Stable bundles with small c_2 over 2-dimensional complex tori

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1 Introduction

Let X be a compact complex surface and E a topological complex vector bundle on X of rank r and Chern classes $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. When X is algebraic E admits a holomorphic structure if and only if c_1 lies in the Néron-Severi group $NS(X)$ of X (i.e. the image of c_1 in $H^2(X, \mathbb{R})$ is of type (1,1)). If moreover c_2 is large enough as compared to r and c_1^2 then E admits stable holomorphic structures with respect to any fixed polarization on X and their moduli spaces have nice geometric properties. In particular they admit natural projective compactifications.

The situation changes if we let X be non-algebraic. For the existence of holomorphic structures in E, the condition $c_1 \in NS(X)$ is still necessary but no longer sufficient. In fact it was proved by Bănică and Le Potier that if E admits a holomorphic structure then

$$2rc_2 - (r-1)c_1^2 \geq 0$$

and that holomorphic structures exist when c_2 is large enough with respect to r and c_1. For small values of c_2 however, the existence problem remains in general open. This situation is strikingly similar to that of the stable structures. Notice that the inequality above is exactly the Bogomolov inequality which is satisfied by the topological invariants of any stable vector bundle. In this paper we show how this similarity can be made more precise for two-dimensional tori.

We use deformations of the complex structure of a torus keeping a suitable Riemannian metric fixed in order to switch between stable structures in E over algebraic tori and so called irreducible structures in E over non-algebraic tori. This enables us to solve the existence problems for rank-two holomorphic vector bundles on non-algebraic tori and for stable rank-two vector bundles of degree zero on any two-dimensional torus. More precisely we prove:

Theorem 1.1 *A topological rank 2 complex vector bundle E on a non-algebraic two-dimensional complex torus X admits some holomorphic structure if and only if*

$$c_1(E) \in NS(X) \text{ and } 4c_2(E) - c_1(E)^2 \geq 0.$$

Theorem 1.2 *Let X be a complex 2-dimensional torus and ω a Kähler class on X. Let $c_1 \in NS(X)$ such that $c_1 \cdot \omega = 0$. Suppose that*

$$c_1^2 = \max\{(c_1 + 2b)^2 \mid b \in NS(X),\ b \cdot \omega = 0\}.$$

Then a topological rank 2 vector bundle with Chern classes c_1, c_2 admits a holomorphic structure stable with respect to ω of and only if

$$4c_2 - c_1^2 \geq 0,$$

except when

$$c_1 = 0 \text{ and } c_2 \in \{0, 1\}, \text{ or}$$
$$c_1^2 = -2 \text{ and } c_2 = 0.$$

In the excepted cases the holomorphic structures on E are unstable with respect to any polarization ω such that $c_1 \cdot \omega = 0$.

Remark that $c_1 \cdot \omega = 0$ implies $c_1^2 \leq 0$ by Hodge index and that the condition

$$c_1^2 = \max\{(c_1 + 2a) \mid a \in NS(X),\ a \cdot \omega = 0\}$$

can always be fulfilled by twisting E with a suitable line bundle. Neither the stability nor the invariant $4c_2(E) - c_1^2(E)$ are modified by such twists.

Remark also that if $c_1^2 < 0$ or $c_1 = 0$ there exist polarizations ω such that $c_1 \cdot \omega = 0$.

For "large c_2" one can always construct locally free sheaves as extensions of coherent sheaves of smaller rank. This method cannot work for all c_2. In fact, for c_2 below a certain bound all existing holomorphic structures are "irreducible" i.e. do not admit coherent subsheaves of lower rank (cf.[1]). Some holomorphic structures for E with "small c_2" have been constructed in [5]. We fill in the gaps left by [5] in the following way. We consider two suitable invariant metrics on our torus X and perform two "quaternionic deformations" (see 2.1.1 for the definition) such that the deformed torus has a convenient algebraic structure (see section 3). We construct a stable holomorphic structure in E with respect to this new complex structure of the base and use anti-self-dual connections to get a holomorphic structure for E over our original X.

To prove Theorem 1.2 we perform again a quaternionic deformation, this time changing the structure of X into a non-algebraic one. It is enough then to know which bundles admit here irreducible structures. So we reduce ourselves to

Theorem 1.3 *When X is a non-algebraic 2-dimensional torus and E a topological rank 2 vector bundle having $c_1(E) \in NS(X)$ such that $c_1(E)^2 = \max\{(c_1(E)+2a)^2 \mid a \in NS(X)\}$, then E admits an irreducible holomorphic structure if and only if*

$$4c_2(E) - c_1(E)^2 \geq 0,$$

unless

$$c_1(E)^2 = 0 \text{ and } c_2(E) \in \{0,1\} \text{ or,}$$
$$c_1(E)^2 = -2 \text{ and } c_2(E) = 0.$$

In the excepted cases all holomorphic structures are reducible.

