DISTANCES ON A MASURE

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Abstract. A masure (also known as an affine ordered hovel) $\mathcal{I}$ is a generalization of the Bruhat–Tits building that is associated with a split Kac–Moody group G over a nonarchimedean local field. This is a union of affine spaces called apartments. When G is a reductive group, $\mathcal{I}$ is a building and there is a G-invariant distance inducing a norm on each apartment. In this paper, we study distances on $\mathcal{I}$ inducing the affine topology on each apartment. We construct distances such that each element of G is a continuous automorphism of $\mathcal{I}$ and we study their properties (completeness, local compactness, ...).

Contents

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1. Introduction

If G is a split Kac–Moody group over a nonarchimedean local field, Stéphane Gaussent and Guy Rousseau introduced a space $\mathcal{I}$ on which G acts, and they called this set a "masure" (or an "affine ordered hovel"); see [GR08], [Rou17]. This construction generalizes the construction of the Bruhat–Tits building associated with a split reductive group over a field equipped with a nonarchimedean valuation made by François Bruhat and Jacques Tits; see [BT72] and [BT84]. A masure is an object similar to a building. It is a union of subsets called "apartments", each one having a structure of a finite dimensional real-affine space and an additional structure defined by hyperplanes (called walls) of this affine space. The group G acts transitively on the set of apartments. It induces affine maps on each apartment, sending walls on walls. We can also define sectors and retractions from $\mathcal{I}$ onto apartments with center a sector-germ, as in the case of Bruhat–Tits buildings. However there can be two points of $\mathcal{I}$ which do not belong to a common apartment. Studying $\mathcal{I}$ enables one to get information on G and this is one reason to study masures.

In this paper, we assume the valuation of the valued field to be discrete. Each Bruhat–Tits building BT associated with a split reductive group H over a field equipped with a discrete nonarchimedean valuation is equipped with a distance d such that H acts isometrically on BT and such that the restriction of d to each apartment is a euclidean distance. These distances are important tools in the study of buildings. We will show that we cannot equip masures which are not buildings with distances having these properties, but it seems natural to ask whether we can define distances on a masure which:

- induce the topology of finite-dimensional real-affine space on each apartment,
- are compatible with the action of G,
- are compatible with retractions centered at a sector-germ.

We show that under the assumption of continuity of retractions, the metric space we have is never complete nor locally compact (see Subsection 3.3). We show that there is no distance on $\mathcal{I}$ such that the restriction to each apartment is a norm. However, for each sector-germ $\mathfrak{s}$ of $\mathcal{I}$, we construct distances having the following properties (Corollary 37, Lemma 26, Corollary 38 and Theorem 41):

- the topology induced on each apartment is the affine topology,

- each retraction with the center $\mathfrak{s}$ is 1-Lipschitz,
- each retraction with center a sector-germ of the same sign as $\mathfrak{s}$ is Lipschitz,
- each $g \in G$ is Lipschitz when we regard it as an automorphism of $\mathcal{I}$.

We call them *distances of positive* or *negative type*, depending on the sign of $\mathfrak{s}$. We prove that all distances of positive type on a masure (resp. of negative type) are equivalent, where two distances d_1 and d_2 are said to be equivalent if there exist $k, \ell \in \mathbb{R}_{>0}$ such that $kd_1 \leq d_2 \leq \ell d_1$ (this is Theorem 36). We thus get a *positive topology* $\mathscr{T}_+$ and a *negative topology* $\mathscr{T}_-$ defined by distances of $\pm$ types. We prove (Corollary 45) that these topologies are different when $\mathcal{I}$ is not a building. When $\mathcal{I}$ is a building, these topologies agree with the usual topology on a building (Proposition 42).

Let $\mathcal{I}_0$ be the G-orbit in $\mathcal{I}$ of some special vertex. If $\mathcal{I}$ is not a building, $\mathcal{I}_0$ is not discrete for both $\mathscr{T}_-$ and $\mathscr{T}_+$. We also prove that if ρ is a retraction centered at a negative (resp. positive) sector-germ, ρ is not continuous for $\mathscr{T}_+$ (resp. $\mathscr{T}_-$); see Proposition 44. For these reasons we introduce *mixed distances*, which are sums of a distance of positive type with a distance of negative type. We then have the following (Theorem 46): all the mixed distances on $\mathcal{I}$ are equivalent, moreover if d is a mixed distance and $\mathcal{I}$ is equipped with d then:

- each $g : \mathcal{I} \to \mathcal{I} \in G$ is Lipschitz,
- each retraction centered at a sector-germ is Lipschitz,
- the topology induced on each apartment is the affine topology,
- the set $\mathcal{I}_0$ is discrete.

The topology $\mathscr{T}_m$ associated with mixed distances is the initial topology with respect to the retractions of $\mathcal{I}$ (see Corollary 50).

We prove that $\mathcal{I}$ is contractible for $\mathscr{T}_+$, $\mathscr{T}_-$ and $\mathscr{T}_m$.

Let us explain how to define distances of positive or negative type. Let $\mathbb{A}$ be the standard apartment of $\mathcal{I}$ and C_f^v be the fundamental chamber of $\mathbb{A}$. Let $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$. After applying some $g \in G$ to A, we may assume $A = \mathbb{A}$ and that $\mathfrak{s}$ is the germ $+\infty$ of C_f^v (or of $-C_f^v$, but this case is similar). Fix a norm $|\ .\ |$ on $\mathbb{A}$. For every $x \in \mathcal{I}$, there exists an apartment A_x containing x and $+\infty$ (which means that A_x contains a subsector of C_f^v). For $u \in \overline{C_f^v}$, we define $x + u$ as the translate of x by u in A_x. If u is chosen to be sufficiently dominant, $x + u \in C_f^v$. Therefore, for all $x, x' \in \mathcal{I}$, there exist $u, u' \in C_f^v$ such that $x + u = x' + u'$. We then define $d(x, x')$ to be the minimum of the $|u| + |u'|$ for such couples u, u'.

We thus obtain a distance for each sector-germ and for each norm $|\ .\ |$ on $\mathbb{A}$.

This paper is organized as follows.

In Section 2, we review basic definitions and set up the notation.

In Section 3, we show that if $\mathfrak{s}$ is a sector-germ of $\mathcal{I}$, we can write each apartment as a finite union of closed convex subsets, each of which is contained in an apartment A containing $\mathfrak{s}$. The most important case for us is when A contains a sector-germ adjacent to $\mathfrak{s}$. We then can write A as the union of two half-apartments, each contained in an apartment containing $\mathfrak{s}$. We conclude Section 3 with a series of properties that distances on $\mathcal{I}$ cannot satisfy.

In Section 4, we construct distances of positive and negative type on $\mathcal{I}$. We prove that all the distances of positive type (resp. negative type) are equivalent. We then study them.

In Section 5, we first show that when $\mathcal{I}$ is not a building, $\mathscr{T}_+$ and $\mathscr{T}_-$ are different. Then we define mixed distances and study their properties.

In Section 6, we show that $\mathcal{I}$ is contractible for the topologies $\mathscr{T}_+$, $\mathscr{T}_-$ and $\mathscr{T}_m$.

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2. Masures

In this section, we review the theory of masures. We restrict our study to semi-discrete masures which are thick of finite thickness and such that there exists a group acting strongly transitively on them (we define these notions at the end of the section). These properties are satisfied by masures associated with split Kac–Moody groups over nonarchimedean local fields (see [Rou16]). To avoid introducing too much notation, we do not treat the case of almost split Kac–Moody groups (see [Rou17]). By adapting Lemma 3, one can prove that our results remain valid in the almost split case.

We begin by defining the standard apartment. References for this section are [Kac94, Chap. 1 and 3], [GR08, Sect. 2], and [GR14, Sect. 1].

2.1. Root generating system

A Kac–Moody matrix (or generalized Cartan matrix) is a square matrix $C = (c_{i,j})_{i,j\in I}$ with integer coefficients, indexed by a finite set I and satisfying:

(1) $\forall i \in I,\ c_{i,i} = 2$,
(2) $\forall (i,j) \in I^2 | i \neq j,\ c_{i,j} \leq 0$,
(3) $\forall (i,j) \in I^2,\ c_{i,j} = 0 \Leftrightarrow c_{j,i} = 0$.

A root generating system is a 5-tuple $\mathcal{S} = (C, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac–Moody matrix C indexed by I, of two dual free $\mathbb{Z}$-modules X (of characters) and Y (of co-characters) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i\in I}$ (of simple coroots) in Y. They have to satisfy the following compatibility condition: $c_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i\in I}$ (resp. $(\alpha_i^\vee)_{i\in I}$) freely generates a $\mathbb{Z}$-submodule of X (resp. of Y)).

We now fix a Kac–Moody matrix C and a root generating system with the matrix C.

Let $\mathbb{A} = Y \otimes \mathbb{R}$. We equip $\mathbb{A}$ with the topology defined by its structure of a finite-dimensional real-vector space. Every element of X induces a linear form on $\mathbb{A}$. We will regard X as a subset of the dual $\mathbb{A}^*$ of $\mathbb{A}$: the α_i, $i \in I$ are viewed as linear forms on $\mathbb{A}$. For $i \in I$, we define an involution r_i of $\mathbb{A}$ by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in \mathbb{A}$. Its fixed points set is $\ker \alpha_i$. The subgroup of $\mathrm{GL}(\mathbb{A})$ generated by the r_i, $i \in I$ is denoted by W^v and is called the *Weyl group* of $\mathcal{S}$. The system $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$. The groups Q and $Q^\vee$ are called the *root lattice* and the *coroot-lattice*.

One defines an action of the group W^v on $\mathbb{A}^*$ as follows: if $x \in \mathbb{A}$, $w \in W^v$ and $\alpha \in \mathbb{A}^*$ then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i | (w,i) \in W^v \times I\}$ be the set of *real roots*. Then $\Phi \subset Q$. Let $Q^+ = \{\sum_{i\in I} n_i\alpha_i^\vee \mid (n_i) \in \mathbb{N}^I\} \subset Q$,

$Q^- = -Q^+$, $\Phi^+ = \Phi \cap Q^+$ and $\Phi^- = \Phi \cap Q^-$. Then $\Phi = \Phi^+ \cup \Phi^-$. The elements of Φ^+ (resp. Φ^-) are called the *real positive roots* (resp. *real negative roots*). Let $W^a = Q^\vee \rtimes W^v \subset \mathrm{GA}(\mathbb{A})$ be the *affine Weyl group* of $\mathcal{S}$, where $\mathrm{GA}(\mathbb{A})$ is the group of affine automorphisms of $\mathbb{A}$.

For α and $k \in \mathbb{R}$, one sets $D(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k = 0\}$, $D^\circ(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k > 0\}$ and $M(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k = 0\}$. One also sets $D(\alpha, +\infty) = D^\circ(\alpha, +\infty) = \mathbb{A}$ and $M(\alpha, +\infty) = \varnothing$. A *wall* (resp. a half-apartment) of $\mathbb{A}$ is a hyperplane (resp. a half-space) of the form $M(\alpha, k)$ (resp. $D(\alpha, k)$) for some $\alpha \in \Phi$ and $k \in \mathbb{R}$. The wall (resp. half-apartment) is said to be a *true* wall (resp. a true half-apartment) if $k \in \mathbb{Z}$, and a *ghost* wall if $k \notin \mathbb{Z}$. This choice of true walls means that the apartment (or the masure) is semidiscrete.

2.2. Vectorial faces and Tits preorder

Vectorial faces. Define $C_f^v = \{v \in \mathbb{A} \mid \alpha_i(v) > 0,\ \forall i \in I\}$. We call it the *fundamental chamber*. For $J \subset I$, one sets $F^v(J) = \{v \in \mathbb{A} \mid \alpha_i(v) = 0\ \forall i \in J, \alpha_i(v) > 0\ \forall i \in J \backslash I\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the subsets $F^v(J)$ for $J \subset I$. The *positive* (resp. *negative*) *vectorial faces* are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A *vectorial face* is either a positive vectorial face or a negative vectorial face. We call *a positive chamber* (resp. *negative*) every cone of the form $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). By [Rou11, Sect. 1.3], the action of W^v on the set of positive chambers is simply transitive. The *Tits cone* $\mathcal{T}$ is defined as the convex cone $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. We also consider the negative cone $-\mathcal{T}$.

Tits preorder on $\mathbb{A}$. One defines a W^v-invariant relation $\leq$ on $\mathbb{A}$ by: $x \leq y \iff y - x \in \mathcal{T}$. This preorder need not be a partial order. For example, if W^v is finite (i.e., when the Kac–Moody matrix C defining $\mathcal{S}$ is a Cartan matrix), then $\mathcal{T} = \mathbb{A}$ and thus $x \leq y$ for all $x, y \in \mathbb{A}$. For an arbitrary Kac–Moody matrix C, every element x of $\bigcap_{i \in I} \ker(\alpha_i)$ satisfies $0 \leq x \leq 0$, and in particular when C is not invertible, $\leq$ is not a partial order.

Let $x, y \in \mathbb{A}$ be such that $x \neq y$. The ray with the base point x and containing y (or an interval $(x, y]$, (x, y), $\ldots$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $y - x \in \pm\mathring{\mathcal{T}}$, the interior of $\pm\mathcal{T}$.

2.3. Metric properties of W^v

In this subsection, we prove that when W^v is infinite there does not exist a W^v-invariant norm on $\mathbb{A}$ and we also establish a density property of the walls of $\mathbb{A}$.

Two true walls M_1 and M_2 are said to be *consecutive* if they are of the form $\alpha^{-1}(\{k\})$, $\alpha^{-1}(\{k \pm 1\})$ for some $\alpha \in \Phi$ and some $k \in \mathbb{Z}$.

Proposition 1.

(1) *Suppose that there exists a W^v-invariant norm on $\mathbb{A}$. Then W^v is finite.*
(2) *Let $|\ .\ |$ be a norm on $\mathbb{A}$, d be the induced distance on $\mathbb{A}$ and suppose that W^v is infinite. Then for every $\epsilon > 0$ there exists a vectorial wall M_0 such that for all consecutive true walls M_1 and M_2 parallel to M_0, $d(M_1, M_2) < \epsilon$.*

Proof. Let $B \subset Y^{\dim \mathbb{A}}$ be a $\mathbb{Z}$-basis of Y. Then the map $W^v \to Y^{\dim \mathbb{A}}$ sending each w to $w.B$ is injective. Thus if W^v is infinite, $\{w.B \mid w \in W^v\}$ is unbounded. Part 1 follows.

Suppose that W^v is infinite. Let $(\beta_n) \in \Phi_+^{\mathbb{N}}$ be an injective sequence. Let $\epsilon > 0$ and $u \in C_f^v$ be such that $|u| < \epsilon$. For $n \in \mathbb{N}$, write $\beta_n = \sum_{i \in I} \lambda_{i,n}\alpha_i$, with $\lambda_{i,n} \in \mathbb{N}$ for all $(i,n) \in I \times \mathbb{N}$. One has

$$\beta_n(u) = \sum_{i \in I} \lambda_{i,n}\alpha_i(u) \geq \big(\min_{i \in I} \alpha_i(u)\big) \sum_{i \in I} \lambda_{i,n} \to +\infty.$$

Let $n \in \mathbb{N}$ be such that $\beta_n(u) \geq 1$ and $M_0 = \beta_n^{-1}(\{0\})$. Then for all consecutive true walls M_1 and M_2 parallel to M_0, $d(M_1, M_2) < \epsilon$, which proves the proposition. $\square$

2.4. Filters and enclosure

Filters. A filter on a set $\mathcal{E}$ is a nonempty set $\mathscr{F}$ of nonempty subsets of $\mathcal{E}$ such that, for all subsets E, E' of $\mathcal{E}$, one has:

- E, $E' \in \mathscr{F}$ implies $E \cap E' \in \mathscr{F}$,
- $E' \subset E$ and $E' \in \mathscr{F}$ implies $E \in \mathscr{F}$.

If $\mathcal{E}$ is a set and $\mathscr{F}, \mathscr{F}'$ are filters on $\mathcal{E}$, we define $\mathscr{F} \Cup \mathscr{F}'$ to be the filter $\{E \cup E' \mid (E, E') \in \mathscr{F} \times \mathscr{F}'\}$.

If $\mathscr{F}$ is a filter on a set $\mathcal{E}$, and E is a subset of $\mathcal{E}$, one says that $\mathscr{F}$ contains E if every element of $\mathscr{F}$ contains E. We denote it by $\mathscr{F} \Supset E$. If E is nonempty, the principal filter on $\mathcal{E}$ associated with E is the filter $\mathscr{F}_{E,\mathcal{E}}$ of subsets of $\mathcal{E}$ containing E.

A filter $\mathscr{F}$ is said to be contained in another filter $\mathscr{F}'$: $\mathscr{F} \Subset \mathscr{F}'$ (resp. in a subset Z in $\mathcal{E}$: $\mathscr{F} \Subset Z$) if every set in $\mathscr{F}'$ is in $\mathscr{F}$ (resp. if $Z \in \mathscr{F}$).

These definitions of containment are inspired by the following facts. Let $\mathcal{E}$ be a set, $\mathscr{F}$ be a filter on $\mathcal{E}$ and $E, E' \subset \mathcal{E}$. Then:

- $E \subset E'$ if and only if $\mathscr{F}_{E,\mathcal{E}} \Subset \mathscr{F}_{E',\mathcal{E}}$,
- $E \Subset \mathscr{F}$ if and only if $\mathscr{F}_{E,\mathcal{E}} \Subset \mathscr{F}$,
- $E \Supset \mathscr{F}$ if and only if $\mathscr{F}_{E,\mathcal{E}} \Supset \mathscr{F}$.

If $\mathscr{F}$ is a filter on a finite-dimensional real-affine space $\mathcal{E}$, its closure $\overline{\mathscr{F}}$ (resp. its convex hull) is the filter of subsets of $\mathcal{E}$ containing the closure (resp. the convex hull) of some element of $\mathscr{F}$. The *support* of a filter $\mathscr{F}$ on $\mathcal{E}$ is the minimal affine subspace containing $\mathscr{F}$.

Enclosure of a filter. Let Δ be the set of all roots of the root generating system $\mathcal{S}$ defined in Chapter 1 of [Kac94]. We only recall that $\Delta \subset \mathbb{A}^*$ and that $\Delta \cap \mathbb{R}\Phi = \Phi$.

