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Moduli of Kähler Manifolds Equipped with Hermite-Einstein Vector Bundles

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Dedicated to the memory of Constantin Bănică and Manuela Stoia

Introduction

Irreducible Hermite-Einstein (or equivalently, stable) holomorphic vector bundles over a Kähler compact manifold admit analytic moduli spaces (cf. [I1],[Kim], [Kob], [L-O], [F-S1],...). These were endowed with a natural Kähler metric by Atiyah and Bott ([A-B]) when the base is a Riemann-surface, by Itoh ([I2]) for a 2-dimensional base, and in higher dimensions by Kim ([Kim]) and Kobayashi ([Kob]).

On the other side for polarized non-uniruled compact Kähler manifolds moduli spaces have been constructed in [F2] and [S1]. Generalizing the Petersson-Weil metric on the Teichmüller space Koiso introduced in [K] a Kähler metric on the moduli space of Kähler-Einstein manifolds.

In this paper we consider the same problems for pairs of polarized compact Kähler manifolds and stable holomorphic vector bundles (see the main text for the definitions). In fact we prove:

Theorem 1 *Non-uniruled polarized Kähler manifolds equipped with isomorphism classes of stable holomorphic bundles admit coarse moduli spaces.*

Theorem 2 *There exist a natural Kähler metric (Petersson-Weil metric) on the regular part of the moduli space of polarized pairs of Kähler-Einstein manifolds (under extraconditions when $c_1 > 0$) and projectively flat vector bundles.*

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1 Deformation theory

Definitions We denote by $(X, \widetilde{E})$ a pair consisting of a compact complex manifold of dimension n, X, and a holomorphic vector bundle on X of rank r, $\widetilde{E}$.

An *isomorphism of the pairs* $(X, \widetilde{E})$, $(X', \widetilde{E}')$ is a pair of maps (φ, Φ) such that $\varphi : X \to X'$ is an isomorphism of complex manifolds and $\Phi : \widetilde{E} \to \widetilde{E}'$ is an isomorphism of holomorphic vector bundles over φ, i.e. we have a commutative diagram

$$\begin{array}{ccc} \widetilde{E} & \stackrel{\widetilde{\Phi}}{\longrightarrow} & \widetilde{E}' \\ \downarrow & & \downarrow \\ X & \stackrel{\widetilde{\varphi}}{\longrightarrow} & X' \end{array}$$

The notation E will be used for the isomorphism class of $\widetilde{E}$.

A *family* of pairs over a reduced analytic space with distinguished point $(S, 0)$ is a pair $(\mathcal{X}, \widetilde{\mathcal{E}})$ consisting of a reduced complex space $\mathcal{X}$ and a holomorphic vector bundle $\widetilde{\mathcal{E}}$ on $\mathcal{X}$ together with a smooth (i.e. open and with smooth fibers) morphism $f : \mathcal{X} \to S$. If its central fiber $(f^{-1}(0), \widetilde{\mathcal{E}} \mid_{f^{-1}(0)})$ is isomorphic to a given pair $(X, \widetilde{E})$ then $(\mathcal{X}, \widetilde{\mathcal{E}}, S, 0)$ will be also called a deformation of $(X, \widetilde{E})$. We'll denote this situation by a diagram

$$\begin{array}{cccl} \widetilde{E} & \longrightarrow & \widetilde{\mathcal{E}} & \\ \downarrow & & \downarrow & \\ X & \longrightarrow & \mathcal{X} & \\ \downarrow & & \downarrow & f \\ 0 & \longrightarrow & S & \end{array}$$

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1.1 Infinitesimal theory

Let Σ_X denote the sheaf of infinitesimal automorphisms of the pair $(X, \widetilde{E})$. It is the middle term of the Atiyah sequence on X:

$$0 \longrightarrow \mathcal{E}nd(\widetilde{E}) \longrightarrow \Sigma_X \longrightarrow \Theta_X \longrightarrow 0 \qquad (1)$$

where Θ_X is the holomorphic tangent bundle of X (cf. [A; thm.1]). This can be described as follows.

Let V be an open subset of X, and $\mathcal{U} = \{\mathcal{U}_i\}_i$ an open covering of X such that $\widetilde{E} \mid_{\mathcal{U}_i}$ are trivial.

Let $f_i : \widetilde{E} \mid_{\mathcal{U}_i} \to \mathcal{O}^r_{\mathcal{U}_i}$ be trivializations and $\gamma_{ij} = f_i f_j^{-1} \mid_{\mathcal{U}_{ij}}$ corresponding transition functions. Then an element in $\Sigma_X(V)$ is given by a family of pairs $((\lambda_i), v)_i \in \mathcal{E}nd(\mathcal{O}^r_{\mathcal{U}_i})(V \cap \mathcal{U}_i) \times \Theta_X(V)$ such that

$$\lambda_j = \gamma_{ij}^{-1} \lambda_i \gamma_{ij} + \gamma_{ij}^{-1}(d\gamma_{ij} \cup v) \qquad (2)$$

where $\cup$ denotes contraction.

A similar law holds for elements in $\mathcal{A}^{p,q}(V, \Sigma_X)$.

A C^∞-connection of E gives a C^∞ splitting of (1) since in the given trivializations it is expressed by a family $(\omega^{(i)})_i \in \mathcal{A}^1(\mathcal{E}nd(\mathcal{O}^r_{\mathcal{U}_i}))$ satisfying

$$\omega^{(j)} = \gamma_{ij}^{-1}\omega^{(i)}\gamma_{ij} + \gamma_{ij}^{-1}d\gamma_{ij}.$$

Thus we associate to a given element $((\lambda_i), v)_i \in \mathcal{A}^{p,q}(V, \Sigma)$ as above (i.e. $\lambda_i \in \mathcal{A}^{p,q}(V, \mathcal{E}nd(\mathcal{O}^r_{\mathcal{U}_i})(V \cap \mathcal{U}_i))$, $v \in \mathcal{A}^{p,q}(V, \Theta_X)$ such that (2) holds) a pair $(\mu, v) \in \mathcal{A}^{p,q}(V, \mathcal{E}nd(E)) \times \mathcal{A}^{p,q}(V, \Theta_X))$ where $\mu_i := f_i \mu f_i^{-1}$ satisfy

$$\mu_i = \lambda_i - \omega^{(i)} \cup v \tag{3}$$

Conversely one can start with (μ, v) satisfying (3) to get $((\lambda_i), v)$ with (2).

