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New decay results for a viscoelastic-type Timoshenko system with infinite memory

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Abstract. This paper is concerned with the following memory-type Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \int\limits_0^{+\infty} g(s)\psi_{xx}(t-s)\mathrm{d}s = 0, \end{cases}$$

with Dirichlet boundary conditions, where g is a positive nonincreasing function satisfying, for some nonnegative functions ξ and G,

$$g'(t) \le -\xi(t)G(g(t)), \qquad \forall t \ge 0.$$

Under appropriate conditions on ξ and G, we establish some new decay results that generalize and improve many earlier results in the literature such as Mustafa (Math Methods Appl Sci 41(1): 192–204, 2018), Messaoudi et al. (J Integral Equ Appl 30(1): 117–145, 2018) and Guesmia (Math Model Anal 25(3): 351–373, 2020). We consider the equal speeds of propagation case, as well as the nonequal-speed case. Moreover, we delete some assumptions on the boundedness of initial data used in many earlier papers in the literature.

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Keywords. Timoshenko system, General decay, Infinite memory, Relaxation function, Viscoelasticity.

1. Introduction

In this paper, we consider the following viscoelastic-type Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \int\limits_0^{+\infty} g(s)\psi_{xx}(t-s)\mathrm{d}s = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \end{cases} \tag{1.1}$$

where $(x,t) \in (0,L)\times(0,+\infty)$, L, b, K, ρ_1, ρ_2 are positive physical constants, $\varphi_0, \varphi_1, \psi_0, \psi_1$ are given data and g is a relaxation function satisfying some conditions to be specified in the next section.
In 1921, Timoshenko [4] introduced the following system of hyperbolic partial differential equations as a model to describe the dynamics of a thick beam:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x + \psi)_x = 0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) = 0 & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{1.2}$$

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where ϕ is the transverse displacement, ψ is the rotational angle of the filament of the beam and ρ_1, ρ_2, b and K are fixed positive physical constants. For almost a century, a great number of researchers have devoted a considerable amount of time an effort studying this model. As a product, many results concerning the well-posedness and long-time behavior of the system have been established. For this matter of various types of dissipation, such as boundary and/or internal feedback, heat or thermoelasticity, infinite memory and Kelvin–Voigt damping have been utilized. See, for example, [5–17]. It is well known that the exponential stability of system (1.2) is achieved in the presence of linear damping mechanisms on both equations of (1.2) without imposing any condition on the speeds of wave propagation. But if the damping effect is acting on only one equation, the system is exponentially stable if and only if it has equal speeds of wave propagation, that is,

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{1.3}$$

The reader is advised to consult the above-cited references for detailed discussion on the stability analysis of Timoshenko systems.

1.1. Finite memory

Now, we concentrate on the stabilization of a viscoelastic-type Timoshenko system which is the main topic of this work. Viscoelastic-type Timoshenko system had received a considerable attention since the work of Ammar-Khodja et al. [18] in which the authors studied the following system:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x + \psi)_x = 0 & \text{in } (0, L) \times (0, +\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(u_x + \psi) + \int_0^t g(t-s)\psi_{xx}(s)\mathrm{d}s = 0 & \text{in } (0, L) \times (0, +\infty), \\ \phi(0, t) = \phi(L, t) = \psi(0, t) = \psi(L, t) = 0 & \text{for } t \geq 0, \end{cases} \tag{1.4}$$

where g is a positive nonincreasing differentiable L^1 function defined on $\mathbb{R}_+$. They established the uniform stability of the system if and only if identity (1.3) holds. For the rate of decay, they obtained exponential and polynomial stability of the system for the relaxation functions g decaying exponentially and polynomially, respectively. Guesmia and Messaoudi [19] proved the same decay result of [18] by weakening some of the assumptions on g. Precisely, they assumed that g satisfies, for some constants $k_0 > 0$ and $1 \leq p < \frac{3}{2}$,

$$g'(t) \leq -k_0 g^p(t), \qquad \forall\, t \geq 0. \tag{1.5}$$

Messaoudi and Mustafa [20] investigated the same system under the more general relation

$$g'(t) \leq -\xi(t)g(t), \qquad \forall\, t \geq 0, \tag{1.6}$$

where ξ is a positive nonincreasing differentiable function defined on $\mathbb{R}_+$. They proved for the first time a general decay result from which the exponential and polynomial stabilities are only special cases. The assumption (1.6) allows a wider class of relaxation functions. However, the "optimality" of the polynomial decay is not guaranteed. Very recently, Messaoudi and Hassan [2] analyzed system (1.4) under the following assumption on the relaxation function: for some $1 \leq p < \frac{3}{2}$ and for a ξ a positive nonincreasing differentiable defined function on $\mathbb{R}_+$,

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall\, t \geq 0. \tag{1.7}$$

They established more general decay results in the case of equal and nonequal speeds of wave propagation. This class of relaxation functions includes those of Ammar-Khodja et al. [18], Guesmia and Messaoudi

[19] and Messaoudi and Mustafa [20] as special cases. This latter result guarantees the optimal polynomial decay result; that is, the rate of decay of energy is exactly the rate of decay of the relaxation function.

1.2. Infinite memory

Giorgi et al. [21] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^3$

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)\mathrm{d}s + g(u) = f \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1.8}$$

with $K(0), K(+\infty) > 0$ and $K' \leq 0$ and proved the existence of global attractors for the solutions. Conti and Pata [22] considered the following semilinear hyperbolic equation:

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)\mathrm{d}s + g(u) = f \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1.9}$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(+\infty) > 0$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonlinear term of at most cubic growth satisfying some conditions. They proved the existence of a regular global attractor. In [23], Appleby et al. studied the linear integro-differential equation

$$u_{tt} + Au(t) + \int_{-\infty}^{t} K(t-s)Au(s)\mathrm{d}s = 0 \quad \text{for} \quad t > 0, \tag{1.10}$$

and established an exponential decay result for strong solutions in a Hilbert space. Pata [24] discussed the decay properties of the semigroup generated by the following equation:

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)\mathrm{d}s = 0 \quad \text{for} \quad t > 0, \tag{1.11}$$

where A is a strictly positive self-adjoint linear operator and $\alpha > 0, \beta \geq 0$ and the memory kernel μ is a decreasing function satisfying specific conditions. Subsequently, they established necessary as well as the sufficient conditions for the exponential stability. In [25], Guesmia considered

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)\mathrm{d}s = 0 \quad \text{for} \quad t > 0, \tag{1.12}$$

and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history kernels satisfies the following condition

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}\mathrm{d}s + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty, \tag{1.13}$$

such that

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty, \tag{1.14}$$

where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function. Using this approach, Guesmia and Messaoudi [26] later looked into

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)\mathrm{div}(a_1(x)\nabla u(s))\mathrm{d}s + \int_0^{+\infty} g_2(s)\mathrm{div}(a_2(x)\nabla u(t-s))\mathrm{d}s = 0,$$

in a bounded domain and under suitable conditions on a_1 and a_2 and for a wide class of relaxation functions g_1 and g_2 that are not necessarily decaying polynomially or exponentially and established a general decay result from which the usual exponential and polynomial decay rates are only special cases. For Timoshenko systems with infinite memory, Rivera et al. [16] considered vibrating systems of Timoshenko type with past history acting only in one equation. They showed that the dissipation given by the history term is strong enough to produce exponential stability if and only if the equations have the same wave speeds. In the case that the wave speeds of the equations are different, they showed that the solution decays polynomially to zero if the corresponding system does not decay exponentially as time goes to infinity, with rates that can be improved depending on the regularity of the initial data. Guesmia et al. [10] have adopted the method introduced in [25] with some necessary modifications to establish a general decay of the solution for a vibrating system of Timoshenko type in a one-dimensional bounded domain with an infinite history acting in the equation of the rotation angle. Guesmia and Messaoudi [27] discussed a Timoshenko system in the presence of an infinite memory, where the relaxation function satisfies (1.6) and established some general decay results for the equal and nonequal speed propagation cases. Recently, Guesmia [3] adapted the approach of [1] to two models of wave equations with infinite memory and proved, under (2.2) (below) relations between the decay rate of solutions and the growth of g at infinity. Al-Mahdi [28,29] also adapted the approach of [1] to some viscoelastic plate equations with relaxation functions satisfy the condition (2.2). The results of [3,28,29] improved and generalized the ones of [25,30–33] by getting a better decay rate and deleted some assumptions on the boundedness of initial data.
In the present work, we study the asymptotic behavior of solutions of (1.1) under the general assumption (2.2) (below) instead of the once in [25,30–34]. Furthermore, our class of admissible initial data is larger than the one considered in [25,31–34] because we do not assume any boundedness condition on ψ_{0x} by adapting the arguments of [1,3] to the case of Timoshenko system (1.1).
The rest of this paper is organized as follows. In Sect. 2, we present some assumptions and material needed for our work. Some technical lemmas are presented and proved in Sect. 3. Finally, we state and prove our main decay results and provide some examples in Sect. 4.

