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BOUNDEDNESS FOR SOME SPECIAL FAMILIES OF EMBEDDED MANIFOLDS

PALTIN IONESCU AND MATEI TOMA

ABSTRACT. We are studying boundedness and classification problems for some special r-folds embedded in $\mathbb{P}^{2r}$ and $\mathbb{P}^{2r-1}$ (e.g. Fano fibrations over curves, scrolls over a surface).

It is conjectured that the Hilbert scheme of non-general type r-dimensional manifolds embedded in $\mathbb{P}^N$ for $N \leqslant 2r$ has only a finite number of components, or equivalently, that the degree of such manifolds is bounded. This was proved by Ellingsrud and Peskine for $r = 2$, $N = 4$ (cf. [4]), and subsequently in [3] for $r = 3$, $N = 5$. The proofs seem to depend on the fact that in these cases the codimension equals two. In [9] Schneider proves the conjecture for $N \leqslant 2r - 2$ by using the positivity of the Schur polynomials for the normal bundle and the fact that in this situation the canonical bundle is induced.

In this paper, we are using Schneider's method in order to prove boundedness for some special classes of embedded manifolds, e.g. Fano manifolds if $N = 2r$ (Proposition 1) and Fano fibrations over curves if $N = 2r-1$ (Proposition 2). A different argument shows boundedness for scrolls over a surface, in case $N = 2r-1$ (Proposition 4). We also give some existence results and a partial classification for the two special classes mentioned above when $N = 2r-1$ (Propositions 3, 4 and 5).

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Let X be an r-dimensional manifold embedded in $\mathbb{P}^N(\mathbb{C})$ with degree d.

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If K_X, H_X denote the canonical and the hyperplane class on X respectively, and a is a nonnegative integer we shall say that X *is in class* C_a if

$$K_X \cdot H_X^{r-1} \leqslant a \cdot d,$$

(or equivalently, if $2g-2 \leqslant (a+r-1)d$, where $g = \frac{1}{2}(K \cdot H^{r-1} + (r-1)H^r) + 1$ is the sectional genus of X).

PROPOSITION 1. *If X is in C_a, $X \subset \mathbb{P}^{2r}$, then*

$$d < (a+2r+1)^r.$$

In particular if X is an r-dimensional Fano manifold (i.e. $-K_X$ is ample) embedded in $\mathbb{P}^{2r}$ or if K_X is numerically trivial and $X \subset \mathbb{P}^{2r}$ then d is bounded by $(2r+1)^r$.

Proof. Let $N_{X/\mathbb{P}^N}$ be the normal bundle of X in $\mathbb{P}^N$. By the self-intersection formula (cf. [7]) we have

$$c_{N-r}(N_{X/\mathbb{P}^N}) = d \cdot H_X^{N-r}.$$

In our case

$$c_r(N_{X/\mathbb{P}^{2r}}) = d^2.$$

Since $N_{X/\mathbb{P}^N}$ is ample the Schur polynomials $(c_{k-1} \cdot c_1 - c_k)(N_{X/\mathbb{P}^N})$ are positive by the Fulton-Lazarsfeld theorem (cf. e.g. [5]) hence one gets

$$\begin{aligned} d^2 = c_r(N_{X/\mathbb{P}^{2r}}) &< c_1(N_{X/\mathbb{P}^{2r}})^r \\ &= (K_X + (2r+1)H_X)^r. \end{aligned}$$

We now apply the generalized Hodge index theorem ([2]) which asserts that for D nef and H ample on X and $j = 1, \ldots, r-1$ one has

$$(H^{j+1} \cdot D^{r-j-1})(H^{j-1} \cdot D^{r-j+1}) \leqslant (H^j \cdot D^{r-j})^2.$$

An easy consequence of it is that

$$D^r \leqslant \frac{(H^{r-1} \cdot D)^r}{(H^r)^{r-1}},$$

(cf. [2]).

