ON L² MODULUS OF CONTINUITY OF BROWNIAN LOCAL TIMES AND RIESZ POTENTIALS

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This article is concerned with modulus of continuity of Brownian local times. Specifically, we focus on 3 closely related problems: (a) Limit theorem for a Brownian modulus of continuity involving Riesz potentials, where the limit law is an intricate Gaussian mixture. (b) Central limit theorems for the projections of L^2 modulus of continuity for a 1-dimensional Brownian motion. (c) Extension of the second result to a 2-dimensional Brownian motion. Our proofs rely on a combination of stochastic calculus and Malliavin calculus tools, plus a thorough analysis of singular integrals.

1. Introduction. Let $\{B_t,\ 0 \leq t \leq 1\}$ be a standard linear Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. In the sequel, we denote by $L_t(x)$ the local time of B at a given point $x \in \mathbb{R}$, defined for $t \in [0, 1]$. A nice combination of stochastic calculus, stochastic analysis and evaluation of singularities associated with heat kernels have recently led to a number of interesting limit theorems for quantities related to the family $\{L_t(x); t \in [0, 1], x \in \mathbb{R}\}$. Let us quote for instance the use of Malliavin and stochastic calculus tools in order to get suitably normalized limits for L^2 modulus of continuity (see [7, 15]) or third moment in space (cf [8]) of Brownian local time. Malliavin calculus tools have also been essential in order to generalize the notion of self-intersection local time [6, 9] and to obtain central limit theorems for additive functionals [10] of fractional Brownian motion.

The current article proposes to take another step into the relationships between Brownian local time and stochastic analysis. Specifically, we shall handle the following problems:

(1) One of the motivation alluded to in [15] for the renormalization of L^2

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modulus of continuity of local times comes from the study of the Hamiltonian

(1)
$$H_t^h(B) = \int_{\mathbb{R}} \left[L_t(x+h) - L_t(x)\right]^2 dx = \int_{\mathbb{R}} \left[\int_0^t \left(\delta_{x+h}(B_u) - \delta_x(B_u)\right) du\right]^2 dx,$$

which is involved in the definition of some non-folding polymers. However, one might wish to consider a slightly weaker repelling self-interaction of the polymer by introducing the following family of Hamiltonians indexed by $\gamma \in (0,1)$:

(2)
$$H_t^{h,\gamma}(B) = \int_{\mathbb{R}} \left[\int_0^t \left(|B_v + x + h|^{-\gamma} - |B_v + x|^{-\gamma}\right) dv\right]^2 dx.$$

For this modified Hamiltonian, we shall prove the following limiting theorem:

THEOREM 1.1. *Consider* $\gamma \in (3/4, 1)$ *and the family of Hamiltonians* $\{H_t^{h,\gamma}(B); \, t \in [0,1]\}$ *defined by* (2). *Then one has, as h tends to zero,*

(3)
$$\frac{H^{h,\gamma}(B) - \mathbf{E}\left[H^{h,\gamma}(B)\right]}{c_\gamma \, h^{7/2 - 2\gamma}} \xrightarrow{(d)} W_\alpha$$

in the space $\mathcal{C}([0,1]; \mathbb{R})$ *of real continuous functions on* $[0,1]$. *In relation* (3), c_γ *stands for a deterministic positive constant depending only on* γ, W *is a standard Brownian motion independent of* B *and* α *is the self-intersection local time of* B, *i.e. (formally)*

(4)
$$\alpha_t := \int_0^t dv \int_0^v du \, \delta_0(B_v - B_u),$$

where δ_0 *is the Dirac delta function concentrated at* 0.

Theorem 1.1 turns out to be interesting for several reasons:

• The Hamiltonian $H^{h,\gamma}(B)$ quantifies a weak self-interaction of the Brownian path, detecting if the path self intersects (products of the form $|B_{v_1} + x|^{-\gamma}|B_{v_2} + x|^{-\gamma}$) or has a fold with amplitude h (products of the form $|B_{v_1} + x + h|^{-\gamma}|B_{v_2} + x|^{-\gamma}$). It can thus be related to the polymer model studied in [5], where a discrete time random walk S_n on $\mathbb{Z}$ is weighted according to the following Hamiltonian:

$$H_n = \sum_{i,j=1}^n \mathbf{1}_{\{S_i = S_j\}} - \sum_{i,j=1}^n \mathbf{1}_{\{|S_i - S_j| = 1\}}.$$

This relation was also the motivation behind the central limit theorem given in [15], and other physically relevant models for self-interacting continuous paths include Brownian filaments (see [3] for a detailed definition of these objects), motivated by turbulent fluids. We thus hope that the scaling limit for our quantity $H^{h,\gamma}(B)$ can shed some light on the aforementioned models.

• Theorem 1.1 also exhibits an interesting phenomenon in terms of limiting behavior. Indeed, the reader can easily observe that the limiting process in the right-hand side of (3) does not depend on the parameter γ in $(3/4, 1)$, the only difference lying in the normalizing quantity $c_\gamma \, h^{7/2-2\gamma}$. Furthermore, it was shown in [7, 15] that relation (3) still holds true in the limiting case $\gamma = 1$. This means that the process W_α, which can be seen as a Gaussian mixture, might also be considered as a rather canonical object.

• At a methodological level, our proof of Theorem 1.1 is another example of the interest of stochastic calculus techniques with respect to the method of moments in this context. We should compare our methodology e.g to the computationally demanding paper [2]. The advantage of stochastic calculus methods had already been highlighted in [7, 15], but our proof combines this approach with an extensive use of Fourier analysis techniques.

(2) Go back now to the Hamiltonian $H_t^h(B)$ defined by (1) and related to L^2 modulus of continuity of the Brownian local time. As mentioned above, it has been shown in [7, 15] that $h^{-3/2}(H^h(B) - \mathbf{E}[H^h(B)])$ converges in law to $c_1 W_\alpha$ for a universal constant c_1, that is relation (3) is still formally satisfied for $\gamma = 1$. This non central limit theorem indicates that an interesting phenomenon might occur as far as limiting behavior of the renormalized quantity $h^{-3/2}(H^h(B) - \mathbf{E}[H^h(B)])$ on chaoses is concerned. We shall specify this with the following result:

THEOREM 1.2.　*Let* $\{H_t^h(B);\ t \in [0, 1]\}$ *be the process defined by* (1). *For a given random variable* $F \in L^2(\Omega)$ *and for all* $n \geq 0$, *we set* $J_n(F)$ *for the projection of* F *on the* n^{th} *chaos of* B, *and subsequently define* $X_t^{n,h} \equiv J_n(H_t^h(B))$. *Then*

(i) *For all* $m \geq 0$ *and all* $t \in [0, 1]$, $h > 0$ *we have* $X_t^{2m+1,h} = 0$.

(ii) *For all* $m \geq 1$ *we have, as* h *tends to zero,*

$$\frac{X^{2m,h}}{h^2[\ln(1/h)]^{1/2}} \xrightarrow{(d)} \sigma_m \, W, \quad with \quad \sigma_m^2 = \frac{c\,(2m-2)!}{2^{2m}[(m-1)!]^2},$$

where W *stands for a Brownian motion independent of* B *and where the convergence takes place in the space* $\mathcal{C}([0, 1]; \mathbb{R})$ *of real continuous functions on* $[0, 1]$.

(iii) *In particular, the series $\sum_{m \geq 1} \sigma_m^2$ is divergent.*

Putting together the results of [7] and our Theorem 1.2, we thus get the following picture: on the one hand one can renormalize the process $H^h(B)$ by $h^{3/2}$ in order to get a limit which is a mixture of Gaussian processes (a non central type limit theorem). On the other hand, each projection $J_n(H^h(B))$ can be properly renormalized (by $h^2[\ln(1/h)]^{1/2}$) so as to obtain a limiting object which is a weighted Brownian motion (corresponding to a central limit theorem). Nevertheless the sum of the weights σ_n^2 obtained by projection is divergent. To the best of our knowledge, this interesting limiting behavior is exhibited here for the first time. Note that it contrasts for instance with the situation described in [6, Theorem 3] (and more specifically in the applications of this result), where, under appropriate variance assumptions, the normal convergence in each chaos guarantees the normal convergence of the sum.

(3) Finally we consider a suitable generalization of Theorem 1.2 to a 2-dimensional Brownian motion B. Namely, we shall obtain the following convergence result:

THEOREM 1.3.   *Let $\{H_t^h(B); t \in [0,1]\}$ be the process defined by (1), for a 2-dimensional Brownian motion B. Like in Theorem 1.2, we define $X_t^{n,h}$ as the projection on the n-th chaos of $H_t^h(B)$. Then the assertions (i)–(iii) of Theorem 1.2 are still valid in this situation, with (ii) replaced with the following statement:*

(ii-2d) *For all $m \geq 1$ we have, as h tends to zero,*

$$\frac{X^{2m,h}}{|h|} \xrightarrow{(d)} \sigma_m W,$$

where W stands for a linear Brownian motion independent of B, where the exact expression of σ_m will be specified at Section 4.3 and where the convergence takes place in the space $\mathcal{C}([0,1]; \mathbb{R})$ of $\mathbb{R}$-valued continuous functions on $[0,1]$.

It is worthwhile noting that the equivalent of the main result of [8], namely the convergence in law of a suitably renormalized version of $H_t^h(B)$, is not available in the 2-dimensional case. Indeed, one can formally show that $|h|^{-2}(H_t^h(B) - \mathbf{E}[H_t^h(B)])$ converges to a random variable of the form $c_2 W_\alpha$, with α defined by (4) and a universal constant c_2. Nevertheless, α is a divergent quantity in the 2-dimensional case and the convergence of $h^{-3/2}(H_t^h(B) - \mathbf{E}[H_t^h(B)])$ is in fact an empty statement.

In spite of this lack of convergence, the analysis of projections on chaoses is still a valuable information for two main reasons: (a) It indicates that a sort of convergence is at least possible for $H_t^h(B)$. (b) We are able to show that the series $\sum_{m \geq 1} \sigma_m^2$ is divergent just as in the 1-dimensional case, which seems to indicate that a non central limit theorem is to be expected for the quantity $(H_t^h(B) - \mathbf{E}[H_t^h(B)])$.

The methodology we have followed in order to get the results mentioned above is based on 3 main ingredients: (a) Stochastic calculus is obviously important in this Brownian context, and Itô formulae of backward type are invoked in order to control terms of the form $\int_0^r e^{\imath \xi (B_r - B_u)} du$ (throughout the paper, we will write $\imath$ for the complex number $(-1)^{1/2}$). Theorem 1.1 will also be a consequence of limit theorems for martingales according to the behavior of their bracket process. (b) An important contribution comes from stochastic analysis techniques: our chaos decompositions are obtained through repeated applications of Stroock's formula and we use representations of Brownian local times by means of Watanabe distributions. We also derive central limit theorems on chaoses by analyzing contractions of kernels for multiple Wiener integrals, as assessed in [12, 13]. (c) After application of the high level tools mentioned above, our results are reduced to rather elementary (though intricate) computations, for which we resort to Fourier analysis and thorough analysis of singularities for integrals defined on simplexes. All those ingredients are detailed in the corresponding sections.

In the remainder of the paper, each section is devoted to the proof of one of the Theorems given above. Specifically, Section 2 handles the non central limit Theorem 1.1 for Riesz type potentials. Section 3 is concerned with the central limit Theorems 1.2 for L^2 modulus of 1-dimensional local time on chaoses, while Section 4 deals with generalizations (Theorem 1.3) to the 2-dimensional case.

2. L^2 modulus of continuity of Brownian Riesz potentials. This section is devoted to the proof of Theorem 1.1. We shall first reduce our problem thanks to an application of Clark-Ocone's formula, and then identify the limiting process with a combination of Fourier analysis and stochastic calculus tools.

2.1. *Reduction of the problem.* In order to proceed with our computations, let us first settle some useful notation:

NOTATION 2.1. *The Gaussian heat kernel on $\mathbb{R}$ is denoted by $p_t(x)$,*

namely

$$(5) \qquad p_t(z) = (2\pi)^{-1/2} \exp\left(-\frac{z^2}{2}\right), \quad z \in \mathbb{R}.$$

For $\beta \in (0,1)$, we call $f_\beta : \mathbb{R}^* \to \mathbb{R}^*$ the function defined by $f_\beta(x) = |x|^{-\beta}$. For $\beta \in (0,1)$ and $0 \le r \le t \le 1$, we also consider the quantity
(6)
$$Q_{t,r}^{h,\beta} = \int_r^t ds \int_0^r du \left[K_{s-r}^\beta(B_r - B_u + h) + K_{s-r}^\beta(B_r - B_u - h) - 2K_{s-r}^\beta(B_r - B_u) \right],$$

where K_u^β stands for the (convolved) Riesz kernel $K_u^\beta := f_\beta * p_u'$ for all $u \ge 0$.

With these notations in mind, the Hamiltonian $H_t^{h,\gamma}(B)$ can be expressed as follows:

LEMMA 2.2. *For $t \in [0,1]$, consider the quantity $H_t^{h,\gamma}(B)$ defined by (2). Then*
(7)
$$H_t^{h,\gamma}(B) = c_\gamma \int_{[0,t]^2} [2f_\beta(B_v - B_u) - f_\beta(B_v - B_u + h) - f_\beta(B_v - B_u - h)]\, dudv,$$

with $\beta = 2\gamma - 1$.

PROOF. Start from expression (2) and write $H_t^{h,\gamma}(B)$ as

$$\int_{\mathbb{R}} \left(\int_{[0,t]^2} [f_\gamma(B_v + x + h) - f_\gamma(B_v + x)] [f_\gamma(B_u + x + h) - f_\gamma(B_u + x)]\, dudv \right) dx.$$

Next expand the product inside the integral, apply Fubini in order to integrate with respect to the variable x first and apply the identity $f_\gamma * f_\gamma = c_\gamma f_{2\gamma-1}$. Our claim is easily deduced from these elementary manipulations. □

We shall now see that Theorem 1.1 can be reduced to the following:

THEOREM 2.3. *For every $\beta \in (1/2, 1]$, consider the process $Q^{h,\beta}$ defined by (6). Then the following limit as h tends to zero holds true in the space $\mathcal{C}([0,1];\mathbb{R})$ of real continuous functions on $[0,1]$:*

$$(8) \qquad \frac{\tilde{Q}^h}{h^{5/2-\beta}} \xrightarrow{(d)} c_\beta W_\alpha, \quad where \quad \tilde{Q}_t^h := \int_0^t Q_{t,r}^{h,\beta}\, dB_r.$$

Here, c_β is a deterministic constant depending only on β, and the process W_α has been introduced at equation (3).

PROOF OF THE EQUIVALENCE BETWEEN THEOREM 1.1 AND THOREM 2.3. Following expression (7), set

$$H_t^\beta(B) = -\int_{[0,t]^2} [2f_\beta(B_v - B_u) - f_\beta(B_v - B_u + h) - f_\beta(B_v - B_u - h)] \, dudv.$$

Then Lemma 2.2 asserts that Theorem 1.1 is proved once we can show that the process $h^{-(7/2-2\gamma)}(H^\beta(B) - \mathbf{E}[H^\beta(B)])$ converges in law to $c_\beta W_\alpha$ for a strictly positive constant c_β. It is obviously easier to express everything in terms of $\beta = 2\gamma - 1$, so that we are reduced to show that $h^{-(5/2-\beta)}(H^\beta(B) - \mathbf{E}[H^\beta(B)])$ converges in law to $c_\beta W_\alpha$. It should also be observed that if $\gamma \in (3/4, 1)$ then β lies into $(1/2, 1)$.

Now along the same lines as in [7], a direct application of Clark-Ocone formula enables to express $H_t^\beta(B)$ in the following way:

$$H_t^\beta(B) - \mathbf{E}\left[H_t^\beta(B)\right] = \int_0^t Q_{t,r}^{h,\beta} \, dB_r,$$

where the process $Q^{h,\beta}$ is defined at Notation 2.1. This finishes the proof of our equivalence.

$\square$

With this equivalence in hand, the remainder of the section is now devoted to the proof of Theorem 2.3. As mentioned in the introduction, our strategy to show this result makes use of some convenient simplifications offered by a Fourier-transform version of the problem. As a last preliminary step, let us thus write an alternative expression for the quantity $Q_{t,r}^{h,\beta}$:

LEMMA 2.4. *Let $\beta \in (1/2, 1)$ and $0 \le r \le t \le 1$. Then*

$$(9) \qquad Q_{t,r}^{h,\beta} = \frac{4\imath}{\pi} \int_{\mathbb{R}} \left[\left(1 - e^{-\frac{1}{2}\xi^2(t-r)}\right) \psi(h\xi) \frac{\xi}{|\xi|^{3-\beta}} \int_0^r e^{\imath\xi(B_r - B_u)} du \right] d\xi,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ stands for the function defined by $\psi(\xi) := \sin^2(\xi/2)$.