2. Preliminaries

In this section, we present some materials needed for the proof of our results and state the existence result of the problem. We use the standard Lebesgue space $L^2(0,L)$ and Sobolev space $H_0^1(0,L)$ with their usual scalar products and norms and assume the following hypotheses

(A) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 nonincreasing function satisfying, for some $\beta_0 > 0$,

$$-\beta_0 g(s) \le g'(s), \; g(t) > 0 \text{ and } \quad b - \int_0^{+\infty} g(s)\mathrm{d}s := \ell > 0, \tag{2.1}$$

and there exists a C^1 function $G : \mathbb{R}_+ \to \mathbb{R}_+$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0,r]$ for some $r > 0$ with $G(0) = G'(0) = 0$, $\lim_{s\to+\infty} G'(s) = +\infty$, $s \mapsto sG'(s)$ and $s \mapsto s\,(G')^{-1}(s)$ are convex on $(0,r]$. Moreover, there exists a positive nonincreasing differentiable function ξ such that

$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall t \geq 0, \tag{2.2}$$

where ξ is satisfying $\int\limits_0^{+\infty} \xi(s)\mathrm{d}s = +\infty$.

Remark 2.1. [1] If G is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is strictly increasing and strictly convex C^2 function on $\mathbb{R}_+$. For instance, if $G(r) = a, G'(r) = b$ and $G''(r) = c$, we can define $\overline{G}$, for $t > r$, by

$$\overline{G}(t) = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right). \tag{2.3}$$

For simplicity, in the rest of this paper, we use G instead of $\overline{G}$

Remark 2.2. [1] Since G is strictly convex on $(0, r]$ and $G(0) = 0$, then

$$G(\theta z) \leq \theta G(z),\ 0 \leq \theta \leq 1 \text{ and } z \in (0, r]. \tag{2.4}$$

Remark 2.3. [1] For any $0 < \alpha < 1$, we define the following

$$C_\alpha = \int\limits_0^{+\infty} \frac{g^2(s)}{\alpha g(s) - g'(s)}\mathrm{d}s \qquad \text{and} \qquad h(t) = \alpha g(t) - g'(t). \tag{2.5}$$

Using the fact that $\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$ and recalling the Lebesgue dominated convergence theorem, we can easily deduce that

$$\alpha C_\alpha = \int\limits_0^{+\infty} \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)}\mathrm{d}s \to 0 \text{ as } \alpha \to 0. \tag{2.6}$$

For completeness, we state, without proof, the global existence and regularity result which can be established by the semigroup theory (see [10,27] where some arguments of [35] are used).

Proposition 2.4. *Let $(\varphi_0, \varphi_1), (\psi_0(., 0), \psi_1) \in H_0^1(0, L) \times L^2(0, L)$ be given. Assume that g satisfies hypothesis (A). Then, problem (1.1) has a unique global (weak) solution $\varphi, \psi \in C(\mathbb{R}_+; H_0^1(0, L)) \cap C^1(\mathbb{R}_+; L^2(0, L))$. Moreover, if $(\varphi_0, \varphi_1), (\psi_0(., 0), \psi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$, then the problem has a unique classical solution $\varphi, \psi \in C(\mathbb{R}_+; H^2(0, L) \cap H_0^1(0, L)) \cap C^1(\mathbb{R}_+; H_0^1(0, L)) \cap C^2(\mathbb{R}_+; L^2(0, L))$.*

We introduce the "modified" energy associated to problem (1.1)

$$E(t) := \frac{1}{2}\left(\rho_1||\varphi_t^2||_2 + \rho_2||\psi_t^2||_2 + \ell||\psi_x^2||_2 + K||(\varphi_x + \psi)^2||_2\right) + \frac{1}{2}(g \circ \psi_x)(t), \tag{2.7}$$

where $||\cdot||_2 = ||\cdot||_{L^2(0,L)}$ and for any $u \in L^2\left(\mathbb{R}_+; L^2(0, L)\right)$,

$$(gou)(t) = \int\limits_0^{+\infty} g(s)||u(t) - u(t - s)||_2^2\mathrm{d}s.$$

Direct differentiation, using (1.1), leads to

$$E'(t) = \frac{1}{2}(g'o\psi_x)(t) \leq 0, \quad \forall t \geq 0. \tag{2.8}$$

3. Technical lemmas

In this section, we state and establish several lemmas needed for the proof of our main result. We assume that (A) holds and take $(\varphi_0, \varphi_1), (\psi_0(.,0), \psi_1) \in H_0^1(0,L) \times L^2(0,L)$.

Lemma 3.1. *There exists a positive constant M_1 such that*

$$\int_t^{+\infty} g(s)||\psi_x(t) - \psi_x(t-s)||_2^2 \mathrm{d}s \leq M_0 h_0(t), \tag{3.1}$$

and

$$\int_t^{+\infty} g(s)||\psi_{xt}(t) - \psi_{xt}(t-s)||_2^2 \mathrm{d}s \leq M_1 h_1(t), \tag{3.2}$$

where $h_0(t) = \int_0^{+\infty} g(t+s)\left(1 + ||\psi_{0x}(s)||^2\right) \mathrm{d}s$ *and* $h_1(t) = \int_0^{+\infty} g(t+s)\left(1 + ||\psi_{0xt}(s)||^2\right) \mathrm{d}s$.

Proof. The proof of (3.1) is identical to the one in [3]. Indeed, we have

$$\begin{aligned}
&\int_t^{+\infty} g(s)||\psi_x(t) - \psi_x(t-s)||_2^2 \mathrm{d}s \leq 2||\psi_x(t)||^2 \int_t^{+\infty} g(s)\mathrm{d}s + 2\int_t^{+\infty} g(s)||\psi_x(t-s)||^2 \mathrm{d}s \\
&\leq 2\sup_{s\geq 0}||\psi_x(s)||^2 \int_0^{+\infty} g(t+s)\mathrm{d}s + 2\int_0^{+\infty} g(t+s)||\psi_x(-s)||^2 \mathrm{d}s \\
&\leq \frac{4E(s)}{\ell} \int_0^{+\infty} g(t+s)\mathrm{d}s + 2\int_0^{+\infty} g(t+s)||\psi_{0x}(s)||^2 \mathrm{d}s \\
&\leq \frac{4E(0)}{\ell} \int_0^{+\infty} g(t+s)\mathrm{d}s + 2\int_0^{+\infty} g(t+s)||\psi_{0x}(s)||^2 \mathrm{d}s \\
&\leq M_0 \int_0^{+\infty} g(t+s)\left(1 + ||\psi_{0x}(s)||^2\right) \mathrm{d}s.
\end{aligned} \tag{3.3}$$

where $M_0 = \max\left\{2, \frac{4E(0)}{\ell}\right\}$. To prove (3.2), we use the same arguments in the proof of (3.1). $\square$

Lemma 3.2. *The following functionals*

$$F(t) := -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s))\mathrm{d}s\mathrm{d}x,$$

$$I(t) := \varepsilon I_1(t) + I_2(t) + I_3(t),$$

$$I_1(t) := -\int_0^L (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t)\mathrm{d}x,$$

$$I_2(t) := \rho_2 \int_0^L \psi_t(\varphi_x + \psi)\mathrm{d}x + \frac{b\rho_1}{K}\int_0^L \varphi_t \psi_x dx - \frac{\rho_1}{K}\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_x(t-s)\mathrm{d}s\mathrm{d}x,$$

$$I_3(t) := \frac{\rho_2}{4\varepsilon}\int_0^L m(x)\psi_t \left(b\psi_x - \int_0^{+\infty} g(s)\psi_x(t-s)\mathrm{d}s \right)\mathrm{d}x + \varepsilon \frac{\rho_1}{K}\int_0^L m(x)\varphi_t\varphi_x \mathrm{d}x$$

and

$$J(t) := \int_0^L (\rho_1 w\varphi_t + \rho_2 \psi\psi_t)\mathrm{d}x,$$

satisfy, along the solution of (1.1)*, the following estimates*

$$\begin{aligned} F'(t) \le & -\rho_2 (b - \ell - \delta) \int_0^L \psi_t^2 \mathrm{d}x + \delta K \int_0^L (\varphi_x + \psi)^2 \mathrm{d}x \\ & + \delta \int_0^L \psi_x^2 \mathrm{d}x + \frac{c}{\delta}(C_\alpha + 1)(h \circ \psi_x)(t), \end{aligned} \tag{3.4}$$

$$\begin{aligned} I'(t) \le & -\frac{K}{2}\int_0^L (\varphi_x + \psi)^2 \mathrm{d}x - c_1\rho_1 \int_0^L \varphi_t^2 \mathrm{d}x \\ & + c\rho_2 \int_0^L \psi_t^2 \mathrm{d}x + c\int_0^L \psi_x^2 \mathrm{d}x \\ & + c(C_\alpha + 1)(h \circ \psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt} \mathrm{d}x, \end{aligned} \tag{3.5}$$

and

$$J'(t) \le \varepsilon_0 \rho_1 \int_0^L \varphi_t^2 \mathrm{d}x + \frac{c}{\varepsilon_0}\rho_2 \int_0^L \psi_t^2 \mathrm{d}x - \frac{l}{2}\int_0^L \psi_x^2 \mathrm{d}x + \frac{C_\alpha}{2l}(h \circ \psi_x)(t), \quad \forall\, \varepsilon_0 > 0. \tag{3.6}$$

