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# A NEW APPROACH OF STABILIZATION OF NONDISSIPATIVE DISTRIBUTED SYSTEMS*

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**Abstract.** In this paper we propose a new approach to prove the nonlinear (internal or boundary) stabilization of certain nondissipative distributed systems (the usual energy is not decreasing). This approach leads to decay estimates (known in the dissipative case) when the integral inequalities method due to Komornik [*Exact Controllability and Stabilization. The Multiplier Method*, Masson, Paris, John Wiley, Chichester, UK, 1994] cannot be applied due to the lack of dissipativity.

First we study the stability of a semilinear wave equation with a nonlinear damping based on the equation

$$u'' - \Delta u + h(\nabla u) + f(u) + g(u') = 0.$$

We consider the general case with a function $h$ satisfying a smallness condition, and we obtain uniform decay of strong and weak solutions under weak growth assumptions on the feedback function and without any control of the sign of the derivative of the energy related with the above equation.

In the second part we consider the case $h(\nabla u) = -\nabla\phi \cdot \nabla u$ with $\phi \in W^{1,\infty}(\Omega)$. We prove some precise decay estimates (exponential or polynomial) of equivalent energy without any restriction on $\phi$.

The same results will be proved in the case of boundary feedback.

Finally, we comment on some applications of our approach to certain nondissipative distributed systems.

Some results of this paper were announced without proof in [A. Guesmia, *C. R. Acad. Sci. Paris Sér.* I *Math.*, 332 (2001), pp. 633–636].

**Key words.** stabilizability by a nonlinear feedback, partial differential equation, wave equation, Petrovsky system, elasticity, integral inequalities

**AMS subject classifications.** 35B40, 35L70, 35B37

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## 1. Introduction.

Consider the semilinear wave equation with a nonlinear internal dissipative term,

$$\text{(P)} \qquad \begin{cases} u'' - \Delta u + h(\nabla u) + f(u) + g(u') = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases}$$

and the nonlinear boundary feedback,

$$\text{(P}'\text{)} \qquad \begin{cases} u'' - \Delta u + h(\nabla u) + f(u) = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \\ \partial_\nu u + g(u') = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^*$) is an open bounded domain with smooth boundary $\Gamma$ and $f$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuous nonlinear functions satisfying some

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general properties (see Assumptions 2.1–2.5 below). In (P′), $\nu$ represents the outward unit normal to $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0$ and $\Gamma_1$ are closed and disjoint. In this paper $\Delta$ and $\nabla$ stand, respectively, for the Laplacian and the gradian with respect to the spatial variables, $'$ denotes the derivative with respect to time $t$, and $\mathbb{R}^+ = [0, \infty[$.

The main goal of this paper is to show that strong and weak solutions to problems (P) and (P′) decay to zero when $t \to \infty$ and give some precise decay properties.

When $h \equiv 0$ the bibliography of works in this direction is truly long. We can cite, for instance, the works of Nakao [18, 21, 22], Kawashima, Nakao, and Ono [11], Nakao and Narazaki [19], Nakao and Ono [20], Haraux and Zuazua [10], Pucci and Serrin [23], and Zuazua [27], among others.

In [21], Nakao considered the following initial boundary value problem:

$$\text{(P1)} \qquad \begin{cases} u'' - \Delta u + \rho(u') + f(u) = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases}$$

where $\rho(v) = |v|^\beta v$, $\beta > -1$, $f(u) = bu|u|^\alpha$, $\alpha$, $b > 0$ (in this paper $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$ and $\mathbb{R}^n$), and $\Omega$ is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\Gamma := \partial\Omega$. He showed that (P1) has a unique global weak solution if $0 \leq \alpha \leq 2/(n-2)$, $n \geq 3$, and a global unique strong solution if $\alpha > 2/(n-2)$, $n \geq 3$ (of course if $n = 1$ or 2, then there is no restriction on $\alpha$). In addition to global existence the issue of the decay rate was addressed. In both cases, it has been shown that the energy of the solution decays algebraically if $\beta > 0$ and it decays exponentially if $\beta = 0$. This improves an earlier result obtained by the author in [22], where he studied the problem in an abstract setting and established a theorem concerning the decay of the solution energy only for the case $\alpha \leq 2/(n-2)$, $n \geq 3$. Later on, in a joint work with Ono [20], this result has been extended to the Cauchy problem for the equation

$$u'' - \Delta u + \lambda^2(x)u + \rho(u') + f(u) = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+,$$

where $\rho(u')$ behaves like $|u'|^\beta u'$ and $f(u)$ behaves like $-bu|u|^\alpha$. In this case the authors required that the initial data be small enough in $H^1 \times L^2$ norm and of compact support.

Pucci and Serrin [23] discussed the stability of the problem

$$\text{(P2)} \qquad \begin{cases} u'' - \Delta u + Q(x,t,u,u') + f(x,u) = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega \end{cases}$$

and proved that the energy of the solution is a Liapunov function. Although they did not discuss the issue of the decay rate, they did show that in general the energy goes to zero as $t$ approaches infinity. They also considered an important special case of (P2), which occurs when $Q(x,t,u,u') = a(t)t^\alpha u'$ and $f(x,u) = V(x)u$, and showed that the behavior of the solutions depends crucially on the parameter $\alpha$. If $|\alpha| \leq 1$, then the rest field is asymptotically stable. On the other hand, when $\alpha < -1$ or $\alpha > 1$ there are solutions that do not approach zero or approach nonzero functions $\phi(x)$ as $t \to \infty$.

Messaoudi [16] discussed an initial boundary value problem related to the equation

$$u'' - \Delta u + a(1 + |u'|^{m-2})u' + bu|u|^{p-2} = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

where $a, b > 0$, $m \geq 2$, $p > 2$, and proved that the energy of the solution decays exponentially. The proof of this result is based on a direct method used in [3] and [5].

Concerning the boundary feedback case, problem (P′) with $h \equiv 0$ has attracted considerable attention in the literature and, in recent years, important progress has been obtained in this context. New techniques were developed which allow us to stabilize a system through its boundary or control it from an initial to a final state (controllability). There is a large body of literature regarding boundary stabilization with linear feedback; we refer the reader to the following works: Lagnese [13], Russell [24], Triggiani [25], and You [26]. Now when the boundary feedback is nonlinear we can cite the works of Zuazua [28], Lasiecka and Tataru [14], Komornik [12], and Guesmia [5], among others. For such cases, the main purpose is to obtain the same stabilization results when a boundary feedback of the form

$$\partial_\nu u + a(x)u + b(x)g(u') = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+$$

is applied on a part $\Gamma_1$ of the boundary $\Gamma$ of $\Omega$ which satisfies certain geometric conditions and $a$, $b$, and $g$ are given functions, whereas no feedback is applied on the other part of the boundary, i.e.,

$$u = 0 \quad \text{on} \quad (\Gamma \setminus \Gamma_1) \times \mathbb{R}^+.$$

However, when $h \not\equiv 0$ very little is known in the literature; more general and recent results in this direction were obtained in [2]. In this paper the authors established well-posedness of the following large class of hyperbolic equations:

$$K(x,t)u'' - \Delta u + F(x,t,u,u',\nabla u) = f(x)$$

with boundary conditions and initial data as in (P′), where $K$, $F$, and $f$ are given functions satisfying some hypotheses.

However, to obtain exponential stability of solutions using classical multipliers and integral inequalities, they assumed some additional hypotheses on $F$ which require, in particular, that $F$ is global Lipschitz with respect to its last variable, where the Lipschitz constant is a function on $t$ and converges exponentially to 0 at $\infty$. This is a strong hypothesis which is not satisfied if, for example, the function $F$ does not depend on time $t$, as in our case.

Hyperbolic-parabolic equations are interesting from the point of view of not only the general theory of PDEs but also to applications in mechanics. For instance, the transonic Karman equation

$$u'u'' - \Delta u = 0$$

models flows of compressible gas in the transonic region where the velocity of gas varies from subsonic values to supersonic ones (see [2] and the references therein).

We note that stability of problems with the nonlinear term $h(\nabla u)$ requires careful treatment because we have any information neither about the influence of the integral $\int_\Omega h(\nabla u)u'\,dx$ on the norm

$$\|(u,u')\|^2_{H_0^1(\Omega)\times L^2(\Omega)} = \int_\Omega \left(|u'(x,t)|^2 + |\nabla u(x,t)|^2\right) dx$$

nor about the sign of its derivative; that is, the energy $E$ defined by (2.7) is not necessary decreasing (see identities (3.2) and (5.1)). Decrease of energy plays a crucial role in studying the asymptotic stability of the solution, as it was considered in the prior literature, in particular, in the works cited above.

We also observe that our problem deals with nonlinearity, which involves the gradient combined with a nonlinear feedback. This situation was not previously considered and leads to new difficulties. In order to overcome these difficulties and obtain energy decay estimates, we give a new and direct approach based on a combination of some ideas given by Guesmia in [3, 4] and the multiplier technique.

In the case where $h$ is linear we introduce a nonincreasing equivalent energy (see (2.14)) and then, by the use of appropriate multipliers and a well-known lemma due to Haraux–Komornik (see [12, Theorem 9.1]), the exponential and polynomial decay estimates are proved. In the case where $h$ is nonlinear, the introduction of a such equivalent energy seems to be not possible. In this case, the main ingredient for proving the exponential stability is to obtain a generalized integral inequalities of the form

$$(*)\qquad \begin{cases} \int_S^T E(t)dt \leq a_1(E(S)+E(T)) + a_2(E(S)-E(T)) \qquad \forall 0 \leq S \leq T < \infty, \\ E'(t) \leq a_3 E(t) \qquad \forall t \geq 0, \end{cases}$$

where $a_i$, $i = 1, 2, 3$, are nonnegative constants and where $E$ stands for the classical energy (2.7). Then we show that if, in addition, $2a_1a_3 < 1$ or $a_1 \leq a_2$, $E$ must converge exponentially to 0 at $\infty$.

Notice that a positive function satisfying $(*)$ does not necessarily converge to 0 at $\infty$; if $a_1a_3 \geq 1 + a_2a_3$, then the function $E(t) = e^{a_3 t}$ satisfies $(*)$. As an open question, it would be interesting to know what happens if $a_1a_3 \in [\frac{1}{2}, 1 + a_2a_3[$ and $a_1 > a_2$.

The integral result $(*)$ gives a generalization to the Haraux–Komornik lemma, which concerns nonincreasing functions (that is, $a_3 = 0$).

The rest of this paper is organized as follows. In section 2 we establish assumptions and state our main results. In section 3 we obtain the uniform stability of (P). In section 4 we consider the case $h(\nabla u) = -\nabla\phi \cdot \nabla u$, where $\phi \in W^{1,\infty}(\Omega)$ and $\cdot$ denotes the scalar product in $\mathbb{R}^n$, and we prove some decay estimates of equivalent energy of (P). In sections 5 and 6 we prove the same results for (P$'$). Finally, in the last section we give some applications of our approach to Petrovsky, coupled, and elasticity systems.