Taking in our case $D = K_X + (2r+1)H_X$ and $H = H_X$ we get

$$\begin{aligned} (K_X + (2r+1)H_X)^r &\leqslant \frac{[H_X^{r-1} \cdot (K_X + (2r+1)H_X)]^r}{(H_X^r)^{r-1}} \\ &\leqslant \frac{(a+2r+1)^r d^r}{d^{r-1}} = (a+2r+1)^r d \end{aligned}$$

and the conclusion follows. □

REMARKS. 1. It is not difficult to give examples of r-dimensional Fano manifolds in $\mathbb{P}^{2r}$, e.g. complete intersections of type $(2, 2, \ldots, 2)$.

2. Let C denote the curve obtained by intersecting $X \subset \mathbb{P}^{2r}$ by an $(r+1)$-dimensional generic linear variety. If one assumes $h^1(\mathcal{O}_C(H)) = 0$ it follows $h^0(\mathcal{O}_C(H)) = d + 1 - g > 0$ so by the above proposition the degree of X is bounded.

The next manifolds we investigate are r-dimensional fibrations X over smooth curves, embedded in $\mathbb{P}^{2r-1}$. By Barth's theorem, [1], X is regular hence the base curve must be $\mathbb{P}^1$.

PROPOSITION 2. *If X is an r-dimensional manifold embedded in $\mathbb{P}^{2r-1}$ with degree d, admitting a fibering over $\mathbb{P}^1$ with a smooth fiber in class C_a, then*

$$d < (a+2r)^{r-1}.$$

In particular if $X \subset \mathbb{P}^{2r-1}$ is an r-dimensional Fano fibration over $\mathbb{P}^1$ then

$$d < (2r)^{r-1}.$$

Proof. Let F be a smooth fiber in class C_a of our fibration $X \longrightarrow \mathbb{P}^1$. The normal bundle sequence

$$0 \longrightarrow N_{F/X} \longrightarrow N_{F/\mathbb{P}^{2r-1}} \longrightarrow N_{X/\mathbb{P}^{2r-1}}|_F \longrightarrow 0$$

implies

$$c_i(N_{F/\mathbb{P}^{2r-1}}) = c_i(N_{X/\mathbb{P}^{2r-1}}|_F), \quad \text{for } 0 < i < r.$$

Making again use of the self-intersection formula for X, the positivity of the Schur polynomials and the generalized Hodge index theorem on F we obtain

$$\begin{aligned} d \cdot \deg F = d \cdot H_F^{r-1} &= c_{r-1}(N_{X/\mathbb{P}^{2r-1}}|_F) \\ &< c_1^{r-1}(N_{X/\mathbb{P}^{2r-1}}|_F) = c_1^{r-1}(N_{F/\mathbb{P}^{2r-1}}) \\ = (K_F + 2rH_F)^{r-1} &\leqslant \frac{[H_F^{r-2} \cdot (K_F + 2rH_F)]}{(H_F^{r-1})^{r-2}} \\ &\leqslant \frac{[(a+2r)\deg F]^{r-1}}{(\deg F)^{r-2}} = (a+2r)^{r-1} \deg F \end{aligned}$$

and the conclusion follows. □

We now wish to illustrate the preceding result by a class of examples.

An r-dimensional manifold embedded in $\mathbb{P}^N$ is said to be a *(degree f)-hyper surface fibration over* $\mathbb{P}^1$ if it admits a map $p : X \to \mathbb{P}^1$ such that any of its fibers is a hypersurface of degree f in some r-plane of $\mathbb{P}^N$.

We shall show the existence of such fibrations for $N = 2r-1$. Remark that any smooth fiber is in class C_f, but in this case one easily gets a precise relation connecting d, f and r. Indeed, letting F be a smooth fiber of p we have as in the proof of Proposition 2

$$\begin{aligned} df = dH_F^{r-1} &= c_{r-1}(N_{X/\mathbb{P}^{2r-1}}|_F) \\ = c_{r-1}(N_{F/\mathbb{P}^{2r-1}}) &= c_{r-1}(\mathcal{O}_F(f) \oplus \mathcal{O}_F(1)^{\oplus(r-1)}) = f\,[(r-1)f+1], \end{aligned}$$

hence $d = (r-1)f + 1$.