Remark again that the intersection form on $NS(X)$ is negative semidefinite if X is a non-algebraic surface and that the condition

$$c_1(E)^2 = \max\{(c_1(E) + 2a)^2 \mid a \in NS(X)\}$$

is always fulfilled after a suitable twist of E.

In a forthcoming paper ([6]) we show that the moduli spaces of stable sheaves with "small c_2" over a non-algebraic torus are compact.

2 Preliminary Material

2.1 Self-duality and complex structures

We recall here some simple basic facts about 4-dimensional geometry.

2.1.1 Self-duality

Let V be a 4-dimensional oriented real vector space. Further let $g : V \times V \longrightarrow \mathbb{R}$ be a metric on V. The Hodge operator restricted to $\Lambda^2 V^*$ is involutive. We denote by Λ^+, Λ^- its eigenspaces belonging to the eigenvalues ± 1. The elements of Λ^+ and Λ^- are called **self-dual**, respectively **anti-self-dual** forms.

Λ^+, Λ^- are maximal positive, respectively negative orthogonal subspaces of $\Lambda^2 V^*$ for the "intersection" form:

$$(\alpha, \beta) \longmapsto \alpha \wedge \beta / \nu = \alpha \cdot \beta$$

where $\nu \in \Lambda^4 V^*$ is the canonical volume form on V. Conversely, one can show that starting with the oriented 4-dimensional space V and with an orthogonal decomposition of $\Lambda^2 V^*$ into maximal positive and negative subspaces for the intersection form, there is a metric g on V, unique up to a constant, such that the given subspaces of $\Lambda^2 V^*$ coincide with the eigenspaces of the associated Hodge operator.

2.1.2 Complex structures and quaternionic deformations

We further consider a complex structure I on V which is compatible with the orientation and with the metric and denote by V_I the complex vector space thus obtained. To g and I one can associate an element $\omega_I \in \Lambda^2 V^*$ by $\omega_I(u, v) := g(u, Iv)$. One sees easily that $\omega_I \cdot \omega_I = g(\omega_I, \omega_I) = 2$, so ω_I belongs to the sphere of radius $\sqrt{2}$ in Λ^+.

One verifies that conversely, each element of this sphere is associated to a unique complex structure on V compatible with the orientation and with the metric. In fact these complex structures turn V into a module over the quaternions. (If ω_I and ω_J are orthogonal, the product $K := IJ$ is a new complex structure corresponding to $\omega_I \times \omega_J$ in Λ^+.) We shall therefore say that two such complex structures are **quaternionic deformations** of each other (with respect to the fixed metric g).

2.1.3 Type decomposition and the positive cone

The complex structure I induces decompositions $V_{\mathbb{C}} := V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$, $V^*_{\mathbb{C}} := V^* \otimes \mathbb{C} = V^{*1,0} \oplus V^{*0,1}$ into eigenspaces of the extension of I to $V_{\mathbb{C}}$ and $V^*_{\mathbb{C}}$, and further decompositions into type $\Lambda^r_{\mathbb{C}} V^*_{\mathbb{C}} = \bigoplus_{p+q=r} \Lambda^{p,q}$, where $\Lambda^{p,q} := \Lambda^p_{\mathbb{C}}(V^{*1,0}) \otimes \Lambda^r_{\mathbb{C}}(V^{*0,1})$. In particular we get $\Lambda^2 V^* = (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \Lambda^{1,1}_{\mathbb{R}}$ where the $\mathbb{R}$-index denotes intersection of the corresponding space with $\Lambda^2 V^*$.

The property $\omega_I(Iu,\, Iv) = \omega_I(u,v)$ means that $\omega_I \in \Lambda^{1,1}_{\mathbb{R}}$. The two orthogonal decompositions of $\Lambda^2 V^*$,

$$\begin{aligned} \Lambda^2 V^* &= \Lambda^+ \oplus \Lambda^- \text{ and} \\ \Lambda^2 V^* &= (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \Lambda^{1,1}_{\mathbb{R}}, \end{aligned}$$

compare in the following way:

$$\begin{aligned} \Lambda^+ &= (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \langle \omega_I \rangle \\ \Lambda^{1,1}_{\mathbb{R}} &= \langle \omega_I \rangle \oplus \Lambda^-, \end{aligned}$$

where $\langle \omega_I \rangle$ is the line spanned by ω_I.

The intersection form on $\Lambda^{1,1}_{\mathbb{R}}$ has type (1,3) and thus the set $\{\eta \in \Lambda^{1,1}_{\mathbb{R}} \mid \eta \cdot \eta > 0\}$ has two components. The condition $\eta(Iu,\, u) > 0$ for one, or equivalently for all $u \neq 0,\ u \in V$, distinguishes one of these components which we call the **positive cone**, $\mathcal{C} := \{\eta \in \Lambda^{1,1}_{\mathbb{R}} \mid \eta \cdot \eta > 0,\ \eta(Iu,\, u) > 0$ for some $u \neq 0,\ u \in V\}$. The above facts now show that $\mathcal{C}$ and the set of metrics on V compatible with a fixed complex structure I are in a natural bijective correspondence.

