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On multiplication in q-Wiener chaoses

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Abstract

We pursue the investigations initiated by Donati-Martin [9] and Effros-Popa [10] regarding the multiplication issue in the chaoses generated by the q-Brownian motion ($q \in (-1,1)$), along two directions: (i) We provide a fully-stochastic approach to the problem and thus make a clear link with the standard Brownian setting; (ii) We elaborate on the situation where the kernels are given by symmetric functions, with application to the study of the q-Brownian martingales.

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1 Introduction: the q-Brownian motion

The object at the center of this study is the so-called q-Brownian motion ($q \in (-1,1)$), which stands for one of the most celebrated family of *non-commutative processes*. For the sake of the clarify, let us thus start our presentation with a very short reminder on the setting of non-commutative probability theory.

First, we call a non-commutative (n.-c.) process any path with values in a n.-c. probability space, that is a von Neumann algebra $\mathcal{A}$ equipped with a weakly continuous, positive and faithful trace φ playing the role of the "expectation" in this framework. Based on this specific structure, we can then define "$L^p(\varphi)$"-spaces along the expected procedure, that is by considering the norm

$$\|X\|_{L^p(\varphi)} := \varphi(|X|^p)^{1/p} \quad (|X| := \sqrt{XX^*}) .$$

When $p = 2$, this naturally gives rise to a Hilbert space (which will be the main setting of our study), with scalar product given by $\langle X, Y\rangle_{L^2(\varphi)} := \varphi\big(XY^*\big)$. Now recall that, according to a fundamental spectral result, any self-adjoint operator X in $\mathcal{A}$ can be associated with a probability measure μ_X on $\mathbb{R}$ satisfying the following property: for any real polynomial P, $\int_{\mathbb{R}} P(x)\mathrm{d}\mu_X(x) = \varphi(P(X))$. In other words, μ_X and X share the same "moments". With this property in mind, self-adjoint operators in $\mathcal{A}$ are called (non-commutative) random variables, and the above measure μ_X is called the law (or distribution) of X. By extension, we define the law of a given family $\{X^{(i)}\}_{i\in I}$ of random variables in $(\mathcal{A}, \varphi)$ as the set of all of its joint moments $\varphi\big(X^{(i_1)} \cdots X^{(i_r)}\big)$, $i_1, \dots, i_r \in I$, $r \in \mathbb{N}$.

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The definition of the q-Brownian can now be introduced along the following combinatorial description:

Definition 1.1. *1. Let r be an even integer. We call a* pairing *of $\{1,\ldots,r\}$ any partition of $\{1,\ldots,r\}$ into $r/2$ disjoint subsets, each of cardinality 2. We denote by $\mathcal{P}_2(\{1,\ldots,r\})$ or $\mathcal{P}_2(r)$ the set of all pairings of $\{1,\ldots,r\}$.*
2. When $\pi\in\mathcal{P}_2(\{1,\ldots,r\})$, we call a crossing *in π any set of the form $\{\{x_1,y_1\},\{x_2,y_2\}\}$ with $\{x_i,y_i\}\in\pi$ and $x_1<x_2<y_1<y_2$. The number of such crossings is denoted by $\mathrm{Cr}(\pi)$.*

Definition 1.2. *For any fixed $q\in(-1,1)$, we call a q-Brownian motion in a n.-c. probability space $(\mathcal{A},\varphi)$ any collection $\{X_t\}_{t\geq 0}$ of random variables in $(\mathcal{A},\varphi)$ such that, for every integer $r\geq 1$ and all $t_1,\ldots,t_r\geq 0$, one has*

$$\varphi\big(X_{t_1}\cdots X_{t_r}\big)=\sum_{\pi\in\mathcal{P}_2(\{1,\ldots,r\})} q^{\mathrm{Cr}(\pi)}\prod_{\{p,q\}\in\pi} t_p\wedge t_q\ . \tag{1.1}$$

The existence issue and the basic stochastic properties of the q-Brownian motion were investigated in a series of remarkable papers by Bożejko, Kümmerer and Speicher [4, 5, 6]. Using (1.1), it can be checked in particular that for all $t_1,\ldots,t_r$ and $Y:=X_{t_3}\cdots X_{t_r}$, $\varphi\big(X_{t_1}X_{t_2}Y\big)-\varphi\big(X_{t_2}X_{t_1}Y\big)=(1-q)\,P_{t_1,\ldots,t_r}(q)$, for some polynomial $P_{t_1,\ldots,t_r}(q)$, which, to some extent, illustrates the following idea: the further q to 1, the less "commutative" the process. This presentation of the q-Brownian model as a possible way to quantify some non-commutativity phenomenon can actually be traced back to a paper by Frisch and Bourret [11].

Let us now consider the two following particular situations:

• When letting q tend to 1 in the right-hand side of (1.1), one recovers the classical Wick description of the joint moments of the standard (commutative) Brownian motion;

• Taking $q=0$, we get that the only terms with non-zero contribution in the right-hand side of (1.1) are those for which $\mathrm{Cr}(\pi)=0$, i.e. the non-crossing pairings. The 0-Brownian motion thus corresponds to a so-called semicircular family (see [13, Lecture 8]), and the process in this case actually coincides with the celebrated free Brownian motion, whose freely-independent increments are known to be closely related to the asymptotic behaviour of large random matrices (see [16]).

At the level of its distribution, the q-Brownian model can thus be seen as a natural extension of two of the most central processes in probability theory. It is then natural to examine whether the classical stochastic properties satisfied by each of these two processes can also be extended to every $q\in(-1,1)$. A first example in this direction is given by the result of [4, Theorem 1.10] about the marginal-distribution of the process, that is the law of X_t for fixed $t\geq 0$. More interestingly, we can study the robustness of this interpolation with respect to the general *stochastic integration* issue. For instance, it has been shown by Donati-Martin in [9] that one could also define a natural "Itô integral" with respect to the q-Brownian motion, which extends both the usual Itô integral associated with the standard Brownian and the free Itô integral exhibited by Biane and Speicher in [3] when $q=0$ (see also [8] for a rough-path-type approach to this stochastic integration issue).

In our study, we will more specifically be interested in the analysis of the so-called q-Wiener chaos, that is the "q-analog" of the usual Wiener chaos generated by the standard Brownian motion. The construction of the chaoses is also due to Donati-Martin (in the aforementionned paper), and it extends the procedure initiated by Biane and Speicher in the free case (for the sake of completeness, we shall briefly review these arguments in Section 2.2). Some first results about the multiplication properties prevailing in q-Wiener chaoses can be found in [9, 10] (let us also quote the q-extension of the fourth-moment phenomenon in [7], at least for $q \in [0,1)$). Our aim in these notes is to go deeper into this analysis, with a twofold objective in mind:

(i) We intend to emphasize the rich combinatorial machinery governing the behaviour of q-Wiener chaoses, and in the same time offer a fully-stochastic approach to the multiplication issue. Our main result here will be the exhibition of a *full q-Wick product formula* for multiple integrals (Theorem 3.1). Although similar formulas can already be found in the specific q-Fock-space framework (see for instance [10, Theorem 3.3]), our proof of the result will only rely on probabilistic arguments, i.e. it will only depend on the law of the process (as given by formula (1.1)) and not on its representation (just as classical probability theory builds upon the law of the Brownian motion and not upon its representation). In brief, our strategy can be regarded as a q-extension of the procedure displayed in [14, Section 1.1.2] for the classical commutative case, which hopefully will make the proof (and accordingly the result) very accessible to a probabilist audience.

To achieve our purpose, we will be led to rephrase the central concept of contraction in terms of q-contraction along possibly “incomplete” pairings (see Section 2.1). Thanks to this representation, many of our arguments can then be conveniently illustrated through basic pictures, and in that sense, our formulation is somehow related to the nice combinatorial approach developed by Nica and Speicher in the free case [13].

