Analytic Geometry/ Géométrie analytique

Stable bundles on non-algebraic surfaces giving rise to compact moduli spaces

Matei Toma

Abstract

We prove the existence of a class of holomorphic vector bundles on some non-algebraic surfaces, especially on tori and primary Kodaira surfaces. This entails the existence of non-empty moduli spaces of stable bundles over non-Kähler surfaces which admit compactifications as constructed in [4].

Fibrés vectoriels stables sur les surfaces non-algébriques engendrant des espaces de modules compacts

Résumé

Nous montrons l'existence d'une certaine classe de fibrés vectoriels holomorphes sur des surfaces non-algébriques, notamment sur les tores et sur les surfaces de Kodaira primaires. On en déduit l'existence des espaces non-vides de modules des fibrés stables sur des surfaces non-kählériennes qui admettent les compactifications construites dans [4].

Version française abrégée

Soient X une surface compacte complexe, ω la forme de K"ahler d'une métrique de Gauduchon sur X, r un entier supérieur à 1, L un élément dans $Pic(X)$ et c un entier arbitraire. Il existe alors un espace de modules $M^\omega(r, L, c)$ des classes d'isomorphisme des fibrés vectoriels sur X stables par rapport à ω, de rang r, déterminant L et seconde classe de Chern c. Lorsque ω est une forme de Hodge on peut compactifier $M^\omega(r, L, c)$ d'une façon naturelle. Pour ω arbitraire Buchdahl propose dans [4] une autre compactification dans le cas $b_1(X)$ pair et $r = 2$ ou bien lorsque $(r, c_1(L), c)$ vérifie la condition suivante :

(*) tout fibré vectoriel E semistable sur X ayant $rang(E) = r$, $c_1(E) = c_1(L)$ et $c_2(E) \leq c$ est stable.

Lorsque $b_1(X)$ est impair (*) equivaut à

(**) tout faisceau cohérent sans torsion $\mathcal{F}$ sur X ayant $rang(\mathcal{F}) = r$, $c_1(\mathcal{F}) = c_1(L)$ et $c_2(\mathcal{F}) \leq c$ est **irréductible** (c.-à-d. $\mathcal{F}$ n'admet pas de sous-faisceau cohérent en rang intermédiaire).

On appellera un faisceau sans torsion $\mathcal{F}$ dont le rang et classes de Chern vérifient (**) **stablement irréductible**.

Le but de cette note est de montrer que de tels faisceaux existent sur des surfaces non-kählériennes. En particulier certains espaces de modules de fibrés stables sur des surfaces non-kählériennes qui admettent des compactifications naturelles seront non-vides.

Plus précisément on a

Théorème. Tout fibré vectoriel topologique (complexe) E sur un tore complexe 2-dimensionnel ou sur une surface de Kodaira primaire ayant $c_1(E) \in NS(X)$ et discriminant entier non-négatif admet une structure holomorphe.

Ici le **discriminant** de E est par définition le nombre rationnel

$$\triangle(E) := c_2(E) - \frac{rang(E) - 1}{2 \cdot rang(E)} \cdot c_1^2(E).$$

Corollaire. Soit X un tore complexe 2-dimensionnel ou une surface de Kodaira primaire et $NS(X)/TorsNS(X)$ soit libre engendré par un element $a \in NS(X)$ tel que $a^2 = -2rn$ pour des entiers positifs r et n, $r > 1$. Alors les fibrés vectoriels topologiques sur X de $rang$ r, première classe de Chern a et discriminant $\triangle \in [0, \frac{n}{r-1}[$ admettent des structures holomorphes stablement irréductibles.

Pour la démonstration du Théorème on construit d'abord des fibrés vectoriels holomorphes à discriminant nul. Ces fibrés seront des images directes des faisceaux inversibles par des recouvrements non-ramifiés de la base. Ensuite on fait croître leur discriminant en leur ajoutant des singularités et en les déformant en des faisceaux localement libres.

1 Introduction

Any hermitian metric on a compact complex surface X is conformally equivalent to a Gauduchon metric (i.e. such that its Kähler form ω is $\partial\overline{\partial}$-closed). With respect to such an ω one may define the degree and hence (slope-)stability for holomorphic vector bundles on X. Isomorphism classes of stable bundles E with fixed rank, $rank(E) = r > 1$, determinant, $\det E = L$, and second Chern class, $c_2(E) = c$, are known to admit complex moduli spaces $M^\omega(r, L, c)$; cf.[6].

When ω is a Hodge form on X, $M^\omega(r, L, c)$ admits a natural compactification, [5]. A different compactification for arbitrary polarizations ω was constructed by Buchdahl in [4] when $b_1(X)$ even and $r = 2$ or when $(r, c_1(L), c)$ satisfies the following condition :

(*) every semistable bundle E on X with $rank(E) = r$, $c_1(E) = c_1(L)$ and $c_2(E) \leq c$ is stable.

As remarked in [4], this condition is satisfied when $b_1(X)$ is even, $c_1(L)$ is not a torsion class in $H^2(X, \mathbb{Z}_r)$ and ω is generic.