PROOF. It is well known that for all $x \in \mathbb{R}^*$ we have

$$K_t^\beta(x) = -\frac{\imath}{2\pi} \int_{\mathbb{R}} e^{\imath\xi x} \frac{\xi}{|\xi|^{1-\beta}} e^{-\frac{t\xi^2}{2}} \, d\xi.$$

Plugging this identity into (6) and applying Fubini's theorem we get

$$-\frac{\imath}{2\pi} \int_{\mathbb{R}} \left[\left(\int_r^t e^{-\frac{(s-r)\xi^2}{2}} ds \right) \int_0^r \frac{\xi}{|\xi|^{1-\beta}} e^{\imath\xi(B_r - B_u)} \left(e^{\imath\xi h} + e^{-\imath\xi h} - 2 \right) du \right] d\xi,$$

from which identity (9) is easily deduced.

$\square$

We now start by identifying the main contribution in the quantity $\int_0^t Q_{t,r}^{h,\beta}\, dB_r$ appearing in (8) by means of our Fourier representation (9).

2.2. *Elimination of some negligible terms.* The first term which might yield a negligible contribution in $\tilde{Q}^h$ is given by the small exponential term $e^{-\frac{(t-r)\xi^2}{2}}$ in expression (9). We thus set $Q_{t,r}^{h,\beta} = Q_r^{h,\beta,1} - A_{t,r}^h$, with

$$(10) \quad Q_r^{h,\beta,1} \;=\; \frac{4\imath}{\pi} \int_{\mathbb{R}} \left[\psi\left(h\xi\right) \frac{\xi}{|\xi|^{3-\beta}} \int_0^r e^{\imath\xi(B_r - B_u)} du \right] d\xi$$

$$(11) \qquad A_{t,r}^h \;=\; \frac{4\imath}{\pi} \int_{\mathbb{R}} \left[e^{-\frac{1}{2}\xi^2(t-r)} \psi\left(h\xi\right) \frac{\xi}{|\xi|^{3-\beta}} \int_0^r e^{\imath\xi(B_r - B_u)} du \right] d\xi.$$

Then the following proposition identifies a first vanishing term:

PROPOSITION 2.5. *Let A^h be the process defined by (11), and for $t \in [0,1]$ set*

$$\tilde{A}_t^h := \frac{1}{h^{5/2-\beta}} \int_0^t A_{t,r}^h\, dB_r.$$

Then we have:

(i) *For every fixed $t \in [0,1]$, $\tilde{A}_t^h \to 0$ in $L^2(\Omega)$ as h tends to zero.*

(ii) *There exists $p \geq 1$ and $\alpha > 0$ such that for all $0 \leq s < t \leq 1$ and every $h \in (0,1)$,*

$$\mathbf{E}\big[|\tilde{A}_t^h - \tilde{A}_s^h|^{2p}\big] \leq c_p\, h^{2p(\beta-\frac{1}{2})}\, |t-s|^{1+\alpha}\,,$$

for some constant c_p depending only on p.

(iii) *As a consequence, we have $\tilde{A}^h \xrightarrow{(d)} 0$ in $\mathcal{C}([0,1];\mathbb{R})$ as h tends to zero.*

PROOF. Let us prove the three items separately:

(i) Consider a given $t \in [0,1]$. One has

$$\mathbf{E}\left[\left(\int_0^t A_{t,r}^h\, dB_r \right)^2 \right] = 2 \int_0^t dr \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} d\eta \int_0^r dv \int_0^v du$$

$$\frac{\xi\, e^{-\frac{1}{2}\xi^2(t-r)} \psi(h\xi)}{|\xi|^{3-\beta}}\, \frac{\eta\, e^{-\frac{1}{2}\eta^2(t-r)} \psi(h\eta)}{|\eta|^{3-\beta}}\, \mathbf{E}\big[e^{\imath(\xi+\eta)(B_r - B_v) + \imath\eta(B_v - B_u)} \big].$$

Furthermore, for $u < v < r < t$ we have

$$0 \leq \mathbf{E}\left[e^{\imath(\xi+\eta)(B_r - B_v) + \imath\eta(B_v - B_u)} \right] = e^{-\frac{(\xi+\eta)^2}{2}(r-v)}\, e^{-\frac{\eta^2}{2}(v-u)} \leq e^{-\frac{\eta^2}{2}(v-u)}.$$

Now integrate this inequality in u and invoke the fact that $\psi(z) \le c\, z^2$ in order to get

$$(12) \qquad \mathbf{E}\left[\left(\int_0^t A_{t,r}^h \, dB_r\right)^2\right] \le c\, h^4 \int_0^t dr \int_0^r dv \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} d\eta\, \ell_{r,v}^t(\xi, \eta),$$

where

$$\ell_{r,v}^t(\xi, \eta) \equiv e^{-\frac{1}{2}\xi^2(t-r)} e^{-\frac{1}{2}\eta^2(t-r)} \frac{|\xi|^\beta}{|\eta|^{2-\beta}} \{1 - e^{-\frac{\eta^2}{2}v}\}.$$

To see that the integral in the right-hand side of (12) is indeed finite, observe first that $\int_{\mathbb{R}} e^{-\frac{a}{2}\xi^2} |\xi|^\beta \, d\xi = c_\beta a^{-\frac{1+\beta}{2}}$ for any $a > 0$ and $\beta \in (0,1)$. Thus

$$\int_{\mathbb{R}} d\xi \int_{-1}^1 d\eta\, \ell_{r,v}^t(\xi, \eta) \le c \int_{\mathbb{R}} e^{-\frac{1}{2}\xi^2(t-r)} |\xi|^\beta \, d\xi \int_{-1}^1 |\eta|^\beta d\eta$$
$$\le c \int_{\mathbb{R}} d\xi\, e^{-\frac{1}{2}\xi^2(t-r)} |\xi|^\beta \le \frac{c}{|t-r|^{\frac{1+\beta}{2}}}.$$

In the same way, since $\beta \in (0,1)$ we also have

$$\int_{\mathbb{R}} d\xi \int_{|\eta| \ge 1} d\eta\, \ell_{r,v}^t(\xi, \eta) \le c \int_{|\eta| \ge 1} |\eta|^{-(2-\beta)} d\eta \int_{\mathbb{R}} e^{-\frac{1}{2}\xi^2(t-r)} |\xi|^\beta \, d\xi \le \frac{c}{|t-r|^{\frac{1+\beta}{2}}}.$$

Plugging these estimates into (12) and taking into account the fact that $\beta \in (0,1)$, we end up with

$$\mathbf{E}\left[\left(\int_0^t A_{t,r}^h \, dB_r\right)^2\right] \le c_{t,\beta}\, h^4 \int_0^t \frac{dr}{|t-r|^{\frac{1+\beta}{2}}} \le c_{t,\beta}\, h^4,$$

which yields our first claim (i).

(ii) In order to bound the increment $\tilde{A}_t^h - \tilde{A}_s^h$, set $k_{h,t}(\xi) := e^{-\frac{1}{2}\xi^2 t} \psi(h\xi) \frac{\xi}{|\xi|^{3-\beta}}$. Then it is readily checked that

$$(13) \quad \tilde{A}_t^h - \tilde{A}_s^h = \frac{1}{h^{5/2-\beta}} \int_s^t dB_r \int_{\mathbb{R}} d\xi\, k_{h,t-r}(\xi) \int_0^r e^{\imath\xi(B_r - B_u)} du$$
$$+ \frac{1}{h^{5/2-\beta}} \int_0^s dB_r \int_{\mathbb{R}} d\xi\, [k_{h,t-r}(\xi) - k_{h,s-r}(\xi)] \int_0^r e^{\imath\xi(B_r - B_u)} du := \tilde{A}_{s,t}^{h,1} + \tilde{A}_{s,t}^{h,2}.$$

Consider first $\tilde{A}_{s,t}^{h,1}$ and write

$$\tilde{A}_{s,t}^{h,1} = \frac{1}{h^{5/2-\beta}} \int_s^t H_{t,r} \, dB_r, \quad \text{with} \quad H_{t,r} := \int_{\mathbb{R}} d\xi\, k_{h,t-r}(\xi) \int_0^r e^{\imath\xi(B_r - B_u)} du.$$

By using successively Burkholder-Davies-Gundy and Cauchy-Schwarz inequalities, we get

$$(14) \qquad \mathbf{E}\big[|\tilde{A}_{s,t}^{h,1}|^{2p}\big] \leq \frac{c_p}{h^{(5-2\beta)p}} \bigg(\int_s^t \mathbf{E}\big[|H_{t,r}|^{2p}\big]^{\frac{1}{p}} \, dr \bigg)^p,$$

with

$$\mathbf{E}\big[|H_{t,r}|^{2p}\big] = c_p \int_{\mathbb{R}^{2p}} d\xi_1 \cdots d\xi_{2p} \int_{0<u_1<\ldots<u_{2p}<r} du_1 \cdots du_{2p} \prod_{j=1}^{2p} k_{h,t-r}(\xi_j)\, \mathbf{E}\left[e^{\imath \xi_j (B_r - B_{u_j})}\right],$$

which can also be expressed as

$$\mathbf{E}\big[|H_{t,r}|^{2p}\big] = c_p \int_{\mathbb{R}^{2p}} d\xi_1 \cdots d\xi_{2p} \int_{0<u_1<\ldots<u_{2p}<r} du_1 \cdots du_{2p}$$

$$\prod_{j=1}^{2p} k_{h,t-r}(\xi_j)\, e^{-\frac{1}{2}\xi_1^2(u_2-u_1)} e^{-\frac{1}{2}(\xi_1+\xi_2)^2(u_3-u_2)} \cdots e^{-\frac{1}{2}(\xi_1+\cdots+\xi_{2p})^2(r-u_{2p})}.$$

We can then rely on the uniform estimate

$$|k_{h,t-r}(\xi_i)| \leq c\, h^2 e^{-\frac{1}{2}\xi_i^2(t-r)} |\xi_i|^\beta \leq c\, \frac{h^2}{|t-r|^{\beta/2}},$$

and the fact that

$$\int_{\mathbb{R}} d\xi_1\, e^{-\frac{1}{2}\xi_1^2(u_2-u_1)} \int_{\mathbb{R}} d\xi_2\, e^{-\frac{1}{2}(\xi_1+\xi_2)^2(u_3-u_2)} \cdots \int_{\mathbb{R}} d\xi_{2p}\, e^{-\frac{1}{2}(\xi_1+\cdots+\xi_{2p})^2(r-u_{2p})}$$

$$= \int_{\mathbb{R}} d\xi_1\, e^{-\frac{1}{2}\xi_1^2(u_2-u_1)} \int_{\mathbb{R}} d\xi_2\, e^{-\frac{1}{2}\xi_2^2(u_3-u_2)} \cdots \int_{\mathbb{R}} d\xi_{2p}\, e^{-\frac{1}{2}\xi_{2p}^2(r-u_{2p})}$$

$$= c_p\, (u_2-u_1)^{-1/2}(u_3-u_2)^{-1/2} \cdots (r-u_{2p})^{-1/2}$$

in order to get

$$\mathbf{E}\big[|H_{t,r}|^{2p}\big] \leq \frac{c_p\, h^{4p}\, r^p}{|t-r|^{\beta p}}.$$

Plugging this estimate into (14) we end up with

$$(15) \qquad E\big[|\tilde{A}_{s,t}^{h,1}|^{2p}\big] \leq c_p\, h^{2p(\beta-1/2)}\, |t-s|^{(1-\beta)p}.$$

The bound for $\tilde{A}_{s,t}^{h,2}$ can be derived from a similar procedure. Observe for instance that

$$|k_{h,t-r}(\xi) - k_{h,s-r}(\xi)| \leq h^2\, |e^{-\frac{1}{2}\xi^2(t-r)} - e^{-\frac{1}{2}\xi^2(s-r)}||\xi|^\beta \leq c\, h^2\, |t-s|^\varepsilon\, |s-r|^{(1-\varepsilon)/2},$$

and invoking this bound for $\varepsilon := (1 - \beta)/3$ one obtains that inequality (15) also holds true for $\tilde{A}_{s,t}^{h,2}$. Going back to (13), we see that the bounds on $\tilde{A}_{s,t}^{h,1}$ and $\tilde{A}_{s,t}^{h,2}$ easily yield our claim (ii). The assertion (iii) is now a standard consequence of (i) and (ii).

$\square$

Let us go back to expression (9), as well as the decomposition (10) and (11) for $Q^{h,\beta}$. Proposition 2.5 allows to reduce our study to an analysis of $\tilde{Q}^{h,\beta,1}$ defined by $\tilde{Q}_t^{h,\beta,1} = h^{-(5/2-\beta)} \int_0^t Q_r^{h,\beta,1} \, dB_r$, where $Q^{h,\beta,1}$ is given by (10). In order to identify another negligible term within $\tilde{Q}^{h,\beta,1}$, let us resort to Itô's formula applied to the (backward) Brownian motion $\hat{B}^r = \{B_r - B_u; \, 0 \le u \le r\}$ and $f(x) := e^{\imath \xi x}$. This gives

$$(16) \qquad \int_0^r e^{\imath \xi (B_r - B_u)} du = -\frac{2\left(e^{\imath \xi B_r} - 1\right)}{\xi^2} + \frac{2\imath}{\xi} \int_0^r e^{\imath (B_r - B_u)} \, d\hat{B}_u^r,$$

and plugging this identity into (10) we get $Q_r^{h,\beta,1} = D_r^h - Q_r^{h,\beta,2}$, with

$$(17) \qquad D_r^h \;=\; \frac{8\imath}{\pi} \int_{\mathbb{R}} \left[\frac{\xi \, \psi(h\xi)}{|\xi|^{5-\beta}} \left(e^{\imath \xi B_r} - 1\right) \right] d\xi$$

$$(18) \qquad Q_r^{h,\beta,2} \;=\; \frac{8}{\pi} \int_{\mathbb{R}} \left[\frac{\psi(h\xi)}{|\xi|^{3-\beta}} \int_0^r e^{\imath \xi (B_r - B_u)} \, d\hat{B}_u^r \right] d\xi.$$

We now prove the following Proposition:

PROPOSITION 2.6.   *Let D^h be the process defined by (17), and for $t \in [0,1]$ set*

$$\tilde{D}_t^h := \frac{1}{h^{5/2-\beta}} \int_0^t D_r^h \, dB_r.$$

Then the conclusions of Proposition 2.5 hold true for $\tilde{D}^h$.

PROOF. The proof goes along the same lines as for Proposition 2.5, and is left to the reader for the sake of conciseness. Let us just highlight the following decomposition:

$$\mathbf{E}\left[\left(\tilde{D}_t^h\right)^2\right] \le c \, h^{2\beta-1} \int_0^t \mathbf{E}^2\left[B_r^2\right] dr \left(\int_{-1}^1 \frac{d\xi}{|\xi|^{1-\beta}}\right)^2 + c \, h^{2\beta-1} \left(\int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{2-\beta}}\right)^2,$$

which allows us to conclude that $\lim_{h\to 0} \mathbf{E}[(\tilde{D}_t^h)^2] = 0$ since $1/2 < \beta < 1$.

$\square$

REMARK 2.7. With Proposition 2.5 and Proposition 2.6 in hand, Theorem 2.3 now boils down to the following property:

$$(19) \qquad \frac{M^h}{h^{5/2-\beta}} \xrightarrow{(d)} c_\beta W_\alpha \quad \text{in } \mathcal{C}([0,1];\mathbb{R}), \quad \text{with} \quad M_t^h := \int_0^t Q_r^{h,\beta,2} \, dB_r,$$

where $Q^{h,\beta,2}$ is the process defined by (18). It should be observed that M^h is now a Brownian martingale, for which specific limit theorems are available.

2.3. *Study of the martingale term.* Similar to the argument used in [7, 8, 15], our strategy towards (19) is now based on the martingale convergence criterion summed up in [4, Theorem A.1]. Using the latter result, the proof of (19) reduces to showing that, as $h \to 0$, we have simultaneously

$$(20) \qquad \frac{\langle M^h, B \rangle_t}{h^{5/2-\beta}} \to 0 \quad \text{and} \quad \frac{\langle M^h \rangle_t}{h^{5-2\beta}} \to c_\beta \, \alpha_t$$

in $L^2(\Omega)$ for every fixed $t \in [0,1]$, with α_t defined by (4).

To this aim, let us start by recasting M^h in a suitable way. Indeed, thanks to a stochastic Fubini theorem we have:

$$(21) \qquad \frac{Q_r^{h,\beta,2}}{h^{5/2-\beta}} = \int_0^r g_h(B_r - B_u) \, d\hat{B}_u^r,$$

where

$$(22) \qquad g_h = g_h^\beta := \mathcal{F}(f_h), \quad \text{with} \quad f_h(\xi) = f_h^\beta(\xi) := \frac{1}{h^{5/2-\beta}} \frac{\psi(h\xi)}{|\xi|^{3-\beta}}.$$

In the course of the reasoning, we shall appeal to the following key properties of g_h:

LEMMA 2.8. *It holds that:*

(i) *For some c_β independent of h, we have $\int_{\mathbb{R}} g_h(x)^2 \, dx = c_\beta > 0$.*

(ii) *Recalling that p_t stands for the Gaussian heat kernel defined by (5), we have for every $t \in (0,1]$:*

$$(23) \qquad \int_{\mathbb{R}} g_h(x) p_t(x) \, dx \le \frac{c \, h^{\beta-1/2}}{t^{\beta/2}}.$$

(iii) *The function g_h can also be written as*

$$(24) \qquad g_h(x) = \frac{c}{h^{5/2-\beta}} \int_{x-h}^{x+h} \frac{(h - |x-y|)}{|y|^\beta} \, dy.$$

In particular, $g_h(-x) = g_h(x)$ and $g_h(x) \geq 0$ for all $x \in \mathbb{R}$.