Proof. The proof of this lemma can be done by following the calculations in [2]. □

Lemma 3.3. [2] *There exist strictly positive constants* N, N_1, N_2 *and* ε *such that the functional*

$$L(t) = NE(t) + N_1 F(t) + I(t) + N_2 J(t)$$

satisfies, for all $t \in \mathbb{R}_+$,

$$L \sim E, \tag{3.7}$$

and

$$
\begin{aligned}
L'(t) \leq & -\frac{K}{4}||(\varphi_x+\psi)^2||_2 - \frac{\rho_1}{4}||\varphi_t^2||_2 - \frac{\rho_2}{4}||\psi_t^2||_2 - 4(b-l)||\psi_x^2||_2 \\
& + \frac{1}{4}(g\circ\psi_x)(t) + \left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}\mathrm{d}x, \qquad \forall\, t\geq 0,
\end{aligned} \tag{3.8}
$$

Proof. It is a routine computation to establish that $L(t)\sim E(t)$. To prove (3.8), combining (2.8), (3.4), (3.5), (3.6), recalling that $g'=\alpha g-h$ and setting $\delta=\frac{1}{4N_1}$, we obtain, for all $t\geq t_0$,

$$
\begin{aligned}
L'(t) \leq & -\frac{K}{4}\int_0^L(\varphi_x+\psi)^2\mathrm{d}x - (c_1-N_2\varepsilon_0)\rho_1\int_0^L\varphi_t^2\mathrm{d}x \\
& - \left[(b-\ell)N_1 - \frac{1}{4} - c\left(1+\frac{1}{\varepsilon_0}N_2\right)\right]\rho_2\int_0^L\psi_t^2\mathrm{d}x \\
& - \left(\frac{l}{2}N_2 - \frac{1}{4} - c\right)\int_0^L\psi_x^2\mathrm{d}x + \frac{\alpha}{2}N(g\circ\psi_x)(t) \\
& - \left[\frac{1}{2}N - c(4N_1^2+1) - C_\alpha\left(\frac{1}{2l}N_2 + c + 4cN_1^2\right)\right](h\circ\psi_x)(t) \\
& + \left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L\varphi_t\psi_{xt}\mathrm{d}x.
\end{aligned}
$$

We start by choosing N_2 large enough so that

$$
\frac{l}{2}N_2 - \frac{1}{4} - c > 4(b-l),
$$

then pick ε_0 so small that

$$
c_1 - N_2\varepsilon_0 > \frac{1}{4}.
$$

Next, we select N_1 so large that

$$
(b-\ell)N_1 - \frac{1}{4} + c(1+\frac{1}{\varepsilon_0}N_2) > \frac{1}{4}.
$$

As $\dfrac{\alpha g^2(s)}{\alpha g(s)-g'(s)} < g(s)$, it follows from the Lebesgue Dominated Convergence Theorem that

$$
\lim_{\alpha\to 0^+}\alpha C_\alpha = \lim_{\alpha\to 0^+}\int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s)-g'(s)}\mathrm{d}s = 0.
$$

Consequently, there exists $0<\alpha_0<1$ such that if $\alpha<\alpha_0$, then

$$
\alpha C_\alpha < \frac{1}{8\left[\frac{1}{2l}N_2 + c(1+4N_1^2)\right]}.
$$

Now, choose N large enough so that

$$
N > \max\left\{4c(4N_1^2+1),\ \frac{1}{2\alpha_0}\right\}
$$

and set

$$\alpha = \frac{1}{2N}.$$

So

$$\frac{1}{4}N - c(4N_1^2 + 1) > 0 \qquad \text{and} \qquad \alpha = \frac{1}{2N} < \alpha_0.$$

This gives

$$\begin{aligned} \frac{1}{2}N - c(4N_1^2 + 1) - C_\alpha \left[\frac{1}{2l}N_2 + c(1 + 4N_1^2)\right] &> \frac{1}{2}N - c(4N_1^2 + 1) - \frac{1}{8\alpha} \\ &= \frac{1}{4}N - c(4N_1^2 + 1) > 0. \end{aligned}$$

Hence, we arrive at the required estimate. □

Lemma 3.4. *The functional*

$$N_3(t) = \int_0^t p(t-s)||\psi_x(s)||_2^2 \mathrm{d}s,$$

satisfies, along the solution of (1.1), *the estimate*

$$N_3'(t) \le -\frac{1}{2}(g \circ \psi_x)(t) + 3(b - \ell)||\psi_x(t)||_2^2 \mathrm{d}x + \frac{1}{2}\int_t^{+\infty} g(s)||\psi_x(t) - \psi_x(t-s)||_2^2 \mathrm{d}s, \tag{3.9}$$

where $p(t) = \int_t^{+\infty} g(s)\mathrm{d}s$.

Proof. In fact, we have the following

$$p'(t) = -g(t), \ \int_0^t g(t-s)\mathrm{d}s = \int_0^t g(s)\mathrm{d}s = \int_0^\infty g(s)\mathrm{d}s - \int_t^\infty g(s)\mathrm{d}s = p(0) - p(t). \tag{3.10}$$

Now, direct differentiation of N_3 leads to

$$\begin{aligned} N_3'(t) &= p(0)||\psi_x(t)||_2^2 + \int_0^t p'(t-s)||\psi_x(s)||_2^2 \mathrm{d}s \\ &= p(0)||\psi_x(t)||_2^2 - \int_0^t g(t-s)||\psi_x(s)||_2^2 \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= p(0)||\psi_x(t)||_2^2 - \int_0^t g(t-s)||\psi_x(s)-\psi_x(t)+\psi_x(t)||_2^2 \mathrm{d}s \\
&= p(0)||\psi_x(t)||_2^2 - \int_0^t g(t-s)||\psi_x(s)-\psi_x(t)||_2^2 - 2\int_0^t g(t-s)||\psi_x(s)-\psi_x(t)||_2||\psi_x(t)||_2 \\
&\quad - \int_0^t g(t-s)||\psi_x(t)||_2^2 \mathrm{d}s \\
&= p(t)||\psi_x(t)||_2^2 + 2\int_0^t g(t-s)||\psi_x(t)-\psi_x(s)||_2||\psi_x(t)||_2 - \int_0^t g(t-s)||\psi_x(t)||_2^2 \mathrm{d}s \\
&\le p(t)||\psi_x(t)||_2^2 - \int_0^t g(t-s)||\psi_x(t)||_2^2 \mathrm{d}s \\
&\quad + 2(b-\ell)||\psi_x(t)||_2^2 + \frac{\int_0^t g(s)\mathrm{d}s}{2(b-\ell)} \int_0^t g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s \\
&\le 3(b-\ell)||\psi_x(t)||_2^2 - \int_0^t g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s + \frac{1}{2}\int_0^t g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\le 3(b-\ell)||\psi_x(t)||_2^2 - \frac{1}{2}\int_0^t g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s \\
&\le 3(b-\ell)||\psi_x(t)||_2^2 - \frac{1}{2}\int_0^\infty g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s + \frac{1}{2}\int_t^\infty g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s \qquad (3.12) \\
&\le 3(b-\ell)||\psi_x(t)||_2^2 - \frac{1}{2}\left(g\circ\psi_x\right)(t) + \frac{1}{2}\int_t^\infty g(t-s)||\psi_x(t)-\psi_x(s)||_2^2 \mathrm{d}s.
\end{aligned}
$$

Then, (3.9) is established. □

Lemma 3.5. *Assume that (1.3) holds. Then, the energy functional satisfies, for all $t \in \mathbb{R}^+$ and for some positive constant $\tilde{m}$, the following estimate*

$$
\int_0^t E(s)\mathrm{d}s < \tilde{m} f(t), \tag{3.13}
$$

where $f(t) = 1 + \int_0^t h_0(s)\mathrm{d}s$ and h_0 is defined in Lemma 3.1.

