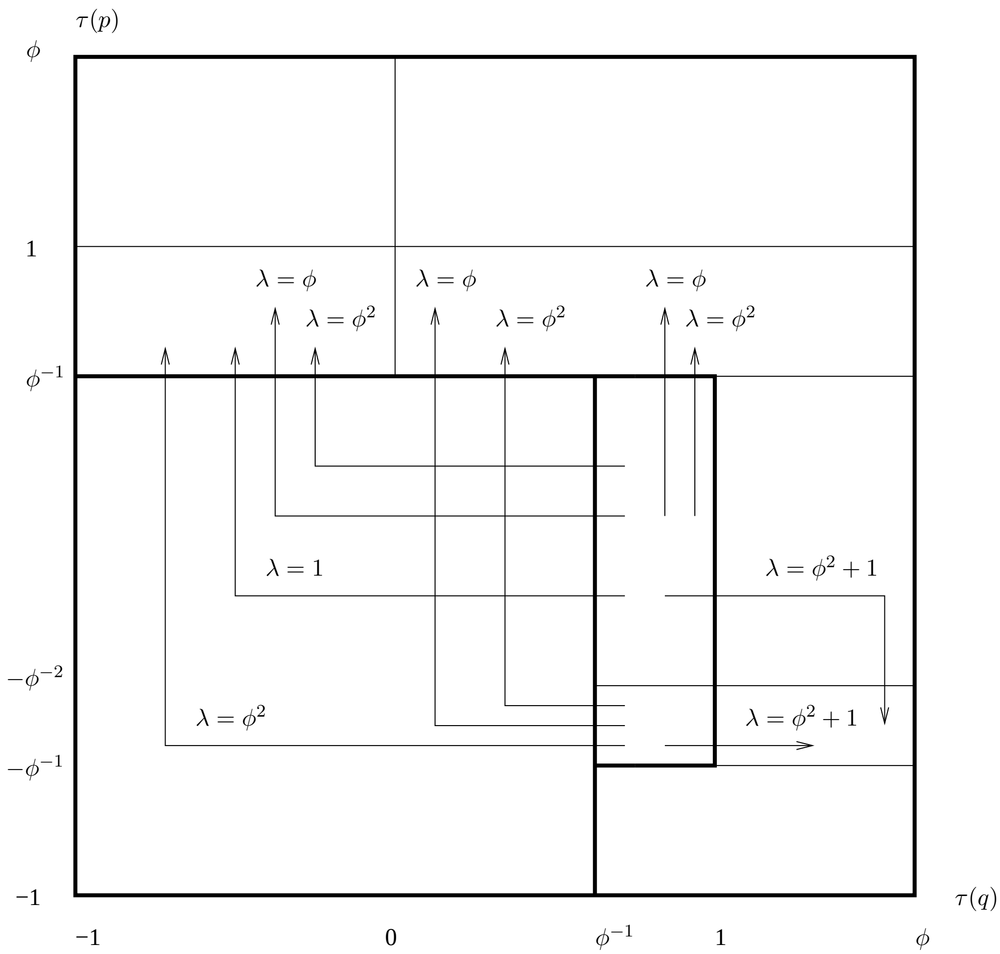

Figure 8: Graph associated to Case 4.

We have considered all the possibilities that the algorithm A may eventually encounter when we obtain by iteration of the algorithm A a ultimately periodic sequence $(p_i, q_i)_{i \in \mathbb{N}}$ that is not finite. This is equivalent to the possibility of constructing a ultimately periodic continued ϕ-fraction.

Now, let us study more closely the possible cycles in the graph obtained when we stack the graphs depicted by Figures 5, 6, 7 and 8. We obtain in this way a graph $\mathcal{G}'$, whose vertices are the intersection of the vertices of the graphs depicted by Figures 5, 6, 7 and 8. The edges of $\mathcal{G}'$ are obtained by splitting the edges in any of these Figures, that is, if there exists an edge (E_j, E_k) indexed by λ in any of the graphs depicted by Figures 5, 6, 7 or 8, and if $\{F_h, h \in [\![1, \ldots, N_j]\!]\}$ and $\{F_l, l \in [\![1, \ldots, N_k]\!]\}$ are respectively partitions of E_j and E_k which consists of vertices of $\mathcal{G}'$, we create in $\mathcal{G}'$ the edges (F_h, F_l) indexed by λ for any $(h, l) \in [\![1, \ldots, N_j]\!] \times [\![1, \ldots, N_k]\!]$. We gather then all possible edges, removing some of them thanks to the following remarks.

1. For any vertex of $\mathcal{G}'$, there may exist at least one incoming edge and one outgoing edge. Otherwise, this vertex cannot be used by any cycle. Thus, we remove the vertices of $\mathcal{G}'$ that are not used in any connected subgraph, and we remove the edges that use any of these vertices as well.