(iv) *For every $\varepsilon > 0$ such that $\beta > 1/2 + \varepsilon$, every $h \leq 1/4$ and every $|x| \geq \sqrt{h}$,*

$$(25) \qquad g_h^\beta(x) \leq c\, h^{\varepsilon/2} g_h^{\beta-\varepsilon}(x).$$

PROOF. By Fourier isometry,

$$\|g_h\|_{L^2}^2 = \|f_h\|_{L^2}^2 = \frac{1}{h^{5-2\beta}} \int_{\mathbb{R}} \frac{\psi^2(h\xi)}{|\xi|^{6-2\beta}}\, d\xi = \int_{\mathbb{R}} \frac{\psi^2(\xi)}{|\xi|^{6-2\beta}}\, d\xi,$$

which gives (i). In order to prove (ii) use Fourier isometry again, which according to (22) yields

$$\int_{\mathbb{R}} g_h(x)p_t(x)\, dx = \frac{c}{h^{5/2-\beta}} \int_{\mathbb{R}} \frac{\psi(h\xi)}{|\xi|^{3-\beta}}\, e^{-\frac{t\xi^2}{2}}\, d\xi \leq c\, h^{\beta-1/2} \int_{\mathbb{R}} \frac{e^{-\frac{t\xi^2}{2}}}{|\xi|^{1-\beta}}\, d\xi \leq \frac{c\, h^{\beta-1/2}}{t^{\beta/2}}.$$

For (iii), observe that

$$f_h(\xi) = h^{1/2}\varphi(h\xi), \quad \text{with} \quad \varphi(u) = \frac{\text{sinc}^2(u)}{|u|^{1-\beta}},$$

where the sinc function refers to $\text{sinc}(x) = \frac{\sin(x)}{x}$. Thus, using the fact $\mathcal{F}(\text{sinc}^2(\cdot))(\xi) = \mathbf{1}_{[-1,1]}(\xi)(1 - |\xi|)$, we get

$$g_h(\xi) = \mathcal{F}(f_h)(\xi) = \frac{1}{h^{1/2}}\, \mathcal{F}(\varphi)\Big(\frac{\xi}{h}\Big) = \frac{1}{h^{1/2}}\Big[\mathcal{F}\big(|\cdot|^{-1+\beta}\big) * \mathcal{F}\big(\text{sinc}^2(\cdot)\big)\Big]\Big(\frac{\xi}{h}\Big)$$

$$= \frac{c}{h^{1/2}} \int_{\frac{\xi}{h}-1}^{\frac{\xi}{h}+1} \frac{dy}{|y|^\beta}\Big(1 - \Big|\frac{\xi}{h} - y\Big|\Big),$$

which clearly leads to (24).

Now we can use (24) in order to prove (iv): for $x > \sqrt{h}$, write

$$g_h^\beta(x) = c\, h^\varepsilon \frac{1}{h^{5/2-(\beta-\varepsilon)}} \int_{x-h}^{x+h} \frac{h - |x-y|}{|y|^\varepsilon |y|^{\beta-\varepsilon}}\, dy \leq \frac{c\, h^\varepsilon g_h^{\beta-\varepsilon}(x)}{|x-h|^\varepsilon} \leq c\, h^{\varepsilon/2} g_h^{\beta-\varepsilon}(x),$$

since $|x - h| \geq \frac{1}{2}\sqrt{h}$. By symmetry of g_h, this finishes our proof. $\qquad \square$

Let us develop now the strategy for the convergence of the martingale term, which has been summarized in (20). We shall prove the first claim of (20), namely:

PROPOSITION 2.9.   *For all $t \in [0, 1]$, the martingale term M^h satisfies*

$$\frac{\mathbf{E}\big[\langle M^h, B\rangle_t^2\big]}{h^{5-2\beta}} \leq c_t\, h^{\beta-1/2},$$

where c_t is a uniformly bounded function of $t \in [0, 1]$.

PROOF. According to (19) and (21), one has

$$\frac{\langle M^h, B\rangle_t}{h^{5/2-\beta}} = \frac{\int_0^t Q_r^{\beta,h,2}\, dr}{h^{5/2-\beta}} = \int_0^t dr \int_0^r d\hat{B}_u^r\, g_h(B_r - B_u).$$

Hence

$$\frac{\mathbf{E}\left[\langle M^h, B\rangle_t^2\right]}{h^{5-2\beta}} = 2 \int_0^t dr_1 \int_0^{r_1} dr_2 \int_0^{r_2} du\, \mathbf{E}\big[g_h(B_{r_1} - B_u)g_h(B_{r_2} - B_u)\big],$$

and furthermore

$$\mathbf{E}\big[g_h(B_{r_1} - B_u)g_h(B_{r_2} - B_u)\big] = \mathbf{E}\big[g_h * p_{r_1-r_2}(B_{r_2} - B_u)g_h(B_{r_2} - B_u)\big]$$

$$= \int_{\mathbb{R}} d\xi\, [g_h * p_{r_1-r_2}](\xi)\, g_h(\xi)\, p_{r_2-u}(\xi) \leq c\|g_h * p_{r_1-r_2}\|_\infty \frac{h^{\beta-1/2}}{\sqrt{r_2 - u}},$$

thanks to (23). In addition, $\|g_h * p_{r_1-r_2}\|_\infty \leq \|g_h\|_{L^2}\|p_{r_1-r_2}\|_{L^2} \leq c|r_1 - r_2|^{-1/4}$, and thus

$$\frac{\mathbf{E}\big[\langle M^h, B\rangle_t^2\big]}{h^{5-2\beta}} \leq c\, h^{\beta-1/2} \int_0^t dr_1 \int_0^{r_1} dr_2\, |r_1 - r_2|^{-1/4} \int_0^{r_2} du\, |r_2 - u|^{-1/2},$$

from which our claim is easily deduced.

□

Before we proceed with the proof of (20), let us label a technical lemma on Brownian local times:

LEMMA 2.10.   *Let $\{L_t(a);\, t \in [0, 1], a \in \mathbb{R}\}$ be the local time process of Brownian motion on the interval $[0, 1]$. Then there exists $\varepsilon > 0$ and a strictly positive constant c such that*

$$\sup_{x \in \mathbb{R}, t \in [0,1]} \mathbf{E}[|L_t(x+B_t)|^2] \leq c, \quad and \quad \sup_{t \in [0,1]} \mathbf{E}\left[\sup_{|x-y|<h^{1/2}} |L_t(x) - L_t(y)|^2\right] \leq c\, h^{\varepsilon}.$$

PROOF. By applying Tanaka's formula to the backward Brownian motion $\hat{B}$, we get, for all $x \in \mathbb{R}$,

$$(26) \qquad |L_t(x + B_t)| \leq 2|B_t| + 2\left|\int_0^t \mathbf{1}_{\{\hat{B}_s^t < -x\}} d\hat{B}_s^t\right|,$$

and the first assertion immediately follows. The second assertion of our Lemma can be derived from [1, item (ii)].

$\square$

We are now ready to prove the second part of assertion (20), that is

PROPOSITION 2.11. *Let t be an arbitrary time in $[0,1]$. Then we have:*

$$(27) \qquad L^2(\Omega) - \lim_{h \to 0} \frac{\langle M^h \rangle_t}{h^{5-2\beta}} = c_\beta \, \alpha_t,$$

where α is the self-intersection local time defined by (4).

PROOF. Let us start by applying again the backward Itô formula (16) in order to get the decomposition

$$\langle M^h \rangle_t = \int_0^t dr \left(\int_0^r d\hat{B}_u^r \, g_h(B_r - B_u) \right)^2 := N_t^{h,1} + N_t^{h,2},$$

with

$$\begin{aligned}
N_t^{h,1} &= \int_0^t dr \int_0^r du \, [g_h(B_r - B_u)]^2 \\
N_t^{h,2} &= 2 \int_0^t dr \int_0^r d\hat{B}_u^r \left(g_h(B_r - B_u) \int_u^r d\hat{B}_s^r \, g_h(B_r - B_s) \right).
\end{aligned}$$

We shall now divide our proof in two steps.

Step 1: $N^{h,2}$ vanishes as $h \to 0$. Specifically, we shall prove that $L^2(\Omega) - \lim_{h \to 0} N_t^{h,2} = 0$. Indeed, it is readily checked that

$$\mathbf{E}\left[\left(\int_0^t dr \int_0^r d\hat{B}_u^r \left(g_h(B_r - B_u) \int_u^r d\hat{B}_s^r \, g_h(B_r - B_s) \right) \right)^2 \right]$$
$$= 2 \int_0^t ds \int_s^t (28) \int_u^t dr_1 \int_u^{r_1} dr_2 \, \mathbf{E}\left[g_h(B_{r_1} - B_s) g_h(B_{r_1} - B_u) g_h(B_{r_2} - B_s) g_h(B_{r_2} - B_u) \right].$$

Furthermore, using the fact that g_h is positive (Lemma 2.8 item (iii)), we have, for fixed $0 < s < u < r_2 < r_1 < t$,

$$\begin{aligned}
\mathbf{E}\left[g_h(B_{r_1} - B_s) g_h(B_{r_1} - B_u) | \mathcal{F}_{r_2} \right] &= \int_{\mathbb{R}} g_h(x + B_{r_2} - B_s) g_h(x + B_{r_2} - B_u) p_{r_1 - r_2}(x) \, dx \\
&\leq \|p_{r_1 - r_2}\|_\infty \|g_h\|_{L^2}^2 \leq \frac{c}{\sqrt{r_1 - r_2}},
\end{aligned}$$

where we have used Lemma 2.8 item (i), and

$$
\begin{aligned}
\mathbf{E}\big[g_h(B_{r_2} - B_s)g_h(B_{r_2} - B_u)\big] &= \mathbf{E}\big[g_h(B_{r_2} - B_u)(g * p_{u-s})(B_{r_2} - B_u)\big] \\
&\leq \|g_h * p_{u-s}\|_\infty \int_{\mathbb{R}} g_h(x) p_{r_2-u}(x)\, dx \\
&\leq c\, \|g_h\|_{L^2} \|p_{u-s}\|_{L^2} \frac{h^{\beta-1/2}}{\sqrt{r_2-u}},
\end{aligned}
$$

with the help of Lemma 2.8 item (ii). Going back to (28), the result easily follows.

Step 2: Limit of $N^{h,1}$. We will show the following property:

$$
(29) \qquad \int_0^t dr \int_0^r du\, [g_h(B_r - B_u)]^2 \xrightarrow{h \to 0} c_\beta \int_0^t dr\, L_r(B_r) \quad \text{in } L^2(\Omega),
$$

where c_β is the constant defined at Lemma 2.8. To this aim, observe that according to the occupation density formula we have

$$
\Delta_h := \int_0^t dr \int_0^r du\, [g_h(B_r - B_u)]^2 - c_\beta \int_0^t dr\, L_r(B_r) = \int_0^t \left(\int_{\mathbb{R}} Z_r(x)\, dx \right) dr,
$$

where Z is the process defined by

$$
Z_r(x) = g_h(B_r - x)^2 \big[L_r(x) - L_r(B_r)\big].
$$

Next we decompose Δ_h as $\Delta_h^1 + \Delta_h^2$, where

$$
\Delta_h^1 = \int_0^t \left(\int_{|x-B_r| < h^{1/2}} Z_r(x)\, dx \right) dr, \quad \text{and} \quad \Delta_h^2 = \int_0^t \left(\int_{|x-B_r| \geq h^{1/2}} Z_r(x)\, dx \right) dr.
$$

We now estimate those two terms separately.

The term Δ_h^1 can be bounded as follows: owing to Lemma 2.8 item (i), we have

$$
\Delta_h^1 \leq c \int_0^t \sup_{|x-y| < h^{1/2}} |L_r(x) - L_r(y)|\, dr.
$$

Owing to Lemma 2.10, we thus get

$$
\mathbf{E}\left[|\Delta_h^1|^2\right] \leq c \sup_{t \in [0,1]} \mathbf{E}\left[\sup_{|x-y| < h^{1/2}} |L_t(x) - L_t(y)|^2 \right] \leq c\, h^\varepsilon,
$$

for some constant $\varepsilon \in (0,1)$.

As far as Δ_h^2 is concerned, invoke Lemma 2.8 item (iv) in order to conclude that for any $\varepsilon > 0$ such that $\beta > \frac{1}{2} + \varepsilon$ and every $h \leq 1/4$, we have

$$
\mathbf{E}\left[|\Delta_h^2|^2\right] \leq c\, h^\varepsilon \int_0^t \mathbf{E}\left[\left(\int_{|x-B_r|\geq h^{1/2}} |g_h^{\beta-\varepsilon}(x-B_r)|^2 \,|L_r(x) - L_r(B_r)| \ dx\right)^2\right] dr
$$

$$
\leq c\, h^\varepsilon \int_0^t \mathbf{E}\left[\int_{\mathbb{R}} |g_h^{\beta-\varepsilon}(x-B_r)|^2 \,|L_r(x) - L_r(B_r)|^2 \ dx\right] dr \leq c\, h^\varepsilon,
$$

where we have appealed to Lemma 2.10 for the last inequality.

Step 3: Conclusion. Putting together the bounds on Δ_h^1 and Δ_h^2, we have proved our assertion (29), which easily yields

$$
L^2(\Omega) - \lim_{h\to 0} \frac{\langle M^h\rangle_t}{h^{5-2\beta}} = c_\beta \int_0^t L_r(B_r)\, dr.
$$

In order to prove (27), we now just have to observe that

$$
\int_0^t L_r(B_r)\, dr = \int_0^t \left(\int_0^r \delta_{B_r}(B_u)\, du\right) dr = \int_0^t \left(\int_0^r \delta_0(B_r - B_u)\, du\right) dr = \alpha_t.
$$

This concludes our proof.

$\qquad\square$

3. L^2 modulus of 1-dimensional local time on chaoses. In this section, we go back to the study of the L^2 modulus of the Brownian local time, that is to the study of the quantity $H_t^h(B)$ defined by (1) with the global aim of proving Theorem 1.2. Before we go on with the proof, let us introduce some additional notation:

NOTATION 3.1. *For any $t > 0$ and $n \geq 1$ we write $\mathcal{S}_t^n$ for the simplex of order n on $[0,t]$, i.e., $\mathcal{S}_t^n = \{(t_1,\ldots,t_n) \in [0,t]^n : t_1 < \ldots < t_n\}$. For every $n \geq 2$ and every $h > 0$, we also define a function $\Phi_h(t_1, t_2)$ as*

$$
\Phi_h(t_1, t_2) = \Phi_{h,n}(t_1, t_2) := \int_0^h p_{t_2-t_1}^{(n-2)}(y)(h-y)\, dy \quad,\quad 0 \leq t_1 \leq t_2 \leq t.
$$

From the classical uniform estimate $\sup_{y\in\mathbb{R}} |p_t^{(2m)}(y)| \leq c_m t^{-m-\frac{1}{2}}$, we can already derive the following bounds on $\Phi_{h,2m}$, which will be used in the course of our reasoning:

LEMMA 3.2. *Fix $m \geq 1$. Then there exists a constant c_m such that for every $h \in (0,1)$ and all $0 \leq t_1 < s < t < t_2$, one has*

(30) $$
|\Phi_{h,2m}(s,t)| \leq c_m h^2 \,|t-s|^{-m+\frac{1}{2}}
$$

and for any $\lambda \in (0,1)$,

$$(31) \qquad |\Phi_{h,2m}(t_1, t) - \Phi_{h,2m}(t_1, s)| \leq c_m h^2 |t - s|^\lambda |s - t_1|^{-m + \frac{1}{2} - \lambda},$$

$$(32) \qquad |\Phi_{h,2m}(t, t_2) - \Phi_{h,2m}(s, t_2)| \leq c_m h^2 |t - s|^\lambda |t_2 - t|^{-m + \frac{1}{2} - \lambda}.$$

The proof of Theorem 1.2 is decomposed in four main steps: after some preliminary material, we write an explicit chaos decomposition for each $H_t^h(B)$. Then we study the asymptotic behavior of the variance in each chaos, and the central limit theorem for the finite-dimensional distributions of $J_n(H_\cdot^h(B))$ is obtained by analyzing the contractions of its sequence of kernels. Finally, we study the tightness of the process $\{J_n(H_t^h(B)); \, t \in [0, 1]\}$ properly normalized.

