Z. Angew. Math. Phys. (2016) 67:124

0044-2275/16/050001-39
published online September 19, 2016
DOI 10.1007/s00033-016-0719-y

Zeitschrift für angewandte Mathematik und Physik ZAMP

Uniform and weak stability of Bresse system with two infinite memories

Aissa Guesmia and Mokhtar Kirane

Abstract. In this paper, we consider one-dimensional linear Bresse systems in a bounded open domain under Dirichlet–Neumann–Neumann boundary conditions with two infinite memories acting only on two equations. First, we establish the well-posedness in the sense of semigroup theory. Then, we prove two (uniform and weak) decay estimates depending on the speeds of wave propagations, the smoothness of initial data and the arbitrarily growth at infinity of the two relaxation functions.

Mathematics Subject Classification. 35B40, 35L45, 74H40, 93D20, 93D15.

Keywords. Bresse system, Infinite memory, Well-posedness, Asymptotic behavior, Semigroup theory, Energy method.

1. Introduction

In this paper, we consider the Bresse system in one-dimensional open bounded domain under the homogeneous Dirichlet–Neumann–Neumann boundary conditions and with two infinite memories acting on the second and third equations

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \int_0^{+\infty} g_2(s)\psi_{xx}(x, t-s)\,\mathrm{d}s = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \int_0^{+\infty} g_3(s)w_{xx}(x, t-s)\,\mathrm{d}s = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \\ w(x,-t) = w_0(x,t),\ w_t(x,0) = w_1(x) \end{cases} \tag{1}$$

or on the first and third equations

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \int\limits_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\,\mathrm{d}s = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \int\limits_0^{+\infty} g_3(s)w_{xx}(x,t-s)\,\mathrm{d}s = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ w(x,-t) = w_0(x,t),\ w_t(x,0) = w_1(x) \end{cases} \tag{2}$$

or on the first two equations

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \int\limits_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\,\mathrm{d}s = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \int\limits_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)\,\mathrm{d}s = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{3}$$

where $(x,t) \in]0,L[\times\mathbb{R}_+$, $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions, and L, l, ρ_i, k_i are positive constants. The infinite integrals in systems (1)–(3) represent the infinite memories, and the state (unknown) is

$$(\varphi,\psi,w) :]0,L[\times]0,+\infty[\to \mathbb{R}^3.$$

The derivative of a generic function f with respect to a variable y is noted f_y or $\partial_y f$. If f has only one variable, its derivative is noted f'. For simplicity of notation, the space x and/or the time t and s variables are used only when it is necessary to avoid ambiguity.

Our goal is to study the well-posedness and the asymptotic stability of these systems in terms of the growth at infinity of g_i, the smoothness of initial data $(\varphi_0,\psi_0,w_0,\varphi_1,\psi_1,w_1)$ and the speeds of wave propagations defined by

$$s_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad s_2 = \sqrt{\frac{k_2}{\rho_2}} \quad \text{and} \quad s_3 = \sqrt{\frac{k_3}{\rho_1}}. \tag{4}$$

The Bresse system is known as the circular arch problem and is given by the following equations:

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases}$$

where

$$N = k_0(w_x - l\varphi), \quad Q = k(\varphi_x + lw + \psi) \quad \text{and} \quad M = b\psi_x,$$

ρ_1, ρ_2, l, k, k_0 and b are positive constants, N, Q and M denote, respectively, the axial force, the shear force and the bending moment, and w, φ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements. Here

$$\rho_1 = \rho A, \quad \rho_2 = \rho I, \quad k_0 = EA, \quad k = k'GA, \quad b = EI \quad \text{and} \quad l = R^{-1},$$

where ρ, E, G, k', A, I and R are positive constants and denote, respectively, the density, the modulus of elasticity, the shear modulus, the shear factor, the cross-sectional area, the second moment of area of the cross section and the radius of curvature. Finally, by F_i we are denoting external forces in $]0, L[\times]0, +\infty[$ together with initial and boundary conditions. For more details, we refer to Lagnese et al. [14] and [15].

If we consider

$$(F_1, F_2, F_3) = (0, -\gamma\psi_t, 0) \tag{5}$$

with $\gamma > 0$, we obtain the system considered by Bresse [3] consisting of three coupled wave equations.

The most important asymptotic behavior result of the Bresse system is due to Liu and Rao [16] obtained for a thermoelastic Bresse system which consists of the Bresse system with

$$(F_1, F_2, F_3) = (0, 0, 0) \tag{6}$$

and two heat equations coupled in a certain manner, where the two wave equations about the longitudinal and shear angle displacements are effectively globally damped by the dissipation from the two heat equations. They proved that the norm of solutions in the energy space decays exponentially to zero at infinity if and only if

$$s_1 = s_2 = s_3. \tag{7}$$

Otherwise, the norm of solutions decays polynomially to zero with rates depending on the regularity of the initial data. For the classical solutions, these rates are $t^{-\frac{1}{4}+\epsilon}$ or $t^{-\frac{1}{8}+\epsilon}$ provided that the boundary conditions are of Dirichlet–Neumann–Neumann or Dirichlet–Dirichlet–Dirichlet type, respectively, where ϵ is an arbitrary positive constant. Very similar results to the ones of [16] are obtained in [8] for the Bresse system (in case (6)) coupled with only one heat equation in a certain manner, where the obtained decay rate for classical solutions when (7) is not satisfied is $t^{-\frac{1}{6}+\epsilon}$ in general and $t^{-\frac{1}{3}+\epsilon}$ when $s_1 \neq s_2$ and $s_1 = s_3$. Najdi and Wehbe [17] extended the results of [8] to the case where the thermal dissipation is locally distributed, and improved the polynomial stability estimate when (7) is not satisfied by getting the decay rate $t^{-\frac{1}{2}}$.

Concerning the stability of Bresse systems with (local or global) frictional dampings, we mention here the most known stability results in the literature. Alabau-Boussouira et al. [1] considered the case (5) and proved that the exponential stability is equivalent to (7). Otherwise, they got the same two decay rates as in [8]. The results of [1] were extended and improved in [18] by considering a locally distributed dissipation (that is, γ in (5) is replaced by a nonnegative function $a :]0, L[\to \mathbb{R}_+$ which is positive only on some part of $]0, L[$); the authors of [18] obtained a better decay rate when (7) does not hold. The exponential stability result of [1] was also proved in [21] when $\gamma = a(x)$ and $a :]0, L[\to \mathbb{R}$ has a positive average on $]0, L[$ such that

$$\left\| a - \int_0^L a(x)\,\mathrm{d}x \right\|_{L^2(]0,L[)}$$

is small enough. This implies that a is allowed to have some negative values on $]0, L[$, and in such situation, $a\psi_t$ is called indefinite damping. Also, some optimal polynomial decay rates for Bresse systems in case (5) were proved in [7] when (7) does not hold.

In [23] and [24], the authors studied the stability of Bresse systems damped by two locally frictional dampings

$$(F_1, F_2, F_3) = (0, -a_1(x)\psi_t, -a_2(x)w_t),$$

where $a_i :]0, L[\to \mathbb{R}_+$ are nonnegative functions which can vanish on some part of $]0, L[$. They established that the exponential stability remains valid if and only if $s_1 = s_2$. When $s_1 \neq s_2$, a general decay rate depending on the regularity of the initial data is obtained, where, in case of classical solutions, this rate is $t^{-\frac{1}{2}+\epsilon}$.

When only the first and second equations are controlled by linear frictional dampings; that is,

$$(F_1, F_2, F_3) = (-\gamma_1\varphi_t, -\gamma_2\psi_t, 0)$$

with $\gamma_i > 0$, the equivalence between the exponential stability and the equality $s_1 = s_3$ was proved in [2]. When $s_1 \neq s_3$, the polynomial stability was also showed in [2], where the decay rate depends on the regularity of the initial data. In the particular case of classical solutions, the polynomial decay rate of [2] is $t^{-\frac{1}{2}}$ and it is optimal.

In his PhD thesis [24], Youssef treated the case where the three equations of the Bresse system are all controlled by (linear or nonlinear) frictional dampings; that is,

$$(F_1, F_2, F_3) = (-h_1(\varphi_t), -h_2(\psi_t), -h_3(w_t)),$$

where $h_i : \mathbb{R} \to \mathbb{R}$ are given functions having a linear or a polynomial growth at zero and infinity, and obtained, respectively, the exponential and polynomial stability for any weak solution. These results are proved regardless to s_i. The results of [24] were generalized in [4] and [20] to the case

$$(F_1, F_2, F_3) = (-a_1(x)h_1(\varphi_t), -a_2(x)h_2(\psi_t), -a_3(x)h_3(w_t)),$$

where the nonnegative functions a_i are (all or some of them) effective only on some part of $]0, L[$, and the functions h_i can have a general growth at zero (not necessarily of linear or polynomial type). The case of three frictional dampings was also considered in [22] but in the whole space $\mathbb{R}$ (instead of $]0, L[$), and some polynomial stability estimates were obtained.

Concerning the stability of Bresse systems with memories, there are only a few results. When the three equations are controlled via infinite memories of the form

$$F_1 = -\int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s)\,\mathrm{d}s, \quad F_2 = -\int_0^{+\infty} g_2(s)\psi_{xx}(x, t-s)\,\mathrm{d}s$$

and

$$F_3 = -\int_0^{+\infty} g_3(s)w_{xx}(x, t-s)\,\mathrm{d}s,$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable, nonincreasing and integrable functions on $\mathbb{R}_+$, the stability was proved in [12] regardless to s_i. The obtained decay estimate given in [12] depends only on the growth at infinity of $s \mapsto g_i(s)$, which is allowed to have a decay rate at infinity arbitrarily close to $\frac{1}{s}$.

As far as we know, the more recent stability results for Bresse systems with memories are those in [6] under only one infinite memory considered in the second equation

$$(F_1, F_2, F_3) = \left(0, -\int_0^{+\infty} g(s)\psi_{xx}(x, t-s)\,\mathrm{d}s, 0\right), \tag{8}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable, nonincreasing and integrable function on $\mathbb{R}_+$ satisfying

$$\exists\, \alpha_1, \alpha_2 > 0 : \; -\alpha_2 g(s) \leq g'(s) \leq -\alpha_1 g(s), \quad \forall s \in \mathbb{R}_+. \tag{9}$$

In [6], it was proved that the exponentially stability holds if and only if (7) is satisfied. Otherwise, the polynomial stability with a decay rate of type $t^{-\frac{1}{2}}$ and its optimality were shown. The condition (9) implies that

$$g(0)\mathrm{e}^{-\alpha_2 s} \le g(s) \le g(0)\mathrm{e}^{-\alpha_1 s}, \quad \forall s \in \mathbb{R}_+, \tag{10}$$

which implies that g converges exponentially to zero at infinity.

The proof of the results cited above is based on the spectral theory, the frequency domain method and the multipliers technique.

Our goal is to study the well-posedness and asymptotic stability of systems (1)–(3) in terms of the arbitrary growth at infinity of the kernels g_i, the smoothness of initial data $(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1)$ and the speeds of wave propagations (4). We prove that these systems are well posed and their energy converges to zero when time tends to infinity, and we provide two general decay estimates: a strong decay estimate under some restrictions on v_i and a weak decay one in general. The proof is based on the semigroup theory for the well-posedness. For the decay estimates, we use the energy method and some differential and/or integral equalities.

The paper is organized as follows. In Sect. 2, we present our assumptions on the functions g_i state and prove the well-posedness of (1)–(3). In Sect. 3, we consider some assumptions on the growth of g_i at infinity and state our stability results. Finally, the proof of our uniform and weak decay estimates are given, respectively, in Sects. 4 and 5.

2. Well-posedness

In this section, we discuss the well-posedness of (1)–(3) using the semigroup approach. Following the method of [5], we consider these new functionals

$$\begin{cases} \text{Cases (2) and (3)}: \ \eta_1(x,t,s) = \varphi(x,t) - \varphi(x,t-s) & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \text{Cases (1) and (3)}: \ \eta_2(x,t,s) = \psi(x,t) - \psi(x,t-s) & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \text{Cases (1) and (2)}: \ \eta_3(x,t,s) = w(x,t) - w(x,t-s) & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+. \end{cases} \tag{11}$$

These functionals satisfy

$$\begin{cases} \partial_t\eta_1 + \partial_s\eta_1 - \varphi_t = 0 & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \partial_t\eta_2 + \partial_s\eta_2 - \psi_t = 0 & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \partial_t\eta_3 + \partial_s\eta_3 - w_t = 0 & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \eta_1(0,t,s) = \eta_1(L,t,s) = 0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \partial_x\eta_2(0,t,s) = \partial_x\eta_2(L,t,s) = 0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \partial_x\eta_3(0,t,s) = \partial_x\eta_3(L,t,s) = 0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \eta_i(x,t,0) = 0 & \text{in }]0,L[\times\mathbb{R}_+. \end{cases} \tag{12}$$

Let $\eta_i^0(x,s) = \eta_i(x,0,s)$,

$$\begin{cases} U_1^0 = \left(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \eta_2^0, \eta_3^0\right)^T, \\ U_2^0 = \left(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \eta_1^0, \eta_3^0\right)^T, \\ U_3^0 = \left(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \eta_1^0, \eta_2^0\right)^T \end{cases} \tag{13}$$

and

$$\begin{cases} U_1 = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \eta_2, \eta_3)^T, \\ U_2 = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \eta_1, \eta_3)^T, \\ U_3 = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \eta_1, \eta_2)^T. \end{cases} \tag{14}$$

Then, the system (i), $i = 1, 2, 3$, is equivalent to the following abstract one:

$$\begin{cases} \partial_t U_i = \mathcal{A}_i U_i, \\ U_i(t = 0) = U_i^0, \end{cases} \tag{15}$$

where $\mathcal{A}_i$ is the linear operator defined by

$$\mathcal{A}_1 U_1 = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ \frac{k_1}{\rho_1}\varphi_{xx} - \frac{l^2 k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{l}{\rho_1}(k_1 + k_3) w_x \\ -\frac{k_1}{\rho_2}\varphi_x + \frac{1}{\rho_2}\left(k_2 - g_2^0\right)\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{l k_1}{\rho_2} w + \frac{1}{\rho_2}\int_0^{+\infty} g_2 \partial_{xx}\eta_2 \,\mathrm{d}s \\ -\frac{l}{\rho_1}(k_1 + k_3)\varphi_x - \frac{l k_1}{\rho_1}\psi + \frac{1}{\rho_1}\left(k_3 - g_3^0\right) w_{xx} - \frac{l^2 k_1}{\rho_1} w + \frac{1}{\rho_1}\int_0^{+\infty} g_3 \partial_{xx}\eta_3 \,\mathrm{d}s \\ \psi_t - \partial_s \eta_2 \\ w_t - \partial_s \eta_3 \end{pmatrix},$$

$$\mathcal{A}_2 U_2 = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ \frac{1}{\rho_1}\left(k_1 - g_1^0\right)\varphi_{xx} - \frac{l^2 k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{l}{\rho_1}(k_1 + k_3) w_x + \frac{1}{\rho_1}\int_0^{+\infty} g_1 \partial_{xx}\eta_1 \,\mathrm{d}s \\ -\frac{k_1}{\rho_2}\varphi_x + \frac{k_2}{\rho_2}\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{l k_1}{\rho_2} w \\ -\frac{l}{\rho_1}(k_1 + k_3)\varphi_x - \frac{l k_1}{\rho_1}\psi + \frac{1}{\rho_1}\left(k_3 - g_3^0\right) w_{xx} - \frac{l^2 k_1}{\rho_1} w + \frac{1}{\rho_1}\int_0^{+\infty} g_3 \partial_{xx}\eta_3 \,\mathrm{d}s \\ \varphi_t - \partial_s \eta_1 \\ w_t - \partial_s \eta_3 \end{pmatrix}$$

and

$$\mathcal{A}_3 U_3 = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ \frac{1}{\rho_1}\left(k_1 - g_1^0\right)\varphi_{xx} - \frac{l^2 k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{l}{\rho_1}(k_1+k_3)w_x + \frac{1}{\rho_1}\int\limits_0^{+\infty} g_1 \partial_{xx}\eta_1 \,\mathrm{d}s \\ -\frac{k_1}{\rho_2}\varphi_x + \frac{1}{\rho_2}\left(k_2 - g_2^0\right)\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{lk_1}{\rho_2}w + \frac{1}{\rho_2}\int\limits_0^{+\infty} g_2\partial_{xx}\eta_2 \,\mathrm{d}s \\ -\frac{l}{\rho_1}(k_1+k_3)\varphi_x - \frac{lk_1}{\rho_1}\psi + \frac{k_3}{\rho_1}w_{xx} - \frac{l^2 k_1}{\rho_1}w \\ \varphi_t - \partial_s\eta_1 \\ \psi_t - \partial_s\eta_2 \end{pmatrix}.$$