Let $\mathscr{F}$ be a filter on $\mathbb{A}$. The enclosure $\mathrm{cl}(\mathscr{F})$ is the filter on $\mathbb{A}$ defined as follows. A set E is in $\mathrm{cl}(\mathscr{F})$ if there exists $(k_\alpha) \in (\mathbb{Z} \cup \{+\infty\})^\Delta$ satisfying

$$E \supset \bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha) \Supset \mathscr{F}.$$

Suppose that we are in the reductive case: i.e., that $\mathcal{S}$ is associated with a Cartan matrix or equivalently that Φ is finite. Then $\Delta = \Phi$. Let $E \subset \mathbb{A}$ and E' be the intersection of the true half-apartments containing E (E' is the enclosure of E in the definition of [BT72]). Then $\mathrm{cl}(E) = \mathscr{F}_{E',\mathbb{A}}$.

2.5. Faces, sector-faces, chimneys and germs

Sector-faces, sectors. A *sector-face* f of $\mathbb{A}$ is a set of the form $x+F^v$ for some vectorial face F^v and some $x\in\mathbb{A}$. The point x is its *base point* and F^v is its *direction.* The *germ at infinity* $\mathfrak{F}=\mathrm{germ}_\infty(f)$ of f is the filter composed of all the subsets of $\mathbb{A}$ which contain an element of the form $x+u+F^v$ for some $u\in\overline{F^v}$.

When F^v is a vectorial chamber, one calls f a *sector.* The intersection of two sectors of the same direction is a sector of the same direction. A *sector-germ* of $\mathbb{A}$ is a filter which is the germ at infinity of some sector of $\mathbb{A}$. We denote by $\pm\infty$ the germ of $\pm C^v_f$.

The sector-face f is said to be spherical if $F^v\cap\pm\mathring{\mathcal{T}}$ is nonempty. A *sector-panel* is a sector-face contained in a wall and spanning it as an affine space. Sectors and sector-panels are spherical.

Let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be two sector-germs of the same sign. Let C^v_1, C^v_2 be the two vectorial chambers such that $\mathfrak{q}_1=\mathrm{germ}_\infty(C^v_1)$ and $\mathfrak{q}_2=\mathrm{germ}_\infty(C^v_2)$. We say that $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are adjacent if $\overline{C^v_1}\cap\overline{C^v_2}$ contains some sector-panel.

Let $\mathfrak{q},\mathfrak{q}'$ be two sector-germs of the same sign. A *gallery* between $\mathfrak{q}$ and $\mathfrak{q}'$ is a sequence of sector-germs $\Gamma=(\mathfrak{q}_1,\ldots,\mathfrak{q}_n)$ such that $n\in\mathbb{N}$, $\mathfrak{q}_1=\mathfrak{q}$, $\mathfrak{q}_n=\mathfrak{q}'$ and for all $i\in[\![1,n-1]\!]$, $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ are adjacent. The *length* of Γ is n. For every two sector germs $\mathfrak{q}$ and $\mathfrak{q}'$ of the same sign, there exists a gallery joining $\mathfrak{q}$ and $\mathfrak{q}'$. Indeed, let C^v and C'^v be the vectorial chambers such that $\mathfrak{q}=\mathrm{germ}_\infty(C^v)$ and $\mathfrak{q}'=\mathrm{germ}_\infty(C'^v)$. Let $w\in W^v$ be such that $C'^v=w.C^v$. Let $w=r_{i_1}\ldots r_{i_k}$ be a writing of w, with $i_1,\ldots,i_k\in I$. Then

$$\mathrm{germ}_\infty(C^v),\ \mathrm{germ}_\infty(r_{i_1}.C^v),\ \mathrm{germ}_\infty(r_{i_1}r_{i_2}.C^v),\ldots,\mathrm{germ}_\infty(r_{i_1}\ldots r_{i_k}.C^v)$$

is a gallery from $\mathfrak{q}$ to $\mathfrak{q}'$.

Faces. Let $x\in\mathbb{A}$ and let F^v be a vectorial face of $\mathbb{A}$. The *face* $F(x,F^v)$ is the filter defined as follows: a set $E\subset\mathbb{A}$ is an element of $F(x,F^v)$ if, and only if, there exist $(k_\alpha),(k'_\alpha)\in(\mathbb{Z}\cup\{+\infty\})^\Delta$ and a neighborhood Ω of x in $\mathbb{A}$ such that $E\supset\bigcap_{\alpha\in\Delta}\big(D(\alpha,k_\alpha)\cap D^\circ(\alpha,k'_\alpha)\big)\supset\Omega\cap(x+F^v)$. A *face* of $\mathbb{A}$ is a filter F that can be written as $F=F(x,F^v)$, for some $x\in\mathbb{A}$ and some vectorial face F^v.

A *chamber* is a face whose support is $\mathbb{A}$. A *panel* is a face whose support is a wall.

In the reductive case (i.e., when Φ is finite), we obtain the usual notion of faces: the faces for the definition we gave are exactly the $\mathscr{F}_{F,\mathbb{A}}$, where F is a face of $\mathbb{A}$ equipped with its structure of a simplicial complex.

Chimneys. Let F be a face of $\mathbb{A}$ and F^v be a vectorial face of $\mathbb{A}$. The *chimney* $\mathfrak{r}(F,F^v)$ is the filter $\mathrm{cl}(\mathscr{F}_{F+F^v,\mathbb{A}})$. A *chimney* $\mathfrak{r}$ is a filter on $\mathbb{A}$ of the form $\mathfrak{r}=\mathfrak{r}(F,F^v)$ for some face F and some vectorial face F^v. The enclosure of a sector-face is thus a chimney. The vectorial face F^v is uniquely determined by $\mathfrak{r}$ (this is not necessarily the case of the face F) and one calls it the *direction* of $\mathfrak{r}$.

Let $\mathfrak{r}$ be a chimney and F^v be its direction. One says that $\mathfrak{r}$ is *splayed* if F^v is spherical (or equivalently if F^v contains a generic ray; see Subsection 2.2). One says that $\mathfrak{r}$ is *solid* if the pointwise stabilizer in W^v of the direction of the support of $\mathfrak{r}$ is finite. A splayed chimney is solid.

Let $\mathfrak{r} = \mathfrak{r}(F, F^v)$ be a chimney. A shortening of $\mathfrak{r}$ is a chimney of the form $\mathfrak{r}(F+u, F^v)$, for some $u \in \overline{F^v}$. The *germ at infinity* $\mathfrak{R} = \mathrm{germ}_\infty(\mathfrak{r})$ of $\mathfrak{r}$ is the filter composed of all subsets of $\mathbb{A}$ which contain a shortening of $\mathfrak{r}$. A sector-germ is an example of a germ of a splayed chimney.

2.6. Masure

Let $\alpha \in \Phi$. We can write $\alpha = w.\alpha_i$ for some $i \in I$ and $w \in W^v$. Then $w.\alpha_i^\vee$ does not depend on the choice of w and one denotes it as $\alpha^\vee$. An *automorphism* of $\mathbb{A}$ is an affine bijection $\phi : \mathbb{A} \to \mathbb{A}$ stabilizing the set $\{\big(M(\alpha, k), \alpha^\vee\big) \mid (\alpha, k) \in \Phi \times \mathbb{Z}\}$. One has $W^a \subset W^v \ltimes Y \subset \mathrm{Aut}(\mathbb{A})$, where $\mathrm{Aut}(\mathbb{A})$ is the group of automorphisms of $\mathbb{A}$.

An apartment of type $\mathbb{A}$ is a set A with a nonempty set $\mathrm{Isom}^w(\mathbb{A}, A)$ of bijections (called Weyl isomorphisms) such that if $f_0 \in \mathrm{Isom}^w(\mathbb{A}, A)$ then $f \in \mathrm{Isom}^w(\mathbb{A}, A)$ if and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism (resp. a Weyl isomorphism, a vectorially Weyl isomorphism) between two apartments $\phi : A \to A'$ is a bijection such that for every $f \in \mathrm{Isom}^w(\mathbb{A}, A)$ and $f' \in \mathrm{Isom}^w(\mathbb{A}, A')$, one has $f' \circ \phi \circ f^{-1} \in \mathrm{Aut}(\mathbb{A})$ (resp. $f' \circ \phi \circ f^{-1} \in W^a$, $f' \circ \phi \circ f^{-1} \in (W^v \ltimes \mathbb{A}) \cap \mathrm{Aut}(\mathbb{A})$).

Each apartment A of type $\mathbb{A}$ can be equipped with the structure of an affine space by using an isomorphism of apartments $\phi : \mathbb{A} \to A$. We equip each apartment with its topology defined by its structure of a finite-dimensional real-affine space.

We extend all the notions that are preserved by $\mathrm{Aut}(\mathbb{A})$ to each apartment. In particular, enclosures, sector-faces, faces, chimneys, germs of chimneys, ... are well defined in each apartment of type $\mathbb{A}$. If A is an apartment of type $\mathbb{A}$ and $x, y \in A$, then we denote by $[x, y]_A$ the closed segment of A between x and y.

We say that an apartment contains a filter if it contains at least one element of this filter. We say that a map fixes a filter if it fixes at least one element of this filter.

We now give the definition of masures. These objects were introduced by Gaussent and Rousseau in [GR08] (they were initially called "hovels"). This axiomatic definition was introduced by Rousseau in [Rou11].

Definition 1. A masure of type $\mathbb{A}$ is a set $\mathcal{I}$ endowed with a covering $\mathcal{A}$ by subsets called apartments such that:

(MA1) Each $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$.

(MA2) If F is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists a Weyl isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

(MA3) If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains $\mathfrak{R}$ and F.

(MA4) If two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then there exists a Weyl isomorphism from A to A' fixing $\mathrm{cl}_A(\mathfrak{R} \Cup F)$.

(MAO) If x, y are two points contained in two apartments A and A', and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.

We assume that there exists a group G acting strongly transitively on $\mathcal{I}$, which means that:

- G acts on $\mathcal{I}$,
- $g.A$ is an apartment for every $g \in G$ and every apartment A,
- for every $g \in G$ and every apartment A, the map $A \to g.A$ is an isomorphism of apartments,
- all isomorphisms involved in the above axioms are induced by elements of G.

We choose in $\mathcal{I}$ a "fundamental" apartment, that we identify with $\mathbb{A}$. As G acts strongly transitively on $\mathcal{I}$, the apartments of $\mathcal{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ induces a group $\nu(N)$ of affine automorphisms of $\mathbb{A}$ and we assume that $\nu(N) = W^v \ltimes Y$.

All the isomorphisms that we will consider in this paper will be vectorially Weyl isomorphisms and we will say "isomorphism" instead of "vectorially Weyl isomorphism".

Throughout the paper, we will only consider masures $\mathcal{I}$ which are thick of finite thickness, that is masures satisfying the following axiom:

(MAT) for each panel P, the number of chambers whose closure contains P is finite and greater than 2.

This definition coincides with the usual one when $\mathcal{I}$ is a building.

An example of such a masure $\mathcal{I}$ is the masure associated with a split Kac–Moody group over a field equipped with a nonarchimedean discrete valuation constructed in [GR08] and in [Rou16].

A masure $\mathcal{I}$ is a building if and only if W^v is finite; see [Rou11, 2.2 6]).

2.7. Retractions centered at sector-germs

If A and B are two apartments, and $\phi : A \to B$ is an isomorphism of apartments fixing some filter $\mathcal{X}$, one writes $\phi : A \overset{\mathcal{X}}{\to} B$. If A and B share a sector-germ $\mathfrak{s}$, there exists a unique isomorphism of apartments $\phi : A \to B$ fixing $A \cap B$. Indeed, by (MA4), there exists an isomorphism $\psi : A \to B$ fixing $\mathfrak{s}$. Let $x \in A \cap B$. By (MA4), $A \cap B$ contains the convex hull $\mathrm{Conv}(x, \mathfrak{s})$ in A of x and $\mathfrak{s}$ and there exists an isomorphism of apartments $\psi' : A \to B$ fixing $\mathrm{Conv}(x, \mathfrak{s})$. Then $\psi'^{-1} \circ \psi : A \to A$ is an isomorphism of affine spaces fixing $\mathfrak{s}$: $\psi' = \psi$. By definition $\psi'(x) = x$ and thus ψ fixes $A \cap B$. The uniqueness is a consequence of the fact that the only affine morphism fixing some nonempty open set of A is the identity. One denotes by $A \xrightarrow{A \cap B} B$ or by $A \overset{\mathfrak{s}}{\to} B$ the unique isomorphism of apartments from A to B fixing $\mathfrak{s}$.

Fix a sector-germ $\mathfrak{s}$ of $\mathcal{I}$ and an apartment A containing $\mathfrak{s}$. Let $x \in \mathcal{I}$. By (MA3), there exists an apartment A_x of $\mathcal{I}$ containing x and $\mathfrak{s}$. Let $\phi : A_x \overset{\mathfrak{s}}{\to} A$ fixing $\mathfrak{s}$. By [Rou11] 2.6, $\phi(x)$ does not depend on the choices we made and thus we define $\rho_{A,\mathfrak{s}}(x) = \phi(x)$.

The map $\rho_{A,\mathfrak{s}}$ is a retraction from $\mathcal{I}$ onto A. It only depends on $\mathfrak{s}$ and A and we call it the *retraction onto A centered at $\mathfrak{s}$*. We denote by $\mathcal{I} \overset{\mathfrak{s}}{\to} A$ the retraction onto A fixing $\mathfrak{s}$. We denote by $\rho_{\pm\infty}$ the retraction onto $\mathbb{A}$ centered at $\pm\infty$.

2.8. Parallelism in $\mathcal{I}$

Let us explain briefly the notion of parallelism in $\mathcal{I}$. This is done in detail in [Rou11, Sect. 3].

Let us begin with rays. Let δ and δ' be two generic rays in $\mathcal{I}$. By (MA3) and [Rou11, 2.2 3]) there exists an apartment A containing subrays of δ and δ', and we say that δ and δ' are *parallel* if these subrays are parallel in A. Parallelism is an equivalence relation. The parallelism class of a generic ray δ is denoted δ_∞ and is called its *direction.*

We now review the notion of parallelism for sector-faces. We refer to [Rou11, 3.3.4] for the details.

Twin-building $\mathcal{I}^\infty$ at infinity. If f and f' are two spherical sector-faces in $\mathcal{I}$, there exists an apartment B containing their germs $\mathfrak{F}$ and $\mathfrak{F}'$. One says that f and f' are parallel if $\mathfrak{F} = \mathrm{germ}_\infty(x + F^v)$ and $\mathfrak{F}' = \mathrm{germ}_\infty(y + F^v)$ for some $x, y \in B$ and for some vectorial face F^v of B. Parallelism is an equivalence relation. The parallelism class of a sector-face germ $\mathfrak{F}$ is denoted $\mathfrak{F}_\infty$ and is called its *direction.* We denote by $\mathcal{I}^\infty$ the set of directions of spherical faces of $\mathcal{I}$. If $\mathfrak{s}$ is a sector, all the sectors having the germ at infinity $\mathfrak{s}$ have the same direction. We denote it $\mathfrak{s}$ by abuse of notation. If M is a wall of $\mathcal{I}$, its direction $M^\infty \subset \mathcal{I}^\infty$ is defined to be the set of germs at infinity $\mathfrak{F}_\infty$ such that $\mathfrak{F} = \mathrm{germ}_\infty(f)$, with f a spherical sector-face contained in M.

Let $\mathfrak{F}_\infty \in \mathcal{I}^\infty$ (resp. let δ_∞ be the direction of a generic ray) and A be an apartment. One says that A contains $\mathfrak{F}_\infty$ (resp. δ_∞) if A contains some sector-face f (resp. generic ray δ) whose direction is $\mathfrak{F}_\infty$ (resp. is δ_∞).

Proposition 2.

(1) *Let $x \in \mathcal{I}$ and $\mathfrak{F}_\infty \in \mathcal{I}^\infty$ (resp. δ_∞ be a generic ray direction). Then there exists a unique sector-face $x + \mathfrak{F}_\infty$ (resp. $x + \delta_\infty$) based at x and whose direction is $\mathfrak{F}_\infty$ (resp. δ_∞).*

(2) *Let A_x be an apartment containing x and $\mathfrak{F}_\infty$ (resp. δ_∞) (which exists by (MA3)). Let f (resp. δ') be a sector-face (resp. a generic ray) of A_x whose direction is $\mathfrak{F}_\infty$ (resp. δ_∞). Then $x + \mathfrak{F}_\infty$ (resp. $x + \delta_\infty$) is the sector-face (resp. generic ray) of A_x parallel to f (resp. δ') and based at x.*

(3) *Let B be an apartment containing $\mathfrak{F}_\infty$ (resp. δ_∞). Then for all $x \in B$, $x + \mathfrak{F}_\infty \subset B$ (resp. $x + \delta_\infty \subset B$).*

Proof. The points 1 and 2 for sector-faces are Proposition 4.7.1) of [Rou11] and its proof. Point 3 is a consequence of 2. The statement for rays is analogous (see Lemma 3.2 of [Héb17]). □

Let f, f' be sector-faces. One says that f dominates f' (resp. f and f' are opposite) if $\mathrm{germ}_\infty(f) = \mathrm{germ}_\infty(x + F^v)$, $\mathrm{germ}_\infty(f') = \mathrm{germ}_\infty(x' + F'^v)$ for some $x, x' \in \mathcal{I}$ and F^v, F'^v two vectorial faces of a same apartment of $\mathcal{I}$ such that $\overline{F^v} \supset F'^v$ (resp. such that $F'^v = -F^v$). By [Rou11, Prop. 3.2 2) and 3)], these notions extend to $\mathcal{I}^\infty$.

3. Splitting of apartments

3.1. Splitting of apartments in two half-apartments

The aim of this section is to show that if A is an apartment, M is a wall of A, $\mathfrak{F}$ is a sector-panel of M^∞, and $\mathfrak{s}$ is a sector-germ dominating $\mathfrak{F}_\infty$, then there exist two opposite half-apartments D_1 and D_2 of A such that their common wall is parallel to M and such that for both $i \in \{1,2\}$, D_i and $\mathfrak{s}$ are contained in some apartment. This is Lemma 8. This property is called "sundial configuration" in [BS14, Sect. 2]. This section will enable us to show that for each choice of sign, the distances of positive types and of negative types are equivalent.

For simplicity, we assume that Φ is reduced. This assumption can be dropped with minor changes to the next lemma.

Lemma 3. *Let $\alpha \in \Phi$ and $k \in \mathbb{R}$. Then* $\mathrm{cl}(D(\alpha,k)) = \mathscr{F}_{D(\alpha,\lceil k \rceil),\mathbb{A}}$.

Proof. By definition of cl, $D(\alpha,\lceil k\rceil) \in \mathrm{cl}\big(D(\alpha,k)\big)$ and hence we have $\mathrm{cl}\big(D(\alpha,k)\big) \Subset \mathscr{F}_{D(\alpha,\lceil k\rceil),\mathbb{A}}$.