Proof. As in [3], let $F(t) = L(t) + N_3(t)$; then using (3.8) and (3.9), we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} F'(t) \leq & -\frac{K}{4}||(\phi_x+\psi)^2||_2 - \frac{\rho_1}{4}||\phi_t^2||_2 - \frac{\rho_2}{4}||\psi_t^2||_2 - (b-l)||\psi_x^2||_2 - \frac{1}{4}(g\circ\psi_x)(t) \\ & + \frac{1}{2}\int_0^L \int_t^{+\infty} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)^2 \mathrm{d}s\mathrm{d}x. \end{aligned} \tag{3.14}$$

From (2.7) and (3.14), we obtain for all $t \in \mathbb{R}_+$,

$$F'(t) \leq -mE(t) + \frac{1}{2}\int_t^{+\infty} g(s)||\psi_x(t)-\psi_x(t-s)||_2^2 \mathrm{d}s,$$

where m is some positive constant. Therefore, using (3.1), we obtain

$$\begin{aligned} m\int_0^t E(s)\mathrm{d}s &\leq F(0) - F(t) + \frac{M_0}{2}\int_0^t \int_0^{+\infty} g(\tau+s)\left(1+|\psi_{x0}(s)|\mathrm{d}s\right)^2 \mathrm{d}\tau\mathrm{d}s \\ &\leq F(0) + \frac{M_0}{2}\int_0^t h_0(s)\mathrm{d}s. \end{aligned} \tag{3.15}$$

Hence, we get

$$\int_0^t E(s)\mathrm{d}s \leq \frac{F(0)}{m} + \frac{M_0}{2m}\int_0^t h_0(s)\mathrm{d}s \leq \tilde{m}\left(1+\int_0^t h_0(s)\mathrm{d}s\right), \tag{3.16}$$

where $\tilde{m} = \max\left\{\frac{F(0)}{m}, \frac{M_0}{2m}\right\}$. □

Corollary 3.6. *There exists $0 < q_0 < 1$ such that, for all $t \geq 0$, we have the following estimate:*

$$\int_0^t g(s)||\psi_x(t)-\psi_x(t-s)||_2^2 \mathrm{d}s \leq \frac{1}{q(t)} G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) \tag{3.17}$$

where G is defined in Remark (2.1),

$$\mu(t) := -\int_0^t g'(s)||\psi_x(t)-\psi_x(t-s)||_2^2 \mathrm{d}s \leq -cE'(t) \tag{3.18}$$

and

$$q(t) := \frac{q_0}{f(t)}. \tag{3.19}$$

Proof. As in [3], using (2.7) and (3.13), we have

$$\begin{aligned} \int_0^t ||\psi_x(t)-\psi_x(t-s)||_2^2 \mathrm{d}s &\leq 2\int_0^L \int_0^t \left(|\psi_x(t)|^2 + |\psi_x(t-s)|^2\right) \mathrm{d}s\mathrm{d}x \\ &\leq \frac{4}{\ell}\int_0^t \left(E(t)+E(t-s)\right)\mathrm{d}s \\ &\leq \frac{8}{\ell}\int_0^t E(s)\mathrm{d}s \leq \frac{8}{\ell}\tilde{m}f(t), \ \forall\, t \in \mathbb{R}_+. \end{aligned} \tag{3.20}$$

Thanks to (3.13), we have for all $t \geq 0$ and for $0 < q_0 < \min\left\{1, \frac{\ell}{8\tilde{m}}\right\}$, $q(t) < 1$ and

$$q(t) \int_0^t ||\psi_x(t) - \psi_x(t-s)||_2^2 \mathrm{d}s < 1.$$

So, the proof of (3.17) can be archived as the one given in [1]. □

4. A decay result for equal speeds of wave propagation

In this section, we state and prove a new general decay result in the case of equal speeds of wave propagation (1.3). As in [3], we introduce the following functions:

$$G_1(t) := \int_t^1 \frac{1}{sG'(s)} \mathrm{d}s, \tag{4.1}$$

$$G_2(t) = tG'(t), \quad G_3(t) = t(G')^{-1}(t), \quad G_4(t) = G_3^*(t), \tag{4.2}$$

where G_3^* is the convex conjugate of G_3. Further, we introduce the class S of functions $\chi : \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfying for fixed $c_1, c_2 > 0$ [should be selected carefully in (4.16)]:

$$\chi \in C^1(\mathbb{R}_+),\ \chi \leq 1,\ \chi' \leq 0, \tag{4.3}$$

and

$$c_2 G_4\left[\frac{c}{d} q(t) h_0(t)\right] \leq c_1 \left(G_2\left(\frac{G_5(t)}{\chi(t)}\right) - \frac{G_2\left(G_5(t)\right)}{\chi(t)}\right), \tag{4.4}$$

where $d > 0$, c is a generic positive constant which may change from line to line, h_0 and q are defined in Lemma 3.1 and Corollary 3.6 and

$$G_5(t) = G_1^{-1}\left(c_1 \int_0^t \xi(s) \mathrm{d}s\right). \tag{4.5}$$

Remark 4.1. [3] According to the properties of G introduced in (A), G_2 is convex increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, G_1 is decreasing defines a bijection from $(0, 1]$ to $\mathbb{R}_+$, and G_3 and G_4 are convex and increasing functions on $(0, r]$. Then, the set S is not empty because it contains $\chi(s) = \varepsilon G_5(s)$ for any $0 < \varepsilon \leq 1$ small enough. From (4.1) and (4.5), we notice that (4.3) is satisfied. On the other hand, we have $q(t)h_0(t)$ is nonincreasing, $0 < G_5 \leq 1$, and G' and G_4 are increasing, then (4.4) is satisfied if

$$c_2 G_4\left[\frac{c}{d} q_0 h_0(0)\right] \leq \frac{c_1}{\varepsilon}\left(G'\left(\frac{1}{\varepsilon}\right) - G'(1)\right)$$

which holds, for $0 < \varepsilon \leq 1$ small enough, since $\lim_{t \to +\infty} G'(t) = +\infty$.

Theorem 4.2. *Assume that (A) and (1.3) hold, then there exists a strictly positive constant C such that the solution of (1.1) satisfies, for all $t \geq 0$,*

$$E(t) \leq \frac{CG_5(t)}{\chi(t) q(t)}. \tag{4.6}$$

Remark 4.3. The stability estimate (4.6) holds for any χ satisfying (4.3) and (4.4). But (4.6) does not lead in general to the asymptotic stability $\lim_{t \to \infty} E(t) = 0$ (like in case of the choice $\chi = \varepsilon G_5$ indicated in Remark 4.1, where (4.6) becomes just an upper bound estimate for E). The idea is to choose χ satisfying (4.3) and (4.4) such that (4.6) gives the best possible decay rate for E. This choice can be done by taking χ satisfying (4.3) and (4.4) such that the decay rate of the function in the right-hand side of (4.4) has the closet decay rate to the one of the function in the left-hand side of (4.4). So, such choice of χ can be seen from each specific considered functions g and ψ_{0x} (see the particular example considered below).