3.1. *Stochastic analysis preliminaries.* We will consider here the Brownian motion B as an isonormal process $B \equiv \{B(h); \, h \in \mathcal{H}\}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$, with $\mathcal{H} = L^2([0, 1])$. Recall that it means that B is a centered Gaussian family with covariance function $\mathbf{E}[B(h_1)B(h_2)] = \langle h_1, h_2 \rangle_\mathcal{H}$. We also assume that $\mathcal{F}$ is generated by B.

At this point, we can introduce the Malliavin derivative operator on the Wiener space $(\Omega, \mathcal{H}, \mathbf{P})$. Namely, we first let $\mathcal{S}$ be the family of smooth functionals F of the form

$$F = f(B(h_1), \ldots, B(h_n)),$$

where $h_1, \ldots, h_n \in \mathcal{H}$, $n \geq 1$, and f is a smooth function having polynomial growth together with all its partial derivatives. Then the Malliavin derivative of such a functional F is the $\mathcal{H}$-valued random variable defined by

$$\mathcal{D}F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n))h_i.$$

For all $p > 1$, it is known that the operator $\mathcal{D}$ is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$. We still denote by $\mathcal{D}$ the closure of this operator, whose domain is usually denoted by $\mathbb{D}^{1,p}$ and is defined as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \left(\mathbf{E}\left[|F|^p\right] + \mathbf{E}\left[\|\mathcal{D}F\|_\mathcal{H}^p\right] \right)^{\frac{1}{p}}.$$

We shall also denote by $\mathbb{D}^{\infty,p}$ the intersection $\cap_{k \geq 1} \mathbb{D}^{k,p}$.

Consider the nth Hermite polynomial H_n defined on $\mathbb{R}$, that is

$$(33) \qquad H_n(x) = \frac{(-1)^n}{n!} \, e^{\frac{x^2}{2}} \, \partial_x^n e^{-\frac{x^2}{2}},$$

and let $\mathcal{H}_n$ be the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B(h)); \, h \in \mathcal{H}, \, \|h\|_{\mathcal{H}} = 1\}$. Then $\mathcal{H}_n$ is called Wiener chaos of order n, and $L^2(\Omega)$ can be decomposed into the orthogonal sum of the $\mathcal{H}_n$: we have $L^2(\Omega, \mathcal{F}, \mathbf{P}) = \oplus_{n=0}^{\infty} \mathcal{H}_n$ (see [11, Theorem 1.1.1]). In the sequel we denote by $J_n(F)$ the projection of a given random variable $F \in L^2(\Omega)$ onto $\mathcal{H}_n$ for $n \geq 0$, with $J_0(F) = \mathbf{E}[F]$. In this context Stroock's formula (see [16]) states that, whenever $F \in \mathbb{D}^{\infty,2}$, one can compute $J_n(F)$ explicitly as follows for $n \geq 1$:

$$(34) \qquad J_n(F) = I_n(f_n), \quad \text{with} \quad f_n(t_1, \ldots, t_n) = \frac{\mathbf{E}\left[\mathcal{D}_{t_1,\ldots,t_n} F\right]}{n!},$$

where $I_n(f_n)$ stands for the multiple Itô-Wiener integral of f_n with respect to B. We also label the value of $H_{2m}(0)$ here for further use: for $m \geq 1$ we have

$$(35) \qquad H_{2m}(0) = \frac{(-1)^m}{2^m m!}.$$

Let now f_n be a symmetric function in $L^2([0,1]^n)$. The contraction of order p of f_n is the function defined on $[0,1]^{2(n-p)}$ as follows:

$$(36) \quad [f_n \otimes_p f_n](t_1, \ldots, t_{2(n-p)})$$
$$= \int_{[0,1]^p} f_n(u_1, \ldots, u_p, t_1, \ldots, t_{n-p}) \, f_n(u_1, \ldots, u_p, t_{n-p+1}, \ldots, t_{2(n-p)}) \, du_1 \cdots du_p.$$

With this definition in hand, let us state the following theorem (borrowed from [12]), which will be crucial in order to establish the convergence of our renormalized local times:

PROPOSITION 3.3.   *Let $\{F_h = I_n(f_{n,h}); \, h > 0\}$ be a family of random variables belonging to a fixed Wiener chaos $\mathcal{H}_n$, for which we assume that the kernels $f_{n,h}$ are symmetric. We also suppose that*

(i) *We have $\lim_{h \to 0} \mathbf{E}[F_h^2] = \sigma^2 > 0$.*

(ii) *For all $p \in \{1, \ldots, n-1\}$ the relation $\lim_{h \to 0} \|f_{n,h} \otimes_p f_{n,h}\|_{\mathcal{H}^{\otimes 2(n-p)}} = 0$ holds true.*

Then F_h converges in law to a Gaussian random variable $\mathcal{N}(0, \sigma^2)$ as $h \to 0$.

In order to obtain convergence in law for processes we shall also invoke a CLT for multidimensional vectors in a fixed chaos, originally proved in [13]:

PROPOSITION 3.4.   *Consider a family of d-dimensional random variables* $\{F_h; h > 0\}$ *with* $F_h = (F_h^1, \ldots, F_h^d)$, *such that* F_h^j *belongs to a fixed Wiener chaos* $\mathcal{H}_n$ *for each* $j \in \{1, \ldots, d\}$ *and* $h > 0$. *Suppose furthermore that for a symmetric matrix* Γ *we have:*

(i) *Each* F_h^j *converges in law to a Gaussian random variable* $\mathcal{N}(0, \Gamma(i, i))$ *as* $h \to 0$.

(ii) *For each* $(i, j) \in \{1, \ldots, d\}^2$ *one has* $\lim_{h \to 0} \mathbf{E}[F_h^i F_h^j] = \Gamma(i, j)$.

Then F_h *converges in law to a Gaussian random variable* $\mathcal{N}(0, \Gamma)$ *as* $h \to 0$.

3.2. *Chaos decomposition of* $\mathbf{H_t^h(B)}$.   In order to compute the chaos decomposition of $H_t^h(B)$, we first recall a relation taken from [7], whose proof is similar to our identity (7): we have

(37)
$$H_t^h(B) = \int_{[0,t]^2} \left[\delta_0(B_v - B_u + h) + \delta_0(B_v - B_u - h) - 2\delta_0(B_v - B_u) \right] \, du dv,$$

where $\delta_0(B_v - B_u + h)$ has to be understood as a distribution on the Wiener space in the sense of Watanabe (see [17]). One can also show that the right-hand side of (37) is the $L^2(\Omega)$-limit of the sequence obtained by replacing δ_0 with the Gaussian approximating kernel p_ε (see [7, Section 2] for further details).

Let us also give an elementary yet useful lemma:

LEMMA 3.5.   *Let* p_t *be the Gaussian kernel defined by* (5), *and* N *be a real valued random variable such that* $N \sim \mathcal{N}(h, \sigma^2)$ *with* $h \in \mathbb{R}$ *and* $\sigma^2 > 0$. *Then for all* $n \geq 0$ *we have*

(38)
$$\mathbf{E}\left[p_t^{(n)}(N) \right] = p_{t+\sigma^2}^{(n)}(h).$$

PROOF. Taking into account the analytic form of expected values with respect to N, we have $\mathbf{E}[p_t^{(n)}(N)] = [p_t^{(n)} * p_{\sigma^2}](h)$. Furthermore, elementary relations for convolutions and the semi-group property for p yield:

$$p_t^{(n)} * p_{\sigma^2} = [p_t * p_{\sigma^2}]^{(n)} = p_{t+\sigma^2}^{(n)},$$

from which relation (38) is easily deduced.

$\square$

Recall now that the projection $J_n(F)$ of a L^2 random variable F onto a fixed chaos $\mathcal{H}_n$ has been defined at Section 3.1. For our Hamiltonian $H_t^h(B)$ we get the following:

PROPOSITION 3.6. *For every $n \geq 1$ and every $h > 0$, recall that we have set $X_t^{n,h} = J_n(H_t^h(B))$ for the projection of $H_t^h(B)$ onto the n-th Wiener chaos. Then we have*

(39)
$$X_t^{n,h} = 0 \quad \text{if } n \text{ is odd} \quad , \quad X_t^{n,h} = \frac{16}{n!} I_n\big((f_h + g_{h,t}) \cdot \mathbf{1}_{[0,t]^n}\big) \quad \text{if } n \text{ is even,}$$

where $f_h \in L^2(\mathbb{R}_+^n)$, $g_{h,t} \in L^2([0,t]^n)$ are the symmetric functions defined by

(40) $\qquad f_h(t_1, \ldots, t_n) := \Phi_h(\min(t_1, \ldots, t_n), \max(t_1, \ldots, t_n)),$

(41)
$$g_{h,t}(t_1, \ldots, t_n) := -\Phi_h(\min(t_1, \ldots, t_n), t) + \Phi_h(0, t) - \Phi_h(0, \max(t_1, \ldots, t_n)),$$

and where we recall that the function Φ_h has been defined at Notation 3.1.

PROOF. We divide this proof in two steps:

Step 1: Computation of the projection. Let us first compute the chaos decomposition of $\delta_0(B_v - B_u + h)$. To this aim, recall that, as a distribution on the Wiener space (see [17]), we have $\delta_0(B_v - B_u + h) = \lim_{\varepsilon \to 0} p_\varepsilon(B_v - B_u + h)$ for the Gaussian kernel p_ε defined at (5). Furthermore, according to Stroock's formula (34) we have $J_n(p_\varepsilon(B_v - B_u + h)) = I_n(\varphi_n^\varepsilon)$ with

$$\varphi_n^\varepsilon(t_1, \ldots, t_n) = \frac{1}{n!} \mathbf{E}\left[\mathcal{D}_{t_1, \ldots, t_n} p_\varepsilon(B_v - B_u + h)\right] = \frac{1}{n!} \mathbf{E}\left[p_\varepsilon^{(n)}(B_v - B_u + h)\right] \prod_{i=1}^n \mathbf{1}_{[u,v]}(t_i).$$

We now compute $\mathbf{E}[p_\varepsilon^{(n)}(B_v - B_u + h)]$ by means of formula (38), which yields

$$\varphi_n^\varepsilon(t_1, \ldots, t_n) = \frac{p_{v-u+\varepsilon}^{(n)}(h)}{n!} \prod_{i=1}^n \mathbf{1}_{[u,v]}(t_i).$$

Taking limits as $\varepsilon \to 0$ we end up with $J_n(\delta_0(B_v - B_u + h)) = I_n(\varphi_n)$, where

$$\varphi_n(t_1, \ldots, t_n) = \frac{p_{v-u}^{(n)}(h)}{n!} \prod_{i=1}^n \mathbf{1}_{[u,v]}(t_i).$$

The same kind of computations is valid for $\delta_0(B_v - B_u - h)$ and $\delta_0(B_v - B_u)$ and thus going back to (37), we have obtained:

$$X_t^{n,h} = J_n(H_t^h(B)) = \frac{2}{n!} I_n\left(\int_{\mathcal{S}_t^2} \prod_{i=1}^n \mathbf{1}_{[u,v]}(t_i) \left[p_{v-u}^{(n)}(h) + p_{v-u}^{(n)}(-h) - 2p_{v-u}^{(n)}(0)\right] dudv\right),$$

where we recall that $\mathcal{S}_t^2$ stands for the simplex of order 2 on $[0, t]$ (see Notation 3.1). Moreover, observe that $p_{v-u}^{(n)}(h) + p_{v-u}^{(n)}(-h) - 2p_{v-u}^{(n)}(0) \equiv 0$ when n is odd, which yields the first claim in (39). Therefore only even ns are considered from now on.

Step 2: Simplification of the expression for the projection. Notice first that, since we are dealing with a linear Brownian motion B, one can write $X_t^{n,h}$ as

$$X_t^{n,h} = 2 \int_{\mathcal{S}_t^n} \left(\int_{\mathcal{S}_t^2} \prod_{i=1}^n \mathbf{1}_{[u,v]}(t_i) \left[p_{v-u}^{(n)}(h) + p_{v-u}^{(n)}(-h) - 2p_{v-u}^{(n)}(0) \right] \, dudv \right) dB_{t_1} \cdots dB_{t_n}$$

(42)

$$= 2 \int_{\mathcal{S}_t^n} \left(\int_0^{t_1} \int_{t_n}^t \left[p_{v-u}^{(n)}(h) + p_{v-u}^{(n)}(-h) - 2p_{v-u}^{(n)}(0) \right] \, dvdu \right) dB_{t_1} \cdots dB_{t_n}.$$

Let us transform now the expression $p_{v-u}^{(n)}(h) + p_{v-u}^{(n)}(-h) - 2p_{v-u}^{(n)}(0)$. First, since n is an even number and p is symmetric, we have

$$p_{v-u}^{(n)}(h) + p_{v-u}^{(n)}(-h) - 2p_{v-u}^{(n)}(0) = 2 \left[p_{v-u}^{(n)}(h) - p_{v-u}^{(n)}(0) \right].$$

Then write

$$p_{v-u}^{(n)}(h) - p_{v-u}^{(n)}(0) = \int_0^h p_{v-u}^{(n+1)}(x) \, dx = \int_0^h \int_0^x p_{v-u}^{(n+2)}(y) \, dydx = 2 \int_0^h \int_0^x \partial_v p_{v-u}^{(n)}(y) \, dydx,$$

which yields

$$\int_0^{t_1} du \int_{t_n}^t dv \left[p_{v-u}^{(n)}(h) - p_{v-u}^{(n)}(0) \right] = 2 \int_0^{t_1} du \int_{t_n}^t dv \int_0^h dx \int_0^x dy \, \partial_v p_{v-u}^{(n)}(y)$$

$$= 2 \int_0^{t_1} du \int_0^h dx \int_0^x dy \left[p_{t-u}^{(n)}(y) - p_{t_n-u}^{(n)}(y) \right]$$

$$= -4 \int_0^{t_1} du \int_0^h dx \int_0^x dy \left[\partial_u p_{t-u}^{(n-2)}(y) - \partial_u p_{t_n-u}^{(n-2)}(y) \right]$$

$$= -4 \int_0^h \left[p_{t-t_1}^{(n-2)}(y) - p_{t_n-t_1}^{(n-2)}(y) - p_t^{(n-2)}(y) + p_{t_n}^{(n-2)}(y) \right] (h-y) \, dy.$$

Plugging this expression into (42) and symmetrizing again, relation (39) easily follows.   □

3.3. *Asymptotic behavior of the variance.*   In this section we compute the correct amount of normalization needed for the convergence of each $X_t^{2m,h} = J_{2m}(H_t(B))$ for $m \geq 1$. This will be obtained thanks to an asymptotic

analysis of the variance of those random variables and recall that we have shown that $X_t^{2m,h} = \frac{16}{(2m)!} I_{2m}\big((f_h + g_{h,t}) \cdot \mathbf{1}_{[0,t]^{2m}}\big)$, which means in particular that

$$(43) \quad \mathbf{E}\big[X_t^{2m,h} X_s^{2m,h}\big] = \frac{16^2}{(2m)!} \langle (f_h + g_{h,t}) \cdot \mathbf{1}_{[0,t]^{2m}}, (f_h + g_{h,s}) \cdot \mathbf{1}_{[0,s]^{2m}} \rangle_{L^2(\mathbb{R}_+^{2m})}.$$

Our aim is to prove the following:

PROPOSITION 3.7.   *Fix $m \geq 1$. Then for all $0 \leq s \leq t \leq 1$, it holds that*

$$(44) \qquad \lim_{h \to 0} \frac{\mathbf{E}\big[X_t^{2m,h} X_s^{2m,h}\big]}{h^4 \ln(1/h)} = \sigma_m^2 s \qquad with \qquad \sigma_m^2 = \frac{c\,(2m-2)!}{2^{2m}[(m-1)!]^2}$$

for some strictly positive universal constant c.

The strategy for the proof of Proposition 3.7 is rather simple. Namely, with the expression (43) in mind, our calculations will be decomposed into the following facts:

- The norm $\|g_{h,t}\|_{L^2([0,t]^{2m})}^2$ is of order at most h^4 as h tends to 0, and thus is negligible with respect to $h^4 \ln(1/h)$.
- The quantity $\langle f_h \cdot \mathbf{1}_{[0,t]^{2m}}, f_h \cdot \mathbf{1}_{[0,s]^{2m}} \rangle_{L^2(\mathbb{R}_+^{2m})}$ scales as in relation (44).

Let us thus start by identifying the negligible terms:

LEMMA 3.8.   *Fix $m \geq 1$, and recall that for every $t > 0$, $g_{h,t} = g_{h,t,2m}$ is defined by (41). Then there exists a constant c_m such that for every $h > 0$,*

$$\sup_{t \in [0,1]} \|g_{h,t}\|_{L^2([0,t]^{2m})}^2 \leq c_m\, h^4.$$

PROOF. Write

$\|g_{h,t}\|_{L^2([0,t]^{2m})}^2$

$= (2m)! \int_{\mathcal{S}_t^{2m}} \big[\Phi_h(t_1, t) - \Phi_h(0, t) + \Phi_h(0, t_{2m})\big]^2 dt_1 \cdots dt_{2m}$

$\leq c_m \Big\{ \int_{[0,t]} (t - t_1)^{2m-1} \Phi_h(t_1, t)^2\, dt_1 + t^{2m} \Phi_h(0, t)^2 + \int_{[0,t]} t_{2m}^{2m-1} \Phi_h(0, t_{2m})^2\, dt_{2m} \Big\}$

and the bound is then easily derived from (30).