Let $E \in \mathrm{cl}(D(\alpha,k))$. By definition, there exists $(k_\beta) \in (\mathbb{Z}\cup\infty)^\Delta$ such that $E \supset \bigcap_{\beta\in\Delta} D(\beta,k_\beta) \supset D(\alpha,k)$. Let $\beta \in \Delta\setminus\{\alpha\}$. As $\beta \notin \mathbb{R}_+\alpha$, $D(\beta,\ell) \not\supset D(\alpha,k)$ for all $\ell \in \mathbb{Z}$. Hence $k_\beta = +\infty$.

As the family $\big(D(\alpha,\ell)\big)_{\ell\in\mathbb{R}}$ is ordered by inclusion, $k_\alpha \geq \lceil k \rceil$.

Therefore $\bigcap_{\beta\in\Delta} D(\beta,k_\beta) = D(\alpha,k_\alpha) \supset D(\alpha,\lceil k\rceil)$. Consequently, $\mathscr{F}_{D(\alpha,\lceil k\rceil),\mathbb{A}} \Subset \mathrm{cl}\big(D(\alpha,k)\big)$ and thus $\mathrm{cl}(D(\alpha,k)) = \mathscr{F}_{D(\alpha,\lceil k\rceil),\mathbb{A}}$. □

Lemma 4. *Let A, B be two distinct apartments of $\mathcal{I}$ containing a half-apartment D. Then $A\cap B$ is a true half-apartment.*

Proof. Using isomorphisms of apartments, we may assume $A = \mathbb{A}$. Let $\alpha \in \Phi$ and $k \in \mathbb{R}$ be such that $D = D(\alpha,k)$. Set $M_0 = \alpha^{-1}(\{0\})$. Let S be a sector of $\mathbb{A}$ based at 0 and dominating some sector-panel $f \subset M_0$. Let $f' = -f$ and $\mathfrak{s}$, $\mathfrak{F}_\infty$ and $\mathfrak{F}'_\infty$ be the directions of S, f and f'. Let $x \in \mathbb{A}\cap B$. Then by Proposition 2 (3), $\mathbb{A}\cap B \supset x+\mathfrak{s}$ and $\mathbb{A}\cap B \supset x+\mathfrak{F}'_\infty$. As $\mathrm{germ}_\infty(x+\mathfrak{s})$, $\mathrm{germ}_\infty(x+\mathfrak{F}'_\infty)$ are the germs of splayed chimneys, we can apply (MA4) and we get that $\mathbb{A}\cap B \Supset \mathrm{cl}\big(\mathrm{germ}_\infty(x+\mathfrak{s}) \Cup \mathrm{germ}_\infty(x+\mathfrak{F}'_\infty)\big)$. But

$$\mathrm{cl}\big(\mathrm{germ}_\infty(x+\mathfrak{s}) \Cup \mathrm{germ}_\infty(x+\mathfrak{F}'_\infty)\big) = \mathrm{cl}\Big(\overline{\mathrm{Conv}}\big(\mathrm{germ}_\infty(x+\mathfrak{s}) \Cup \mathrm{germ}_\infty(x+\mathfrak{F}'_\infty)\big)\Big),$$

where $\overline{\mathrm{Conv}}$ denotes the closure of the convex hull. Therefore,

$$\mathrm{cl}\big(\mathrm{germ}_\infty(x+\mathfrak{s}) \Cup \mathrm{germ}_\infty(x+\mathfrak{F}'_\infty)\big) = \mathrm{cl}\big(D(\alpha,-\alpha(x)\big) = \mathscr{F}_{D(\alpha,\lceil-\alpha(x)\rceil),\mathbb{A}}$$

(by Lemma 3). Thus $\mathbb{A}\cap B \supset D(\alpha,\lceil-\alpha(x)\rceil) \ni x$. Consequently,

$$\mathbb{A}\cap B \supset \bigcup_{x\in\mathbb{A}\cap B} D(\alpha,\lceil-\alpha(x)\rceil) \supset \mathbb{A}\cap B.$$

Hence $\mathbb{A}\cap B = D(\alpha,\ell)$, where $\ell = \max_{x\in\mathbb{A}\cap B}\lceil-\alpha(x)\rceil \in \mathbb{Z}$, and the lemma follows. □

From now on, unless otherwise stated, "a half-apartment" (resp. "a wall") will implicitly refer to "a true half-apartment" (resp. "a true wall").

Lemma 5. *Let M be a wall of $\mathbb{A}$ and $w \in W^v \ltimes Y$ be an element fixing M. Then $w \in \{\mathrm{Id}, s\}$, where s is the reflection of $W^v \ltimes Y$ with respect to M.*

Proof. One writes $w = \tau \circ u$, with $u \in W^v$ and τ a translation of $\mathbb{A}$. Then $u(M)$ is a wall parallel to M. Let M_0 be the wall parallel to M containing 0. Then $u(M_0)$ is a wall parallel to M_0 and containing 0: $u(M_0) = M_0$. Let C be a vectorial chamber adjacent to M_0. Then $u(C)$ is a chamber adjacent to C: $u(C) \in \{C, s_0(C)\}$, where s_0 is the reflection of W^v with respect to M_0. After composing u with s_0, we may assume that $u(C) = C$ and thus $u = \mathrm{Id}$ (because the action of W^v on the set of chambers is simply transitive). □

If A is an apartment and D, D' are half-apartments of A, we say that D and D' are opposite if $D \cap D'$ is a wall and one says that D and D' have opposite directions if their walls are parallel and $D \cap D'$ is not a half-apartment.

Lemma 6. *Let A_1, A_2, A_3 be distinct apartments. Suppose that $A_1 \cap A_2$, $A_1 \cap A_3$ and $A_2 \cap A_3$ are half-apartments such that $A_1 \cap A_3$ and $A_2 \cap A_3$ have opposite directions. Let M be the wall of $A_1 \cap A_3$.*

(1) *One has $A_1 \cap A_2 \cap A_3 = M$ where M is the wall of $A_1 \cap A_3$, and for all $(i, j, k) \in \{1, 2, 3\}^3$ such that $\{i, j, k\} = \{1, 2, 3\}$, $A_i \cap A_j$ and $A_i \cap A_k$ are opposite.*

(2) *Let $s : A_3 \to A_3$ be the reflection with respect to M, $\phi_1 : A_3 \xrightarrow{A_1 \cap A_3} A_1$, $\phi_2 : A_3 \xrightarrow{A_2 \cap A_3} A_2$ and $\phi_3 : A_2 \xrightarrow{A_1 \cap A_2} A_1$. Then the following diagram is commutative:*

$$\begin{tikzcd} A_3 \arrow[r, "s"] \arrow[d, "\phi_2"'] & A_3 \arrow[d, "\phi_1"] \\ A_2 \arrow[r, "\phi_3"] & A_1 \end{tikzcd} .$$

Proof. Point 1 is a consequence of "Propriété du Y" and of its proof ([Rou11, Sect. 4.9]).

Let $\phi = \phi_1^{-1} \circ \phi_3 \circ \phi_2 : A_3 \to A_3$. Then ϕ fixes M. Let $D_1 = A_2 \cap A_3$, $D_2 = A_1 \cap A_3$ and $D_3 = A_1 \cap A_2$. One has $\phi_3(A_2) = A_1 = D_2 \cup D_3$ and thus $\phi_3(D_1) = D_2$. One has $\phi_1^{-1}(D_2) = D_2$. Thus $\phi(D_1) = D_2$. We conclude with Lemma 5. □

Lemma 7. *Let $\mathfrak{s}$, $\mathfrak{s}'$ be two opposite sector-germs of $\mathcal{I}$. Then there exists a unique apartment containing $\mathfrak{s}$ and $\mathfrak{s}'$.*

Proof. The existence is a particular case of (MA3). Let A and A' be apartments containing $\mathfrak{s} \Cup \mathfrak{s}'$. Let $x \in A \cap A'$. Then by Proposition 2 (3), $A = \bigcup_{y \in x+\mathfrak{s}} y + \mathfrak{s}' \subset A \cap A'$, thus $A \subset A'$ and the lemma follows by symmetry. □

Recall the definition of $\mathcal{I}^\infty$ and of the direction M^∞ of a wall M from Subsection 2.8. The following lemma is similar to [Rou11, Prop. 2.9.1)]. This is analogous to the *sundial configuration* of [BS14, Sect. 2].

Lemma 8. *Let A be an apartment, M be a wall of A, and M^∞ be its direction. Let $\mathfrak{F}_\infty$ be the direction of a sector-panel of M^∞ and $\mathfrak{s}$ be a sector-germ dominating $\mathfrak{F}_\infty$ and not contained in A. Then there exists a unique pair $\{D_1, D_2\}$ of half-apartments of A such that:*

- *D_1 and D_2 are opposite with the common wall M' parallel to M,*
- *for all $i \in \{1,2\}$, D_i and $\mathfrak{s}$ are in some apartment A_i.*

Moreover,

- *D_1 and D_2 are true half-apartments*
- *such apartments A_1 and A_2 are unique and if D is the half-apartment of A_1 opposite to D_1, then $D \cap D_2 = D_1 \cap D_2 = M'$ and $A_2 = D_2 \cup D$.*

Proof. Let us first show the existence of D_1 and D_2. Let $\mathfrak{F}'_\infty$ be the sector-panel of M^∞ opposite to $\mathfrak{F}_\infty$. Let $\mathfrak{s}'_1$ and $\mathfrak{s}'_2$ be the sector-germs of A containing $\mathfrak{F}'_\infty$. For $i \in \{1,2\}$, let A_i be an apartment of $\mathcal{I}$ containing $\mathfrak{s}'_i$ and $\mathfrak{s}$ which exists by (MA3). Let $i \in \{1,2\}$ and $x \in A \cap A_i$. Then by Proposition 2(3), $x + \mathfrak{s}'_i \subset A \cap A_i$ and the open half-apartment $E_i = \bigcup_{y \in x+\mathfrak{s}'_i} y + \mathfrak{F}_\infty \subset A \cap A_i$ is contained in A and A_i.

Suppose $A_1 = A_2$. Then $A_1 \supset \bigcup_{x \in E_1} x + \mathfrak{s}'_2 = A$ and thus $A_1 = A \Supset \mathfrak{s}$. This is absurd and thus $A_1 \neq A_2$.

The apartments A_1, A_2 contain $\mathfrak{F}'_\infty$ and $\mathfrak{s}$. Take $x \in A_1 \cap A_2$. Then by Proposition 2(3), $A_1 \cap A_2$ contains the open half-apartment $\bigcup_{y \in x+\mathfrak{s}} y + \mathfrak{F}'_\infty$. By Lemma 4, $A_1 \cap A_2$ is a half-apartment. Thus we can apply Lemma 6: $A_1 \cap A_2 \cap A = M'$, where M' is a wall of A parallel to M. Set $D_i = A \cap A_i$ for all $i \in \{1,2\}$. Then $\{D_1, D_2\}$ fulfills the requirements of the lemma.

Let D'_1, D'_2 be another pair of opposite half-apartments of A such that for all $i \in \{1,2\}$, D'_i and $\mathfrak{s}$ are contained in some apartment A'_i and such that $D'_1 \cap D'_2$ is parallel to M.

We can assume $D'_i \Supset \mathfrak{s}'_i$ for both $i \in \{1,2\}$. Let $\mathfrak{s}'$ be the sector-germ of A'_i opposite to $\mathfrak{s}$. Then $\mathfrak{s}'$ dominates $\mathfrak{F}'_\infty$ and is contained in D'_i. Therefore $\mathfrak{s}' = \mathfrak{s}_i$. By Lemma 7, $A'_i = A_i$, which proves the uniqueness of $\{D_1, D_2\}$ and $\{A_1, A_2\}$.

Moreover, by [Rou11, Prop. 2.9 2)], $D \cup D_2$ is an apartment. As $D \cup D_2 \Supset \mathfrak{s} \Cup \mathfrak{s}_2$, one has $D \cup D_2 = A_2$, which concludes the proof of the lemma. $\square$

3.2. Splitting of apartments

In this subsection, we mainly generalize Lemma 8. We show that if $\mathfrak{s}$ is a sector-germ of $\mathcal{I}$ and if A is an apartment of $\mathcal{I}$, then A is the union of a finite number of convex closed subsets P_i of A such that for all i, P_i and $\mathfrak{s}$ are contained in some apartment. This is Proposition 9.

Let $\mathfrak{s}, \mathfrak{s}'$ be two sector-germs of the same sign. Let A be an apartment containing $\mathfrak{s}$ and $\mathfrak{s}'$, which exists by (MA3). Let $d(\mathfrak{s}, \mathfrak{s}')$ be the length of a minimal gallery from $\mathfrak{s}$ to $\mathfrak{s}'$ (see Subsection 2.5 for the definition of a gallery). By (MA4), $d(\mathfrak{s}, \mathfrak{s}')$ does not depend on the choice of A.

Let $\mathfrak{s}$ be a sector-germ and A be an apartment of $\mathcal{I}$. Let $d_{\mathfrak{s}}(A)$ be the minimum of the $d(\mathfrak{s}, \mathfrak{s}')$, where $\mathfrak{s}'$ runs over the sector-germs of A of the same sign as $\mathfrak{s}$. Let $\mathcal{D}_A$ be the set of half-apartments of A. One sets $\mathcal{P}_{A,0} = \{A\}$ and for all $n \in \mathbb{N}^*$, $\mathcal{P}_{A,n} = \{\bigcap_{i=1}^n D_i \mid (D_i) \in (\mathcal{D}_A)^n\}$. The following proposition is very similar to [Cha10, Prop. 4.3.1].

Proposition 9. *Let A be an apartment of $\mathcal{I}$, $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$ et $n = d_{\mathfrak{s}}(A)$. Then there exist $P_1, \ldots, P_k \in \mathcal{P}_{A,n}$, with $k \leq 2^n$ such that $A = \bigcup_{i=1}^k P_i$ and for each $i \in [\![1,k]\!]$, P_i and $\mathfrak{s}$ are contained in some apartment A_i such that there exists an isomorphism $f_i : A_i \xrightarrow{P_i} A$.*

Proof. We prove the proposition by the induction on n. This is clear if $n = 0$. Let $n \in \mathbb{N}_{>0}$. Suppose this is true for every apartment B such that $d_{\mathfrak{s}}(B) \leq n-1$.

Let B be an apartment such that $d_{\mathfrak{s}}(B) = n$. Let $\mathfrak{t}$ be a sector-germ of B such that there exists a minimal gallery $\mathfrak{t} = \mathfrak{s}_0, \dots, \mathfrak{s}_{n-1} = \mathfrak{s}$ from $\mathfrak{t}$ to $\mathfrak{s}$. By Lemma 8, there exist opposite half-apartments D_1, D_2 of B such that for both $i \in \{1,2\}$, D_i and $\mathfrak{s}_1$ are contained in an apartment B_i. Let $i \in \{1,2\}$. One has $d_{\mathfrak{s}}(B_i) = n-1$ and thus $B_i = \bigcup_{j=1}^{k_i} P_j^{(i)}$, with $k_i \leq 2^{n-1}$, for all $j \in [\![1, k_i]\!]$, $P_j^{(i)} \in \mathcal{P}_{B_i, n-1}$ and $\mathfrak{s}, P_j^{(i)}$ is contained in some apartment $A_j^{(i)}$. One has

$$B = D_1 \cup D_2 = B_1 \cap D_1 \cup B_2 \cap D_2 = \bigcup_{i \in \{1,2\}, j \in [\![1,k_i]\!]} P_j^{(i)} \cap D_i.$$

Let $i \in \{1,2\}$, $j \in [\![1, k_i]\!]$ and $\phi_i : B_i \xrightarrow{B \cap B_i} B$. Then $P_j^{(i)} \cap D_i = \phi_i(P_j^{(i)} \cap D_i) \in \mathcal{P}_{B,n}$ and $B_i \supset (P_j^{(i)} \cap D_i), \mathfrak{s}$.

Let $f_i^{(j)} : A_i^{(j)} \xrightarrow{P_i^{(j)}} B_i$ and $f = \phi_i \circ f_i^{(j)}$. Then $f : A_j^{(i)} \xrightarrow{P_j^{(i)} \cap D_i} B$ and the proposition follows. $\square$

We deduce from the previous proposition a corollary which was already known for masures associated with split Kac–Moody groups over fields equipped with a nonarchimedean discrete valuation by [GR08, Sect. 4.4].

Corollary 10. *Let $\mathfrak{s}$ be a sector-germ, A be an apartment, and $x, y \in A$. Then there exists $x = x_1, \dots, x_k = y \in [x,y]_A$ such that $[x,y]_A = \bigcup_{i=1}^{k-1} [x_i, x_{i+1}]_A$ and such that for every $i \in [\![1, k-1]\!]$, $\mathfrak{s}$ and $[x_i, x_{i+1}]_A$ are contained in an apartment A_i such that there exists an isomorphism $f_i : A \xrightarrow{[x_i, x_{i+1}]_{A_i}} A_i$.*

3.3. Restrictions on the distances

In this subsection, we show that some properties cannot be satisfied by distances on masures. If A is an apartment of $\mathcal{I}$, we show that there exist apartments branching at every wall of A (this is Lemma 11). This implies that if $\mathcal{I}$ is not a building, the interior of each apartment is empty for the distances we study. We write $\mathcal{I}$ as a countable union of apartments and then use Baire's Theorem to show that under a rather weak assumption of regularity for retractions, a masure cannot be complete nor locally compact for the distances we study.

Let us show a slight refinement of [Rou11, Cor. 2.10]:

Lemma 11. *Let A be an apartment of $\mathcal{I}$ and D be a half-apartment of A. Then there exists an apartment B such that $A \cap B = D$.*

Proof. Let M be the wall of D, P be a panel of M, and C be a chamber whose closure contains P and which is not contained in A. By [Rou11, Prop. 2.9(1)], there exists an apartment B containing D and C. By Lemma 4, $A \cap B = D$, which proves the lemma. $\square$

Proposition 12. *Assume that there exists a distance $d_{\mathcal{I}}$ on $\mathcal{I}$ such that for every apartment A, $d_{\mathcal{I}}|_{A^2}$ is induced by some norm. Then $\mathcal{I}$ is a building and $d_{\mathcal{I}}|_{\mathbb{A}^2}$ is W^a-invariant.*

Proof. Let $\mathfrak{s}$ be a sector-germ and A, B be two apartments containing $\mathfrak{s}$. Let $\phi : A \xrightarrow{A\cap B} B$. Let us first prove that $\phi : (A, d_{\mathcal{I}}) \to (B, d_{\mathcal{I}})$ is an isometry. Let $d' : A \times A \to \mathbb{R}_+$ be defined by $d'(x,y) = d_{\mathcal{I}}(\phi(x), \phi(y))$ for all $x, y \in A$. Then d' is induced by some norm. Moreover $d'|_{(A\cap B)^2} = d_{\mathcal{I}}|_{(A\cap B)^2}$. As $A \cap B$ has a nonempty interior, we deduce that $d' = d_{\mathcal{I}}$ and thus $\phi : (A, d_{\mathcal{I}}) \to (B, d_{\mathcal{I}})$ is an isometry.