The element $(f_j^{-1} d\gamma_{ij} f_i)_{ij} \in Z^1(\mathcal{U}, \Omega^1(\mathcal{E}nd(\widetilde{E})))$ gives a cohomology class in $H^{1,1}(X, \mathcal{E}nd(\widetilde{E}))$ which we denote by $\Omega(E)$. Now (2) implies

Proposition 1 *a) The equivalence class of the extension (1) is* $\Omega(E) \in H^1(X, \Omega^1(\mathcal{E}nd(\widetilde{E}))) \cong \mathrm{Ext}^1(X; \Theta_X, \mathcal{E}nd(\widetilde{E}))$.

b) The edge homomorphisms of the long exact sequence associated to (1),

$$\delta^q : H^q(X, \Theta_X) \longrightarrow H^{q+1}(X, \mathcal{E}nd(\widetilde{E}))$$

are given by the cup product with $\Omega(E)$.

Consider now a deformation situation:

$$\begin{array}{ccc} \widetilde{E} & \longrightarrow & \widetilde{\mathcal{E}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & S \end{array}$$

and define $\Sigma_{\mathcal{X}/S} := \mathrm{Ker}(\Sigma_{\mathcal{X}} \to f^*\Theta_S)$. Then we get a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{E}nd(\widetilde{\mathcal{E}}) & = & \mathcal{E}nd(\widetilde{\mathcal{E}}) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Sigma_{\mathcal{X}/S} & \longrightarrow & \Sigma_{\mathcal{X}} & \longrightarrow & f^*\Theta_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Theta_{\mathcal{X}/S} & \longrightarrow & \Theta_{\mathcal{X}} & \longrightarrow & f^*\Theta_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{4}$$

We can describe now the Kodaira-Spencer map

$$\rho : T_0S \longrightarrow H^1(X, \Sigma_X)$$

in the usual way: take a local section of Θ_S, lift it to $\Theta_{\mathcal{X}}$ (when dealing with Kähler-Einstein manifolds this will be done in a canonical way, see §3), then to $\Sigma_{\mathcal{X}}$ differentiably in the middle column of (4), apply $\bar{\partial}$ and get a closed form in $\mathcal{A}^{0,1}(\Sigma_{\mathcal{X}/S})$ which restricted to X gives us the wanted cohomology class in $H^1(X, \Sigma_X)$. (Thus ρ is the restriction to X of the natural map $f_*f^*\Theta_S \longrightarrow R^1 f_*\Sigma_{\mathcal{X}/S}$). For the space of infinitesimal deformations of $(X, \widetilde{E})$, $H^1(X, \Sigma_X)$, we have:

$$0 \longrightarrow H^1(X, \mathcal{E}nd(E))/H^0(X, \Theta_X) \longrightarrow H^1(X, \Sigma_X) \longrightarrow$$

$$\longrightarrow \mathrm{Ker}(\delta^1 : H^1(X, \Theta_X) \longrightarrow H^2(X, \mathcal{E}nd(E))) \longrightarrow 0$$

where $H^1(X, \mathcal{E}nd(E))/H^0(X, \Theta_X)$ describes the infinitesimal deformation space of E modulo $\mathrm{Aut}(X)$ and $\mathrm{Ker}\delta^1$ the space of infinitesimal deformations of X which come from deformations of the pair (X, E).

For the automorphism space of the pair $(X, \widetilde{E})$, $H^0(X, \Sigma_X)$, we have:

$$0 \to H^0(X, \mathcal{E}nd(E)) \to H^0(X, \Sigma_X) \to \mathrm{Ker}(\delta^0 : H^0(X, \Theta_X) \to H^1(X, \mathcal{E}nd(E))) \to 0.$$

In paticular this shows that if E is simple and $H^0(X, \Theta_X) = 0$ then $h^0(X, \Sigma_X)$ is constant in a family hence any versal deformation of the pair $(X, \widetilde{E})$ is universal since the Schlessinger conditions are satisfied in this case. æ

1.2 Existence of versal deformations for pairs $(X, \widetilde{E})$

Although this has been proved in [B] and [S-T] we give here an argument based on results of Flenner.

For a given pair $(X, \widetilde{E})$ we consider the trivial extension space $Y := X[\widetilde{E}]$. Then there exists a versal deformation of the sequence of natural maps $X \to Y \to X$, (cf. [F]), which we represent by

$$\begin{array}{ccccc} & X & \longrightarrow & X_1 & \\ \alpha & \downarrow & & \downarrow & \alpha_1 \\ & Y & \longrightarrow & Y_1 & \\ \beta & \downarrow & & \downarrow & \beta_1 \\ & X & \longrightarrow & X_1 & \\ & \downarrow & & \downarrow & \\ & 0 & \longrightarrow & S_1 & \end{array}$$

The equation $(\beta_1 \circ \alpha_1) \mid_{X_{1,s}} = \mathrm{id}_{X_{1,s}}$ determines a subspace $S_2 \subset S_1$ and an induced diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X_2 \\
\downarrow & & \downarrow \alpha_2 \\
Y & \longrightarrow & Y_2 \\
\downarrow & & \downarrow \beta_2 \\
X & \longrightarrow & X_2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S_2
\end{array}
$$

for which $\beta_2 \circ \alpha_2 = \mathrm{id}$. Now if Z is a subspace of a complex space Q such that $\mathcal{J}_Z^2 = 0$ then there is an extension of sheaves of $\mathbf{C}$-algebras:

$$0 \longrightarrow \mathcal{J}_Z \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and this is trivial if and only if there is a morphism $\sigma : Q \to Z$ such that $\sigma \mid_Z = \mathrm{id}_Z$. So we look at the subspace S of S_2 where $\mathcal{J}_{X_2}^2 \mid_{Y_s} = 0$ for $s \in S$. This exists and has the natural universal property by [P], Prop. 1.

We get in the end

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Y = X[\widetilde{E}] & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S
\end{array}
$$

with $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}/\mathcal{J}$ and $\mathcal{J}^2 = 0$, hence $\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}[\widetilde{\mathcal{E}}]$ is the trivial extension of a coherent sheaf $\widetilde{\mathcal{E}}$ which can be assumed locally free for S small. Thus

$$
\begin{array}{ccc}
X[\widetilde{E}] & \longrightarrow & \mathcal{X}[\widetilde{\mathcal{E}}] \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S
\end{array}
$$

gives us the versal deformation of the pair $(X, \widetilde{E})$. æ

2 Families of pairs of polarized compact Kähler manifolds and stable vector bundles

Definitions We consider pairs (X, E) consisting of polarized compact Kähler manifolds (X, λ), (i.e. X is a Kähler manifold and λ is a fixed Kähler class in $H^2(X, \mathbf{R})$), and isomorphism classes E of stable vector bundles on X with respect to this polarization. Recall that E is *stable* with respect to λ if for any coherent subsheaf $\mathcal{F} \subset E$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} E$ we have

$$\frac{c_1(\mathcal{F}) \cup \lambda^{n-1}}{\operatorname{rank} \mathcal{F}} < \frac{c_1(E) \cup \lambda^{n-1}}{\operatorname{rank} E},$$

where $n = \dim X$. From now on we shall not mention the polarization λ when it is understood. An isomorphism of such pairs (X,E), (X',E') comes down to an isomorphism of polarized Kähler manifolds $\varphi : (X,\lambda) \to (X',\lambda')$, $(\varphi^*\lambda' = \lambda)$, such that $\varphi^*E' = E$. Hence in the moduli space of the pairs, $(X,\widetilde{E})$ and $(X,\varphi^*\widetilde{E})$ will be identified for $\varphi \in \mathrm{Aut}(X,\lambda)$.