For $b_1(X)$ odd however, (*) is a much stronger restriction :

Remark. When $b_1(X)$ is odd (*) is equivalent to

(**) every torsion-free coherent sheaf $\mathcal{F}$ on X with $rank(\mathcal{F}) = r$, $c_1(\mathcal{F}) = c_1(L)$ and $c_2(\mathcal{F}) \leq c$ is **irreducible** (i.e. it does not admit coherent subsheaves of intermediate rank).

It is clear that (**) implies (*). For the converse take a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$ by coherent subsheaves such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ are torsion-free and irreducible for $1 \leq i \leq m$. For $b_1(X)$ odd $\deg_\omega : Pic^0(X) \to \mathbb{R}$ is surjective, so suitable twists by line bundles $\mathcal{L}_i$ from $Pic^0(X)$ yield a semistable vector bundle $\bigoplus_{i=1}^m ((\mathcal{F}_i/\mathcal{F}_{i-1})^{\vee\vee} \otimes L_i)$. This vector bundle is unstable precisely when $m > 1$, i.e. when $\mathcal{F}$ is reducible. □

We shall call a torsion-free coherent sheaf $\mathcal{F}$ **stably irreducible** if $(rank(\mathcal{F}), c_1(\mathcal{F}), c_2(\mathcal{F}))$ satisfies (**).

The only general existence result for holomorphic vector bundles on non-algebraic surfaces covers only the case of filtrable holomorphic structures ([1]). Thus it is a priori not clear whether stably irreducible bundles exist. They don't for instance when $b_2(X) = 0$. (Use the fact that on non-algebraic surfaces $2rc_2(E) - (r-1)c_1^2(E) \geq 0$ and remark that when $b_1(X)$ is odd (*) implies $c < 0$).

In this note we prove that a certain class of topological vector bundles over non-algebraic surfaces, especially over tori and over primary Kodaira surfaces, admit holomorphic structures. This will provide examples of stably irreducible vector bundles over primary Kodaira surfaces. (Some examples have been known only in the case of tori, [7]).

In a forthcoming paper we shall show that, under the condition (**), $M^\omega(r, L, c)$ admits a different compactification, more natural from the view-point of analytic geometry. This will be the moduli space of simple sheaves on X with invariants (r, L, c). We thus obtain examples of compact moduli spaces of stable sheaves over non-Kähler surfaces.

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Our main existence result is

Theorem Any topological (complex) vector bundle E on a 2-dimensional torus or on a primary Kodaira surface X with $c_1(E) \in NS(X)$ and non-negative integer discriminant admits some holomorphic structure.

We define the **discriminant** of E to be the rational number

$$\triangle(E) := c_2(E) - \frac{(rank(E) - 1)}{2 \cdot rank(E)} \cdot c_1^2(E).$$

The theorem is a consequence of the following Propositions :

Proposition 1. Any topological vector bundle E on a 2-dimensional torus or on a primary Kodaira surface X with $c_1(E) \in NS(X)$ and $\triangle(E) = 0$ admits some holomorphic structure.

Proposition 2. Let X be a compact complex surface with $kod(X) = -\infty$ or with $kod(X) = 0$ and $p_g(X) = 1$. Let E be a holomorphic vector bundle on X whose rank exceeds 1 and n

a positive integer. Excepting the case when X is K3, $a(X) = 0$, E is a twist of the trivial bundle by some line bundle and $n = 1$, there exists a holomorphic vector bundle F on X with $rank(F) = rank(E)$, $c_1(F) = c_1(E)$ and $c_2(F) = c_2(E) + n$.

For the proof of Proposition 1 in case X is a torus see [7], [8]. In the same way, by taking direct images of invertible sheaves through unramified covering maps, we prove the other case once we have :

Lemma. Let X be a primary Kodaira surface, $a \in NS(X)$ and p a prime divisor of $\frac{1}{2}a^2$. There exists then an unramified covering $q : X' \to X$ of degree p and an element $a' \in NS(X')$ such that

$$p \cdot a' = q^*(a).$$

Proof of the Lemma. A primary Kodaira surface X is a topologically nontrivial elliptic bundle over an elliptic curve B. We denote by $f : X \to B$ the projection and by F its fiber. Applying the Leray spectral sequence to the locally constant sheaf $\mathbb{Z}_X$, one obtains an exact sequence

$$0 \to H^2(B,\mathbb{Z}) \xrightarrow{\cdot m} H^2(B,\mathbb{Z}) \xrightarrow{f^*} H^2(X,\mathbb{Z}) \to H^1(F,\mathbb{Z}) \otimes H^1(B,\mathbb{Z}) \to 0$$

for some positive integer m ; cf. [2]. Choosing positively oriented frames for $H^1(F,\mathbb{Z})$ and $H^1(B,\mathbb{Z})$ and representing the image of an element $a \in H^2(X,\mathbb{Z})$ in $H^1(F,\mathbb{Z}) \otimes H^1(B,\mathbb{Z})$ by a 2×2-matrix A we see that the intersection form computes as