(ii) Our second line of investigations regarding q-Wiener chaoses will consist of a close examination of the *fully-symmetric situation*, that is the situation where the integrals involve fully-symmetric kernels. In this case, it turns out that the coefficients in the product formula can be expressed in terms of classical q-combinatorial numbers, making the link with the classical commutative case even more clear (Theorem 4.2). We will also provide an explicit description of the stochastic dynamics governing the so-called q-Hermite martingales (Corollary 4.4), a central family of processes whose definition is based on fully-symmetric kernels. These procedures will again give us the opportunity to offer a glimpse on the rich combinatorics associated with the q-Brownian process.

The study is organized as follows. In Section 2, we introduce the main combinatorial ingredients at the core of our analysis and recall the definition of the q-Wiener chaos. Section 3 is then devoted to the presentation of our stochastic proof of the general multiplication formula. The specificities of the fully-symmetric situation are finally highlighted in Section 4.

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2 Generalities on q-Wiener chaoses

As a first step of our analysis, let us introduce an alternative and as-graphical-as-possible approach to the notion of q-contraction, in comparison with the definition provided in [9].

2.1 Arrangement along a partition and q-contraction

We have seen through formula (1.1) that the law of a q-Brownian motion can be conveniently described as a sum over pairings. In order to extend such formulas at the level of the processes (and not only their laws), we will need to involve more general “incomplete pairings” into the procedure:

Definition 2.1. (i) *For $n \geq 1$, we denote by $\mathcal{P}_{\leq 2}(\{1,\dots,n\})$ or $\mathcal{P}_{\leq 2}(n)$ the set of partitions π of $\{1,\dots,n\}$ with blocks of one or two elements, and for every $0 \leq k \leq \frac{n}{2}$, we denote $\mathcal{P}^k_{\leq 2}$ the set of partitions $\pi \in \mathcal{P}_{\leq 2}$ containing exactly k pairs.*

(ii) *For $n_1,\dots,n_r \geq 1$, we denote by $\mathcal{P}_{\leq 2}(n_1\otimes\cdots\otimes n_r)$ the set of partitions $\pi\in\mathcal{P}_{\leq 2}(n_1+\cdots+n_r)$ respecting $n_1\otimes\cdots\otimes n_r$, i.e., with no pair contained in a same block $\{1,\dots,n_1\}$, $\{n_1+1,\dots,n_1+n_2\}$,... We define $\mathcal{P}_2(n_1\otimes\cdots\otimes n_r)$ and $\mathcal{P}^k_{\leq 2}(n_1\otimes\cdots\otimes n_r)$ along the same lines.*

(iii) *Given $\pi\in\mathcal{P}_{\leq 2}(n)$, a crossing in π is any set of the form $\{(a_1,b_1),(a_2,b_2)\}$ with $(a_i,b_i)\in\pi$ and $a_1<a_2<b_1<b_2$, or of the form $\{(a,b),(c)\}$ with $(a,b)\in\pi$, $(c)\in\pi$ and $a<c<b$. We will denote by $Cr(\pi)$ the number of crossings in π.*

Now observe that any partition $\pi\in\mathcal{P}^k_{\leq 2}(n)$ can be described in a unique way as

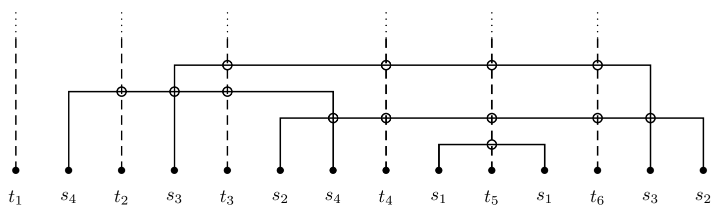
$$\pi=\{(a_i,b_i), i=1,\dots,k,\ a_k<a_{k-1}<\ldots<a_1\}\\ \cup\{(c_i),\ i=1,\dots,n-2k,\ c_1<c_2<\ldots<c_{n-2k}\}\,. \quad (2.1)$$

With this notation in mind, and for all $\mathbf{s}=(s_1,\dots,s_k)\in\mathbb{R}^k_+$ and $\mathbf{t}=(t_1,\dots,t_{n-2k})\in\mathbb{R}^{n-2k}_+$ with $1\leq k\leq\frac{n}{2}-\frac{1}{2}$, we define the *arrangement* $\pi(\mathbf{s},\mathbf{t})\in\{\mathbf{s},\mathbf{t}\}^n$ as follows: for $l=1,\dots,n$,

$$\pi(\mathbf{s},\mathbf{t})_l:=\begin{cases} s_i & \text{if } l=a_i \text{ or } l=b_i\ ,\\ t_i & \text{if } l=c_i\ .\end{cases}$$

When $k=0$, respectively $2k=n$, we naturally extend the latter definition by setting $\pi(\mathbf{t}):=\mathbf{t}$ for every $\mathbf{t}\in\mathbb{R}^n_+$, respectively $\pi(\mathbf{s})_l:=s_i$ if $l=a_i$ or $l=b_i$, for every $\mathbf{s}\in\mathbb{R}^{n/2}_+$.

Example 2.2. Consider the partition $\pi\in\mathcal{P}^4_{\leq 2}(14)$ drawn below, with singletons represented by dashed lines "extending to infinity" and pairs by continuous lines. Such a representation implies in particular that each pair $(a_i,b_i)\in\pi$ satisfying $a_i<c_j<b_i$, for some singleton $(c_j)\in\pi$, necessarily crosses the "line" (c_j). It is then easy to visualize that in this case $\pi(\mathbf{s},\mathbf{t})=(t_1,s_4,t_2,s_3,t_3,s_2,s_4,t_4,s_1,t_5,s_1,t_6,s_3,s_2)$ and $\mathrm{Cr}(\pi)=13$.



The suitable definition of a contraction in our setting now reads as follows:

Definition 2.3. (i) *Given $f\in L^2(\mathbb{R}^n_+)$ and $\pi\in\mathcal{P}^k_{\leq 2}(n)$, we call* integral of f along π, *and denote by $\int_\pi f$, the function defined for every $\mathbf{t}\in\mathbb{R}^{n-2k}_+$ by*

$$\Big[\int_\pi f\Big](\mathbf{t}):=\int_{\mathbb{R}^k_+}\mathrm{d}\mathbf{s}\, f(\pi(\mathbf{s},\mathbf{t}))\ ,$$

where $\pi(\mathbf{s},\mathbf{t})$ stands for the above-defined arrangement along π.

(ii) *For all $f\in L^2(\mathbb{R}^m_+)$, $g\in L^2(\mathbb{R}^n_+)$ and $\pi\in\mathcal{P}_{\leq 2}(m\otimes n)$, we define the* contraction of f and g along π *by the formula $f\otimes_\pi g:=\int_\pi f\otimes g$. Finally, for all $q\in[0,1)$ and $k\in\{0,\dots,m\wedge n\}$, we define the* q-contraction of f and g of order k *as the sum*

$$f\otimes^q_k g:=\sum_{\pi\in\mathcal{P}^k_{\leq 2}(m\otimes n)} q^{Cr(\pi)} f\otimes_\pi g\ .$$

This definition can be seen as a natural q-extension of the standard contracting procedure along a Feynman diagram (as defined in [12, Section 7.2]). It actually coincides with the notion of a q-contraction introduced in [9], which can be recovered by comparing [9, Proposition 4.1] with the subsequent Proposition 3.4 and applying the isometry property (2.3).