2. Due to Lemma 4.3, if $d_\phi(\lambda_i)$ admits 1 as a suffix, then $\lambda_{i+1} \geqslant \phi$. This means that there is no cycle in $\mathcal{G}'$ constituted by two consecutives edges (V_i, V_{i+1}) and (V_{i+1}, V_{i+2}) such that $s_\phi(\lambda_i) = \lambda_i + \phi^{-1}$ and $\lambda_{i+1} = 1$.

It is possible to remove other edges in $\mathcal{G}'$. For instance, we note that the conditions $\tau(p_{i+1}) \in]-1, -\phi^{-2}[$ and $\tau(q_{i+1}) \in]1, \phi[$ may be satisfied in only two cases:

1. $(p_{i+1}, q_{i+1}) = A(p_i, q_i)$, with $\lambda_i = \phi^2 + 1$ and $(p_i, q_i) \in E_2$;

2. $(p_{i+1}, q_{i+1}) = A(p_i, q_i)$, with $\lambda_i = 1$ and $(p_i, q_i) \in E_3$.

1. If $(p_{i+1}, q_{i+1}) = A(p_i, q_i)$, with $\lambda_i = \phi^2 + 1$ and $(p_i, q_i) \in E_2$, then $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$ and $\tau(q_{i+1}) \in]1, 1 + \phi^{-3}[$.

2. If $(p_{i+1}, q_{i+1}) = A(p_i, q_i)$, with $\lambda_i = 1$ and $(p_i, q_i) \in E_3$, then $(\tau(p_i), \tau(q_i)) \in]-1, -\phi^{-1}[\times]\phi^{-1}, 1[$ implies $\tau(p_i - \lambda_i q_i) \in]-\phi-\phi^{-2}, -1[$. Thus, we get $\tau(q_{i+1}) \in]1, 1+\phi^{-3}[$ and $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$.

   However, if $\tau(p_{i+1}) \in]-\phi^{-1}, -\phi^{-2}[$ and $\tau(q_{i+1}) \in]1, 1+\phi^{-3}[$, then, with $\lambda_{i+1} = \phi^2+1$, we obtain $\tau(p_{i+1} - \lambda_{i+1} q_{i+1}) > -\phi^{-1} - (1+\phi^{-3})(1+\phi^{-2})$. Thus, $\tau(p_{i+1} - \lambda_{i+1} q_{i+1}) > -\phi^2$.

We have proven that the edge indexed ϕ^2+1, having its initial vertex in $]-1, -\phi^{-2}[\times]1, \phi[$ cannot be preceded by any edge among the remaining ones. Thus, this edge cannot be used in any cycle.

Using the same method, we remark that the subset defined by $(\tau(p), \tau(q)) \in]\phi^{-1}, 1[\times]0, \phi[$ contains only one initial vertex among the remaining edges, and this vertex is included in fact in $]\phi^{-1}, 1[\times]1, \phi[$. Since any pair of ϕ-integers (p_i, q_i) satisfying $\tau(p_i) \in]-\phi^{-2}, 0[$, $\tau(q_i) \in]\phi^{-1}, 1[$ and $\tau(\lambda) \in [-1, 1]$ is sent under the action of the algorithm A on (p_{i+1}, q_{i+1}) such that $\tau(q_{i+1}) < 1$ holds, we get another simplification of possible edges. Thus, there are only three possible cycles, depicted in Figure 9.

There exists three possible cycles that may represent the iteration of A starting from a ϕ-fractionary expansion (p_0, q_0) of $x \in \mathbb{Q}(\phi)^+$. Among these cycles, one of them is associated to a sequence of partial ϕ-convergents $(\lambda_i)_{i \in \mathbb{N}}$ such that, from an index N on, the value of λ_i is alternately 1 or ϕ^2. This sequence provides the continued ϕ-fraction of the positive real number y which satisfies $y = \phi^2 + \frac{1}{1+\frac{1}{y}}$. However, y is quadratic over $\mathbb{Q}(\phi)$ but does not belong to $\mathbb{Q}(\phi)$. The two remaining cycles define cases for which $(\lambda_i)_{i \in \mathbb{N}}$ is stationary. However, one