Proof. To prove Theorem 4.2, we start by combining (2.8), (3.1), (3.8), (3.17) and (1.3), then, for some $m>0$ and for any $t\geq 0$, we have

$$L'(t)\leq -mE(t)+\frac{c}{q(t)}G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)+ch_0(t). \tag{4.7}$$

Without loss of generality, one can assume that $E(0)>0$. For $\varepsilon_0<r$, let the functional $\mathcal{F}$ defined by

$$\mathcal{F}(t):=G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L(t),$$

which satisfies $\mathcal{F}\sim E$. By noting that $G''\geq 0$, $q'\leq 0$ and $E'\leq 0$, we get

$$\begin{aligned}\mathcal{F}'(t)&=\varepsilon_0\frac{(qE)'(t)}{E(0)}G''\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L(t)+G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L'(t)\\&\leq -mE(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{c}{q(t)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)\\&\quad+ch_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.8}$$

Let G^* be the convex conjugate of G in the sense of Young (see [36]), then

$$G^*(s)=s(G')^{-1}(s)-G\left[(G')^{-1}(s)\right],\quad \text{if}\quad s\in(0,G'(r)] \tag{4.9}$$

and G^* satisfies the following generalized Young inequality

$$AB\leq G^*(A)+G(B),\qquad \text{if}\quad A\in(0,G'(r)],\quad B\in(0,r]. \tag{4.10}$$

So, with $A=G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$ and $B=G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)$ and using (2.8) and (4.8)–(4.10), we arrive at

$$\begin{aligned}\mathcal{F}'(t)&\leq -mE(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{c}{q(t)}G^*\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)+c\left(\frac{\mu(t)q(t)}{\xi(t)}\right)\\&\quad+ch_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\&\leq -mE(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\left(\frac{\mu(t)q(t)}{\xi(t)}\right)\\&\quad+ch_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.11}$$

So, multiplying (4.11) by $\xi(t)$ and using (3.18) and the fact that $\varepsilon_0\frac{E(t)q(t)}{E(0)}<r$ and $G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)=G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$ give

$$\begin{aligned}\xi(t)\mathcal{F}'(t)&\leq -m\xi(t)E(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c\xi(t)\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\&\quad+c\mu(t)q(t)+c\xi(t)h_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\&\leq -\varepsilon_0(\frac{mE(0)}{\varepsilon_0}-c)\xi(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)-cE'(t)+c\xi(t)h_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right).\end{aligned}$$

Consequently, recalling the definition of G_2 and choosing ε_0 so that $k = (\frac{mE(0)}{\varepsilon_0} - c) > 0$, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} \mathcal{F}_1'(t) &\leq -k\xi(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + c\xi(t)h_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) \\ &= -k\frac{\xi(t)}{q(t)}G_2\left(\frac{E(t)q(t)}{E(0)}\right) + c\xi(t)h_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right), \end{aligned} \tag{4.12}$$

where $\mathcal{F}_1 = \xi\mathcal{F} + cE \sim E$ and satisfies for some $\alpha_1, \alpha_2 > 0$.

$$\alpha_1\mathcal{F}_1(t) \leq E(t) \leq \alpha_2\mathcal{F}_1(t). \tag{4.13}$$

Since $G_2'(t) = G'(t) + tG''(t)$, then, using the strict convexity of G on $(0, r]$, we find that $G_2'(t), G_2(t) > 0$ on $(0, r]$.

Using the general Young inequality (4.10) on the last term in (4.12) with $A = G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$ and $B = [\frac{c}{d}h_0(t)]$, we have for $d > 0$

$$\begin{aligned} ch_0(t)G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) &= \frac{d}{q(t)}\left[\frac{c}{d}q(t)h_0(t)\right]\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right) \\ &\leq \frac{d}{q(t)}G_3\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}G_3^*\left[\frac{c}{d}q(t)h_0(t)\right] \\ &\leq \frac{d}{q(t)}\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\left(G'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)h_0(t)\right] \\ &\leq \frac{d}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)h_0(t)\right]. \end{aligned} \tag{4.14}$$

Now, combining (4.12) and (4.14) and choosing d small enough so that $k_1 = (k - d) > 0$, we arrive at

$$\begin{aligned} \mathcal{F}_1'(t) &\leq -k\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)h_0(t)\right] \\ &\leq -k_1\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)h_0(t)\right]. \end{aligned} \tag{4.15}$$

Using the equivalent property in (4.13) and the increasing of G_2, we have

$$G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) \geq G_2\Big(d_0\mathcal{F}_1(t)q(t)\Big).$$

Letting $\mathcal{F}_2(t) := d_0\mathcal{F}_1(t)q(t)$ and recalling $q' \leq 0$, then we arrive at, for some $c_1, c_2 > 0$,

$$\mathcal{F}_2'(t) \leq -c_1\xi(t)G_2(\mathcal{F}_2(t)) + c_2\xi(t)G_4\left[\frac{c}{d}q(t)h_0(t)\right]. \tag{4.16}$$

$d_0q(t)$ is nonincreasing. Using the equivalent property $\mathcal{F}_1 \sim E$ implies that there exists $b_0 > 0$ such that $\mathcal{F}_2(t) \geq b_0E(t)q(t)$. Let $t \in \mathbb{R}_+$ and $\chi(t)$ satisfying (4.3) and (4.4).

If

$$b_0q(t)E(t) \leq 2\frac{G_5(t)}{\chi(t)}, \tag{4.17}$$

then, we have

$$E(t) \leq \frac{2}{b_0}\frac{G_5(t)}{\chi(t)q(t)}. \tag{4.18}$$

If

$$b_0q(t)E(t) > 2\frac{G_5(t)}{\chi(t)}, \tag{4.19}$$

then, for any $0 \leq s \leq t$, we get

$$b_0 q(s)E(s) > 2\frac{G_5(t)}{\chi(t)}, \tag{4.20}$$

since $q(t)E(t)$ is nonincreasing function. Therefore,

$$\mathcal{F}_2(s) > 2\frac{G_5(t)}{\chi(t)}, \tag{4.21}$$

for any $0 \leq s \leq t$. Recalling the definition of G_2, using the fact that G_2 is convex, $G_2(0) = 0$ and $0 < \chi \leq 1$, we have, for any $0 \leq s \leq t$ and $0 < \epsilon_1 \leq 1$,

$$\begin{aligned} G_2\Big(\epsilon_1\chi(s)\mathcal{F}_2(s) - \epsilon_1 G_5(s)\Big) &\leq \epsilon_1\chi(s)G_2\Big(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\Big) \\ &\leq \epsilon_1\chi(s)\mathcal{F}_2(s)G'\Big(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\Big) - \epsilon_1\chi(s)\frac{G_5(s)}{\chi(s)}G'\Big(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\Big) \\ &\leq \epsilon_1\chi(s)\mathcal{F}_2(s)G'\Big(\mathcal{F}_2(s)\Big) - \epsilon_1\chi(s)\frac{G_5(s)}{\chi(s)}G'\Big(\frac{G_5(s)}{\chi(s)}\Big). \end{aligned} \tag{4.22}$$

Now, we let

$$\mathcal{F}_3(s) = \epsilon_1\chi(s)\mathcal{F}_2(s) - \epsilon_1 G_5(s), \tag{4.23}$$

where ϵ_1 small enough so that $\mathcal{F}_3(0) \leq 1$. Then, (4.22) becomes, for any $0 \leq s \leq t$,

$$G_2\Big(\mathcal{F}_3(s)\Big) \leq \epsilon_1\chi(t)G_2\Big(\mathcal{F}_2(s)\Big) - \epsilon_1\chi(t)G_2\Big(\frac{G_5(s)}{\chi(s)}\Big). \tag{4.24}$$

Further, we have

$$\mathcal{F}_3'(t) = \epsilon_1\chi'(t)\mathcal{F}_2(t) + \epsilon_1\chi(s)\mathcal{F}_2'(t) - \epsilon_1 G_5'(t). \tag{4.25}$$

Since $\chi' \leq 0$ and using (4.16), then for any $0 \leq s \leq t$, $0 < \epsilon_1 \leq 1$, we obtain

$$\begin{aligned} \mathcal{F}_3'(t) &\leq \epsilon_1\chi(s)\mathcal{F}_2'(t) - \epsilon_1 G_5'(t) \\ &\leq -c_1\epsilon_1\xi(t)\chi(t)G_2(\mathcal{F}_2(t)) + c_2\epsilon_1\xi(t)\chi(s)G_4\left[\frac{c}{d}q(t)h_0(t)\right] - \epsilon_1 G_5'(t). \end{aligned} \tag{4.26}$$

Then, using (4.4) and (4.24), we get

$$\begin{aligned} \mathcal{F}_3'(t) \leq -c_1\xi(t)G_2(\mathcal{F}_3(t)) - c_1\epsilon_1\xi(t)\chi(t)G_2\Big(\frac{G_5(s)}{\chi(s)}\Big) \\ + c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_0(t)\right] - \epsilon_1 G_5'(t). \end{aligned} \tag{4.27}$$

Thus,

$$\begin{aligned} \mathcal{F}_3'(t) \leq -c_1\xi(t)G_2(\mathcal{F}_3(t)) + c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_0(t)\right] \\ - c_1\epsilon_1\xi(t)\chi(t)G_2\Big(\frac{G_5(s)}{\chi(s)}\Big) - \epsilon_1 G_5'(t). \end{aligned} \tag{4.28}$$