$\square$

We can now turn to the proof of the main proposition of this section:

PROOF OF PROPOSITION 3.7. Thanks to Lemma 3.8, we only have to focus on

$$A_h(s,t) \equiv \langle f_h \cdot \mathbf{1}_{[0,t]^{2m}}, f_h \cdot \mathbf{1}_{[0,s]^{2m}} \rangle_{L^2(\mathbb{R}_+^{2m})}.$$

An easy integration over the simplex gives

$$
\begin{aligned}
A_h(s,t) &= (2m)! \int_{\mathcal{S}_s^{2m}} [\Phi_{h,2m}(t_1,t_{2m})]^2 dt_1 \cdots dt_{2m} \\
&= (2m)! \int_{\mathcal{S}_s^2} \frac{(t_{2m}-t_1)^{2m-2}}{(2m-2)!} [\Phi_{h,2m}(t_1,t_{2m})]^2 dt_1 dt_{2m}.
\end{aligned}
$$

Then, using the classical formula for the $2m$-th derivative of p_t, that is

$$(45) \qquad p_t^{(2m)}(y) = (2m)!\, t^{-m} p_t(y) H_{2m}\left(\frac{y}{t^{1/2}}\right),$$

where H_{2m} is defined by (33), we deduce that

$$A_h(s,t) = (2m)!(2m-2)! \int_{\mathcal{S}_s^2} (t_2-t_1)^{2m-2}$$

$$\times \left[\int_0^h (t_2-t_1)^{-(m-1)} \frac{e^{-\frac{y^2}{2(t_2-t_1)}}}{(2\pi(t_2-t_1))^{1/2}} H_{2m-2}\left(\frac{y}{(t_2-t_1)^{1/2}}\right) (h-y)\, dy \right]^2 dt_1 dt_2$$

$$= (2m)!(2m-2)! \int_{\mathcal{S}_s^2} \left[\int_0^h \frac{e^{-\frac{y^2}{2(t_2-t_1)}}}{(2\pi(t_2-t_1))^{1/2}} H_{2m-2}\left(\frac{y}{(t_2-t_1)^{1/2}}\right) (h-y)\, dy \right]^2 dt_1 dt_2.$$

Perform the change of variable $t_2 - t_1 = \tau$ and $t_1 = \sigma$, which yields

$$A_h(s,t) = (2m)!(2m-2)! \int_0^s (s-\tau) \left[\int_0^h \frac{e^{-\frac{y^2}{2\tau}}}{(2\pi\tau)^{1/2}} H_{2m-2}\left(\frac{y}{\tau^{1/2}}\right) (h-y)\, dy \right]^2 d\tau.$$

Now set $y/\tau^{1/2} = z$ in order to get

$$A_h(s,t) = \frac{(2m)!(2m-2)!}{2\pi} h^2 \int_0^s (s-\tau) \left[\int_0^{h/\tau^{1/2}} e^{-\frac{z^2}{2}} H_{2m-2}(z) \left(1 - \frac{\tau^{1/2} z}{h}\right) dz \right]^2 d\tau.$$

Finally let $u = h/\tau^{1/2}$, so that we end up with $A_h(s,t) = \frac{1}{\pi}(2m)!(2m-2)!\, h^4\, a(h)$, where

$$a(h) \equiv \int_{h/s^{1/2}}^\infty u^{-3} \left(s - \frac{h^2}{u^2}\right) \left[\int_0^u e^{-\frac{z^2}{2}} H_{2m-2}(z) \left(1 - \frac{z}{u}\right) dz \right]^2 du.$$

It is now readily checked that the main singularity in the integral defining $a(h)$ is due to a term $u^{-3}u^2 = u^{-1}$ integrated close to 0, so that for small h, $a(h)$ is of order $\ln(1/h)$.

In order to quantify this fact, let us apply l'Hopital's rule to $a(h)/\ln(1/h)$. We get

$$\lim_{h \to 0} \frac{a(h)}{\ln(1/h)} = \lim_{h \to 0} \frac{b(h)}{h^{-2}}, \quad \text{with} \quad b(h) = 2 \int_{h/s^{1/2}}^{\infty} u^{-5} \left[\int_0^u e^{-\frac{z^2}{2}} H_{2m-2}(z) \left(1 - \frac{z}{u} \right) dz \right]^2 du.$$

It is now easily seen that $b'(h)$ is equivalent to $-\frac{s}{2} h^{-3}[H_{2m-2}(0)]^2$ in a neighborhood of the origin, so that a second application of l'Hopital's rule to $b(h)/h^{-2}$ yields

$$\lim_{h \to 0} \frac{a(h)}{\ln(1/h)} = \frac{s}{4} [H_{2m-2}(0)]^2.$$

In order to conclude recall that $A_h(s,t) = \frac{1}{\pi}(2m)!(2m-2)! h^4 a(h)$, and thus with the value of $H_{2m-2}(0)$ in mind (see (35)) we end up with

$$\lim_{h \to 0} \frac{A_h(s,t)}{h^4 \ln(1/h)} = \frac{s}{4} [H_{2m-2}(0)]^2 (2m)!(2m-2)! = \frac{(2m)!(2m-2)!}{2^{2m-1}[(m-1)!]^2} s,$$

which finishes the proof of relation (44) since

$$\lim_{h \to 0} \frac{\mathbf{E}[X_t^{2m,h} X_s^{2m,h}]}{h^4 \ln(1/h)} = \frac{128}{\pi(2m)!} \lim_{h \to 0} \frac{A_h(s,t)}{h^4 \ln(1/h)}.$$

$\square$

REMARK 3.9.   The fact that $\sum \sigma_m^2 = \infty$, mentioned at Theorem 1.2 item (iii), follows at once from relation (44). Indeed, using Stirling's formula, we can easily conclude that σ_m^2 is asymptotically equivalent to $\frac{c}{\sqrt{m}}$ for some constant $c > 0$.

3.4. *Contractions.*   In this section we shall prove that for a fixed $t \in [0,1]$ the random variable $H_t(B)/[h^2 \ln(1/h)^{1/2}]$ converges in law to a Gaussian random variable as h goes to 0. Owing to Proposition 3.3 and with Proposition 3.7 in hand, this boils down to the study of contractions for the functions f_h, g_h involved in the definition of $J_n(H_t(B))$ given at (39). Those contractions are evaluated in the following proposition:

PROPOSITION 3.10.   *Fix $n = 2m \geq 2$, and recall that $f_h, g_{h,t}$ also depend on n as highlighted in (40)-(41). Then for every $r \in \{1, \ldots, n-1\}$, one has*

$$(46) \qquad \frac{1}{h^8 \ln^2(1/h)} \|(f_h + g_{h,t}) \otimes_r (f_h + g_{h,t})\|_{L^2([0,t]^{2n-2r})}^2 \to 0$$

as h tends to 0.

PROOF. Due to Lemma 3.8, the proof of Proposition 3.7 and thanks to the fact that

$$\|f_h \otimes_r g_{h,t}\|_{L^2([0,t]^{2n-2r})} \le \|f_h\|_{L^2([0,t]^{2m})}\|g_{h,t}\|_{L^2([0,t]^{2m})},$$

it is readily checked that as h tends to 0,

$$\frac{1}{h^8 \ln^2(1/h)}\|(f_h+g_{h,t})\otimes_r(f_h+g_{h,t})\|^2_{L^2([0,t]^{2n-2r})} = \frac{1}{h^8 \ln^2(1/h)}\|f_h\otimes_r f_h\|^2_{L^2([0,t]^{2n-2r})}+o(1).$$

We are thus reduced to prove that

$$(47) \qquad \lim_{h\to 0} \frac{\|f_h \otimes_r f_h\|^2_{L^2([0,t]^{2n-2r})}}{h^8 \ln^2(1/h)} = 0.$$

In order to compute $\|f_h \otimes_r f_h\|^2_{L^2([0,t]^{2n-2r})}$, let us consider the following general problem: fix an integrable function φ defined on $\mathcal{S}_t^2$ and compute the contraction norm:

$$R_{n,r}(\varphi) = \int_{[0,t]^{2(n-r)}} \left(\int_{[0,t]^r} \varphi(\max(\mathbf{s},\mathbf{t}^1),\min(\mathbf{s},\mathbf{t}^1))\,\varphi(\max(\mathbf{s},\mathbf{t}^2),\min(\mathbf{s},\mathbf{t}^2))\,d\mathbf{s}\right)^2 d\mathbf{t}^1\,d\mathbf{t}^2,$$

where we have set

$$\max(\mathbf{s},\mathbf{t}) = \max(s_1,\ldots,s_r,t_1,\ldots,t_{n-r}), \qquad \min(\mathbf{s},\mathbf{t}) = \min(s_1,\ldots,s_r,t_1,\ldots,t_{n-r}).$$

Note that $R_{n,r}(\varphi)$ can also be written as

$$R_{n,r}(\varphi) = \int_{[0,t]^{2(n-r)}} \int_{[0,t]^{2r}} \prod_{i,j=1}^{2} \varphi\left(\max(\mathbf{s}^i,\mathbf{t}^j),\min(\mathbf{s}^i,\mathbf{t}^j)\right) d\mathbf{s}^1\,d\mathbf{s}^2\,d\mathbf{t}^1\,d\mathbf{t}^2.$$

In order to evaluate this integral, the following simple transformations can be performed: *(i)* Replace $\max(\mathbf{s}^i,\mathbf{t}^j)$ by $\max(\mathbf{s}^i)\vee\max(\mathbf{t}^j)$. *(ii)* Integrate on simplexes such as $0 < s_1 < \cdots < s_r < t$. For $2 \le r \le n-2$, this simplifies the above expression into

$$R_{n,r}(\varphi) = [(n-r)!r!]^2 \int_{(\mathcal{S}_t^2)^2} \int_{(\mathcal{S}_t^2)^2} \prod_{i,j=1}^{2} \varphi\left(\max(\sigma_2^i,\tau_2^j),\min(\sigma_1^i,\tau_1^j)\right)$$

$$\times \prod_{k=1}^{2} \left[\frac{(\sigma_2^k-\sigma_1^k)^{r-2}(\tau_2^k-\tau_1^k)^{n-r-2}}{(r-2)!(n-r-2)!}\right] d\sigma_1^k\,d\sigma_2^k\,d\tau_1^k\,d\tau_2^k,$$

that is

$$(48) \quad R_{n,r}(\varphi) = P_r(n) \int_{(\mathcal{S}_t^2)^2} \int_{(\mathcal{S}_t^2)^2} \prod_{i,j=1}^{2} \varphi\left(\max(\sigma_2^i, \tau_2^j), \min(\sigma_1^i, \tau_1^j)\right)$$
$$\times \prod_{k=1}^{2} (\sigma_2^k - \sigma_1^k)^{r-2}(\tau_2^k - \tau_1^k)^{n-r-2} \, d\sigma_1^k \, d\sigma_2^k \, d\tau_1^k \, d\tau_2^k,$$

where we have set $P_r(n) = [(n-r)(n-r-1)r(r-1)]^2$.

We now recall that f_h is defined by (40), which means that we shall apply identity (48) to the function $\varphi = \Phi_h$ where Φ_h is introduced at Notation 3.1. Towards this aim, observe that one can write $\Phi_h(u,v) = \ell_{n,h}(v-u)$ with $\ell_{n,h} : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\ell_{n,h}(w) := \int_0^h p_w^{(n-2)}(y) \, (h-y) \, dy$. Thanks to the expression (45) we have already recalled for $p_w^{(n-2)}$ we thus get

$$\ell_{n,h}(w) = \frac{(-1)^n (n-2)!}{\sqrt{2\pi}} \int_0^h \frac{e^{-\frac{y^2}{2w}}}{w^{(n-1)/2}} H_{n-2}\left(\frac{y}{w^{1/2}}\right) (h-y) \, dy \le c_n \frac{h^2}{w^{(n-1)/2}}.$$

Plugging this relation into (48), we obtain that for $2 \le r \le n-2$,

$$\|f_h \otimes_r f_h\|_{L^2([0,t]^{2n-2r})}^2$$
$$\le \quad c_{n,r} \, h^8 \int_{(\mathcal{S}_t^2)^2} \int_{(\mathcal{S}_t^2)^2} \prod_{i,j=1}^{2} \left(\max(\sigma_2^i, \tau_2^j) - \min(\sigma_1^i, \tau_1^j)\right)^{-(n-2)/2}$$
$$\times \prod_{k=1}^{2} (\sigma_2^k - \sigma_1^k)^{r-2}(\tau_2^k - \tau_1^k)^{n-r-2} \, d\sigma_1^k \, d\sigma_2^k \, d\tau_1^k \, d\tau_2^k$$
$$\le \quad c_{n,r} h^8 \int_{(\mathcal{S}_t^2)^2} \int_{(\mathcal{S}_t^2)^2} \prod_{i,j=1}^{2} \left(\max(\sigma_2^i, \tau_2^j) - \min(\sigma_1^i, \tau_1^j)\right)^{-3/2} \prod_{k=1}^{2} d\sigma_1^k \, d\sigma_2^k \, d\tau_1^k \, d\tau_2^k. \quad (49)$$

This kind of integral will be handled in Lemma 5.1, which allows to conclude that $\|f_h \otimes_r f_h\|_{L^2([0,t]^{2n-2r})}^2 \le c_{n,r} \, h^8$. Hence relation (47) obviously holds true, which in turn implies (46).

We have thus proved relation (46) for $n \ge 4$ and $2 \le r \le n-2$. The remaining possibilities can be treated applying the same reasoning: in the case $(n \ge 4, \ r \in \{1, n-1\})$ we have

$$\|f_h \otimes_r f_h\|_{L^2([0,t]^{2n-2r})}^2$$
$$\le c_{r,n} h^8 \int_{(\mathcal{S}_t^2)^2} \int_{[0,t]^2} \prod_{i,j=1}^{2} \left(\max(\sigma^i, \tau_2^j) - \min(\sigma^i, \tau_1^j)\right)^{-1} d\sigma^1 d\sigma^2 \prod_{k=1}^{2} d\tau_1^k d\tau_2^k$$

and we recognize here the second (finite) integral involved in Lemma 5.1. Finally, the case ($n = 2$, $r = 1$) reduces to

$$\|f_h \otimes_r f_h\|^2_{L^2([0,t]^{2n-2r})}$$

$$\leq c_{r,n} h^8 \int_{[0,t]^2} \int_{[0,t]^2} \prod_{i,j=1}^{2} \big(\max(\sigma^i, \tau^j) - \min(\sigma^i, \tau^j) \big)^{-1/2} d\sigma^1 d\sigma^2 d\tau^1 d\tau^2,$$

so that we can conclude with Lemma 5.1 as well.

□

Summarizing our considerations up to now, we have obtained the following convergence in law for the finite-dimensional distributions of $X^{2m,h}$:

PROPOSITION 3.11. *Taking up the notations of Theorem 1.2, consider* $t_1, \ldots, t_d \in [0,1]$ *and* $m \geq 1$. *Then as* $h \to 0$ *we have*

$$\frac{1}{h^2[\ln(1/h)]^{1/2}} (X^{2m,h}_{t_1}, \ldots, X^{2m,h}_{t_d}) \xrightarrow{(d)} \sigma_m \mathcal{N}(0, \Gamma) \quad where \quad \sigma_m^2 = \frac{c\,(2m-2)!}{2^{2m}[(m-1)!]^2}$$

and $\mathcal{N}(0, \Gamma)$ *is the centered Gaussian law in* $\mathbb{R}^d$ *with covariance matrix* $\Gamma(i,j) = \min(t_i, t_j)$.

PROOF. We shall simultaneously apply Propositions 3.3 and 3.4 to the random vector $h^{-2}[\ln(1/h)]^{-1/2}(X^{2m,h}_{t_1}, \ldots, X^{2m,h}_{t_d})$, which is of course a sequence of random vectors in $(\mathcal{H}_{2m})^d$. Moreover:

(i) According to Proposition 3.7, we have for every t_i, t_j,

$$\lim_{h \to 0} \frac{\mathbf{E}[X^{2m,h}_{t_i} X^{2m,h}_{t_j}]}{h^4 \ln(1/h)} = \sigma_m^2 \, \Gamma(i,j).$$

(ii) For each fixed t_i, one can write $X^{2m,h}_{t_i} = I_{2m}(k_{2m,h})$ with $k_{2m,h} = \frac{16}{(2m)!}(f_h + g_{h,t_i}) \cdot \mathbf{1}_{[0,t_i]^{2m}}$. Then Proposition 3.10 asserts that, for all $r \in \{1, \ldots, 2m-1\}$,

$$\lim_{h \to 0} \frac{1}{h^4 \ln(1/h)} \|k_{2m,h} \otimes_r k_{2m,h}\|_{\mathcal{H}^{\otimes 2(n-r)}} = 0.$$

We can thus combine Propositions 3.3 and 3.4 so as to conclude.