A holomorphic *family* of pairs of polarized compact Kähler manifolds and isomorphism classes of stable vector bundles will be a family $(\mathcal{X},\widetilde{\mathcal{E}},S,0)$ as in §1 together with the prescription of an element $\widetilde{\lambda} \in R^2f_*\mathbf{R}$ such that all restrictions $\widetilde{\lambda} \mid_{X_s} \in H^2(X_s,\mathbf{R})$ are Kähler classes on X_s (thus $(\mathcal{X} \to S, \widetilde{\lambda})$ is a *polarized family*) and that $\widetilde{E}_s$ are stable with respect to λ_s.

For S small around 0, $\widetilde{\lambda}$ is constant and moreover there exists a locally $\partial\bar{\partial}$-exact real $(1,1)$-form $\omega_{\mathcal{X}}$ such that its restriction to each fiber $\omega_s := \omega_{\mathcal{X}} \mid_{\mathcal{X}_s}$ are Kähler forms of class λ_s (see [F-S2]).

An isomorphism of families of pairs as above over S is given in a natural way by a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{E}} & \longrightarrow & \widetilde{\mathcal{E}}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}\,' \\ \searrow & & \swarrow \\ & S & \end{array}$$

which also preserves the polarizations.

In order to get the existence of versal deformations for pairs of polarized compact Kähler manifolds and stable vector bundles out of that for pairs as in §1 we need the following stability property.

Theorem 3 *Let $(\mathcal{X} \to S, \widetilde{\lambda})$ be a polarized family of compact Kähler manifolds and $\mathcal{E}$ a holomorphic vector bundle on $\mathcal{X}$ such that $\mathcal{E}_0 := \mathcal{E} \mid_{\mathcal{X}_0}$ is stable for $0 \in S$. Then there is a neighbourhood $\mathcal{U}$ of 0 in S such that for all $s \in \mathcal{U}, \mathcal{E}_s := \mathcal{E} \mid \mathcal{X}_s$ are stable.*

For the proof we shall rephrase Theorem 3 in terms of Hermite-Einstein vector bundles.

Definition If X is a compact Kähler manifold with Kähler metric g and (E,h) a holomorphic Hermitian vector bundle with Ω and R associated curvature form and tensor of the Hermitian connection, then (E,h) is called *Hermite-Einstein* if

$$\sqrt{-1}\Lambda\Omega = c \cdot \mathrm{Id}_E \tag{5}$$

i.e.

$$g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} = c \cdot \mathrm{Id}_E$$

with respect to local coordinates (z^α) on X, where c is a real constant; (the sumation convention is used). If, moreover, (E,h) admits no orthogonal holomorphic direct sum decomposition it is called *irreducible Hermite-Einstein.*

By the solution of the Kobayashi-Hitchin conjecture there is a 1-1 correspondence between stable and irreducible Hermite-Einstein bundles. Thus we get the following equivalent statement:

Theorem 4 *Let $(\mathcal{X} \to S, \omega_{\mathcal{X}})$ be a holomorphic family of compact complex manifolds with $\omega_{\mathcal{X}}$ a locally $\partial\bar{\partial}$-exact $(1,1)$-form such that $\omega_{\mathcal{X}} \mid_{\mathcal{X}_s}$ is a Kähler form on $\mathcal{X}_s$ for all $s \in S$. Let $\mathcal{E}$ be a holomorphic vector bundle on $\mathcal{X}$ such that $\mathcal{E}_0$ admits some metric h_0 making it an irreducible Hermite-Einstein bundle for $0 \in S$. Then there exist a neighbourhood $\mathcal{U}$ of 0 in S such that for all $s \in \mathcal{U}$, $\mathcal{E}_s$ are irreducible Hermite-Einstein for suitable metrics h_s.*

Proof. We can restrict ourselves to the case when S is smooth by passing to a resolution of singularities for instance.

Let $A''(s) : \mathcal{A}^0(\mathcal{E}_s) \to \mathcal{A}^{0,1}(\mathcal{E}_s)$ be a family of integrable semiconnections on the underlying C^∞-complex vector bundle E corresponding to the holomorphic structures $\mathcal{E}_s$ (see [Kob] for the definitions). One can get such a family by extending the metric h_0 smoothly to the whole $\mathcal{E}$ and then considering the induced Hermitian connections on $(\mathcal{E}_s, h_s)$. We denote by $\Omega(A''(s)) \in \mathcal{A}^{1,1}(\mathcal{X}_s, \mathcal{E}nd(\mathcal{E}_s))$ the associated curvature forms of these connections.

Assume $A''(0)$ is irreducible Hermite-Einstein. Then $\mathcal{E}_0$ is simple i.e. any holomorphic endomorphism is a constant multiple of the identity. By semicontinuity this is also true in the neighbouring fibers. So it will be enough to show that the Einstein condition (5) holds for suitable connections on $\mathcal{E}_s$.

The idea of finding such connections is standard: one replaces the given connections using the action of the gauge group. Let $GL(E)$ be the group of C^∞ vector bundle automorphisms of E. $GL(E)$ acts on the space of integrable connections

$(A'', f) \mapsto A''^f := f^{-1} \circ A'' \circ f = A'' + f^{-1}\bar{\partial}_{A''} f$

such that two integrable connections induce isomorphic holomorphic structures on E if and only if they lie in the same orbit of $GL(E)$.

Consider the subgroup of $GL(E)$ consisting of constant multiples of the identity and let $G := GL(E)/\mathbf{C}^* \cdot \mathrm{id}$. We get an action of G on the space of connections.