Notation 2.4. *Given* $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$*, we will denote by* $\mathbf{t}^*$ *the vector* $(t_n, \ldots, t_1)$*. We recall that for every* $f \in L^2(\mathbb{R}_+^n)$*, the* mirror-symmetric *of* f*, denoted by* f^**, is the function in* $L^2(\mathbb{R}_+^n)$ *defined by* $f^*(\mathbf{t}) = f(\mathbf{t}^*)$*. Besides, given* $\pi^* \in \mathcal{P}_{\leq 2}^k(m \otimes n)$*, we define the* mirror-symmetrized π^* *of* π *as the element of* $\mathcal{P}_{\leq 2}^k(n \otimes m)$ *obtained by taking the symmetric of* π *along a vertical axis between the two blocks* $\{1, \ldots, m\}$ *and* $\{m+1, \ldots, m+n\}$*. Otherwise stated, if* π *is given by (2.1), then*

$$\pi^* := \{(m+n+1-b_i, m+n+1-a_i), (m+n+1-c_j)\} \ .$$

For further use, let us also label the following readily-checked relations between mirror-symmetry and q-contractions:

Lemma 2.5. *For all* $f \in L^2(\mathbb{R}_+^m)$*,* $g \in L^2(\mathbb{R}_+^n)$ *and* $\pi \in \mathcal{P}_{\leq 2}(m \otimes n)$*, it holds that*

$$(f \otimes_\pi g)^* = g^* \otimes_{\pi^*} f^* \ .$$

Besides, $Cr(\pi^*) = Cr(\pi)$ *and accordingly* $(f \otimes_k^q g)^* = g^* \otimes_k^q f^*$ *for every* $k \in \{0, \ldots, m \wedge n\}$*.*

2.2 Construction of the q-Wiener chaos

With the above "graphical" approach in mind, and also for the sake of completeness, let us briefly go back here to the definition of the q-Wiener chaos.

From now on and for the rest of the paper, we fix $q \in (-1, 1)$ and consider a q-Brownian motion $(X_t)_{t \geq 0} = (X^{(q)})_{t \geq 0}$ in some non-commutative probability space $(\mathcal{A}, \varphi)$. With the notations of Section 2.1, observe that the joint moments of $(X_t)_{t \geq 0}$ can also be expressed as

$$\varphi(X_{t_1} \ldots X_{t_n}) = \sum_{\pi \in \mathcal{P}_2(n)} q^{\mathrm{Cr}(\pi)} \int_\pi (\mathbf{1}_{[0,t_1]} \otimes \cdots \otimes \mathbf{1}_{[0,t_n]}) \ . \tag{2.2}$$

The construction of the multiple integrals of X, which give birth to the q-Wiener chaoses, can then be made along a similar procedure as in the classical commutative case (see [14]). Consider first the set $\mathcal{E}_m$ of simple functions, that is the set of functions f of the form $f = \sum_{i_1, \ldots, i_m = 1}^p \lambda_{i_1, \ldots, i_m} \mathbf{1}_{A_{i_1}} \otimes \cdots \otimes \mathbf{1}_{A_{i_m}}$, for pairwise disjoint intervals A_i and coefficients $\lambda_{i_1, \ldots, i_m}$ vanishing on diagonals. For such a function $f \in \mathcal{E}_m$, we set naturally $I_m^q(f) := \sum_{i_1, \ldots, i_m = 1}^p \lambda_{i_1, \ldots, i_m} X(\mathbf{1}_{A_{i_1}}) \cdots X(\mathbf{1}_{A_{i_m}})$, using the convention that for any interval $A := [\ell_1, \ell_2]$, $X(\mathbf{1}_A) := X_{\ell_2} - X_{\ell_1}$. With this definition, observe in particular that $I_m^q(f)^* = I_m^q(f^*)$ for every $f \in \mathcal{E}_m$. Then, just as in the commutative case, the extension of $I_m^q(f)$ to any real-valued function $f \in L^2(\mathbb{R}_+^m)$ relies on two ingredients: the density of $\mathcal{E}_m$ within $L^2(\mathbb{R}_+^m)$ on the one hand, an isometry property on the other, which, in this setting, reads as follows:

Proposition 2.6. *For* $f \in \mathcal{E}_m$ *and* $g \in \mathcal{E}_n$*, it holds that*

$$\langle I_m^q(f), I_n^q(g) \rangle_{L^2(\varphi)} = \delta_{m,n} \langle f, P_q(g) \rangle_{L^2(\mathbb{R}_+^n)} \ , \tag{2.3}$$

where P_q *is the* q-symmetrization operator *defined for all* $f \in L^2(\mathbb{R}_+^m)$ *and* $\mathbf{s} \in \mathbb{R}_+^m$ *by*

$$P_q(f)(\mathbf{s}) := \sum_{\pi \in \mathcal{P}_2(m \otimes m)} q^{Cr(\pi)} f(\pi(\mathbf{s})_{m+1::2m}) \ , \ \text{with } \pi(\mathbf{s})_{m+1::2m} := (\pi(\mathbf{s})_{m+1}, \ldots, \pi(\mathbf{s})_{2m}).$$

In particular, $\|P_q(f)\|_{L^2(\mathbb{R}_+^m)} \leq m! \|f\|_{L^2(\mathbb{R}_+^m)}$*.*

The resulting extension of the integral to any $f \in L^2(\mathbb{R}_+^m)$ will still be denoted by $I_m^q(f)$, as usual. It satisfies the isometry property (2.3), as well as $\varphi(I_m^q(f)) = 0$ for $m \geq 1$.

Proof of Proposition 2.6. Consider the case where

$$f := \sum_{i_1,\dots,i_m=1}^{p} \lambda_{i_1\dots i_m} \mathbf{1}_{A_{i_1}} \otimes \cdots \otimes \mathbf{1}_{A_{i_m}} \ , \ g := \sum_{i_1,\dots,i_n=1}^{p} \beta_{i_1\dots i_n} \mathbf{1}_{A_{i_1}} \otimes \cdots \otimes \mathbf{1}_{A_{i_n}} \ ,$$

for disjoint intervals (A_i) and coefficients λ, β vanishing on diagonals. Then by (2.2)

$$\varphi\big(I_m^q(f)^* I_n^q(g)\big) = \sum_{i_1,\dots,i_m=1}^{p} \sum_{j_1,\dots,j_n=1}^{p} \lambda_{i_1\dots i_m} \beta_{j_1\dots j_m} \sum_{\pi \in \mathcal{P}_2(m+n)} q^{\mathrm{Cr}(\pi)} \int_\pi \big(\mathbf{1}_{A_{i_m}} \otimes \cdots \otimes \mathbf{1}_{A_{i_1}} \otimes \mathbf{1}_{A_{j_1}} \otimes \cdots \otimes \mathbf{1}_{A_{j_n}}\big) \ . \tag{2.4}$$

Now, for pairwise distinct $i_1, .., i_m \in \{1, \dots, p\}$ and pairwise distinct $j_1, .., j_n \in \{1, \dots, p\}$, the integral

$$\int_\pi \big(\mathbf{1}_{A_{i_m}} \otimes \cdots \otimes \mathbf{1}_{A_{i_1}} \otimes \mathbf{1}_{A_{j_1}} \otimes \cdots \otimes \mathbf{1}_{A_{j_n}}\big)$$

clearly vanishes for every $\pi \in \mathcal{P}_2(m+n)$ if $n \neq m$, and for every $\pi \in \mathcal{P}_2(2n) \backslash \mathcal{P}_2(n \otimes n)$ if $n = m$. Thus,

$$\begin{aligned}
&\sum_{\pi \in \mathcal{P}_2(m+n)} q^{\mathrm{Cr}(\pi)} \int_\pi \big(\mathbf{1}_{A_{i_m}} \otimes \cdots \otimes \mathbf{1}_{A_{i_1}} \otimes \mathbf{1}_{A_{j_1}} \otimes \cdots \otimes \mathbf{1}_{A_{j_n}}\big) \\
&= \delta_{m,n} \sum_{\pi \in \mathcal{P}_2(m \otimes m)} q^{\mathrm{Cr}(\pi)} \int_{\mathbb{R}_+^m} d\mathbf{s}\, (\mathbf{1}_{A_{i_m}} \otimes \cdots \otimes \mathbf{1}_{A_{i_1}})(\mathbf{s}^*)(\mathbf{1}_{A_{j_1}} \otimes \cdots \otimes \mathbf{1}_{A_{j_m}})(\pi(\mathbf{s})_{m+1::2m}) \\
&= \delta_{m,n} \int_{\mathbb{R}_+^m} d\mathbf{s}\, (\mathbf{1}_{A_{i_1}} \otimes \cdots \otimes \mathbf{1}_{A_{i_m}})(\mathbf{s})\, P_q(\mathbf{1}_{A_{j_1}} \otimes \cdots \otimes \mathbf{1}_{A_{j_m}})(\mathbf{s}) \ .
\end{aligned}$$