Here, for $i = 1, 2, 3$,

$$g_i^0 = \int\limits_0^{+\infty} g_i(s)\,\mathrm{d}s. \tag{16}$$

Let

$$\begin{cases} L_1 = \left\{ v:\, \mathbb{R}_+ \to H_0^1(]0,L[),\ \int\limits_0^L \int\limits_0^{+\infty} g_1 v_x^2 \,\mathrm{d}s\,\mathrm{d}x < +\infty \right\}, \\ L_2 = \left\{ v:\, \mathbb{R}_+ \to H_*^1(]0,L[),\ \int\limits_0^L \int\limits_0^{+\infty} g_2 v_x^2 \,\mathrm{d}s\,\mathrm{d}x < +\infty \right\}, \\ L_3 = \left\{ v:\, \mathbb{R}_+ \to H_*^1(]0,L[),\ \int\limits_0^L \int\limits_0^{+\infty} g_3 v_x^2 \,\mathrm{d}s\,\mathrm{d}x < +\infty \right\} \end{cases} \tag{17}$$

and

$$\begin{cases} \mathcal{H}_1 = H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2 \times L^2(]0,L[) \times \left(L_*^2(]0,L[)\right)^2 \times L_2 \times L_3, \\ \mathcal{H}_2 = H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2 \times L^2(]0,L[) \times \left(L_*^2(]0,L[)\right)^2 \times L_1 \times L_3, \\ \mathcal{H}_3 = H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2 \times L^2(]0,L[) \times \left(L_*^2(]0,L[)\right)^2 \times L_1 \times L_2, \end{cases} \tag{18}$$

where

$$L_*^2(]0,L[) = \left\{ v \in L^2(]0,L[),\ \int\limits_0^L v\,\mathrm{d}x = 0 \right\} \tag{19}$$

and

$$H_*^1(]0,L[) = \left\{ v \in H^1(]0,L[),\ \int\limits_0^L v\,\mathrm{d}x = 0 \right\}. \tag{20}$$

The domain $D(\mathcal{A}_i)$ of $\mathcal{A}_i$ is defined by

$$D(\mathcal{A}_i) = \Big\{V = (v_1, \ldots, v_8)^T \in \mathcal{H}_i,\ \mathcal{A}_i V \in \mathcal{H}_i,\ v_7(0) = v_8(0) = 0,\ \partial_x v_2(0) = 0,$$
$$\partial_x v_3(0) = \partial_x v_2(L) = \partial_x v_3(L) = 0,\ \partial_x v_j(\cdot, 0) = \partial_x v_j(\cdot, L) = 0,\ j = 7, 8 \text{ if } i = 1,\ j = 8 \text{ if } i = 2, 3\Big\}; \tag{21}$$

that is, according to the definition of $\mathcal{H}_i$ and $\mathcal{A}_i$,

$$D(\mathcal{A}_i) = \Big\{(v_1, \ldots, v_8)^T \in \mathcal{H}_i :\ (v_1, \ldots, v_6)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2 \times H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2,$$
$$v_7(0) = v_8(0) = 0,\ \partial_x v_2(0) = \partial_x v_3(0) = \partial_x v_2(L) = \partial_x v_3(L) = 0\Big\} \cap D_i,$$

where

$$D_1 = \Big\{(v_1, \ldots, v_8)^T \in \mathcal{H}_i :\ \partial_s v_7 \in L_2,\ \partial_s v_8 \in L_3,\ v_1 \in H^2(]0, L[),$$
$$\partial_x v_7(\cdot, 0) = \partial_x v_8(\cdot, 0) = \partial_x v_7(\cdot, L) = \partial_x v_8(\cdot, L) = 0,$$
$$\left(k_2 - g_2^0\right) \partial_{xx} v_2 + \int_0^{+\infty} g_2 \partial_{xx} v_7 \,\mathrm{d}s \in L_*^2(]0, L[),\ \left(k_3 - g_3^0\right) \partial_{xx} v_3 + \int_0^{+\infty} g_3 \partial_{xx} v_8 \,\mathrm{d}s \in L_*^2(]0, L[)\Big\},$$
$$D_2 = \Big\{(v_1, \ldots, v_8)^T \in \mathcal{H}_i :\ \partial_s v_7 \in L_1,\ \partial_s v_8 \in L_3,\ v_2 \in H^2(]0, L[),\ \partial_x v_8(\cdot, 0) = \partial_x v_8(\cdot, L) = 0,$$
$$\left(k_1 - g_1^0\right) \partial_{xx} v_1 + \int_0^{+\infty} g_1 \partial_{xx} v_7 \,\mathrm{d}s \in L^2(]0, L[),\ \left(k_3 - g_3^0\right) \partial_{xx} v_3 + \int_0^{+\infty} g_3 \partial_{xx} v_8 \,\mathrm{d}s \in L_*^2(]0, L[)\Big\}$$

and

$$D_3 = \Big\{(v_1, \ldots, v_8)^T \in \mathcal{H}_i :\ \partial_s v_7 \in L_1,\ \partial_s v_8 \in L_2,\ v_3 \in H^2(]0, L[),\ \partial_x v_8(\cdot, 0) = \partial_x v_8(\cdot, L) = 0,$$
$$\left(k_1 - g_1^0\right) \partial_{xx} v_1 + \int_0^{+\infty} g_1 \partial_{xx} v_7 \,\mathrm{d}s \in L^2(]0, L[),\ \left(k_2 - g_2^0\right) \partial_{xx} v_2 + \int_0^{+\infty} g_2 \partial_{xx} v_8 \,\mathrm{d}s \in L_*^2(]0, L[)\Big\}.$$

More generally, for $n \in \mathbb{N}$,

$$D(\mathcal{A}_i^n) = \begin{cases} \mathcal{H}_i & \text{if } n = 0, \\ D(\mathcal{A}_i) & \text{if } n = 1, \\ \left\{V \in D(\mathcal{A}_i^{n-1}),\ \mathcal{A}_i V \in D(\mathcal{A}_i^{n-1})\right\} & \text{if } n = 2, 3, \ldots \end{cases}$$

endowed with the graph norm

$$\|V\|_{D(\mathcal{A}_i^n)} = \sum_{k=0}^{n} \|\mathcal{A}_i^k V\|_{\mathcal{H}_i},$$

where $\|\cdot\|_{\mathcal{H}_i}$ is defined in (43).

Remark 2.1. By integrating on $]0, L[$ the second and third equations in (1)–(3), and using the boundary conditions, we obtain

$$\partial_{tt}\left(\int_0^L \psi \,\mathrm{d}x\right) + \frac{k_1}{\rho_2} \int_0^L \psi \,\mathrm{d}x + \frac{lk_1}{\rho_2} \int_0^L w \,\mathrm{d}x = 0 \tag{22}$$

and

$$\partial_{tt}\left(\int_0^L w\,\mathrm{d}x\right)+\frac{l^2k_1}{\rho_1}\int_0^L w\,\mathrm{d}x+\frac{lk_1}{\rho_1}\int_0^L \psi\,\mathrm{d}x=0. \tag{23}$$

Therefore, (22) implies that

$$\int_0^L w\,\mathrm{d}x=-\frac{\rho_2}{lk_1}\partial_{tt}\left(\int_0^L \psi\,\mathrm{d}x\right)-\frac{1}{l}\int_0^L \psi\,\mathrm{d}x. \tag{24}$$

Substituting (24) into (23), we get

$$\partial_{tttt}\left(\int_0^L \psi\,\mathrm{d}x\right)+\left(\frac{k_1}{\rho_2}+\frac{l^2k_1}{\rho_1}\right)\partial_{tt}\left(\int_0^L \psi\,\mathrm{d}x\right)=0. \tag{25}$$

Let $l_0=\sqrt{\frac{k_1}{\rho_2}+\frac{l^2k_1}{\rho_1}}$. Then, solving (25), we find

$$\int_0^L \psi\,\mathrm{d}x=\tilde{c}_1\cos(l_0t)+\tilde{c}_2\sin(l_0t)+\tilde{c}_3t+\tilde{c}_4, \tag{26}$$

where $\tilde{c}_1,\ldots,\tilde{c}_4$ are real constants. By combining (24) and (26), we get

$$\int_0^L w\,\mathrm{d}x=\tilde{c}_1\left(\frac{\rho_2l_0^2}{lk_1}-\frac{1}{l}\right)\cos(l_0t)+\tilde{c}_2\left(\frac{\rho_2l_0^2}{lk_1}-\frac{1}{l}\right)\sin(l_0t)-\frac{\tilde{c}_3}{l}t-\frac{\tilde{c}_4}{l}. \tag{27}$$

Let

$$(\tilde{\psi}_0(x),\tilde{w}_0(x))=\begin{cases}(\psi_0(x,0),w_0(x,0)) & \text{in case (1)},\\ (\psi_0(x),w_0(x,0)) & \text{in case (2)},\\ (\psi_0(x,0),w_0(x)) & \text{in case (3)}.\end{cases}$$

Using the initial data of ψ and w in (1)–(3), we see that

$$\begin{cases}\tilde{c}_1=\frac{k_1}{\rho_2l_0^2}\int_0^L\tilde{\psi}_0\,\mathrm{d}x+\frac{lk_1}{\rho_2l_0^2}\int_0^L\tilde{w}_0\,\mathrm{d}x,\\ \tilde{c}_2=\frac{k_1}{\rho_2l_0^3}\int_0^L\psi_1\,\mathrm{d}x+\frac{lk_1}{\rho_2l_0^3}\int_0^L w_1\,\mathrm{d}x,\\ \tilde{c}_3=\left(1-\frac{k_1}{\rho_2l_0^2}\right)\int_0^L\psi_1\,\mathrm{d}x-\frac{lk_1}{\rho_2l_0^2}\int_0^L w_1\,\mathrm{d}x,\\ \tilde{c}_4=\left(1-\frac{k_1}{\rho_2l_0^2}\right)\int_0^L\tilde{\psi}_0\,\mathrm{d}x-\frac{lk_1}{\rho_2l_0^2}\int_0^L\tilde{w}_0\,\mathrm{d}x.\end{cases}$$

Let

$$\tilde{\psi}=\psi-\frac{1}{L}\left(\tilde{c}_1\cos(l_0t)+\tilde{c}_2\sin(l_0t)+\tilde{c}_3t+\tilde{c}_4\right) \tag{28}$$

and

$$\tilde{w} = w - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\cos(l_0 t) + \tilde{c}_2\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\sin(l_0 t) - \frac{\tilde{c}_3}{l}t - \frac{\tilde{c}_4}{l}\right). \tag{29}$$

Then, from (26) and (27) one can check that

$$\int_0^L \tilde{\psi}\,\mathrm{d}x = \int_0^L \tilde{w}\,\mathrm{d}x = 0, \tag{30}$$

and, hence,

$$\int_0^L \tilde{\eta}_2\,\mathrm{d}x = \int_0^L \tilde{\eta}_3\,\mathrm{d}x = 0, \tag{31}$$

where

$$\begin{cases} \text{Cases (1) and (3)}: \ \tilde{\eta}_2(x,t,s) = \tilde{\psi}(x,t) - \tilde{\psi}(x,t-s) & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \text{Cases (1) and (2)}: \ \tilde{\eta}_3(x,t,s) = \tilde{w}(x,t) - \tilde{w}(x,t-s) & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+. \end{cases}$$

Therefore, the Poincaré's inequality

$$\exists\, c_0 > 0 : \int_0^L v^2\,\mathrm{d}x \le c_0 \int_0^L v_x^2\,\mathrm{d}x, \quad \forall v \in H_*^1(]0,L[) \tag{32}$$

is applicable for $\tilde{\psi}$, $\tilde{w}$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$, provided that $\tilde{\psi}, \tilde{w} \in H^1(]0,L[)$. In addition, $(\varphi, \tilde{\psi}, \tilde{w})$ satisfies the boundary conditions and the first three equations in (1)–(3) with initial data

$$\psi_0 - \frac{1}{L}(\tilde{c}_1 + \tilde{c}_4), \quad \psi_1 - \frac{1}{L}(l_0\tilde{c}_2 + \tilde{c}_3),$$

$$w_0 - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_4}{l}\right) \quad \text{and} \quad w_1 - \frac{1}{L}\left(\tilde{c}_2 l_0\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_3}{l}\right)$$

instead of ψ_0, ψ_1, w_0 and w_1, respectively. In the sequel, we work with $\tilde{\psi}$, $\tilde{w}$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$ instead of ψ, w, η_2 and η_3, but, for simplicity of notation, we use ψ, w, η_2 and η_3 instead of $\tilde{\psi}$, $\tilde{w}$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$, respectively.

Now, to get the well-posedness of (15), we assume the following hypothesis:

(H1) The function $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, nonincreasing and integrable on $\mathbb{R}_+$ such that there exists a positive constant k_0 such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2,$$

we have

$$k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right)\mathrm{d}x \le \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right)\mathrm{d}x$$

$$- \begin{cases} \displaystyle\int_0^L \left(g_2^0\psi_x^2 + g_3^0 w_x^2\right)\mathrm{d}x & \text{in case (1)}, \\ \displaystyle\int_0^L \left(g_1^0\varphi_x^2 + g_3^0 w_x^2\right)\mathrm{d}x & \text{in case (2)}, \\ \displaystyle\int_0^L \left(g_1^0\varphi_x^2 + g_2^0 \psi_x^2\right)\mathrm{d}x & \text{in case (3)}. \end{cases} \tag{33}$$

Moreover, there exists a positive constant β such that

$$-\beta g_j(s) \leq g_j'(s), \quad \forall s \in \mathbb{R}_+, \tag{34}$$

where $j \in \{1,2,3\} \setminus \{i\}$ for system (i), $i = 1,2,3$.

Remark 2.2. 1. The set of functions satisfying **(H1)** is very large; indeed, if, for example, the constants L and l satisfy

$$lL \neq m\pi, \quad \forall m \in \mathbb{N}^*, \tag{35}$$

then, by contradiction arguments, we see that there exists a positive constant $\bar{k}_0$ such that, for any $(\varphi,\psi,w)^T \in H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2$,

$$\bar{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x \leq \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) \mathrm{d}x. \tag{36}$$

To prove (36) in case (35), it is sufficient to prove that, if the right-hand side of (36) vanishes, then

$$(\varphi,\psi,w) = (0,0,0). \tag{37}$$

But (37) can be directly deduced from (30), (35) and the boundary conditions on φ. Therefore, if (35) holds and

$$\max_{j\in\{1,2,3\}\setminus\{i\}} \left\{g_j^0\right\} < \bar{k}_0 \quad \text{in case}\,(i), \quad i = 1,2,3, \tag{38}$$

then (33) is satisfied with

$$k_0 = \bar{k}_0 - \begin{cases} \max\left\{g_2^0, g_3^0\right\} & \text{in case (1)}, \\ \max\left\{g_1^0, g_3^0\right\} & \text{in case (2)}, \\ \max\left\{g_1^0, g_2^0\right\} & \text{in case (3)}. \end{cases}$$

2. Thanks to (32) applied for ψ and w, and the Poincaré's inequality

$$\exists\, \tilde{c}_0 > 0 : \int_0^L v^2\, \mathrm{d}x \leq \tilde{c}_0 \int_0^L v_x^2\, \mathrm{d}x, \quad \forall v \in H_0^1(]0,L[) \tag{39}$$

applied for φ, there exists a positive constant $\tilde{k}_0$ such that, for any

$$(\varphi,\psi,w)^T \in H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2,$$

we have

$$\int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) \mathrm{d}x \leq \tilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x. \tag{40}$$

Thus, from (36) and (40), we deduce that the right-hand side of the inequality (36) defines a norm on $H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2$ for (φ,ψ,w) equivalent to the usual norm of $\left(H^1(]0,L[)\right)^3$.