Take the map

$$\begin{array}{rccc} \Phi & : & G \times S & \longrightarrow & \mathcal{A}_0^0(X, \mathcal{E}nd(E)) \\ & & (f, s) & \longmapsto & \sqrt{-1}\Lambda_s \Omega(A''(s)^f) - c \cdot \mathrm{id} \end{array}$$

where

$$\mathcal{A}_0^0(X, \mathcal{E}nd(E)) = \{\varphi \in \mathcal{A}^0(X, \mathcal{E}nd(E)) \mid \int \mathrm{tr}\varphi \; gdv = 0\} \qquad (6)$$

is the space of C^∞ endomorphisms of E of trace zero, c is the constant given by the Einstein condition (5) for $A''(0)$, and the curvatures $\Omega(A''(S)^f))$ are taken with respect to the hermitian connections given by $A''(s)^f$ and h_s.

Now both the Kähler class of $\mathcal{X}_s$ and the first Chern class of $\mathcal{E}_s$ are constant in s so firstly one can take the volume element $gdv = \omega_s^n/n!$ in (6) with respect to any of the metrics ω_s, and secondly the constants needed for the Einstein condition for $A''(s)$ will be the same c.

Thus we have to solve the implicit function equation

$$\Phi(f(s), s) = 0.$$

The tangent space at id in G is naturally identified to $\mathcal{A}^0_0(X, \mathcal{E}nd(E))$ and we get the following first partial derivative

$$D_1\Phi \mid_{\substack{s=0\\ f=\mathrm{id}}} : \begin{array}{ccc} \mathcal{A}^0_0(X, \mathcal{E}nd(E)) & \longrightarrow & \mathcal{A}^0_0(X, \mathcal{E}nd(E)) \\ \beta & \longmapsto & \overline{\partial}^*\overline{\partial}\beta = \Box\beta \end{array}$$

(see [Kob],ch.VII§4, for related computations in the absolute case), which is bijective since $\mathcal{E}_0$ is simple. Extending now Φ to the Sobolev completions we apply the implicit function theorem in this context and remark that the solution is smooth since the linearized equation is elliptic.

Corollary *There exist versal deformations of pairs of polarized compact Kähler manifolds and stable vector bundles.*

We define in the usual way the isomorphism functor $\mathbf{Isom}_S((\mathcal{X}, \mathcal{E}), (\mathcal{X}', \mathcal{E}'))$ of two families over S of pairs of polarized compact Kähler manifolds and isomorphism by associating to each (reduced) analytic space S' over S the set of isomorphisms of the pull-backs of the families to S'.

Using the representability of the isomorphism functor $\mathbf{Isom}_S(\mathcal{X}, \mathcal{X}')$ of the underlying polarised families (and also for pairs of spaces $(\mathcal{X}, \mathcal{X}[\mathcal{E}])$, [F1]) one gets that $\mathbf{Isom}_S((\mathcal{X}, \mathcal{E}), (\mathcal{X}', \mathcal{E}'))$ is representable by an analytic set $Isom_S((\mathcal{X}, \mathcal{E}), (\mathcal{X}', \mathcal{E}'))$ over S.

For the existence of the moduli space of pairs of polarized non-uniruled compact Kähler manifolds and isomorphism classes of stable vector bundles it is now enough to show that $Isom_S((\mathcal{X}, \mathcal{E}), (\mathcal{X}', \mathcal{E}'))$ is proper over S, by a general criterion, cf.[S3].

Proposition 2 *For two families over S of pairs of polarized non-uniruled Kähler manifolds and isomorphism classes of stable vector bundles $(\mathcal{X}, \mathcal{E})$, $(\mathcal{X}', \mathcal{E}')$ the natural map,*

$$\alpha : Isom_S((\mathcal{X}, \mathcal{E}), (\mathcal{X}', \mathcal{E}')) \longrightarrow S$$

is proper.

Proof. One has a factorization of α

$$Isom_S((\mathcal{X}, \mathcal{E}), (\mathcal{X}', \mathcal{E}')) \xrightarrow{\beta} Isom_S(\mathcal{X}, \mathcal{X}') \xrightarrow{\gamma} S$$

where γ is known to be proper (see [F2], [S1]).

Let now $(s_\nu)_\nu$ be a sequence in S converging to $0 \in S$ and let $\varphi_\nu : \mathcal{X}_{s_\nu} \xrightarrow{\sim} \mathcal{X}'_{s_\nu}$ be isomorphisms such that $\mathcal{E}_{s_\nu} = \varphi^*_{s_\nu}\mathcal{E}'_{s_\nu}$. Since γ is proper, a subsequence $\varphi_{\nu(\mu)}$ converges to some $\varphi_0 : \mathcal{X}_0 \xrightarrow{\sim} \mathcal{X}'_0$. Take Hermite-Einstein connections ω_ν, ω'_ν on $\mathcal{E}_{s_\nu}$ and $\mathcal{E}'_{s_\nu}$ respectively. Then $\varphi^*_\nu\omega'_\nu = \omega_\nu$, $(\omega'_\nu)_\nu$ converges to ω'_0 on $\mathcal{E}'_0$ and $(\omega_\nu)_\nu$ converges to ω_0 on $\mathcal{E}_0$ by the uniqueness of the Hermite-Einstein connections. Hence we have

$$\varphi^*_{\nu(\mu)}\omega'_{\nu(\mu)} \longrightarrow \varphi^*_0\omega'_0$$

and

$$\varphi^*_{\nu(\mu)}\omega'_{\nu(\mu)} = \omega_{\nu(\mu)} \longrightarrow \omega_0 \quad \text{, for } \mu \to \infty.$$

Thus $\varphi_0^*\omega_0' = \omega_0$ showing that $\mathcal{E}_0 = \varphi_0^*\mathcal{E}_0'$ and proving the claim.

Theorem 1 is now proved. æ

3 Representatives in terms of Kähler-Einstein and Hermite-Einstein metrics for the Kodaira-Spencer map

We recall that a Kähler compact manifold (X, g) is called *Kähler-Einstein* if its tangent bundle with the metric g is Hermite-Einstein i.e.

$$\mathrm{Ric}(g) = k \cdot \omega_X$$

where

$$\mathrm{Ric}(g) = \sqrt{-1}\bar{\partial}\partial \lg(\det(g_{\alpha\bar{\beta}}))$$

is the Ricci form of g, ω_X the Kähler form and k some real constant.

Taking cohomology classes this gives

$$2\pi c_1(X) = k\lambda_X$$

where $\lambda = \lambda_X = [\omega_X]$. k may be normalized to ± 1 or 0, and as such we have the cases:

a) $c_1(X)$ negative definite, $(k = -1)$, i.e. the canonical bundle K_X is ample. For canonically polarized manifolds moduli spaces were constructed from Hilbert schemes. Also the existence and uniqueness of a Kähler-Einstein metric was proved.

b) $c_1(X) = 0$, $(k = 0)$. The construction of the moduli space of polarized such manifolds is based upon Yau's solution of the Calabi problem. (For the Petersson-Weil metric in this case cf. also [S2]).

c) $c_1(X)$ positive definite $(k = 1)$. According to [F-S2] there exist moduli spaces of Kähler-Einstein manifolds with positive curvature whose automorphism groups are finite. For the existence of Kähler-Einstein metrics cf. [Siu], [T1], [T2].