Going back to (2.4), we get the desired conclusion for f ang g, and the general result follows by linearity. □

3 Multiplication formulas in q-Wiener chaoses

With the formalism of Section 2 in hand, we can now state our main result about q-Wiener chaos, namely the extension, to every $q \in (-1, 1)$, of the full Wick product formula:

Theorem 3.1. *Let $n_1, \dots, n_r \geq 1$ and for every $i \in \{1, \dots, r\}$, let $f_i \in L^2(\mathbb{R}_+^{n_i})$. Then, with the notations of Section 2.1, it holds that*

$$I_{n_1}^q(f_1) \cdots I_{n_r}^q(f_r) = \sum_{\pi \in \mathcal{P}_{\leq 2}(n_1 \otimes \cdots \otimes n_r)} q^{Cr(\pi)} I^q\bigg(\int_\pi f_1 \otimes \cdots \otimes f_r\bigg) \ . \tag{3.1}$$

By letting q tend to 1 in (3.1) (at least at some heuristic level), we indeed recover the exact expression of the classical full Wick product formula for the standard Bm (see [12, Theorem 7.33]). Observe also that, by combining (3.1) and the fact that $\varphi(I_m^q(f)) = 0$ for $m \geq 1$, we recover the result of [7, Theorem 2.7]:

Corollary 3.2. *In the setting of Theorem 3.1, it holds that*

$$\varphi(I_{n_1}^q(f_1) \cdots I_{n_r}^q(f_r)) = \sum_{\pi \in \mathcal{P}_2(n_1 \otimes \cdots \otimes n_r)} q^{Cr(\pi)} \int_\pi f_1 \otimes \cdots \otimes f_r \ .$$

The rest of this section is devoted to the proof of Theorem 3.1, which will consist in a natural three-step procedure: we first show the formula when $(r=2, n_1=1, n_2\geq 1)$, then extend the result to the case where $(r=2, n_1, n_2 \geq 1)$, and finally turn to the general situation. At each step, our strategy will actually be based on the (non-trivial) q-extension of the classical arguments used in the commutative framework.

Proposition 3.3. *For all $f\in L^2(\mathbb{R}_+)$ and $g\in L^2(\mathbb{R}_+^n)$, it holds that*

$$I_1^q(f)I_n^q(g) = I_{n+1}^q(f\otimes g) + I_{n-1}^q(f\otimes_1^q g)\ . \tag{3.2}$$

Proof. For any interval A, let us write $X(A)$ for $X(\mathbf{1}_A)$. Using bilinearity and a density argument, it is readily checked that we can focus on the following situation:

$$f=\mathbf{1}_A \quad \text{and} \quad g=\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_{k-1}}\otimes\mathbf{1}_B\otimes\mathbf{1}_{A_k}\otimes\cdots\otimes\mathbf{1}_{A_{n-1}}\ ,$$

where $k\in\{1,\ldots,n\}$, the intervals A_i are disjoint and A,B are both intervals disjoint from the A_i's such that $A=B$ or $A\cap B=\emptyset$.

If $A\cap B=\emptyset$, then clearly $f\otimes_1^q g=0$ and the formula is trivially satisfied, so we assume from now on that $A=B$. Then, denoting by $\mu(A)$ the length of A, it is easy to see (as in Figure 1) that $f\otimes_1^q g = q^{k-1}\mu(A)\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_{n-1}}$ and accordingly $I_{n-1}^q(f\otimes_1^q g) = q^{k-1}\mu(A)X(A_1)\cdots X(A_{n-1})$.

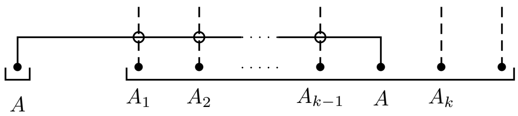


Figure 1: The only partition with non-zero contribution in the computation of $f\otimes_1^q g$.

For $\varepsilon>0$, pick disjoint intervals $B_1,\ldots,B_l$ such that $A=B_1\cup\ldots\cup B_l$ and $\mu(B_i)<\varepsilon$. Then write

$$\begin{aligned}
I_1(f)I_n(g) &= X(A)X(A_1)\cdots X(A_{k-1})X(A)X(A_k)\cdots X(A_{n-1})\\
&= \sum_{i\neq j} X(B_i)X(A_1)\cdots X(A_{k-1})X(B_j)X(A_k)\cdots X(A_{n-1})\\
&\quad + \sum_{i=1}^{l}\big\{X(B_i)X(A_1)\cdots X(A_{k-1})X(B_i)X(A_k)\cdots X(A_{n-1})\\
&\qquad\quad -q^{k-1}\mu(B_i)X(A_1)\cdots X(A_{n-1})\big\} + q^{k-1}\mu(A)X(A_1)\cdots X(A_{n-1})\\
&= I_{n+1}^q(h_\varepsilon) + \sum_{i=1}^{l} R_{\varepsilon,i} + I_{n-1}^q(f\otimes_1^q g)\ ,
\end{aligned}$$

where we have set $h_\varepsilon := \sum_{i\neq j}\mathbf{1}_{B_i}\otimes\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_{k-1}}\otimes\mathbf{1}_{B_j}\otimes\mathbf{1}_{A_k}\otimes\cdots\otimes\mathbf{1}_{A_{n-1}}$. At this point, observe that due to (2.3), one has

$$\|I_{n+1}^q(h_\varepsilon) - I_{n+1}^q(f\otimes g)\|_{L^2(\varphi)} \leq (n+1)!\,\|h_\varepsilon - f\otimes g\|_{L^2(\mathbb{R}_+^{n+1})}\ ,$$

which tends to 0 as $\varepsilon\to 0$. Therefore, it remains us to check that $\|\sum_{i=1}^{l} R_{\varepsilon,i}\|_{L^2(\varphi)}\to 0$ as $\varepsilon\to 0$, and to this end, decompose $\|\sum_{i=1}^{l} R_{\varepsilon,i}\|_{L^2(\varphi)}^2$ as

$$\|\sum_{i=1}^{l} R_{\varepsilon,i}\|_{L^2(\varphi)}^2 = \sum_{i\neq j}\varphi\big(R_{\varepsilon,i}^* R_{\varepsilon,j}\big) + \sum_{i=1}^{l}\varphi\big(R_{\varepsilon,i}^* R_{\varepsilon,i}\big)\ . \tag{3.3}$$