3. From (33), we conclude that, in case (i), $i = 1,2,3$,

$$k_0 + g_j^0 - k_j \leq 0, \quad \forall j \in \{1,2,3\} \setminus \{i\}. \tag{41}$$

Indeed, for the choice $\varphi = w = 0$, (33) in cases (1) and (3) gives

$$\left(k_0 + g_2^0 - k_2\right) \int_0^L \psi_x^2\, \mathrm{d}x \leq k_1 \int_0^L \psi^2\, \mathrm{d}x, \quad \forall \psi \in H_*^1(]0,L[).$$

This inequality implies, for $\psi(x) = \cos(\lambda x) - \frac{1}{\lambda L}\sin(\lambda L)$ and $\lambda \in]0, +\infty[$ (notice that $\psi \in H^1_*(]0, L[)$),

$$\left(k_0 + g_2^0 - k_2\right)\left(L - \frac{1}{2\lambda}\sin(2\lambda L)\right) \leq \frac{k_1}{\lambda^2}\left(L + \frac{1}{2\lambda}\sin(2\lambda L) - \frac{2}{\lambda^2 L}\sin^2(\lambda L)\right), \quad \forall \lambda > 0.$$

By letting λ go to $+\infty$, we deduce (41), for $j = 2$. In the same way, using the choices

$$(\varphi(x), \psi(x), w(x)) = \left(\sin\left(\frac{m\pi}{L}x\right), 0, 0\right) \text{ and } (\varphi(x), \psi(x), w(x)) = \left(0, 0, \cos(\lambda x) - \frac{1}{\lambda L}\sin(\lambda L)\right),$$

for $m \in \mathbb{N}$ and $\lambda \in]0, +\infty[$ (notice that $\varphi \in H_0^1(]0, L[)$ and $w \in H^1_*(]0, L[)$), and letting m and λ go to $+\infty$, we conclude (41), for $j = 1$ and $j = 3$, respectively.

According to Remark 2.2, we notice that, under the hypothesis **(H1)**, the sets L_i and $\mathcal{H}_i$ are Hilbert spaces equipped, respectively, with the inner products that generate the norms, for $v \in L_i$ and $V = (v_1, \ldots, v_8)^T \in \mathcal{H}_i$, $i = 1, 2, 3$,

$$\|v\|_{L_i}^2 = \int_0^L \int_0^{+\infty} g_i v_x^2 \, \mathrm{d}s \, \mathrm{d}x \tag{42}$$

and

$$\|V\|_{\mathcal{H}_i}^2 = \int_0^L \left(k_2(\partial_x v_2)^2 + k_1(\partial_x v_1 + v_2 + l v_3)^2 + k_3(\partial_x v_3 - l v_1)^2\right) \mathrm{d}x$$
$$+ \int_0^L \left(\rho_1 v_4^2 + \rho_2 v_5^2 + \rho_1 v_6^2\right) \mathrm{d}x + \begin{cases} \|v_7\|_{L_2}^2 + \|v_8\|_{L_3}^2 - \int_0^L \left(g_2^0(\partial_x v_2)^2 + g_3^0(\partial_x v_3)^2\right) \mathrm{d}x & \text{if } i = 1, \\ \|v_7\|_{L_1}^2 + \|v_8\|_{L_3}^2 - \int_0^L \left(g_1^0(\partial_x v_1)^2 + g_3^0(\partial_x v_3)^2\right) \mathrm{d}x & \text{if } i = 2, \\ \|v_7\|_{L_1}^2 + \|v_8\|_{L_2}^2 - \int_0^L \left(g_1^0(\partial_x v_1)^2 + g_2^0(\partial_x v_2)^2\right) \mathrm{d}x & \text{if } i = 3. \end{cases} \tag{43}$$

Now, the domain of $D(\mathcal{A}_i)$ is dense in $\mathcal{H}_i$, and a simple computation implies that, for any $V = (v_1, \ldots, v_8)^T \in D(\mathcal{A}_i)$,

$$\langle \mathcal{A}_i V, V \rangle_{\mathcal{H}_i} = \frac{1}{2}\begin{cases} \int_0^L \int_0^{+\infty} \left(g_2'(\partial_x v_7)^2 + g_3'(\partial_x v_8)^2\right) \mathrm{d}s \, \mathrm{d}x & \text{if } i = 1, \\ \int_0^L \int_0^{+\infty} \left(g_1'(\partial_x v_7)^2 + g_3'(\partial_x v_8)^2\right) \mathrm{d}s \, \mathrm{d}x & \text{if } i = 2, \\ \int_0^L \int_0^{+\infty} \left(g_1'(\partial_x v_7)^2 + g_2'(\partial_x v_8)^2\right) \mathrm{d}s \, \mathrm{d}x & \text{if } i = 3. \end{cases} \tag{44}$$

Since g_i is nonincreasing, we deduce from (44) that

$$\langle \mathcal{A}_i V, V \rangle_{\mathcal{H}_i} \leq 0. \tag{45}$$

This implies that A_i is dissipative. Notice that, according to (34) and the fact that g_i is nonincreasing, we see that, for $v \in L_i$,

$$\begin{aligned}\left|\int_0^L \int_0^{+\infty} g_i' v_x^2 \,\mathrm{d}s\,\mathrm{d}x\right| &= -\int_0^L \int_0^{+\infty} g_i' v_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ &\le \beta \int_0^L \int_0^{+\infty} g_i v_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ &\le \beta \|v\|_{L_i}^2 \\ &< +\infty,\end{aligned}$$

so the integrals in the right-hand side of (44) are well defined.

Next, we prove that $Id - \mathcal{A}_i$ is surjective, where Id is the identity operator. Let $F = (f_1, \ldots, f_8)^T \in \mathcal{H}_i$. We prove the existence of $V = (v_1, \ldots, v_8)^T \in D(\mathcal{A}_i)$ solution of the equation

$$(Id - \mathcal{A}_i)V = F. \tag{46}$$

Let us consider the case $\mathcal{A}_1$ (the cases $\mathcal{A}_2$ and $\mathcal{A}_3$ can be treated similarly). The first three equations in (46) reduce to

$$\begin{cases} v_4 = v_1 - f_1, \\ v_5 = v_2 - f_2, \\ v_6 = v_3 - f_3. \end{cases} \tag{47}$$

Using (47), the last two equations in (46) are equivalent to

$$\begin{cases} \partial_s v_7 + v_7 = v_2 + f_7 - f_2, \\ \partial_s v_8 + v_8 = v_3 + f_8 - f_3. \end{cases} \tag{48}$$

By integrating the two differential equations in (48) and using the fact that $v_7(0) = v_8(0) = 0$ (from (21)), we get

$$\begin{cases} v_7(s) = (1 - \mathrm{e}^{-s})(v_2 - f_2) + \displaystyle\int_0^s \mathrm{e}^{\tau - s} f_7(\tau)\,\mathrm{d}\tau, \\ v_8(s) = (1 - \mathrm{e}^{-s})(v_3 - f_3) + \displaystyle\int_0^s \mathrm{e}^{\tau - s} f_8(\tau)\,\mathrm{d}\tau. \end{cases} \tag{49}$$

We see that, from (47), if $(v_1, v_2, v_3) \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$, then $(v_4, v_5, v_6) \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$. On the other hand, using Fubini theorem, Hölder's inequality and noting that $f_7 \in L_2$, we get

$$\int_0^L \int_0^{+\infty} g_2(s) \left(\mathrm{e}^{-s} \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\,\mathrm{d}\tau \right)^2 \mathrm{d}s\,\mathrm{d}x \le \int_0^L \int_0^{+\infty} \mathrm{e}^{-2s} g_2(s) \left(\int_0^s \mathrm{e}^{\tau}\,\mathrm{d}\tau \right) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}x$$
$$\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-s}(1-\mathrm{e}^{-s}) g_2(s) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}x$$
$$\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-s} g_2(s) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}x$$
$$\le \int_0^L \int_0^{+\infty} \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2 \int_\tau^{+\infty} \mathrm{e}^{-s} g_2(s)\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}x$$
$$\le \int_0^L \int_0^{+\infty} \mathrm{e}^{\tau} g_2(\tau) (\partial_x f_7(\tau))^2 \int_\tau^{+\infty} \mathrm{e}^{-s}\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}x$$
$$\le \int_0^L \int_0^{+\infty} g_2(\tau) (\partial_x f_7(\tau))^2\,\mathrm{d}\tau\,\mathrm{d}x$$
$$\le \|f_7\|_{L_2}^2 < +\infty,$$

then

$$s \mapsto \mathrm{e}^{-s} \int_0^s \mathrm{e}^{\tau} f_7(\tau)\,\mathrm{d}\tau \in L_2,$$

and therefore, (49) implies that $v_7 \in L_2$. Moreover, $\partial_s v_7 \in L_2$ since (48). Similarly, we have v_8, $\partial_s v_8 \in L_3$. So, to prove that (46) (for $i = 1$) admits a solution $V \in D(\mathcal{A}_1)$, it is enough to prove that

$$\partial_x v_7(\cdot, 0) = \partial_x v_8(\cdot, 0) = \partial_x v_7(\cdot, L) = \partial_x v_8(\cdot, L) = 0 \tag{50}$$

and (v_1, v_2, v_3) exists and satisfies the required regularity and boundary conditions in $D(\mathcal{A}_1)$; that is,

$$(v_1, v_2, v_3)^T \in \left(H^2(]0, L[) \cap H_0^1(]0, L[)\right) \times H_*^1(]0, L[) \times H_*^1(]0, L[), \tag{51}$$

$$\left(k_2 - g_2^0\right) \partial_{xx} v_2 + \int_0^{+\infty} g_2 \partial_{xx} v_7\,\mathrm{d}s,\ \left(k_3 - g_3^0\right) \partial_{xx} v_3 + \int_0^{+\infty} g_3 \partial_{xx} v_8\,\mathrm{d}s \in L_*^2(]0, L[) \tag{52}$$

and

$$\partial_x v_2(0) = \partial_x v_3(0) = \partial_x v_2(L) = \partial_x v_3(L) = 0. \tag{53}$$

Let us assume that (50)–(53) hold. Multiplying the fourth, fifth and sixth equations in (46) by $\rho_1 \tilde{v}_1$, $\rho_2 \tilde{v}_2$ and $\rho_1 \tilde{v}_3$, respectively, integrating their sum over $]0, L[$, using the boundary conditions (50) and (53), and inserting (47) and (49), we get

$$a_1\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = \tilde{a}_1\left((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right), \tag{54}$$

for any $(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^T \in H_0^1(]0,L[)\times\left(H_*^1(]0,L[)\right)^2$, where

$$
\begin{aligned}
&a_1\left((v_1,v_2,v_3)^T,(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^T\right)\\
&=\int_0^L \left(k_1(\partial_x v_1+v_2+lv_3)(\partial_x\tilde{v}_1+\tilde{v}_2+l\tilde{v}_3)+k_3(\partial_x v_3-lv_1)(\partial_x\tilde{v}_3-l\tilde{v}_1)\right)\mathrm{d}x\\
&+\int_0^L \left(\rho_1 v_1\tilde{v}_1+\rho_2 v_2\tilde{v}_2+\rho_1 v_3\tilde{v}_3+(k_2-\tilde{g}_2^0)\partial_x v_2\partial_x\tilde{v}_2-\tilde{g}_3^0\partial_x v_3\partial_x\tilde{v}_3\right)\mathrm{d}x,
\end{aligned}
\tag{55}
$$

$\tilde{g}_i^0=\int_0^{+\infty}\mathrm{e}^{-s}g_i(s)\,\mathrm{d}s$ and

$$
\begin{aligned}
\tilde{a}_1\left((\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^T\right)&=\int_0^L\left(\rho_1(f_1+f_4)\tilde{v}_1+\rho_2(f_2+f_5)\tilde{v}_2+\rho_1(f_3+f_6)\tilde{v}_3\right)\mathrm{d}x\\
&+\int_0^L\left((g_2^0-\tilde{g}_2^0)\partial_x f_2\partial_x\tilde{v}_2+(g_3^0-\tilde{g}_3^0)\partial_x f_3\partial_x\tilde{v}_3\right)\mathrm{d}x\\
&-\int_0^L\left(\int_0^{+\infty}\mathrm{e}^{-s}g_2(s)\int_0^s\mathrm{e}^{\tau}\partial_x f_7(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right)\partial_x\tilde{v}_2\,\mathrm{d}x\\
&-\int_0^L\left(\int_0^{+\infty}\mathrm{e}^{-s}g_3(s)\int_0^s\mathrm{e}^{\tau}\partial_x f_8(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right)\partial_x\tilde{v}_3\,\mathrm{d}x.
\end{aligned}
\tag{56}
$$

We remark that, using Fubini's theorem, Hölder's inequality and noting that $f_7\in L_2$,

$$
\begin{aligned}
\int_0^L\left(\int_0^{+\infty}\mathrm{e}^{-s}g_2(s)\int_0^s\mathrm{e}^{\tau}\partial_x f_7(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right)^2\mathrm{d}x&\le\int_0^L\left(\int_0^{+\infty}\mathrm{e}^{-s}g_2(s)\int_0^s\mathrm{e}^{\tau}|\partial_x f_7(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\le\int_0^L\left(\int_0^{+\infty}\mathrm{e}^{\tau}|\partial_x f_7(\tau)|\int_\tau^{+\infty}g_2(s)\mathrm{e}^{-s}\,\mathrm{d}s\,\mathrm{d}\tau\right)^2\mathrm{d}x\\
&\le\int_0^L\left(\int_0^{+\infty}g_2(\tau)\mathrm{e}^{\tau}|\partial_x f_7(\tau)|\int_\tau^{+\infty}\mathrm{e}^{-s}\,\mathrm{d}s\,\mathrm{d}\tau\right)^2\mathrm{d}x\\
&\le\int_0^L\left(\int_0^{+\infty}g_2(\tau)|\partial_x f_7(\tau)|\,\mathrm{d}\tau\right)^2\mathrm{d}x\\
&\le\int_0^L\left(\int_0^{+\infty}g_2(\tau)\,\mathrm{d}\tau\right)\left(\int_0^{+\infty}g_2(\tau)|\partial_x f_7(\tau)|^2\,\mathrm{d}\tau\right)\mathrm{d}x\\
&\le g_2^0\|f_7\|_{L_2}^2<+\infty,
\end{aligned}
$$

which implies that

$$x \mapsto \int_0^{+\infty} \mathrm{e}^{-s} g_2(s) \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \in L^2(]0, L[).$$

Similarly, we have

$$x \mapsto \int_0^{+\infty} \mathrm{e}^{-s} g_3(s) \int_0^s \mathrm{e}^{\tau} \partial_x f_8(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \in L^2(]0, L[).$$

On the other hand, $\tilde{g}_2^0 \leq g_2^0 < k_2$ (since (41)) and $\tilde{g}_3^0 \leq g_3^0$. Then, by virtue of (33) and (40), we have a_1 is a bilinear, continuous and coercive form on

$$\left(H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2\right) \times \left(H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2\right),$$

and $\tilde{a}_1$ is a linear and continuous form on $H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$. Consequently, using Lax–Milgram's theorem, we deduce that (54) has a unique solution

$$(v_1, v_2, v_3)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2.$$

Therefore, using classical elliptic regularity arguments, we conclude that the forth, fifth and sixth equations in (46) are satisfied with $(v_1, v_2, v_3)^T$ satisfying (51) and (53), and, using (47) and (49), v_7 and v_8 satisfy (50) and (52). Thus, we deduce that (46) (for $i = 1$) admits a unique solution $V \in D(\mathcal{A}_1)$, and then, $Id - \mathcal{A}_1$ is surjective.

Finally, thanks to the Lumer–Phillips theorem (see [19]), we deduce from (45) and (46) that $\mathcal{A}_i$ generates a C_0-semigroup of contractions in $\mathcal{H}_i$. This gives the following well-posedness results of (15) (see [13] and [19]).