Suppose now $(\mathcal{X} \to S, \tilde{\lambda})$ is an effective polarized family of Kähler-Einstein manifolds. Then one gets a natural Kähler metric (Petersson-Weil metric) on the regular part of S by taking the inner product of harmonic representatives of Kodaira-Spencer classes. This can be visualized in the following way.

First one has locally with respect to S a real $(1-1)$-form $\omega_{\mathcal{X}}$ on $\mathcal{X}$ inducing the Kähler-Einstein forms when restricted to the fibers, $\omega_{\mathcal{X}} \mid_{\mathcal{X}_s} = \omega_s$. For $k \neq 0$ one just takes

$$\omega_{\mathcal{X}} = \frac{\sqrt{-1}}{k}\partial_{\mathcal{X}}\bar{\partial}_{\mathcal{X}}\log g,$$

where $g = g(s) = \det(g_{\alpha\bar{\beta}}(s))_{\alpha\bar{\beta}}$. (We denote by $(s^i)_i$, $(z^\alpha)_\alpha$ local coordinates on S, and on the fibers, respectively).

Then one lifts tangent vector fields $\frac{\partial}{\partial s_i} \in \Theta(S)$ horizontally with respect to $\omega_{\mathcal{X}}$ to $\Theta(\mathcal{X})$. Locally the lifts have the form $\frac{\partial}{\partial s_i} + a_i^\alpha \frac{\partial}{\partial z^\alpha}$, with $a^\alpha = a_i^\alpha = -g_{i\bar{\beta}} g^{\bar{\beta}\alpha}$ where $(g^{\bar{\beta}\alpha})$ is the inverse of $(g_{\alpha\bar{\beta}})_{\alpha\bar{\beta}}$. Let $A^\alpha_{\bar{\beta}} = A^\alpha_{i\bar{\beta}} := \frac{\partial}{\partial z^{\bar{\beta}}}(a_i^\alpha) \mid_{X_0}$. Then $A^\alpha_{i\bar{\beta}} \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}} \in \mathcal{A}^{0,1}(X_0, \Theta_{x_0})$ will be the harmonic representative for the Kodaira-Spencer class $\rho(\frac{\partial}{\partial s_i}) \in H^1(X_0, \Theta_{x_0})$.

Let's compute now the representatives for Kodaira-Spencer classes in $H^1(X, \Sigma_X)$ for an effective local family of pairs $(f : \mathcal{X} \to S, \mathcal{E})$ of polarized Kähler-Einstein manifolds and Hermite-Einstein bundles, in terms of these metrics. Let $\omega_{\mathcal{X}}$, $h_{\mathcal{E}}$ be global metrics inducing the given metrics on the fibers. Consider as before a tangent vector field $\frac{\partial}{\partial s^i} \in \Theta_S(S)$, its horizontal lift

$$v_i = \frac{\partial}{\partial s^i} \ + \ a_i^\alpha \frac{\partial}{\partial z^\alpha} \in \Theta_{\mathcal{X}}(\mathcal{X})$$

with respect to $\omega_{\mathcal{X}}$, and then the lifting to $\Sigma_{\mathcal{X}}$ in (4) of §1, given by the splitting induced by the associated holomorphic connection of $h_{\mathcal{E}}$, $(\omega^{(j)})_j$, for a covering $(\mathcal{U}_j)_j$ of $\mathcal{X}$. In the notations of §1 we get a family

$$((\lambda_j), v_i) \in \mathcal{A}^0(\mathcal{U}_j, \mathcal{O}^r_{\mathcal{U}_j}) \times \mathcal{A}^0(\mathcal{X}, \Theta_{\mathcal{X}})$$

where

$$\lambda_j = \omega^{(j)} \cup v_i = \omega_i^{(j)} + a_i^\alpha \cdot \omega_\alpha^{(j)}$$

by (3), since we want $\mu = 0$. Applying $\bar{\partial}$ to this element and restricting to the central fiber $X = X_0$ we get the element

$$(\Omega \cup v_i \mid_X + (A^\alpha_{\bar{\beta}} \cup \omega_\alpha^{(j)}) dz^{\bar{\beta}}, A^\alpha_{\bar{\beta}} \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}})$$

in $\mathcal{A}^{0,1}(X, \Sigma_X)$, where Ω denotes now the curvature form of the holomorphic connection $(\omega^{(j)})_j$ on $\mathcal{E}$ associated to h.

Using the C^∞-splitting again this is identified to the pair

$$(\Omega \cup v_i \mid_X, A^\alpha_{\bar{\beta}} \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}}) \in \mathcal{A}^{0,1}(X, \mathcal{E}ndE) \times \mathcal{A}^{0,1}(X, \Theta_X)$$

which represents $\rho(\frac{\partial}{\partial s^i}) \in H^1(X, \Sigma_X)$.

Its norm with respect to h_0 and ω_0 gives the looked for expression of the Petersson-Weil metric:

$$\| \frac{\partial}{\partial s^i} \|^2_{PW} = \| A^\alpha_{i\bar{\beta}} \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}} \|^2 + \| (R_{i\bar{\beta}} + a_i^\alpha R_{\alpha\bar{\beta}}) dz^{\bar{\beta}} \|^2 . \tag{7}$$

æ

4 Kähler property of the Petersson-Weil metric

In this paragraph we prove theorem 2.

We consider simple projectively flat hermitian vector bundles of rank r, (E, h), on a polarized n-dimensional Kähler manifold (X, λ), (i.e. the associated projectified bundle $P(E)$ is induced by an irreducible representation $\pi_1(X) \to PU(r) := U(r)/U(1)I_r$)

Definition In the above situation $(X, \lambda; E, h)$ shall be called a *polarized pair* (with factor c) if moreover

$$\frac{2n\pi}{r} c_1(E) = c \cdot \lambda \tag{8}$$

for some (real) constant c.

Proposition 3 *Let $(X, \lambda; E, h)$ be a polarized pair with factor c and $\omega_X = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ representing the polarization λ of X. Then a conformal change of h will be Hermite-Einstein with respect to ω and in this case the associated curvature is expressed locally by*

$$R_{\alpha\bar{\beta}} = \frac{c}{n} \cdot g_{\alpha\bar{\beta}} \cdot \mathrm{Id}_E \tag{9}$$

Proof. Since (E, h) is projectively flat the curvature tensor associated to its hermitian connection has locally the form

$$R_{\alpha\bar{\beta}} = \varphi_{\alpha\bar{\beta}} \mathrm{Id}_E \quad .$$

h will be adapted to be Hermite-Einstein by the following standard argument.