PROPOSITION 3. *For any integers $r \geqslant 2$, $f \geqslant 1$, there exist r-dimensional (degree f)-hypersurface fibrations X over $\mathbb{P}^1$ embedded in $\mathbb{P}^{2r-1}$.*

Proof. We may assume $f \geqslant 2$ since for $f = 1$ a solution is given by the Segre embedding $\mathbb{P}^{r-1} \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2r-1}$.

Let $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)}$ (we make no distinction between a vector bundle and its associated sheaf of sections), $Y = \mathbb{P}_{\mathbb{P}^1}(E)$ the projectivized bundle over $\mathbb{P}^1$, $\pi : Y \longrightarrow \mathbb{P}^1$ the projection, $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and $A = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$.

Our X will be a general member of $|fL + A|$. First notice that since $(fL + A)^{r+1} = f^{r+1}(r-1) + (r+1)f^r > 0$ it follows by Bertini's theorem that X is smooth and connected.

Let $\varphi_L : Y \to \mathbb{P}^{2r-1}$ be the morphism induced by $|L|$ (L is generated by sections). Since φ_L embeds the fibres of π as linear subspaces in $\mathbb{P}^{2r-1}$, $p := \pi|_X : X \longrightarrow \mathbb{P}^1$ will be a (degree f)-hypersurface fibration if we prove that $\varphi_L|_X : X \longrightarrow \mathbb{P}^{2r-1}$ is an embedding.

Let $E' = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)}$, $E'' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $S = \mathbb{P}_{\mathbb{P}^1}(E'')$. Then S is naturally embedded in Y and $\varphi_L(S)$ is a line ℓ in $\mathbb{P}^{2r-1}$. In fact $p_1 := \pi|_S$ and $p_2 := \varphi_L|_S$ are the two projections of $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, by general facts on projective bundles, $\varphi_L(Y)$ is a cone with vertex ℓ over $\varphi_L(\mathbb{P}_{\mathbb{P}^1}(E'))$ and $\varphi_L|_{Y\setminus S} : Y \setminus S \longrightarrow \mathbb{P}^{2r-1} \setminus \ell$ is an embedding.

So the fact that $\varphi_L|_X$ is an embedding remains to be checked only around S.

Let on S, $F_i := p_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $i = 1, 2$. Then $(fL + A)|_S = fF_2 + F_1$. Since $(fF_2 + F_1)^2 = 2f > 0$ and $(fF_2 + F_1)F_2 = 1$ we may choose a smooth X in $|fL + A|$ such that its intersection with S is smooth again and $\varphi_L|_{X\cap S} : X \cap S \to \ell$ is an isomorphism.

It follows that $\varphi_L|_X$ is injective. In order to see that it is also immersive we shall verify that the only tangent vectors to Y killed by $d\varphi_L$ are tangent to the fibers of $S \to \ell$. (Such vectors cannot be tangent to X too).

Consider an open subset U of $\mathbb{P}^1$ over which E is trivial and coordinates $(t, (x_0 : \cdots : x_r))$ on $Y_U \cong U \times \mathbb{P}^r$. Let $S_U \cong U \times \mathbb{P}_1 = \{x_2 = \cdots = x_r = 0\} \subset Y_U$. With respect to the chosen coordinates φ_L is given by $(t, (x_0 : \cdots : x_r)) \mapsto (x_0 : \cdots : x_r : tx_2 : tx_3 : \cdots : tx_r)$. Since we are interested to compute everything around S, we may further assume $x_0 = 1$ thus getting the map $(t, x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, tx_2, \cdots, tx_r)$. A direct computation of its Jacobian proves now our claim. □