Theorem 2.3. *Assume that* **(H1)** *holds. For any $U_i^0 \in D(\mathcal{A}_i^n)$, $n \in \mathbb{N}$, (15) has a unique solution*

$$U_i \in \cap_{k=0}^{n} C^{n-k}\left(\mathbb{R}_+; D\left(\mathcal{A}_i^k\right)\right). \tag{57}$$

3. Stability

In this section, we study the stability of (15), where the obtained two (uniform and weak) decay rates of solution depend on the speeds of wave propagations (4), the smoothness of initial data U_i^0, defined in (13), and the growth of g_i at infinity characterized by the following additional hypothesis:

(H2) Assume that $g_j(0) > 0$, and there exist a positive constant α and an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty$$

such that

$$g_j'(s) \leq -\alpha g_j(s), \quad \forall s \in \mathbb{R}_+ \tag{58}$$

or

$$\int_0^{+\infty} \frac{g_j(s)}{G^{-1}(-g_j'(s))}\, \mathrm{d}s + \sup_{s \in \mathbb{R}_+} \frac{g_j(s)}{G^{-1}(-g_j'(s))} < +\infty, \tag{59}$$

where $j \in \{1, 2, 3\} \setminus \{i\}$, for system (i), $i = 1, 2, 3$.

We start by considering systems (1)–(3) in the case where the speeds of wave propagations (4) satisfy

$$\begin{cases} s_1 = s_2 & \text{in case (1)}, \\ s_2 = s_1 & \text{in case (2)}, \\ s_3 = s_1 & \text{in case (3)}. \end{cases} \tag{60}$$

Theorem 3.1. *Assume that* **(H1)**, **(H2)** *and* $(60)_i$ *are satisfied such that*

$$\begin{cases} g_3^0 \text{ is small enough} & \text{in case (1)}, \\ l < \frac{2k_1}{\sqrt{k_2 k_3}} \text{ and } g_1^0 \text{ and } g_3^0 \text{ are small enough} & \text{in case (2)}, \\ g_1^0 \text{ is small enough} & \text{in case (3)}. \end{cases} \tag{61}$$

Let $U_i^0 \in \mathcal{H}_i$ *be such that, for any* $j \in \{1, 2, 3\} \setminus \{i\}$,

$$(58) \text{ holds} \quad \text{or} \quad \sup_{t \in \mathbb{R}_+} \int_t^{+\infty} \frac{g_j(s)}{G^{-1}(-g_j'(s))} \int_0^L \left(\partial_x \eta_j^0(x, s-t)\right)^2 \, dx \, ds < +\infty. \tag{62}$$

Then, there exist positive constants c_i' *and* c_i'' *such that the solution of (15) satisfies*

$$\|U_i(t)\|^2_{\mathcal{H}_i} \le c_i'' \tilde{G}^{-1}(c_i' t), \quad \forall t \in \mathbb{R}_+, \tag{63}$$

where $\tilde{G}(s) = \int_s^1 \frac{1}{G_0(\tau)} \, d\tau$ $(s \in]0, 1])$ *and*

$$G_0(s) = \begin{cases} s & \text{if (58) holds for any } j \in \{1, 2, 3\}/\{i\}, \\ sG'(s) & \text{otherwise.} \end{cases} \tag{64}$$

When $(60)_i$ does not hold, we prove the following weaker stability result for (15).

Theorem 3.2. *Assume that* **(H1)**, **(H2)** *and (61) are satisfied. Let* $n \in \mathbb{N}^*$ *and* $U_i^0 \in D(\mathcal{A}_i^n)$ *be such that, for any* $j \in \{1, 2, 3\} \setminus \{i\}$,

$$(58) \text{ holds} \quad \text{or} \quad \sup_{t \in \mathbb{R}_+} \max_{k=0,\dots,n} \int_t^{+\infty} \frac{g_j(s)}{G^{-1}(-g_j'(s))} \int_0^L \left(\partial_s^k \partial_x \eta_j^0(x, s-t)\right)^2 \, dx \, ds < +\infty. \tag{65}$$

Then, there exists a positive constant $c_{i,n}$ *such that*

$$\|U_i(t)\|^2_{\mathcal{H}_i} \le c_{i,n} G_n\left(\frac{c_{i,n}}{t}\right), \quad \forall t > 0, \tag{66}$$

where $G_m(s) = G_1(sG_{m-1}(s))$, *for* $m = 2, \dots, n$ *and* $s \in \mathbb{R}_+$, $G_1 = G_0^{-1}$ *and* G_0 *is defined in (64).*

Remark 3.3. 1. Estimates (63) and (66) imply the strong stability of (15); that is,

$$\lim_{t \to +\infty} \|U_i(t)\|^2_{\mathcal{H}_i} = 0. \tag{67}$$

2. If (58) holds, for any $j \in \{1, 2, 3\} \setminus \{i\}$, then (63) and (66) give, respectively,

$$\|U_i(t)\|^2_{\mathcal{H}_i} \le c_i'' \mathrm{e}^{-c_i' t}, \quad \forall t \in \mathbb{R}_+ \tag{68}$$

and

$$\|U_i(t)\|^2_{\mathcal{H}_i} \le \frac{c_{i,n}}{t^n}, \quad \forall t > 0. \tag{69}$$

The estimates (68) and (69) give the best decay rates which can be obtained from (63) and (66), respectively.

3. Condition (58) implies that g_j converges exponentially to zero at infinity. However, condition (59) (introduced in [10]) allows $s \mapsto g_j(s)$ to have a decay rate arbitrarily close to $\frac{1}{s}$, which represents the

critical limit, since g_j is integrable on $\mathbb{R}_+$. For specific examples of g_j and η_j satisfying (59), (62) and (65), and the corresponding decay rates given by (63) and (66), see [10] and [11].

4. The abstract systems considered in [10] and [11] do not include (1)–(3) because the operator B in [10] is assumed to be positive definite, and the operator $\tilde{B}$ in [11] is assumed to be bounded.

To prove (63) and (66), we will consider suitable multipliers and construct Lyapunov functionals satisfying some differential inequalities, for any $U_i^0 \in D(\mathcal{A}_i)$ and $t \in \mathbb{R}_+$; so all the calculations are justified. By integrating these differential inequalities, we get (63) and (66) (for $n = 1$). By simple density arguments ($D(\mathcal{A}_i)$ is dense in $\mathcal{H}_i$) and induction on n, (63) remains valid, for any $U_i^0 \in \mathcal{H}_i$, and (66) holds, for any $n \in \mathbb{N}^*$.

We will use c, throughout the rest of this paper, to denote a generic positive constant which depends continuously on the initial data U_i^0 and the fixed parameters in (1)–(3), (32) and (39) and can be different from line to line. When c depends on some new constants $y_1, y_2, \ldots$, introduced in the proof, the constant c is noted $c_{y_1}, c_{y_1,y_2}, \ldots$

Let us consider the energy functional E_i associated with (15) defined by

$$E_i(t) = \frac{1}{2}\|U_i(t)\|^2_{\mathcal{H}_i}. \tag{70}$$

From (15) and (44), we see that

$$E_i'(t) = \frac{1}{2}\begin{cases} \int_0^L \int_0^{+\infty} \left(g_2'(\partial_x\eta_2)^2 + g_3'(\partial_x\eta_3)^2\right) \mathrm{d}s\,\mathrm{d}x & \text{if } i = 1, \\ \int_0^L \int_0^{+\infty} \left(g_1'(\partial_x\eta_1)^2 + g_3'(\partial_x\eta_3)^2\right) \mathrm{d}s\,\mathrm{d}x & \text{if } i = 2, \\ \int_0^L \int_0^{+\infty} \left(g_1'(\partial_x\eta_1)^2 + g_2'(\partial_x\eta_2)^2\right) \mathrm{d}s\,\mathrm{d}x & \text{if } i = 3. \end{cases} \tag{71}$$

Recalling that g_i is nonincreasing, (71) implies that E_i is nonincreasing, and consequently, (15) is dissipative. If no infinite memory is considered, then $E_i' \equiv 0$; thus, (15) is a conservative system. This fact shows that the infinite memories generate the unique dissipation in (15). On the other hand, if $E_i(t_0) = 0$, for some $t_0 \in \mathbb{R}_+$, then $E_i(t) = 0$, for all $t \geq t_0$, and therefore, (63) and (66) hold. Consequently, without loss of generality, we can assume that $E_i(t) > 0$, for all $t \in \mathbb{R}_+$.

4. Proof of uniform decay (63)

First, we consider the following functionals:

$$\text{Cases (2) and (3)}: \quad I_1(t) = -\rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)\eta_1 \,\mathrm{d}s\,\mathrm{d}x, \tag{72}$$

$$\text{Cases (1) and (3)}: \quad I_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)\eta_2 \,\mathrm{d}s\,\mathrm{d}x \tag{73}$$

and

$$\text{Cases (1) and (2)}: \quad I_3(t) = -\rho_1 \int_0^L w_t \int_0^{+\infty} g_3(s)\eta_3 \,\mathrm{d}s\,\mathrm{d}x. \tag{74}$$

Lemma 4.1. *For any $\delta_0 > 0$, there exists $c_{\delta_0} > 0$ such that*

$$I_1'(t) \leq -\rho_1 \left(g_1^0 - \delta_0\right) \int_0^L \varphi_t^2 \, dx + \delta_0 \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2\right) \, dx$$
$$+ c_{\delta_0} \int_0^L \int_0^{+\infty} (g_1(s) - g_1'(s)) \, (\partial_x \eta_1)^2 \, ds \, dx, \tag{75}$$

$$I_2'(t) \leq -\rho_2 \left(g_2^0 - \delta_0\right) \int_0^L \psi_t^2 \, dx + \delta_0 \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2\right) \, dx$$
$$+ c_{\delta_0} \int_0^L \int_0^{+\infty} (g_2(s) - g_2'(s)) \, (\partial_x \eta_2)^2 \, ds \, dx \tag{76}$$

and

$$I_3'(t) \leq -\rho_1 \left(g_3^0 - \delta_0\right) \int {}_0^L w_t^2 \, dx + \delta_0 \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2\right) \, dx$$
$$+ c_{\delta_0} \int_0^L \int_0^{+\infty} (g_3(s) - g_3'(s)) \, (\partial_x \eta_3)^2 \, ds \, dx. \tag{77}$$

Proof. First, noticing that

$$\partial_t \int_0^{+\infty} g_1(s)\eta_1 \, \mathrm{d}s = \partial_t \int_{-\infty}^t g_1(t-s)(\varphi(t) - \varphi(s)) \, \mathrm{d}s$$
$$= \int_{-\infty}^t g_1'(t-s)(\varphi(t) - \varphi(s)) \, \mathrm{d}s + \left(\int_{-\infty}^t g_1(t-s) \, \mathrm{d}s \right) \varphi_t;$$

that is,

$$\partial_t \int_0^{+\infty} g_1(s)\eta_1 \, \mathrm{d}s = \int_0^{+\infty} g_1'(s)\eta_1 \, \mathrm{d}s + g_1^0 \varphi_t. \tag{78}$$

Similarly,

$$\partial_t \int_0^{+\infty} g_2(s)\eta_2 \, \mathrm{d}s = \int_0^{+\infty} g_2'(s)\eta_2 \, \mathrm{d}s + g_2^0 \psi_t \tag{79}$$

and

$$\partial_t \int_0^{+\infty} g_3(s)\eta_3 \, \mathrm{d}s = \int_0^{+\infty} g_3'(s)\eta_3 \, \mathrm{d}s + g_3^0 w_t. \tag{80}$$

Second, using Young's and Hölder's inequalities, we get the following: For all $\lambda > 0$, there exists $c_\lambda > 0$ such that, for any $v \in L^2(]0, L[)$ and $\eta \in \{\eta_i, \partial_x \eta_i\}$, $i = 1, 2, 3$,

$$\left| \int_0^L v \int_0^{+\infty} g_i(s)\eta \, \mathrm{d}s \, \mathrm{d}x \right| \leq \lambda \int_0^L v^2 \, \mathrm{d}x + c_\lambda \int_0^L \int_0^{+\infty} g_i(s)\eta^2 \, \mathrm{d}s \, \mathrm{d}x. \tag{81}$$

Similarly,

$$\left|\int_0^L v \int_0^{+\infty} g_i'(s)\eta \,\mathrm{d}s\,\mathrm{d}x\right| \leq \lambda \int_0^L v^2 \,\mathrm{d}x - c_\lambda \int_0^L \int_0^{+\infty} g_i'(s)\eta^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{82}$$

Now, direct computations, using the first equation in (2) or (3), integrating by parts and using the boundary conditions and (78), yield

$$\begin{aligned} I_1'(t) = &-\rho_1 g_1^0 \int_0^L \varphi_t^2 \,\mathrm{d}x + \int_0^L \left(\int_0^{+\infty} g_1(s)\partial_x\eta_1 \,\mathrm{d}s\right)^2 \mathrm{d}x \\ &+ k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g_1(s)\partial_x\eta_1 \,\mathrm{d}s\,\mathrm{d}x - lk_3 \int_0^L (w_x - l\varphi) \int_0^{+\infty} g_1(s)\eta_1 \,\mathrm{d}s\,\mathrm{d}x \\ &- \rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1'(s)\eta_1 \,\mathrm{d}s\,\mathrm{d}x - g_1^0 \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x\eta_1 \,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Using (81) and (82) for the last four terms of this equality, Poincaré's inequality (39) for η_1 and (36) and Hölder's inequality to estimate

$$\int_0^L \varphi_x^2 \,\mathrm{d}x \quad \text{and} \quad \left(\int_0^{+\infty} g_1(s)\partial_x\eta_1 \,\mathrm{d}s\right)^2,$$

respectively, we get (75).

Similarly, using the second equation in (1) or (3), the third equation in (1) or (2), (79), (80) and (32) (instead of (39)), we find (76) and (77). □

Lemma 4.2. *Let*

$$\begin{aligned} \mathit{Case}\ (1): \quad J_1(t) = \rho_2 \int_0^L (\varphi_x + \psi + lw)\psi_t \,dx + \frac{k_2\rho_1}{k_1} \int_0^L \psi_x\varphi_t \,dx \\ - \tfrac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^{+\infty} g_2(s)\psi_x(t-s) \end{aligned} \tag{83}$$

hboxds dx,

$$\begin{aligned} \mathit{Case}\ (2): \quad J_2(t) = -\rho_2 \int_0^L (\varphi_x + \psi + lw)\psi_t \,dx - \frac{k_2\rho_1}{k_1} \int_0^L \psi_x\varphi_t \,dx \\ + \tfrac{\rho_2}{k_1} \int_0^L \psi_t \int_0^{+\infty} g_1(s)\varphi_x(t-s) \end{aligned} \tag{84}$$

hboxds dx

and

$$\text{Case (3)}: \quad J_3(t) = -\rho_1 \int_0^L (\varphi_x + \psi + lw) w_t \, dx - \frac{k_3 \rho_1}{k_1} \int_0^L (w_x - l\varphi) \varphi_t \, dx + \tfrac{\rho_1}{k_1} \int_0^L w_t \int_0^{+\infty} g_1(s) \varphi_x(t-s) \, dx. \tag{85}$$

Then, for any δ_0, ϵ_0, ϵ_1, $\epsilon_2 > 0$, there exists c_{δ_0}, $c_{\epsilon_0} > 0$ such that

$$\begin{aligned} J_1'(t) \leq & -k_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \left(\delta_0 + \frac{lk_3\epsilon_1}{2k_1}\left(k_2 - g_2^0\right)\right) \int_0^L (w_x - l\varphi)^2 \, dx \\ & + \delta_0 \int_0^L \varphi_t^2 \, dx + \frac{lk_3}{2\epsilon_1 k_1}\left(k_2 - g_2^0\right) \int_0^L \psi_x^2 \, dx + \int_0^L \left(\frac{3\rho_2}{2}\psi_t^2 + \frac{l^2\rho_2}{2} w_t^2\right) dx \\ & + \left(\rho_2 - \tfrac{k_2\rho_1}{k_1}\right) \int_0^L \psi_t \varphi_{xt} \, dx + c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_2(s) - g_2'(s)\right) (\partial_x \eta_2)^2 \, ds \, dx, \end{aligned} \tag{86}$$

$$\begin{aligned} J_2'(t) \leq & \left(k_1 + \delta_0 + \tfrac{g_1^0 \epsilon_1}{2}\right) \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \frac{lk_2k_3\epsilon_2}{2k_1} \int_0^L (w_x - l\varphi)^2 \, dx \\ & + \tfrac{g_1^0}{2\epsilon_1} \int_0^L \varphi_x^2 \, dx + \frac{lk_2k_3}{2k_1\epsilon_2} \int_0^L \psi_x^2 \, dx + (-\rho_2 + \delta_0 + \epsilon_0) \int_0^L \psi_t^2 \, dx + c_{\epsilon_0} \int_0^L w_t^2 \, dx \\ & + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_t \varphi_{xt} \, dx + c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_1(s) - g_1'(s)\right) (\partial_x \eta_1)^2 \, ds \, dx \end{aligned} \tag{87}$$

and

$$\begin{aligned} J_3'(t) \leq & \left(lk_1 + \delta_0 + \tfrac{lg_1^0\epsilon_1}{2}\right) \int_0^L (\varphi_x + \psi + lw)^2 \, dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 \, dx \\ & + \tfrac{lg_1^0}{2\epsilon_1} \int_0^L \varphi_x^2 \, dx + c_{\epsilon_0} \int_0^L \left(\varphi_t^2 + \psi_t^2\right) dx + (-l\rho_1 + \delta_0 + \epsilon_0) \int_0^L w_t^2 \, dx \\ & + \rho_1 \left(\tfrac{k_3}{k_1} - 1\right) \int_0^L w_t \varphi_{xt} \, dx + c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_1(s) - g_1'(s)\right) (\partial_x \eta_1)^2 \, ds \, dx. \end{aligned} \tag{88}$$