A conformal change of h, i.e. replacing h by $h' = a \cdot h$ for a positive smooth function a, induces a corresponding curvature tensor

$$R'_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}} + \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} (\log a) \mathrm{Id}_E =: \varphi'_{\alpha\bar{\beta}} \mathrm{Id}_E \quad ,$$

hence $g^{\alpha\bar{\beta}} \varphi'_{\alpha\bar{\beta}} = g^{\alpha\bar{\beta}} \varphi_{\alpha\bar{\beta}} + \Box \log a$.

How (8) implies $\int_X (c - g^{\alpha\bar{\beta}} \varphi_{\alpha\bar{\beta}}) g dv = 0$ which shows that there is a solution of $\Box \log a = c - g^{\alpha\bar{\beta}} \varphi_{\alpha\bar{\beta}}$. For such an a we have $g^{\alpha\bar{\beta}} \varphi_{\alpha\bar{\beta}} = c$ showing that (E, h') is Hermite-Einstein.

In order to prove (9) remark that by (8) the forms $n\sqrt{-1} \varphi'_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ and $c \cdot \omega_X$ are cohomologous and hence they differ by a $\partial\bar{\partial}$-exact $(1,1)$-form. There exists thus a real C^∞ function χ on X such that

$$n \varphi'_{\alpha\bar{\beta}} = c \cdot g_{\alpha\bar{\beta}} + \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} \chi \quad .$$

Taking trace with respect to g we get $\Box \chi = 0$ and hence (9).

In proving the Kähler property for moduli spaces of pairs of polarized Kähler-Einstein manifolds and stable holomorphic bundles we shall have to restrict ourselves to isomorphism classes of polarized pairs as above.

Remark that these form an open and closed subset in the whole moduli space since they can be described by topological conditions. For this, recall that a Hermite-Einstein vector bundle E of rank r is projectively flat if and only if $[2rc_2(E) - (r-1)c_1(E)^2] \cup \lambda^{n-2} = 0$ where λ is the Kähler class of the polarized manifold. (cf.[Kob]).

Remark also that for the Riemann surface case the whole interesting range will be still covered.

For the proof of theorem 2 we consider $(\mathcal{X} \xrightarrow{f} S, \mathcal{E})$ an effective local family of polarized pairs of Kähler-Einstein manifolds and simple projectively flat vector bundles. We use the conventions and notations of §3.

For $\frac{\partial}{\partial s^i} \in \Theta_S(S)$ and $v_i = \frac{\partial}{\partial s_i} + a_i^\sigma \frac{\partial}{\partial z^\sigma}$ its horizontal lift, set

$$\eta_i := v_i \cup \Omega = (R_{i\bar\beta} + a_i^\sigma R_{\sigma\bar\beta}) dz^{\bar\beta} + (R_{i\bar\jmath} + a_i^\sigma R_{\sigma\bar\jmath}) ds^{\bar\jmath}$$

and $\eta_{\bar\imath} := (\eta_i)^* =^t (\bar\eta_i)$.

Since the Kähler property is known for the contribution in the $\mathcal{X}$-direction of the Petersson-Weil metric (i.e. for the first term of its expression (7)), we only have to deal with the $\mathcal{E}$-contribution which is

$$G_{i\bar\jmath}^{PW}(s) := (\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j})_{PW} := \int_{\mathcal{X}/S} \mathrm{tr}(\eta_i \wedge \eta_{\bar\jmath}^*) \wedge \omega_{\mathcal{X}}^{n-1} \quad .$$

These integrals are

$$G_{i\bar\jmath}^{PW}(s) = \int_{\mathcal{X}_s} \mathrm{tr} g^{\bar\beta\alpha} (\eta_{i\bar\beta} \wedge \eta_{\bar\jmath\alpha}^*) \; g dv \quad .$$

We shall compute $\frac{\partial}{\partial s^k} G_{i\bar\jmath}$ with respect to Lie derivatives L_{v_k} in the v_k-direction. First we need some lemmas. The semi-colon stands for covariant derivative in fiber direction.

Lemma 1 $L_{v_k}(v_i) = 0$

Lemma 2 $L_{v_k}(v_{\bar\jmath}^*) = -(v_k, v_j)_{;\bar\beta} g^{\bar\beta\alpha} \frac{\partial}{\partial z^\alpha} - (v_k, v_j)_{;\alpha} g^{\bar\beta\alpha} \frac{\partial}{\partial z^{\bar\beta}}$ *in local coordinates on* $\mathcal{X}$.

Proof. Since coordinate fields on S commute we will have only fiber direction components for $L_{v_k}(v_i) = [v_k, v_i]$ and $L_{v_k}(v_{\bar\jmath}^*) = [v_k, v_{\bar\jmath}]$. For simplicity we denote $\partial_i := \frac{\partial}{\partial s^i}$, $\partial_\alpha := \frac{\partial}{\partial z^\alpha}$, ...

1.

$$\begin{aligned} L_{v_k}(v_i) &= [\partial_k + a_k^\sigma \partial_\sigma, \partial_i + a_i^\sigma \partial_\sigma] = \\ &= \left\{ \partial_k(a_i^\sigma) + a_k^\lambda \partial_\lambda(a_i^\sigma) - \partial_i(a_k^\sigma) - a_i^\lambda \partial_\lambda(a_k^\sigma) \right\} \partial_\sigma. \end{aligned}$$

Now

$$\begin{aligned}
\partial_k(a_i^\sigma) &= \partial_k(g^{\bar\tau\sigma}a_{i\bar\tau}) = \\
&= -g^{\bar\tau\nu}g^{\bar\nu\sigma}\partial_k(g_{\mu\bar\nu})\cdot a_{i\bar\tau} + g^{\bar\tau\sigma}\partial_k(a_{i\bar\tau}) = \\
&= g^{\bar\tau\mu}g^{\bar\nu\sigma}\partial_\mu(a_{k\bar\nu})\cdot a_{i\bar\tau} - g^{\bar\tau\sigma}\partial_k(g_{i\bar\tau}) = \\
&= a_i^\mu\cdot a_{k;\mu}^\sigma - g^{\bar\tau\mu}\partial_\mu(g^{\bar\nu\sigma})a_{k\bar\nu}\cdot a_{i\bar\tau} - g^{\bar\tau\sigma}\partial_k(g_{i\bar\tau})
\end{aligned}$$

where the last term is symmetric in i and k. Thus $\partial_k(a_i^\sigma)-\partial_i(a_k^\sigma) = a_i^\mu\cdot a_{k;\mu}^\sigma - a_k^\mu\cdot a_{i;\mu}^\sigma$ which replaced in the expression of $L_{v_k}(v_i)$ proves Lemma 1.