2.2 Line bundle cohomology on complex tori

Consider a $2g$-dimensional real vector space V endowed with a complex structure I, a lattice $\Gamma \subset V$ and the complex torus $X = X_I := V_I/\Gamma$.

Using translation invariant differential forms on X we get the following natural isomorphisms for the de Rham and Dolbeault cohomology groups of

X:

$$H^r(X,\mathbb{R}) \cong \Lambda^r V^*$$
$$H^{p,q} \cong \Lambda^{p,q}$$
$$H^r(X,\mathbb{Z}) \cong \Lambda^r \Gamma^*,$$

where $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \subset V^*$.

The first Chern class of a holomorphic line bundle on X is an element of

$$H^{1,1} \cap H^2(X,\mathbb{Z}) \cong \Lambda^{1,1} \cap \Lambda^2 \Gamma^*$$

and thus it is represented by a real skew-symmetric bilinear form E on V taking integer values on $\Gamma \times \Gamma$ and such that $E(Iu,\, Iv) = E(u,v),$ for all $u, v \in V$. To E one associates a hermitian form H on V_I such that $E = \mathrm{Im}\, H$. The pairs (H, α) consisting of a hermitian form H on V_I such that $E := \mathrm{Im}\, H$ is integer valued on $\Gamma \times \Gamma$ and a map $\alpha : \Gamma \times \Gamma \to U(1)$ satisfying $\alpha(\gamma_1 + \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)(-1)^{E(\gamma_1,\gamma_2)}$ are called **Appell-Humbert data**. Addition on the first component and multiplication on the second induce a group structure on the set of Appell-Humbert data. There is a natural way to construct a holomorphic line bundle $L(H,\alpha)$ on X out of the data (H,α) and this gives an isomorphism from the group of Appell-Humbert data to the Picard group of X. Moreover, through this isomorphism $c_1(L(H,\alpha))$ corresponds to $E := \mathrm{Im}\, H$. Let

$$\begin{aligned} \mathrm{Ker}\, H :&= \{u \in V \mid H(u,v) = 0, \quad \forall\ v \in V\} \\ &= \{u \in V \mid E(u,v) = 0, \quad \forall\ v \in V\}, \end{aligned}$$

$k = \dim \mathrm{Ker}\, H$ and n the number of negative eigenvalues of H. We denote by $pf(E)$ the Pfaffian of E: for an oriented symplectic basis $(u_i)_{1\le i\le 2g}$ of V with respect to E (i.e. such that $E(u_i, u_j) = 0$ for $|j-i| \neq g$) $pf(E) = \det(E(u_i, u_{j+g}))_{0\le i,j\le g}$. Remark that there always exists such a basis which is also a basis for Γ over $\mathbb{Z}$. Moreover this can be chosen in such a way that $d_i | d_{i+1}$ where $d_i := E(u_i, u_{i+g})$. In this case we shall call the sequence $(d_1, d_2, \ldots, d_g)$ the **type** of $L(H,\alpha)$. We can now state the results we shall need on line bundle cohomology on tori (cf. [2]).

Theorem 2.1 *(Mumford-Kempf)*

(a) $H^i(X, L(H,\alpha)) = 0$ *for* $i < n$ *or* $i > n+k$.

(b) $H^{n+i}(X, L(H,\alpha)) \cong H^n(X, L(H,\alpha)) \otimes H^{0i}(\mathrm{Ker}\, H/\Gamma \cap \mathrm{Ker}\, H)$ *for* $0 \le i \le k$

(c) $H^n(X, L(H,\alpha)) = 0$ *if and only if* $\alpha\big|_{\Gamma\cap\operatorname{Ker} H} \not\equiv 1$.

Theorem 2.2 *(Riemann-Roch)*

$$\begin{aligned}\chi(L(H,\alpha)) &:= \sum_{i=0}^{g}(-1)^i \dim H^i(X, L(H,\alpha) \\ &= \frac{1}{g!}c_1(L(H,\alpha))^g = \frac{1}{g!}pf(E).\end{aligned}$$

When $g = 2$ we distinguish the following cases:

- $c_1(L(H,\alpha))^2 > 0$
  implies H or $-H$ is positive definite (which is equivalent to saying that $L(H,\alpha)$ or $L(H,\alpha)^{-1}$ is ample) and according to this, the cohomology of $L(H,\alpha)$ is concentrated in degree 0 or 2.
- $c_1(L(H,\alpha))^2 < 0$
  implies H is indefinite and the cohomology of $L(H,\alpha)$ is concentrated in degree one
- $c_1(L(H,\alpha))^2 = 0$ and $H \neq 0$
  imply H or $-H$ is positive semi-definite and thus the cohomology of $L(H,\alpha)$ in degree 2, respectively 0, must vanish; for suitable $\alpha - s$ all cohomology groups will vanish in this case.