Proof. First, notice that

$$\partial_t \int_0^{+\infty} g_1(s)\varphi_x(t-s)\,\mathrm{d}s = \partial_t \int_{-\infty}^{t} g_1(t-s)\varphi_x(s)\,\mathrm{d}s$$

$$= g_1(0)\varphi_x(t) + \int_{-\infty}^{t} g_1'(t-s)\varphi_x(s)\,\mathrm{d}s$$

$$= -\int_0^{+\infty} g_1'(s)\varphi_x(t)\,\mathrm{d}s + \int_0^{+\infty} g_1'(s)\varphi_x(t-s)\,\mathrm{d}s;$$

that is,

$$\partial_t \int_0^{+\infty} g_1(s)\varphi_x(t-s)\,\mathrm{d}s = -\int_0^{+\infty} g_1'(s)\partial_x\eta_1\,\mathrm{d}s. \tag{89}$$

Similarly,

$$\partial_t \int_0^{+\infty} g_2(s)\psi_x(t-s)\,\mathrm{d}s = -\int_0^{+\infty} g_2'(s)\partial_x\eta_2\,\mathrm{d}s. \tag{90}$$

Now, by exploiting the first two equations in (1), integrating by parts, recalling (90) and using the boundary conditions, we get

$$J_1'(t) = -k_1\int_0^L (\varphi_x+\psi+lw)^2\,\mathrm{d}x + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x$$
$$+\rho_2\int_0^L \psi_t^2\,\mathrm{d}x + \rho_2 l\int_0^L \psi_t w_t\,\mathrm{d}x + \frac{lk_3}{k_1}\left(k_2-g_2^0\right)\int_0^L (w_x-l\varphi)\psi_x\,\mathrm{d}x$$
$$+\tfrac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^{+\infty} g_2'(s)\partial_x\eta_2\,\mathrm{d}s\,\mathrm{d}x + \frac{lk_3}{k_1}\int_0^L (w_x-l\varphi)\int_0^{+\infty} g_2(s)\partial_x\eta_2\,\mathrm{d}s\,\mathrm{d}x.$$

By applying (81), (82) and the Young's inequality for the last four terms of the above equality and noting that $k_2 - g_2^0 > 0$ (by virtue of (41)), we deduce (86).

Similarly, using the first two equations in (2), and the first and third equations in (3), and exploiting (89), we find

$$J_2'(t) = k_1\int_0^L (\varphi_x+\psi+lw)^2\,\mathrm{d}x - g_1^0\int_0^L (\varphi_x+\psi+lw)\varphi_x\,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x$$
$$-\rho_2\int_0^L \psi_t^2\,\mathrm{d}x - l\rho_2\int_0^L \psi_t w_t\,\mathrm{d}x - \frac{lk_2k_3}{k_1}\int_0^L (w_x-l\varphi)\psi_x\,\mathrm{d}x$$
$$-\tfrac{\rho_2}{k_1}\int_0^L \psi_t\int_0^{+\infty} g_1'(s)\partial_x\eta_1\,\mathrm{d}s\,\mathrm{d}x + \int_0^L (\varphi_x+\psi+lw)\int_0^{+\infty} g_1(s)\partial_x\eta_1\,\mathrm{d}s\,\mathrm{d}x$$

and

$$J_3'(t) = lk_1 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x - lg_1^0 \int_0^L (\varphi_x + \psi + lw)\varphi_x \,\mathrm{d}x + \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L w_t \varphi_{xt} \,\mathrm{d}x$$
$$- \tfrac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 \,\mathrm{d}x - \int_0^L \left(l\rho_1 w_t^2 + \rho_1 \psi_t w_t - \frac{l\rho_1 k_3}{k_1} \varphi_t^2 \right) \mathrm{d}x$$
$$- \tfrac{\rho_1}{k_1} \int_0^L w_t \int_0^{+\infty} g_1'(s) \partial_x \eta_1 \,\mathrm{d}s \,\mathrm{d}x + l \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g_1(s) \partial_x \eta_1 \,\mathrm{d}s \,\mathrm{d}x.$$

Then, by proceeding as for (86), we conclude (87) and (88). □

Lemma 4.3. *Let*

$$D(t) = \begin{cases} 0 & \textit{in cases (1) and (3)}, \\ \rho_2 \int_0^L \psi_x \int_0^x \psi_t(y,t)\, dy\, dx & \textit{in case (2)}. \end{cases} \tag{91}$$

Then, for any $\delta_1 > 0$, we have

$$D'(t) \le -\rho_2 \int_0^L \psi_t^2 \, dx + \left(\frac{k_1 \delta_1}{2} + k_2 \right) \int_0^L \psi_x^2 \, dx + \frac{k_1 \tilde{c}_0}{2\delta_1} \int_0^L (\varphi_x + \psi + lw)^2 \, dx \tag{92}$$

in case (2), and $D'(t) = 0$ in cases (1) and (3).

Proof. By exploiting the second equation in (2), integrating by parts and using the boundary conditions, we get

$$D'(t) = \int_0^L \left(-\rho_2 \psi_t^2 + k_2 \psi_x^2 \right) \mathrm{d}x - k_1 \int_0^L \psi_x \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t)) \,\mathrm{d}y \,\mathrm{d}x \tag{93}$$

in case (2), and $D'(t) = 0$ in cases (1) and (3). Now, noting that the function

$$x \mapsto \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t)) \,\mathrm{d}y$$

vanishes at 0 and L (because of (30)), then, applying (39), we have

$$\int_0^L \left(\int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t)) \,\mathrm{d}y \right)^2 \mathrm{d}x \le \tilde{c}_0 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x. \tag{94}$$

By applying Young's inequality for the last term in (93) and recalling (94), we conclude (92). □

Lemma 4.4. *Let*

$$\text{Case (1)}: \quad P_1(t) = -\rho_1 k_3 \int_0^L (w_x - l\varphi) \int_0^x w_t(y,t)\,dy\,dx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\varphi_x + \psi + lw)(y,t)\,dy\,dx, \tag{95}$$

$$\text{Case (2)}: \quad P_2(t) = P_1(t) \tag{96}$$

and

$$\text{Case (3)}: \quad P_3(t) = -P_1(t). \tag{97}$$

Then, for any ϵ_0, δ_0, δ_1, δ_2, $\delta_3 > 0$, *there exists* c_{ϵ_0}, $c_{\delta_0} > 0$ *such that*

$$\begin{aligned} P_1'(t) \le{}& k_1^2 \int_0^L (\varphi_x + \psi + lw)^2\,dx + \left(\frac{k_3 g_3^0 \delta_1}{2} + \delta_0 - k_3^2\right) \int_0^L (w_x - l\varphi)^2\,dx \\ &+ (-\rho_1 k_1 + \epsilon_0) \int_0^L \varphi_t^2\,dx + c_{\epsilon_0} \int_0^L \left(\psi_t^2 + w_t^2\right)\,dx + \frac{k_3 g_3^0}{2\delta_1} \int_0^L w_x^2\,dx \\ &+ c_{\delta_0} \int_0^L \int_0^{+\infty} g_3(s)(\partial_x \eta_3)^2\,ds\,dx, \end{aligned} \tag{98}$$

$$\begin{aligned} P_2'(t) \le{}& \left(k_1^2 + \tfrac{k_1 g_1^0 \delta_2}{2} + \delta_0\right) \int_0^L (\varphi_x + \psi + lw)^2\,dx + \left(\frac{k_3 g_3^0 \delta_3}{2} + \delta_0 - k_3^2\right) \int_0^L (w_x - l\varphi)^2\,dx \\ &+ \epsilon_0 \int_0^L \psi_t^2\,dx + c_{\epsilon_0} \int_0^L \left(\varphi_t^2 + w_t^2\right)\,dx + \frac{k_1 g_1^0}{2\delta_2} \int_0^L \varphi_x^2\,dx + \frac{k_3 g_3^0}{2\delta_3} \int_0^L w_x^2\,dx \\ &+ c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_3(s)(\partial_x \eta_3)^2\right)\,ds\,dx \end{aligned} \tag{99}$$

and

$$\begin{aligned} P_3'(t) \le{}& + \left(\tfrac{k_1 g_1^0 \delta_1}{2} + \delta_0 - k_1^2\right) \int_0^L (\varphi_x + \psi + lw)^2\,dx + k_3^2 \int_0^L (w_x - l\varphi)^2\,dx \\ &+ (-\rho_1 k_3 + \epsilon_0) \int_0^L w_t^2\,dx + c_{\epsilon_0} \int_0^L \left(\varphi_t^2 + \psi_t^2\right)\,dx + \frac{k_1 g_1^0}{2\delta_1} \int_0^L \varphi_x^2\,dx \\ &+ c_{\delta_0} \int_0^L \int_0^{+\infty} g_1(s)(\partial_x \eta_1)^2\,ds\,dx. \end{aligned} \tag{100}$$

Proof. By exploiting the first and third equations in (1)–(3), integrating by parts and using (30), (31) and the boundary conditions, we find

$$
\begin{aligned}
P_i'(t) = {} & \rho_1 k_3 \int_0^L w_t^2 \,\mathrm{d}x - \rho_1 k_1 \int_0^L \varphi_t^2 \,\mathrm{d}x - k_3^2 \int_0^L (w_x - l\varphi)^2 \,\mathrm{d}x \\
& + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x + l\rho_1 k_3 \int_0^L \varphi_t \int_0^x w_t(y,t)\,\mathrm{d}y\,\mathrm{d}x \\
& - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\psi_t(y,t) + l w_t(y,t))\,\mathrm{d}y\,\mathrm{d}x \\
& + k_3 g_3^0 \int_0^L (w_x - l\varphi) w_x \,\mathrm{d}x - k_3 \int_0^L (w_x - l\varphi) \int_0^{+\infty} g_3(s)\partial_x \eta_3 \,\mathrm{d}s\,\mathrm{d}x \\
& + \begin{cases} 0 & \text{if } i = 1, \\ -k_1 g_1^0 \int_0^L (\varphi_x + \psi + lw)\varphi_x \,\mathrm{d}x + k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g_1(s)\partial_x \eta_1 \,\mathrm{d}s\,\mathrm{d}x & \text{if } i = 2 \end{cases}
\end{aligned}
\tag{101}
$$

and

$$
\begin{aligned}
P_3'(t) = {} & -\rho_1 k_3 \int_0^L w_t^2 \,\mathrm{d}x + \rho_1 k_1 \int_0^L \varphi_t^2 \,\mathrm{d}x + k_3^2 \int_0^L (w_x - l\varphi)^2 \,\mathrm{d}x \\
& - k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x - l\rho_1 k_3 \int_0^L \varphi_t \int_0^x w_t(y,t)\,\mathrm{d}y\,\mathrm{d}x \\
& + \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\psi_t(y,t) + l w_t(y,t))\,\mathrm{d}y\,\mathrm{d}x \\
& + k_1 g_1^0 \int_0^L (\varphi_x + \psi + lw)\varphi_x \,\mathrm{d}x - k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g_1(s)\partial_x \eta_1 \,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
\tag{102}
$$

Now, noting that the functions

$$
x \mapsto \int_0^x \psi_t(y,t)\,\mathrm{d}y \quad \text{and} \quad x \mapsto \int_0^x w_t(y,t)\,\mathrm{d}y
$$

vanish at 0 and L (because of (30)), then, applying (39), we have

$$
\int_0^L \left(\int_0^x \psi_t(y,t)\,\mathrm{d}y \right)^2 \mathrm{d}x \le \tilde{c}_0 \int_0^L \psi_t^2 \,\mathrm{d}x \tag{103}
$$

and

$$
\int_0^L \left(\int_0^x w_t(y,t)\,\mathrm{d}y \right)^2 \mathrm{d}x \le \tilde{c}_0 \int_0^L w_t^2 \,\mathrm{d}x. \tag{104}
$$

By applying Young's inequality and (81) in (101) and (102), and recalling (103) and (104), we deduce (98)–(100). □

Lemma 4.5. *Let*

$$\text{Cases (1)–(3)}: \quad R(t) = \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t)\, dx. \tag{105}$$

Then, for any $\delta_0 > 0$, there exists $c_{\delta_0} > 0$ such that

$$R'(t) \le \int_0^L \left(-k_2 \psi_x^2 - k_1(\varphi_x + \psi + lw)^2 - k_3(w_x - l\varphi)^2 + \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2\right) dx$$
$$+ \begin{cases} \int_0^L \left(\left(\delta_0 + g_2^0\right)\psi_x^2 + \left(\delta_0 + g_3^0\right) w_x^2\right) dx + c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_2(s)(\partial_x \eta_2)^2 + g_3(s)(\partial_x \eta_3)^2\right) ds\, dx & \text{in case (1)}, \\ \int_0^L \left(\left(\delta_0 + g_1^0\right)\varphi_x^2 + \left(\delta_0 + g_3^0\right) w_x^2\right) dx + c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_3(s)(\partial_x \eta_3)^2\right) ds\, dx & \text{in case (2)}, \\ \int_0^L \left(\left(\delta_0 + g_1^0\right)\varphi_x^2 + \left(\delta_0 + g_2^0\right) \psi_x^2\right) dx + c_{\delta_0} \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_2(s)(\partial_x \eta_2)^2\right) ds\, dx & \text{in case (3)}. \end{cases} \tag{106}$$

Proof. By using the first three equations in (1)–(3) and the boundary conditions, we obtain

$$R'(t) = \int_0^L \left(-k_2 \psi_x^2 - k_1(\varphi_x + \psi + lw)^2 - k_3(w_x - l\varphi)^2 + \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2\right) \mathrm{d}x$$
$$+ \begin{cases} \int_0^L \left(g_2^0 \psi_x^2 + g_3^0 w_x^2\right) \mathrm{d}x - \int_0^L \psi_x \int_0^{+\infty} g_2(s) \partial_x \eta_2 \,\mathrm{d}s\,\mathrm{d}x - \int_0^L w_x \int_0^{+\infty} g_3(s) \partial_x \eta_3 \,\mathrm{d}s\,\mathrm{d}x & \text{in case (1)}, \\ \int_0^L \left(g_1^0 \varphi_x^2 + g_3^0 w_x^2\right) \mathrm{d}x - \int_0^L \varphi_x \int_0^{+\infty} g_1(s) \partial_x \eta_1 \,\mathrm{d}s\,\mathrm{d}x - \int_0^L w_x \int_0^{+\infty} g_3(s) \partial_x \eta_3 \,\mathrm{d}s\,\mathrm{d}x & \text{in case (2)}, \\ \int_0^L \left(g_1^0 \varphi_x^2 + g_2^0 \psi_x^2\right) \mathrm{d}x - \int_0^L \varphi_x \int_0^{+\infty} g_1(s) \partial_x \eta_1 \,\mathrm{d}s\,\mathrm{d}x - \int_0^L \psi_x \int_0^{+\infty} g_2(s) \partial_x \eta_2 \,\mathrm{d}s\,\mathrm{d}x & \text{in case (3)}. \end{cases} \tag{107}$$

By applying (81) for the last two terms in (107), we conclude (106). □

Let N, N_1, N_2, N_3, $N_4 \ge 0$ and, for $i = 1, 2, 3$,

$$F_i := N E_i + J_i + N_2 D + N_3 P_i + N_4 R + N_1 \sum_{j \in \{1,2,3\} \setminus \{i\}} I_j. \tag{108}$$