$$a^2 = -2 \det A.$$

When $2p$ divides a^2 we can choose a frame for $H^1(B,\mathbb{Z})$ such that a column of A is a multiple of p. The other column will become divisible by p too after a suitable covering $q : B' \to B$ (given by the obvious sublattice of $H_1(B,\mathbb{Z})$). Letting $X' := X \times_B B'$ and denoting by $q_X : X' \to X$ the projection we obtain an element $a' \in H^2(X',\mathbb{Z})$ such that $q_X^*(a) - p \cdot a' = q_X^*(a - (q_X)_! a')$ is torsion. But the torsion classes in $H^2(X',\mathbb{Z})$ coming from $H^2(X,\mathbb{Z})$ are divisible by p. Thus $q_X^*(a) = p \cdot a''$ for some $a'' \in H^2(X',\mathbb{Z})$, and a'' is in $NS(X')$ when a is in $NS(X)$. □

Proof of Proposition 2. We may assume that for fixed rank and first Chern class E is chosen with minimal second Chern class.

Consider a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$ by coherent subsheaves such that E_i/E_{i-1} are irreducible of positive ranks r_i, $1 \leq i \leq m$. From the minimality assumption it follows that E_i/E_{i-1} are locally free.

When all r_i equal 1 the assertion follows from [1].

If some $r_{i_0} > 1$ we first perform surgery on E_{i_0}/E_{i_0-1} to obtain a simple coherent sheaf $\mathcal{F}_0$ with $c_2(\mathcal{F}_0) = c_2(E_{i_0}/E_{i_0-1}) + n$ by gluing singularities at $x_1, \ldots, x_n \in X$ which locally look like $\mathcal{I}_{x_j} \oplus \mathcal{O}^{r_{i_0}-1}$. Our assumptions on X imply that for any simple sheaf $\mathcal{G}$, $H^2(X, End_0(\mathcal{G}))$ vanishes (cf. [1], 5.11), and thus the moduli spaces of simple sheaves on X are smooth of the expected dimensions (or empty). By a dimension count it follows that some deformation $\mathcal{F}$ of $\mathcal{F}_0$ is locally free. Now $\bigoplus_{i \neq i_0} (E_i/E_{i-1}) \oplus \mathcal{F}$ gives the looked for holomorphic vector bundle. □

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We now apply the Theorem in order to prove existence for stably irreducible bundles. For a non-algebraic surface X, $a \in NS(X)$ and r a positive integer, set

$$\begin{aligned} s(r,a) &:= -\frac{1}{2r} \sup_{\mu \in NS(X)} (a - r\mu)^2 \\ t(r,a) &:= \inf_{1 \le k \le r-1} \frac{s(r,ka)}{k(r-k)}. \end{aligned}$$

Remark 1. (cf. [9], II. 1.3) A holomorphic vector bundle E on a non-algebraic surface having $\triangle(E) < t(rank(E), c_1(E))$ is stably irreducible.

Corollary. Suppose X is a complex 2-torus or a primary Kodaira surface and $NS(X)/Tors(NS(X))$ is freely generated by some $a \in NS(X)$ with $a^2 = -2rn$ for positive integers r and n, $r > 1$. Then the topological vector bundles of rank r, first Chern class a and discriminant $\triangle$ with $0 \le \triangle < \frac{n}{r-1}$ admit holomorphic stably irreducible structures.

Remark 2. For any positive integers n, r both tori and Kodaira surfaces as in the Corollary exist. (See [9] I. 1.3 for examples of tori and use the explicit description of $NS(X)/Tors(NS(X))$ from [3] in the case of Kodaira surfaces).

Remark 3. The moduli spaces $M^\omega(r, L, c)$ in the situation of the Corollary are smooth of dimension $2r\triangle$ and independent of the polarization ω.

Références

[1] C. Bănică and J. Le Potier, Sur l'existence des fibrés vectoriels holomorphes sur les surfaces non algébriques, J. reine angew. Math. 378, (1987), 1-31.

[2] V. Brînzănescu, Neron-Severi group for nonalgebraic elliptic surfaces I, manuscripta math. 79, (1993), 187-195.

[3] V. Brînzănescu, The Picard group of a primary Kodaira surface, Math. Ann. 296, (1993), 725-738.

[4] N.P. Buchdahl, Blowups and gauge fields, preprint 1995.

[5] D. Gieseker, On the moduli of vector bundles on an algebraic surface, Ann. Math. 106, (1977), 45-60.

[6] M. Lübke and A. Teleman, The Kobayashi-Hitchin correspondence, World Scientific, Singapore, 1995.

[7] M. Toma, Une classe de fibrés vectoriels holomorphes sur les 2-tores complexes, C.R. Acad. Sci. Paris 311, Série I, (1990), 257-258.

[8] M. Toma, A class of holomorphic vector bundles on two-dimensional tori, Rev. Roumaine Math. Pures Appl. 36, (1991), 309-317.

[9] M. Toma, Holomorphic vector bundles on non-algebraic surfaces, Dissertation, Bayreuth, 1992.

Universität Osnabrück, Fachbereich Mathematik/Informatik, 49069 Osnabrück, Germany and

Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, RO-70700 Bucharest, Romania.