3 Vector bundles on non-algebraic 2-tori

Let X be a 2 dimensional non-algebraic torus, and E a differential complex vector bundle of rank 2 on X having $c_1(E) \in NS(X)$. As proved in [1] a necessary condition for E to admit some holomorphic structure is that

$$\Delta(E) := \frac{1}{2}(c_2(E) - \frac{1}{4}c_1(E)^2) \geq 0.$$

We shall show that this condition is also sufficient. If 4 divides $c_1(E)^2$ this has been proved in [4], [5]. The statement holds also when $c_1(E)^2 = -2$ since we can construct a filtrable vector bundle E as a direct sum of holomorphic line bundles having

$$\Delta(E) = \frac{1}{2}\Big(0 - \frac{1}{4}(-2)\Big) = \frac{1}{4},$$

and then increase c_2 by applying the following

Proposition 3.1 *([5], [6]) Let X be a compact complex surface with Kodaira dimension* $\mathrm{kod}(X) = -\infty$ *or with* $\mathrm{kod}(X) = 0$ *and* $p_g(X) = 1$. *Let E be a holomorphic vector bundle on X whose* rank *exceeds 1 and n a positive integer. Then there exists a holomorphic vector bundle F on X with*

$$\mathrm{rank}(F) = \mathrm{rank}(E),\ c_1(F) = c_1(E),\ c_2(F) = c_2(E) + n.$$

excepting the case when X is $K3$ without non-constant meromorphic functions, E is a twist of the trivial line bundle by some line bundle and $n = 1$.

Thus we only have to deal with the case

$$c_1(E)^2 = -2(4k \pm 1),$$

k a positive integer. (Recall that the self intersection of an element of $NS(X)$ is non positive since X is non-algebraic). If L is a holomorphic line bundle on X then

$$\begin{aligned} c_1(E \otimes L) &= c_1(E) + 2c_1(L) \\ \Delta(E \otimes L) &= \Delta(E), \end{aligned}$$

so it will be enough to solve the existence problem for some vector bundle E' of rank 2 with $c_1(E') \in c_1(E) + 2NS(X)$ and

$$\Delta(E') = \Delta(E).$$

In particular we may always suppose that $c_1(E)$ is a primitive element in $NS(X)$. X will be considered as the quotient $X_I = V_I/\Gamma$ of a fixed real 4-dimensional vector space V endowed with a complex structure I through the fixed lattice Γ. Let a be a primitive element in

$$NS(X_I) \cong H^2(X, \mathbb{Z}) \cap H^{1,1} \cong \Lambda^2\Gamma^* \cap \Lambda_I^{1,1}.$$

We first connect the complex structure I to a new complex structure K by two quaternionic deformations such that $NS(X_K)$ is generated by a and an ample class of a special type.

Lemma 3.2 *Let X_I be a complex 2-dimensional torus and $a \in NS(X)$ a primitive element such that $a^2 < 0$.*

(a) *There exists an invariant hermitian metric h on X_I which makes a (seen as an invariant 2-form) anti-self-dual. Moreover this metric may be chosen such that all integer elements in Λ^- are multiples of a.*

(b) *If h is chosen as above and g is its real part, then there exists a dense open set of quaternionic deformations J of I with respect to g, such that $NS(X_J)$ is cyclic generated by a.*

(c) *If $a^2 = -2(4k \pm 1)$ there exists $\eta \in H^2(X, \mathbb{Z})$ of type $(1, k)$ with $a \cdot \eta = 0$.*

For a suitably chosen complex structure J as above there exists a quaternionic deformation K of it, with respect to a possibly new metric on X_J, such that η is proportional to the imaginary part ω_K of the associated hermitian metric and $NS(X_K)$ is generated over $\mathbb{Q}$ by a and η.

Proof

(a) By 2.1.3 it is enough to find an element $\omega_I \in \mathcal{C}$ with $\omega_I \cdot a = 0$ but $\omega_I \cdot b \neq 0$ if $b \in NS(X)$ is not a multiple of a. For the existence of such an ω_I we just remark that the intersection form on the orthogonal of a in $\Lambda^{1,1}_{\mathbb{R}}$ has type (1,2), since $a^2 < 0$.

(b) Using 2.1.2 one sees that it suffices to take ω_J on the sphere of radius $\sqrt{2}$ in Λ^+ and not belonging to a line of the form $(\langle b \rangle \oplus \Lambda^-) \cap \Lambda^+$, where $b \in H^2(X, \mathbb{Z}) \backslash \langle a \rangle$.