□

3.5. *Tightness.*   Now endowed with Proposition 3.11, the proof of Theorem 1.2 reduces to showing that the sequence of processes $\{h^{-2}\ln(1/h)^{-1/2}X_t^{2m,h}, \, t \in [0,1]\}$ is tight. This is the contents of the following proposition:

PROPOSITION 3.12.   *Fix $m \geq 1$. Then:*

(i) *There exists $\lambda > 0$ and a constant c_m such that for all $0 \leq s \leq t \leq 1$,*

$$(50) \qquad \sup_{h\in(0,1)} \frac{1}{h^4\ln(1/h)}\mathbf{E}\big[|X_t^{2m,h} - X_s^{2m,h}|^2\big] \leq c_m\,|t-s|^\lambda\,.$$

(ii) *The family $\{h^{-2}\ln(1/h)^{-1/2}X^{2m,h}; \, h > 0\}$ is tight in $\mathcal{C}([0,1])$.*

In order to prove Proposition 3.12, recall that $X_t^{2m,h} = \frac{16}{(2m)!}I_{2m}((f_h + g_{h,t}) \cdot \mathbf{1}_{[0,t]^{2m}})$ with f_h, g_h defined by (40)-(41). We will also use the following additional property of g_h, which can be readily checked with the help of (31)-(32), as in the proof of Lemma 3.8 :

LEMMA 3.13.   *Fix $m \geq 1$, and recall that we write $g_{h,t}$ instead of $g_{h,t,2m}$ for notational sake. Then there exists $\lambda > 0$ and a constant c_m such that for all $0 \leq s \leq t \leq 1$, one has*

$$\sup_{h\in(0,1)} \|g_{h,t} - g_{h,s}\|_{L^2([0,s]^{2m})}^2 \leq c_m h^4\,|t-s|^\lambda\,.$$

We can now turn to the proof of the main proposition of this section:

PROOF OF PROPOSITION 3.12.   We prove the two claims of the proposition separately:

Step 1: Proof of assertion (i). Let us write

$$X_t^{2m,h} - X_s^{2m,h} = c_m\Big\{I_n\big((f_h+g_{h,t})\cdot\{\mathbf{1}_{[0,t]^{2m}}-\mathbf{1}_{[0,s]^{2m}}\}\big)+I_n\big((g_{h,t}-g_{h,s})\cdot\mathbf{1}_{[0,s]^{2m}}\big)\Big\}.$$

The second term of this decomposition can be treated with Lemma 3.13. As for the first term, we clearly have

$$\mathbf{E}\left[\Big(I_n\big((f_h + g_{h,t}) \cdot \{\mathbf{1}_{[0,t]^{2m}} - \mathbf{1}_{[0,s]^{2m}}\}\big)\Big)^2\right] \leq c_m\left(A_{s,t}^h + B_{s,t}^h\right),$$

with

$$
\begin{aligned}
A_{s,t}^h &= \int_{\substack{0<t_1<\ldots<t_{2m}\\ s<t_{2m}<t}} \Phi_h(t_1,t_{2m})^2\,dt_1\cdots dt_{2m}\\
B_{s,t}^h &= \int_{\substack{0<t_1<\ldots<t_{2m}\\ s<t_{2m}<t}} g_{h,t}(t_1,\ldots,t_{2m})^2\,dt_1\cdots dt_{2m}.
\end{aligned}
$$

We now bound those two terms: first, split up $A_{s,t}^h$ into $A_{s,t}^h = c_m\{A_{s,t}^{h,1} + A_{s,t}^{h,2}\}$ with

$$A_{s,t}^{h,1} \equiv \int_{0<t_1<s<t_{2m}<t} (t_{2m} - t_1)^{2m-2} \Phi_h(t_1, t_{2m})^2 \, dt_1 \, dt_{2m},$$

and

$$\begin{aligned} A_{s,t}^{h,2} &\equiv \int_{s<t_1<t_{2m}<t} (t_{2m} - t_1)^{2m-2} \Phi_h(t_1, t_{2m})^2 \, dt_1 \, dt_{2m} \\ &= \int_{0<t_1<t_{2m}<t-s} (t_{2m} - t_1)^{2m-2} \Phi_h(t_1, t_{2m})^2 \, dt_1 \, dt_{2m}. \end{aligned}$$

Then by (30), one has for any small $\varepsilon > 0$,

$$\begin{aligned} A_{s,t}^{h,1} &\leq c_m h^4 \int_{0<t_1<s<t_{2m}<t} (t_{2m} - t_1)^{-1} \, dt_1 \, dt_{2m} \\ &\leq c_m h^4 \int_{0<t_1<s} (s - t_1)^{-1+\varepsilon} \, dt_1 \int_{s<t_{2m}<t} (t_{2m} - t_1)^{-\varepsilon} \, dt_{2m} \leq c_m h^4 |t - s|^{1-\varepsilon}. \end{aligned}$$

As far as $A_{s,t}^{h,2}$ is concerned, we can follow the lines of the proof of Proposition 3.7 and conclude that $\lim \frac{1}{h^4 \ln(1/h)} A_{s,t}^{h,2} = c_m |t - s|$ as h tends to zero, which gives us a proper estimate.

Finally, the bound for $B_{s,t}^h$ is easily derived as follows: first notice that, according to the definition (41) of $g_{h,t}$, we have

$$B_{s,t}^h \leq c \int_{\substack{0<t_1<\ldots<t_{2m} \\ s<t_{2m}<t}} \left[\Phi_h^2(t_1, t) + \Phi_h^2(0, t) + \Phi_h^2(0, t_{2m}) \right] \, dt_1 \cdots dt_{2m}.$$

The three terms above are handled easily, and along the same lines, thanks to (30). For the first one, we get for instance

$$\begin{aligned} \int_{\substack{0<t_1<t_{2m}<t \\ s<t_{2m}<t}} &(t_{2m} - t_1)^{2m-2} \, \Phi_h^2(t_1, t) \, dt_1 \cdots dt_{2m} \\ &\leq c_m h^4 \int_{\substack{0<t_1<t_{2m}<t \\ s<t_{2m}<t}} (t_{2m} - t_1)^{2m-2} (t - t_1)^{-2m+1} \, dt_1 \, dt_{2m} \\ &\leq c_m h^4 (t - s) \int_{0<t_1<s} (t - t_1)^{-1} \, dt_1 + c_m h^4 \int_{s<t_1<t} (t - t_1)^{2m-1} (t - t_1)^{-2m+1} \, dt_1 \\ &\leq c_m h^4 |t - s|^{1-\varepsilon} \end{aligned}$$

for any small $\varepsilon > 0$. Gathering now our estimates on $A_{s,t}^h$ and $B_{s,t}^h$, we have proved our claim (50).

Step 2: Proof of assertion (ii). With inequality (50) in hand, the tightness result is easily deduced. Indeed, the random variable $X_t^{2m,h} - X_s^{2m,h}$ living in a finite chaos, we are in a position to use hypercontractivity (see [11]) and assert that for all $p \geq 1$,

$$\sup_{h \in (0,1)} \frac{1}{h^{4p} \ln(1/h)^p} \mathbf{E}\big[|X_t^{2m,h} - X_s^{2m,h}|^{2p}\big] \leq c_{m,p} \, |t-s|^{\lambda p} .$$

As we have done before, Kolmogorov's tightness criterion is therefore verified for every p such that $\lambda p > 1$, which finishes our proof.

$\square$

4. L^2 modulus of 2-dimensional local time on chaoses.

We now carry on the task of proving Theorem 1.3 for projections of the quantity $H_t^h(B)$ defined by (1) when B is a 2-dimensional Brownian motion. For the sake of simplicity, we shall take up most of the notations introduced at Section 3, starting from the fact that our Hamiltonian is written $H_t^h(B)$ independently of the fact that B is a one-dimensional or a two-dimensional Brownian motion. Like in [8], we shall also invoke the following important representation formula for $H_t^h(B)$:

(51)
$$H_t^h(B) = \int_{[0,t]^2} [\delta_0(B_v - B_u + h) + \delta_0(B_v - B_u - h) - 2\delta_0(B_v - B_u)] \, dudv.$$

REMARK 4.1. The reader should be aware of the fact that expression (51) is formal, since the self-intersection local time is a divergent quantity for a 2-dimensional Brownian motion. Notice however that only projections on fixed chaoses will be considered in the sequel, and all projections of the random distribution defined by (51) are well-defined.

Next we introduce the equivalent of the functions Φ introduced at Notation 3.1. In the 2-d case, we will let this set of functions appear in a Fourier transform procedure, as follows:

NOTATION 4.2. *For every $n \geq 2$, every $\mathbf{i} = (i_1, \ldots, i_n) \in \{1,2\}^n$ and every $h \in \mathbb{R}^2$, we define a function $\Phi_{\mathbf{i}}(t_1, t_2)$ as*

$$\Phi_{\mathbf{i}}(t,s) = \Phi_{\mathbf{i},h}(t,s) := \int_{\mathbb{R}^2} d\xi \, \big(\prod_{k=1}^n \xi_{i_k}\big) \frac{\{1 - \cos(\langle h, \xi \rangle)\}}{|\xi|^4} e^{-(t-s)|\xi|^2/2}.$$

REMARK 4.3. In order to draw a link between $\Phi_{\mathbf{i}}$ and the function $\Phi = \Phi^{1\text{-d}}$ introduced at Notation 3.1, observe that, at least for $n = 2m$ even (the

only cases of interest in our study), one can also write $\Phi^{\text{1-d}}$ as

$$
\begin{aligned}
\Phi^{\text{1-d}}_{h,2m}(t,s) &= \int_0^h p^{(2m-2)}_{t-s}(y)(h-y)\,dy \\
&= p^{(2m-2)}_{t-s}(h) - p^{(2m-2)}_{t-s}(0) \\
&= \frac{1}{2}\{p^{(2m-2)}_{t-s}(h) + p^{(2m-2)}_{t-s}(-h) - 2p^{(2m-2)}_{t-s}(0)\} \\
&= c\int_{\mathbb{R}} d\xi\,\xi^{2m-2}\{1-\cos(h\xi)\}e^{-(t-s)\xi^2/2},
\end{aligned}
$$

where we have used the Fourier representation $p_{t-s}(x) = c\int_{\mathbb{R}} d\xi\,e^{\imath x\xi}e^{-(t-s)\xi^2/2}$.

The continuity properties of the functions $\Phi_{\mathbf{i}}$, mimicking (30)–(32), are summarized below:

LEMMA 4.4. *Fix $m \geq 1$ and $\alpha \in (0,2)$. Then there exists a constant $c_{m,\alpha}$ such that for every $h \in \mathbb{R}^2$ and all $0 \leq t_1 < s < t < t_2$,*

$$
\text{(52)} \qquad \max_{\mathbf{i}\in\{1,2\}^{2m}} |\Phi_{\mathbf{i},h}(t,s)| \leq c_{m,\alpha}\,|h|^\alpha\,|t-s|^{-m+1-\frac{\alpha}{2}}.
$$

$$
\text{(53)} \quad \max_{\mathbf{i}\in\{1,2\}^{2m}} |\Phi_{\mathbf{i},h}(t,t_1) - \Phi_{\mathbf{i},h}(s,t_1)| \leq c_{m,\alpha}\,|h|^\alpha\,|t-s|^\lambda\,|s-t_1|^{-m+1-\frac{\alpha}{2}-\lambda}.
$$

$$
\text{(54)} \quad \max_{\mathbf{i}\in\{1,2\}^{2m}} |\Phi_{\mathbf{i},h}(t_2,t) - \Phi_{\mathbf{i},h}(t_2,s)| \leq c_{m,\alpha}\,|h|^\alpha\,|t-s|^\lambda\,|t_2-t|^{-m+1-\frac{\alpha}{2}-\lambda}.
$$

The strategy of the proof for Theorem 1.3 is now similar to the one-dimensional case of Theorem 1.2: exact computation of the chaos decomposition, analysis of the variance and contraction properties for $H_t^h(B)$. This is why we shall skip some details below, and mainly stress the differences between the 1-d and the 2-d case.

4.1. *Stochastic analysis in dimension 2.* The Malliavin calculus setting we shall use in this section is very similar to the one explained at Section 3.1. However, we stress here some differences between stochastic analysis for 1-d and 2-d Brownian motions.

Notice first that our standing Wiener space is now the space of $\mathbb{R}^2$-valued continuous functions $\mathcal{C}(\mathbb{R}_+;\mathbb{R}^2)$, while the related Hilbert space is $\mathcal{H} \equiv (L^2([0,1]))^2$. We set $B = (B^1, B^2)$ for the 2-dimensional Wiener process and for $h = (h^1, h^2) \in \mathcal{H}$ we define $B(h) = B^1(h^1) + B^2(h^2)$. Starting from this

definition of Wiener integral the Malliavin derivatives and Sobolev spaces are defined along the same lines as in Section 3.1.

Stroock's formula takes the following form in the 2-dimensional situation: designate by $\mathbf{i} = (i_1, \ldots, i_n)$ a generic element of $\{1, 2\}^n$. Then for a functional $F \in \mathbb{D}^{\infty, 2}$ we have $J_n(F) = I_n(f_n)$, with

(55)

$$I_n(f_n) = \sum_{\mathbf{i} \in \{1,2\}^n} \int_{[0,1]^n} f_{\mathbf{i}}(t_1, \ldots, t_n) dB_{t_1}^{i_1} \cdots dB_{t_n}^{i_n}, \quad f_{\mathbf{i}}(t_1, \ldots, t_n) = \frac{\mathbf{E}\left[\mathcal{D}_{t_1, \ldots, t_n}^{\mathbf{i}} F\right]}{n!}.$$

Finally Propositions 3.3 and 3.4 are still valid in our 2-d Wiener space context, except for the fact that the expression for the r-th contraction ($r \in \{1, \ldots, n\}$) of a given kernel f reads as follows: for $\mathbf{k}^1, \mathbf{k}^2 \in \{1, 2\}^{n-r}$ and $\mathbf{t}^1, \mathbf{t}^2 \in [0, 1]^{n-r}$,

(56) $\quad (f \otimes_r f)_{(\mathbf{k}^1, \mathbf{k}^2)}(\mathbf{t}^1, \mathbf{t}^2) := \sum_{\mathbf{l} \in \{1,2\}^r} \int_{[0,1]^r} d\mathbf{s} \, f_{(\mathbf{k}^1, \mathbf{l})}(\mathbf{t}^1, \mathbf{s}) f_{(\mathbf{k}^2, \mathbf{l})}(\mathbf{t}^2, \mathbf{s}).$

4.2. *Chaos decomposition of $\mathbf{H_t^h(B)}$.* We are now ready to compute the projections $X^{n,h}$ of $H^h(B)$ on chaoses, which is the analogous statement to Proposition 3.6 in the 2-d situation.

PROPOSITION 4.5. *For every $n \geq 1$ and every non-zero $h \in \mathbb{R}^2$, recall that we have set $X_t^{n,h} = J_n(H_t^h(B))$ for the projection of $H_t^h(B)$ onto the n-th Wiener chaos. Then we have $X_t^{n,h} = 0$ if n is odd and*
(57)
$$X_t^{n,h} = \frac{c}{n!} \sum_{\mathbf{i} \in \{1,2\}^n} \int_{[0,t]^n} \{f_{\mathbf{i}}(t_1, \ldots, t_n) + g_{\mathbf{i},t}(t_1, \ldots, t_n)\} \, dB_{t_1}^{i_1} \cdots dB_{t_n}^{i_n} \quad \text{if } n \text{ is even,}$$

for some universal constant c. In the previous equation, the symmetric functions $f_{\mathbf{i}} \in L^2(\mathbb{R}_+^n)$ and $g_{\mathbf{i},t} \in L^2([0,t]^n)$ are defined for each $\mathbf{i} \in \{1,2\}^n$ by

(58) $\quad f_{\mathbf{i}}(t_1, \ldots, t_n) = f_{\mathbf{i}}^h(t_1, \ldots, t_n) := \Phi_{\mathbf{i}}(\min(t_1, \ldots, t_n), \max(t_1, \ldots, t_n)),$

and

(59) $\quad g_{\mathbf{i},t}(t_1, \ldots, t_n) = g_{\mathbf{i},t}^h(t_1, \ldots, t_n)$
$$\qquad\qquad := -\Phi_{\mathbf{i}}(\min(t_1, \ldots, t_n), t) + \Phi_{\mathbf{i}}(0, t) - \Phi_{\mathbf{i}}(0, \max(t_1, \ldots, t_n)),$$

where we recall that the functions $\Phi_{\mathbf{i}}$ are introduced at Notation 4.2.