From the definition of G_1 and G_5, we have

$$G_1\left(G_5(s)\right) = c_1\int_0^s \xi(\tau)\mathrm{d}\tau;$$

hence,

$$G_5'(s) = -c_1\xi(s)G_2\left(G_5(s)\right). \tag{4.29}$$

Now, we have

$$\begin{aligned} & c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1\epsilon_1\xi(t)\chi(t)G_2\left(\frac{G_5(s)}{\chi(s)}\right) - \epsilon_1 G_5'(t) \\ &= c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_0(t)\right] - \epsilon_1\xi(t)\chi(t)G_2\left(\frac{G_5(s)}{\chi(s)}\right) + c\epsilon_1\xi(t)G_2\left(G_5(t)\right) \\ &= \epsilon_1\xi(t)\chi(t)\left(c_2G_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1G_2\left(\frac{G_5(s)}{\chi(s)}\right) + c_1\frac{G_2\left(G_5(t)\right)}{\chi(t)}\right). \end{aligned} \tag{4.30}$$

Then, according to (4.4), we get

$$\epsilon_1\xi(t)\chi(t)\left(c_2G_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1G_2\left(\frac{G_5(s)}{\chi(s)}\right) - c_1\frac{G_2\left(G_5(t)\right)}{\chi(t)}\right) \leq 0$$

Then, (4.28) gives

$$\mathcal{F}_3'(t) \leq -c_1\xi(t)G_2(\mathcal{F}_3(t)). \tag{4.31}$$

Thus, from (4.31) and the definition of G_1 and G_2 in (4.1) and (4.2), we obtain

$$\Big(G_1\left(\mathcal{F}_3(t)\right)\Big)' \geq c_1\xi(t). \tag{4.32}$$

Integrating (4.32) over $[0,t]$, we get

$$G_1\left(\mathcal{F}_3(t)\right) \geq c_1\int_0^t \xi(s)\mathrm{d}s + G_1\left(\mathcal{F}_3(0)\right). \tag{4.33}$$

Since G_1 is decreasing, $\mathcal{F}_3(0) \leq 1$ and $G_1(1) = 0$, then

$$\mathcal{F}_3(t) \leq G_1^{-1}\left(c_1\int_0^t \xi(s)\mathrm{d}s\right) = G_5(t). \tag{4.34}$$

Recalling that $\mathcal{F}_3(t) = \epsilon_1\chi(t)\mathcal{F}_2(t) - \epsilon_1 G_5(t)$, we have

$$\mathcal{F}_2(t) \leq \frac{(1+\epsilon_1)}{\epsilon_1}\frac{G_5(t)}{\chi(t)}, \tag{4.35}$$

Similarly, recall that $\mathcal{F}_2(t) := d_0\mathcal{F}_1(t)q(t)$, then

$$\mathcal{F}_1(t) \leq \frac{(1+\epsilon_1)}{d_0\epsilon_1}\frac{G_5(t)}{\chi(t)q(t)}, \tag{4.36}$$

Since $\mathcal{F}_1 \sim E$, then for some $b > 0$, we have $E(t) \leq b\mathcal{F}_1$, which gives

$$E(t) \leq \frac{b(1+\epsilon_1)}{d_0\epsilon_1}\frac{G_5(t)}{\chi(t)q(t)}, \tag{4.37}$$

From (4.18) and (4.37), we obtain the following estimate

$$E(t) \leq c_3\left(\frac{G_5(t)}{\chi(t)q(t)}\right), \tag{4.38}$$

where $c_3 = \max\{\frac{2}{b_0}, \frac{b(1+\epsilon_1)}{d_0\epsilon_1}\}$. □

Example 4.4. [3]: Let $g(t) = \frac{a}{(1+t)^\nu}$, where $\nu > 1$ and $0 < a < \nu - 1$ so that (A) is satisfied. In this case $\xi(t) = \nu a^{\frac{-1}{\nu}}$ and $G(t) = t^{\frac{\nu+1}{\nu}}$. Then, there exist positive constants $a_i (i = 1, ..., 5)$ depending only on a, ν such that

$$G_3(t) = a_3 t^{\nu+1},\ G_4(t) = a_4 t^{\frac{\nu+1}{\nu}},\ G_2(t) = a_2 t^{\frac{\nu+1}{\nu}},\ G_1(t) = a_1(t^{\frac{-1}{\nu}} - 1),\ G_5(t) = (a_5 t + 1)^{-\nu}. \tag{4.39}$$

We will discuss two cases:

Case 1 If

$$m_0(1+t)^r \leq 1+||\psi_{0x}||^2 \leq m_1(1+t)^r \tag{4.40}$$

where $0<r<\nu-1$ and $m_0, m_1>0$, then we have, for some positive constants $a_i (i=6,...,9)$ depending only on a, ν, m_0, m_1, r, the following:

$$a_6(1+t)^{-\nu+1+r} \leq h_0(t) \leq a_7(1+t)^{-\nu+1+r}, \tag{4.41}$$

$$\frac{q_0}{q(t)} \geq a_8 \begin{cases} 1+\ln(1+t), & \nu-r=2; \\ 2, & \nu-r>2; \\ (1+t)^{-\nu+r+2}, & 1<\nu-r<2. \end{cases} \tag{4.42}$$

$$\frac{q_0}{q(t)} \leq a_9 \begin{cases} 1+\ln(1+t), & \nu-r=2; \\ 2, & \nu-r>2; \\ (1+t)^{-\nu+r+2}, & 1<\nu-r<2. \end{cases} \tag{4.43}$$

We notice that condition (4.4) is satisfied if

$$(t+1)^\nu q(t)h_0(t)\chi(t) \leq a_{10}\left(1-(\chi)^{\frac{1}{\nu}}\right)^{\frac{\nu}{\nu+1}}. \tag{4.44}$$

where $a_{10}>0$ depending on a, ν, c_1 and c_2. Choosing $\chi(t)$ as the following

$$\chi(t) = \lambda \begin{cases} (1+t)^{-p}, & p=r+1 \quad \nu-r\geq 2; \\ (1+t)^{-p}, & p=\nu-1, \ 1<\nu-r<2. \end{cases} \tag{4.45}$$

so that (4.3) is valid. Moreover, using (4.41) and (4.42), we see that (4.44) is satisfied if $0<\lambda\leq 1$ is small enough, and then (4.4) is satisfied. Hence (4.6) and (4.43) imply that, for any $t\in\mathbb{R}_+$

$$E(t) \leq a_{11} \begin{cases} \left(1+\ln(1+t)\right)(1+t)^{-(\nu-r-1)}, & \nu-r=2; \\ (1+t)^{-(\nu-r-1)}, & \nu-r>2; \\ (1+t)^{-(\nu-r-1)}, & 1<\nu-r<2. \end{cases} \tag{4.46}$$

Thus, estimate (4.46) gives $\lim_{t\to+\infty} E(t)=0$. We notice that estimate (4.46) extends and improves the decay rate $(t+1)^p$ (for some $0<p$ small enough) obtained in [30] for $\nu>2$.
Case 2 if $m_0 \leq 1+||\psi_{0x}||^2 \leq m_1$. That is, $r=0$ in (4.40) (as it was assumed in [25,31–33], then (4.46) holds with $r=0$, which gives a better decay rate than the ones $(1+t)^{-p}$ (for any $0<p<\frac{\nu-1}{2}$), $(1+t)^{-p}$(for some $0<p$ small enough) and $(1+t)^{\frac{-\nu^2-\nu-1}{\nu}}$ obtained in [25,31–33], respectively.

5. A decay result for nonequal speeds of wave propagation

In this section, we give an estimate to the decay rate in the case of nonequal speeds of wave propagation. We start by stating, under assumption (A) and for $(\varphi_0,\varphi_1),(\psi_0(.,0),\psi_1) \in H_0^1(0,L)\times L^2(0,L)$, some lemmas that are necessary for the proof of our second main result.

First, we introduce the second energy functional

$$E_*(t) := \frac{1}{2}\left(\rho_1||\varphi_{tt}^2||_2^2 + \rho_2||\psi_{tt}^2||_2^2 + \ell||\psi_{xt}^2||_2^2 + K||(\varphi_{xt}+\psi_t)^2\right)||_2^2 + \frac{1}{2}(g\circ\psi_{xt})(t). \tag{5.1}$$

Then, we have the following lemma.

Lemma 5.1. [37] *Let (φ,ψ) be the strong solution of (1.1). Then, the second energy functional satisfies, for all $t\geq 0$,*

$$E_*'(t) = \frac{1}{2}(g'\circ\psi_{xt}) \leq 0 \tag{5.2}$$

and

$$E_*(t) \leq E_*(0). \tag{5.3}$$

Next, we have the following estimate for the last term in the right-hand side of (3.8).