(c) We choose a symplectic integer basis for a. Since a was primitive the associated matrix will have the form:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(4k \pm 1) \\ -1 & 0 & 0 & 0 \\ 0 & (4k \pm 1) & 0 & 0 \end{pmatrix}.$$

Any element $\eta \in H^2(X, \mathbb{Z})$ is represented in this basis by an integer-valued skew-symmetric matrix

$$S = \begin{pmatrix} 0 & \theta & \alpha & \beta \\ -\theta & 0 & \gamma & \delta \\ -\alpha & -\gamma & 0 & \tau \\ -\beta & -\delta & -\tau & 0 \end{pmatrix}.$$

The condition $a \cdot \eta = 0$ becomes:

$$\delta = \alpha(4k \pm 1).$$

η is of type $(1,k)$ if there exists some basis of Γ with respect to which its associated matrix is:

$$\Xi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \\ -1 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{pmatrix}.$$

We thus need a transformation matrix $M \in SL(4,\mathbb{Z})$ changing Ξ into S by

$$S = M^t \Xi M$$

such that for S, $\delta = \alpha(4k \pm 1)$. It is easy to see that such an M exists. (In fact one can show that the $\eta - s$ satisfying the given conditions span $\langle a \rangle^\perp$ over $\mathbb{R}$.)

So let now η be of type $(1,k)$ with $\eta \cdot a = 0$. Let Λ^+ be the space of invariant self-dual forms with respect to the metric which was fixed in (a).

If $\eta \in \Lambda^+$ we just take ω_K proportional to η on the sphere of radius $\sqrt{2}$ in Λ^+. For the new complex structure K, $NS(X_K)$ will be generated over $\mathbb{Q}$ by a and η. Let indeed $b = c \cdot \eta + \mu \in NS(X_K)$ with $c \in \mathbb{R}$ and $\mu \in \Lambda^-$. Since $\eta \cdot b = c\eta^2$ is an integer, c must be rational and μ lies in $\Lambda^- \cap H^2(X,\mathbb{Q}) = \mathbb{Q} \cdot a$.

When $\eta \notin \Lambda^+$ we have to change the metric on X. We do this as follows. $\eta^\perp \cap \Lambda^+$ is a 2-dimensional subspace of Λ^+. Let ω be an element of the sphere of radius $\sqrt{2}$ in Λ^+ around 0, which is orthogonal to $\eta^\perp \cap \Lambda^+$. In fact ω is proportional to the projection of η on Λ^+. We choose ω_J as in (b) but close to ω. We claim that the intersection form is positive definite on $(\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_\mathbb{R} \oplus \langle \eta \rangle$. Indeed, let $\omega_K := \frac{\sqrt{2} \cdot \eta}{\eta^2}$, $\nu \in (\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_\mathbb{R} = \omega_J^\perp \cap \Lambda^+$ of norm $\sqrt{2}$ and consider an element $s\omega_K + t\nu$ in $(\Lambda_J^{2,0} \oplus \Lambda_J^{0,2}) \oplus \langle \eta \rangle$; $s,t \in \mathbb{R}$. We have $(s\omega_K + t\nu)^2 = 2s^2 + 2t^2 + 2st\omega_K \cdot \nu = 2s^2 + 2t^2 + 2st\,\omega_K^+ \cdot \nu$, where $\omega_K = \omega_K^+ + \omega_K^-$ is the decomposition in self-dual and anti-self-dual parts of ω_K. There exists a real number c depending on η and Λ^+ such that $\omega_K^+ = c\omega$. When ω_J is sufficiently close to ω we have $|\omega_K^+ \cdot \nu| = |c\omega \cdot \nu| < \frac{1}{2}$ for all $\nu - s$, hence our claim. Thus we may view $(\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_\mathbb{R} \oplus \langle \eta \rangle$ as the space of self-dual forms with respect to a new Riemannian metric g' which is compatible with J.

Since $NS(X_J)$ is generated by a, a will span $\Lambda_{g'}^- \cap H^2(X,\mathbb{Z})$ too. The complex structure K corresponding to ω_K is the quaternionic deformation of J (with respect to g') we have been looking for. $\square$

The next move is to construct a stable vector bundle E on X_K with $c_1(E) = a$ and smallest possible $c_2(E)$, i.e. such that $\Delta(E) = \frac{1}{4}$.