PROOF. By applying Stroock's formula (55) to expression (51) in a similar manner as in the proof of Proposition 3.6, we obtain that $X_t^{n,h}$ is equal to

$$= \frac{c}{n!} \sum_{\mathbf{i} \in \{1,2\}^n} \int_{[0,t]^n} \left(\int_{\mathcal{S}_t^2} \left[\frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(h) + \frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(-h) - 2\frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(0) \right] \prod_{i=1}^n \mathbf{1}_{[u,v]}(t_i) \, dudv \right)$$

$$\times \ dB_{t_1}^{i_1} \cdots dB_{t_n}^{i_n}$$

$$= c \sum_{\mathbf{i} \in \{1,2\}^n} \int_{\mathcal{S}_t^n} \left(\int_0^{t_1} \int_{t_n}^t \left[\frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(h) + \frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(-h) - 2\frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(0) \right] dudv \right) dB_{t_1}^{i_1} \cdots dB_{t_n}^{i_n},$$

where $p_t(x)$ stands here for the 2-dimensional Gaussian kernel and where we have set $\partial x_{\mathbf{i}} := \partial x_{i_1} \cdots \partial x_{i_n}$. Observe first that $\frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(h) + \frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(-h) - 2\frac{\partial^n p_{v-u}}{\partial x_{\mathbf{i}}}(0)$ vanishes when n is odd, which yields our claim $X_t^{n,h} = 0$ in this situation. In the case where n is even, use the Fourier representation

$$p_{v-u}(x) = c \int_{\mathbb{R}^2} e^{\iota \langle x, \xi \rangle} e^{-(v-u)|\xi|^2/2} \, d\xi$$

and Fubini's theorem in order to derive

$$X_t^{n,h} = c \sum_{\mathbf{i} \in \{1,2\}^n} \int_{\mathcal{S}_t^n} \left[\Phi_{\mathbf{i}}(t_n, t_1) - \Phi_{\mathbf{i}}(t_1, t) + \Phi_{\mathbf{i}}(0, t) - \Phi_{\mathbf{i}}(t_n, 0) \right] dB_{t_1}^{i_1} \cdots dB_{t_n}^{i_n}.$$

Formula (57) follows by symmetrization.

$\square$

4.3. *Asymptotic behavior of the variance.* With expression (57) in hand, we now proceed as for the one-dimensional case, and compute $\mathbf{E}[X_t^{2m,h} X_s^{2m,h}]$ in order to see how this kind of quantity scales in h. Let us first label the following analytic lemma which will feature in our future computations:

LEMMA 4.6. *Fix $m \geq 1$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$(60) \qquad\qquad |\varphi(x,y)| \leq c \left\{ |x|^{1-\varepsilon}|y|^{1-\varepsilon} + |x|^{1+\varepsilon}|y|^{1+\varepsilon} \right\}$$

for some small $\varepsilon > 0$. For every non-zero $e \in \mathbb{R}^2$, set

$$(61) \ \ L_{2m,e}^\varphi := \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^2} d\eta \, \frac{\langle \xi, \eta \rangle^{2m}}{|\xi|^4 |\eta|^4} \, \varphi(\langle \xi, e \rangle, \langle \eta, e \rangle) \exp\left(-\frac{1}{2}(|\xi|^2 + |\eta|^2) \right).$$

Then $L_{2m,e}^\varphi$ is well-defined and for all unit vectors $e, \tilde{e} \in \mathbb{R}^2$, one has $L_{2m,e}^\varphi = L_{2m,\tilde{e}}^\varphi$. We denote by L_{2m}^φ this common quantity.

PROOF. The fact that $L^{\varphi}_{2m,e}$ is well-defined can be easily checked using (60). As for the second assertion, introduce the rotation A which sends e to $\tilde{e}$ and then use the isometric change of variables $\xi = A^*\tilde{\xi}$, $\eta = A^*\tilde{\eta}$, so as to turn $L^{\varphi}_{2m,e}$ into $L^{\varphi}_{2m,\tilde{e}}$.

$\square$

We will also make use of the following uniform estimate for $g^h_{\mathbf{i},t}$, which (as in the proof of Lemma 3.8) can be easily derived from the bound (52):

LEMMA 4.7.   *Fix $m \geq 1$, and recall that for every $t > 0$ and every $\mathbf{i} \in \{1,2\}^{2m}$, $g_{\mathbf{i},t}$ is defined by (41). Then there exists a constant c_m and a small $\varepsilon > 0$ such that for every $h \in \mathbb{R}^2$,*

$$\sup_{t\in[0,1],\mathbf{i}\in\{1,2\}^{2m}} \|g^h_{\mathbf{i},t}\|^2_{L^2([0,t]^{2m})} \leq c_m\,|h|^{2+\varepsilon}.$$

We can now compute the correct order of $\mathbf{E}[X^{2m,h}_t X^{2m,h}_s]$ as follows:

PROPOSITION 4.8.   *Fix $m \geq 1$. Then for all $0 \leq s \leq t \leq 1$, it holds that*

$$\lim_{h\to 0} |h|^{-2}\mathbf{E}\big[X^{2m,h}_t X^{2m,h}_s\big] = \sigma^2_m s \quad with \quad \sigma^2_m = \frac{c\,L^{\varphi}_{2m}}{(2m-2)!},$$

where c is a universal constant, where φ is defined for every $(x,y) \in \mathbb{R}^2$ by

$$\varphi(x,y) := \int_0^\infty \frac{du}{u^3}\{1 - \cos(ux)\}\{1 - \cos(uy)\},$$

and where we recall that L^{φ}_{2m} has been introduced at relation (61).

PROOF. Recall that

$$\mathbf{E}\big[X^{2m,h}_t X^{2m,h}_s\big] = \frac{c}{(2m)!} \sum_{\mathbf{i}\in\{1,2\}^{2m}} \langle (f_{\mathbf{i}}+g_{\mathbf{i},t})\cdot\mathbf{1}_{[0,t]^{2m}}, (f_{\mathbf{i}}+g_{\mathbf{i},s})\cdot\mathbf{1}_{[0,s]^{2m}}\rangle_{L^2(\mathbb{R}^{2m})}$$

and thanks to Lemma 4.7, we only have to focus on the sum of the terms

$$\mathcal{A}^{\mathbf{i}}_{s,t} := \langle f_{\mathbf{i}}\cdot\mathbf{1}_{[0,t]^{2m}}, f_{\mathbf{i}}\cdot\mathbf{1}_{[0,s]^{2m}}\rangle_{L^2(\mathbb{R}^{2m})}.$$

An integration over the simplex gives

$$\mathcal{A}^{\mathbf{i}}_{s,t} = \frac{(2m)!}{(2m-2)!}\int_{\Delta^2_s} \Phi_{\mathbf{i},h}(t_{2m},t_1)^2 (t_{2m}-t_1)^{2m-2}\,dt_1 dt_{2m}.$$

The change of variables $t_{2m} - t_1 = \tau$ and $t_1 = \sigma$ easily leads us to

$$\mathcal{A}_{s,t}^{\mathbf{i}} = \frac{(2m)!}{(2m-2)!} \int_0^s (s-\tau) \Phi_{\mathbf{i}, \frac{h}{\tau^{1/2}}}(1,0)^2 \, d\tau.$$

Setting $e_h \equiv \frac{h}{|h|}$, the change of variable $u = |h|/\tau^{1/2}$ now gives

$$\mathcal{A}_{s,t}^{\mathbf{i}} = \frac{2(2m)!}{(2m-2)!} |h|^2 \int_{|h|/s^{1/2}}^{\infty} u^{-3} \left(s - \frac{|h|^2}{u^2} \right) \Phi_{\mathbf{i}, u e_h}(1,0)^2 \, du.$$

By using (52), one can check that $|h|^2 \int_{|h|/s^{1/2}}^{\infty} u^{-5} \Phi_{\mathbf{i}, u e_h}(1,0)^2 \, du \to 0$ as $h \to 0$, so that the main contribution will come from the terms

$$\hat{\mathcal{A}}_{s,t}^{\mathbf{i}} := \frac{2(2m)!}{(2m-2)!} |h|^2 s \int_{|h|/s^{1/2}}^{\infty} u^{-3} \Phi_{\mathbf{i}, u e_h}(1,0)^2 \, du.$$

Now write

$$\Phi_{\mathbf{i}, u e_h}(1,0)^2 = \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^2} d\eta \, \frac{\prod_{k=1}^{2m} \xi_{i_k} \eta_{i_k}}{|\xi|^4 |\eta|^4} \{1 - \cos(u\langle e_h, \xi \rangle)\}\{1 - \cos(u\langle e_h, \eta \rangle)\} e^{-\frac{1}{2}(|\xi|^2 + |\eta|^2)}$$

and observe that $\sum_{\mathbf{i} \in \{1,2\}^{2m}} \prod_{k=1}^{2m} \xi_{i_k} \eta_{i_k} = \langle \xi, \eta \rangle^{2m}$. Thus, thanks to Lemma 4.6 and using Fubini theorem, we deduce that

$$|h|^{-2} \sum_{\mathbf{i} \in \{1,2\}^{2m}} \hat{\mathcal{A}}_{s,t}^{\mathbf{i}} = \frac{2(2m)!}{(2m-2)!} s \cdot L_{2m}^{\varphi_h},$$

where φ_h is defined as

$$\varphi_h(x,y) := \int_{|h|/s^{1/2}}^{\infty} \frac{\{1 - \cos(ux)\}\{1 - \cos(uy)\}}{u^3} \, du.$$

Finally, the convergence of $L_{2m}^{\varphi_h}$ towards L_{2m}^{φ} easily follows from the fact that φ satisfies relation (60) for some small $\varepsilon > 0$, and this achieves the proof. □

4.4. *Contractions.* We now turn to the contractions estimation for the functions f_h, g_h, where our 2-dimensional contractions are defined by (56). The following is of course an analog of Proposition 3.10 in our 2-d setting:

PROPOSITION 4.9. *For every $r \in \{1, \ldots, n-1\}$, one has*

$$\|(f. \, \mathbf{1}_{[0,t]^n} + g_{.,t}) \otimes_r (f. \, \mathbf{1}_{[0,t]^n} + g_{.,t})\|^2$$
$$= \sum_{\mathbf{i} \in \{1,2\}^{2n-2r}} \|((f. \, \mathbf{1}_{[0,t]^n} + g_{.,t}) \otimes_r (f. \, \mathbf{1}_{[0,t]^n} + g_{.,t}))_{\mathbf{i}}\|_{L^2([0,t]^{2n-2r})}^2 = o(|h|^4).$$

PROOF. Thanks to Lemma 4.7, it suffices to focus on the sum

$$\sum_{\mathbf{k}^1,\mathbf{k}^2\in\{1,2\}^{n-r}}\|((f.\ \mathbf{1}_{[0,t]^n})\otimes_r(f.\ \mathbf{1}_{[0,t]^n}))_{(\mathbf{k}^1,\mathbf{k}^2)}\|^2_{L^2([0,t]^{2n-2r})}.$$

Assume first that $2m\geq 4$ and $2\leq r\leq 2m-2$. Then we can follow the lines of the proof of Proposition 3.10 and deduce that

$$\sum_{\mathbf{k}^1,\mathbf{k}^2\in\{1,2\}^{n-r}}\|((f.\ \mathbf{1}_{[0,t]^n})\otimes_r(f.\ \mathbf{1}_{[0,t]^n}))_{(\mathbf{k}^1,\mathbf{k}^2)}\|^2_{L^2([0,t]^{2n-2r})}$$

$$=\ c_m\sum_{\substack{\mathbf{k}^1,\mathbf{k}^2\in\{1,2\}^{n-r}\\\mathbf{l}^1,\mathbf{l}^2\in\{1,2\}^r}}\int_{(\mathcal{S}_t^2)^2}\int_{(\mathcal{S}_t^2)^2}\prod_{i,j=1}^2\Phi_{(\mathbf{k}^i,\mathbf{l}^j),h}(\max(\sigma_2^i,\tau_2^j),\min(\sigma_1^i,\tau_1^j))$$

$$\times\prod_{k=1}^2(\sigma_2^k-\sigma_1^k)^{r-2}(\tau_2^k-\tau_1^k)^{n-r-2}\,d\sigma_1^k d\sigma_2^k d\tau_1^k d\tau_2^k.$$

Now, plugging the bound (52) (uniformly over $(\mathbf{k}^i,\mathbf{l}^j)$) into the latter expression yields, similarly to (49): for any small $\varepsilon>0$,

$$\sum_{\mathbf{k}^1,\mathbf{k}^2\in\{1,2\}^{n-r}}\|((f.\ \mathbf{1}_{[0,t]^n})\otimes_r(f.\ \mathbf{1}_{[0,t]^n}))_{(\mathbf{k}^1,\mathbf{k}^2)}\|^2_{L^2([0,t]^{2n-2r})}\leq c_m\,|h|^{4+4\varepsilon}\,\mathcal{J}_\varepsilon,$$

with

$$\mathcal{J}_\varepsilon:=\int_{(\Delta_1^2)^2}\int_{(\Delta_1^2)^2}\prod_{i,j=1}^2\big(\max(\sigma_2^i,\tau_2^j)-\min(\sigma_1^i,\tau_1^j)\big)^{-3/2-\varepsilon/2}\prod_{k=1}^2 d\sigma_1^k d\sigma_2^k d\tau_1^k d\tau_2^k.$$

By Lemma 5.1, we know that this integral is finite for $\varepsilon>0$ small enough, which achieves the proof of the proposition in the case $(2m\geq 4,\ 2\leq r\leq 2m-2)$.

The two situations $(2m\geq 4,\ r\in\{1,2m-1\})$ and $(2m=2,\ r=1)$ can also be handled with the same arguments as in the proof of Proposition 3.10 (with the help of Lemma 5.1 as well). Details are left to the reader. □

Like in Section 3.4, by combining Propositions 4.8 and 4.9 we end up with the following convergence in law result for the finite-dimensional distributions of $X^{2m,h}$:

PROPOSITION 4.10.   *Taking up the above notations, consider $t_1, \ldots, t_d \in [0,1]$ and $m \geq 1$. Then as $h \to 0$ we have*

$$\frac{1}{|h|}(X^{2m,h}_{t_1}, \ldots, X^{2m,h}_{t_d}) \xrightarrow{(d)} \sigma_m \mathcal{N}(0,\Gamma) \quad \text{where} \quad \sigma_m^2 = \frac{c\, L^{\varphi}_{2m}}{(2m-2)!}$$

and $\mathcal{N}(0,\Gamma)$ is the centered Gaussian law in $\mathbb{R}^d$ with covariance matrix $\Gamma(i,j) = \min(t_i, t_j)$. Recall that the quantity L^{φ}_{2m} has been defined in Proposition 4.8.

Let us briefly check point *(iii)* of Theorem 1.3, i.e., the divergence of the series of variances, as it is less obvious than in the 1-d case.