Let M be a wall of $\mathbb{A}$, D_1 and D_2 be the half-apartments defined by M, and $s \in W^a$ be the reflection with respect to M. Let A_2 be an apartment of $\mathcal{I}$ such that $A \cap A_2 = D_1$, which exists by Lemma 11. Let D_3 be the half-apartment of B opposite to D_1. Then $D_3 \cap D_2 \subset D_3 \cap \mathbb{A} \subset M$ and thus $D_2 \cap D_3 = M$. By [Rou11, Prop. 2.9(2)], $D_3 \cup D_2$ is an apartment A_1 of $\mathcal{I}$. Let $\phi_2 : \mathbb{A} \xrightarrow{\mathbb{A}\cap A_1} A_1$, $\phi_1 : \mathbb{A} \xrightarrow{A\cap A_2} A_2$ and $\phi_3 : A_1 \xrightarrow{A_1\cap A_2} A_2$. Then by Lemma 6, the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "s"] \arrow[d, "\phi_2"] & \mathbb{A} \arrow[d, "\phi_1"] \\
A_1 \arrow[r, "\phi_3"] & A_2
\end{tikzcd}.$$

By the first part of the proof, s is an isometry of $\mathbb{A}$ and thus W^a is a group of isometries for $d_{\mathcal{I}}|_{\mathbb{A}^2}$. By Proposition 1(1), W^v is finite and by [Rou11, 2.2(6)], $\mathcal{I}$ is a building. $\square$

Lemma 13. *Let $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$ and d be a distance on $\mathcal{I}$ inducing the affine topology on each apartment and such that there exists a continuous retraction ρ of $\mathcal{I}$ centered at $\mathfrak{s}$. Then each apartment containing $\mathfrak{s}$ is closed.*

Proof. Let A be an apartment containing $\mathfrak{s}$ and $B = \rho(\mathcal{I})$. Let $\phi : B \overset{\mathfrak{s}}{\to} A$ and $\rho_A : \mathcal{I} \overset{\mathfrak{s}}{\to} A$. Then $\rho_A = \phi \circ \rho$ is continuous because ϕ is an affine map. Let $(x_n) \in A^{\mathbb{N}}$ be a converging sequence for d and $x = \lim x_n$. Then $x_n = \rho_A(x_n) \to \rho_A(x)$ and thus $x = \rho_A(x) \in A$. $\square$

Proposition 14. *Suppose $\mathcal{I}$ is not a building. Let d be a distance on $\mathcal{I}$ inducing the affine topology on each apartment. Then the interior of each apartment of $\mathcal{I}$ is empty.*

Proof. Let V be a nonempty open set of $\mathcal{I}$. Let A be an apartment of $\mathcal{I}$ such that $A \cap V \neq \varnothing$. By Proposition 1 (2), there exists a wall M of A such that $M \cap V \neq \varnothing$. Let D be a half-apartment delimited by M. Let B be an apartment such that $A \cap B = D$, which exists by Lemma 11. Then $B \cap V$ is an open set of B containing $M \cap V$ and thus $E \cap V \neq \varnothing$, where E is the half-apartment of B opposite to D. Therefore $V \backslash A \neq \varnothing$ and we get the proposition. $\square$

One sets $\mathcal{I}_0 = G.0$ where $0 \in \mathbb{A}$. This is the set of *vertices of type* 0. Recall that $\pm\infty = \mathrm{germ}_\infty(\pm C_f^v)$ and that $\rho_{\pm\infty} : \mathcal{I} \overset{\pm\infty}{\to} \mathbb{A}$.

Lemma 15. *One has $\mathcal{I}_0 \cap \mathbb{A} = Y$.*

Proof. Let $\lambda \in \mathcal{I}_0 \cap \mathbb{A}$. Then $\lambda = g.0$ for some $g \in G$. By (MA2), there exists $\phi : g.\mathbb{A} \to \mathbb{A}$ fixing λ. Then $\lambda = \phi(g.0)$ and $\phi \circ g|_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ is an automorphism of apartments. Let $h \in G$ induce ϕ on $g.\mathbb{A}$. Then $h.g \in N$, hence $(h.g)|_{\mathbb{A}} \in \nu(N) = W^v \ltimes Y$ (by the end of Subsection 2.6) and thus $\lambda = h.g.0 \in Y$. $\square$

Lemma 16. *The set $\mathcal{I}_0$ is countable.*

Proof. For all $\lambda \in \mathcal{I}_0$, $\rho_{-\infty}(\lambda)$, $\rho_{+\infty}(\lambda) \in \mathcal{I}_0$ and thus $\rho_{-\infty}(\lambda)$, $\rho_{+\infty}(\lambda) \in Y$. Therefore $\mathcal{I}_0 = \bigcup_{(\lambda,\mu)\in Y^2} \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})$. By [Héb17, Thm. 5.6], the set $\rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})$ is finite for all $(\lambda,\mu) \in Y^2$, which completes the proof. □

Let $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$. For $\lambda \in \mathcal{I}_0$, choose an apartment $A(\lambda)$ containing $\lambda + \mathfrak{s}$. Let $x \in \mathcal{I}$ and A be an apartment containing x and $\mathfrak{s}$. Then there exists $\lambda \in \mathcal{I}_0 \cap A$ such that $x \in \lambda + \mathfrak{s}$ and thus $x \in A(\lambda)$. Therefore $\mathcal{I} = \bigcup_{\lambda\in\mathcal{I}_0} A(\lambda)$.

Proposition 17. *Let d be a distance on $\mathcal{I}$. Suppose that there exists a sector-germ $\mathfrak{s}$ such that every apartment containing $\mathfrak{s}$ is closed and with empty interior. Then $(\mathcal{I}, d)$ is incomplete and the interior of every compact subset of $\mathcal{I}$ is empty.*

Proof. One has $\mathcal{I} = \bigcup_{\lambda\in\mathcal{I}_0} A(\lambda)$, with $\mathcal{I}_0$ countable by Lemma 16. Thus by Baire's Theorem, $(\mathcal{I}, d)$ is incomplete.

Let K be a compact subset of $\mathcal{I}$. Then $K = \bigcup_{\lambda\in\mathcal{I}_0} K \cap A(\lambda)$ and thus K has empty interior. □

4. Distances of positive type and negative type

4.1. Translation in a direction

Let $\mathfrak{s}$ be a sector-germ. We now define a map $+_{\mathfrak{s}}$ such that for all $x \in \mathcal{I}$ and $u \in \overline{C_f^v}$, $x +_{\mathfrak{s}} u$ is the "translate of x by u in the direction $\mathfrak{s}$". Let $\mathrm{sgn}(\mathfrak{s}) \in \{-,+\}$ be the sign of $\mathfrak{s}$.

Definition/Proposition 18. *Let $\mathfrak{s}$ be a sector-germ. Let $x \in \mathcal{I}$. Let A_1 be an apartment containing $x + \mathfrak{s}$. Let $(x + \overline{\mathfrak{s}})_{A_1}$ be the closure of $x + \mathfrak{s}$ in A_1. Then $(x + \overline{\mathfrak{s}})_{A_1}$ does not depend on the choice of A_1 and we denote it by $x + \overline{\mathfrak{s}}$.*

Proof. Let A_2 be an apartment containing $x+\mathfrak{s}$ and $\phi : A_1 \xrightarrow{A_1\cap A_2} A_2$. By (MA4), ϕ fixes the enclosure of $x + \mathfrak{s}$, which contains $(x + \overline{\mathfrak{s}})_{A_1}$. Therefore $(x + \overline{\mathfrak{s}})_{A_1} \supset (x + \overline{\mathfrak{s}})_{A_2}$ and by symmetry, $(x + \overline{\mathfrak{s}})_{A_1} = (x + \overline{\mathfrak{s}})_{A_2}$. Proposition follows. □

If A and B are apartments and $\psi : A \to B$ is an isomorphism, then ψ induces a bijection still denoted ψ between the sector-germs of A and those of B.

Definition/Proposition 19. *Let $\mathfrak{s}$ be a sector-germ. Let $x \in \mathcal{I}$ and A_1 be an apartment containing $x + \mathfrak{s}$. Let $\mathrm{sgn}(\mathfrak{s}) \in \{-,+\}$ be the sign of $\mathfrak{s}$. Let $u \in \overline{C_f^v}$ and $\psi_1 : \mathbb{A} \to A_1$ be an isomorphism such that $\psi_1(\mathrm{sgn}(\mathfrak{s})\infty) = \mathfrak{s}$. Then $\psi_1\big(\psi_1^{-1}(x + \mathrm{sgn}(\mathfrak{s})u)\big)$ does not depend on the choice of A_1 and ψ_1 and we denote it $x +_{\mathfrak{s}} u$. Moreover, $x +_{\mathfrak{s}} C_f^v = x + \mathfrak{s}$ and $x +_{\mathfrak{s}} \overline{C_f^v} = x + \overline{\mathfrak{s}}$.*

Proof. As the case where $\mathfrak{s}$ is negative is similar, we assume that $\mathfrak{s}$ is positive.

We first prove the independence of the choice of isomorphism. Let $\psi_1' : \mathbb{A} \to A_1$ be an isomorphism such that $\psi_1'(+\infty) = \mathfrak{s}$. Then $\psi_1'^{-1} \circ \psi_1 \in W^v \ltimes Y$ fixes the direction $+\infty$ and thus $\psi_1'^{-1} \circ \psi_1$ is a translation of $\mathbb{A}$. Therefore

$$\psi_1'^{-1} \circ \psi_1\big(\psi_1^{-1}(x) + u\big) = \psi_1'^{-1} \circ \psi_1\big(\psi_1^{-1}(x)\big) + u = \psi_1'^{-1}(x) + u,$$

and thus

$$\psi_1\big(\psi_1^{-1}(x+u)\big) = \psi_1'\big(\psi_1'^{-1}(x+u)\big).$$

Let now A_2 be an apartment containing $x+\mathfrak{s}$ and $\psi_2 : \mathbb{A} \to A_2$ be an isomorphism such that $\psi_2(+\infty) = \mathfrak{s}$. From what has already been proved, we can assume that $\psi_2 \circ \psi_1^{-1} = \phi$, where $\phi : A_1 \xrightarrow{A_1 \cap A_2} A_2$.

As $x \in A_1 \cap A_2$, $\phi(x) = x$ and thus

$$\psi_1^{-1}(x) = \psi_2^{-1}(x).$$

Let $i \in \{1, 2\}$. Then $\psi_i\big(\psi_i^{-1}(x)+C_f^v\big)$ is a sector with the base point x and with the direction $\mathfrak{s}$: $\psi_i\big(\psi_i^{-1}(x)+C_f^v\big) = x+\mathfrak{s}$ (see Proposition 2). Moreover $\psi_i\big(\psi_i^{-1}(x)+\overline{C_f^v}\big)$ is the closure of $\psi_i\big(\psi_i^{-1}(x)+C_f^v\big) = x+\mathfrak{s}$ in A_i and thus $\psi_i\big(\psi_i^{-1}(x)+\overline{C_f^v}\big) = x+\overline{\mathfrak{s}}$.

Consequently $\psi_1\big(\psi_1^{-1}(x+u)\big) \in x+\overline{\mathfrak{s}} \subset A_1 \cap A_2$. Thus

$$\phi\Big(\psi_1\big(\psi_1^{-1}(x+u)\big)\Big) = \psi_1\big(\psi_1^{-1}(x+u)\big) = \psi_2\big(\psi_1^{-1}(x)+u\big) = \psi_2\big(\psi_2^{-1}(x)+u\big),$$

which is our assertion. $\square$

Through the end of this section, we fix a sector-germ $\mathfrak{s}$. As the case when $\mathfrak{s}$ is negative is similar to the case when it is positive, we assume that $\mathfrak{s}$ is positive.

Lemma 20. *Let $x \in \mathcal{I}$ and $u, u' \in \overline{C_f^v}$. Then $(x +_{\mathfrak{s}} u) +_{\mathfrak{s}} u' = x +_{\mathfrak{s}} (u+u')$.*

Proof. Let A be an apartment containing $x+\mathfrak{s}$ and $\psi : \mathbb{A} \to A$ be such that $\psi(+\infty) = \mathfrak{s}$. One has $(x+_{\mathfrak{s}} u)+_{\mathfrak{s}} u'$, $x+_{\mathfrak{s}}(u+u') \in A$. By definition, $\psi^{-1}(x+_{\mathfrak{s}} u) = \psi^{-1}(x)+u$, thus

$$(x +_{\mathfrak{s}} u) +_{\mathfrak{s}} u' = \psi(\psi^{-1}(x +_{\mathfrak{s}} u)+u') = \psi(\psi^{-1}(x)+u+u') = x +_{\mathfrak{s}} (u+u'),$$

which proves the lemma. $\square$

For $x, x' \in \mathcal{I}$, we set $U_{\mathfrak{s}}(x, x') = \Big\{(u, u') \in \overline{C_f^v}^2 \mid x +_{\mathfrak{s}} u = x' +_{\mathfrak{s}} u'\Big\}$.

Lemma 21. *Let $x, x' \in \mathcal{I}$. Then $U_{\mathfrak{s}}(x, x')$ is nonempty.*

Proof. Let A be an apartment containing $\mathfrak{s}$. Choose $a \in (x+\mathfrak{s}) \cap A$ and $a' \in (x'+\mathfrak{s}) \cap A$. Then $a+\mathfrak{s}$ and $a'+\mathfrak{s}$ are sectors of A of the same direction and thus there exists $b \in (a+\mathfrak{s}) \cap (a'+\mathfrak{s})$. By Definition/Proposition 19, there exist $u, u', v, v' \in C_f^v$ such that $a = x +_{\mathfrak{s}} u$, $a' = x' +_{\mathfrak{s}} u'$ and $b = a +_{\mathfrak{s}} v = a' +_{\mathfrak{s}} v'$. By Lemma 20, $(u+v, u'+v') \in U_{\mathfrak{s}}(x, x')$ and the lemma is proved.

4.2. Definition of distances of positive type and negative type

Let Θ_+ (resp. Θ_-) be the set of pairs $(|\ .\ |, \mathfrak{s})$ such that $\mathfrak{s}$ is a positive (resp. negative) sector-germ and $|\ .\ |$ is a norm on $\mathbb{A}$.

We now define the distance on $\mathcal{I}$ associated with $(|\cdot|, \mathfrak{s})$. When $\mathcal{I}$ is a building, the distance on $\mathcal{I}$ is usually defined as follows. One equips $\mathbb{A}$ with a euclidean W^v-invariant norm $|\cdot|$. For $x, y \in \mathbb{A}$, one chooses $g \in G$ such that $g.x, g.y \in \mathbb{A}$ and one sets $d(x,y) = |g.y - g.x|$. The fact that $|\ .\ |$ is W^v invariant implies that this distance is well defined. When $\mathcal{I}$ is a masure which is not a building, there exists no W^v-invariant norm on $\mathbb{A}$ (by Proposition 1) and there can exist pairs of points which are not contained in a common apartment. Thus we have to find another method to define a distance. For each pair of points, there exists a piecewise linear path joining them (where a piecewise linear path is a map $\gamma : [0,1] \to \mathcal{I}$ such that there exists $n \in \mathbb{N}$ and $t_0 = 0 < t_1 < \ldots < t_n = 1$ such that for all $i \in [\![0, n-1]\!]$, $\gamma|_{[t_i, t_{i+1}]}$ takes its values in some apartment A_i and is an affine parametrization of the segment $[\gamma(t_i), \gamma(t_{i+1})]_{A_i}$). Thus we can try to define the distance between two points as the minimal length of a piecewise linear path joining them. However to this end, we need to define a notion of a length for a path and the nonexistence of a W^v-invariant norm on $\mathbb{A}$ makes it difficult (because we cannot simply define it on $\mathbb{A}$ and transport it to each apartment by using isomorphism of apartments). To avoid this problem, we fix a sector $\mathfrak{s}$, which enables us to define a length on every apartment containing $\mathfrak{s}$. As the paths described above are difficult to handle — for example we do not really understand the segments that are not increasing or decreasing for the Tits preorder — we impose very strict conditions on the paths that we measure. In particular, we require $n = 2$ (i.e., at most one break-point in the paths) and we require the directions of the segments to be contained in $\mathfrak{s}$, which leads to the definition below.

Definition/Proposition 22. *Let $\theta = (|\ .\ |, \mathfrak{s}) \in \Theta_+ \cup \Theta_-$. Let $d_\theta : \mathcal{I}^2 \to \mathbb{R}_+$ be defined by $d_\theta(x,x') = \inf\{|u| + |u'| \mid (u,u') \in U_\mathfrak{s}(x,x')\}$ for $x, x' \in \mathcal{I}$. Then d_θ is a distance on $\mathcal{I}$.*

Proof. By Lemma 21, d_θ is well defined. Moreover it is clearly symmetric.

Let us show the triangle inequality. Let $x, x', x'' \in \mathcal{I}$. Let $\epsilon > 0$ and let $(u,u') \in U_\mathfrak{s}(x,x')$, $(v',v'') \in U_\mathfrak{s}(x',x'')$ be such that $|u|+|u'| \leq d_\theta(x,x')+\epsilon$ and $|v'|+|v''| \leq d_\theta(x',x'')+\epsilon$. One has $x +_\mathfrak{s} u = x' +_\mathfrak{s} u'$ and $x' +_\mathfrak{s} v' = x'' +_\mathfrak{s} v''$. Thus $x +_\mathfrak{s} u +_\mathfrak{s} v' = x' +_\mathfrak{s} v' +_\mathfrak{s} u' = x'' +_\mathfrak{s} v'' +_\mathfrak{s} u'$ (by Lemma 20) and hence $(u+v', v''+u') \in U_\mathfrak{s}(x,x'')$. Consequently, $d_\theta(x,x'') \leq |u+v'| + |v''+u'| \leq |u| + |v'| + |v''| + |u''| \leq d_\theta(x,x') + d_\theta(x',x'') + 2\epsilon$, which proves the triangle inequality.