When $i \neq j$, it turns out that $\varphi\big(R^*_{\varepsilon,i} R_{\varepsilon,j}\big) = 0$, as a consequence of the following readily-checked relations (see Figure 2 for an illustration of the first one):

$$\begin{aligned} &\varphi\big([X(A_{n-1}) \cdots X(A_k) X(B_i) X(A_{k-1}) \cdots X(A_1) X(B_i)] \cdot \\ &\qquad [X(B_j) X(A_1) \cdots X(A_{k-1}) X(B_j) X(A_k) \cdots X(A_{n-1})]\big) \\ &\quad = q^{2(k-1)} \mu(B_i)\mu(B_j)\mu(A_1) \cdots \mu(A_{n-1}) \ , \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\varphi\big([X(A_{n-1}) \cdots X(A_k) X(B_i) X(A_{k-1}) \cdots X(A_1) X(B_i)] \cdot [X(A_1) \cdots X(A_{n-1})]\big) \\ &\quad = q^{k-1} \mu(B_i)\mu(A_1) \cdots \mu(A_{n-1}) \ , \end{aligned}$$

and $\varphi\big([X(A_{n-1}) \cdots X(A_1)] \cdot [X(A_1) \cdots X(A_{n-1})]\big) = \mu(A_1) \cdots \mu(A_{n-1})$. As for the second summand in (3.3), it is easy to see that $\varphi\big(R^*_{\varepsilon,i} R_{\varepsilon,i}\big) \leq c\, \mu(B_i)^2$, so $0 \leq \sum_{i=1}^{l} \varphi\big(R^*_{\varepsilon,i} R_{\varepsilon,i}\big) \leq c\,\varepsilon \mu(A)$, which, by letting ε tend to zero, completes the proof of our statement. □

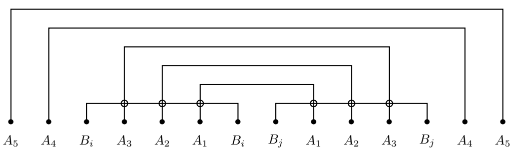


Figure 2: The only partition with non-zero contribution in (3.4) (here, $k = 4, n = 6$).

Let us now extend (3.2) to any function $f \in L^2(\mathbb{R}_+^m)$, $m \geq 1$:

Proposition 3.4. *For all $f \in L^2(\mathbb{R}_+^m)$ and $g \in L^2(\mathbb{R}_+^n)$, it holds that*

$$I_m^q(f) I_n^q(g) = \sum_{k=0}^{m \wedge n} I^q_{m+n-2k}(f \otimes_k^q g) \ . \tag{3.5}$$

The key ingredient towards (3.5) is the following recursion formula satisfied by q-contractions:

Lemma 3.5. *Fix $m \leq n-1$ and let $g \in L^2(\mathbb{R}_+^n)$, $f_1 := \mathbf{1}_{A_1}$, $f_2 := \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_{m+1}}$, for disjoint intervals (A_i). Then the following relations hold true:*

$$(f_1 \otimes f_2) \otimes^q_{m+1} g = f_1 \otimes^q_1 (f_2 \otimes^q_m g),$$

and for every $k = 1, \ldots, m$,

$$(f_1 \otimes f_2) \otimes^q_k g = f_1 \otimes (f_2 \otimes^q_k g) + f_1 \otimes^q_1 (f_2 \otimes^q_{k-1} g) \ .$$

Proof. Given $\mathbf{t} := (t_1, \ldots, t_n) \in \mathbb{R}_+^n$ and $\ell \leq m \in \{1, \ldots, n\}$, let us set

$$\mathbf{t}_{\ell::m} := (t_\ell, t_{\ell+1}, \ldots, t_m) \ .$$

Using this notation, one has, for all $k \in \{0, \ldots, m\}$ and $\mathbf{t} \in \mathbb{R}_+^{m+n-2k-1}$,

$$\begin{aligned} &[f_1 \otimes^q_1 (f_2 \otimes^q_k g)](\mathbf{t}) \\ &\quad = \sum_{\ell=1}^{m+n-2k} q^{\ell-1} \int_{\mathbb{R}^+} \mathrm{d}s\, f_1(s) (f_2 \otimes^q_k g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) \\ &\quad = \sum_{\ell=1}^{m+n-2k} \sum_{\pi \in \mathcal{P}^k_{\leq 2}(m \otimes n)} q^{\ell-1+\mathrm{Cr}(\pi)} \int_{\mathbb{R}^+} \mathrm{d}s\, f_1(s) (f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) \ . \end{aligned}$$

Since f_1 and f_2 have disjoint supports, it is easy to see that if $\ell \in \{1,\ldots,m-k\}$ then for every $\pi \in \mathcal{P}^k_{\leq 2}(m\otimes n)$,

$$\int_{\mathbb{R}^+} \mathrm{d}s\, f_1(s)(f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) = 0\ ,$$

and accordingly the above formula reduces to

$$[f_1 \otimes^q_1 (f_2 \otimes^q_k g)](\mathbf{t}) = \sum_{\ell=m-k+1}^{m+n-2k} \sum_{\pi\in\mathcal{P}^k_{\leq 2}(m\otimes n)} q^{\ell-1+\mathrm{Cr}(\pi)} \int_{\mathbb{R}^+} \mathrm{d}s\, f_1(s)(f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1})\ . \tag{3.6}$$

Now observe that there is a one-to-one correspondance between pairs $(\ell,\pi) \in \{m-k+1,\ldots,m+n-2k\} \times \mathcal{P}^k_{\leq 2}(m\otimes n)$ and partitions $\pi' \in \mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)$ not containing the singleton (1). Namely, given such a pair (ℓ,π), we can construct π' along the following two-step procedure (see Figure 3):

• let the k interactions between the blocks $\{2,\ldots,m+1\}$ and $\{m+2,\ldots,m+n\}$ be governed by π;

• connect the point 1 with the ℓ-th unpaired point of $\{2,\ldots,m+n+1\}$, when counting from the left to the right (in particular, the right-end point of this pair necessarily belongs to $\{m+2,\ldots,m+n+1\}$ due to $\ell \geq m-k+1$).

With these notations, it is readily checked (see again Figure 3) that $\mathrm{Cr}(\pi') = \mathrm{Cr}(\pi) + (\ell-1)$ and

$$\int_{\mathbb{R}^+} \mathrm{d}s\, f_1(s)(f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) = [(f_1\otimes f_2)\otimes_{\pi'} g](\mathbf{t})\ .$$

Going back to (3.6), we deduce that

$$f_1 \otimes^q_1 (f_2 \otimes^q_k g) = \sum_{\substack{\pi\in\mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)\\ (1)\notin\pi}} q^{\mathrm{Cr}(\pi)} (f_1\otimes f_2)\otimes_\pi g\ . \tag{3.7}$$

When $k=m$, (3.7) reduces to

$$f_1 \otimes^q_1 (f_2 \otimes^q_m g) = \sum_{\pi\in\mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)} q^{\mathrm{Cr}(\pi)} (f_1\otimes f_2)\otimes_\pi g = (f_1\otimes f_2)\otimes^q_{m+1} g\ ,$$

which corresponds to the first claim of our statement. Then, for $k\in\{0,\ldots,m-1\}$, one has

$$\begin{aligned}
&(f_1\otimes f_2)\otimes^q_{k+1} g \\
&\quad= \sum_{\pi\in\mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)} q^{\mathrm{Cr}(\pi)} (f_1\otimes f_2)\otimes_\pi g \\
&\quad= \sum_{\substack{\pi\in\mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)\\ (1)\in\pi}} q^{\mathrm{Cr}(\pi)} (f_1\otimes f_2)\otimes_\pi g + \sum_{\substack{\pi\in\mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)\\ (1)\notin\pi}} q^{\mathrm{Cr}(\pi)} (f_1\otimes f_2)\otimes_\pi g \\
&\quad= f_1\otimes (f_2\otimes^q_{k+1} g) + \sum_{\substack{\pi\in\mathcal{P}^{k+1}_{\leq 2}((m+1)\otimes n)\\ (1)\notin\pi}} q^{\mathrm{Cr}(\pi)} (f_1\otimes f_2)\otimes_\pi g\ ,
\end{aligned}$$

and we can conclude by using (3.7) again. □

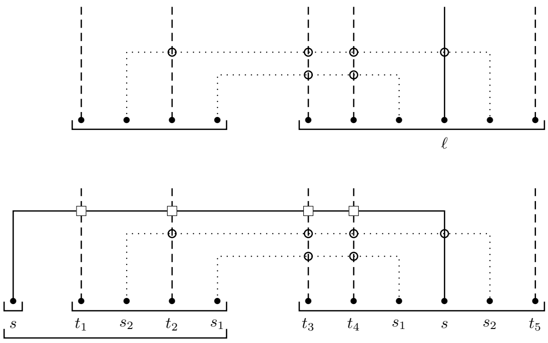


Figure 3: Construction of π' (second line) from π and the ℓ-th unpaired position (first line). Here, $m = 4, n = 6, r = 2$ and $\ell = 5$.