PROPOSITION 4.11.   *With the notations of Proposition 4.10, it holds that $\sum_{m=1}^{\infty} \sigma_m^2 = \infty$.*

PROOF. One has

$$
\begin{aligned}
\sum_{m=1}^{\infty} \sigma_m^2 &= c \sum_{m=1}^{\infty} \frac{L^{\varphi}_{2m}}{(2m-2)!} \\
&= c \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^2} d\eta \, \frac{\langle \xi, \eta \rangle^2}{|\xi|^4 |\eta|^4} \{ e^{\langle \xi, \eta \rangle} + e^{-\langle \xi, \eta \rangle} \} \varphi(\xi_1, \eta_1) e^{-\frac{1}{2}(|\xi|^2 + |\eta|^2)} \\
&\geq c \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^2} d\eta \, \frac{\langle \xi, \eta \rangle^2}{|\xi|^4 |\eta|^4} \varphi(\xi_1, \eta_1) e^{-\frac{1}{2}|\xi - \eta|^2} \\
&\geq c \int_{[R,\infty)^2} d\xi \int_{B_\xi} d\eta \, \frac{\langle \xi, \eta \rangle^2}{|\xi|^4 |\eta|^4} \varphi(\xi_1, \eta_1)
\end{aligned}
$$

(62)

for every $R > 0$ and where the notation B_ξ refers to the unit ball around ξ. Now observe that for R large enough, $\xi \in [R,\infty)^2$ and $\eta \in B_\xi$, one has

$$\varphi(\xi_1, \eta_1) = \xi_1^2 \cdot \varphi\Big(1, \frac{\eta_1}{\xi_1}\Big) \geq \xi_1^2 \cdot c_\varphi \quad \text{with} \quad c_\varphi = \inf_{\frac{1}{2} \leq x \leq 2} \varphi(1, x) > 0,$$

and also $\langle \xi, \eta \rangle^2 \geq \frac{1}{2} |\xi|^4$. Therefore, going back to (62), one has for R large enough and a suitable (finite) $\tilde{R}$,

$$\sum_{m=1}^{\infty} \sigma_m^2 \geq c \int_{[R,\infty)^2} \frac{\xi^2}{|\xi|^4}\, d\xi \geq c \int_{\tilde{R}} \frac{dr}{r},$$

which achieves the proof. $\square$

4.5. *Tightness.* In order to complete the proof of Theorem 1.3, we are now left with the tightness property for the family of processes $\{h^{-1}X_t^{2m,h},\ t \in [0,1]\}$. The following proposition is thus the equivalent of Proposition 3.12 in our 2-d context:

PROPOSITION 4.12. *Fix $m \geq 1$. Then:*

(i) *There exists $\lambda > 0$ and a constant c_m such that for all $0 \leq s \leq t \leq 1$,*

$$(63) \qquad \sup_{|h|\in(0,1)} \frac{1}{|h|^2}\mathbf{E}\big[|X_t^{2m,h} - X_s^{2m,h}|^2\big] \leq c_m\,|t-s|^\lambda.$$

(ii) *The family $\{X^{2m,h};\ |h| > 0\}$ is tight in $\mathcal{C}([0,1])$.*

PROOF. We use the same arguments as in the proof of Proposition 3.12. First, observe that

$$\mathbf{E}\big[|X_t^{2m,h} - X_s^{2m,h}|^2\big] \leq c_m \sum_{\mathbf{i}\in\{1,2\}^{2m}} \big\{A_{s,t}^{\mathbf{i}} + B_{s,t}^{\mathbf{i}} + \|g_{\mathbf{i},t} - g_{\mathbf{i},s}\|^2_{L^2([0,s]^{2m})}\big\},$$

where

$$A_{s,t}^{\mathbf{i}} := \int_{\substack{0<t_1<\ldots<t_{2m}\\ s<t_{2m}<t}} \Phi_{\mathbf{i}}(t_1,t_{2m})^2\,dt_1\cdots dt_{2m},$$

$$B_{s,t}^{\mathbf{i}} := \int_{\substack{0<t_1<\ldots<t_{2m}\\ s<t_{2m}<t}} g_{\mathbf{i},t}(t_1,\cdots,t_{2m})^2\,dt_1\cdots dt_{2m}.$$

By using both (53) and (54), it is readily checked that

$$\max_{\mathbf{i}\in\{1,2\}^{2m}} |h|^{-2}\|g_{\mathbf{i},t} - g_{\mathbf{i},s}\|^2_{L^2([0,s]^{2m})} \leq c\,|t-s|^\lambda$$

for some $\lambda > 0$. Then the treatments of $\sum_{\mathbf{i}\in\{1,2\}^{2m}} A_{s,t}^{\mathbf{i}}$ and $\sum_{\mathbf{i}\in\{1,2\}^{2m}} B_{s,t}^{\mathbf{i}}$, as well as the derivation of assertion *(ii)*, follow the lines of the proof of Proposition 3.12. For the sake of conciseness, we do not repeat the details of the procedure.

$\square$

5. Appendix: a technical lemma.   It only remains to prove the technical result on which the contraction computations of Propositions 3.10 and 4.9 rely.

LEMMA 5.1.   *The three following integrals*

$$(64) \qquad \int_{[0,1]^2} \int_{[0,1]^2} \prod_{i,j=1}^{2} \big(\max(\sigma^i, \tau^j) - \min(\sigma^i, \tau^j) \big)^{-3\delta} d\sigma^1 d\sigma^2 d\tau^1 d\tau^2,$$

$$(65) \qquad \int_{(\Delta_1^2)^2} \int_{[0,1]^2} \prod_{i,j=1}^{2} \big(\max(\sigma^i, \tau_2^j) - \min(\sigma^i, \tau_1^j) \big)^{-5\delta} d\sigma^1 d\sigma^2 \prod_{k=1}^{2} d\tau_1^k d\tau_2^k,$$

and

$$(66) \qquad \int_{(\Delta_1^2)^2} \int_{(\Delta_1^2)^2} \prod_{i,j=1}^{2} \big(\max(\sigma_2^i, \tau_2^j) - \min(\sigma_1^i, \tau_1^j) \big)^{-7\delta} \prod_{k=1}^{2} d\sigma_1^k d\sigma_2^k d\tau_1^k d\tau_2^k$$

are convergent if and only if $\delta < 1/4$.

We only focus on (66), since (64) and (65) can be treated with similar arguments (see Remark 5.3 at the end of the proof). In order to ease notations, we shall also change our time indices and set $(\sigma_1^1, \sigma_2^1) = (x_1, x_5)$, $(\sigma_1^2, \sigma_2^2) = (x_2, x_6)$, $(\tau_1^1, \tau_2^1) = (x_3, x_7)$, $(\tau_1^2, \tau_2^2) = (x_4, x_8)$. Our integral of interest can thus be written as

$$(67) \quad I_\alpha := \int_D [(x_7 \vee x_5) - (x_3 \wedge x_1)]^{-\alpha} [(x_8 \vee x_5) - (x_4 \wedge x_1)]^{-\alpha}$$
$$\times [(x_7 \vee x_6) - (x_3 \wedge x_2)]^{-\alpha} [(x_8 \vee x_6) - (x_4 \wedge x_2)]^{-\alpha} \, dx,$$

where $D = \{x \in [0,1]^8 : x_i < x_{4+i}, \ 1 \le i \le 4\}$. and $\alpha < 7/4$.

The necessity of the condition $\alpha < 7/4$ for the convergence of (67) stems from the following fact: observe that if

$$\mathcal{S} := \{x \in [0,1]^4 : \ 0 < x_1 < x_5 < x_2 < x_6 < x_3 < x_7 < x_4 < x_8 < 1\},$$

one has

$$
\begin{aligned}
I_{7/4} &\ge \int_{\mathcal{S}} (x_7 - x_1)^{-7/4} (x_8 - x_1)^{-7/4} (x_7 - x_2)^{-7/4} (x_8 - x_2)^{-7/4} dx dy \\
&\ge c \int_{[0,1]^3} (u_1 + u_2)^{-7/4} (u_1 + u_2 + u_3)^{-7/4} u_2^{-7/4} (u_2 + u_3)^{-7/4} u_1 u_2^2 u_3 \, du_1 du_2 du_3 \\
&\ge c \int_0^1 \frac{dr}{r},
\end{aligned}
$$

by using spherical coordinates.

In order to prove the convergence of I_α when $\alpha < 7/4$, we propose to rely on some block-type representation of the integral, described as follows. First, given $x \in D$, denote $J_1 := [x_3 \wedge x_1, x_7 \vee x_5]$, $J_2 := [x_4 \wedge x_1, x_8 \vee x_5]$, $J_3 := [x_3 \wedge x_2, x_7 \vee x_6]$, $J_4 := [x_4 \wedge x_2, x_8 \vee x_6]$, so that

$$I_\alpha = \int_D \prod_{i=1}^{4} \ell(J_i)^{-\alpha} \quad \text{where} \quad \ell([a,b]) = b - a.$$

Now and for the rest of the proof, we fix a generic permutation $\sigma \in \mathfrak{S}_8$ and consider the simplex $\mathcal{S}^\sigma$ generated by σ, i.e., $\mathcal{S}^\sigma := \{x \in [0,1]^8 : x_{\sigma(1)} < \ldots < x_{\sigma(8)}\}$, assuming that $\mathcal{S}^\sigma \subset D$. If $J_i = [x_{\sigma(m_i)}, x_{\sigma(n_i)}]$ on $\mathcal{S}^\sigma$ (for $m_i < n_i \in \{1, \ldots, 8\}$ depending on σ as well), we introduce the block $B_i^\sigma := \{m_i, m_i + 1, \ldots, n_i\}$ and set $\mathcal{B}^\sigma := \{B_1^\sigma, \ldots, B_4^\sigma\}$. Then, using an elementary change of variables, it is readily checked that

$$I_{\alpha,\sigma} := \int_{\mathcal{S}^\sigma} \prod_{i=1}^{4} \ell(J_i)^{-\alpha} = \int_{\mathcal{S}^\sigma} \prod_{i=1}^{4} (x_{\sigma(n_i)} - x_{\sigma(m_i)})^{-\alpha} = I_{\alpha,\mathcal{B}^\sigma},$$

where we have used the following general notation:

NOTATION 5.2. *Given* $B_i := \{m_i, m_i + 1, \ldots, n_i\}$ *(i = 1, ..., 4) with* $m_i < n_i \in \{1, \ldots, 8\}$ *and* $\mathcal{B} := \{B_1, \ldots, B_4\}$, *we set*

$$I_{\alpha,\mathcal{B}} := \int_{0 < x_1 < \ldots < x_8 < 1} \prod_{i=1}^{4} (x_{n_i} - x_{m_i})^{-\alpha} \quad \in [0, \infty].$$

Of course, $I_\alpha = \sum_{\sigma : \mathcal{S}^\sigma \subset D} I_{\alpha,\sigma} = \sum_{\sigma : \mathcal{S}^\sigma \subset D} I_{\alpha,\mathcal{B}^\sigma}$. Our key argument to prove that $I_{\alpha,\mathcal{B}^\sigma} < \infty$ for every $\sigma \in \mathfrak{S}_8$ and $\alpha < \frac{7}{4}$ lies in the following three basic observations regarding the four blocks B_i^σ composing $\mathcal{B}^\sigma$:

(i) $\text{Card}(B_i^\sigma) \geq 4$ (J_i involves the min/max over four points);

(ii) $\text{Card}(B_i^\sigma \cup B_j^\sigma) \geq 6$ if $i \neq j$ ($J_i \cup J_j$ involves the min/max over at least six points);

(iii) Each of the extremum points 1 and 8 appears exactly twice in $\mathcal{B}^\sigma$. Indeed, on $\mathcal{S}^\sigma$, the minimum $x_{\sigma(1)}$ (resp. maximum $x_{\sigma(8)}$) appears exactly twice as a left (resp. right) bound in $J_1, \ldots, J_4$.

Let us now discriminate the possible situations for $\mathcal{B}^\sigma$ according to this last condition *(iii)* (see Figure 1 for a representation in each case):

Case 1: 1 and 8 never appear in the same block $B_1^\sigma, \ldots, B_4^\sigma$. Then, by focusing on the possibilities for the two blocks with left-hand side 1 (resp. the two blocks with right-hand side 8), and given the above constraints *(i)-(ii)*, we end up with $I_{\alpha,\mathcal{B}^\sigma} \leq I_{\alpha,\mathcal{B}_0}$ where

$$\mathcal{B}_0 := \{\{1, \ldots, 4\}, \{1, \ldots, 6\}, \{5, \ldots, 8\}, \{3, \ldots, 8\}\}.$$

Case 2: 1 and 8 appear once and only once in a same block (and so each of them appears once 'alone' in another block). Then it remains to pick one block over the points $\{2, \ldots, 7\}$, and given the constraints *(i)-(ii)* on this block, we can easily conclude that there exists $k \in \{1, 2\}$ such that $I_{\alpha,\mathcal{B}^\sigma} \leq I_{\alpha,\mathcal{B}_k}$ where

$$\mathcal{B}_1 := \{\{1, \ldots, 8\}, \{1, \ldots, 4\}, \{5, \ldots, 8\}, \{2, \ldots, 5\}\},$$

$$\mathcal{B}_2 := \{\{1, \ldots, 8\}, \{1, \ldots, 4\}, \{5, \ldots, 8\}, \{3, \ldots, 6\}\}.$$

Case 3: 1 and 8 appear twice in a same block (necessarily $\{1, \ldots, 8\}$). Then we have to pick two blocks over the points $\{2, \ldots, 7\}$, and given the constraints *(i)-(ii)* on these two blocks (note for instance that, given *(ii)*, 2 and 7 are necessarily involved in the union of these blocks), we can easily conclude that there exists $k \in \{1, 2\}$ such that $I_{\alpha,\mathcal{B}^\sigma} \leq I_{\alpha,\mathcal{B}_{2+k}}$, where

$$\mathcal{B}_3 := \{\{1, \ldots, 8\}, \{1, \ldots, 8\}, \{2, \ldots, 5\}, \{4, \ldots, 7\}\},$$

$$\mathcal{B}_4 := \{\{1, \ldots, 8\}, \{1, \ldots, 8\}, \{2, \ldots, 7\}, \{3, \ldots, 6\}\}.$$

As a consequence of this reasoning, the problem is now reduced to the sole consideration of the five 'extremal' integrals $I_{\alpha,\mathcal{B}_k}$ ($k \in \{0, \ldots, 4\}$), which can be very easily done with basic estimates. For instance, if $\alpha = \frac{7}{4} - \varepsilon$ with $\varepsilon > 0$, one has

$$
\begin{aligned}
I_{\alpha,\mathcal{B}_0} &= \int_{0 < x_1 < \cdots < x_8 < 1} dx \, (x_4 - x_1)^{-\alpha} (x_6 - x_1)^{-\alpha} (x_8 - x_5)^{-\alpha} (x_8 - x_3)^{-\alpha} \\
&= c \int_{[0,1]^5} du \, (u_1 + u_2)^{-\alpha} (u_1 + \cdots + u_4)^{-\alpha} (u_4 + u_5)^{-\alpha} (u_2 + \cdots + u_5)^{-\alpha} u_1 u_5 \\
&\leq c \int_{[0,1]^5} du \, u_1^{-1+\varepsilon} u_5^{-1+\varepsilon} u_2^{-1+\frac{2}{3}\varepsilon} u_3^{-1+\frac{2}{3}\varepsilon} u_4^{-1+\frac{2}{3}\varepsilon} \quad < \quad \infty,
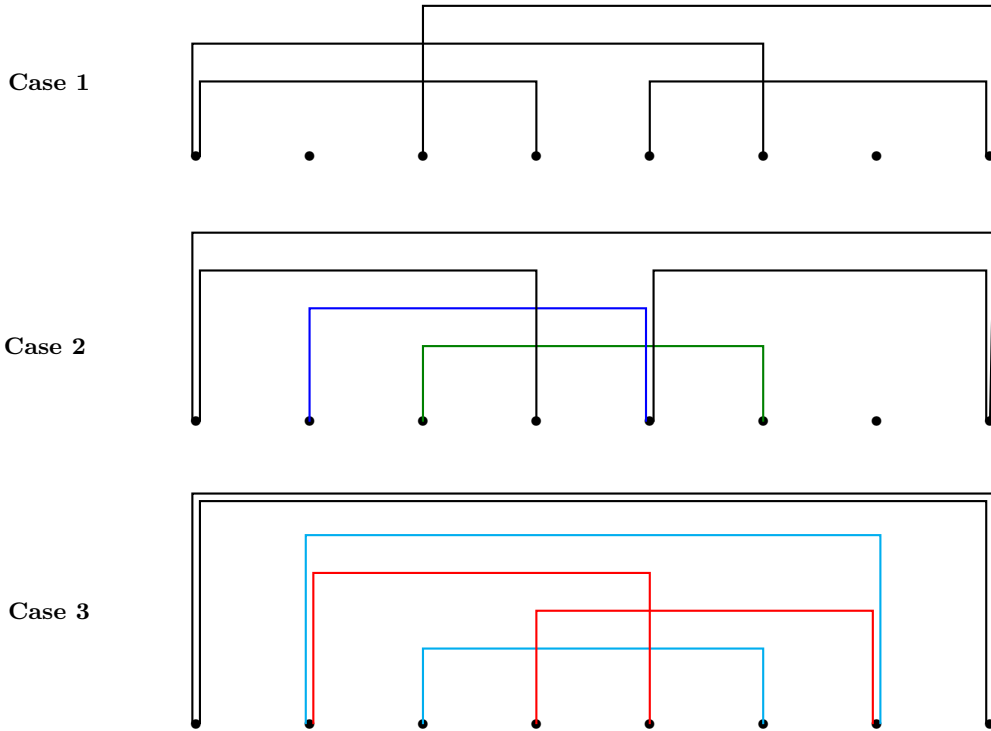\end{aligned}
$$

FIG 1. *Representation of the 'extremal' situations in each case, i.e. the $\mathcal{B}_k$ ($k \in \{0, \ldots, 4\}$). Each line connects the extremities of a block in $\mathcal{B}_k$. In Case 2 (resp. Case 3), the black lines are the ones common to $\mathcal{B}_1$ and $\mathcal{B}_2$ (resp. $\mathcal{B}_3$ and $\mathcal{B}_3$).*

where we have used the elementary bounds

$$(u_1+u_2)^{-\alpha} \leq u_1^{-\alpha} \quad , \quad (u_4+u_5)^{-\alpha} \leq u_5^{-\alpha} \quad , \quad (u_1+\ldots+u_4)^{-\alpha} \leq u_1^{-\alpha+3\kappa} u_2^{-\kappa} u_3^{-\kappa} u_4^{-\kappa}$$

with $\kappa := \frac{1}{2} - \frac{\varepsilon}{3}$.

REMARK 5.3.   This reduction of the problem, based on a block representation of the integral, can be easily adapted to prove the convergence of (64) (resp. (65)), by working with blocks in $\{1, \ldots, 4\}$ (resp. $\{1, \ldots, 6\}$) made of at least two (resp. three) elements. Thus, for relation (64) (resp. (65)), one can check that the situation reduces to the sole consideration of two (resp. three) easy-to-handle integrals on specific simplexes.

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