The simplest class of Fano fibrations is the class of scrolls. $X \subset \mathbb{P}^N$ is called a *scroll* over an s-dimensional manifold B if there exists a surjective morphism $\pi : X \longrightarrow B$ whose fibres are $(r-s)$-dimensional linear subspaces of $\mathbb{P}^N$. Letting $E = \pi_*\mathcal{O}_X(H)$ we may identify X to $\mathbb{P}_B(E)$ and $\mathcal{O}_X(H)$ to the tautological line bundle. It is an easy exercise to see that an r-dimensional scroll in $\mathbb{P}^{2r-1}$ over a curve is necessarily isomorphic to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{r-1}$. Now we would like to investigate r-dimensional scrolls in $\mathbb{P}^{2r-1}$ over a two-dimensional base.

When $r = 3$ Ottaviani ([8]) gave a complete classification of such objects. The following proposition partly extends his results to all dimensions.

PROPOSITION 4. *If $X = \mathbb{P}_B(E) \subset \mathbb{P}^{2r-1}$ is an r-dimensional scroll of degree d*

over the smooth surface B, then:

$$d < \frac{r^6}{2}$$

and X belongs to one of the following five classes:

i) $r = 3$, *B is a cubic in $\mathbb{P}^3$, $d = 7$ (cf. [8]);*
ii) $r \geqslant 3$, *X is the isomorphic projection to $\mathbb{P}^{2r-1}$ of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{r-2} \subset \mathbb{P}^{3r-4}$, having degree $d = \frac{r(r-1)}{2}$;*
iii) $r \geqslant 3$, *$B \cong \mathbb{P}^2$, $d = \frac{r(r+1)}{2}$, X is the locus where a suitable morphism $u : \mathcal{O}^3_{\mathbb{P}^{2r-1}} \longrightarrow \mathcal{O}^{\oplus(r+1)}_{\mathbb{P}^{2r-1}}(1)$ drops rank;*
iv) $r \geqslant 3$, *B is a $K3$ surface, $d = r^2$;*
v) $r \geqslant 7$, *B is a regular surface with ample effective canonical divisor.*

REMARK. Actually we suspect that the last class never occurs. When $r = 3$, Ottaviani, [8], excludes class v) using codimension two arguments. When $r = 4$, R. Braun, (work in progress), rules it out by a divisibility argument, which inspired the proof below for the boundedness. The cases $r = 5$ and $r = 6$ may also be excluded by divisibility manipulations.

Proof. Keeping the above notations, let us consider the standard exact sequences:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2r-1}}(-H) \longrightarrow \mathcal{O}^{2r}_{\mathbb{P}^{2r-1}} \longrightarrow T_{\mathbb{P}^{2r-1}}(-H) \longrightarrow 0$$
$$0 \longrightarrow T_X(-H) \longrightarrow T_{\mathbb{P}^{2r-1}}(-H)|_X \longrightarrow N_{X/\mathbb{P}^{2r-1}}(-H) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \pi^* E^\vee \longrightarrow T_X(-H) \longrightarrow \pi^* T_B(-H) \longrightarrow 0.$$

One infers:

$$c(N')c(\pi^* E^\vee)c(\pi^* T_B(-H)) = 1,$$

where $N' := N_{X/\mathbb{P}^{2r-1}}(-H)$.

Hence $c(N') = c(\pi^* E^\vee)^{-1} c(\pi^* T_B(-H))^{-1}$.

Letting $c_1(\pi^* E^\vee) =: y_1$, $(c_1^2 - c_2)(\pi^* E^\vee) =: y_2$ we can immediately write down:

$$c(\pi^* E^\vee)^{-1} = 1 - y_1 + y_2.$$

We consider now the Segre classes of $\pi^* T_B(-H)$,

$$s(\pi^* T_B(-H)) := c(\pi^* T_B(-H))^{-1}.$$

Then

$$s_t(\pi^* T_B(-H))$$
$$= \binom{t+1}{1} H^t + \binom{t+1}{2} s_1(B) H^{t-1} + \binom{t+1}{3} s_2(B) H^{t-2},$$

cf. [5], p. 49. Denoting $c_1(B) = -s_1(B) =: x_1$, $(c_1^2 - c_2)(B) = s_2(B) =: x_2$, we can express

$$c_t(N') = \binom{t+1}{1} H^t - \left(\binom{t}{1} y_1 + \binom{t+1}{2} x_1\right) H^{t-1} + \left(\binom{t-1}{1} y_2 + \binom{t}{2} x_1 y_1 + \binom{t+1}{3} x_2\right) H^{t-2}.$$