## 2. Assumptions and main results.

We begin this section stating the general hypotheses.

*Assumption* 2.1 (assumptions on $f$). $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function such that $f(0) = 0$ and, deriving from a potential $F$, that is

$$F(s) = \int_0^s f(\sigma)\, d\sigma \qquad \forall s \in \mathbb{R},$$

$$F(s) \geq -as^2 \qquad \forall s \in \mathbb{R}, \tag{2.1}$$

with $0 \leq a < \frac{1}{2c_0}$, where $c_0$ is the smallest positive constant (depending only on $\Omega$) such that (Poincaré's inequality)

$$\int_\Omega |v|^2 dx \leq c_0 \int_\Omega |\nabla v|^2 dx \qquad \forall v \in H_0^1(\Omega). \tag{2.2}$$

Also, there exists $b > 0$ such that

$$2bF(s) \leq sf(s) \qquad \forall s \in \mathbb{R}. \tag{2.3}$$

*Assumption* 2.2 (assumptions on $g$). $g : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function, nondecreasing, $g(0) = 0$, such that

$$g(s)s > 0 \qquad \forall s \neq 0. \tag{2.4}$$

Also, there exist two positive constants $c_1$ and $c_2$ such that

$$c_1|s| \leq |g(s)| \leq c_2|s| \qquad \forall s \in \mathbb{R}. \tag{2.5}$$

*Assumption* 2.3 (assumptions on $h$). $h : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ function such that $\nabla h$ is bounded and there exists $\beta > 0$ such that

$$|h(\zeta)| \leq \beta|\zeta| \qquad \forall \zeta \in \mathbb{R}^n. \tag{2.6}$$

We define the energy of the solution of (P) by the formula

$$E(t) = \int_\Omega \Big(\big|u'\big|^2 + \big|\nabla u\big|^2 + 2F(u)\Big)\, dx, \qquad t \in \mathbb{R}^+. \tag{2.7}$$

*Remarks.* 1. If the function $f$ is increasing and $f(0) = 0$, then (2.1) and (2.3) are satisfied with $a = 0$ and $b = \frac{1}{2}$.

2. Condition (2.1) assures the following inequality:

$$\|(u, u')\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq kE(t) \qquad \forall t \in \mathbb{R}^+, \tag{2.8}$$

where $k = \frac{1}{1-2ac_0} > 0$. Indeed, (2.1) and (2.2) imply that

$$E(t) \geq \int_\Omega \Big(\big|u'\big|^2 + \big|\nabla u\big|^2 - 2a\big|u\big|^2\Big)\, dx$$

$$\geq \int_\Omega \Big(\big|u'\big|^2 + (1 - 2ac_0)\big|\nabla u\big|^2\Big)\, dx$$

$$\geq (1 - 2ac_0)\int_\Omega \Big(\big|u'\big|^2 + \big|\nabla u\big|^2\Big)\, dx = (1 - 2ac_0)\|(u, u')\|^2_{H^1_0(\Omega)\times L^2(\Omega)},$$

which gives (2.8).

3. Under Assumptions 2.1, 2.2, 2.3 and using analogous considerations like the ones used in [2] (we omit the details), we can use Galerkin's method (semigroup theory is not suitable to treat degenerate problems) and prove that problem (P) possesses a unique strong solution, $u : ]0, \infty[ \to \mathbb{R}$, such that

$$u \in L^\infty(]0, \infty[; H^1_0(\Omega) \cap H^2(\Omega)), \qquad u' \in L^\infty(]0, \infty[; H^1_0(\Omega)), \tag{2.9}$$

and

$$u'' \in L^\infty(]0, \infty[; L^2(\Omega)).$$

Moreover, supposing that $\{u_0, u_1\}$ is in $H^1_0(\Omega) \times L^2(\Omega)$ and using density arguments, we can show that (P) has a unique weak solution $u : \Omega\times]0, \infty[ \to \mathbb{R}$ in the space

$$C(]0, \infty]; H^1_0(\Omega)) \cap C^1(]0, \infty[; L^2(\Omega)). \tag{2.10}$$

Now we are in position to state our first main result.

THEOREM 2.1. *Assume that Assumptions* 2.1, 2.2, 2.3 *hold such that* $b < 1$ *and* $\beta$ *satisfies the following smallness hypotheses:*

$$\frac{\beta}{2}\left(\sqrt{c_0 + \left(\frac{2}{c_1}\right)^2} + \sqrt{c_0}\right) + \sqrt{\frac{c_0 c_2 \beta}{2\sqrt{2}}} \le 1 - b,$$

$$\beta < \frac{b}{k^2\sqrt{c_0}}, \quad \text{or} \quad \frac{k\sqrt{c_0}}{2} \le \frac{1}{c_1} + \frac{1}{2}\sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}}.$$

*Then the energy determined by the strong solution* $u$ *decays exponentially. That is, to say for some positive constants* $c, \omega$, *one has*

$$E(t) \le cE(0)e^{-\omega t} \qquad \forall t \in \mathbb{R}^+. \tag{2.11}$$

*Furthermore,* (2.11) *holds for the weak solution* $u$.

*Remark.* If $F$ is positive (for example, $sf(s) \ge 0$ for all $s \in \mathbb{R}$), then $\beta$ and $b$ can be taken such that $b > 0$ and

$$\frac{\beta}{2}\left(\sqrt{c_0 + \left(\frac{2}{c_1}\right)^2} + (1 + 2k^2)\sqrt{c_0}\right) + \sqrt{\frac{c_0 c_2 \beta}{2\sqrt{2}}} < 1, \qquad \beta < \frac{b}{k^2\sqrt{c_0}}$$

or

$$\frac{\beta}{2}\left(\sqrt{c_0 + \left(\frac{2}{c_1}\right)^2} + \sqrt{c_0}\right) + \sqrt{\frac{c_0 c_2 \beta}{2\sqrt{2}}} < 1, \qquad \frac{k\sqrt{c_0}}{2} \le \frac{1}{c_1} + \frac{1}{2}\sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}}.$$

We consider now the case $h(\nabla u) = -\nabla\phi \cdot \nabla u$, where $\phi \in W^{1,\infty}(\Omega)$ and $g$ satisfies a hypothesis weaker than (2.5).

*Assumption* 2.4 (assumptions on $g$). $g : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function, nondecreasing, $g(0) = 0$, such that (2.4) holds and there exist four constants $r$, $p \ge 1$ and $c_1$, $c_2 > 0$ such that

$$c_1 \min\{|s|, |s|^r\} \le |g(s)| \le c_2 \max\{|s|^{\frac{1}{r}}, |s|^p\} \qquad \forall s \in \mathbb{R}, \tag{2.12}$$

$$(n-2)p \le n + 2. \tag{2.13}$$

We have the following stabilization result.

THEOREM 2.2. *Let* $u$ *be a solution of* (P) *in the class* (2.10). *Under Assumptions* 2.1 *and* 2.4, *there exist two positive constants* $\omega$, $c$ *such that the equivalent energy of* (P), *defined by*

$$E(t) = \int_\Omega e^{\phi(x)}\left(|u'|^2 + |\nabla u|^2 + 2F(u)\right)dx, \qquad t \in \mathbb{R}^+, \tag{2.14}$$

*satisfies* (2.11) *if* $r = 1$, *and*

$$E(t) \le c(1+t)^{\frac{-2}{r-1}} \qquad \forall t \in \mathbb{R}^+ \tag{2.15}$$

*if* $r > 1$.

*Remarks.* 1. If we take $g(s) = \alpha s$ for all $s \in \mathbb{R}$ with $\alpha > 0$ (that is, $r = p = 1$), then we find the results obtained in [15]. On the other hand, the case of $g(s) = \alpha(1 + \left|s\right|^{m-2})s$ for all $s \in \mathbb{R}$ with $m > 2$ (that is, $p = m - 1$ and $r = 1$) gives the results obtained in [16].

2. In Theorem 2.1 we can weaken assumption (2.6) by taking $\beta$ as the Lipschitz constant of only the nonlinear part of $h$; that is, we assume that there exists $\bar{\zeta} \in \mathbb{R}^n$ such that

$$|h(\zeta) + \bar{\zeta} \cdot \zeta| \leq \beta|\zeta| \qquad \forall \zeta \in \mathbb{R}^n.$$

To prove this we have only to consider the equivalent energy defined by (2.14) where $\phi(x) = \bar{\zeta} \cdot x$.

3. It is possible to weaken the growth assumption (2.12) as was done for the study of elasticity systems in [3, 7] and the Petrovsky system in [6]. In order to simplify we shall only consider in this paper the case of assumption (2.12).

Now we are concerned by the stability of (P′). In order to obtain the estimates (2.11) and (2.15), the following assumptions are made on $\Gamma$ and $f$. Let $x^0$ be a fixed point in $\mathbb{R}^n$. Then put

$$m = m(x) = x - x^0, \qquad R = \max_{x \in \bar{\Omega}} \left|m(x)\right|$$

and partition the boundary $\Gamma$ into two nonempty sets:

$$\Gamma_0 = \{x \in \Gamma : \ m(x) \cdot \nu(x) \leq 0\}, \qquad \Gamma_1 = \{x \in \Gamma : \ m(x) \cdot \nu(x) \geq \delta > 0\}.$$

*Examples.* Concerning the existence of such a partition of $\Gamma$, we can take $\Omega$ as follows:

1. If $n = 1$, then $\Omega$ is a bounded open interval, say $\Omega = ]x_1, x_2[ \subset \mathbb{R}$, and our geometric hypotheses are satisfied in each of the following two cases:
(i) $\Gamma_0 = \{x_1\},\ \Gamma_1 = \{x_2\},$ and $x^0 \leq x_1$,
(ii) $\Gamma_0 = \{x_2\},\ \Gamma_1 = \{x_1\},$ and $x^0 \geq x_2$.

2. If $n \geq 2$ and $\Omega = \Omega_1 \setminus \bar{\Omega}_0$, where $\Omega_1$ and $\Omega_0$ are two open domains with boundary $\Gamma_1$, and $\Gamma_0$, respectively, $\bar{\Omega}_0 \subset \Omega_1$, and $\Omega_1$ and $\Omega_0$ are star-shaped with respect to some point $x^0 \in \Omega_0$ (a domain $\Omega$ is called star-shaped with respect to $x^0$ if $m \cdot \nu > 0$ on $\partial\Omega$), then our geometric hypotheses are satisfied.

3. If $n \geq 2$ and $\Omega$ is not of the form mentioned in the preceding example, then in general there is no point $x^0$ satisfying simultaneously the geometric hypotheses assumed on $\Gamma_1$ and $\Gamma_0$. By applying an approximational method, one could considerably weaken these geometric hypotheses, at least in dimensions $n = 2$, 3, by adapting an analogous argument given by Komornik–Zuazua for the wave equation (see [12] and the references therein).