Let $x, x' \in \mathcal{I}$ be such that $d_\theta(x,x') = 0$. Then there exists $\big((u_n, u'_n)\big)_{n\in\mathbb{N}} \in U_\mathfrak{s}(x,x')^\mathbb{N}$ such that $u_n \to 0$ and $u'_n \to 0$. Let $n \in \mathbb{N}$. One has $x + \mathfrak{s} \supset x +_\mathfrak{s} u_n + \mathfrak{s} = x' +_\mathfrak{s} u'_n + \mathfrak{s}$ and thus $x + \mathfrak{s} \supset \bigcup_{n\in\mathbb{N}} x' + u'_n + \mathfrak{s} = x' + \mathfrak{s}$. By symmetry, $x' + \mathfrak{s} \supset x + \mathfrak{s}$ and hence $x + \mathfrak{s} = x' + \mathfrak{s}$. Let B be an apartment containing x and $\mathfrak{s}$. By (MA2), $B \Supset \mathrm{cl}(x+\mathfrak{s}) = \mathrm{cl}(x'+\mathfrak{s})$ and thus $x' \in B$. Therefore $x = x'$. □

Thus we have constructed a distance d_θ for all $\theta \in \Theta_+ \cup \Theta_-$. A distance of the form d_{θ_+} (resp. d_{θ_-}) for some $\theta_+ \in \Theta_+$ (resp. $\theta_- \in \Theta_-$) is called a *distance of positive type* (resp. *distance of negative type*). When $\mathcal{I}$ is a tree, we obtain a distance proportional to the distance of a Euclidean building on $\mathcal{I}$: there exists $k \in \mathbb{R}$ such that if $x, y \in \mathbb{A} = \mathbb{R}$, then $d_\theta(x,y) = k|y-x|$ and the action of G on $\mathcal{I}$ is isometric.

The choice of the norm has an influence on the metric on $\mathcal{I}$ but not on the topology defined on $\mathcal{I}$ (see Theorem 36). Independently of the choice of the norm every pair of points is joined by a geodesic and there exists pairs of points joined by infinitely many geodesics (when $\dim \mathbb{A} \geq 2$; see Proposition 30).

Examples. We suppose that $\mathbb{A}$ is two-dimensional and that $\bigcap_{i\in I}\ker\alpha_i=\{0\}$ (thus the root generating system defining $\mathbb{A}$ is associated with a size 2 Kac–Moody matrix). We determine the restriction of d_θ to $\mathbb{A}$ when $\theta=(|\cdot|,+\infty)$ for different choices of a norm on $\mathbb{A}$.

Write $I=\{1,2\}$. Let $u_1,u_2\in\mathbb{A}$ be such that $\alpha_1(u_1)=1$, $\alpha_2(u_1)=0$, $\alpha_1(u_2)=0$ and $\alpha_2(u_2)=1$. Then $\overline{C^v_f}=\{x_1u_1+x_2u_2\mid (x_1,x_2)\in(\mathbb{R}_+)^2\}$.

We begin by determining $d_\theta|_{\mathbb{A}^2}$, when θ is associated with any Euclidean norm. For $x=(x_1,x_2)\in\mathbb{A}$, one sets $|x|_2=\sqrt{x_1^2+x_2^2}$. Let $\theta_2=(|\,.\,|_2,+\infty)$. We now determine the restriction of d_{θ_2} to $\mathbb{A}$.

Proposition 23. *Let $x,y\in\mathbb{A}$. Then:*

$$d_{\theta_2}(x,y)=\begin{cases}|y-x|_2 & \text{if } y-x\in\pm\overline{C^v_f},\\ |\alpha_1(y-x)u_1|_2+|\alpha_2(y-x)u_2|_2 & \text{if } y-x\notin\pm\overline{C^v_f}.\end{cases}$$

Moreover, if $\alpha_1(x)\leq\alpha_1(y)$ and $\alpha_2(y)\leq\alpha_2(x)$, then $(\alpha_1(y-x)u_1,\alpha_2(x-y)u_2)\in U_{+\infty}(x,y)$ and $d_{\theta_2}(x,y)=|\alpha_1(y-x)u_1|_2+|\alpha_2(x-y)u_2|_2$.

For $x\in\mathbb{A}$ and $u\in\mathbb{A}$, we denote by $x+\mathbb{R}_+u$ the set $\{x+tu\mid t\in\mathbb{R}_+\}$.

Lemma 24. *Let $x,y\in\mathbb{A}$. Let $f:\mathbb{A}\to\mathbb{R}_+$ be defined by $f(z)=|z-x|_2+|y-x|_2$, for $z\in\mathbb{A}$. Then the set of points at which f admits a local minimum is the segment $[x,y]$ and* $\min f=|y-x|_2=f(z)$*, for all $z\in[x,y]$.*

Proof. Using a translation and a rotation, we may assume that $x=0$ and $y=(y_1,0)$ for some $y_1\in\mathbb{R}$. The lemma follows by straightforward computations of $\frac{\partial f}{\partial z_2}(z_1,z_2)$ and $z_1\mapsto f(z_1,0)$. $\square$

We now prove Proposition 23. Let $x,y\in\mathbb{A}$ and $f:\mathbb{A}\to\mathbb{R}_+$ be defined by $f(z)=|z-x|_2+|y-x|_2$, for $z\in\mathbb{A}$. Suppose $y-x\in\overline{C^v_f}$. Then $y=x+y-x$ and thus

$$d_{\theta_2}(x,y)\leq|0|_2+|y-x|_2=|y-x|_2=\min f\leq d_{\theta_2}(x,y),$$

which proves that $d_{\theta_2}(x,y)=|y-x|_2$.

Suppose $y-x\notin\pm\overline{C^v_f}$. Then $[x,y]\cap(x+\overline{C^v_f})\cap(y+\overline{C^v_f})$ is empty and thus $\min\{f(z)\mid z\in(x+\overline{C^v_f})\cap(y+\overline{C^v_f})\}>|y-x|_2$. By Lemma 24, the minimum of f on $(x+\overline{C^v_f})\cap(y+\overline{C^v_f})$ is attained on the boundary $\partial\big((x+\overline{C^v_f})\cap(y+\overline{C^v_f})\big)$ of $(x+\overline{C^v_f})\cap(y+\overline{C^v_f})$. Suppose, for example, that $\alpha_1(x)\leq\alpha_1(y)$ and $\alpha_2(y)\leq\alpha_2(x)$. Let $z\in\mathbb{A}$ be such that $\alpha_1(z)=\alpha_1(y)$ and $\alpha_2(z)=\alpha_2(x)$. Then $\partial\big((x+\overline{C^v_f})\cap(y+\overline{C^v_f})\big)=(z+\mathbb{R}_+u_1)\cup(z+\mathbb{R}_+u_2)$. Let $z'\in z+\mathbb{R}_+u_1$. Write $z'=z+tu_1$. Then $f(z')=|z'-x|_2+|z'-y|_2=|z-x|_2+t|u_1|_2+|z+tu_1-y|_2\geq f(z)$. By symmetry we deduce that $\min\{f(z')\mid z'\in(x+\overline{C^v_f})\cap(y+\overline{C^v_f})\}=f(z)=\alpha_1(y-x)|u_1|_2+\alpha_2(x-y)|u_2|_2$, and the proposition follows. $\square$

We now determine $d_\theta|_{\mathbb{A}^2}$, when θ is associated with a certain norm 1 on $\mathbb{A}$.

Proposition 25. *Define $|\cdot|_1:\mathbb{A}\to\mathbb{R}_+$ by $|x|_1=|\alpha_1(x)|+|\alpha_2(x)|$ for $x\in\mathbb{A}$. Let $\theta_1=(|\cdot|_1,+\infty)$. Then $d_{\theta_1}(x,y)=|y-x|_1$ for all $x,y\in\mathbb{A}$.*

Proof. Let $x, y \in \mathbb{A}$. Let $(v, v') \in U_{+\infty}(x, y)$. Then for $i \in \{1, 2\}$, we have $\alpha_i(v) \geq \max\big(0, \alpha_i(y-x)\big)$ and $\alpha_i(v') \geq \max\big(0, \alpha_i(x-y)\big)$. Thus $|v|_1 + |v'|_1 \geq |y-x|_1$ and hence $d_{\theta_1}(x, y) \geq |y-x|_1$.

Suppose $y - x \in \overline{C^v_f}$. Then $(y-x, 0) \in U_{+\infty}(x, y)$, thus $d_{\theta_1}(x, y) \leq |y-x|_1$ and hence $d_{\theta_1}(x, y) = |y-x|_1$.

Suppose $y - x \notin \pm\overline{C^v_f}$. Suppose for example $\alpha_1(x) \geq \alpha_1(y)$ and $\alpha_2(x) \leq \alpha_2(y)$. Then $\big(\alpha_2(y-x)u_2, \alpha_1(x-y)u_1\big) \in U_{+\infty}(x, y)$ and thus $d_{\theta_1}(x, y) \leq |y-x|_1$. Therefore $d_{\theta_1}(x, y) = |y-x|_1$ and the proposition follows. □

4.3. Study on the apartments containing $\mathfrak{s}$

We now study the d_θ, for $\theta \in \Theta_+ \cup \Theta_-$. In order to simplify the notation and by symmetry, we will mainly take $\theta \in \Theta_+$.

Fix $\theta \in \Theta_+$. Write $\theta = (|\ .\ |, \mathfrak{s})$, where $|\ .\ |$ is a norm and $\mathfrak{s}$ is a sector-germ.

Lemma 26. *Let A and B be two apartments containing $\mathfrak{s}$. Set $\rho : \mathcal{I} \xrightarrow{\mathfrak{s}} A$ and $\phi : A \xrightarrow{A \cap B} B$. Then:*

1. *the distance $d_\theta|_{A^2}$ is induced by some norm on A,*
2. *for all $x \in \mathcal{I}$ and $u \in \overline{C^v_f}$, $\rho(x +_{\mathfrak{s}} u) = \rho(x) +_{\mathfrak{s}} u$,*
3. *the retraction $\rho : (\mathcal{I}, d_\theta) \to (A, d_\theta|_{A^2})$ is 1-Lipschitz,*
4. *the map $\phi : (A, d_\theta|_{A^2}) \to (B, d_\theta|_{B^2})$ is an isometry.*

Proof. Let us prove 1. Let $\psi : \mathbb{A} \to A$ be such that $\psi(+\infty) = \mathfrak{s}$. Let $|\ .\ |' : \mathbb{A} \to \mathbb{R}^+$ be defined by $|a|' = d_\theta\big(\psi(a), \psi(0)\big)$ for $a \in \mathbb{A}$.

For $a_1, a_2 \in \mathbb{A}$, set $V(a_1, a_2) = \big\{(u_1, u_2) \in \overline{C^v_f}^2 \mid a_1 - a_2 = u_2 - u_1\big\}$. Let $(a_1, a_2) \in A$. Let $i \in \{1, 2\}$ and $u_i \in \overline{C^v_f}$. Then $a_i +_{\mathfrak{s}} u_i = \psi(\psi^{-1}(a_i) + u_i)$ and thus

$$U_{\mathfrak{s}}(a_1, a_2) = V\big(\psi^{-1}(a_1), \psi^{-1}(a_2)\big).$$

Let a_1, a_2. Then $U_{\mathfrak{s}}(a_1, a_2) = V\big(\psi^{-1}(a_1), \psi^{-1}(a_2)\big) = V\big(\psi^{-1}(a_1) - \psi^{-1}(a_2), 0\big)$. Consequently $d_\theta(a_1, a_2) = |\psi^{-1}(a_1) - \psi^{-1}(a_2)|'$. It remains to prove that $|\ .\ |'$ is a norm on $\mathbb{A}$. Let $a_1, a_2 \in \mathbb{A}$. Then

$$|a_1 + a_2|' = d_\theta\big(\psi(a_1 + a_2), \psi(0)\big) \leq d_\theta\big(\psi(a_1 + a_2), \psi(a_1)\big) + d_\theta\big(\psi(a_1), \psi(0)\big)$$

by Definition/Proposition 22. As $V(a_1 + a_2, a_1) = V(a_2, 0)$, we deduce the equality $d_\theta\big(\psi(a_1 + a_2), \psi(a_1)\big) = d_\theta\big(\psi(a_2), \psi(0)\big)$ and hence $|a_1 + a_2|' \leq |a_1|' + |a_2|'$.

Let $t \in \mathbb{R}$ and $a \in \mathbb{A}$. As $V(0, ta) = tV(0, a)$, we deduce that $|ta|' = |t||a|'$, which proves (1).

Let us prove (2). Let $x \in \mathcal{I}$ and A_x be an apartment containing $x + \mathfrak{s}$. Let $\phi : A_x \xrightarrow{A_x \cap A} A$. Let $\psi_x : \mathbb{A} \to A_x$ be such that $\psi_x(+\infty) = \mathfrak{s}$ and $\psi_A = \phi \circ \psi_x$. Then $\psi_A(+\infty) = \mathfrak{s}$. Let $u \in \overline{C^v_f}$. Then by Definition/Proposition 19, $A_x \ni x +_{\mathfrak{s}} u$ and $A \ni \rho(x) +_{\mathfrak{s}} u$. Therefore

$$\rho(x +_{\mathfrak{s}} u) = \phi(x +_{\mathfrak{s}} u) = \phi \circ \psi_x(\psi_x^{-1}(x) + u) = \psi_A(\psi_x^{-1}(x) + u)$$

and

$$\rho(x) +_{\mathfrak{s}} u = \psi_A\Big(\psi_A^{-1}\big(\phi(x)\big) + u\Big) = \psi_A\big(\psi_x^{-1}(x) + u\big) = \rho(x +_{\mathfrak{s}} u),$$

which proves (2).

By (2), for all $x, x' \in \mathcal{I}$, $U_{\mathfrak{s}}\big(\rho(x), \rho(x')\big) \supset U_s(x, x')$, which proves (3). By (3), $\phi^{-1} : (B, d_\theta|_{B^2}) \to (A, d_\theta|_{A^2})$ is 1-Lipschitz. By symmetry, $\phi : (A, d_\theta|_{A^2}) \to (B, d_\theta|_{B^2})$ is 1-Lipschitz, which proves (4). $\square$

Lemma 27. *Let d' be a distance on $\mathbb{A}$ induced by some norm on $\mathbb{A}$. Define $d_{\theta,d'} : \mathcal{I} \times \overline{C_f^v} \to \mathbb{R}_+$ by $d_{\theta,d'}\big((x,u),(x',u')\big) = d_\theta(x,x') + d'(u,u')$ for $(x,u),(x',u') \in \mathcal{I} \times \overline{C_f^v}$. Then the continuous map $(\mathcal{I} \times \overline{C_f^v}, d_{\theta,d'}) \to (\mathcal{I}, d_\theta)$ defined by $(x,u) \mapsto x +_{\mathfrak{s}} u$ is Lipschitz.*

Proof. Using isomorphisms of apartments, we may assume that $\mathfrak{s}$ is contained in $\mathbb{A}$. Since all norms on $\mathbb{A}$ are equivalent, it suffices to prove the assertion for a particular choice of d'. We choose $d' = d_\theta|_{\mathbb{A}^2}$, which is possible by Lemma 26(1). We regard $\overline{C_f^v}$ as a subset of $\mathcal{I}$.

Let $(x,u),(x',u') \in \mathcal{I} \times \overline{C_f^v}$. Let $\epsilon > 0$. Let $(u,u') \in U_{\mathfrak{s}}(x,x')$ and $(v,v') \in U_{\mathfrak{s}}(u,u')$ be such that $|u| + |u'| \leq d_\theta(x,x') + \epsilon$ and $|v| + |v'| \leq d_\theta(u,u') + \epsilon$. By Lemma 20, $(u+v, u'+v') \in U_{\mathfrak{s}}(x +_{\mathfrak{s}} u, x' +_{\mathfrak{s}} u')$, thus $d_\theta(x+u, x'+u') \leq |u| + |v| + |u'| + |v'| \leq d_\theta(x,x') + d_\theta(u,u') + 2\epsilon$ and hence $d_\theta(x+u, x'+u') \leq d_\theta(x,x') + d_\theta(u,u') = d_{\theta,d'}\big((x,u),(x',u')\big)$. Lemma follows. $\square$

Lemma 28. *For all $x, x' \in \mathcal{I}$, there exists $(u,u') \in U_{\mathfrak{s}}(x,x')$ such that $d_\theta(x,x') = |u| + |u'|$.*

Proof. Let $x, x' \in \mathcal{I}$ and let $\big((u_n, u_n')\big) \in U_{\mathfrak{s}}(x,x')^{\mathbb{N}}$ be such that $|u_n| + |u_n'| \to d_\theta(x,x')$. Then $(|u_n|), (|u_n'|)$ are bounded and thus after extraction we can assume that (u_n) and (u_n') converge in $(\overline{C_f^v}, |\ .\ |)$. Lemma 27 implies that $(\lim u_n, \lim u_n') \in U_{\mathfrak{s}}(x,x')$, which proves our assertion. $\square$

4.4. Geodesics in $\mathcal{I}$

Fix $\theta = (|\ .\ |, \mathfrak{s}) \in \Theta_+$. We now prove that for all $x_1, x_2 \in \mathcal{I}$, there exists a geodesic for d_θ between x_1 and x_2. However we prove that when $\dim \mathbb{A} \geq 2$, such a geodesic is not unique (even if θ is associated with a Euclidean norm). The non-uniqueness already appeared on the examples of Subsection 4.2 (see Propositions 23 and 25). As a comparison, Euclidean buildings equipped with their usual metrics are CAT(0) and thus uniquely geodesic.

Using isomorphisms of apartments, we may assume that $\mathfrak{s} = +\infty$. For all $x \in \mathbb{A}$ and $u \in \overline{C_f^v}$, $x +_{+\infty} u = x + u$. To simplify the notation we write $+$ instead of $+_{+\infty}$.

Lemma 29.