We have a first equation

$$c_r(N') = 0. \tag{1}$$

Using the Hirsch-Leray formula

$$H^{r-1} + c_1(E^\vee)H^{r-2} + c_2(E^\vee)H^{r-3} = 0$$

we get

$$y_2 H^{r-2} = d, \quad x_1 y_1 H^{r-2} = -x_1 H^{r-1}, \quad y_1^2 H^{r-2} = -y_1 H^{r-1}. \tag{$*$}$$

Introducing all these in (1) one has

$$2d - (rx_1 + y_1)H^{r-1} + \frac{r^2-1}{6} x_2 H^{r-2} = 0. \tag{1'}$$

Next one notices that

$$c_{r-1}(N_{X/\mathbb{P}^{2r-1}}) = \sum_{t=0}^{r-1} c_t(N')H^{r-1-t},$$

hence

$$c_{r-1}(N_{X/\mathbb{P}^{2r-1}}) = \binom{r+1}{2} H^{r-1} - \left(\binom{r}{2} y_1 + \binom{r+1}{3} x_1\right) H^{r-2} + \left(\binom{r-1}{2} y_2 + \binom{r}{3} x_1 y_1 + (r+1) x_2\right) H^{r-3},$$

by applying the formula $\sum_{t=0}^{s} \binom{t+k}{k} = \binom{s+k+1}{k+1}$.

Multiplying the self-intersection formula,

$$c_{r-1}(N_{X/\mathbb{P}^{2r-1}}) = d \cdot H^{r-1},$$

by H and replacing the entities from $(*)$ we get a second equation

$$d^2 = (r^2 - r + 1)d - \binom{r}{2} y_1 H^{r-1} - \frac{r(r-1)(2r-1)}{6} x_1 H^{r-1} + \binom{r+1}{4} x_2 H^{r-2} \tag{2}$$

The third needed equation is obtained by multiplying the self-intersection formula by y_1:

$$\left[(d-r^2)y_1 - \binom{r+1}{3} x_1\right] \cdot H^{r-1} = 0. \tag{3}$$

If we let $L := \det E$ and $K := K_B$, it follows from (1′), (2), (3):

$$L^2 = \frac{2(r^2-1)d(2d-r^2-2)}{2(r^2+2)d - r^2(r^2+5)},$$

$$K^2 = \left[\frac{d-r^2}{\binom{r+1}{3}}\right]^2 \cdot L^2,$$

$$K \cdot L = \frac{d-r^2}{\binom{r+1}{3}} \cdot L^2,$$

$$\chi(\mathcal{O}_B) = \frac{1}{12}(c_1^2 + c_2)(B)$$

$$= 1 + \frac{[2d - r(r+1)][2d - r(r-1)][6d^2 - 6(r^2+1)d + r^2(r^2+5)]}{r^2(r^2-1)[2(r^2+2)d - r^2(r^2+5)]}.$$

We now may write $L^2 = \frac{P(d)}{Q(d)}$, where

$$P := (r^2-1)d(2d-r^2-2),$$

$$Q := (r^2+2)d - \frac{1}{2}r^2(r^2+5)$$

seen as polynomials in d. Since L^2 is an integer, $Q(d)$ divides $\frac{1}{2}r^2(r^2-4)(r^2-1)(r^2+5)$ which is the remainder of the division of $(r^2+2)^2 P$ by Q. Assuming $Q(d) > 0$ we have

$$Q(d) \leqslant \frac{1}{2}r^2(r^2-4)(r^2-1)(r^2+5)$$

which gives the desired bound

$$d < \frac{r^6}{2}.$$

If $Q(d) \leqslant 0$ one has an even better bound.