2. We compute the $\bar\beta$- component the other one being similar.

$$\begin{aligned}
L_{v_k}(v_{\bar\jmath}^*)^{\bar\beta} &= \left[\partial_k + a_k^\sigma\partial_\sigma, \partial_{\bar\jmath} + a_{\bar\jmath}^{\bar\tau}\partial_{\bar\tau}\right]^{\bar\beta} = \\
&= \partial_k(a_{\bar\jmath}^{\bar\beta}) + a_k^\sigma\cdot a_{\bar\jmath;\sigma}^{\bar\beta} = \\
&= \partial_k(g^{\bar\beta\sigma}a_{\bar\jmath\sigma}) + a_k^\sigma\cdot a_{\bar\jmath;\sigma}^{\bar\beta} = \\
&= g^{\bar\beta\gamma}a_{k;\gamma}^\sigma\cdot a_{\bar\jmath\sigma} - g^{\bar\beta\sigma}\partial_k(g_{\sigma\bar\jmath}) + a_k^\sigma\cdot a_{\bar\jmath;\sigma}^{\bar\beta} = \\
&= -g^{\bar\beta\sigma}\cdot g_{k\bar\jmath;\sigma} + a_k^{\sigma;\bar\beta}\cdot a_{\bar\jmath\sigma} + a_k^\sigma\cdot a_{\bar\jmath\sigma}{}^{;\bar\beta} = \\
&= -g_{k\bar\jmath}{}^{;\bar\beta} + (a_k^\sigma\cdot a_{\bar\jmath\sigma})^{;\bar\beta}
\end{aligned}$$

On the other side

$$\begin{aligned}
(v_k, v_j) &= (\partial_k + a_i^\sigma\partial_\sigma, \partial_{\bar\jmath} + a_{\bar\jmath}^{\bar\tau}\partial_{\bar\tau}) = \\
&= g_{k\bar\jmath} + g_{k\bar\tau}a_{\bar\jmath}^{\bar\tau} + a_k^\sigma\cdot g_{\sigma\bar\jmath} + a_k^\sigma a_{\bar\jmath}^{\bar\tau}g_{\sigma\bar\tau} = \\
&= g_{k\bar\jmath} - a_{k\bar\tau}a_{\bar\jmath}^{\bar\tau} - a_k^\sigma a_{\bar\jmath\sigma} + a_{k\bar\tau}a_{\bar\jmath}^{\bar\tau} = \\
&= g_{k\bar\jmath} - a_k^\sigma a_{\bar\jmath\sigma}
\end{aligned}$$

proving the claim of Lemma 2.

Lemma 3 $L_{v_k}(\Omega) = \partial(v_k\cup\Omega)$

Proof. We show the equality for the $\alpha\bar\beta$-component since the others follow in the same way. We now use covariant derivative with respect to the hermitian connection of $\mathcal{E}$.

$$\begin{aligned}
[v_k, R]_{\alpha\bar\beta} &= [\partial_k + a_k^\sigma\partial_\sigma, R]_{\alpha\bar\beta} = \\
&= R_{\alpha\bar\beta;k} + a_k^\sigma R_{\alpha\bar\beta;\sigma} + R_{\sigma\bar\beta}a_{k;\alpha}^\sigma = \\
&= (R_{k\bar\beta} + a_k^\sigma R_{\sigma\bar\beta})_{;\alpha} = \\
&= (\eta_{k\bar\beta})_{;\alpha}
\end{aligned}$$

Lemma 4 $v_i\cup L_{v_k}(\Omega) = v_k\cup L_{v_i}(\Omega)$

Proof. We use "$\equiv$" for an equality holding modulo symmetric terms in i and k.

$$\begin{aligned}
v_i \cup L_{v_k}(\Omega) &= (\partial_i + a_i^\sigma \partial_\sigma) \cup \partial \eta_k = \\
&= \eta_{k\bar\beta;i} + a_i^\sigma \eta_{k\bar\beta;\sigma} = \\
&= R_{k\bar\beta;i} + (a_k^\sigma R_{\sigma\bar\beta})_{;i} + + a_i^\sigma R_{k\beta;\sigma} + a_i^\sigma a_k^\lambda R_{\lambda\bar\beta;\sigma} + a_i^\sigma a_{k;\sigma}^\lambda R_{\lambda\bar\beta} \equiv \\
&\equiv \partial_i(a_k^\sigma) R_{\sigma\bar\beta} + (a_k^\sigma R_{\sigma\bar\beta;i} + a_i^\sigma R_{\sigma\bar\beta;k}) + a_i^\sigma a_{k;\sigma}^\lambda R_{\lambda\bar\beta} \equiv \\
&\equiv \partial_i(g^{\bar\tau\sigma}) \cdot a_{k\bar\tau} R_{\sigma\bar\beta} + g^{\bar\tau\sigma} \partial_i(a_{k\bar\tau}) R_{\sigma\bar\beta} + a_i^\sigma a_{k;\sigma}^\lambda R_{\lambda\bar\beta} \equiv \\
&\equiv a_{i;\alpha}^\sigma a_k^\alpha R_{\sigma\bar\beta} + a_{k;\sigma}^\sigma R_{\lambda\bar\beta} \equiv \\
&\equiv 0
\end{aligned}$$

Lemma 5 $(L_{v_k}(\omega_{\mathcal{X}}))_{\alpha\bar\beta} = 0$

Proof.

$$\begin{aligned}
[\partial_k + a_k^\sigma \partial_\sigma, g]_{\alpha\bar\beta} &= \partial_k g_{\alpha\bar\beta} + a_k^\sigma \partial_\sigma (g_{\alpha\bar\beta}) + g_{\sigma\bar\beta} \partial_\alpha (a_k^\sigma) = \\
&= \partial_\alpha(-a_{k\bar\beta}) + \partial_\alpha (g_{\sigma\bar\beta} a_k^\sigma) = \\
&= 0
\end{aligned}$$

Lemma 6 $\bar\partial^* \eta_i = \bar\partial^*(v_i \cup \Omega) = 0$

Proof. Here we shall make use of the Einstein property.