Lemma 3.3 *Let X_K be a complex 2-dimensional torus whose Néron-Severi group is generated over $\mathbb{Q}$ by a and η where a is primitive with $a^2 = -2(4k \pm 1)$, k a positive integer and η is an ample class of type $(1,k)$ orthogonal to a. Then there exists a* rank 2 *vector bundle E on X_K, stable with respect to η and having $c_1(E) = a,\ c_2(E) = \frac{1}{4}(a^2+2)$.*

Proof

We first prove the existence of a holomorphic rank 2 vector bundle E on X_K with the given invariants which is simple. We begin with the case $a^2 = -2(4k+1)$. Let A, L be line bundles on X_K having $c_1(A) = a,\ c_1(L) = \eta$. We have $\chi(A^{-1} \otimes L^{-2}) = \frac{1}{2}(a+2\eta)^2 = -1$ by Riemann-Roch hence $\mathrm{Ext}^1(X; L \otimes A,\ L^{-1}) \cong H^1(X_K,\ A^{-1} \otimes L^{-2}) \neq 0$ and there exists a nontrivial extension

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow A \otimes L \longrightarrow 0.$$

Notice that E has the required Chern classes. The fact the E is simple is implied by the vanishing of $\mathrm{Hom}(L^{-1},\ A \otimes L)$ and of $\mathrm{Hom}(A \otimes L,\ L^{-1})$ as one can easily check.

Let now $a^2 = -2(4k-1),\ k \geq 1$. As before we consider two line bundles A and L on X_K having $c_1(A) = a$ and $c_1(L) = \eta$. Since L is ample and $(2\eta + a)^2 = 2 > 0$, the bundle $L^2 \otimes A$ will have a nontrivial section vanishing on a divisor, say D. For numerical reasons D must have a reduced component. We may then choose a point p on the regular part of D, seen as a subvariety of X_K. Let Z be the reduced subspace of X_K consisting of the point p.

We want to construct E as the middle term of an extension

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow \mathcal{I}_Z \otimes A \otimes L \longrightarrow 0.$$

Such an extension is given by an element $\theta \in \mathrm{Ext}^1(X_K; \mathcal{I}_Z \otimes A \otimes L,\ L^{-1})$. By a criterion of Serre E is locally free if and only if the image of θ through the canonical mapping

$$\mathrm{Ext}^1(X_K;\ \mathcal{I}_Z \otimes A \otimes L,\ L^{-1}) \longrightarrow H^0(X_K, \mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L,\ L^{-1}))$$

generates the sheaf $\mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L,\ L^{-1}))$; cf. [3] I.5.

From the exact sequence of the first terms of the Ext spectral sequence.

$$\begin{aligned}
0 &\longrightarrow H^1(X_K;\mathcal{H}om(\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}))\\
&\longrightarrow \mathrm{Ext}^1(X_K;\,\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}) \longrightarrow H^0(X_K,\mathcal{E}xt^1(\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}))\\
&\longrightarrow H^2(X_K,\mathcal{H}om(\mathcal{I}_Z\otimes A\otimes L,\, L^{-1})) \longrightarrow \mathrm{Ext}^2(X_K;\,\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}),
\end{aligned}$$

we see that in our situation

$$\mathrm{Ext}^1(X_K,\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}) \longrightarrow H^0(X_K,\mathcal{E}xt^1(\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}))$$

is an isomorphism, and since

$$\mathcal{E}xt^1(\mathcal{I}_Z\otimes A\otimes L,\, L^{-1}) \cong \mathcal{O}_Z,$$

a non-zero θ will give a locally free middle term E.

Let then E be such a locally free sheaf. We shall prove that E is simple. The first term of the exact sequence

$$0 \longrightarrow \mathrm{Hom}(E,L^{-1}) \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{Hom}(E,\mathcal{I}_Z\otimes A\otimes L)$$

vanishes, so it will be enough to prove that

$$\mathrm{Hom}(E,\mathcal{I}_Z\otimes A\otimes L) \cong H^0(X_K,E^\vee\otimes\mathcal{I}_Z\otimes A\otimes L) \cong H^0(X_K,E\otimes\mathcal{I}_Z\otimes L)$$

is one-dimensional. But this holds since

$$H^0(X_K,\mathcal{I}_Z\otimes\mathcal{I}_Z\otimes L^2\otimes A) \cong H^0(X_K,\mathcal{I}_Z\otimes\mathcal{I}_Z(D))$$

is one-dimensional; (use the fact that p is a simple point on the regular part of D and that D does not move, so $\mathcal{I}_Z^2(D)$ can have no global section).

We now turn to the proof of the stability of E (in both cases).

Suppose E were not stable. There would exist then a locally free subsheaf B of rank 1 of E having $\deg_\eta B \geq 0$. Let

$$b := c_1(B) = s\eta + ta; \quad s,t\in\mathbb{Q}.$$

Since η and a are primitive, s and t must be either both in $\mathbb{Z}$ or both in $\frac{1}{2}\cdot\mathbb{Z}$. Moreover, $\deg B\geq 0$ is equivalent to $s\geq 0$. We may also suppose that E/B is torsion-free.