Then, by combining (75)–(77), (86)–(88), (92), (98)–(100) and (106), we obtain

$$
\begin{aligned}
F_i'(t) \leq & \int_0^L \left(l_1\varphi_t^2 + l_2\psi_t^2 + l_3 w_t^2 + l_4\psi_x^2 + \left(l_5 + g_3^0\tilde{l}_5\right)(w_x - l\varphi)^2 + \left(l_6 + g_1^0\tilde{l}_6\right)(\varphi_x + \psi + lw)^2\right) \mathrm{d}x \\
& + \int_0^L \left(g_1^0 l_7\varphi_x^2 + g_3^0 l_8 w_x^2\right) \mathrm{d}x + NE_i'(t) + c_{N_1,N_3,N_4,\delta_0} \sum_{j\in\{1,2,3\}\setminus\{i\}} \int_0^L \int_0^{+\infty} \left(g_j(s) - g_j'(s)\right)(\partial_x\eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x \\
& + \delta_0 c_{N_1,N_3,N_4} \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2 + \varphi_t^2 + \psi_t^2 + w_t^2\right) \mathrm{d}x \\
& + \begin{cases}
c_{N_3,\epsilon_0} \int_0^L \left(\psi_t^2 + w_t^2\right) \mathrm{d}x + \epsilon_0 c_{N_3} \int_0^L \varphi_t^2 \,\mathrm{d}x + \left(\rho_2 - \dfrac{k_2\rho_1}{k_1}\right) \int_0^L \psi_t\varphi_{xt} \,\mathrm{d}x & \text{if } i = 1, \\
c_{N_3,\epsilon_0} \int_0^L \left(\varphi_t^2 + w_t^2\right) \mathrm{d}x + \epsilon_0 c_{N_3} \int_0^L \psi_t^2 \,\mathrm{d}x + \left(\dfrac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_t\varphi_{xt} \,\mathrm{d}x & \text{if } i = 2, \\
c_{N_3,\epsilon_0} \int_0^L \left(\varphi_t^2 + \psi_t^2\right) \mathrm{d}x + \epsilon_0 c_{N_3} \int_0^L w_t^2 \,\mathrm{d}x + \rho_1 \left(\dfrac{k_3}{k_1} - 1\right) \int_0^L w_t\varphi_{xt} \,\mathrm{d}x & \text{if } i = 3,
\end{cases}
\end{aligned}
\tag{109}
$$

where

$$
l_1 = \begin{cases} -\rho_1 k_1 N_3 + \rho_1 N_4 & \text{if } i = 1, \\ -\rho_1 g_1^0 N_1 + \rho_1 N_4 & \text{if } i = 2, \\ -\rho_1 g_1^0 N_1 + \rho_1 N_4 & \text{if } i = 3, \end{cases} \quad
l_2 = \begin{cases} -\rho_2 g_2^0 N_1 + \rho_2 N_4 & \text{if } i = 1, \\ -\rho_2 N_2 + \rho_2 N_4 - \rho_2 & \text{if } i = 2, \\ -\rho_2 g_2^0 N_1 + \rho_2 N_4 & \text{if } i = 3, \end{cases}
$$

$$
l_3 = \begin{cases} -\rho_1 g_3^0 N_1 + \rho_1 N_4 & \text{if } i = 1, \\ -\rho_1 g_3^0 N_1 + \rho_1 N_4 & \text{if } i = 2, \\ -\rho_1 k_3 N_3 + \rho_1 N_4 - l\rho_1 & \text{if } i = 3, \end{cases} \quad
l_4 = \begin{cases} -\left(k_2 - g_2^0\right) N_4 + \frac{lk_3\left(k_2 - g_2^0\right)}{2k_1\epsilon_1} & \text{if } i = 1, \\ \left(\frac{k_1\delta_1}{2} + k_2\right) N_2 - k_2 N_4 + \frac{lk_2k_3}{2k_1\epsilon_2} & \text{if } i = 2, \\ -\left(k_2 - g_2^0\right) N_4 & \text{if } i = 3, \end{cases}
$$

$$
l_5 = \begin{cases} -k_3^2 N_3 - k_3 N_4 + \frac{lk_3\left(k_2 - g_2^0\right)\epsilon_1}{2k_1} & \text{if } i = 1, \\ -k_3^2 N_3 - k_3 N_4 + \frac{lk_2k_3\epsilon_2}{2k_1} & \text{if } i = 2, \\ k_3^2 N_3 - k_3 N_4 - \frac{lk_3^2}{k_1} & \text{if } i = 3, \end{cases} \quad
\tilde{l}_5 = \begin{cases} \frac{k_3\delta_1}{2} N_3 & \text{if } i = 1, \\ \frac{k_3\delta_3}{2} N_3 & \text{if } i = 2, \\ 0 & \text{if } i = 3, \end{cases}
$$

$$
l_6 = \begin{cases} k_1^2 N_3 - k_1 N_4 - k_1 & \text{if } i = 1, \\ \frac{k_1\tilde{c}_0}{2\delta_1} N_2 + k_1^2 N_3 - k_1 N_4 + k_1 & \text{if } i = 2, \\ -k_1^2 N_3 - k_1 N_4 + lk_1 & \text{if } i = 3, \end{cases} \quad
\tilde{l}_6 = \begin{cases} 0 & \text{if } i = 1, \\ \frac{k_1\delta_2}{2} N_3 + \frac{\epsilon_1}{2} & \text{if } i = 2, \\ \frac{k_1\delta_1}{2} N_3 + \frac{l\epsilon_1}{2} & \text{if } i = 3, \end{cases}
$$

$$
l_7 = \begin{cases} 0 & \text{if } i = 1, \\ \frac{k_1}{2\delta_2} N_3 + N_4 + \frac{1}{2\epsilon_1} & \text{if } i = 2, \\ \frac{k_1}{2\delta_1} N_3 + N_4 + \frac{l}{2\epsilon_1} & \text{if } i = 3 \end{cases} \quad \text{and} \quad
l_8 = \begin{cases} \frac{k_3}{2\delta_1} N_3 + N_4 & \text{if } i = 1, \\ \frac{k_3}{2\delta_3} N_3 + N_4 & \text{if } i = 2, \\ 0 & \text{if } i = 3. \end{cases}
$$

Using (33), (43), (70) and (71), we get from (109) that

$$F_i'(t) \leq \int_0^L \left(l_1\varphi_t^2 + l_2\psi_t^2 + l_3 w_t^2 + l_4\psi_x^2 + \left(l_5 + g_3^0\tilde{l}_5\right)(w_x - l\varphi)^2 + \left(l_6 + g_1^0\tilde{l}_6\right)(\varphi_x + \psi + lw)^2\right) \mathrm{d}x$$
$$+ \int_0^L \left(g_1^0 l_7\varphi_x^2 + g_3^0 l_8 w_x^2\right) \mathrm{d}x + \delta_0 c_{N_1,N_3,N_4} E_i(t)$$
$$+ (N - c_{N_1,N_3,N_4,\delta_0})E_i'(t) + c_{N_1,N_3,N_4,\delta_0} \textstyle\sum_{j\in\{1,2,3\}\setminus\{i\}} \displaystyle\int_0^L \int_0^{+\infty} g_j(s)(\partial_x\eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x$$
$$+ \begin{cases} c_{N_3,\epsilon_0}\displaystyle\int_0^L \left(\psi_t^2 + w_t^2\right) \mathrm{d}x + \epsilon_0 c_{N_3}\int_0^L \varphi_t^2 \,\mathrm{d}x + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x & \text{if } i = 1, \\ c_{N_3,\epsilon_0}\displaystyle\int_0^L \left(\varphi_t^2 + w_t^2\right) \mathrm{d}x + \epsilon_0 c_{N_3}\int_0^L \psi_t^2 \,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x & \text{if } i = 2, \\ c_{N_3,\epsilon_0}\displaystyle\int_0^L \left(\varphi_t^2 + \psi_t^2\right) \mathrm{d}x + \epsilon_0 c_{N_3}\int_0^L w_t^2 \,\mathrm{d}x + \rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x & \text{if } i = 3. \end{cases} \tag{110}$$

At this point, we choose carefully the constants N, N_i, δ_i, ϵ_i to get suitable values of l_i.

Case $i = 1$. We choose

$$\delta_1 = \sqrt{\frac{k_3}{k_0 + g_3^0}}, \quad \epsilon_1 = \sqrt{\frac{(k_1 + k_3)k_3}{\left(k_2 - g_2^0\right)k_1}}$$

(notice that ϵ_1 is well defined thanks to (41)),

$$N_3 > \frac{lk_3}{2k_1^2\epsilon_1} \quad \text{and} \quad \max\left\{k_1N_3 - 1, -k_3N_3 + \frac{l(k_2 - g_2^0)\epsilon_1}{2k_1}, \frac{lk_3}{2k_1\epsilon_1}\right\} < N_4 < k_1N_3.$$

We remark that N_4 exists according to the choice of N_3. By virtue of the choice of δ_1, ϵ_1, N_3 and N_4, we see that

$$\max\{l_1, l_4, l_5, l_6\} < 0.$$

On the other hand, from (33) (in case (1)), we find that

$$g_3^0 l_8 \int_0^L w_x^2 \,\mathrm{d}x \leq \frac{g_3^0 l_8}{k_0 + g_3^0}\int_0^L \left((k_2 - g_2^0 - k_0)\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) \mathrm{d}x. \tag{111}$$

Because $\tilde{l}_6 = l_7 = 0$ and l_4, l_5, $\tilde{l}_5$, l_6 and l_8 do not depend on g_3^0, then, if g_3^0 is small enough so that

$$\lambda_0 := \max\left\{l_4 + \frac{\left(k_2 - g_2^0 - k_0\right)g_3^0 l_8}{k_0 + g_3^0}, l_5 + g_3^0\tilde{l}_5 + \frac{k_3 g_3^0 l_8}{k_0 + g_3^0}, l_6 + \frac{k_1 g_3^0 l_8}{k_0 + g_3^0}\right\} < 0, \tag{112}$$

we get

$$\int_0^L \left(l_4\psi_x^2 + \left(l_5 + g_3^0\tilde{l}_5\right)(w_x - l\varphi)^2 + \left(l_6 + g_1^0\tilde{l}_6\right)(\varphi_x + \psi + lw)^2 + g_1^0 l_7\varphi_x^2 + g_3^0 l_8 w_x^2\right) \mathrm{d}x$$
$$\leq \lambda_0 \int_0^L \left(\psi_x^2 + (w_x - l\varphi)^2 + (\varphi_x + \psi + lw)^2\right) \mathrm{d}x. \tag{113}$$

After, we choose $\epsilon_0 > 0$ small enough such that $\epsilon_0 c_{N_3} + l_1 < 0$ and N_1 large enough so that

$$\max\{l_2 + c_{N_3,\epsilon_0}, l_3 + c_{N_3,\epsilon_0}\} < 0.$$

Finally, we choose $\delta_0 > 0$ small enough such that

$$\tilde{c}_1 := -\max\left\{\frac{2}{\rho_1}(l_1 + \epsilon_0 c_{N_3}), \frac{2}{\rho_2}(l_2 + c_{N_3,\epsilon_0}), \frac{2}{\rho_1}(l_3 + c_{N_3,\epsilon_0}), \frac{2\lambda_0}{k_2}, \frac{2\lambda_0}{k_3}, \frac{2\lambda_0}{k_1}\right\} - \delta_0 c_{N_1,N_3,N_4} > 0.$$

Then, using (43) and (70), we deduce from (110) that the estimate

$$F_i'(t) \leq -\tilde{c}_1 E_i(t) + (N - c)E_i'(t) + c\sum\nolimits_{j\in\{1,2,3\}\backslash\{i\}} \int_0^L \int_0^{+\infty} g_j(s)(\partial_x \eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x$$
$$+ \begin{cases} \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right) \int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x & \text{if } i = 1, \\ \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x & \text{if } i = 2, \\ \rho_1\left(\frac{k_3}{k_1} - 1\right) \int_0^L w_t\varphi_{xt}\,\mathrm{d}x & \text{if } i = 3 \end{cases} \tag{114}$$

is satisfied, for $i = 1$.

Case $i = 2$. As in the previous case, we choose

$$\delta_1 > \frac{\tilde{c}_0}{2}, \quad \epsilon_2 = \frac{2k_1}{lk_2},$$
$$\frac{(lk_2\epsilon_2 - 2k_1)\delta_1}{2(k_1 + k_3)\delta_1 - \tilde{c}_0 k_3} < N_2 < \frac{2k_2}{k_1\delta_1}\left(1 - \frac{lk_3}{2k_1\epsilon_2}\right), \quad \frac{lk_2\epsilon_2 - 2k_1(N_2+1)}{2k_1k_3} < N_3 < \frac{(2\delta_1 - \tilde{c}_0)N_2}{2k_1\delta_1},$$
$$\max\left\{\left(\frac{k_1\delta_1}{2k_2} + 1\right)N_2 + \frac{lk_3}{2k_1\epsilon_2}, -k_3N_3 + \frac{lk_2\epsilon_2}{2k_1}, k_1N_3 + \frac{\tilde{c}_0 N_2}{2\delta_1} + 1\right\} < N_4 < N_2 + 1,$$

ϵ_1, δ_2 and δ_3 are any positive numbers.

By virtue of the choice of δ_1 and ϵ_2 and the hypothesis on l in (61), we see that N_2, N_3 and N_4 exist and

$$\max\{l_2, l_4, l_5, l_6\} < 0.$$

On the other hand, from (33) (in case (2)), we find, similarly to (111),

$$g_1^0 l_7 \int_0^L \varphi_x^2\,\mathrm{d}x \leq \frac{g_1^0 l_7}{k_0 + g_1^0} \int_0^L \left((k_2 - k_0)\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) \mathrm{d}x \tag{115}$$

and

$$g_3^0 l_8 \int_0^L w_x^2 \,\mathrm{d}x \le \frac{g_3^0 l_8}{k_0+g_3^0} \int_0^L \left((k_2-k_0)\psi_x^2 + k_1(\varphi_x+\psi+lw)^2 + k_3(w_x-l\varphi)^2\right) \mathrm{d}x. \tag{116}$$

Therefore, if g_1^0 and g_3^0 are small enough so that

$$\begin{aligned}&\lambda_0 := \max\left\{ l_4 + (k_2-k_0)\left(\tfrac{g_1^0 l_7}{k_0+g_1^0} + \tfrac{g_3^0 l_8}{k_0+g_3^0}\right), l_5 + g_3^0 \tilde{l}_5 + k_3\left(\tfrac{g_1^0 l_7}{k_0+g_1^0} + \tfrac{g_3^0 l_8}{k_0+g_3^0}\right),\right.\\ &\left. l_6 + g_1^0 \tilde{l}_6 + k_1\left(\tfrac{g_1^0 l_7}{k_0+g_1^0} + \tfrac{g_3^0 l_8}{k_0+g_3^0}\right)\right\} < 0\end{aligned} \tag{117}$$

(notice that l_4, l_5, $\tilde{l}_5$, l_6, $\tilde{l}_6$, l_7 and l_8 do not depend neither on g_1^0 nor on g_3^0), then (113) holds. After, we choose $\epsilon_0 > 0$ small enough such that $\epsilon_0 c_{N_3} + l_2 < 0$ and N_1 large enough so that

$$\max\{l_1 + c_{N_3,\epsilon_0}, l_3 + c_{N_3,\epsilon_0}\} < 0.$$

Finally, we choose $\delta_0 > 0$ small enough such that

$$\tilde{c}_1 := -\max\left\{\frac{2}{\rho_1}(l_1 + c_{N_3,\epsilon_0}), \frac{2}{\rho_2}(l_2 + \epsilon_0 c_{N_3}), \frac{2}{\rho_1}(l_3 + c_{N_3,\epsilon_0}), \frac{2\lambda_0}{k_2}, \frac{2\lambda_0}{k_3}, \frac{2\lambda_0}{k_1}\right\} - \delta_0 c_{N_1,N_3,N_4} > 0,$$

and we deduce (114), for $i = 2$, from (43), (70) and (110).