(1) *Let $x_1, x_2 \in \mathcal{I}$ and let $(u_1, u_2) \in U_{+\infty}(x_1, x_2)$ be such that $d_\theta(x_1, x_2) = |u_1| + |u_2|$. Then for both $i \in \{1,2\}$ and all $t, t' \in [0,1]$,*

$$d_\theta(x_i + tu_i, x_i + t'u_i) = |t' - t||u_i|$$

and

$$d_\theta(x_1 + tu_1, x_2 + t'u_2) = (1-t)|u_1| + (1-t')|u_2|.$$

(2) *Let $x \in \mathbb{A}$ and $(u_1, u_2) \in U_{+\infty}(0, x)$ be such that $d_\theta(0, x) = |u_1| + |u_2|$. Then for all $t_1, t_1', t_2, t_2' \in [0,1]$ such that $t_1 \leq t_1'$ and $t_2 \leq t_2'$,*

$$d_\theta(t_1 u_1 - t_2 u_2, t_1' u_1 - t_2' u_2) = (t_1' - t_1)|u_1| + (t_2' - t_2)|u_2|.$$

Proof. Let $t,t'\in[0,1]$. We assume $t\leq t'$. Let $i\in\{1,2\}$ and let j be such that $\{i,j\}=\{1,2\}$. As $x_i+u_i=x_j+u_j$,

$$\begin{aligned}d_\theta(x_1,x_2)\leq d_\theta(x_i,x_i+tu_i)+&d_\theta(x_i+tu_i,x_i+t'u_i)\\&+d_\theta(x_i+t'u_i,x_i+u_i)+d_\theta(x_j+u_j,x_j).\end{aligned}$$

By the definition of d_θ, $d_\theta(x_i,x_i+tu_i)\leq t|u_i|$, $d_\theta(x_i+tu_i,x_i+t'u_i)\leq(t'-t)|u_i|$, $d_\theta(x_i+t'u_i,x_i+u_i)\leq(1-t')|u_i|$ and $d_\theta(x_j+u_j,x_j)\leq|u_j|$.

As $d_\theta(x_1,x_2)=|u_1|+|u_2|=t|u_i|+(t'-t)|u_i|+(1-t')|u_i|+|u_j|$, we deduce that $d_\theta(x_i,x_i+tu_i)=t|u_i|$, $d_\theta(x_i+tu_i,x_i+t'u_i)=(t'-t)|u_i|$, $d_\theta(x_i+t'u_i,x_i+u_i)=(1-t')|u_i|$ and $d_\theta(x_j+u_j,x_j)=|u_j|$.

We no longer assume $t\leq t'$. One has

$$\begin{aligned}d_\theta(x_1+tu_1,x_2+t'u_2)&\geq d_\theta(x_1,x_2)-d_\theta(x_1,x_1+tu_1)-d_\theta(x_2,x_2+t'u_2)\\&=(1-t)|u_1|+(1-t')|u_2|.\end{aligned}$$

Moreover,

$$\begin{aligned}d_\theta(x_1+tu_1,x_2+t'u_2)&\leq d_\theta(x_1+tu_1,x_1+u_1)+d_\theta(x_2+u_2,x_2+t'u_2)\\&=(1-t)|u_1|+(1-t')|u_2|,\end{aligned}$$

which proves (1). A similar argument proves (2). $\square$

Remark 1. When we are in the situation of Proposition 23, then the u_1 and u_2 of Lemma 29 are in the boundary of $\overline{C^v_f}$.

Proposition 30. *Equip $\mathcal{I}$ with d_θ. For all $x_1,x_2\in\mathcal{I}$, there exists a geodesic from x_1 to x_2. Moreover, if $\dim\mathbb{A}\geq 2$, there exists a pair $(x_1,x_2)\in\mathcal{I}^2$ such that there are infinitely many geodesics from x_1 to x_2.*

Proof. Let $x_1,x_2\in\mathcal{I}$ be such that $x_1\neq x_2$. Let $(u_1,u_2)\in U_{+\infty}(x_1,x_2)$ be such that $|u_1|+|u_2|=d_\theta(x_1,x_2)$. Let $a_1=\frac{|u_1|}{|u_1|+|u_2|}$ and $a_2=1-a_1$. Set $\frac{1}{0}u_1=\frac{1}{0}u_2=0$.

Let $\gamma:[0,1]\to\mathcal{I}$ be defined by $\gamma(t)=x_1+\frac{t}{a_1}u_1$ if $t\in[0,a_1]$ and $\gamma(a_1+t)=x_2+(1-\frac{t}{a_2})u_2$ if $t\in[0,a_2]$. Then by Lemma 29 (1), for all $t,t'\in[0,1]$, $d_\theta(\gamma(t),\gamma(t'))=|t'-t|(|u_1|+|u_2|)$ and hence γ is a geodesic from x_1 to x_2.

Let now $x\in\mathbb{A}\backslash(\overline{C^v_f}\cup\overline{-C^v_f})$. Let us construct infinitely many geodesics joining 0 to x. Let $(u_1,u_2)\in U_{+\infty}(0,x)$ be such that $d_\theta(0,x)=|u_1|+|u_2|$. One has $x=u_1-u_2$ and thus $u_1,u_2\neq 0$. By Lemma 29, for $z\in[0,1]$, one has $d_\theta(0,zu_1)+d_\theta(zu_1,x)=d_\theta(0,x)$. Thus our idea is to concatenate a geodesic from 0 to zu_1 and a geodesic from zu_1 to x.

Let $z\in[0,1]$. Set $t_z=z|u_1|/(|u_1|+|u_2|)$. Let $\gamma_z:[0,1]\to\mathbb{A}$ be defined by:

$$\gamma_z(t)=\begin{cases}t\dfrac{|u_1|+|u_2|}{|u_1|}u_1 & \text{for } t\in[0,t_z],\\[2ex] zu_1+\dfrac{|u_1|+|u_2|}{(1-z)|u_1|+|u_2|}(t-t_z)\big((1-z)u_1-u_2\big) & \text{for } t\in[t_z,1].\end{cases}$$

Let $t, t' \in [0,1]$. First assume $0 \leq t \leq t' \leq t_z$. Then by Lemma 29 (2),

$$d\big(\gamma_z(t), \gamma_z(t')\big) = d\left(t\frac{|u_1|+|u_2|}{|u_1|}u_1, t'\frac{|u_1|+|u_2|}{|u_1|}u_1\right) = (t'-t)(|u_1|+|u_2|).$$

Assume $t_z \leq t \leq t' \leq 1$. Let

$$\tilde{t} = \frac{|u_1|+|u_2|}{(1-z)|u_1|+|u_2|}(t-t_z) \quad \text{and} \quad \tilde{t}' = \frac{|u_1|+|u_2|}{(1-z)|u_1|+|u_2|}(t'-t_z).$$

Then by Lemma 29(2),

$$\begin{aligned} d\big(\gamma_z(t), \gamma_z(t')\big) &= d\big((\tilde{t}(1-z)+z)u_1 - \tilde{t}u_2, (\tilde{t}'(1-z)+z)u_1 - \tilde{t}'u_2\big) \\ &= (\tilde{t}'-\tilde{t})((1-z)|u_1|+|u_2|) \\ &= (t'-t)(|u_1|+|u_2|). \end{aligned}$$

Assume $t \leq t_z \leq t'$. Then by Lemma29 (2),

$$\begin{aligned} d\big(\gamma_z(t), \gamma_z(t')\big) &= d\big(t\frac{|u_1|+|u_2|}{|u_1|}u_1, (\tilde{t}'(1-z)+z)u_1 - \tilde{t}'u_2\big) \\ &= \big(\tilde{t}'(1-z)+z-t\frac{|u_1|+|u_2|}{|u_1|}\big)|u_1| + \tilde{t}'|u_2| \\ &= (t'-t)(|u_1|+|u_2|). \end{aligned}$$

Therefore, γ_z is a geodesic from 0 to x. Moreover, as $x \notin \overline{C_f^v} \cup \overline{-C_f^v}$, $\mathbb{R}u_1 \neq \mathbb{R}u_2$, thus $\gamma_z([0,1]) \neq \gamma_{z'}([0,1])$ for all $z \neq z'$ and the proposition is proved. □

4.5. Equivalence of the distances of positive type

The aim of this section is to show that if $\theta_1, \theta_2 \in \Theta_+$, then d_{θ_1} and d_{θ_2} are equivalent. Fix a norm $|\ .\ |$ on $\mathbb{A}$.

Fix two adjacent positive sector-germs $\mathfrak{s}$ and $\mathfrak{s}'$ and set $\theta = (|\ .\ |, \mathfrak{s})$ and $\theta' = (|\ .\ |, \mathfrak{s})$. We begin by proving the existence of $\ell \in \mathbb{R}_+$ such that $d_{\theta'} \leq \ell d_\theta$ (see Lemma 35).

Fix an apartment A_0 containing $\mathfrak{s}$ and $\mathfrak{s}'$, which exists by (MA3). Let $\rho_{\mathfrak{s}} : \mathcal{I} \xrightarrow{\mathfrak{s}} A_0$ and $\rho_{\mathfrak{s}'} : \mathcal{I} \xrightarrow{\mathfrak{s}'} A_0$.

Lemma 31. *There exists $\ell_0 \in \mathbb{R}_{>0}$ such that for every apartment B containing $\mathfrak{s}$ and $\mathfrak{s}'$, for all $x, x' \in B$, $d_{\theta'}(x,x') \leq \ell_0 d_\theta(x,x')$.*

Proof. By Lemma 26(1) and the fact that all the norms on $\mathbb{A}$ are equivalent, there exists $\ell_0 \in \mathbb{R}_{>0}$ such that for all $x, x' \in A$, $d_{\theta'}(x,x') \leq \ell_0 d_\theta(x,x')$. Let B be an apartment containing $\mathfrak{s}$ and $\mathfrak{s}'$. Let $x, x' \in B$. By Lemma 26 (4), $d_\theta(x,x') = d_\theta(\rho_{\mathfrak{s}}(x), \rho_{\mathfrak{s}}(x'))$ and $d_{\theta'}(\rho_{\mathfrak{s}'}(x), \rho_{\mathfrak{s}'}(x')) = d_{\theta'}(x,x')$. Moreover $\rho_{\mathfrak{s}'|B} = \rho_{\mathfrak{s}|B}$, which proves the lemma. □

We now fix an apartment B_0 containing $\mathfrak{s}$ but not $\mathfrak{s}'$. Let $\mathfrak{F}_\infty$ be the sector-panel direction dominated by $\mathfrak{s}$ and $\mathfrak{s}'$. Using Lemma 8, one writes $B_0 = D_1 \cup D_2$, where D_1 and D_2 are two opposite half-apartments whose wall contains $\mathfrak{F}_\infty$ and such that $D_i \cup \mathfrak{s}$ is contained in some apartment B_i for both $i \in \{1,2\}$. We assume that $D_1 \supset \mathfrak{s}$.

Let M_0 be a wall of A_0 containing $\mathfrak{F}_\infty$ and $t_0 : A_0 \to A_0$ be the reflection with respect to M_0.

Lemma 32. *One has:*

$$\begin{cases} \rho_{\mathfrak{s}}(x) = \rho_{\mathfrak{s}'}(x) & \text{if } x \in D_1, \\ \rho_{\mathfrak{s}}(x) = \tilde{t} \circ \rho_{\mathfrak{s}'}(x) & \text{if } x \in D_2, \end{cases}$$

where $\tilde{t} = \tau \circ t_0$, for some translation τ of A_0.

Proof. Let $\rho_{\mathfrak{s},B_1} : \mathcal{I} \xrightarrow{\mathfrak{s}} B_1$ and $\rho_{\mathfrak{s}',B_1} : \mathcal{I} \xrightarrow{\mathfrak{s}'} B_1$.

Let $\phi_i : B_0 \xrightarrow{B_0 \cap B_i} B_i$, for $i \in \{1,2\}$ and $\phi : B_2 \xrightarrow{B_1 \cap B_2} B_1$. Let t be the reflection of B_1 with respect to $D_1 \cap D_2$. By Lemma 6, the following diagram is commutative:

$$\begin{tikzcd} B_0 \arrow[r, "\phi_2"] \arrow[d, "\phi_1"'] & B_2 \arrow[d, "\phi"] \\ B_1 \arrow[r, "t"] & B_1. \end{tikzcd} .$$

Let $x \in D_1$. Then $\rho_{\mathfrak{s},B_1}(x) = x = \rho_{\mathfrak{s}',B_1}(x)$. Let $\phi_3 : B_1 \xrightarrow{B_1 \cap A_0} A_0$. Then $\rho_{\mathfrak{s}}(x) = \phi_3(\rho_{\mathfrak{s},B_1}(x)) = \phi_3(\rho_{\mathfrak{s}',B_1}(x)) = \rho_{\mathfrak{s}'}(x)$.

Let $x \in D_2$. One has $\rho_{\mathfrak{s},B_1}(x) = \phi_1(x)$ and $\rho_{\mathfrak{s}',B_1}(x) = \phi(x)$ and thus $\rho_{\mathfrak{s},B_1}(x) = t \circ \rho_{\mathfrak{s}',B_1}(x)$. Let $\tilde{t}$ be such that the following diagram commutes:

$$\begin{tikzcd} B_1 \arrow[r, "t"] \arrow[d, "\phi_3"'] & B_1 \arrow[d, "\phi_3"] \\ A_0 \arrow[r, "\tilde{t}"] & A_0. \end{tikzcd} .$$

Then $\rho_{\mathfrak{s}}(x) = \tilde{t} \circ \rho_{\mathfrak{s}'}(x)$.

Moreover $\tilde{t}$ fixes $\phi_3(D_1 \cap D_2)$, which contains $\mathfrak{F}_\infty$. Thus $\tilde{t} = \tau \circ t_0$ for some translation τ of A_0 (by Lemma 5). $\square$

By Lemma 26 (1) and since every affine map on A_0 is Lipschitz, there exists $\ell_1 \in \mathbb{R}_+$ such that $t_0 : (A_0, d_{\theta'}) \to (A_0, d_{\theta'})$ is ℓ_1-Lipschitz. As t_0 is an involution, $\ell_1 \geq 1$.

Lemma 33. *Let ℓ_0 be as in Lemma 31. Then for all $x, x' \in B_0$, $d_{\theta'}(x,x') \leq \ell_0 \ell_1 d_\theta(x,x')$.*

Proof. Let $i \in \{1,2\}$ and $x, x' \in D_i$. By Lemma 26(4), $d_\theta(x,y) = d_\theta(\rho_{\mathfrak{s}}(x), \rho_{\mathfrak{s}}(x'))$ and $d_{\theta'}(x,x') = d_{\theta'}(\rho_{\mathfrak{s}'}(x), \rho_{\mathfrak{s}'}(x'))$. By Lemma 32, for all $x, x' \in D_i$, $d_{\theta'}(x,x') \leq \ell_0 \ell_1 d_\theta(x,x')$.

Let $x, x' \in B_0$. Assume that $x \in D_1$ and $x' \in D_2$. Let $m \in [x,x']_{B_0} \cap D_1 \cap D_2$. Then $d_{\theta'}(x,x') \leq d_{\theta'}(x,m) + d_{\theta'}(m,x') \leq \ell_0 \ell_1 \big(d_\theta(x,m) + d_\theta(m,x')\big)$. By Lemma 26(1), $d_\theta(x,m) + d_\theta(m,x') = d_\theta(x,x')$ and the lemma follows. $\square$

Lemma 34. *Let (X, d_X) be a metric space, $f : (\mathcal{I}, d_\theta) \to (X, d_X)$ be a map and $k \in \mathbb{R}_+$. Then f is k-Lipschitz if and only if for every apartment A containing $\mathfrak{s}$, $f|_A$ is k-Lipschitz.*

Proof. One implication is clear. Assume that for every apartment A containing $\mathfrak{s}$, $f|_A$ is k-Lipschitz. Let $x, x' \in \mathcal{I}$ and $A_x, A_{x'}$ be apartments containing $x + \mathfrak{s}$ and $x' + \mathfrak{s}$. Let $(u, u') \in U_{\mathfrak{s}}(x, x')$ be such that $|u| + |u'| = d_\theta(x, x')$, which exists by Lemma 28. Then $x +_{\mathfrak{s}} u \in A_x$ and $x' +_{\mathfrak{s}} u' \in A_{x'}$. One has

$$\begin{aligned} d_X\big(f(x), f(x')\big) &\leq d_X\big(f(x), f(x +_{\mathfrak{s}} u)\big) + d_X\big(f(x' +_{\mathfrak{s}} u'), f(x')\big) \\ &\leq k(|u| + |u'|) \leq k d_\theta(x, x'). \qquad \square \end{aligned}$$

Lemma 35. *One has* $d_{\theta'} \leq \ell_0 \ell_1 d_\theta$.

Proof. By Lemmas 34, 31, and 33, $\mathrm{Id} : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_{\theta'})$ is $\ell_0\ell_1$-Lipschitz, which proves the lemma. □

Theorem 36. *Let* $\theta_1, \theta_2 \in \Theta_+$. *Then the metrics* d_{θ_1} *and* d_{θ_2} *are equivalent.*

Proof. For $i \in \{1, 2\}$, write $\theta_i = (|\ .\ |_i, \mathfrak{s}_i)$. As all the norms on $\mathbb{A}$ are equivalent, we may assume $|\ .\ |_1 = |\ .\ |_2 = |\ .\ |$. Let $\mathfrak{t}^0 = \mathfrak{s}_1, \ldots, \mathfrak{t}^n = \mathfrak{s}_2$ be a gallery between $\mathfrak{s}_1$ and $\mathfrak{s}_2$. For $i \in [\![0, n]\!]$ set $\theta^i = (|\ .\ |, \mathfrak{t}^i)$. By an induction using Lemma 35, there exists $\ell \in \mathbb{R}_{>0}$ such that $d_{\theta_1} = d_{\theta^0} \leq \ell d_{\theta^n} = K d_{\theta_2}$. Theorem follows by symmetry. □

We thus obtain (at most) two topologies on $\mathcal{I}$: the topology $\mathscr{T}_+$ induced by d_{θ_+}, for each $\theta_+ \in \Theta_+$ and the topology $\mathscr{T}_-$ induced by d_{θ_-}, for each $\theta_- \in \Theta_-$. We will see that when $\mathcal{I}$ is not a building, these topologies are different (see Corollary 45).

Corollary 37. *Let A be an apartment of $\mathcal{I}$. Then the topology on A induced by $\mathscr{T}_+$ is the affine topology on A.*

Proof. By Theorem 36, this topology is induced by $d_{(|\ .\ |, \mathfrak{t})}$ for some positive sector-germ $\mathfrak{t}$ of A. Then Lemma 26(1) concludes the proof. □

Corollary 38. *Let ρ be a retraction centered at a positive sector-germ, $A = \rho(\mathcal{I})$, B be an apartment and d_A (resp. d_B) be a distance on A (resp. B) induced by a norm. Then:*

1. *for all $\theta \in \Theta_+$, $\rho : (\mathcal{I}, d_\theta) \to (A, d_A)$ is Lipschitz,*
2. *the map $\rho_{|B} : (B, d_B) \to (A, d_A)$ is Lipschitz.*

Proof. By Theorem 36, we may assume $\theta = (|\ .\ |, \mathfrak{t})$, where $\mathfrak{t}$ is the center of ρ. Then by Lemma 26(3), $\rho : (\mathcal{I}, d_\theta) \to (A, d_\theta)$ is Lipschitz and Lemma 26(1) completes the proof. □

Corollary 39. *Let A, B be two apartments of $\mathcal{I}$. Then $A \cap B$ is a closed subset of A (seen as an affine space).*

Proof. By Lemma 13, A and B are closed for $\mathscr{T}_+$ and thus $A \cap B$ is closed for $\mathscr{T}_+$. Consequently it is closed for the topology induced by $\mathscr{T}_+$ on A, and Corollary 37 completes the proof. □

Remark 2. Suppose that $\mathcal{I}$ is not a building. Then by Proposition 17, for all $\theta_+ \in \Theta_+$, $(\mathcal{I}, d_{\theta_+})$ is incomplete.