Thus there exists a 2-codimensional subspace Y of X_K such that E sits in an exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow \mathcal{I}_Y\otimes A\otimes B^{-1} \longrightarrow 0.$$

Hence

$$\frac{1}{4}(a^2+2) = c_2(E) = b(a-b) + c_2(\mathcal{I}_Y) = b(a-b) + h^0(\mathcal{O}_Y).$$

On the other side, since E is simple

$$0 = \mathrm{Hom}(\mathcal{I}_Y \otimes A \otimes B^{-1},\, B) = H^0(A^{-1} \otimes B^2)$$

and thus $(2b-a)^2 \leq 0$. This further gives $4h^0(\mathcal{O}_Y) \leq 2$, hence $Y = \emptyset$ and $(2b-a)^2 = -2$.

Using the two exact sequences we have for E and again the fact that E is simple we get

$$H^0(L \otimes A \otimes B^{-1}) \neq 0 \text{ and } H^0(L^{-1} \otimes A^{-1} \otimes B) = 0,$$

hence $s \in \{0, \frac{1}{2}\}$. But then b cannot fulfill the condition $(2b-a)^2 = -2$. □

Proof of Theorem 1.1

We start with a non-algebraic torus X_I, a primitive element $a \in NS(X_I)$ with $a^2 = -2(4k \pm 1)$, $k \geq 1$, and a differentiable vector bundle E on X_I with $c_1(E) = a$ and

$$\Delta(E) := \frac{1}{2}\Big(c_2(E) - \frac{c_1(E)^2}{4}\Big) = \frac{1}{4}.$$

We shall show that E admits a holomorphic structure.

Consider deformations X_J, X_K of X_I as in Lemma 3.2. Now Lemma 3.3 shows the existence over (X_K, η) of a stable holomorphic structure on E. By the Kobayashi-Hitchin correspondence we get an anti-self-dual connection on E with respect to the Riemannian metric which corresponds to η. The curvature of this connection remains of type (1,1) when considered on X_J, and thus gives a holomorphic structure E_J on E over X_J. Now

$$\frac{1}{4} = \Delta(E) < \frac{c_1(E)^2}{8} = k \pm \frac{1}{4}.$$

By [1] E_J must be irreducible (i.e. it will admit no coherent subsheaf of rank 1). Thus E_J is stable with respect to any polarization on X_J. In particular E_J has an anti-self-dual connection with respect to ω_J too.

By deforming back to X_I we obtain a holomorphic structure E_I on E over X_I in the same way as on X_J.

By Proposition 3.1 any rank 2 topological vector bundle with $c_1(E) = a$ and $\Delta(E) \geq 0$ will admit some holomorphic structure. □

4 Stability versus irreducibility

Irreducible vector bundles are stable with respect to any polarization on the base surface. In particular they admit anti-self-dual connections if their associated determinant bundles do.

In this paragraph we construct stable vector bundles of rank two on any polarized 2-dimensional torus X using the anti-self-dual connections which exist on irreducible vector bundles over some non-algebraic quaternionic deformation of X. In order to do this we first prove Theorem 1.3 which tells us which rank 2 vector bundles on a non-algebraic torus admit irreducible structures.

We start with a non-existence result.

Lemma 4.1 *Let X be a 2-dimensional complex torus, ω a Kähler class on it and $a \in NS(X)$. When $a = 0$ and $c \in \{0, 1\}$ or when $a^2 = -2$, $a \cdot \omega = 0$ and $c = 0$, there exists no stable (with respect to ω)* rank 2 *vector bundle E on X with $c_1(E) = a$ and $c_2(E) = c$.*

Proof

Suppose that E is a stable rank 2 vector bundle on X of degree zero with respect to ω. We consider its Fourier-Mukai transform:

Let $\mathcal{P}$ be the Poincaré line-bundle on $X \times \mathrm{Pic}^0(X)$ where $\mathrm{Pic}^0(X)$ denotes the variety of topologically trivial holomorphic line bundles on X. Let $p_1 : X \times \mathrm{Pic}^0(X) \to X$, $p_2 : X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ be the projections and $E^\wedge := R^1 p_{2*}(p_1^*(E) \otimes \mathcal{P})$. Since $H^0(X; E \otimes L)$ and $H^2(X; E \otimes L)$ vanish for all $L \in \mathrm{Pic}^0(X)$ it follows that $E^\wedge$ is locally free of rank $-\chi(E) = c_2(E) - \frac{1}{2}c_1(E)^2$.

When one computes the Chern classes of $E^\wedge$ with Grothendieck-Riemann-Roch, one gets:

$$\begin{aligned} \mathrm{rank}(E^\wedge) &= c_2(E) - \frac{1}{2}c_1(E)^2, \\ \mathrm{rank}\, E &= c_2(E^\wedge) - \frac{1}{2}c_1(E^\wedge)^2 \\ &= c_2(E^\wedge) - \frac{1}{2}c_1(E)^2, \end{aligned}$$

hence $c_2(E^\wedge) = 2 + \frac{1}{2}c_1(E)^2$.