Case $i = 3$. Similarly to the above two cases, we choose

$$\max\left\{l - k_1 N_3, k_3 N_3 - \frac{l k_3}{k_1}\right\} < N_4 < k_3 N_3 + l,$$
$$N_3,\ \delta_1 \text{ and } \epsilon_1 \text{ are any positive numbers.}$$

We have $\tilde{l}_5 = l_8 = 0$ and thanks to the choice of N_4 we see that

$$\max\{l_3, l_4, l_5, l_6\} < 0.$$

On the other hand, similarly to (111), we get from (33) (in case (3)) that

$$g_1^0 l_7 \int_0^L \varphi_x^2 \,\mathrm{d}x \le \frac{g_1^0 l_7}{k_0+g_1^0} \int_0^L \left(\left(k_2 - g_2^0 - k_0\right)\psi_x^2 + k_1(\varphi_x+\psi+lw)^2 + k_3(w_x-l\varphi)^2\right) \mathrm{d}x. \tag{118}$$

Therefore, if g_1^0 is small enough so that

$$\lambda_0 := \max\left\{ l_4 + \frac{g_1^0 l_7\left(k_2 - g_2^0 - k_0\right)}{k_0+g_1^0}, l_5 + \frac{k_3 g_1^0 l_7}{k_0+g_1^0}, l_6 + g_1^0\tilde{l}_6 + \frac{k_1 g_1^0 l_7}{k_0+g_1^0}\right\} < 0, \tag{119}$$

then (113) holds (the condition (119) holds, since l_4, l_5, l_6, $\tilde{l}_6$ and l_7 do not depend on g_1^0). After, we choose $\epsilon_0 > 0$ small enough such that $\epsilon_0 c_{N_3} + l_3 < 0$ and N_1 large enough so that

$$\max\{l_1 + c_{N_3,\epsilon_0}, l_2 + c_{N_3,\epsilon_0}\} < 0.$$

Finally, we choose $\delta_0 > 0$ small enough such that

$$\tilde{c}_1 := -\max\left\{\frac{2}{\rho_1}(l_1 + c_{N_3,\epsilon_0}), \frac{2}{\rho_2}(l_2 + c_{N_3,\epsilon_0}), \frac{2}{\rho_1}(l_3 + \epsilon_0 c_{N_3}), \frac{2\lambda_0}{k_2}, \frac{2\lambda_0}{k_3}, \frac{2\lambda_0}{k_1}\right\} - \delta_0 c_{N_1,N_3,N_4} > 0,$$

Then, (43), (70) and (110) give (114), for $i = 3$.

Now, we estimate the integral of $g_j(\partial_x\eta_j)^2$ in (114), for $j \in \{1,2,3\}/\{i\}$. When (58) holds, we see that, by virtue of (71),

$$\int_0^L \int_0^{+\infty} g_j(s)(\partial_x\eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x \le -\frac{2}{\alpha}E_i'(t). \tag{120}$$

When (59) holds, we apply Lemma 3.6 [11] (in the particular case $B = -\partial_{xx}$ and $\|\cdot\| = \|\cdot\|_{L^2(]0,L[)}$) to get the following inequality.

Lemma 4.6. *There exists a positive constant c such that, for any $\tau_0 > 0$, we have*

$$G'(\tau_0 E_i(t)) \int_0^L \int_0^{+\infty} g_j(s)(\partial_x \eta_j)^2 \, ds \, dx \leq -cE_i'(t) + c\tau_0 E_i(t) G'(\tau_0 E_i(t)). \tag{121}$$

Proof. See Lemma 3.6 [11]. □

Using (120) and (121), we get in both cases (58) and (59)

$$\begin{aligned} \tfrac{G_0(\tau_0 E_i(t))}{\tau_0 E_i(t)} \textstyle\sum_{j\in\{1,2,3\}\setminus\{i\}} \displaystyle\int_0^L \int_0^{+\infty} g_j(s)(\partial_x \eta_j)^2 \, \mathrm{d}s \, \mathrm{d}x &\leq cG_0(\tau_0 E_i(t)) - cE_i'(t) \\ &\quad - c\tfrac{G_0(\tau_0 E_i(t))}{\tau_0 E_i(t)} E_i'(t), \end{aligned} \tag{122}$$

where G_0 is defined in (64). By multiplying (114) by $\frac{G_0(\tau_0 E_i(t))}{E_i(t)}$ and combining with (122), we obtain

$$\begin{aligned} \tfrac{G_0(\tau_0 E_i(t))}{E_i(t)} F_i'(t) &\leq -(\tilde{c}_1 - c\tau_0) G_0(\tau_0 E_i(t)) + \left((N-c)\tfrac{G_0(\tau_0 E_i(t))}{E_i(t)} - c\tau_0\right) E_i'(t) \\ &\quad + \tfrac{G_0(\tau_0 E_i(t))}{E_i(t)} \begin{cases} \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right) \displaystyle\int_0^L \psi_t \varphi_{xt} \, \mathrm{d}x & \text{if } i = 1, \\ \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \displaystyle\int_0^L \psi_t \varphi_{xt} \, \mathrm{d}x & \text{if } i = 2, \\ \rho_1 \left(\frac{k_3}{k_1} - 1\right) \displaystyle\int_0^L w_t \varphi_{xt} \, \mathrm{d}x & \text{if } i = 3. \end{cases} \end{aligned} \tag{123}$$

On the other hand, from (33), (43) and (70), we deduce that there exists a positive constant γ_i (not depending on N) satisfying

$$\left| J_i + N_2 D + N_3 P + N_4 R + N_1 \sum_{j\in\{1,2,3\}\setminus\{i\}} I_j \right| \leq \gamma_i E_i,$$

which, combined with (108), implies that

$$(N - \gamma_i) E_i \leq F_i \leq (N + \gamma_i) E_i. \tag{124}$$

Choosing N so that

$$N \geq c \quad \text{and} \quad N > \gamma_i$$

(c is the constant in (123)) and using (123), (124) and $E_i' \leq 0$, we deduce that $F_i \sim E_i$ and

$$\frac{G_0(\tau_0 E_i(t))}{E_i(t)} F_i'(t) \leq -(\tilde{c}_1 - c\tau_0) G_0(\tau_0 E_i(t)) - c\tau_0 E_i'(t) + \frac{G_0(\tau_0 E_i(t))}{E_i(t)} \begin{cases} \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right) \int_0^L \psi_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 1, \\ \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 2, \\ \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L w_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 3. \end{cases} \tag{125}$$

Let $\tilde{\tau}_i > 0$ and

$$\tilde{F}_i = \tilde{\tau}_i \left(\frac{G_0(\tau_0 E_i(t))}{E_i(t)} F_i + c\tau_0 E_i(t) \right). \tag{126}$$

Because $\frac{G_0(\tau_0 E_i(t))}{E_i(t)}$ is nonincreasing, then, thanks to (124),

$$c\tilde{\tau}_i \tau_0 E_i \leq \tilde{F}_i \leq \tilde{\tau}_i \left((N + \gamma_i) \frac{G_0(\tau_0 E_i(0))}{E_i(0)} + c\tau_0 \right) E_i. \tag{127}$$

Let us choose $\tilde{\tau}_i > 0$ such that

$$\tilde{F}_i \leq \tau_0 E_i \quad \text{and} \quad \tilde{F}_i(0) \leq 1. \tag{128}$$

We have, using (125), (126) and the fact that $\frac{G_0(\tau_0 E_i)}{E_i}$ is nonincreasing,

$$\tilde{F}_i'(t) \leq -\tilde{\tau}_i(\tilde{c}_1 - c\tau_0) G_0(\tau_0 E_i(t)) + \tilde{\tau}_i \frac{G_0(\tau_0 E_i(t))}{E_i(t)} \begin{cases} \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right) \int_0^L \psi_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 1, \\ \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 2, \\ \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L w_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 3. \end{cases} \tag{129}$$

According to (60), the coefficients of the integrals in (129) vanish, and hence, by choosing $\tau_0 > 0$ small enough such that $\tilde{c}_1 - c\tau_0 > 0$ and using the first inequality in (128), we get, for $c_i' = \tilde{\tau}_i(\tilde{c}_1 - c\tau_0)$,

$$\tilde{F}_i' \leq -c_i' G_0(\tilde{F}_i), \tag{130}$$

whereupon

$$(\tilde{G}(\tilde{F}_i))' \geq c_i', \tag{131}$$

where $\tilde{G}(t) = \int_t^1 \frac{1}{G_0(s)}\, \mathrm{d}s$. Integrating (131) over $[0, t]$ yields

$$\tilde{G}(\tilde{F}_i(t)) \geq c_i' t + \tilde{G}(\tilde{F}_i(0)). \tag{132}$$

Because $\tilde{F}_i(0) \leq 1$ (from (128)), $\tilde{G}(1) = 0$ and $\tilde{G}$ is decreasing, we obtain from (132) that

$$\tilde{G}(\tilde{F}_i(t)) \geq c_i' t,$$

which implies that

$$\tilde{F}_i(t) \leq \tilde{G}^{-1}(c_i' t).$$

Then, (70) and (127) give (63).

5. Proof of weak decay (66)

In this section, we treat the case when (60) does not hold which is more realistic from the physical point of view. We need to estimate the integrals in (125) using the following systems resulting from differentiating, respectively, (1)–(3) with respect to time t:

$$\begin{cases} \rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + l w_t)_x - l k_3 (w_{xt} - l\varphi_t) = 0, \\ \rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t + l w_t) + \int\limits_0^{+\infty} g_2(s) \psi_{xxt}(x, t-s)\, \mathrm{d}s = 0, \\ \rho_1 w_{ttt} - k_3 (w_{xt} - l\varphi_t)_x + l k_1 (\varphi_{xt} + \psi_t + l w_t) + \int\limits_0^{+\infty} g_3(s) w_{xxt}(x, t-s)\, \mathrm{d}s = 0, \\ \varphi_t(0,t) = \psi_{xt}(0,t) = w_{xt}(0,t) = \varphi_t(L,t) = \psi_{xt}(L,t) = w_{xt}(L,t) = 0, \end{cases} \tag{133}$$

$$\begin{cases} \rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + l w_t)_x - l k_3 (w_{xt} - l\varphi_t) + \int\limits_0^{+\infty} g_1(s) \varphi_{xxt}(x, t-s)\, \mathrm{d}s = 0, \\ \rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t + l w_t) = 0, \\ \rho_1 w_{ttt} - k_3 (w_{xt} - l\varphi_t)_x + l k_1 (\varphi_{xt} + \psi_t + l w_t) + \int\limits_0^{+\infty} g_3(s) w_{xxt}(x, t-s)\, \mathrm{d}s = 0, \\ \varphi_t(0,t) = \psi_{xt}(0,t) = w_{xt}(0,t) = \varphi_t(L,t) = \psi_{xt}(L,t) = w_{xt}(L,t) = 0 \end{cases} \tag{134}$$

and

$$\begin{cases} \rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + l w_t)_x - l k_3 (w_{xt} - l\varphi_t) + \int\limits_0^{+\infty} g_1(s) \varphi_{xxt}(x, t-s)\, \mathrm{d}s = 0, \\ \rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t + l w_t) + \int\limits_0^{+\infty} g_2(s) \psi_{xxt}(x, t-s)\, \mathrm{d}s = 0, \\ \rho_1 w_{ttt} - k_3 (w_{xt} - l\varphi_t)_x + l k_1 (\varphi_{xt} + \psi_t + l w_t) = 0, \\ \varphi_t(0,t) = \psi_{xt}(0,t) = w_{xt}(0,t) = \varphi_t(L,t) = \psi_{xt}(L,t) = w_{xt}(L,t) = 0. \end{cases} \tag{135}$$

Systems (133)–(135) are well posed for initial data $U_i^0 \in D(\mathcal{A}_i^n)$, $i = 1, 2, 3$, respectively, and $n \in \mathbb{N}^*$ thanks to Theorem 2.3, where

$$\partial_t U_i \in \cap_{k=0}^{n-1} C^{n-1-k}\left(\mathbb{R}_+; D\left(\mathcal{A}_i^k\right)\right).$$

Let $U_i^0 \in D(\mathcal{A}_i)$ and $\tilde{E}_i$, $i = 1, 2, 3$, be the energy of (133)–(135), respectively, defined by

$$\tilde{E}_i(t) = \frac{1}{2}\|\partial_t U_i(t)\|^2_{\mathcal{H}_i}. \tag{136}$$

As for (71), we have

$$\tilde{E}_i'(t) = \frac{1}{2}\begin{cases} \int_0^L \int_0^{+\infty} \left(g_2'(\partial_{xt}\eta_2)^2 + g_3'(\partial_{xt}\eta_3)^2\right) \mathrm{d}s\, \mathrm{d}x & \text{if } i = 1, \\ \int_0^L \int_0^{+\infty} \left(g_1'(\partial_{xt}\eta_1)^2 + g_3'(\partial_{xt}\eta_3)^2\right) \mathrm{d}s\, \mathrm{d}x & \text{if } i = 2, \\ \int_0^L \int_0^{+\infty} \left(g_1'(\partial_{xt}\eta_1)^2 + g_2'(\partial_{xt}\eta_2)^2\right) \mathrm{d}s\, \mathrm{d}x & \text{if } i = 3, \end{cases} \tag{137}$$

so $\tilde{E}_i$ is nonincreasing. Let $\tilde{\tau}_i = 1$ in (126). Then, (125) leads to

$$\tilde{F}_i'(t) \le -(\tilde{c}_1 - c\tau_0)G_0(\tau_0 E_i(t)) + \tfrac{G_0(\tau_0 E_i(t))}{E_i(t)} \begin{cases} \left(\rho_2 - \tfrac{k_2\rho_1}{k_1}\right) \int_0^L \psi_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 1, \\ \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 2, \\ \rho_1 \left(\tfrac{k_3}{k_1} - 1\right) \int_0^L w_t \varphi_{xt}\, \mathrm{d}x & \text{if } i = 3. \end{cases} \tag{138}$$

We use an idea introduced in [9] to get the following lemma.