Now let us remark that

$$K^2 \cdot L^2 = (KL)^2,$$

by the preceding formulae.

The Hodge index theorem implies that K and L are numerically proportional. Since L is the determinant of an ample vector bundle, it is itself ample so we distinguish the following three cases:

a) $-K$ is ample, i.e. B is a Del Pezzo surface,

b) K is numerically trivial,

c) K is ample.

Next we show that case a) leads to one of the classes i), ii), iii), b) leads to class iv) and c) to v). Remark that, since X lies in $\mathbb{P}^{2r-1}$ it must be regular by Barth's theorem so B is also regular.

Assume now that $-K$ is ample. By the well-known classification of Del Pezzo surfaces we must have $1 \leqslant K^2 \leqslant 9$ and $K^2 = 9$ holds only if $B \cong \mathbb{P}^2$. Knowing that $\chi(\mathcal{O}_B) = 1$ we get from the above expression of $\chi(\mathcal{O}_B)$ the following equation:

$$(4) \qquad [2d - r(r+1)]\,[2d - r(r-1)]\,[6d^2 - 6(r^2+1)d + r^2(r^2+5)] = 0.$$

Two of its solutions are $d = \frac{r(r+1)}{2}$ and $d = \frac{r(r-1)}{2}$. For these values of d, $K^2 = 9$ so $B \cong \mathbb{P}^2$.

For $d = \frac{r(r+1)}{2}$ we are in case iii) of the proposition; we postpone the argument that X is a determinantal variety until the next proposition.

When $d = \frac{r(r-1)}{2}$ one verifies that $\deg E = \operatorname{rank} E = r - 1$. Since the restriction of E to any line of $\mathbb{P}^2$ is ample, this must be $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)}$. By [10], $E \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus(r-1)}$ and we are in case ii).

Still assuming that we are in case a) we may suppose that d satisfies the remaining degree two equation from (4). Putting $\lambda := K^2$, we consider the above expression of K^2 as an equation of degree four in d with parameter λ. The remainder of its division by $6d^2 - 6(r^2+1)d + r^2(r^2+5)$ is

$$-2(r^2-1)[(6-\lambda)r^2 + 2(12-\lambda)]d + (r^2-1)r^2(r^2+5)(8-\lambda),$$

hence $d = \frac{r^2(r^2+5)(8-\lambda)}{2[(6-\lambda)r^2+2(12-\lambda)]}$. Since this value satisfies $6d^2 - 6(r^2+1)d + r^2(r^2+5) = 0$, we get a quadratic equation in r^2. Giving to the parameter λ values from 1 to 9 we see that the only solutions are: $r = \lambda = 3$, $d = 7$, which is case i) of the proposition, and $\lambda = 9$, $r = 3$, $d = 3$ which is a special case of ii).

Let now K be numerically trivial. Since $L^2 > 0$ and $K^2 = 0$ we get from our formulas $d = r^2$ and $\chi(\mathcal{O}_B) = 2$. By surface classification B must be $K3$ and we are in case iv).

We come to the last case when K is ample. Since now $KL > 0$ we get $d > r^2$ and $\chi(\mathcal{O}_B) > 2$. Taking into account the remark following the proposition, the proof is complete. □

When $r = 3$ the existence of classes i) – iv) was proved in [8].

PROPOSITION 5. *The class* ii) *from Proposition 4 occurs for all* $r \geqslant 3$. *Any scroll in class* iii) *is the degeneracy locus of a morphism* $u\colon \mathcal{O}^3_{\mathbb{P}^{2r-1}} \longrightarrow \mathcal{O}^{\oplus(r+1)}_{\mathbb{P}^{2r-1}}(1)$ *and moreover, by taking* u *to be generic their existence follows for all* $r \geqslant 3$.