*Assumption* 2.5 (assumptions on $f$). $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function such that (2.3) and

$$F(s) \geq 0 \qquad \forall s \in \mathbb{R}. \tag{2.16}$$

The well-posedness of the problem (P′) can be established by standard Galerkin's method (see [15]); we do not discuss this point here. We use the notations

$$V = \{v \in H^1(\Omega) : \ v = 0 \ \text{ on } \Gamma_0\} \quad \text{and} \quad W = H^2(\Omega) \cap V;$$

we have the following:

1. For all $(u_0, u_1) \in W \times V$ such that $\partial_\nu u_0 + g(u_1) = 0$ on $\Gamma_1$, problem (P′) has a unique strong solution, $u : ]0, \infty[ \to \mathbb{R}$, such that

$$u \in L^\infty(]0,\infty[; W), \quad u' \in L^\infty(]0,\infty[; V), \text{ and } u'' \in L^\infty(]0,\infty[; L^2(\Omega)).$$

2. If $\{u_0, u_1\}$ is in $V \times L^2(\Omega)$, then (using density arguments) the solution is weak: $u : \Omega \times ]0, \infty[ \to \mathbb{R}$ in the space

$$C(]0,\infty]; V) \cap C^1(]0,\infty[; L^2(\Omega)). \tag{2.17}$$

THEOREM 2.3. *Let $u$ be a solution of* (P′) *in the class* (2.17). *Assume, moreover, that Assumptions* 2.2, 2.3, 2.5 *hold with $\beta$ small enough and $b > 1$ or $f$ is linear. Then the energy of $u$, defined by* (2.7), *decays exponentially to zero in the sense of* (2.11).

We consider now the case $h(\nabla u) = -\nabla\phi \cdot \nabla u$, where $\phi \in W^{1,\infty}(\Omega)$.

We have the following stabilization result for (P′).

THEOREM 2.4. *Let $u$ be a solution of* (P′) *in the class* (2.17). *Under Assumptions* 2.5, 2.4 *with $p = 1$, $R\|\nabla\phi\|_\infty < \min\{2, n\}$, and $b > \frac{n+R\|\nabla\phi\|_\infty}{n-R\|\nabla\phi\|_\infty}$ or $f$ is linear and $\|\nabla\phi\|_\infty$ is small enough, where $\|\nabla\phi\|_\infty = \max_{x\in\bar{\Omega}} |\nabla\phi(x)|$, the results of Theorem* 2.2 *hold true.*

*Remarks.* 1. As an example of a function $f$ satisfying Assumption 2.5, we can take $f(s) = \gamma s\,|s|^{q-1}$ with $\gamma \geq 0$ and $q \geq 1$. Condition (2.3) is satisfied for all $b \leq \frac{q+1}{2}$.

2. We have many possibilities to take the function $g$ such that condtions (2.12) and (2.13) are satisfied, for example, $g(s) = \gamma|s|^{r-1}s$ if $|s| \leq 1$, and $g(s) = \gamma s$ if $|s| \geq 1$, where $\gamma > 0$.

3. Thanks to (2.16), the function $F$ is positive, and then the usual energy (2.7) satisfies

$$\int_\Omega \left(\left|u'\right|^2 + \left|\nabla u\right|^2\right) dx \leq E(t). \tag{2.18}$$

The quantity $(\int_\Omega \left|\nabla u\right|^2 dx)^{\frac{1}{2}}$ defines a norm on $V$ equivalent to the usual norm induced by $H^1(\Omega)$; consequently, $V$ is a Hilbert space with this norm.

4. If $h$ is nonlinear and $r > 1$, we do not know if the energy of (P) and (P′) decays polynomially to zero.

5. In the case of uniform stability (Theorem 2.1 and Theorem 2.3), our proof allows us to obtain explicit constants $c$ and $\omega$ in (2.11).

6. Theorem 2.1, Theorem 2.3, and Theorem 2.4 probably remain valid without the smallness conditions assumed on $\beta$, but we could not prove them.

**3. Uniform decay: Proof of Theorem 2.1.** To justify all the computations that follow, we assume first that the solution is strong, and by a standard density argument we deduce the result for weak solutions.

We are going to prove that the energy defined by (2.7) satisfies the estimate

$$E(S + T_0) \leq dE(S) \qquad \forall S \in \mathbb{R}^+ \tag{3.1}$$

with $0 < d < 1$ and $T_0 > 0$. (This will be fixed later in the course of the proof.) Using (3.1), inequality (3.9) below gives (2.11).

We start this section by giving an explicit formula for the derivative of the energy. A simple computation shows that

$$E'(t) = -2\int_\Omega u' g(u')\,dx - 2\int_\Omega u' h\left(\nabla u\right)\,dx. \tag{3.2}$$

Multiplying the first equation in (P) by $u$ and integrating the obtained result over $\Omega \times [S,T]$, we obtain

$$\begin{aligned} 0 &= \int_S^T \int_\Omega u\left(u'' - \Delta u + h(\nabla u) + f(u) + g(u')\right) dx\,dt \\ &= \left[\int_\Omega uu' dx\right]_S^T + \int_S^T \int_\Omega \Big(-\left|u'\right|^2 + \left|\nabla u\right|^2 + uf(u)\Big)\,dx\,dt \\ &\quad + \int_S^T \int_\Omega ug(u')\,dxdt + \int_S^T \int_\Omega uh\left(\nabla u\right) dx\,dt. \end{aligned} \tag{3.3}$$

Hence, from (3.3), making use of the Cauchy–Schwarz inequality and taking assumption (2.6) and property (2.2) into account, we infer

$$\begin{aligned} &\int_S^T \int_\Omega \Big(\left|u'\right|^2 + \left|\nabla u\right|^2 + uf(u)\Big)\,dx\,dt \\ &\quad\le -\left[\int_\Omega uu' dx\right]_S^T + \int_S^T \int_\Omega \Big(2\left|u'\right|^2 - ug(u')\Big)\,dx\,dt \\ &\qquad + \frac{\beta}{2\sqrt{c_0}} \int_S^T \int_\Omega \left|u\right|^2 dx\,dt + \frac{\sqrt{c_0}}{2\beta} \int_S^T \int_\Omega \left|h\left(\nabla u\right)\right|^2 dx\,dt \\ &\quad\le -\left[\int_\Omega uu' dx\right]_S^T + \int_S^T \int_\Omega \Big(2\left|u'\right|^2 - ug(u')\Big)\,dx\,dt \\ &\qquad + \frac{\beta\sqrt{c_0}}{2} \int_S^T \int_\Omega \left|\nabla u\right|^2 dx\,dt + \frac{\beta\sqrt{c_0}}{2} \int_S^T \int_\Omega \left|\nabla u\right|^2 dx\,dt. \end{aligned}$$

Then, taking assumption (2.3) into account, from this inequality we deduce

$$\begin{aligned} &\int_S^T \int_\Omega \Big(\left|u'\right|^2 + (1-\beta\sqrt{c_0})\left|\nabla u\right|^2 + 2bF(u)\Big)\,dx\,dt \\ &\quad\le -\left[\int_\Omega uu' dx\right]_S^T + \int_S^T \int_\Omega \Big(2\left|u'\right|^2 - ug(u')\Big)\,dx\,dt. \end{aligned} \tag{3.4}$$

Using (2.2), (2.8), and the Cauchy–Schwarz inequality, we can easily get

$$\begin{aligned} \Big|\int_\Omega uu' dx\Big| &\le \frac{1}{2}\int_\Omega \left(\sqrt{c_0}\left|u'\right|^2 + \frac{1}{\sqrt{c_0}}\left|u\right|^2\right) dx \\ &\le \frac{\sqrt{c_0}}{2}\int_\Omega \Big(\left|u'\right|^2 + \left|\nabla u\right|^2\Big)\,dx \le \frac{k\sqrt{c_0}}{2}E(t); \end{aligned}$$

then

$$-\left[\int_\Omega uu'dx\right]_S^T \leq \frac{k\sqrt{c_0}}{2}\left(E(S)+E(T)\right).$$

Next, we insert this inequality into (3.4); it follows that

$$\int_S^T\int_\Omega\left(|u'|^2+(1-\beta\sqrt{c_0})\,|\nabla u|^2+2bF(u)\right)dx\,dt \tag{3.5}$$

$$\leq \frac{k\sqrt{c_0}}{2}\left(E(S)+E(T)\right)+\int_S^T\int_\Omega\left(2\,|u'|^2-ug(u')\right)dx\,dt.$$

Next, we want to majorize the last term in the right-hand side of (3.5).

**Estimate for $\int_S^T\int_\Omega\left(2|u'|^2-ug(u')\right)dx\,dt$.** Using (3.2) and the Cauchy–Schwarz inequality and taking the assumptions (2.4), (2.5), and (2.6) into account, it holds that

$$2\int_S^T\int_\Omega|u'|^2dx\,dt \leq \frac{2}{c_1}\int_S^T\int_\Omega u'g(u')dx\,dt$$

$$= \frac{1}{c_1}\int_S^T\left(-E'(t)-2\int_\Omega u'h(\nabla u)dx\right)dt$$

$$\leq \frac{1}{c_1}\left(E(S)-E(T)\right)+\frac{1}{c_1}\int_S^T\int_\Omega\left(\epsilon|u'|^2+\frac{\beta^2}{\epsilon}|\nabla u|^2\right)dx\,dt;$$

we choose $\epsilon>0$ such that $\frac{\beta^2}{\epsilon c_1}=\frac{\epsilon}{c_1}-\beta\sqrt{c_0}$, that is, $\epsilon=\frac{\beta}{2}(\sqrt{{c_1}^2c_0+4}+c_1\sqrt{c_0})$; then we deduce

$$2\int_S^T\int_\Omega|u'|^2dx\,dt \leq \frac{1}{c_1}\left(E(S)-E(T)\right) \tag{3.6}$$

$$+\beta\int_S^T\int_\Omega\left(\frac{1}{2}\left(\sqrt{c_0+\left(\frac{2}{c_1}\right)^2}+\sqrt{c_0}\right)|u'|^2+\frac{1}{2}\left(\sqrt{c_0+\left(\frac{2}{c_1}\right)^2}-\sqrt{c_0}\right)|\nabla u|^2\right)dx\,dt.$$