Proof of Proposition 3.4. Assume that formula (3.5) holds true for all $m \leq n$, $f \in L^2(\mathbb{R}_+^m)$ and $g \in L^2(\mathbb{R}_+^n)$. Then for $m \geq n$, it holds that

$$I_m^q(f)I_n^q(g) = \big(I_n^q(g^*)I_m^q(f^*)\big)^* = \Big(\sum_{k=0}^{m\wedge n} I_{n+m-2k}^q(g^* \otimes_k^q f^*)\Big)^* = \sum_{k=0}^{m\wedge n} I_{m+n-2k}^q(f \otimes_k^q g) ,$$

where we have used Lemma 2.5 to derive the last equality.

Therefore, we can stick to an induction procedure on $m \geq 1$ for $n \geq m$. If $m = 1$, then (3.5) is nothing but the result of Proposition 3.3. Assume that the decomposition holds true for some $m \geq 1$ and every $n \geq m$, and let $n \geq m+1$. By a density argument, we can take $f \in L^2(\mathbb{R}_+^{m+1})$ of the form $f = f_1 \otimes f_2$ with $f_1 = \mathbf{1}_{A_1}$ and $f_2 = \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_{m+1}}$, for disjoint intervals (A_i). Then $I_{m+1}^q(f) = I_1^q(f_1)I_m^q(f_2)$ and by the induction hypothesis

$$I_{m+1}^q(f)I_n^q(g) = I_1^q(f_1) \cdot \big[I_m^q(f_2)I_n^q(g)\big] = \sum_{k=0}^{m} I_1^q(f_1)I_{m+n-2k}^q(f_2 \otimes_k^q g) ,$$

which, by Proposition 3.3, yields that

$$\begin{aligned}
&I_{m+1}^q(f)I_n^q(g)\\
&\quad= \sum_{k=0}^{m}\big[I_{m+n+1-2k}^q\big(f_1 \otimes (f_2 \otimes_k^q g)\big) + I_{m+n-1-2k}^q\big(f_1 \otimes_1^q (f_2 \otimes_k^q g)\big)\big]\\
&\quad= I_{m+n+1}^q(f_1 \otimes f_2 \otimes g)\\
&\qquad\quad+ \sum_{k=1}^{m} I_{m+n+1-2k}^q\big(f_1 \otimes (f_2 \otimes_k^q g) + f_1 \otimes_1^q (f_2 \otimes_{k-1}^q g)\big) + I_{n-m-1}^q\big(f_1 \otimes_1^q (f_2 \otimes_m^q g)\big) .
\end{aligned}$$

The conclusion immediately follows from the two identities exhibited in Lemma 3.5. $\square$

We can finally turn to the proof of the general formula.

Proof of Theorem 3.1. We proceed by induction on $r \geq 1$. For $r = 1$, the result only amounts to saying that $q^{\mathrm{Cr}(\pi)} \int_\pi f_1 = f_1$ when $\pi = \{(1), \ldots, (n_1)\}$, which is obvious. Assume that the relation holds true up to $r-1$ and let $f_i \in L^2(\mathbb{R}_+^{n_i})$ for $i = 1, \ldots, r$. Then, using Proposition 3.4, we have

$$I^q_{n_1}(f_1)\cdots I^q_{n_r}(f_r) = \sum_{k=0}^{n_1\wedge n_2} \sum_{\pi_1\in\mathcal{P}^k_{\leq 2}(n_1\otimes n_2)} q^{\mathrm{Cr}(\pi_1)} I^q(f_1\otimes_{\pi_1} f_2) I^q_{n_3}(f_3)\cdots I^q_{n_r}(f_r)$$

$$= \sum_{k=0}^{n_1\wedge n_2} \sum_{\pi_1\in\mathcal{P}^k_{\leq 2}(n_1\otimes n_2)} \sum_{\pi_2\in\mathcal{P}_{\leq 2}((n_1+n_2-2k)\otimes n_3\otimes\cdots\otimes n_r)} q^{\mathrm{Cr}(\pi_1)+\mathrm{Cr}(\pi_2)} I^q\Big(\int_{\pi_2} [f_1\otimes_{\pi_1} f_2]\otimes f_3\otimes\cdots\otimes f_r\Big) . \quad (3.8)$$

At this point, and in a similar way as in the proof of Lemma 3.5, observe that there is a one-to-one correspondance between the set of triplets (k,π_1,π_2) in the above sum and the set of partitions $\pi\in\mathcal{P}_{\leq 2}(n_1\otimes\cdots\otimes n_r)$. Namely, given such a triplet (k,π_1,π_2), we can construct π along the following 2-step procedure (see Figure 4):

• the first two blocks $\{1,\ldots,n_1\}$ and $\{n_1+1,\ldots,n_1+n_2\}$ are connected by k pairs, and these interactions are described through π_1;

• then π_2 governs the interactions between the n_1+n_2-2k unpaired points of $\{1,\ldots,n_1+n_2\}$ and the set $\{n_1+n_2+1,\ldots,n_1+\cdots+n_r\}$.

With these notations, it only remains to observe (see Figure 4 again) that $\mathrm{Cr}(\pi)=\mathrm{Cr}(\pi_1)+\mathrm{Cr}(\pi_2)$ and

$$\int_{\pi_2} [f_1\otimes_{\pi_1} f_2]\otimes f_3\otimes\cdots\otimes f_r = \int_{\pi} f_1\otimes f_2\otimes f_3\otimes\cdots\otimes f_r .$$

Going back to (3.8), we get the desired conclusion. □

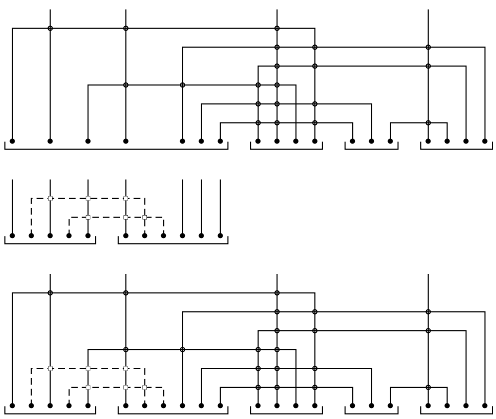


Figure 4: Construction of π (third line) from π_1 (second line) and π_2 (first line).

4 Multiplication in the fully-symmetric case

Let us now elaborate on the specific situation where the kernels involved within the multiple integrals under consideration are given by *fully-symmetric* functions. Recall that a function $f:\mathbb{R}^n_+\to\mathbb{R}$ is said to be fully-symmetric if for all times $t_1,\ldots,t_n\in\mathbb{R}_+$ and every permutation σ of $\{1,\ldots,n\}$, one has $f(t_1,\ldots,t_n)=f(t_{\sigma(1)},\ldots,t_{\sigma(n)})$.

It is important to notice here that, contrary to the classical commutative case (with multiple integrals generated by a standard Bm), the fully-symmetric assumption must be regarded as highly restrictive in the non-commutative situation. Otherwise stated, considering any generic function $f \in L^2(\mathbb{R}_+^n)$ ($n \geq 2$) and its symmetrization

$$\tilde{f}(t_1, \ldots, t_n) := \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}) \ ,$$

there is no reason in general for the two multiple integrals $I_n^q(f)$ and $I_n^q(\tilde{f})$ to be equal.