Proof. For ii) it is enough to see that the secant variety of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{r-2} \subset \mathbb{P}^{3r-4}$ is $(2r-1)$ -dimensional. We give the following (certainly classical) argument for the reader's convenience. The secant variety of the restricted scroll over any line in $\mathbb{P}^2$ is a $(2r-3)$-plane in $\mathbb{P}^{3r-4}$. But any secant to

$\mathbb{P}^2 \times \mathbb{P}^{r-2}$ is a secant to such a restricted scroll over a line. Hence by counting parameters, we get the assertion.

Let us prove now the existence for class iii). Consider a generic morphism

$$u : \mathcal{O}^3_{\mathbb{P}^{2r-1}} \longrightarrow \mathcal{O}^{\oplus(r+1)}_{\mathbb{P}^{2r-1}}(1).$$

By [6] the locus where u has rank $\leqslant 1$ (resp. $\leqslant 2$) is empty (resp. smooth connected of dimension r). Let X be the locus where u drops rank and $\mathcal{K} = \operatorname{Ker} u|_X$, which is a rank 1 subbundle of $\mathcal{O}^3_X$. Taking the associated projective bundles (this time lines instead of hyperplanes!) we get

$$X = \mathbb{P}(\mathcal{K}) \hookrightarrow \mathbb{P}(\mathcal{O}^3_X) = X \times \mathbb{P}^2$$

which gives a natural map $\pi : X \longrightarrow \mathbb{P}^2$ by composing with the second projection. By construction there are $r+1$ linear forms vanishing on each fiber of π. The genericity assumption on u implies that for all fibers of π the corresponding $r+1$ linear forms are independent. So $\pi : X \longrightarrow \mathbb{P}^2$ is a scroll and an easy application of Porteous' formula, (cf. [5]), gives $\deg X = \frac{r(r+1)}{2}$.

Conversely, consider $\pi : X = \mathbb{P}(E) \longrightarrow \mathbb{P}^2$ an r-dimensional scroll of degree $\frac{r(r+1)}{2}$ embedded in $\mathbb{P}^{2r-1}$. Letting $\ell \subset \mathbb{P}^2$ be any line and $D = \pi^{-1}(\ell)$, we first show that $D \subset \mathbb{P}^{2r-1}$ is nondegenerate and linearly normal. If D were contained in a hyperplane, the self-intersection formula would give

$$(\deg D - (r-1))(\deg D - r) = 0.$$

But in our case $\deg D = \deg E_{|\ell} = r+1$, by the formulas given in the proof of Proposition 4. So D is nondegenerate. Next, from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(H-D) \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{O}_D(H) \longrightarrow 0$$

it follows

$$2r \leqslant h^0(\mathcal{O}_X(H)) \leqslant h^0(\mathcal{O}_D(H)) = h^0(E_{|\ell}) = 2r.$$

So both D and X ar linearly normal.

Let us recall a classical determinantal description of D as a rational normal scroll. Fixing a point $P_0 \in \ell$, any hyperplane containing $\pi^{-1}(P_0)$ lies in an unique pencil of hyperplanes sweeping out D fiberwise; this corresponds to the natural map

$$H^0(\ell, \mathcal{O}(1)) \otimes H^0(\ell, E_{|\ell}(-1)) \rightarrow H^0(\ell, E_{|\ell}).$$

Thus, letting $F_0 := \pi^{-1}(P_0)$, $F_1 := \pi^{-1}(P_1)$ be two fibers and $s_{0,1}, \dots, s_{0,r+1}$ a basis of $H^0(D, \mathcal{O}(H-F_0)) \subset H^0(D, \mathcal{O}(H)) \cong H^0(\mathbb{P}^{2r-1}, \mathcal{O}(H))$, one obtains a basis $s_{1,1}, \dots, s_{1,r+1}$ for $H^0(D, \mathcal{O}(H-F_1)) \subset H^0(\mathbb{P}^{2r-1}, \mathcal{O}(H))$ by multiplying with a linear form vanishing at P_1. The classical fact is that D is precisely the degeneracy locus of the matrix of linear forms $(s_{ij})_{\substack{i=0,1;\\ j=1,\dots,r+1}}$.