Similary we have

$$-\int_S^T\int_\Omega ug(u')dx\,dt \leq \frac{1}{2}\int_S^T\int_\Omega\left(\frac{1}{\epsilon}g^2(u')+\epsilon|u|^2\right)dx\,dt$$

$$\leq \frac{1}{2}\int_S^T\int_\Omega\left(\frac{c_2}{\epsilon}u'g(u')+\epsilon c_0|\nabla u|^2\right)dx\,dt$$

$$= \frac{c_2}{2\epsilon}\int_S^T\left(-\frac{1}{2}E'(t)-\int_\Omega u'h(\nabla u)dx\right)dt+\frac{\epsilon c_0}{2}\int_S^T\int_\Omega|\nabla u|^2dx\,dt$$

$$\leq \frac{c_2}{4\epsilon}\left(E(S)-E(T)\right)+\frac{\epsilon c_0}{2}\int_S^T\int_\Omega |\nabla u|^2 dx\,dt$$

$$+\frac{c_2}{2\epsilon}\int_S^T\int_\Omega \left(\frac{\epsilon'\beta^2}{2}|\nabla u|^2+\frac{1}{2\epsilon'}|u'|^2\right)dx\,dt;$$

we choose $\epsilon=\beta\sqrt{\frac{c_2\epsilon'}{2c_0}}$ and $\epsilon'=\frac{1}{\sqrt{2}\beta}$. It follows that

$$-\int_S^T\int_\Omega ug(u')dx\,dt \leq \sqrt{\frac{c_0c_2\beta}{2\sqrt{2}}}\int_S^T\int_\Omega \left(|u'|^2+|\nabla u|^2\right)dx\,dt + \frac{1}{2}\sqrt{\frac{c_0c_2}{\sqrt{2}\beta}}\left(E(S)-E(T)\right). \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we conclude that

$$\left(1-\frac{\beta}{2}\left(\sqrt{c_0+\left(\frac{2}{c_1}\right)^2}+\sqrt{c_0}\right)-\sqrt{\frac{c_0c_2\beta}{2\sqrt{2}}}\right)\int_S^T\int_\Omega\left(|u'|^2+|\nabla u|^2\right)dx\,dt + b\int_S^T\int_\Omega 2F(u)dx\,dt \leq \left(\frac{k\sqrt{c_0}}{2}+\frac{1}{c_1}+\frac{1}{2}\frac{\sqrt{c_0c_2}}{\sqrt{2}\beta}\right)E(S)+\left(\frac{k\sqrt{c_0}}{2}-\frac{1}{c_1}-\frac{1}{2}\frac{\sqrt{c_0c_2}}{\sqrt{2}\beta}\right)E(T). \tag{3.8}$$

Hence, if we take $\beta$ small enough so that $\frac{\beta}{2}(\sqrt{c_0+(\frac{2}{c_1})^2}+\sqrt{c_0})+\sqrt{\frac{c_0c_2\beta}{2\sqrt{2}}}\leq 1-b$ as it is assumed in Theorem 2.1, then, from (3.8) and making use of definition (2.7) of energy, we arrive at

$$\int_S^T E(t)dt \leq \frac{k\sqrt{c_0}}{2b}\left(E(S)+E(T)\right)+\frac{1}{b}\left(\frac{1}{c_1}+\frac{1}{2}\sqrt{\frac{c_0c_2}{\sqrt{2}\beta}}\right)\left(E(S)-E(T)\right). \tag{3.9}$$

If $F$ is positive, then we assume that $\frac{\beta}{2}(\sqrt{c_0+(\frac{2}{c_1})^2}+\sqrt{c_0})+\sqrt{\frac{c_0c_2\beta}{2\sqrt{2}}}<1$ and we obtain (3.9) with $b$ replaced by

$$\bar{b}=\min\left\{b, 1-\frac{\beta}{2}\left(\sqrt{c_0+\left(\frac{2}{c_1}\right)^2}+\sqrt{c_0}\right)-\sqrt{\frac{c_0c_2\beta}{2\sqrt{2}}}\right\}.$$

Now we return to equality (3.2). Using (2.4), (2.6), (2.8), and the Cauchy–Schwarz inequality, we infer

$$E'(t)\leq -2\int_\Omega u'h(\nabla u)dx \leq \int_\Omega\left(\beta|u'|^2+\frac{1}{\beta}|h(\nabla u)|^2\right)dx$$

$$\leq \beta \int_\Omega \left(|u'|^2 + |\nabla u|^2\right) dx \leq \beta k E(t);$$

then

$$E'(t) \leq \beta k E(t). \tag{3.10}$$

We may assume in the rest of this section that $E(t) > 0$ for all $t \geq 0$. Otherwise if $E(t_0) = 0$ for some $t_0 \geq 0$, then from (2.8) we have $u(t_0, x) = u'(t_0, x) = 0$ in $\Omega$; hence $v(t, x) := u(t + t_0, x)$ solves (P) with $(0, 0)$ as initial data. By the uniqueness of solution we conclude that $v = v' = 0$; hence $E(t) = 0$ for all $t \geq t_0$ and then we have nothing to prove.

Now by Gronwall's lemma, we conclude from (3.10) that

$$E(t) \leq e^{\beta k(t-\tau)} E(\tau) \qquad \forall\, 0 \leq \tau \leq t < \infty. \tag{3.11}$$

On the other hand, (3.10) implies that

$$E(t) \geq \frac{1}{\beta k} \frac{\partial}{\partial t} \Big( (1 - e^{-\beta k(t-\tau)}) E(t) \Big) \qquad \forall\, 0 \leq \tau \leq t < \infty. \tag{3.12}$$

Now we distinguish two cases (corresponding to the hypothesis assumed on $\beta$ in Theorem 2.1).

*Case* 1. $\beta < \frac{b}{k^2 \sqrt{c_0}}$. We fix

$$T_0 > \frac{-1}{\beta k} \ln \left( 1 - \frac{\beta k^2 \sqrt{c_0}}{b} \right). \tag{3.13}$$

From (3.12) with $\tau = S$ we have

$$\int_S^{S+T_0} E(t) dt \geq \frac{1}{\beta k} (1 - e^{-\beta k T_0}) E(S + T_0).$$

Combining this inequality and (3.9) with $T = S + T_0$, we arrive at

$$\left( \frac{1}{\beta k} (1 - e^{-\beta k T_0}) + \frac{1}{b} \left( \frac{1}{c_1} + \frac{1}{2} \sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}} \right) - \frac{k\sqrt{c_0}}{2b} \right) E(S + T_0)$$

$$\leq \left( \frac{k\sqrt{c_0}}{2b} + \frac{1}{b} \left( \frac{1}{c_1} + \frac{1}{2} \sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}} \right) \right) E(S).$$

Thanks to our choice (3.13) of $T_0$, we have

$$\frac{1}{\beta k} (1 - e^{-\beta k T_0}) > \frac{k\sqrt{c_0}}{b};$$

then we obtain (3.1) with

$$d = \frac{\frac{1}{b} \left( \frac{1}{c_1} + \frac{1}{2} \sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}} \right) + \frac{k\sqrt{c_0}}{2b}}{\frac{1}{\beta k} (1 - e^{-\beta k T_0}) + \frac{1}{b} \left( \frac{1}{c_1} + \frac{1}{2} \sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}} \right) - \frac{k\sqrt{c_0}}{2b}} \in ]0, 1[.$$

We note that if a nonnegative function $E : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the estimate (3.1), then it also satisfies (2.11). Indeed, let $t \in \mathbb{R}^+$; then $t = mT_0 + t_0$ with $0 \le t_0 < T_0$ and $m \in \mathbb{N}$. From (3.1) and taking (3.11) with $t = t_0$ and $\tau = 0$ into account, it holds that

$$E(t) \le dE((m-1)T_0 + t_0) \le \cdots \le d^m E(t_0)$$

$$\le d^{\frac{1}{T_0}(t-t_0)} e^{\beta k t_0} E(0) \le \frac{e^{\beta k T_0}}{d} E(0) e^{\frac{\ln d}{T_0} t};$$

then we deduce (2.11), where $c = \frac{e^{\beta k T_0}}{d}$ and $\omega = -\frac{\ln d}{T_0}$.

*Case* 2. $\frac{k\sqrt{c_0}}{2} \le \frac{1}{c_1} + \frac{1}{2}\sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}}$. Inequality (3.9) implies that

$$\int_S^T E(t)dt \le a_0 E(S) \qquad \forall 0 \le S \le T < \infty,$$

where $a_0 = \frac{k\sqrt{c_0}}{2b} + \frac{1}{b}(\frac{1}{c_1} + \frac{1}{2}\sqrt{\frac{c_0 c_2}{\sqrt{2}\beta}})$. Let $T$ go to $\infty$; we deduce

$$\int_S^\infty E(t)dt \le a_0 E(S) \qquad \forall S \ge 0. \tag{3.14}$$

Introduce the function

$$\psi(S) = \int_S^\infty E(t)dt, \qquad S \ge 0.$$

It is positive and nonincreasing. Differentiating and using (3.14), we find that

$$\psi'(S) \le -\frac{1}{a_0}\psi(S),$$

hence $(\ln(\psi(S)))' \le -\frac{1}{a_0}$. Integrating in $[0, S]$ and using (3.14) again, we obtain that

$$\psi(S) \le a_0 E(0) e^{-\frac{1}{a_0}S} \qquad \forall S \ge 0. \tag{3.15}$$

On the other hand, $E$ being nonnegative and satisfying (3.12) (with $\tau = S$), $\psi(S)$ may be estimated as follows: let $T_0 > 0$,

$$\psi(S) \ge \int_S^{S+T_0} E(t)dt \ge \int_S^{S+T_0} \frac{1}{\beta k}\frac{\partial}{\partial t}\Big((1 - e^{-\beta k(t-S)})E(t)\Big)\, dt$$

$$= \frac{1 - e^{-\beta k T_0}}{\beta k} E(S + T_0).$$

Therefore, taking $t = S + T_0$ and choosing $T_0 = \frac{1}{\beta k}\ln(1 + \beta k a_0)$ (for which the quantity $\frac{e^{T_0/a_0}}{1 - e^{-\beta k T_0}}$ reachs its minimum), hence we deduce from (3.15) the estimate

$$E(t) \le (1 + \beta k a_0)^{1 + \frac{1}{\beta k a_0}} E(0) e^{-\frac{1}{a_0}t} \qquad \forall t \ge T_0. \tag{3.16}$$

This inequality holds, in fact, also for $t \in [0, T_0]$. Indeed, by (3.11) with $\tau = 0$, we have

$$E(t) \le e^{\beta k t} E(0) \le e^{(\beta k + \frac{1}{a_0})T_0} E(0) e^{-\frac{1}{a_0}t} = (1 + \beta k a_0)^{1 + \frac{1}{\beta k a_0}} E(0) e^{-\frac{1}{a_0}t}.$$

Then (3.16) holds true for all $t \ge 0$ and hence the inequality (2.11) follows with $c = (1 + \beta k a_0)^{1 + \frac{1}{\beta k a_0}}$ and $\omega = \frac{1}{a_0}$.

This concludes the proof of Theorem 2.1.

**4. Energy decay estimates: Proof of Theorem 2.2.** For the proof of Theorem 2.2 which concerns the stability of (P) in the particular case $h(\nabla u) = -\nabla\phi \cdot \nabla u$, with $\phi \in W^{1,\infty}(\Omega)$, we are going to prove that the equivalent energy $E$ defined by (2.14) satisfies, for any $0 \leq S < \infty$,

$$\int_S^\infty E^{\frac{r+1}{2}}(t)dt \leq cE(S). \tag{4.1}$$

Here and in what follows we shall denote by $c$ diverse positive constants, by $\epsilon$ diverse positive constants small enough, and by $c_\epsilon$ diverse positive constants depending on $\epsilon$. (All these constants do not depend on $S$.) The inequality (4.1) gives (2.11) and (2.15) (see [12, Theorem 9.1]).