Thus in the considered cases we get $\operatorname{rank}(E^\wedge) \leq 1$ and $c_2(E^\wedge) \neq 0$ which contradicts the locally freeness of $E^\wedge$. $\square$

Next we need a reformulation of part (b) of Lemma 3.2 which will ensure the existence of a convenient quaternionic deformation of I.

Lemma 4.2 *Let (X_I, ω_I) be a polarized 2-dimensional torus. There exists then a quaternionic deformation J of I such that $NS(X_J) \subset \Lambda^-$.*

The proof is the same as for part (b) of Lemma 1: just take ω_J on the sphere of radius $\sqrt{2}$ in Λ^+ and not on lines of the type $(\langle b\rangle \oplus \Lambda^-) \cap \Lambda^+$ with $b \in H^2(X, \mathbb{Z}) \setminus \Lambda^-$.

With these preparations the proofs of theorems 1.3 and 1.2 will follow from our Theorem 1.1 and the argument in [1] § 5 where irreducible vector bundles are found in the versal deformation of reducible ones provided that c_2 is sufficiently large.

More precisely, if X is a non-algebraic torus and E is a holomorphic rank 2 vector bundle on X with $c_1(E)$ primitive in $NS(X)$, then it is proved in [1] 5.10 that there exist irreducible vector bundles in the versal deformation of E provided

$$\Delta(E) \geq 1 + \frac{1}{8} c_1(E)^2.$$

Now let E be a topological vector bundle of rank 2 on a non-algebraic torus X with $c_1(E) \in NS(X)$ and $\Delta(E) \geq 0$. We investigate when E admits an irreducible holomorphic structure. We may suppose that $c_1(E)$ is a primitive element in $NS(X)$, otherwise we twist by a line bundle, etc. By Theorem 1.1 E admits a holomorphic structure which we denote also by E. By the above mentioned result from [1] we find an irreducible structure in the versal deformation of E as soon as

$$\Delta(E) \geq 1 + \frac{1}{8} c_1(E)^2.$$

On the other side it is easy to check that for

$$\Delta(E) < -\frac{1}{8} c_1(E)^2.$$

every holomorphic structure of E is irreducible (cf. [1]). It remains to consider the case when

$$-\frac{1}{8} c_1(E)^2 \leq \Delta(E) < 1 + \frac{1}{8} c_1^2(E).$$

But now

$$c_1(E)^2 = 0 \text{ and } c_2(E) \in \{0,1\} \text{ or } c_1(E)^2 = -2 \text{ and } c_2(E) = 0.$$

By Lemma 3.2 there is a Kähler class ω on X such that $c_1(E) \cdot \omega = 0$ and by Lemma 4.1 E cannot be stable with respect to ω. Thus this is exactly the case when E cannot admit irreducible structures. The proof of theorem 1.3 is completed.

We can now prove Theorem 1.2. Let X_I be a complex torus and ω_I a Kähler class on it. Let $a \in NS(X_I)$ such that $a \cdot \omega_I = 0$ and

$$a^2 = \max\{(a+2b)^2 \mid b \in NS(X),\ b \cdot \omega_I = 0\}.$$

(Recall that the intersection form is negative definite on the orthogonal of ω_I in $H^{1,1}$).

Let $c \in \mathbb{Z}$ be such that

$$\begin{aligned} c &\geq 2 \text{ if } a = 0, \\ c &\geq 1 \text{ if } a^2 = -2 \text{ and} \\ c &\geq \frac{a^2}{4} \text{ if } a^2 \leq -4. \end{aligned}$$

We have to show that stable rank 2 vector bundles E with $c_1(E) = a$ and $c_2(E) = c$ exist on (X_I, ω_I). (The other implication follows from Lemma 4.1 already.)

For this we consider a quaternionic deformation J as in Lemma 4.2. X_J is non-algebraic and $\Lambda_I^- = \Lambda_J^-$. Hence

$$\begin{aligned} a^2 &= \max\{(a+2b)^2 \mid b \in NS(X_I),\ b \cdot \omega_I = 0\} \\ &= \max\{(a+2b)^2 \mid b \in NS(X_J)\} \end{aligned}$$

and by Theorem 1.3 there exists an irreducible rank 2 vector bundle E_J on X_J with $c_1(E_J) = a,\ c_2(E_J) = c$. In particular E_J is stable with respect to ω_J and thus admits an anti-self-dual connection. This connection remains anti-self-dual on (X_I, ω_I), as the Riemannian structure does not change, and induces a holomorphic structure E_I over X_I on the underlying differentiable bundle. By the Kobayashi-Hitchin correspondence E_I is stable or splits into a sum of line bundles of degree zero which are invariant under the connection. In the last case we should get a splitting of E_J as a direct sum of holomorphic line bundles on X_J, which is not the case. Thus E_I is stable and Theorem 1.2 is proved.

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