4.1 Multiplication formula

Given two fully-symmetric functions $f_1 \in L^2(\mathbb{R}_+^{n_1}), f_2 \in L^2(\mathbb{R}_+^{n_2})$ and a parameter $k \in \{1, \ldots, n_1 \wedge n_2\}$, it is readily checked that the contraction $f_1 \otimes_\pi f_2$ (see Definition 2.3) does not depend on the choice of $\pi \in \mathcal{P}_{\leq 2}^k(n_1 \otimes n_2)$: we denote by $f_1 \otimes_k f_2$ this common value. For $q \in (-1, 1)$, the q-contraction of f_1 and f_2 (see again Definition 2.3) is then equal to $f_1 \otimes_k^q f_2 = C_{n_1,n_2}^{(k)}(q) f \otimes_k g$, where

$$C_{n_1,n_2}^{(k)}(q) := \sum_{\pi \in \mathcal{P}_{\leq 2}^k(n_1 \otimes n_2)} q^{\mathrm{Cr}(\pi)} \ .$$

We propose to show that the latter coefficients can be conveniently expressed in terms of standard q-binomial coefficients. To this end, let us first recall the definition of the classical coefficients associated with q-combinatorics: $[n]_q := 1 + q + \cdots + q^{n-1}$, $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$,

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} \qquad , \qquad \binom{n}{n_1, \ldots, n_p}_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_p]_q!} \ ,$$

for all $n_1, \ldots, n_p \geq 0$ such that $n_1 + \cdots + n_p = n$.

Proposition 4.1. *For all $m, n \geq 0$ and $k \in \{0, \ldots, m \wedge n\}$, it holds that*

$$C_{m,n}^{(k)}(q) = [k]_q! \binom{n}{k}_q \binom{m}{k}_q \ . \tag{4.1}$$

Injecting (4.1) into (3.5) immediately gives rise to the following combinatorial description of the product of any two multiple integrals build upon fully-symmetric kernels. Note that, when compared for instance with [14, Proposition 1.1.3], this formulation makes the transition with the classical commutative case even more clear.

Theorem 4.2. *Let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be fully-symmetric functions. Then it holds that*

$$I_n^q(f) I_m^q(g) = \sum_{k=0}^{n \wedge m} [k]_q! \binom{n}{k}_q \binom{m}{k}_q I_{n+m-2r}^q(f \otimes_k g) \ .$$

Proof of Proposition 4.1. Observe that any partition $\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)$ can be entirely described through two elements (see Figure 5):

- the positions $\{m+1-i_k < \ldots < m+1-i_1\}$ in $\{1, \ldots, m\}$ (resp. $\{m+j_1 < \ldots < m+j_k\}$ in $\{m+1, \ldots, m+n\}$) corresponding to the left-end (resp. right-end) points of the pairs in π;
- the interactions in π between these paired points, which can be summed through a unique $\pi' \in \mathcal{P}_2(k \otimes k)$.

Setting $\mathbf{i} := (i_1, \ldots, i_k)$ and $\mathbf{j} := (j_1, \ldots, j_k)$, the difference $\mathrm{Cr}(\mathbf{i}, \mathbf{j}) := \mathrm{Cr}(\pi) - \mathrm{Cr}(\pi')$ is then given by the number of crossings between a pair and a singleton in π. We can easily compute this quantity as follows (see again Figure 5):

$$\begin{aligned}&\mathrm{Cr}(\mathbf{i},\mathbf{j})\\ &=\big[k\times(i_1-1)+(k-1)\times(i_2-i_1-1)+\cdots+1\times(i_k-i_{k-1}-1)+0\times(m-i_k)\big]\\ &\quad+\big[k\times(j_1-1)+(k-1)\times(j_2-j_1-1)+\cdots+1\times(j_k-j_{k-1}-1)+0\times(n-j_k)\big]\ ,\end{aligned}$$

which in fact reduces to $\mathrm{Cr}(\mathbf{i},\mathbf{j})=\sum_{l=1}^{k}(i_l+j_l)-k(k+1)$. As a consequence,

$$\begin{aligned}\sum_{\pi\in\mathcal{P}^k_{\leq 2}(m\otimes n)} q^{\mathrm{Cr}(\pi)} &= \Bigg(\sum_{\pi'\in\mathcal{P}_2(k\otimes k)} q^{\mathrm{Cr}(\pi')}\Bigg)\Bigg(\sum_{\substack{1\leq i_1<\ldots<i_k\leq m\\ 1\leq j_1<\ldots<j_k\leq n}} q^{\mathrm{Cr}(\mathbf{i},\mathbf{j})}\Bigg)\\ &= q^{-k(k+1)}\Bigg(\sum_{\pi'\in\mathcal{P}_2(k\otimes k)} q^{\mathrm{Cr}(\pi')}\Bigg)\Bigg(\sum_{1\leq i_1<\ldots<i_k\leq m} q^{i_1+\cdots+i_k}\Bigg)\Bigg(\sum_{1\leq j_1<\ldots<j_k\leq n} q^{j_1+\cdots+j_k}\Bigg)\ .\end{aligned}$$

It now remains us to apply Corollary 1.2 and Corollary 1.4 of [1, Section 1.6], i.e. the respective classical formulas

$$\sum_{1\leq i_1<\ldots<i_k\leq n} q^{i_1+\cdots+i_k} = q^{\frac{k(k+1)}{2}}\binom{n}{k}_q \qquad \text{and} \qquad \sum_{\pi\in\mathcal{P}_2(n\otimes n)} q^{\mathrm{Cr}(\pi)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!\ ,$$

where $\mathfrak{S}_n$ refers to the group of permutations of $\{1,\ldots,n\}$ and $\mathrm{inv}(\sigma)$ to the number of inversions in σ.

$\square$

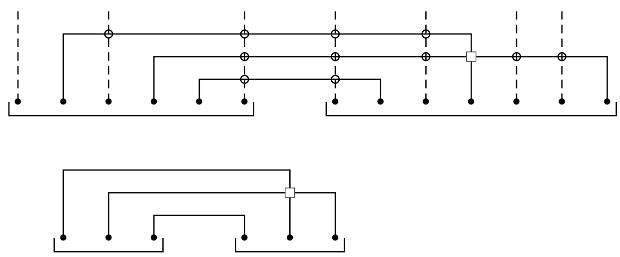


Figure 5: A partition $\pi\in\mathcal{P}^3_{\leq 2}(6\times 7)$ (first line) with its associated partition $\pi'\in\mathcal{P}_2(3\otimes 3)$ (second line). With the notations of the proof, the positions of the pairs in π correspond to $i_1=2, i_2=3, i_3=5$, and $j_1=2, j_2=4, j_3=7$.

4.2 Stochastic dynamics of the q-Hermite martingales

A well-known example of multiple integrals involving fully-symmetric kernels is provided by the sequence of the *q-Hermite martingales*

$$M_n^{(q)}(t) := I_n^q\big(\mathbf{1}_{[0,t]}^{\otimes n}\big)\ ,\ n\geq 1\ ,$$

which corresponds to the q-analog of the classical sequence of Hermite martingales derived from Hermite polynomials. The study of the martingale properties of $M_n^{(q)}$, as well as its connections with the q-Hermite polynomials, can be found in [4, Proposition 2.9 and Corollary 4.7].