Using the first equation of (P) and the boundary condition, we can easily prove that the equivalent energy $E$ satisfies

$$E'(t) = -2\int_\Omega e^{\phi(x)}u'g(u')dx, \qquad t \in \mathbb{R}^+. \tag{4.2}$$

Assumption (2.4) implies that the equivalent energy is nonincreasing. Given $0 \leq S \leq T < \infty$ arbitrarily, integrate (4.2) between $S$ and $T$ to get

$$\int_S^T \int_\Omega e^{\phi(x)}u'g(u')dx = \frac{1}{2}\left(E(S) - E(T)\right). \tag{4.3}$$

We multiply the first equation of (P) by $E^{\frac{r-1}{2}}(t)e^{\phi(x)}u$ and integrate over $\Omega\times[S,T]$ to get

$$\int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(\left|u'\right|^2 + \left|\nabla u\right|^2 + uf(u)\right)dxdt \tag{4.4}$$

$$= \int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(2\left|u'\right|^2 - ug(u')\right)dxdt$$

$$+ \frac{r-1}{2}\int_S^T \int_\Omega E^{\frac{r-3}{2}}(t)E'(t)e^{\phi(x)}uu'dxdt - \left[\int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}uu'dxdt\right]_S^T.$$

The last two terms of (4.4) can be easily majorized by $cE^{\frac{r+1}{2}}(S)$ (see [3] and [5]). We follow now the proof given in [5]. We note $q = p+1$,

$$\Omega^+ = \{x \in \Omega : \left|u'\right| > 1\}, \quad \text{and} \quad \Omega^- = \Omega \setminus \Omega^+.$$

We exploit the Cauchy–Schwarz, Hölder, and Young inequalities and the Sobolev imbedding $H_0^1(\Omega) \subset L^q(\Omega)$ to get

$$-\int_S^T \int_{\Omega^+} E^{\frac{r-1}{2}}(t)e^{\phi(x)}ug(u')dxdt$$

$$\leq \int_S^T E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(\int_{\Omega^+}\left|u\right|^q dx\right)^{\frac{1}{q}}\left(\int_{\Omega^+}\left|g(u')\right|^{1+\frac{1}{p}}dx\right)^{\frac{p}{p+1}}dt$$

$$\leq \int_S^T E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(\epsilon\int_{\Omega^+}\left|u\right|^q dx + c_\epsilon\int_{\Omega^+}\left|g(u')\right|^{1+\frac{1}{p}}dx\right)dt$$

$$\leq \epsilon\int_S^T E^{\frac{r+q-1}{2}}(t)dt + c_\epsilon E^{\frac{r-1}{2}}(S)\int_S^T\int_{\Omega^+} e^{\phi(x)}u'g(u')dxdt$$

$$\leq \epsilon\int_S^T E^{\frac{r+1}{2}}(t)dt + c_\epsilon\left(E^{\frac{r+1}{2}}(S) - E^{\frac{r+1}{2}}(T)\right).$$

On the other hand, using the growth assumption (2.12) and Poincaré's inequality, we have

$$-\int_S^T\int_{\Omega^-} E^{\frac{r-1}{2}}(t)e^{\phi(x)}ug(u')dxdt$$

$$\leq \int_S^T E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(\epsilon\int_{\Omega^-}\left|u\right|^2 dx + c_\epsilon\int_{\Omega^-} g^2(u')dx\right)dt$$

$$\leq \epsilon\int_S^T E^{\frac{r-1}{2}}(t)\int_{\Omega^-} e^{\phi(x)}\left|\nabla u\right|^2 dxdt + c_\epsilon\int_S^T\int_{\Omega^-} E^{\frac{r-1}{2}}(t)\left(e^{\phi(x)}u'g(u')\right)^{\frac{2}{r+1}}dxdt$$

$$\leq \epsilon\int_S^T E^{\frac{r+1}{2}}(t)dt + c_\epsilon\left(E(S) - E(T)\right).$$

Taking the sum of the last two inequalities and substituting it into the right-hand side of (4.4), using (2.3), and choosing $\epsilon \in ]0, b[$, we obtain that

$$\int_S^T E^{\frac{r+1}{2}}(t)dt \leq c\left(E^{\frac{r+1}{2}}(S) + E(S)\right) + c\int_S^T\int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|u'\right|^2 dxdt. \tag{4.5}$$

Using another time (2.12) and (4.3), we have

$$\int_S^T\int_{\Omega^+} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|u'\right|^2 dxdt \leq cE^{\frac{r-1}{2}}(S)\int_S^T\int_{\Omega^+} e^{\phi(x)}u'g(u')dxdt$$

$$\leq c\left(E^{\frac{r+1}{2}}(S) - E^{\frac{r+1}{2}}(T)\right).$$

In the same way, using Young's inequality, we get

$$\int_S^T\int_{\Omega^-} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|u'\right|^2 dxdt \leq c\int_S^T\int_{\Omega^-} E^{\frac{r-1}{2}}(t)\left(e^{\phi(x)}u'g(u')\right)^{\frac{2}{r+1}}dxdt$$

$$\leq \epsilon\int_S^T E^{\frac{r+1}{2}}(t)dt + c_\epsilon\int_S^T\int_{\Omega^-} e^{\phi(x)}u'g(u')dxdt$$

$$\leq \epsilon\int_S^T E^{\frac{r+1}{2}}(t)dt + c_\epsilon\left(E(S) - E(T)\right).$$

Substituting the sum of these two estimates into the right-hand side of (4.5), choosing $\epsilon$ small enough, and letting $T$ go to $\infty$, we obtain

$$\int_S^\infty E^{\frac{r+1}{2}}(t)dt \leq c\left(1 + E^{\frac{r-1}{2}}(0)\right)E(S) \leq cE(S);$$

then (4.1) follows, which gives (2.11) and (2.15) and finishes the proof of Theorem 2.2.

**5. Uniform decay: Proof of Theorem 2.3.** In this section we prove the exponential decay of energy (2.7) for strong solutions of (P′), and by a density argument we obtain the same results for weak solutions.

The proof is similar to the one given in section 3.

Using the first equation in (P′) and the boundary conditions, we can easily prove that

$$E'(t) = -2\int_{\Gamma_1} u' g(u')\,dx - 2\int_\Omega u' h\left(\nabla u\right)\,dx. \tag{5.1}$$

Using Assumptions 2.2, 2.3, and 2.5, from (5.1) it holds that (see section 3)

$$E'(t) \le -2\int_\Omega u' h\left(\nabla u\right)\,dx \le \beta\int_\Omega \Big(\left|u'\right|^2 + \left|\nabla u\right|^2\Big)\,dx \le \beta E(t);$$

then $E$ satisfies (3.11) and (3.12) with $k = 1$ (see (2.18)). Following the proof given in section 3, it is sufficient to prove that, for all $0 \le S \le T < \infty$,

$$\int_S^T E(t)dt \le \bar{a}(E(S) + E(T)) + \hat{a}(E(S) - E(T)) \tag{5.2}$$

with $\bar{a}$, $\hat{a} > 0$ and $2\beta\bar{a} < 1$ or $\bar{a} \le \hat{a}$. Then the proof can be completed as in section 3.

To prove (5.2), let $\epsilon_0 \in ]0, 1[$ (will be chosen later in the course of the proof); we multiply the first equation in (P′) by

$$2m \cdot \nabla u + (n - \epsilon_0)u,$$

integrating the obtained result over $\Omega \times [S, T]$ and using the boundary conditions. We are going to estimate the terms of the result formula. We have

$$I_1 := \int_S^T \int_\Omega u''\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dxdt$$

$$= \left[\int_\Omega u'\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dx\right]_S^T - \int_S^T \int_\Omega \Big(m \cdot \nabla(u')^2 + (n - \epsilon_0)\left|u'\right|^2\Big)dxdt$$

$$= \epsilon_0 \int_S^T \int_\Omega \left|u'\right|^2 dxdt - \int_S^T \int_{\Gamma_1} (m \cdot \nu)\left|u'\right|^2 d\Gamma dt$$

$$+ \left[\int_\Omega u'\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dx\right]_S^T.$$

We estimate the last term in this inequality; we have

$$\int_\Omega \Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)^2 dx - \int_\Omega (2m \cdot \nabla u)^2 dx$$

$$= \int_\Omega \Big((n - \epsilon_0)^2 \left|u\right|^2 + 2(n - \epsilon_0)m \cdot \nabla(u)^2\Big)\,dx$$

$$= \int_\Omega \Big((n-\epsilon_0)^2 |u|^2 - 2(n-\epsilon_0)n |u|^2\Big)\, dx + 2(n-\epsilon_0)\int_{\Gamma_1} (m\cdot\nu)\, |u|^2 \, d\Gamma$$

$$= (\epsilon_0 + n)(\epsilon_0 - n)\int_\Omega |u|^2\, dx + 2(n-\epsilon_0)\int_{\Gamma_1} (m\cdot\nu)\, |u|^2\, d\Gamma$$

$$\leq 2(n-\epsilon_0)R\int_{\Gamma_1} |u|^2\, d\Gamma;$$

then

$$\int_\Omega \Big(2m\cdot\nabla u + (n-\epsilon_0)u\Big)^2 dx \leq \int_\Omega (2m\cdot\nabla u)^2 dx + 2(n-\epsilon_0)R\int_{\Gamma_1} |u|^2\, d\Gamma. \tag{5.3}$$

Since, for all $\epsilon > 0$,

$$\Big|\int_\Omega \Big(2m\cdot\nabla u + (n-\epsilon_0)u\Big)u' dx\Big|$$

$$\leq \frac{\epsilon}{2}\int_\Omega |u'|^2\, dx + \frac{1}{2\epsilon}\Big(\int_\Omega (2m\cdot\nabla u)^2 dx + 2(n-\epsilon_0)R\int_{\Gamma_1} |u|^2\, d\Gamma\Big)$$

$$\leq \int_\Omega \left(\frac{\epsilon}{2}|u'|^2 + \frac{2R^2}{\epsilon}|\nabla u|^2\, dx\right) + \frac{R}{\epsilon}(n-\epsilon_0)\bar{c}\int_\Omega |\nabla u|^2\, dx,$$

where $\bar{c}$ is the positive constant satisfying (Poincaré's inequality)

$$\int_{\Gamma_1} |v|^2 d\Gamma \leq \bar{c}\int_\Omega |\nabla v|^2 dx \qquad \forall v\in V.$$