Lemma 5.2. [37] *Let (φ, ψ) be the strong solution of (1.1). Then, for any $\varepsilon > 0$, we have*

$$\left(\frac{\rho_1 b}{K} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \mathrm{d}x \leq \varepsilon E(t) + \frac{c}{\varepsilon}\Big((g \circ \psi_{xt})(t) - E'(t)\Big), \quad \forall t \geq 0. \tag{5.4}$$

Proof. For any $t \geq 0$, we have the following

$$\begin{aligned} \left(\frac{\rho_1 b}{K} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \mathrm{d}x &= \frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g(s) \left(\psi_{xt}(t) - \psi_{xt}(t-s)\right) \mathrm{d}s\mathrm{d}x \\ &\quad + \frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g(s)\psi_{xt}(t-s)\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{5.5}$$

By exploiting Young's inequality, we get for all $\varepsilon > 0$ and $t \geq 0$, the following estimates

$$\begin{aligned} \frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g(s) \left(\psi_{xt}(t) - \psi_{xt}(t-s)\right) \mathrm{d}s\mathrm{d}x &\leq \frac{c\varepsilon}{2} \int_0^L \varphi_t^2 \mathrm{d}x + \frac{c}{\varepsilon} \left(g \circ \psi_{xt}\right) \\ &\leq \frac{\varepsilon}{2} E(t) + \frac{c}{\varepsilon} \left(g \circ \psi_{xt}\right). \end{aligned} \tag{5.6}$$

On the other hand, by using the fact $\psi_{xt}(t-s) = -\psi_{xs}(t-s)$ and integrating by parts, we obtain

$$\begin{aligned} &\frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g(s) \left(\psi_{xt}(t-s)\right) \mathrm{d}s\mathrm{d}x = -\frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g(s) \left(\psi_{xs}(t-s)\right) \mathrm{d}s\mathrm{d}x \\ &= -\frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \left(\Big[0 - g(0)\psi_x(t)\Big] + \int_0^\infty g'(s)\psi_x(t-s)\mathrm{d}s\right)\mathrm{d}x \\ &= \frac{g(0)\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \psi_x(t)\mathrm{d}x - \frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g'(s)\psi_x(t)\mathrm{d}s\mathrm{d}x \\ &- \frac{g(0)\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \left(\int_0^\infty g'(s) \left(\psi_x(t-s) - \psi_x(t)\right) \mathrm{d}s\right)\mathrm{d}x \\ &= \frac{2g(0)\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \psi_x(t)\mathrm{d}x - \frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \left(\int_0^\infty g'(s) \left(\psi_x(t-s) - \psi_x(t)\right) \mathrm{d}s\right)\mathrm{d}x. \end{aligned} \tag{5.7}$$

Using Young's inequality, we get for all $\varepsilon > 0$ and $t \geq 0$,

$$
\begin{aligned}
&-\frac{\left(\frac{\rho_1 b}{K} - \rho_2\right)}{\int_0^\infty g(s)\mathrm{d}s} \int_0^L \varphi_t \int_0^\infty g'(s)\left(\psi_x(t-s)\right) - \psi_x(t)\right)\mathrm{d}s\mathrm{d}x \\
&\leq \frac{\varepsilon}{2} \int_0^L \varphi_t^2 \mathrm{d}x - \frac{c}{\varepsilon}\left(g' \circ \psi_x\right) \\
&\leq \frac{\varepsilon}{2} E(t) - \frac{c}{\varepsilon}\left(g' \circ \psi_x\right),
\end{aligned}
\tag{5.8}
$$

and

$$
c \int_0^L \varphi_t \psi_x(t) \mathrm{d}x \leq \frac{\varepsilon}{2} \int_0^L \varphi_t^2 \mathrm{d}x + \frac{c}{\varepsilon} \int_0^L \psi_x^2 \mathrm{d}x. \tag{5.9}
$$

Combining (5.8) and (5.9), (5.4) is established. □

Lemma 5.3. *We have the following estimate:*

$$
\int_0^t g(s)\left(\|\psi_x(t) - \psi_x(t-s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(t-s)\|_2^2\right)\mathrm{d}s \leq \frac{1}{\gamma(t)} G^{-1}\left(\frac{\gamma(t)\theta(t)}{\xi(t)}\right), \tag{5.10}
$$

for any $t > 0$, where $\gamma(t) := \frac{\gamma_0}{t+1}$, $\gamma_0 \in (0,1)$, and

$$
\theta(t) := -\int_0^t g'(s)\left(\|\psi_x(t) - \psi_x(t-s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(t-s)\|_2^2\right)\mathrm{d}s \leq c\left(E'(t) + E'_*(t)\right). \tag{5.11}
$$

Proof. Let us define the following functional:

$$
\eta(t) := \gamma(t) \int_0^t \left(\|\psi_x(t) - \psi_x(t-s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(t-s)\|_2^2\right)\mathrm{d}s, \quad \forall\, t > 0. \tag{5.12}
$$

The use of (2.7), (2.8), (5.1) and (5.3) gives for any $t \geq 0$,

$$
\begin{aligned}
&\gamma(t) \int_0^t \left(\|\psi_x(t) - \psi_x(t-s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(t-s)\|_2^2\right)\mathrm{d}s \\
&\leq 2\gamma(t) \int_0^t \left(\|\psi_x(t)\|_2^2 + \|\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)\|_2^2 + \|\psi_{xt}(t-s)\|_2^2\right)\mathrm{d}s \\
&\leq \frac{4\gamma(t)}{\ell} \int_0^t \left(E(t) + E(t-s) + E_*(t) + E_*(t-s)\right)\mathrm{d}s \\
&\leq \frac{8\gamma(t)}{\ell} \int_0^t \left[E(0) + c\left(E_*(0)\right)\right]\mathrm{d}s \\
&\leq \frac{8\gamma_0}{\ell}\left[E(0) + c\left(E_*(0)\right)\right] < +\infty, \qquad \forall\, t > 0.
\end{aligned}
\tag{5.13}
$$

This allows us to pick $0 < \gamma_0 < 1$ such that $\eta < 1$. Thus, using Jensen's inequality and (5.12), we obtain, for any $t > 0$,

$$\begin{aligned}
\theta(t) &= -\frac{1}{\eta(t)}\int_0^t \eta(t)g'(s)\big(\|\psi_x(t)-\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\big)\mathrm{d}s \\
&\ge \frac{1}{\eta(t)}\int_0^t \eta(t)\xi(s)G(g(s))\big(\|\psi_x(t)-\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\big)\mathrm{d}s \\
&\ge \frac{\xi(t)}{\eta(t)}\int_0^t G(\eta(t)g(s))\big(\|\psi_x(t)-\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\big)\mathrm{d}s \\
&\ge \frac{\xi(t)}{\gamma(t)}G\left(\gamma(t)\int_0^t g(s)\big(\|\psi_x(t)-\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\big)\mathrm{d}s\right) \\
&\ge \frac{\xi(t)}{\gamma(t)}G\left(\gamma(t)\int_0^t g(s)\big(\|\psi_x(t)-\psi_x(t-s)\|_2^2 + \|\psi_{xt}(t)-\psi_{xt}(t-s)\|_2^2\big)\mathrm{d}s\right),
\end{aligned} \tag{5.14}$$

Then, (5.10) is established. □

Theorem 5.4. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in \big(H^2(0,L) \cap H_0^1(0,L)\big) \times H_0^1(0,L)$. Assume that (A) holds and the relation (1.3) is not satisfied, that is,*

$$\frac{\rho_1}{K} \ne \frac{\rho_2}{b}.$$

Then, there exist a positive constant C such that the solution of (1.1) satisfies, for all $t \ge 0$,

$$E(t) \le C\frac{E(0)}{\gamma(t)}G_2^{-1}\left[\frac{C + \int_0^t \xi(s)G_4\left[C\gamma(s)h_1(s)\right]\mathrm{d}s,}{\int_0^t \xi(s)\mathrm{d}s}\right], \tag{5.15}$$

where γ, h_1, G_2 and G_4 are functions defined earlier in this paper.