This construction may be adapted to X as follows. Let P_0, P_1, P_2 be non-colinear points in $\mathbb{P}^2$, $s_{0,1}, \dots, s_{0,r+1}$ a basis for $H^0(\mathbb{P}^{2r-1}, \mathcal{I}_{F_0}(H)) \subset H^0(\mathbb{P}^{2r-1}, \mathcal{O}(H))$. We transport this basis to a basis $(s_{1,j})_{j=1,\dots,r+1}$ of $H^0(\mathbb{P}^{2r-1}, \mathcal{I}_{F_1}(H))$, (resp. $(s_{2,j})_{j=1,\dots,r+1}$ of $H^0(\mathbb{P}^{2r-1},\mathcal{I}_{F_2}(H))$) along the line P_0P_1 (resp. P_0P_2) by the above procedure. We claim that X is the degeneracy locus of the matrix of linear forms $s := (s_{ij})_{\substack{i=0,1,2;\\ j=1,\dots,r+1}}$.

The main point is that for any choice of points P_0,P_1, P_2 by transporting our basis successively along P_0P_1, P_1P_2, P_2P_0 we get the same basis up to a multiplicative factor. In order to see this, consider the natural morphism

$$\alpha : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathrm{Aut}(\mathbb{P}(H^0(\mathbb{P}^{2r-1},\mathcal{I}_{F_0}(H))))$$

obtained by fixing P_0 and associating to the pair (P_1, P_2) the automorphism induced by our successive transport. α has to be constant since it maps to an affine variety.

This remark enables one to see that any fiber of $\pi : X \longrightarrow \mathbb{P}^2$ appears as the common zero locus of the entries of a suitable linear combination of the three lines of our matrix s. In particular any 3×3 minor of s vanishes on X. If we let $\Delta_i(s) := \{x \in \mathbb{P}^{2r-1} \mid \mathrm{rank}\, s(x) \leqslant i\}$ we have shown that X lies in $\Delta_2(s)$.

Next we notice that $\Delta_1(s)$ is empty. Indeed, throwing out one line of s, say the first, $\Delta_1(s)$ is contained in the degeneracy locus of the remaining matrix which is a rational normal scroll over the line P_1P_2. Since the lines P_0P_1, P_1P_2, P_2P_0 have no common point, $\Delta_1(s)$ is empty. As in the existence proof above this ensures that $\Delta_2(s)$ is an r-dimensional scroll over $\mathbb{P}^2$ and so $X = \Delta_2(s)$. □

REMARKS. 1. The degeneracy locus of a generic morphism

$$\mathcal{O}_{\mathbb{P}^N}^{t+1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{\oplus(N-r+t)}(1)$$

is an r-dimensional scroll over $\mathbb{P}^t$ of degree $\binom{N-r+t}{t}$ if $N \geqslant 2r-1$ and $r \geqslant 2t-1$. The proof is the same as in the case $t = 2$, $N = 2r-1$.

2. Assuming that $X = \mathbb{P}(E) \subset \mathbb{P}^N$ is a scroll over $\mathbb{P}^t$ such that for any line ℓ in $\mathbb{P}^t$ the restricted scroll over ℓ is linearly normal and nondegenerate in $\mathbb{P}^N$, the preceding argument applies to give that X is the degeneracy locus of a morphism as in 1.

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PALTIN IONESCU, UNIVERSITY OF BUCHAREST, ACADEMIEI 14, 70109 BUCHAREST, ROMANIA.
E-mail address: pionescu@roearn.bitnet

MATEI TOMA, INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O.BOX 1–764, 70700 BUCHAREST, ROMANIA.
E-mail address: mtoma@roearn.bitnet