Choosing $\epsilon = 2\sqrt{R(R+\frac{\bar{c}}{2}(n-\epsilon_0))}$, we obtain

$$\Big|\int_\Omega \Big(2m\cdot\nabla u + (n-\epsilon_0)u\Big)u' dx\Big| \leq \sqrt{R\Big(R+\frac{\bar{c}}{2}(n-\epsilon_0)\Big)}E(t) := a_1 E(t).$$

Then we deduce

$$I_1 \geq -a_1(E(S)+E(T)) - R\int_S^T\int_{\Gamma_1} |u'|^2\, d\Gamma dt + \epsilon_0\int_S^T\int_\Omega |u'|^2\, dxdt. \tag{5.4}$$

On the other hand, taking the generalized Green formula and recalling the identity

$$2\nabla u\cdot\nabla(m\cdot\nabla u) = 2|\nabla u|^2 + m\cdot\nabla(|\nabla u|^2)$$

(note also that on $\Gamma_0$ we have $\nabla u = \partial_\nu u\nu$), we infer

$$I_2 := \int_S^T\int_\Omega (-\Delta u)\Big(2m\cdot\nabla u + (n-\epsilon_0)u\Big)dxdt$$

$$= (2-\epsilon_0)\int_S^T\int_\Omega |\nabla u|^2\, dxdt - \int_S^T\int_{\Gamma_0} (m\cdot\nu)\, |\nabla u|^2\, d\Gamma dt$$

$$+ \int_S^T\int_{\Gamma_1} \Big((m\cdot\nu)\, |\nabla u|^2 - (n-\epsilon_0)u\partial_\nu u - 2(m\cdot\nabla u)\partial_\nu u\Big)d\Gamma dt.$$

Using the definition of $\Gamma_0$ and $\Gamma_1$, we deduce

$$I_2 \geq (2-\epsilon_0)\int_S^T\int_\Omega |\nabla u|^2\,dxdt$$

$$+\int_S^T\int_{\Gamma_1}\Big(\delta\,|\nabla u|^2-(n-\epsilon_0)u\partial_\nu u-\delta\,|\nabla u|^2-\frac{R^2}{\delta}(\partial_\nu u)^2\Big)d\Gamma dt;$$

then

$$I_2 \geq (2-\epsilon_0)\int_S^T\int_\Omega |\nabla u|^2\,dxdt-\int_S^T\int_{\Gamma_1}\Big((n-\epsilon_0)u\partial_\nu u+\frac{R^2}{\delta}(\partial_\nu u)^2\Big)d\Gamma dt. \tag{5.5}$$

Similarly, using (2.6), (5.3), and the Cauchy–Schwarz inequality, we have

$$I_3 := \int_S^T\int_\Omega h(\nabla u)\Big(2m\cdot\nabla u+(n-\epsilon_0)u\Big)dxdt$$

$$\geq -\frac{R}{\beta}\int_S^T\int_\Omega h^2(\nabla u)dxdt-\frac{\beta}{4R}\int_S^T\Big(4R^2\int_\Omega|\nabla u|^2\,dx+2(n-\epsilon_0)R\int_{\Gamma_1}|u|^2\,d\Gamma\Big)dt;$$

we conclude that

$$I_3 \geq -2\beta R\int_S^T\int_\Omega|\nabla u|^2\,dxdt-\frac{\beta}{2}(n-\epsilon_0)\int_S^T\int_{\Gamma_1}|u|^2\,d\Gamma dt. \tag{5.6}$$

Using (2.3) and the fact that $F$ is nonnegative and $F(0)=0$, we obtain

$$\begin{aligned}I_4 &:= \int_S^T\int_\Omega f(u)\Big(2m\cdot\nabla u+(n-\epsilon_0)u\Big)dxdt\\
&\geq (n-\epsilon_0)b\int_S^T\int_\Omega 2F(u)dxdt+\int_S^T\int_\Omega 2m\cdot\nabla(F(u))dxdt\\
&\geq ((n-\epsilon_0)b-n)\int_S^T\int_\Omega 2F(u)dxdt+\int_S^T\int_{\Gamma_1}2(m\cdot\nu)F(u)d\Gamma dt;\end{aligned}$$

then we deduce

$$I_4 \geq ((n-\epsilon_0)b-n)\int_S^T\int_\Omega 2F(u)dxdt. \tag{5.7}$$

Now we distinguish two cases.

*Case* 3. If $b>1$, then assuming that $\beta R<1$ and choosing $\epsilon_0=\min\{1-\beta R,\frac{b-1}{b+1}n\}$, we deduce that $\min\{\epsilon_0, 2-\epsilon_0-2\beta R,(n-\epsilon_0)b-n\}=\epsilon_0$. Combining (5.4)–(5.7), taking the fact that $I_1+I_2+I_3+I_4=0$ in account, we obtain

$$\epsilon_0\int_S^T\int_\Omega\Big(|u'|^2+|\nabla u|^2+2F(u)\Big)dxdt\leq a_1(E(S)+E(T))$$

$$+\int_S^T\int_{\Gamma_1}\left(R\left|u'\right|^2+\frac{\beta}{2}(n-\epsilon_0)\left|u\right|^2+(n-\epsilon_0)u\partial_\nu u+\frac{R^2}{\delta}(\partial_\nu u)^2\right)d\Gamma dt.$$

*Case* 4. If $f$ is linear, $f(s)=\alpha s$ for some positive constant $\alpha$, then $b=1$ and we conclude from (5.7) that

$$\begin{aligned} I_4 &\geq -\epsilon_0\int_S^T\int_\Omega 2F(u)dxdt=\epsilon_0\int_S^T\int_\Omega 2F(u)dxdt-2\epsilon_0\int_S^T\int_\Omega 2F(u)dxdt \\ &=\epsilon_0\int_S^T\int_\Omega 2F(u)dxdt-2\epsilon_0\alpha\int_S^T\int_\Omega\left|u\right|^2dxdt \\ &\geq\epsilon_0\int_S^T\int_\Omega 2F(u)dxdt-2\epsilon_0\alpha\hat{c}\int_S^T\int_\Omega\left|\nabla u\right|^2dxdt, \end{aligned}$$

where $\hat{c}$ is the smallest imbedding positive constant satisfying

$$\int_\Omega\left|v\right|^2dx\leq\hat{c}\int_\Omega\left|\nabla v\right|^2dx\qquad\forall v\in V. \tag{5.8}$$

Assuming that $\beta R<1$ and choosing $\epsilon_0=\frac{1-\beta R}{1+\alpha\hat{c}}$, then $\min\{\epsilon_0,2-\epsilon_0-2\beta R-2\epsilon_0\alpha\hat{c}\}=\epsilon_0$ and the same inequality obtained in Case 3 holds true.

We now use the boundary condition on $\Gamma_1$; we have in both previous cases

$$\epsilon_0\int_S^T E(t)dt\leq a_1(E(S)+E(T)) \tag{5.9}$$

$$+\int_S^T\int_{\Gamma_1}\left(R\left|u'\right|^2+\frac{R^2}{\delta}g^2(u')+\frac{\beta}{2}(n-\epsilon_0)\left|u\right|^2-(n-\epsilon_0)ug(u')\right)d\Gamma dt.$$

Using (5.1), the Cauchy–Schwarz inequality and taking the assumptions (2.4), (2.5), and (2.6) into account, it holds that

$$\int_S^T\int_{\Gamma_1}\left(R|u'|^2+\frac{R^2}{\delta}g^2(u')\right)dx\,dt\leq\left(\frac{R}{c_1}+\frac{R^2}{\delta}c_2\right)\int_S^T\int_{\Gamma_1}u'g(u')dx\,dt$$

$$=\frac{1}{2}\left(\frac{R}{c_1}+\frac{R^2}{\delta}c_2\right)\int_S^T\left(-E'(t)-2\int_\Omega u'h(\nabla u)dx\right)dt$$

$$\leq\frac{1}{2}\left(\frac{R}{c_1}+\frac{R^2}{\delta}c_2\right)(E(S)-E(T))+\frac{1}{2}\left(\frac{R}{c_1}+\frac{R^2}{\delta}c_2\right)\beta\int_S^T\int_\Omega\left(|u'|^2+|\nabla u|^2\right)dx\,dt;$$

we note $a_2:=\frac{1}{2}(\frac{R}{c_1}+\frac{R^2}{\delta}c_2)$ and deduce

$$\int_S^T\int_{\Gamma_1}\left(R|u'|^2+\frac{R^2}{\delta}g^2(u')\right)dx\,dt\leq a_2\left(E(S)-E(T)\right)+\beta a_2\int_S^T E(t)dt. \tag{5.10}$$

Similarly, we have

$$(n-\epsilon_0)\int_S^T\int_{\Gamma_1}\left(\frac{\beta}{2}|u|^2-ug(u')\right)dx\,dt$$

$$\leq \frac{1}{2}(n-\epsilon_0)\int_S^T\int_{\Gamma_1}\left(\frac{1}{\epsilon}g^2(u')+(\beta+\epsilon)|u|^2\right)dx\,dt$$

$$\leq \frac{1}{2}(n-\epsilon_0)\int_S^T\int_{\Gamma_1}\left(\frac{c_2}{\epsilon}u'g(u')+(\beta+\epsilon)|u|^2\right)dx\,dt$$

$$=\frac{c_2}{2\epsilon}(n-\epsilon_0)\int_S^T\left(-\frac{1}{2}E'(t)-\int_\Omega u'h(\nabla u)dx\right)dt$$

$$+\frac{1}{2}(\beta+\epsilon)(n-\epsilon_0)\bar{c}\int_S^T\int_\Omega|\nabla u|^2dx\,dt$$

$$\leq \frac{c_2}{4\epsilon}(n-\epsilon_0)(E(S)-E(T))+\frac{1}{2}(\beta+\epsilon)(n-\epsilon_0)\bar{c}\int_S^T\int_\Omega|\nabla u|^2dx\,dt$$

$$+\frac{c_2}{2\epsilon}(n-\epsilon_0)\int_S^T\int_\Omega\left(\frac{\epsilon'\beta^2}{2}|\nabla u|^2+\frac{1}{2\epsilon'}|u'|^2\right)dx\,dt,$$

we choose $\epsilon=\beta\sqrt{\frac{c_2\epsilon'}{2\bar{c}}}$, $\epsilon'=\frac{1}{\beta\sqrt{2}}$, and we note $a_3:=\frac{1}{2}(n-\epsilon_0)\sqrt{\frac{\bar{c}c_2}{\sqrt{2}\beta}}$, $a_4:=(n-\epsilon_0)(\frac{\beta\bar{c}}{2}+\sqrt{\frac{\bar{c}c_2\beta}{2\sqrt{2}}})$. It follows that

$$(n-\epsilon_0)\int_S^T\int_{\Gamma_1}\left(\frac{\beta}{2}|u'|^2-ug(u')\right)dx\,dt \tag{5.11}$$

$$\leq a_4\int_S^T E(t)dt+a_3\left(E(S)-E(T)\right).$$

Combining (5.9), (5.10), and (5.11), we have

$$(\epsilon_0-\beta a_2-a_4)\int_S^T E(t)dt \tag{5.12}$$

$$\leq a_1\left(E(S)+E(T)\right)+(a_2+a_3)\left(E(S)-E(T)\right).$$

If $\beta$ is small enough so that $2\beta a_1<a_5:=\epsilon_0-\beta a_2-a_4$, that is,

$$\beta(2a_1+a_2)+a_4<\epsilon_0=\begin{cases}\min\{1-\beta R,\frac{b-1}{b+1}n\} & \text{if } b>1,\\ \frac{1-\beta R}{1+\alpha\hat{c}} & \text{if } f \text{ is linear}\end{cases}$$