Let $\mathfrak{s}''$ be a positive sector-germ of $\mathcal{I}$, $\theta_+ = (|\ .\ |, \mathfrak{s}'')$ and (S_n) be an increasing sequence of sectors with the germ $\mathfrak{s}''$. One says that (S_n) is converging if there exists a retraction onto an apartment $\rho : \mathcal{I} \overset{\mathfrak{s}''}{\to} \rho(\mathcal{I})$ such that $(\rho(x_n))$ converges, where x_n is the base point of S_n for all $n \in \mathbb{N}$ and we call *limit* of (S_n) the set $\bigcup_{n\in\mathbb{N}} S_n$. One can show that the incompleteness of $(\mathcal{I}, d_\theta)$ implies the existence of a converging sequence of the direction $\mathfrak{s}''$ whose limit is not a sector of $\mathcal{I}$. To prove this, one can associate to each Cauchy sequence (x_n) a sequence (x_n') such that $d_{\theta_+}(x_n', x_n) \to 0$ and such that $x_n' + \mathfrak{s}'' \subset x_{n+1}' + \mathfrak{s}''$ for all $n \in \mathbb{N}$. Then we show that (x_n') converges in $(\mathcal{I}, d_\theta)$ if, and only if the limit of $(x_n' + \mathfrak{s}'')$ is a sector of $\mathcal{I}$.

4.6. Study of the action of G

In this subsection, we show that for every $g \in G$, the induced map $g : \mathcal{I} \to \mathcal{I}$ is Lipschitz for the distances of positive type.

Lemma 40. *Let $g \in G$ and $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$. Then for every $x \in \mathcal{I}$ and $u \in \overline{C_f^v}$, $g.(x +_{\mathfrak{s}} u) = g.x +_{g.\mathfrak{s}} u$.*

Proof. Let $x \in \mathcal{I}$ and $u \in \overline{C_f^v}$. Let A be an apartment containing $x + \mathfrak{s}$. Let $A' = g.A$. Then A' contains $\mathfrak{s}' = g.\mathfrak{s}$. Let $\psi : \mathbb{A} \to A$ be an isomorphism such that $\psi(+\infty) = \mathfrak{s}$. Let $f : A \to A'$ be the isomorphism induced by g. Set $\psi' = f \circ \psi$. Then $\psi'(+\infty) = \mathfrak{s}'$.

As $x +_{\mathfrak{s}} u \in A$,

$$\begin{aligned} g.(x +_{\mathfrak{s}} u) &= f(x +_{\mathfrak{s}} u) = f \circ \psi\big(\psi^{-1}(x) + u\big) \\ &= \psi'\big(\psi^{-1} \circ f^{-1}\big(f(x)\big) + u\big) = g.x +_{g.\mathfrak{s}} u. \qquad \square \end{aligned}$$

Theorem 41. *Let $g \in G$ and $\theta \in \Theta_+$. Then $g : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_\theta)$ is Lipschitz.*

Proof. Write $\theta = (|\ .\ |, \mathfrak{s})$. Let $\theta' = (|\ .\ |, g.\mathfrak{s})$. By Theorem 36, it suffices to prove that $g : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_{\theta'})$ is Lipschitz.

Let $x, x' \in \mathcal{I}$. By Lemma 40, $U_{g.\mathfrak{s}}(g.x, g.x') \supset U_{\mathfrak{s}}(x, x')$, thus $d_{\theta'}(g.x, g.x') \leq d_\theta(x, x')$, which proves the theorem. $\square$

4.7. Case of a building

In this subsection, we assume that $\mathcal{I}$ is a building. We show that the distances of positive type are equivalent to the usual distance.

Let $d_{\mathbb{A}}$ be a distance on $\mathbb{A}$ induced by some W^v-invariant euclidean norm $|\ .\ |$ on $\mathbb{A}$. Let $x, x' \in \mathcal{I}$, A be an apartment containing x, x' and $f : A \to \mathbb{A}$ be an isomorphism of apartments. One sets $d_{\mathcal{I}}(x, x') = d_{\mathbb{A}}\big(f(x), f(x')\big)$. Then $d_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}_+$ is well defined and is a distance on $\mathcal{I}$ (see [Bro89, VI.3] for example). Recall that $\rho_{+\infty} : \mathcal{I} \xrightarrow{+\infty} \mathbb{A}$.

Proposition 42. *Let $\theta \in \Theta_+$. Then $d_{\mathcal{I}}$ and d_θ are equivalent.*

Proof. By Theorem 36, one can assume that $\theta = (|\ .\ |, +\infty)$. Let $k, \ell \in \mathbb{R}_{>0}$ be such that $k d_{\mathcal{I}}|_{\mathbb{A}^2} \leq d_\theta|_{\mathbb{A}^2} \leq \ell d_{\mathcal{I}}|_{\mathbb{A}^2}$, which exists by Lemma 26(1). Let us first show that $\mathrm{Id} : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_{\mathcal{I}})$ is $\frac{1}{k}$-Lipschitz.

Let A be an apartment containing $+\infty$. Let $x, x' \in A$. Then by Lemma 26(4) and the fact that the restriction of $\rho_{+\infty}$ to A is an isometry for $d_{\mathcal{I}}$, $d_\theta(x, x') =$

$d_\theta(\rho_{+\infty}(x),\rho_{+\infty}(x')) \geq kd_{\mathcal{I}}(\rho_{+\infty}(x),\rho_{+\infty}(x')) = kd_{\mathcal{I}}(x,x')$. From Lemma 34, we deduce that $\mathrm{Id} : (\mathcal{I},d_\theta) \to (\mathcal{I},d_{\mathcal{I}})$ is $\frac{1}{k}$-Lipschitz.

Let $x,x' \in \mathcal{I}$. By Corollary 10, there exist $n \in \mathbb{N}_{>0}$ and $x_0 = x, x_1, \ldots, x_n = x' \in [x,x']$ such that $[x,x'] = \bigcup_{i=0}^{n-1}[x_i,x_{i+1}]$ and such that $[x_i,x_{i+1}] \cup +\infty$ is contained in an apartment for all $i \in [\![0,n-1]\!]$. By Lemma 26(4),

$$\begin{aligned} d_\theta(x,x') \leq \sum_{i=0}^{n-1} d_\theta(x_i,x_{i+1}) &= \sum_{i=0}^{n-1} d_\theta(\rho_{+\infty}(x_i),\rho_{+\infty}(x_{i+1})) \\ &\leq \ell \sum_{i=0}^{n-1} d_{\mathcal{I}}(\rho_{+\infty}(x_i),\rho_{+\infty}(x_{i+1})) \\ &= \ell \sum_{i=0}^{n-1} d_{\mathcal{I}}(x_i,x_{i+1}) = \ell d_{\mathcal{I}}(x,x'), \end{aligned}$$

which proves the proposition. □

5. Mixed distances

In this section, we begin by proving that unless $\mathcal{I}$ is a building, if $\mathfrak{s}_-$ is a negative sector-germ, then every retraction centered at $\mathfrak{s}_-$ is not continuous for $\mathscr{T}_+$ (see Subsection 5.1). To prove this, we show that the set of vertices $\mathcal{I}_0$ contains no isolated points when $\mathcal{I}$ is not a building and then we apply finiteness results of [Héb17].

This implies that $\mathscr{T}_+ \neq \mathscr{T}_-$ and motivates the introduction of mixed distances, each of which is the sum of a distance of positive type with a distance of negative type. We then study them.

5.1. Comparison of positive and negative topologies

Fix a norm $|\ .\ |$ on $\mathbb{A}$.

Proposition 43. *Let $\theta \in \Theta$. Then $\mathcal{I}_0$ is discrete in $(\mathcal{I},d_\theta)$ if and only if $\mathcal{I}$ is a building.*

Proof. Assume that $\mathcal{I}$ is a building. By Proposition 42, we can replace d_θ by the usual distance $d_{\mathcal{I}}$ on $\mathcal{I}$. By Lemma 15, $\mathcal{I}_0 \cap \mathbb{A} = Y$, which is a lattice in $\mathbb{A}$. Let $\eta > 0$ be such that for all $\lambda,\lambda' \in Y$, $d_{\mathcal{I}}(\lambda,\lambda') < \eta$ implies $\lambda = \lambda'$. Let $\lambda,\lambda' \in \mathcal{I}_0$ be such that $d_{\mathcal{I}}(\lambda,\lambda') < \eta$. Let A be an apartment of $\mathcal{I}$ containing λ and λ' and $g \in G$ be such that $g.A = \mathbb{A}$. Then $d_{\mathcal{I}}(g.\lambda,g.\lambda') = d_{\mathcal{I}}(\lambda,\lambda') < \eta$, thus $\lambda = \lambda'$ and hence $\mathcal{I}_0$ is discrete in $\mathcal{I}$.

Assume now that $\mathcal{I}$ is not a building and thus W^v is infinite. By Theorem 36, we can assume that $\theta = (|\ .\ |,+\infty)$. Let $\epsilon > 0$. Let us show that there exists $\lambda \in \mathcal{I}_0$ such that $d_\theta(\lambda,0) < 2\epsilon$ and $\lambda \neq 0$. Let M_0 be a wall of $\mathbb{A}$ containing 0 such that for all consecutive walls M_1 and M_2 of the direction M_0, $d_\theta(M_1,M_2) < \epsilon$ (such a direction exists by Proposition 1(2)). Let M be a wall such that $d_\theta(0,M) < \epsilon$ and such that $0 \notin D$, where D is the half-apartment of $\mathbb{A}$ delimited by M and containing $+\infty$. By Lemma 11, there exists an apartment A such that $A \cap \mathbb{A} = D$. Let $\phi : \mathbb{A} \xrightarrow{A\cap\mathbb{A}} A$ and $\mu = \phi(0)$. Let $x \in M$ be such that $d_\theta(0,x) < \epsilon$. Then by

Lemma 26(4), $d_\theta(\lambda, x) = d_\theta(0, x)$ and thus $d(\lambda, 0) < 2\epsilon$. As $\lambda \notin \mathbb{A}$, $\lambda \neq 0$ and we get the proposition. $\square$

Remark 3. In fact, by Theorem 41, and since G acts transitively on $\mathcal{I}_0$, we proved that when $\mathcal{I}$ is not a building, every point of $\mathcal{I}_0$ is a limit point.

If B is an apartment and $(x_n) \in B^{\mathbb{N}}$, one says that (x_n) diverges to ∞, if for some isomorphism $f : B \to \mathbb{A}$, $|f(x_n)| \to +\infty$.

Proposition 44. *Assume that $\mathcal{I}$ is not a building. Let $\mathfrak{s}_-$ be a negative sector-germ of $\mathcal{I}$ and $\theta \in \Theta_+$. Equip $\mathcal{I}$ with d_θ. Let ρ_- be a retraction centered at $\mathfrak{s}_-$ and $(\lambda_n) \in \mathcal{I}_0^{\mathbb{N}}$ be an injective and converging sequence. Then $\rho_-(\lambda_n) \to \infty$ in $\rho_-(\mathcal{I})$. In particular, ρ_- is not continuous.*

Proof. Let $A = \rho_-(\mathcal{I})$ and $\mathfrak{s}_+$ be the sector-germ of A opposite to $\mathfrak{s}_-$. Using Theorem 36, we may assume that $\theta = (|\ .\ |, \mathfrak{s}_+)$. Let $\rho_+ : \mathcal{I} \overset{\mathfrak{s}_+}{\to} A$. Let $\lambda = \lim \lambda_n$ and $\mu = \rho_+(\lambda)$. Then by Corollary 38, $\rho_+(\lambda_n) \to \mu$. Let $Y_A = \mathcal{I}_0 \cap A$. Then Y_A is a lattice in A by Lemma 15. As $\rho_+(\lambda_n) \in Y_A$ for all $n \in \mathbb{N}$, $\rho_+(\lambda_n) = \mu$ for n large enough.

For all $n \in \mathbb{N}$, $\rho_-(\lambda_n) \in Y_A$. By [Héb17, Thm. 5.6], for all $\lambda' \in Y_A$, $\rho_+^{-1}(\{\lambda'\}) \cap \rho_-^{-1}(\{\mu\})$ is finite, and the proposition follows. $\square$

Corollary 45. *If $\mathcal{I}$ is not a building, $\mathscr{T}_+$ and $\mathscr{T}_-$ are different.*

Remark 4. Proposition 44 shows that if $\theta, \theta' \in \Theta$ have opposite signs, then every open subset of $(\mathcal{I}, d_\theta)$ containing a point of $\mathcal{I}_0$ is unbounded with respect to $d_{\theta'}$.

5.2. Mixed distances

In this section, we define and study mixed distances.

Let $\Xi = \Theta_+ \times \Theta_-$. Let $\xi = (\theta_+, \theta_-) \in \Xi$. Set $d_\xi = d_{\theta_+} + d_{\theta_-}$.

Theorem 46. *Let $\xi \in \Xi$. We equip $\mathcal{I}$ with d_ξ. Then:*

1. *For each $\xi' \in \Xi$, d_ξ and $d_{\xi'}$ are equivalent.*
2. *For each $g \in G$, the induced map $g : \mathcal{I} \to \mathcal{I}$ is Lipschitz.*
3. *The topology induced on every apartment is the affine topology.*
4. *Every retraction of $\mathcal{I}$ centered at a sector-germ is Lipschitz.*
5. *The set $\mathcal{I}_0$ is discrete.*

Proof. The assertions 1 to 4 are consequences of Theorems 36, 41, Corollaries 37 and 38. Let us prove (5). Let $\lambda \in \mathcal{I}_0$ and set $\lambda_+ = \rho_{+\infty}(\lambda)$ and $\lambda_- = \rho_{-\infty}(\lambda)$. By [Héb17, Thm. 5.6], $\rho_{+\infty}^{-1}(\{\lambda_+\}) \cap \rho_{-\infty}^{-1}(\{\lambda_-\})$ is finite and thus there exists $r > 0$ such that $B(\lambda, r) \cap \rho_{+\infty}^{-1}(\{\lambda_+\}) \cap \rho_{-\infty}^{-1}(\{\lambda_-\}) = \{\lambda\}$, where $B(\lambda, r)$ is the open ball of the radius r and the center λ. Let $k \in \mathbb{R}_{>0}$ be such that $\rho_{+\infty}$ and $\rho_{-\infty}$ are k-Lipschitz. Let $\eta > 0$ be such that for all $\mu, \mu' \in Y$, $\mu \neq \mu'$ implies $d_\xi(\mu, \mu') \geq \eta$. Let $r' = \min(r, \eta/k)$. Let us prove that $B(\lambda, r') \cap \mathcal{I}_0 = \{\lambda\}$. Let $\mu \in B(\lambda, r') \cap \mathcal{I}_0$. Suppose $\rho_{\sigma\infty}(\mu) \neq \lambda_\sigma$, for some $\sigma \in \{-, +\}$. Then

$$k d_\xi(\mu, \lambda) \geq d_\xi(\rho_{\sigma\infty}(\mu), \rho_{\sigma\infty}(\lambda)) \geq \eta,$$

thus $\lambda \notin B(\lambda, r')$: a contradiction. Therefore $\rho_{+\infty}(\mu) = \lambda_+$ and $\rho_{-\infty}(\mu) = \lambda_-$, hence $\lambda = \mu$ by choice of r, which completes the proof of the theorem. $\square$

We denote by $\mathscr{T}_m$ the topology on $\mathcal{I}$ induced by any d_ξ, $\xi \in \Xi$.

5.3. Link with the initial topology with respect to the retractions

In this subsection, we prove that the topology $\mathscr{T}_m$ agrees with the initial topology with respect to the family of retractions centered at sector-germs (see Corollary 50). To this end, for each $u \in C_f^v$ we introduce a map $T_u : \mathcal{I} \to \mathbb{R}_+$ which, for each $x \in \mathcal{I}$, measures the distance along the ray $x + (\mathbb{R}_+ u)_\infty$ between $x \in \mathcal{I}$ and $\mathbb{A}$. We then use the fact that for all $\lambda \in Y \cap C_f^v$, $T_\lambda \leq \ell(\rho_{+\infty} - \rho_{-\infty})$, for some $\ell \in \mathbb{R}_+$ (see Lemma 47).

Fix a norm $| \, . \, |$ on $\mathbb{A}$.

Definition of y_u and T_u. We now review briefly the results of the paragraph "Definition of y_ν and T_ν" of [Héb17, Sect. 3]. Let $u \in C_f^v$ and $\sigma \in \{-,+\}$. Let $\delta_+ = \mathbb{R}_+ u \subset \mathbb{A}$ and $\delta_- = \mathbb{R}_- u \subset \mathbb{A}$. Then δ_+ and δ_- are generic rays. Let $x \in \mathcal{I}$, then there exists a unique $y_{\sigma u}(x) \in \mathbb{A}$ such that $x + \delta_{\sigma,\infty} \cap \mathbb{A} = y_{\sigma u}(x) + \sigma \mathbb{R}_+ u \subset \mathbb{A}$ and there exists a unique $T_{\sigma u}(x) \in \mathbb{R}_+$ such that

$$y_{\sigma u}(x) = x +_{\sigma\infty} T_{\sigma u}(x).u = \rho_{\sigma\infty}(x) + \sigma T_{\sigma u}(x).u.$$

Then for each $x \in \mathcal{I}$, $x \in \mathbb{A}$ if and only if $y_u(x) = x$, if and only if $T_u(x) = 0$.