We propose here to use the results of the previous sections (and especially identity (4.1)) so as to complete the program initiated in [9, Section 4.2] regarding the stochastic dynamics of the sequence $(M_n^{(q)})$. Let us first recall that in the classical commutative framework, the sequence $(M_n^{(1)})$ of Hermite martingales is known to be governed by the formula (denoting by $X^{(1)}$ the standard Bm)

$$M_{n+1}^{(1)}(t) = (n+1)\int_0^t M_n^{(1)}(s)\,\mathrm{d}X_s^{(1)} \;, \tag{4.2}$$

while in the free situation, that is when $q=0$, the following relation can be found in [2]

$$M_{n+1}^{(0)}(t) = \sum_{0\le k\le n}\int_0^t M_k^{(0)}(s)\,\mathrm{d}X_s^{(0)}\,M_{n-k}^{(0)}(s) \;. \tag{4.3}$$

In order to express our interpolation result between these two formulas, let us consider the family of kernels defined for $\ell, m \ge 1$ and $t, s_1, \ldots, s_\ell \ge 0$ as

$$h_{m,t}^{\ell}(s_1,\ldots,s_\ell) := \mathbf{1}_{[0,t]}^{\otimes \ell}(s_1,\ldots,s_\ell)\prod_{i\neq m}\mathbf{1}_{\{s_i<s_m\}}$$

Proposition 4.3. *For all $m, n \ge 0$ and $t\ge 0$, it holds that*

$$I_{m+n+1}^q\big(h_{m+1,t}^{m+n+1}\big) = \sum_{\ell=0}^{m\wedge n}(-1)^\ell q^{\frac{\ell(\ell+1)}{2}}C_{m,n}^{(\ell)}(q)\int_0^t s^\ell M_{m-\ell}^{(q)}(s)\,\mathrm{d}X_s^{(q)}\,M_{n-\ell}^{(q)}(s) \;, \tag{4.4}$$

where the integral in the right-hand side is understood in Itô's sense (see [9, Corollary 3.2]).

To see that identity (4.4) indeed provides us with the desired interpolation between (4.2) and (4.3), it suffices to observe that, on the one hand,

$$M_{n+1}^{(1)}(t) = I_{n+1}^1\big(\mathbf{1}_{[0,t]}^{\otimes n+1}\big) = (n+1)I_{n+1}^1\big(h_{1,t}^{n+1}\big) \;,$$

which gives (4.2) by applying (4.4) with $m=0$. On the other hand, by writing

$$M_{n+1}^{(q)}(t) = I_{n+1}^q\big(\mathbf{1}_{[0,t]}^{\otimes n+1}\big) = \sum_{k=0}^{n} I_{(n-k)+k+1}^q\big(h_{k+1,t}^{(n-k)+k+1}\big)$$

and then applying (4.4) to each summand, we immediately recover (4.3) for $q=0$, and more generally:

Corollary 4.4. *For all $n\ge 0$ and $t \ge 0$, it holds that*

$$M_{n+1}^{(q)}(t) = \sum_{k=0}^{n}\sum_{\ell=0}^{k\wedge(n-k)}(-1)^\ell q^{\frac{\ell(\ell+1)}{2}}C_{k,n-k}^{(\ell)}(q)\int_0^t s^\ell\, M_{k-\ell}^{(q)}(s)\,\mathrm{d}X_s^{(q)}\,M_{n-k-\ell}^{(q)}(s) \;,$$

where the integral in the right-hand side is understood in Itô's sense.

Proof of Proposition 4.3. For the sake of clarity, we fix q and t for the whole proof and drop the dependence on these two parameters in the notation, that is we write $C_{m,n}^{(k)}$ for $C_{m,n}^{(k)}(q)$, h_m^ℓ for $h_{m,t}^\ell$, and so on. Besides, let us set, for all $\ell, p \ge 0$, $m, n\ge 1$ and $s_1, \ldots, s_\ell \ge 0$,

$$h_m^{\ell,p}(s_1,\ldots,s_\ell) := s_m^p\mathbf{1}_{[0,t]}^{\otimes\ell}(s_1,\ldots,s_\ell)\prod_{i\neq m}\mathbf{1}_{\{s_i<s_m\}} \quad , \quad J_{m,n}^\ell := \int_0^t s^\ell M_m(s)\,\mathrm{d}X_s\,M_n(s) \;,$$

noting in particular that $h_m^\ell = h_m^{\ell,0}$. With these notations in hand, a straightforward application of [9, Proposition 4.2] yields that

$$J_{m,n}^p = I_{m+n+1}\big(h_{m+1}^{m+n+1,p}\big) + \sum_{\ell_1=1}^{m\wedge n} q^{\ell_1}C_{m,n}^{(\ell_1)}I_{m+n+1-2\ell_1}\big(h_{m+1-\ell_1}^{m+n+1-2\ell_1,p+\ell_1}\big) \;.$$

Therefore,

$$I_{m+n+1}\big(h_{m+1}^{m+n+1}\big) = J_{m,n}^0 - \sum_{\ell_1=1}^{m\wedge n} q^{\ell_1} C_{m,n}^{(\ell_1)} I_{m+n+1-2\ell_1}\big(h_{m+1-\ell_1}^{m+n+1-2\ell_1,\ell_1}\big)$$

$$= J_{m,n}^0 - \sum_{\ell_1=1}^{m\wedge n} q^{\ell_1} C_{m,n}^{(\ell_1)}$$

$$\Big[J_{m-\ell_1,n-\ell_1}^{\ell_1} - \sum_{\ell_2=1}^{(m\wedge n)-\ell_1} q^{\ell_2} C_{m-\ell_1,n-\ell_1}^{(\ell_2)} I_{m+n+1-2(\ell_1+\ell_2)}\big(h_{m+1-(\ell_1+\ell_2)}^{m+n+1-2(\ell_1+\ell_2),\ell_1+\ell_2}\big)\Big]$$

and by repeating the procedure, we end up with

$$I_{m+n+1}\big(h_{m+1}^{m+n+1}\big) = J_{m,n} + \sum_{\ell=1}^{m\wedge n} q^{\ell} J_{m-\ell,n-\ell}^{\ell} \sum_{p=1}^{\ell} (-1)^p$$
$$\sum_{\substack{\ell_1+\cdots+\ell_p=\ell\\ \ell_i\geq 1}} \Big[C_{m,n}^{(\ell_1)} C_{m-\ell_1,n-\ell_1}^{(\ell_2)} \cdots C_{m-(\ell_1+\cdots+\ell_{p-1}),n-(\ell_1+\cdots+\ell_{p-1})}^{(\ell_p)}\Big] .$$

Now, using identity (4.1), it is readily checked that for all $\ell_1,\ldots,\ell_p \in \{1,\ldots,m\wedge n\}$ with $\ell_1+\cdots+\ell_p=\ell$,

$$C_{m,n}^{(\ell_1)} C_{m-\ell_1,n-\ell_1}^{(\ell_2)} \cdots C_{m-(\ell_1+\cdots+\ell_{p-1}),n-(\ell_1+\cdots+\ell_{p-1})}^{(\ell_p)} = C_{m,n}^{(\ell)} \binom{\ell}{\ell_1,\ldots,\ell_p}_q ,$$

and thus we obtain

$$I_{m+n+1}\big(h_{m+1}^{m+n+1}\big) = J_{m,n}^0 + \sum_{\ell=1}^{m\wedge n} q^{\ell} J_{m-\ell,n-\ell}^{\ell} C_{m,n}^{(\ell)} \sum_{p=1}^{\ell} (-1)^p \sum_{\substack{\ell_1+\cdots+\ell_p=\ell\\ \ell_i\geq 1}} \binom{\ell}{\ell_1,\ldots,\ell_p}_q .$$

It only remains us to apply the q-combinatorial identity

$$\sum_{p=1}^{\ell} (-1)^p \sum_{\substack{\ell_1+\cdots+\ell_p=\ell\\ \ell_i\geq 1}} \binom{\ell}{\ell_1,\ldots,\ell_p}_q = (-1)^{\ell} q^{\frac{\ell(\ell-1)}{2}} ,$$

the proof of which is left to the reader as an exercise. □

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