(note that $\beta(2a_1+a_2)+a_4$ goes to 0 when $\beta$ goes to 0), we conclude (5.2) with $\bar{a}=\frac{a_1}{a_5}$ and $\hat{a}=\frac{a_2+a_3}{a_5}$. We fix then $T_0>\frac{-1}{\beta}\ln(1-2\beta\bar{a})$. Using (3.12) with $\tau=S$, we have

$$\int_S^{S+T_0}E(t)dt\geq\frac{1}{\beta}(1-e^{-\beta T_0})E(S+T_0).$$

We insert this inequality into (5.2) with $T = S + T_0$ and obtain

$$\left(\frac{1}{\beta}(1 - e^{-\beta T_0}) + \hat{a} - \bar{a}\right) E(S + T_0) \leq (\hat{a} + \bar{a})E(S).$$

Thanks to the hypothesis on $T_0$, we have $\frac{1}{\beta}(1 - e^{-\beta T_0}) > 2\bar{a}$, which implies (3.1) with $d = \frac{\hat{a}+\bar{a}}{\frac{1}{\beta}(1-e^{-\beta T_0})+\hat{a}-\bar{a}}$.

If $\beta a_2 + a_4 < \epsilon_0$ and $a_1 \leq a_2 + a_3$ (that is, $\sqrt{R(R + \frac{\bar{c}}{2}(n - \epsilon_0))} \leq \frac{1}{2}(\frac{R}{c_1} + \frac{R^2}{\delta} c_2) + \frac{1}{2}(n - \epsilon_0)\sqrt{\frac{\bar{c}c_2}{\sqrt{2}\beta}}$), we conclude from (5.12) that (3.14) follows with $a_0 = \frac{a_1+a_2+a_3}{a_5}$.

Then in both cases the proof of Theorem 2.3 can be completed as in section 3.

**6. Decay estimates: Proof of Theorem 2.4.** To prove Theorem 2.4, which concerns the stability of (P′) in the particular case $h(\nabla u) = -\nabla\phi \cdot \nabla u$, with $\phi \in W^{1,\infty}(\Omega)$, it is sufficient to prove that the equivalent energy $E$ defined by (2.14) satisfies (4.1) (see section 4).

In this section, we shall denote by $c$ diverse positive constants, by $\epsilon$ diverse positive constants small enough (which can be changed from a line to another), and by $c_\epsilon$ diverse positive constants depending on $\epsilon$.

A simple computation shows that

$$E'(t) = -2\int_{\Gamma_1} e^{\phi(x)} u' g(u') dx, \qquad t \in \mathbb{R}^+. \tag{6.1}$$

Assumption (2.4) implies that the equivalent energy is nonincreasing.

We fix $\epsilon_0 > 0$ and we multiply the first equation in (P′) by

$$E^{\frac{r-1}{2}}(t)e^{\phi(x)}\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big),$$

integrating the obtained result over $\Omega \times [S, T]$ and using the boundary conditions. We have

$$\begin{aligned}
I_1 &:= \int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}u''\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dxdt \\
&= \left[\int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}u'\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dx\right]_S^T \\
&\quad - \frac{r-1}{2}\int_S^T \int_{\Gamma_1} E^{\frac{r-3}{2}}(t)E'(t)e^{\phi(x)}\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dxdt \\
&\quad - \int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\int_\Omega\Big(m \cdot \nabla(u')^2 + (n - \epsilon_0)\left|u'\right|^2\Big)dxdt
\end{aligned}$$

$$\begin{aligned}
= &\int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}(\epsilon_0 + m \cdot \nabla\phi)\left|u'\right|^2 dxdt - \int_S^T \int_{\Gamma_1} E^{\frac{r-1}{2}}(t)e^{\phi(x)}(m \cdot \nu)\left|u'\right|^2 d\Gamma dt \\
&+ \left[\int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}u'\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dx\right]_S^T \\
&- \frac{r-1}{2}\int_S^T \int_{\Gamma_1} E^{\frac{r-3}{2}}(t)E'(t)e^{\phi(x)}\Big(2m \cdot \nabla u + (n - \epsilon_0)u\Big)dxdt.
\end{aligned}$$

The last two terms of this equality can be easily majorized by $cE^{\frac{r+1}{2}}(S)$; then we deduce

$$I_1 \geq -cE^{\frac{r+1}{2}}(S) - R\int_S^T \int_{\Gamma_1} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|u'\right|^2 d\Gamma dt \tag{6.2}$$

$$+(\epsilon_0 - R\|\nabla\phi\|_\infty)\int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|u'\right|^2 dxdt.$$

On the other hand, taking the generalized Green formula (see section 5), we infer

$$I_2 := \int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}(-\Delta u - \nabla\phi\cdot\nabla u)\Big(2m\cdot\nabla u + (n-\epsilon_0)u\Big)dxdt$$

$$= (2-\epsilon_0)\int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|\nabla u\right|^2 dxdt - \int_S^T \int_{\Gamma_0} E^{\frac{r-1}{2}}(t)e^{\phi(x)}(m\cdot\nu)\left|\nabla u\right|^2 d\Gamma dt$$

$$+\int_S^T \int_{\Gamma_1} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\Big((m\cdot\nu)\left|\nabla u\right|^2 - (n-\epsilon_0)u\partial_\nu u - 2(m\cdot\nabla u)\partial_\nu u\Big)d\Gamma dt.$$

Using the definition of $\Gamma_0$ and $\Gamma_1$, we deduce

$$I_2 \geq (2-\epsilon_0)\int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left|\nabla u\right|^2 dxdt \tag{6.3}$$

$$-\int_S^T \int_{\Gamma_1} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left((n-\epsilon_0)u\partial_\nu u + \frac{R^2}{\delta}(\partial_\nu u)^2\right)d\Gamma dt.$$

Using (2.3) and the fact that $F$ is nonnegative, we obtain

$$I_3 := \int_S^T \int_\Omega E^{\frac{r-1}{2}}(t)e^{\phi(x)}f(u)\Big(2m\cdot\nabla u + (n-\epsilon_0)u\Big)dxdt$$

$$\geq (n-\epsilon_0)b\int_S^T \int_\Omega 2E^{\frac{r-1}{2}}(t)e^{\phi(x)}F(u)dxdt + \int_S^T \int_\Omega 2E^{\frac{r-1}{2}}(t)e^{\phi(x)}m\cdot\nabla(F(u))dxdt$$

$$\geq \int_S^T \int_\Omega((n-\epsilon_0)b - n - m\cdot\nabla\phi)2E^{\frac{r-1}{2}}(t)e^{\phi(x)}F(u)dxdt$$

$$+\int_S^T \int_{\Gamma_1} 2E^{\frac{r-1}{2}}(t)e^{\phi(x)}(m\cdot\nu)F(u)d\Gamma dt;$$

then we conclude that

$$I_3 \geq ((n-\epsilon_0)b - n - R\|\nabla\phi\|_\infty)\int_S^T \int_\Omega 2F(u)dxdt. \tag{6.4}$$

Thanks to the assumptions in Theorem 2.4, we have the following.

*Case* 5. If $R\|\nabla\phi\|_\infty < \min\{2,n\}$ and $b > \frac{n+R\|\nabla\phi\|_\infty}{n-R\|\nabla\phi\|_\infty}$, we can choose $\epsilon_0 \in ]R\|\nabla\phi\|_\infty, \min\{2, n - \frac{n+R\|\nabla\phi\|_\infty}{b}\}[$ and then

$$\min\{\epsilon_0 - R\|\nabla\phi\|_\infty, 2-\epsilon_0, (n-\epsilon_0)b - n - R\|\nabla\phi\|_\infty\} > 0.$$

*Case* 6. If $f$ is linear, $f(s) = \alpha s$ for some positive constant $\alpha$, then $b = 1$ and we conclude from (6.4) that

$$\begin{aligned} I_3 &\geq (-\epsilon_0 - R\|\nabla\phi\|_\infty)\int_S^T\int_\Omega 2F(u)dxdt \\ &= (\epsilon_0 - R\|\nabla\phi\|_\infty)\int_S^T\int_\Omega 2F(u)dxdt - 2\epsilon_0\int_S^T\int_\Omega 2F(u)dxdt \\ &= (\epsilon_0 - R\|\nabla\phi\|_\infty)\int_S^T\int_\Omega 2F(u)dxdt - 2\epsilon_0\alpha\int_S^T\int_\Omega |u|^2\,dxdt \\ &\geq (\epsilon_0 - R\|\nabla\phi\|_\infty)\int_S^T\int_\Omega 2F(u)dxdt - 2\epsilon_0\alpha\hat{c}\int_S^T\int_\Omega |\nabla u|^2\,dxdt, \end{aligned}$$

where $\hat{c}$ is the positive constant defined by (5.8). Then, assuming that $R\|\nabla\phi\|_\infty < \frac{2}{1+2\alpha\hat{c}}$ and taking $\epsilon_0 \in ]R\|\nabla\phi\|_\infty, \frac{2}{1+2\alpha\hat{c}}[$, the quantity $\min\{\epsilon_0 - R\|\nabla\phi\|_\infty, 2-(1+2\alpha\hat{c})\epsilon_0\}$ is positive.