Lemma 5.1. *For any $\epsilon > 0$, (1)–(3) imply that,*

$$\left| \left(\rho_2 - \tfrac{k_2\rho_1}{k_1}\right) \tfrac{G_0(\tau_0 E_1(t))}{E_1(t)} \int_0^L \psi_t \varphi_{xt}\, dx \right| \le c_\epsilon \tfrac{G_0(\tau_0 E_1(t))}{E_1(t)} \int_0^L \int_0^{+\infty} g_2(s)(\partial_{xt}\eta_2)^2\, ds\, dx + \epsilon G_0(\tau_0 E_1(t)) - c_\epsilon \tfrac{G_0(\tau_0 E_1(0))}{E_1(0)} E_1'(t), \tag{139}$$

$$\left| \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right) \tfrac{G_0(\tau_0 E_2(t))}{E_2(t)} \int_0^L \psi_t \varphi_{xt}\, dx \right| \le c_\epsilon \tfrac{G_0(\tau_0 E_2(t))}{E_2(t)} \int_0^L \int_0^{+\infty} g_1(s)(\partial_{xt}\eta_1)^2\, ds\, dx + \epsilon G_0(\tau_0 E_2(t)) - c_\epsilon \frac{G_0(\tau_0 E_2(0))}{E_2(0)} E_2'(t) \tag{140}$$

and

$$\left| \rho_1 \left(\tfrac{k_3}{k_1} - 1\right) \tfrac{G_0(\tau_0 E_3(t))}{E_3(t)} \int_0^L w_t \varphi_{xt}\, dx \right| \le c_\epsilon \tfrac{G_0(\tau_0 E_3(t))}{E_3(t)} \int_0^L \int_0^{+\infty} g_1(s)(\partial_{xt}\eta_1)^2\, ds\, dx + \epsilon G_0(\tau_0 E_3(t)) - c_\epsilon \tfrac{G_0(\tau_0 E_3(0))}{E_3(0)} E_3'(t). \tag{141}$$

Proof. We have, by integrating with respect to x and using the definition of η_2,

$$\begin{aligned}\left(\rho_2-\tfrac{k_2\rho_1}{k_1}\right)\int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x &= -\left(\rho_2-\tfrac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_t\psi_{xt}\,\mathrm{d}x\\ &= -\tfrac{1}{g_2^0}\left(\rho_2-\tfrac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_t\int_0^{+\infty} g_2(s)\partial_{xt}\eta_2\,\mathrm{d}s\,\mathrm{d}x\\ &\quad-\tfrac{1}{g_2^0}\left(\rho_2-\tfrac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_t\int_0^{+\infty} g_2(s)\psi_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}\tag{142}$$

Using (81) (for $\eta=\partial_{xt}\eta_2$ and $v=\varphi_t$) and (70) (for $i=1$), we get, for all $\epsilon>0$,

$$\begin{aligned}\left|\tfrac{1}{g_2^0}\left(\rho_2-\tfrac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_t\int_0^{+\infty} g_2(s)\partial_{xt}\eta_2\,\mathrm{d}s\,\mathrm{d}x\right| &\le \tfrac{\epsilon}{2}E_1(t)\\ &\quad+c_\epsilon\int_0^L\int_0^{+\infty} g_2(s)(\partial_{xt}\eta_2)^2\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}\tag{143}$$

On the other hand, by integrating with respect to s and using the definition of η_2, we obtain

$$\begin{aligned}\int_0^L \varphi_t\int_0^{+\infty} g_2(s)\psi_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x &= -\int_0^L \varphi_t\int_0^{+\infty} g_2(s)\partial_s(\psi_x(t-s))\,\mathrm{d}s\,\mathrm{d}x\\ &= \int_0^L \varphi_t\left(g_2(0)\psi_x(t)+\int_0^{+\infty} g_2'(s)\psi_x(t-s)\,\mathrm{d}s\right)\mathrm{d}x\\ &= -\int_0^L \varphi_t\int_0^{+\infty} g_2'(s)\partial_x\eta_2\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

Therefore, using (82) (for $\eta=\partial_x\eta_2$ and $v=\varphi_t$) and (71) (for $i=1$),

$$\left|\frac{1}{g_2^0}\left(\rho_2-\frac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_t\int_0^{+\infty} g_2(s)\psi_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x\right| \le \frac{\epsilon}{2}E_1(t)-c_\epsilon E_1'(t).\tag{144}$$

Inserting (143) and (144) into (142), multiplying by $\frac{G_0(\tau_0E_1)}{E_1}$ and noting that $\frac{G_0(\tau_0E_1)}{E_1}$ is nonincreasing, we obtain (139). Using the same arguments, we get (140) and (141). □

Now, by combining (138)–(141), and choosing $\epsilon = \frac{\tilde{c}_1}{2}$, we get

$$\tilde{F}_i'(t) \le -\left(\tfrac{\tilde{c}_1}{2} - c\tau_0\right) G_0(\tau_0 E_i(t)) - c\tfrac{G_0(\tau_0 E_i(0))}{E_i(0)} E_i'(t) + c\tfrac{G_0(\tau_0 E_i(t))}{E_i(t)} \begin{cases} \int_0^L \int_0^{+\infty} g_2(s)(\partial_{xt}\eta_2)^2 \,\mathrm{d}s\,\mathrm{d}x & \text{if } i = 1, \\ \int_0^L \int_0^{+\infty} g_1(s)(\partial_{xt}\eta_1)^2 \,\mathrm{d}s\,\mathrm{d}x & \text{if } i = 2 \text{ or } i = 3. \end{cases} \tag{145}$$

Similarly to (120) and (121), using (137), we find, for $j \in \{1,2,3\} \setminus \{i\}$,

$$\int_0^L \int_0^{+\infty} g_j(s)(\partial_{xt}\eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x \le -\frac{2}{\alpha}\tilde{E}_i'(t) \tag{146}$$

when (58) holds. When (59) holds, there exists a positive constant c such that, for any $\tau_0 > 0$, we have as for (121) (see the proof of Lemma 3.6 [11])

$$G'(\tau_0 E_i(t)) \int_0^L \int_0^{+\infty} g_j(s)(\partial_{xt}\eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x \le -c\tilde{E}_i'(t) + c\tau_0 E_i(t) G'(\tau_0 E_i(t)). \tag{147}$$

From (146) and (147), we find that, in both cases (58) and (59),

$$\begin{aligned} \tfrac{G_0(\tau_0 E_i(t))}{E_i(t)} \int_0^L \int_0^{+\infty} g_j(s)(\partial_{xt}\eta_j)^2 \,\mathrm{d}s\,\mathrm{d}x \le{} & c\tau_0 G_0(\tau_0 E_i(t)) - c\tfrac{G_0(\tau_0 E_i(t))}{E_i(t)}\tilde{E}_i'(t) \\ & - c\tau_0 \tilde{E}_i'(t), \ \forall j \in \{1,2,3\} \setminus \{i\}. \end{aligned} \tag{148}$$

Inserting (148) in (145), choosing $\tau_0 > 0$ small enough such that $\frac{\tilde{c}_1}{2} - c\tau_0 > 0$ and using the fact that $\frac{G_0(\tau_0 E_i)}{E_i}$ is nonincreasing, we find, for some $\tilde{c}_2 > 0$,

$$G_0(\tau_0 E_i(t)) \le -\tilde{c}_2 \tilde{F}_i'(t) - c\left(1 + \frac{G_0(\tau_0 E_i(0))}{E_i(0)}\right)\left(E_i'(t) + \tilde{E}_i'(t)\right). \tag{149}$$

By integration with respect to t and using (127), we get, for some $\tilde{c}_3 > 0$,

$$\int_S^T G_0(\tau_0 E_i(t))\,\mathrm{d}t \le \tilde{c}_3\left(1 + \frac{G_0(\tau_0 E_i(0))}{E_i(0)}\right)\left(E_i(S) + \tilde{E}_i(S)\right), \quad \forall T \ge S \ge 0. \tag{150}$$

Choosing $S = 0$ in (150) and using the fact that $G_0(\tau_0 E_i)$ is nonincreasing, we obtain

$$G_0(\tau_0 E_i(T))T \le \int_0^T G_0(E_i(t))\,\mathrm{d}t \le \tilde{c}_3\left(1 + \frac{G_0(\tau_0 E_i(0))}{E_i(0)}\right)\left(E_i(0) + \tilde{E}_i(0)\right). \tag{151}$$

Because G_0^{-1} is increasing, (66) for $n = 1$ is deduced from (70) and (151) with

$$c_{i,1} = \max\left\{\frac{2}{\tau_0}, \tilde{c}_3\left(1 + \frac{G_0(\tau_0 E_i(0))}{E_i(0)}\right)\left(E_i(0) + \tilde{E}_i(0)\right)\right\}.$$

By induction on n, suppose that (66) holds and let $U_i^0 \in D(\mathcal{A}_i^{n+1})$ be such that (65) holds, for $n+1$ instead of n. We have $\partial_t U_i(0) \in D(\mathcal{A}_i^n)$ (thanks to Theorem 2.3) and $\partial_t U_i$ satisfies the first three equations and the boundary conditions of system (i), $i=1,2,3$ (that is, (133)–(135) are satisfied). On the other hand, let $\tilde{\eta}_i = \partial_t \eta_i$, $i=1,2,3$. From (11), we remark that

$$\tilde{\eta}_i(x,t,s) = \begin{cases} \varphi_t(x,t) + \partial_s(\varphi(x,t-s)) & \text{if } i=1, \\ \psi_t(x,t) + \partial_s(\psi(x,t-s)) & \text{if } i=2, \\ w_t(x,t) + \partial_s(w(x,t-s)) & \text{if } i=3. \end{cases}$$

Then,

$$\tilde{\eta}_i^0(x,s) := \tilde{\eta}_i(x,0,s) = \begin{cases} \varphi_1(x) + \partial_s \varphi_0(x,s) & \text{if } i=1, \\ \psi_1(x) + \partial_s \psi_0(x,s) & \text{if } i=2, \\ w_1(x) + \partial_s w_0(x,s) & \text{if } i=3, \end{cases}$$

therefore, for $k=1,\dots,n$,

$$\partial_s^k \tilde{\eta}_i^0(x,s) = \begin{cases} \partial_s^{k+1} \varphi_0(x,s) & \text{if } i=1, \\ \partial_s^{k+1} \psi_0(x,s) & \text{if } i=2, \\ \partial_s^{k+1} w_0(x,s) & \text{if } i=3. \end{cases}$$

But, again from (11),

$$\partial_s^{k+1} \eta_i^0(x,s) = \partial_s^{k+1} \eta_i(x,0,s) = \begin{cases} -\partial_s^{k+1} \varphi_0(x,s) & \text{if } i=1, \\ -\partial_s^{k+1} \psi_0(x,s) & \text{if } i=2, \\ -\partial_s^{k+1} w_0(x,s) & \text{if } i=3. \end{cases}$$

Then,

$$\partial_s^k \tilde{\eta}_i^0(x,s) = -\partial_s^{k+1} \eta_i^0(x,s).$$

Consequently, (65) is satisfied, for $\tilde{\eta}_j^0$ instead of η_j^0, $j \in \{1,2,3\} \setminus \{i\}$ (because (65) is supposed being satisfied, for $n+1$ instead of n). Hence, for some positive constant $\tilde{c}_{i,n}$,

$$\|\partial_t U_i(t)\|_{\mathcal{H}_i}^2 \le \tilde{c}_{i,n} G_n\left(\frac{\tilde{c}_{i,n}}{t}\right), \quad \forall t > 0. \tag{152}$$

Choosing $S = \frac{T}{2}$ in (150) and combining with (66) and (152), we conclude that ($G_0(\tau_0 E_i)$ is nonincreasing)

$$G_0(\tau_0 E_i(T))T \le 2 \int_{\frac{T}{2}}^{T} G_0(\tau_0 E_i(t))\,\mathrm{d}t \le \tilde{c}_3 \left(1 + \frac{G_0(\tau_0 E_i(0))}{E_i(0)}\right) \left(c_{i,n} G_n\left(\frac{2c_{i,n}}{T}\right) + \tilde{c}_{i,n} G_n\left(\frac{2\tilde{c}_{i,n}}{T}\right)\right),$$

which implies that, for

$$c_{i,n+1} = \max\left\{\frac{2}{\tau_0}, \tilde{c}_3(c_{i,n} + \tilde{c}_{i,n})\left(1 + \frac{G_0(\tau_0 E_i(0))}{E_i(0)}\right), 2c_{i,n}, 2\tilde{c}_{i,n}\right\}$$

and any $T > 0$ (notice that G_n is increasing),

$$\begin{aligned} \|U_i(t)\|_{\mathcal{H}_i}^2 &= 2E_i(T) \le c_{i,n+1} G_0^{-1}\left(\frac{c_{i,n+1}}{T} G_n\left(\frac{c_{i,n+1}}{T}\right)\right) = c_{i,n+1} G_1\left(\frac{c_{n+1}}{T} G_n\left(\frac{c_{n+1}}{T}\right)\right) \\ &= c_{i,n+1} G_{n+1}\left(\frac{c_{i,n+1}}{T}\right), \end{aligned}$$

so (66) holds, for $n+1$.

Remark 5.2. 1. It is possible to prove some similar results under other kinds of boundary conditions like the homogeneous Dirichlet–Dirichlet–Dirichlet ones.

2. One of the interesting questions related to our results is the optimality of the decay estimates (63) and (66). In case of linear frictional dampings or infinite memories with kernels satisfying (58), the optimality of (69), when $n = 1$ and (7) does not hold, was proved in [2] and [6].

3. Another interesting question concerns the stability of Bresse systems with only one infinite memory and a kernel having a general growth at infinity (not necessarily of exponential type). The case (8) has been treated in [6] but for kernels g satisfying (9), which implies (10); that is, g converges exponentially to zero.

4. Our stability results (63) and (66) hold under the smallness conditions (61) on l, g_1^0 and g_3^0, and the boundedness conditions (62) and (65) on the initial data η_j^0. It is interesting to drop these purely technical conditions (61), (62) and (65).

Acknowlegments

The second-named author thanks the NNAM group, King Abdulaziz University and Jeddah University for their support.

References

1. Alabau-Boussouira, F., Muñoz Rivera, J.E., Almeida Júnior, D.S.: Stability to weak dissipative Bresse system. J. Math. Anal. Appl. **374**, 481–498 (2011)
2. Alves, M.O., Fatori, L.H., Jorge Silva, M.A., Monteiro, R.N.: Stability and optimality of decay rate for weakly dissipative Bresse system. Math. Meth. Appl. Sci. **38**, 898–908 (2015)
3. Bresse, J.A.C.: Cours de Méchanique Appliquée. Mallet Bachelier, Paris (1859)
4. Charles, W., Soriano, J.A., Nascimentoc, F.A., Rodrigues, J.H.: Decay rates for Bresse system with arbitrary nonlinear localized damping. J. Diff. Equa. **255**, 2267–2290 (2013)
5. Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Rational Mech. Anal. **37**, 297–308 (1970)
6. De Lima Santos, M., Soufyane, A., Da Silva Almeida Júnior, D.: Asymptotic behavior to Bresse system with past history. Quart. Appl. Math. **73**, 23–54 (2015)
7. Fatori, L.H., Monteiro, R.N.: The optimal decay rate for a weak dissipative Bresse system. Appl. Math. Lett. **25**, 600–604 (2012)
8. Fatori, L.H., Muñoz Rivera, J.E.: Rates of decay to weak thermoelastic Bresse system. IMA J. Appl. Math. **75**, 881–904 (2010)
9. Fernández Sare, H.D., Muñoz Rivera, J.E.: Stability of Timoshenko systems with past history. J. Math. Anal. Appl. **339**, 482–502 (2008)
10. Guesmia, A.: Asymptotic stability of abstract dissipative systems with infinite memory. J. Math. Anal. Appl. **382**, 748–760 (2011)
11. Guesmia, A.: Asymptotic behavior for coupled abstract evolution equations with one infinite memory. Applicable Analysis **94**, 184–217 (2015)
12. Guesmia, A., Kafini, M.: Bresse system with infinite memories. Math. Meth. Appl. Sci. **38**, 2389–2402 (2015)
13. Komornik, V.: Exact Controllability and Stabilization. The Multiplier Method. Masson-John Wiley, Paris (1994)
14. Lagnese, J.E., Leugering, G., Schmidt, J.P.: Modelling of dynamic networks of thin thermoelastic beams. Math. Meth. Appl. Scie. **16**, 327–358 (1993)
15. Lagnese, J.E., Leugering, G., Schmidt, E.J.P.G.: Modelling, Analysis and Control of Dynamic Elastic Multi-Link Structures. Systems & Control: Foundations & Applications. Birkhäuser, Boston. ISBN: 0-8176-3705-2 (1994)
16. Liu, Z., Rao, B.: Energy decay rate of the thermoelastic Bresse system. Z. Angew. Math. Phys. **60**, 54–69 (2009)
17. Najdi, N., Wehbe, A.: Weakly locally thermal stabilization of Bresse systems. Elec. J. Diff. Equa. **2014**, 1–19 (2014)
18. Noun, N., Wehbe, A.: Weakly locally internal stabilization of elastic Bresse system. C. R. Acad. Scie. Paris Sér. I **350**, 493–498 (2012)
19. Pazy, A.: Semigroups of linear operators and applications to partial differential equations. Springer, New York (1983)
20. Soriano, J.A., Charles, W., Schulz, R.: Asymptotic stability for Bresse systems. J. Math. Anal. Appl. **412**, 369–380 (2014)

21. Soriano, J.A., Muñoz Rivera, J.E., Fatori, L.H.: Bresse system with indefinite damping. J. Math. Anal. Appl. **387**, 284–290 (2012)
22. Soufyane, A., Said-Houari, B.: The effect of the wave speeds and the frictional damping terms on the decay rate of the Bresse system. Evol. Equa. Cont. Theory **3**, 713–738 (2014)
23. Wehbe, A., Youssef, W.: Exponential and polynomial stability of an elastic Bresse system with two locally distributed feedbacks. J. Math. Phys. **51**, 1–17 (2010)
24. Youssef, W.: Contrôle et stabilisation de systèmes élastiques couplés, PhD thesis, University of Metz, France (2009)

A. Guesmia
Institut Elie Cartan de Lorraine
UMR 7502, Université de Lorraine
Bat. A, Ile du Saulcy
57045 Metz Cedex 01
France
e-mail: aissa.guesmia@univ-lorraine.fr

M. Kirane
LaSIE, Faculté des Sciences
Pôle Sciences et Technologies
Université de la Rochelle
La Rochelle
France
e-mail: mokhtar.kirane@univ-lr.fr

(Received: February 1, 2016; revised: July 6, 2016)