In the following computation ω will denote the connection form of $\mathcal{E}$.

$$\begin{aligned}
0 &= \partial_i(g^{\bar\beta\alpha} R_{\alpha\bar\beta}) = \\
&= \partial_i(g^{\bar\beta\alpha}) R_{\alpha\bar\beta} + g^{\bar\beta\alpha} \partial_i(R_{\alpha\bar\beta}) = \\
&= g^{\bar\beta\gamma} a_{i;\gamma}^\alpha R_{\alpha\bar\beta} + g^{\bar\beta\alpha}(R_{\alpha\bar\beta;i} - \left[\omega_i, R_{\alpha\bar\beta}\right]) = \\
&= g^{\bar\beta\sigma} a_{i;\alpha}^\sigma R_{\sigma\bar\beta} + g^{\bar\beta\alpha} R_{i\bar\beta;\alpha} = \\
&= g^{\bar\beta\alpha}(R_{i\bar\beta;\alpha} + a_{i;\alpha}^\sigma R_{\sigma\bar\beta} + a_i^\sigma R_{\alpha\bar\beta;\sigma}) = \\
&= g^{\bar\beta\alpha}(R_{i\bar\beta} + a_i^\sigma R_{\sigma\bar\beta})_{;\alpha} = \\
&= -\bar\partial^* \eta_i
\end{aligned}$$

Proof of Theorem 2. We need to show that the following partial derivative is symmetric in i and k.

$$\begin{aligned}
\partial_k G_{ij}(s) &= \partial_k \int_{\mathcal{X}/S} \mathrm{tr}\left[(v_i \cup \Omega) \wedge (v_{\bar\jmath}^* \cup \Omega)\right] \wedge \omega_{\mathcal{X}}^{n-1} = \\
&= \int_{\mathcal{X}/S} \mathrm{tr}\left[(L_{v_k}(v_i) \cup \Omega) \wedge (v_{\bar\jmath}^* \cup \Omega)\right] \wedge \omega^{n-1} + \\
&+ \int_{\mathcal{X}/S} \mathrm{tr}\left[(v_i \cup L_{v_k}(\Omega)) \wedge (v_{\bar\jmath}^* \cup \Omega)\right] \wedge \omega^{n-1} + \\
&+ \int_{\mathcal{X}/S} \mathrm{tr}\left[(v_i \cup \Omega) \wedge (L_{v_k}(v_{\bar\jmath}^*) \cup \Omega)\right] \wedge \omega^{n-1} + \\
&+ \int_{\mathcal{X}/S} \mathrm{tr}\left[(v_i \cup \Omega) \wedge (v_{\bar\jmath}^* \cup L_{v_k}(\Omega))\right] \wedge \omega^{n-1} + \\
&+ \int_{\mathcal{X}/S} \mathrm{tr}\left[(v_i \cup \Omega) \wedge (v_{\bar\jmath}^* \cup \Omega)\right] \wedge L_{v_k}(\omega^{n-1}).
\end{aligned}$$

Now the first and last terms are zero by Lemmas 1 and 5 respectively, while the second one is symmetric by Lemma 4.

For the fourth term we need the $(1,0)$-component of

$$\begin{aligned} v_{\bar{\jmath}} \cup L_{v_k}(\Omega) &= v_{\bar{\jmath}} \cup \partial(v_k \cup \Omega). \\ \partial(v_k \cup \Omega) &= \partial(\eta_{k\bar{\beta}} dz^{\bar{\beta}} + \eta_{k\bar{l}} ds^{\bar{l}}) = \\ &= \eta_{k\bar{\beta};\alpha} dz^{\alpha} \wedge dz^{\bar{\beta}} + \eta_{k\bar{\beta};m} ds^m \wedge dz^{\bar{\beta}} + \eta_{k\bar{l};\alpha} dz^{\alpha} \wedge ds^{\bar{l}} + \eta_{k\bar{l};m} ds^m \wedge ds^{\bar{l}} \end{aligned}$$

So

$$\begin{aligned} (v_{\bar{\jmath}} \cup L_{v_k}(\Omega))_{\alpha} &= \eta_{k\bar{\jmath};\alpha} + \eta_{k\bar{\tau};\alpha} a_{\bar{\jmath}}^{\bar{\tau}} = \\ &= (\eta_{k\bar{\jmath}} + \eta_{k\bar{\tau}} a_{\bar{\jmath}}^{\bar{\tau}})_{;\alpha} - \eta_{k\bar{\tau}} \cdot A_{\bar{\jmath}\alpha}^{\bar{\tau}} = \\ &= (v_k \cup v_{\bar{\jmath}} \cup \Omega)_{;\alpha} - \eta_{k\bar{\tau}} \cdot A_{\bar{\jmath}\alpha}^{\bar{\tau}} \end{aligned}$$

Now the fourth term equals at s, using Lemma 6

$$\int_{\mathcal{X}_s} \operatorname{tr}\left[(v_i \cup \Omega) \wedge \partial(v_k \cup v_{\bar{\jmath}} \cup \Omega)\right] \wedge \omega_{\mathcal{X}_s}^{n-1} - \int_{\mathcal{X}_s} \operatorname{tr}(\eta_{i\bar{\beta}} \cdot \eta_{k\bar{\tau}}) A_{\bar{\jmath}}^{\bar{\tau}\bar{\beta}} \cdot g dv =$$
$$= - \int \operatorname{tr}(\eta_{i\bar{\beta}} \cdot \eta_{k\bar{\tau}}) A_{\bar{\jmath}}^{\bar{\tau}\bar{\beta}} \cdot g dv \quad .$$

This is symmetric since $A_{\bar{\jmath}}^{\bar{\tau}\bar{\beta}} = A_{\bar{\jmath}}^{\bar{\beta}\bar{\tau}}$.
For the third term we need the $(1,0)$-component of $L_{v_k}(v_{\bar{\jmath}}) \cup \Omega$. This is

$$L_{vk}(v_{\bar{\jmath}}) \cup \Omega)_{\alpha} = -\left[((v_i, v_j)^{;\alpha} \partial_{\alpha} + (v_k, v_j)^{;\bar{\beta}} \partial_{\bar{\beta}}) \cup \Omega\right]_{\alpha} = (v_k, v_j)^{;\bar{\tau}} R_{\alpha\bar{\tau}}.$$

Then our third term equals at s

$$-\int_{\mathcal{X}_s} \operatorname{tr}\left[\eta_{i\bar{\beta}} \cdot (v_k \cdot v_{\bar{\jmath}})_{;\sigma} g^{\bar{\tau}\sigma} g^{\bar{\beta}\alpha} R_{\alpha\bar{\tau}}\right] g dv = \int \operatorname{tr}\left[\eta_{i\bar{\beta};\sigma} (v_k \cdot v_{\bar{\jmath}}) \cdot R_{\alpha\bar{\tau}} g^{\alpha\bar{\beta}} g^{\sigma\bar{\tau}}\right] g dv.$$

At this point we need the assumption that $(\mathcal{X}_s, \mathcal{E}_s)$ is a polarized pair in order to apply relation (9) and make this last term also vanish.

This ends the proof. æ

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