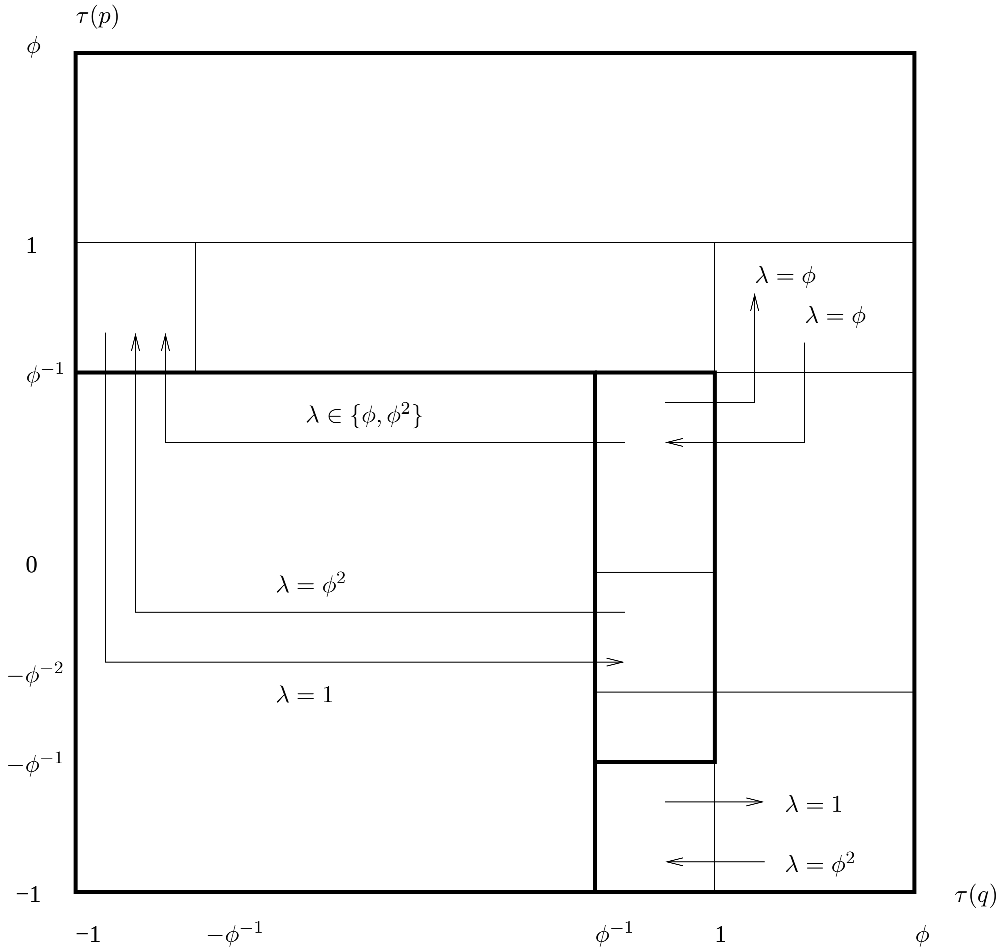

Figure 9: Only 3 possible cycles may occur under the iteration of A

checks that any positive real number whose sequence of partial ϕ-convergents is stationary is quadratic over $\mathbb{Q}(\phi)$.

Since the three possible cycles cannot occur, the sequence $t(p_i, q_i)_{i\in\mathbb{N}}$ cannot have a positive lower bound. It means that this sequence $t(p_i, q_i)_{i\in\mathbb{N}}$ tends to 0, which implies that there exists i_0 such that $p_{i_0} = 0$. This ends the proof of the theorem. □

Remark 5.4. *Let us detail the action of the algorithm A on a particular case for the numeration system introduced in Remark 4.13, that is, when $d_\beta(1) = 0.41$. As we have seen, the continued β-fraction of ϕ is in this case the classical continued fraction $[1; 1^\infty]$. It means that there exists a sequence of pairs of β-integers $((p_i, q_i))_{i\in\mathbb{N}}$ such that $\phi = \frac{p_0}{q_0}$, $A(p_i, q_i) = (p_{i+1}, q_{i+1})$ for all $i \in \mathbb{N}$, and $[\frac{p_i}{q_i}]_\beta = 1$ for all $i \in \mathbb{N}$.*

We check that the following relations hold:

$$\begin{aligned} 3\beta + 1 &= (2\beta) \times 1 + \beta + 1, \\ 2\beta &= (\beta + 1) \times 1 + 3 + \beta^{-1} = \beta^{-1}((\beta^2 + \beta) \times 1 + 3\beta + 1), \\ \beta^2 + \beta &= (3\beta + 1) \times 1 + 2\beta. \end{aligned}$$

This means that $A(3\beta+1,2\beta)=(2\beta,\beta+1)$, $A(2\beta,\beta+1)=(\beta^2+\beta,3\beta+1)$ *and* $A(\beta^2+\beta,3\beta+1)=(3\beta+1,2\beta)$. *The associated values for* M, *which are set at step 2. of the algorithm* A, *are respectively* $0,0$ *and* 1, *since one has to multiply the elements of the pair* $(\beta+1,3+\beta^{-1})$ *by* β *to get a pair of* β*-integers. For all these cases, the corresponding value of* $\lambda=[\frac{p}{q}]_\beta$ *is 1. Hence the corresponding cycle represent* $\frac{3\beta+1}{2\beta}=\frac{2\beta}{\beta+1}=\frac{\beta^2+\beta}{3\beta+1}=\phi$, *whose continued* β*-fraction is* $[1;1^\infty]$; *one may define either* $(p_0,q_0)=(3\beta+1,2\beta),(2\beta,\beta+1)$ *or* $(\beta^2+\beta,3\beta+1)$ *to obtain a sequence of pairs of* β*-integers which produces the continued* β*-fraction of* ϕ *by iteration of the algorithm* A. *Note also that* ϕ *does not admit an unique reduced* β*-fractionary expansion, introduced in Definition 1.5, since* $t(3\beta+1,2\beta)=t(2\beta,\beta+1)=3$.

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