Lemma 47. *Let $\lambda \in Y \cap C_f^v$. Then there exists $\ell_{| \, . \, |} \in \mathbb{R}_{>0}$ such that for all $x \in \mathcal{I}$,*

$$T_\lambda(x), T_{-\lambda}(x) \leq \ell_{| \, . \, |} |\rho_{+\infty}(x) - \rho_{-\infty}(x)|.$$

Proof. By [Héb17, Cor. 4.2 and Rem. 4.3], there exists a linear map $h : \mathbb{A} \to \mathbb{R}$ such that $T_\lambda(x), T_{-\lambda}(x) \leq h\big(\rho_{-\infty}(x) - \rho_{+\infty}(x)\big)$ for all $x \in \mathcal{I}$, which proves the existence of $\ell_{| \, . \, |}$. $\square$

Lemma 48. *Let $\xi \in \Xi$ and $a \in \mathcal{I}$. Let A be an apartment containing a. Let $\mathfrak{s}_-, \mathfrak{s}_+$ be two opposite sector-germs of A and $\rho_+ : \mathcal{I} \overset{\mathfrak{s}_+}{\to} A$, $\rho_- : \mathcal{I} \overset{\mathfrak{s}_-}{\to} A$. Then there exists $k \in \mathbb{R}_{>0}$ such that for all $x \in \mathcal{I}$, $d_\xi(a, x) \leq k\Big(d_\xi\big(a, \rho_-(x)\big) + d_\xi\big(a, \rho_+(x)\big)\Big)$.*

Proof. Using isomorphisms of apartments, we may assume $A = \mathbb{A}$, $\mathfrak{s}_+ = +\infty$ and $\mathfrak{s}_- = -\infty$. By Theorem 46(1), we may assume $\xi = \big((| \, . \, |, \mathfrak{s}_+), (| \, . \, |, \mathfrak{s}_-)\big)$.

Let $\lambda \in C_f^v$. Let $T_+ = T_\lambda : \mathcal{I} \to \mathbb{R}_+$ and $T_- = T_{-\lambda} : \mathcal{I} \to \mathbb{R}_+$. By Lemma 47 and Lemma 26(1), there exists $\ell \in \mathbb{R}_{>0}$ such that $T_\sigma(x) \leq \ell d_\xi(\rho_-(x), \rho_+(x))$ for all $x \in \mathcal{I}$ and both $\sigma \in \{-,+\}$.

Set $d_+ = d_{(| \, . \, |, +\infty)}$ and $d_- = d_{(| \, . \, |, -\infty)}$. Let $x \in \mathcal{I}$ and $\sigma \in \{-,+\}$. One has $x +_{\sigma\infty} T_\sigma(x) u = \rho_\sigma(x) + \sigma T_\sigma(x) u$. Thus,

$$\begin{aligned} d_\sigma(\rho_\sigma(x), x) \leq 2T_\sigma(x)|u| &\leq 2\ell |u| d_\sigma(\rho_-(x), \rho_+(x)) \\ &\leq 2\ell|u|\big(d_\sigma(\rho_-(x), a) + d_\sigma(\rho_+(x), a)\big). \end{aligned}$$

As $d_\sigma(a, x) \leq d\big(a, \rho_\sigma(x)\big) + d\big(\rho_\sigma(x), x\big)$ we deduce that

$$d_\xi(a, x) = d_-(a, x) + d_+(a, x) \leq (4\ell|u| + 2)\big(d_\xi(a, \rho_-(x)) + d_\xi(a, \rho_+(x))\big). \qquad \square$$

Corollary 49. *Let $\xi \in \Xi$. Equip $\mathcal{I}$ with $d = d_\xi$. Then, for $X \subset \mathcal{I}$ the following assertions are equivalent:*

1. *X is bounded.*
2. *For every retraction ρ centered at a sector-germ of $\mathcal{I}$, $\rho(X)$ is bounded.*
3. *There exist two opposite sector-germs $\mathfrak{s}_+$ and $\mathfrak{s}_-$ such that if $\rho_{\mathfrak{s}_-}$ and $\rho_{\mathfrak{s}_+}$ are retractions centered at $\mathfrak{s}_-$ and $\mathfrak{s}_+$, $\rho_{\mathfrak{s}_-}(X)$ and $\rho_{\mathfrak{s}_+}(X)$ are bounded.*

Moreover every bounded subset of $\mathcal{I}_0$ is finite.

Proof. By Theorem 46, (1) implies (2) which clearly implies (3). The fact that (3) implies (1) is a consequence of Lemma 48. The last assertion is a consequence of (3) and of [Héb17, Thm. 5.6]. $\square$

Corollary 50. *The sets $\rho_+^{-1}(V) \cap \rho_-^{-1}(V)$ such that V is an open set of an apartment A and ρ_-, ρ_+ are retractions onto A centered at opposite sector-germs of A form a basis of $\mathscr{T}_m$. In particular, $\mathscr{T}_m$ is the initial topology with respect to the retractions centered at sector-germs.*

Proof. This is a consequence of Lemma 48. $\square$

5.4. A continuity property for the map $+_{+\infty}$

The aim of this subsection is to prove the theorem below, which will be useful to prove the contractibility of $\mathcal{I}$ for $\mathscr{T}_m$. To simplify the notation, we write $+$ instead of $+_{+\infty}$.

Theorem 51. *Let $\xi \in \Xi$ and $u \in C_f^v$. Equip $\mathcal{I}$ with d_ξ. Then the map $\mathcal{I} \times \mathbb{R}_+ \to \mathcal{I}$ defined by $(x,t) \mapsto x + tu$ is continuous.*

To prove this theorem, we prove that if a sequence $\big((x_n),(t_n)\big) \in (\mathcal{I} \times \mathbb{R}_+)^{\mathbb{N}}$ converges to some $(x,t) \in \mathcal{I} \times \mathbb{R}_+$, then $(x_n + t_n u)$ converges to $x + tu$. We first treat the case where $t \neq 0$.

Fix $\xi \in \Xi$ and write $\xi = (\theta_+, \theta_-)$. Fix a norm $|\ .\ |$ on $\mathbb{A}$.

Lemma 52. *Let $u \in C_f^v$. Then $T_u : \mathcal{I} \to \mathbb{R}_+$ and $y_u : \mathcal{I} \to (\mathbb{A}, |\ .\ |)$ are Lipschitz for d_{θ_+} and d_ξ.*

Proof. By Theorems 36 and 46, we can assume $\theta_+ = (+\infty, |\ .\ |)$. Let $\ell \in \mathbb{R}_{>0}$ be such that for all $a \in \mathbb{A}$, $\ell|a|u - a \in C_f^v$. Let $x, x' \in \mathcal{I}$ and $(u, u') \in U_{+\infty}(x,x')$ be such that $d_{\theta_+}(x,x') = |u| + |u'|$, which exists by Lemma 28. Then $x + T_u(x)u \in \mathbb{A}$ and thus $x' + u' + T_u(x)u = x + u + T_u(x)u \in \mathbb{A}$. Therefore,

$$x' + u' + T_u(x)u + (\ell|u'|u - u') = x' + (T_u(x) + \ell|u'|)u \in \mathbb{A}.$$

Hence, $T_u(x') \leq T_u(x) + \ell|u'| \leq T_u(x) + \ell d_{\theta_+}(x,x')$. By symmetry we deduce that T_u is ℓ-Lipschitz for d_{θ_+}. The fact that y_u is Lipschitz for d_{θ_+} is a consequence of the continuity of the map $+$ (Lemma 27) and of the fact that $y_u = \rho_{+\infty} + T_u.u$. As $d_{\theta_+} \leq d_\xi$, the lemma is proved. $\square$

Lemma 53. *Let $u \in C_f^v$ and $(x_n, t_n) \in \mathcal{I} \times \mathbb{R}_{>0}$ be such that $x_n \to x$ for d_ξ and $t_n \to t$, for some $x \in \mathcal{I}$ and $t \in \mathbb{R}_{>0}$. Then $x_n + t_n u \to x + tu$ for d_ξ.*

Proof. First assume $x \in \mathbb{A}$. By Lemma 52, $T_u(x_n) \to T_u(x) = 0$. Consequently, for $n \in \mathbb{N}$ large enough, $x_n + t_n u \in \mathbb{A}$. Write $\xi = (\theta_+, \theta_-)$. By the continuity of the map $+$ for d_{θ_+} (Lemma 27), $x_n + t_n u \to x + tu$ for d_{θ_+}. As the topologies induced by d_{θ_+} and d_ξ on $\mathbb{A}$ agree with the topology induced by its structure of a finite-dimensional real vector-space (by Corollary 37 and Theorem 46(3)), we deduce that $x_n + u \to x + u$ for d_ξ.

We no longer assume that $x \in \mathbb{A}$. Let A be an apartment containing $x + \infty$. Let $\phi : A \xrightarrow{+\infty} \mathbb{A}$. Let $g \in G$ be an automorphism inducing ϕ. By Lemma 40, $x_n + t_n u = g^{-1}.\big(g.(x_n + t_n u)\big) = g^{-1}.(g.x_n + t_n u)$ for all $n \in \mathbb{N}$. As $g.x \in \mathbb{A}$, we deduce that $g.x_n + t_n u \to g.x + tu$ for d_ξ. By the continuity of $g^{-1} : (\mathcal{I}, d_\xi) \to (\mathcal{I}, d_\xi)$ (by Theorem 46(2)), $x_n + u \to x + u$ for d_ξ. $\square$

It remains to prove that if a sequence $(x_n, t_n) \in (\mathcal{I} \times \mathbb{R}_+)$ converges to $(x, 0)$, for some $x \in \mathcal{I}$, then $(x_n + t_n u)$ converges to x. In order to prove this, we first study the map $t \mapsto \rho_{-\infty}(x + tu)$.

Tits preorder on $\mathcal{I}$, vectorial distance on $\mathcal{I}$ and paths. Recall the definition of the Tits preorder $\leq$ on $\mathbb{A}$ from Subsection 2.2. As $\leq$ is invariant under the action of the Weyl group W^v, $\leq$ induces a preorder $\leq_A$ on every apartment A. Let A be an apartment and $x, y \in A$ be such that $x \leq_A y$. Then by [Rou11, Prop. 5.4], if A' is an apartment containing x, y, $x \leq_{A'} y$. This enables us to define the following relation $\leq$ on $\mathcal{I}$: if $x, y \in \mathcal{I}$, one says that $x \leq y$ if there exists an apartment A containing x, y and such that $x \leq_A y$. By [Rou11, Thm. 5.9], this defines a preorder on $\mathcal{I}$ and one calls $\leq$ the *Tits preorder*.

Let $x, x' \in \mathcal{I}$ be such that $x \leq x'$. Let A be an apartment containing x, x' and $f : A \to \mathbb{A}$ be an isomorphism of apartments. Then $f(x') - f(x)$ is in the Tits cone $\mathcal{T}$. Therefore there exists a unique $d^v(x, x')$ in $\overline{C^v_f} \cap W^v.\big(f(x') - f(x)\big)$. One calls d^v the *vectorial distance.*

Let $u \in \overline{C^v_f}$. A u-path is a piecewise linear continuous map $\pi : [0, 1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to $W^v.u$. Let $x, x' \in \mathcal{I}$ be such that $x \leq x'$, A is an apartment containing them and $f : \mathbb{A} \to A$. We define $\pi_{x,x'} : [0, 1] \to \mathcal{I}$ by $t \mapsto f\big((1 - t)f^{-1}(x) + tf^{-1}(x)\big)$. By [Rou11, Prop. 5.4], $\pi_{x,x'}$ does not depend on the choice of A.

Let $x \in \mathcal{I}$ and $u \in \overline{C^v_f}$. Then $d^v(x, x + u) = u$.

Lemma 54. *Let $x \in \mathcal{I}$ and $u \in \overline{C^v_f}$. Then $\rho_{-\infty} \circ \pi_{x,x+u}$ is a u-path.*

Proof. This is a weak version of [GR08, Thm. 6.2] (a Hecke path of the shape u is a u-path satisfying some conditions; see [GR08, Sect. 5] for the definition). $\square$

Recall that the $\alpha_i^\vee$, for $i \in I$, denotes the simple roots. Let $Q^\vee_{\mathbb{R}_+} = \{\sum_{i \in I} x_i \alpha_i^\vee \mid (x_i) \in (\mathbb{R}_+)^I\} \subset \mathbb{A}$.

Lemma 55. *Let $u \in \overline{C^v_f}$ and $\pi : [0, 1] \to \mathbb{A}$ be a u-path. Then $\pi(1) - \pi(0) - u \in Q^\vee_{\mathbb{R}_+}$.*

Proof. Let $w \in W^v$. Then by [Kac94, Prop. 3.12d)], $w.u - u \in -Q^\vee_{\mathbb{R}_+}$. Thus for all t such that $\pi'(t)$ is defined, $\pi'(t) - u \in -Q^\vee_{\mathbb{R}_+}$ and the lemma follows. $\square$

Let $| \ . \ |_0$ be a norm on $\mathbb{A}$ such that for all $q = \sum_{i \in i} q_i \alpha_i^\vee \in \bigoplus_{i \in I} \mathbb{R}\alpha_i^\vee$, $|q|_0 = \sum_{i \in I} |q_i|$.

Lemma 56. *Let $x \in \mathcal{I}$, $u \in \overline{C^v_f}$ and $t, t' \in \mathbb{R}_+$ be such that $t \leq t'$. Then*

$$|\rho_{-\infty}(x+tu) - \rho_{-\infty}(x)|_0 \leq (t+t')|u|_0 + |\rho_{-\infty}(x+t'u) - \rho_{-\infty}(x)|_0.$$

Proof. Write $\rho_{-\infty}(x+tu) - \rho_{-\infty}(x) = tu - q_1$ and $\rho_{-\infty}(x+t'u) - \rho_{-\infty}(x+tu) = (t'-t)u - q_2$, with $q_1, q_2 \in Q^\vee_{\mathbb{R}_+}$, which is possible by Lemmas 54 and 55. Then $\rho_{-\infty}(x+t'u) - \rho_{-\infty}(x) = t'u - q_1 - q_2$. One has $|\rho_{-\infty}(x+tu) - \rho_{-\infty}(x)|_0 \leq t|u|_0 + |q_1|_0$. By choice of $|\ .\ |_0$, $|q_1|_0 \leq |q_1+q_2|_0 = |\rho_{-\infty}(x+t'u) - \rho_{-\infty}(x) - t'u|_0$, and the lemma follows. □

The following lemma completes the proof of Theorem 51.

Lemma 57. *Let $u \in C^v_f$. Let $(x_n) \in \mathcal{I}^{\mathbb{N}}$ and $(t_n) \in \mathbb{R}_+^{\mathbb{N}}$ be such that (x_n) converges for d_ξ and (t_n) converges to 0. Then $(x_n + t_n u)$ converges to $\lim x_n$ for d_ξ.*

Proof. By Theorem 46, we can assume $\xi = \big((|\ .\ |_0, +\infty), (|\ .\ |_0, -\infty)\big)$. Let $x = \lim x_n$. By the same reasoning as in the proof of Lemma 53:

- we can assume $x \in \mathbb{A}$, and
- $x_n + (T_u(x_n) + t_n)u \to x$.

By Lemma 56, for all $n \in \mathbb{N}$,

$$\begin{aligned}|\rho_{-\infty}(x_n + t_n u) &- \rho_{-\infty}(x_n)|_0 \\ &\leq |\rho_{-\infty}(x_n + (T_u(x_n)+t_n)u) - \rho_{-\infty}(x)|_0 + (T_u(x_n) + 2t_n)|u|_0,\end{aligned}$$

and thus $\rho_{-\infty}(x_n + t_n u) \to \rho_{-\infty}(x)$.

By continuity of the map $+$ (Lemma 27) and the continuity of $\rho_{+\infty}$ (Corollary 38) for d_{θ_+}, $\rho_{+\infty}(x_n + t_n u) \to \rho_{+\infty}(x)$. Using Lemma 48, we deduce that $(x_n + t_n u)$ converges to x, which is the desired conclusion. □

6. Contractibility of $\mathcal{I}$

In this section, we prove the contractibility of $\mathcal{I}$ for $\mathscr{T}_+$, $\mathscr{T}_-$ and $\mathscr{T}_m$.

Let $|\ .\ |$ be a norm on $\mathbb{A}$, $\theta = (|\ .\ |, +\infty)$ and $\xi = \big((|\ .\ |, +\infty), (|\ .\ |, -\infty)\big)$. To simplify the notation, we write $+$ instead of $+_{+\infty}$.

Proposition 58. *Let $u \in C^v_f$. We define $\chi_u : \mathcal{I} \times [0,1] \to \mathcal{I}$ by*

$$\begin{cases} \chi_u(x,t) = x + \dfrac{t}{1-t}u & \text{if } \dfrac{t}{1-t} < T_u(x), \\ \chi_u(x,t) = y_u(x) & \text{if } \dfrac{t}{1-t} \geq T_u(x), \end{cases}$$

where we set $\frac{1}{0} = +\infty > t$ for all $t \in \mathbb{R}$. Then χ_u is a strong deformation retract on $\mathbb{A}$ for d_θ and d_ξ.

Proof. Let $x \in \mathbb{A}$ and $t \in [0,1]$. Then $T_u(x) = 0$ and thus $\chi_u(x,t) = y_u(x) = x$. Let $x \in \mathcal{I}$. Then $\chi_u(x,0) = x$ and $\chi_u(x,1) = y_u(x) \in \mathbb{A}$. It remains to show that χ_u is continuous for d_θ and d_ξ. Let $(x_n, t_n) \in (\mathcal{I} \times [0,1])^{\mathbb{N}}$ be a converging sequence for d_θ or d_ξ and $(x,t) = \lim(x_n, t_n)$. Suppose, for example, that $\frac{t}{1-t} < T_u(x)$ (the case $\frac{t}{1-t} = T_u(x)$ and $\frac{t}{1-t} > T_u(x)$ are analogous). Then by the continuity of T_u (Lemma 52), $\frac{t_n}{1-t_n} < T_u(x_n)$ for n large enough and thus by the continuity of the map $+$ (Lemma 27 for d_θ and Theorem 51 for d_ξ), $\chi_u(x_n, t_n) = x_n + \frac{t_n}{1-t_n}u \to x + \frac{t}{1-t}u = \chi_u(x,t)$. Therefore, χ_u is continuous, which concludes the proof. □

Corollary 59. *The masure $\mathcal{I}$ is contractible for $\mathscr{T}_+$, $\mathscr{T}_-$ and $\mathscr{T}_m$.*

Proof. Let $u \in C_f^v$. We define $\Upsilon_u : \mathcal{I} \times [0,1] \to \mathcal{I}$ by

$$\begin{cases} \Upsilon_u(x,t) = \chi_u(x,2t) & \text{if } t \leq \frac{1}{2}, \\ \Upsilon_u(x,t) = 2(1-t)y_u(x) & \text{if t} > \frac{1}{2}. \end{cases}$$

Then Υ_u is a strong deformation retract on $\{0\}$ for d_θ and d_ξ, which proves that $(\mathcal{I}, \mathscr{T}_+)$ and $(\mathcal{I}, \mathscr{T}_m)$ are contractible. By symmetry, $(\mathcal{I}, \mathscr{T}_-)$ is contractible. □

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