Proof. Combining (3.8) and (5.4), we have, for some $m > 0$,

$$\begin{aligned}
L'(t) &\le -mE(t) + c(g \circ \psi_x)(t) + \left(\frac{\rho_1 b}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}\mathrm{d}x \\
&\le -(m-\varepsilon)E(t) + c(g \circ \psi_x)(t) + \frac{c}{\varepsilon}\big(g \circ \psi_{xt}(t) - E'(t)\big), \quad \forall\, t \ge 0.
\end{aligned}$$

After fixing ε small enough, we arrive at

$$L'(t) \le -m_1E(t) + c\big(g \circ \psi_x + g \circ \psi_{xt}\big)(t) - cE'(t), \quad \forall\, t \ge 0,$$

where m_1 is a fixed positive constant. By setting $F := L + cE \sim E$, we obtain, for any $t \ge 0$,

$$F'(t) \le -m_1E(t) + c\big(g \circ \psi_x + g \circ \psi_{xt}\big)(t). \tag{5.16}$$

Combining (3.2), (5.10) and (5.16), we have

$$F'(t) \le -m_1E(t) + \frac{c}{\gamma(t)}G^{-1}\left(\frac{\gamma(t)\theta(t)}{\xi(t)}\right) + ch_1(t), \quad \forall\, t > 0. \tag{5.17}$$

Let $0 < \varepsilon_1 < r$, then define a functional F_1 by

$$F_1(t) := G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) F(t), \qquad \forall\, t > 0.$$

Then, estimate (5.17) together with the facts that $E' \leq 0$, $G' > 0$ and $G'' > 0$ leads to

$$\begin{aligned} F_1'(t) &\leq -m_1 E(t) G'\left(\varepsilon_1 \tfrac{E(t)\gamma(t)}{E(0)}\right) + cG'\left(\varepsilon_1 \tfrac{E(t)\gamma(t)}{E(0)}\right) h_1(t) \\ &\quad + \tfrac{c}{\gamma(t)} G'\left(\varepsilon_1 \tfrac{E(t)\gamma(t)}{E(0)}\right) G^{-1}\left(\tfrac{\gamma(t)\theta(t)}{\xi(t)}\right), \qquad \forall\, t > 0. \end{aligned} \tag{5.18}$$

Let G^* be the convex conjugate of G as in (4.9), set

$$A = G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) \qquad \text{and} \qquad B = G^{-1}\left(\frac{q\theta(t)}{\xi(t)(t)}\right).$$

Combining (4.9), (4.10) and (5.18) and selecting ε_1 small enough, we obtain, $\forall\, t > 0$ and $m_2 > 0$,

$$\begin{aligned} F_1'(t) &\leq -m_2 \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) + c\frac{\theta(t)}{\xi(t)} \\ &\quad + cG'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) h_1(t). \end{aligned} \tag{5.19}$$

Multiplying both sides of (5.19) by $\xi(t)$ and using $\varepsilon_1 \dfrac{E(t)}{E(0)} < r$ and inequality (5.11), we arrive at

$$\begin{aligned} \xi(t) F_1'(t) &\leq -m_2 \xi(t) \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) + c\theta(t) + cG'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) \xi(t) h_1(t) \\ &\leq -m_2 \xi(t) \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) - c\big(E'(t) + E_*'(t)\big) \\ &\quad + cG'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) \xi(t) h_1(t), \qquad \forall\, t > 0. \end{aligned} \tag{5.20}$$

Thus, by setting $F_2 = \xi F_1 + c(E + E_*)$ and noting that $0 \leq G'\left(\varepsilon_1 \cdot \frac{E(t)\gamma(t)}{E(0)}\right) \leq G'(\varepsilon_1)$ and $0 \leq \xi(t) \leq \xi(0)$, we deduce that $F_2 \geq cE_* \geq cE$ and because ξ is nonincreasing, the estimate (5.20) becomes,

$$\begin{aligned} F_2'(t) &\leq -m_2 \xi(t) \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) + c\xi(t) G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) h_1(t) \\ &= -\frac{m_2}{\gamma(t)} \xi(t) G_2\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) + c\xi(t) G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) h_1(t), \qquad \forall\, t > 0. \end{aligned} \tag{5.21}$$

Since $G_2'(t) = G'(\varepsilon_1 t) + \varepsilon_1 t G''(\varepsilon_1 t)$, then, using the strict convexity of G on $(0, r]$, we find that $G_2'(t), G_2(t) > 0$ on $(0, 1]$. Using the generalized Young inequality (4.10) on the last term in (5.21) with $B = \frac{c}{d} h_1(t)$ and $A = G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right)$, we have

$$\begin{aligned} ch_1(t) G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) &= \frac{d}{\gamma(t)} \left[\frac{c}{d}\gamma(t) h_1(t)\right] \left[G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right)\right] \\ &\leq \frac{d}{\gamma(t)} G_3\left(G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right)\right) + \frac{d}{\gamma(t)} G_3^*\left[\frac{c}{d}\gamma(t) h_1(t)\right] \\ &\leq \frac{d}{\gamma(t)} \left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right)\left(G'\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right)\right) + \frac{d}{\gamma(t)} G_4\left[\frac{c}{d}\gamma(t) h_1(t)\right] \\ &\leq \frac{d}{\gamma(t)} G_2\left(\varepsilon_1 \frac{E(t)\gamma(t)}{E(0)}\right) + \frac{d}{\gamma(t)} G_4\left[\frac{c}{d}\gamma(t) h_1(t)\right]. \end{aligned} \tag{5.22}$$

Now, combining (5.21) and (5.22) and choosing d small enough, we arrive at

$$\begin{aligned} F_2'(t) &\leq -m_2\frac{\xi(t)}{\gamma(t))}G_2\left(\varepsilon_1\frac{E(t)\gamma(t)}{E(0)}\right)+\frac{d\xi(t)}{\gamma(t)}G_2\left(\varepsilon_1\frac{E(t)\gamma(t)}{E(0)}\right)+\frac{d\xi(t)}{\gamma(t)}G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right)\\ &\leq -c\frac{\xi(t)}{\gamma(t)}G_2\left(\varepsilon_1\frac{E(t)\gamma(t)}{E(0)}\right)+\frac{c\xi(t)}{\gamma(t)}G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right). \end{aligned} \tag{5.23}$$

Since $E'<0$ and $\gamma'<0$, then $G_2\left(\frac{E(t)q(t)}{E(0)}\right)$ is decreasing functions. Hence, for $0\leq t\leq T$, we have

$$G_2\left(\varepsilon_1\frac{E(T)\gamma(T)}{E(0)}\right)\leq G_2\left(\varepsilon_1\frac{E(t)\gamma(t)}{E(0)}\right). \tag{5.24}$$

Combining (5.23) with (5.24) and multiplying by $\gamma(t)$, we get

$$\gamma(t)F_2'(t)+c\xi(t)G_2\left(\varepsilon_1\frac{E(T)\gamma(T)}{E(0)}\right)\leq c\xi(t)G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right), \tag{5.25}$$

since $\gamma'<0$, then

$$\Big(\gamma(t)F_2\Big)'(t)+c\xi(t)G_2\left(\varepsilon_1\frac{E(T)\gamma(T)}{E(0)}\right)\leq c\xi(t)G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right). \tag{5.26}$$

Integrating (5.26) over $[0,T]$, we have

$$G_2\left(\varepsilon_1\frac{E(T)\gamma(T)}{E(0)}\right)\int_0^T\xi(t)\mathrm{d}t\leq\frac{F_2(0)\gamma(0)}{c}+\int_0^T\xi(t)G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right)\mathrm{d}t, \tag{5.27}$$

and then

$$G_2\left(\varepsilon_1\frac{E(T)\gamma(T)}{E(0)}\right)\leq\left[\frac{\frac{F_2(0)}{c}+\int_0^T\xi(t)G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right)\mathrm{d}t,}{\int_0^T\xi(t)\mathrm{d}t}\right]. \tag{5.28}$$

Thus,

$$\varepsilon_1\frac{E(T)\gamma(T)}{E(0)}\leq G_2^{-1}\left[\frac{\frac{F_2(0)}{c}+\int_0^T\xi(t)G_4\left(\frac{c}{d}\gamma(t)h_1(t)\right)\mathrm{d}t,}{\int_0^T\xi(t)\mathrm{d}t}\right]. \tag{5.29}$$

Then, we obtain

$$E(T)\leq C\frac{E(0)}{\gamma(T)}G_2^{-1}\left[\frac{C+\int_0^T\xi(t)G_4\left(C\gamma(t)h_1(t)\right)\mathrm{d}t,}{\int_0^T\xi(t)\mathrm{d}t}\right], \tag{5.30}$$

where $C=\max\left\{1,\frac{F_2(0)}{c},\frac{c}{d},\frac{1}{\varepsilon_1}\right\}$ □

Remark 5.5. We notice that, for any g, estimate (5.15) does not lead to any stability estimate. In this case, (5.15) becomes just an upper bound estimate for E. However, estimate (5.15) is obtained without the boundedness condition on ψ_{0xt} assumed in the literature such as the one concerned with Timoshenko in [27].

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