Combining (6.2)–(6.4), taking the fact that $I_1 + I_2 + I_3 = 0$ into account, and using the boundary condition on $\Gamma_1$, we obtain in both previous cases

$$\tag{6.5} \begin{aligned} \int_S^T\int_\Omega E^{\frac{r+1}{2}}(t)dt &\leq cE^{\frac{r+1}{2}}(S) \\ &\quad + c\int_S^T\int_{\Gamma_1} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\Big(|u'|^2 + g^2(u') + |ug(u')|\Big)d\Gamma dt. \end{aligned}$$

We now estimate the last term of (6.5). We exploit the Cauchy–Schwarz inequality and the Sobolev imbedding $V \subset L^2(\Gamma_1)$ to get

$$\int_{\Gamma_1}|ug(u')|d\Gamma \leq \epsilon\int_{\Gamma_1}|u|^2 d\Gamma + c_\epsilon\int_{\Gamma_1}g^2(u')d\Gamma \leq \epsilon E(t) + c_\epsilon\int_{\Gamma_1}g^2(u')d\Gamma.$$

Substituting this inequality into the right-hand side of (6.5) and choosing $\epsilon > 0$ small enough, we obtain that

$$\tag{6.6} \begin{aligned} \int_S^T\int_\Omega E^{\frac{r+1}{2}}(t)dt &\leq cE^{\frac{r+1}{2}}(S) \\ &\quad + c\int_S^T\int_{\Gamma_1} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\Big(|u'|^2 + g^2(u')\Big)d\Gamma dt. \end{aligned}$$

We follow now the proof given in section 4. We note

$$\Gamma^+ = \{x \in \Gamma_1 : |u'| > 1\} \quad \text{and} \quad \Gamma^- = \Gamma_1 \setminus \Gamma^+.$$

By (2.12) and (6.1) we have

$$\int_S^T \int_{\Gamma^+} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(\left|u'\right|^2 + g^2(u')\right)dxdt \le cE^{\frac{r-1}{2}}(S)\int_S^T \int_{\Gamma^+} e^{\phi(x)}u'g(u')dxdt$$
$$\le c\left(E^{\frac{r+1}{2}}(S) - E^{\frac{r+1}{2}}(T)\right).$$

In the same way (using Young's inequality), we get

$$\int_S^T\int_{\Gamma^-} E^{\frac{r-1}{2}}(t)e^{\phi(x)}\left(\left|u'\right|^2 + g^2(u')\right)dxdt \le c\int_S^T \int_{\Gamma^-} E^{\frac{r-1}{2}}(t)\left(e^{\phi(x)}u'g(u')\right)^{\frac{2}{r+1}}dxdt$$
$$\le \epsilon\int_S^T E^{\frac{r+1}{2}}(t)dt + c_\epsilon\int_S^T\int_{\Gamma^-} e^{\phi(x)}u'g(u')dxdt$$
$$\le \epsilon\int_S^T E^{\frac{r+1}{2}}(t)dt + c_\epsilon\left(E(S) - E(T)\right).$$

Substituting the sum of these two estimates into the right-hand side of (6.6), choosing $\epsilon$ small enough, and letting $T$ go to $\infty$, we obtain (4.1). This finishes the proof of Theorem 2.4.

*Remark.* Using the method developed above, the same results can be easily obtained if we replace the first equation in (P) by

$$u'' - \Delta u + q_1(x)h(\nabla u) + q_2(x)f(u) + q_3(x)g(u') = 0 \quad \text{in} \quad \Omega\times\mathbb{R}^+,$$

and the first equation and the boundary condition on $\Gamma_1$ in (P′) by

$$\begin{cases} u'' - \Delta u + q_1(x)h(\nabla u) + q_2(x)f(u) = 0 & \text{in} \quad \Omega\times\mathbb{R}^+, \\ \partial_\nu u + q_4(x)u + q_3(x)g(u') = 0 & \text{on} \quad \Gamma_1\times\mathbb{R}^+, \end{cases}$$

where $q_i : \Omega \to \mathbb{R}$ are bounded functions such that $q_2(x) \ge 0$, $q_4(x) \ge 0$, $q_3(x) \ge a_0 > 0$. If $q_4(x) \ge b_0 > 0$, we may take $\Gamma_0 = \emptyset$.

We define the equivalent energy of (P) and (P′), respectively, by

$$E(t) = \int_\Omega e^{\varphi(x)}\left(\left|u'\right|^2 + \left|\nabla u\right|^2 + 2q_2(x)F(u)\right)dx, \tag{6.7}$$

$$E(t) = \int_\Omega e^{\varphi(x)}\left(\left|u'\right|^2 + \left|\nabla u\right|^2 + 2q_2(x)F(u)\right)dx + \int_{\Gamma_1} e^{\varphi(x)}\left|u\right|^2 d\Gamma \tag{6.8}$$

if $h(\nabla u) = -\nabla\phi\cdot\nabla u$ with $\phi \in W^{1,\infty}(\Omega)$, where $\varphi \in W^{1,\infty}(\Omega)$ satisfying $\nabla\varphi = q_1(x)\nabla\phi$.

In the general case, we assume that $\beta\|q_1\|_\infty$ is small enough as in Theorem 2.1 and Theorem 2.3, where $\beta$, $c_1$, and $c_2$ are replaced by $\beta\|q_1\|_\infty$, $a_0c_1$, and $c_2\|q_3\|_\infty$, respectively, and we define the energy of (P) and (P′), respectively, by (6.7) and (6.8) with $\varphi \equiv 0$. In order to get ride of the lower-order term, which is $\int_{\Gamma_1}\left|u\right|^2 d\Gamma$, we use the solution of an auxiliary elliptic problem as an additional multiplier (see [4, Lemma 4.2]).

**7. Some applications of our method.** In [6], we considered the following Petrovsky system:

$$
\begin{cases} u'' + \Delta^2 u + q(x)u + g(u') = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{7.1}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\Gamma$ and $\nu$ is the outward unit normal vector to $\Gamma$. For $g$ continuous, increasing, satisfying $g(0) = 0$, and $q : \Omega \to \mathbb{R}^+$ a bounded function, we proved a global existence and a regularity result. We also established, under suitable growth conditions on $g$, decay results for weak, as well as strong, solutions. Precisely, we showed that the solution decays exponentially if $g$ behaves like a linear function, whereas the decay is of a polynomial order otherwise. Similar results to the above system, coupled with a semilinear wave equation, have been established by Guesmia in [5]. In [17], Messaoudi studied the problem

$$
\begin{cases} u'' + \Delta^2 u + au'\left|u'\right|^{m-2} - bu\left|u\right|^{p-2} = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases}
$$

where $a$, $b > 0$ and $p$, $m > 2$. This is a similar problem to (7.1), which contains a nonlinear source term competing with the damping factor. He established an existence result and showed that the solution continues to exist globally if $m \geq p$; however, it blows up in finite time if $m < p$. In this paper no result of stability was announced.

In [7], we obtained some stabilization results of the following elasticity system:

$$
\begin{cases} u_i'' - \sigma_{ij,j} + g_i(u_i') = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u_i = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u_i(x,0) = u_i^0(x) \quad \text{and} \quad u_i'(x,0) = u_i^1(x) & \text{in} \quad \Omega, \\ i = 1, \dots, n, \end{cases} \tag{7.2}
$$

where the unknown $u = (u_1, \dots, u_n) : \Omega \to \mathbb{R}^n$. Here, $\sigma_{ij,j} = \sum_{j=1}^{j=n} \frac{\partial \sigma_{ij}}{\partial x_j}$, $\sigma_{ij} = \sum_{k,l=1}^{k,l=n} a_{ijkl}\varepsilon_{ij}$, $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, $u_{j,i} = \frac{\partial u_j}{\partial x_i}$, and $a_{ijkl} \in W^{1,\infty}(\Omega)$. We proved some decay estimates which are crucially dependent on the behavior of the damping $g_i$ at the origin and infinity. In [8], we extended these results to the case of localized dissipations; that is, the damping is effective only in a neighborhood of a suitable subset of the boundary.

In [4], we considered the problem of exact controllability and boundary stabilization of elasticity systems with coefficients $a_{ijkl}$ depending also on time $t$. The stabilization results obtained in [4] were generalized in [3] to the nonlinear feedback case. The results obtained in [3] and [4] improve and generalize some ones obtained earlier by Alabau and Komornik [1] in the case where $g_i$ is linear and $a_{ijkl} = \text{const}$.

The decrease of energy plays a crucial role in studying the asymptotic stability of the systems cited above. The situation of nondissipative systems (that is, the energy is not decreasing) was not previously considered.

Using the method developed in previous sections, we can extend Theorems 2.1–2.4 to the following more general nondissipative problems.

**7.1. Petrovsky system.**

$$\begin{cases} u'' + \Delta^2 u + q_1(x)h(\Delta u) + q_2(x)f(u) + q_3(x)g(u') = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases}$$

where $h,\ f,\ g : \mathbb{R} \to \mathbb{R}$ are three given functions satisfying Assumptions 2.1–2.5 and $q_i$ are three given functions defined as in the remark above. Here $c_0 > 0$ is the smallest imbedding positive constant (depending only on $\Omega$) satisfying

$$\int_\Omega |v|^2 dx \leq c_0 \int_\Omega |\Delta v|^2 dx \qquad \forall v \in H_0^2(\Omega).$$

The energy and the equivalent energy are, respectively, defined by

$$E(t) = \int_\Omega \Big( |u'|^2 + |\Delta u|^2 + 2q_2(x)F(u) \Big) dx, \qquad t \in \mathbb{R}^+,$$

in the general case, and

$$E(t) = \int_\Omega e^{\varphi(x)} \Big( |u'|^2 + |\Delta u|^2 + 2q_2(x)F(u) \Big) dx, \qquad t \in \mathbb{R}^+$$

if $h(\Delta u) = -\phi(x)\Delta u$, with $\phi \in L^\infty(\Omega)$, where $\varphi \in W^{2,\infty}(\Omega)$ satisfying $\Delta\varphi = q_1(x)\phi(x)$.

**7.2. Coupled system.** We consider the nonlinear coupled wave equation and Petrovsky system:

$$\begin{cases} u_1'' + \Delta^2 u_1 + q_1(x)h_1(\Delta u_1) + q_2(x)f_1(u_1) \\ \qquad\qquad + q_3(x)g_1(u_1') + a_1(x)u_2 = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u_2'' - \Delta u_1 + l_1(x)h_2(\nabla u_2) + l_2(x)f_2(u_2) \\ \qquad\qquad + l_3(x)g_2(u_2') + a_2(x)u_1 = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u_2 = u_1 = \partial_\nu u_1 = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u_i(x,0) = u_i^0(x) \quad \text{and} \quad u_i'(x,0) = u_i^1(x),\ i = 1,2 & \text{in} \quad \Omega, \end{cases}$$

where $a_1$, $a_2$ are two bounded functions with norms small enough (see [5]) and the $l_i$, $h_i$, $f_i$, and $g_i$ are given functions defined as $q_i$, $h$, $f$, and $g$, respectively.

If $h_1(\Delta u_1) = -\phi_1(x)\Delta u_1$ and $h_2(\nabla u_2) = -\nabla\phi_2 \cdot \nabla u_2$ with $\phi_1 \in L^\infty(\Omega)$ and $\phi_2 \in W^{1,\infty}(\Omega)$, then we assume that $a_1(x)e^{\varphi_1(x)} = a_2(x)e^{\varphi_2(x)}$, where $\varphi_1 \in W^{2,\infty}(\Omega)$ and $\varphi_2 \in W^{1,\infty}(\Omega)$ satisfying $\Delta\varphi_1 = q_1(x)\phi_1(x)$ and $\nabla\varphi_2 = l_1(x)\nabla\phi_2$; we define the equivalent energy by

$$E(t) = \int_\Omega e^{\varphi_1(x)} \Big( |u_1'|^2 + |\Delta u_1|^2 + 2q_2(x)F_1(u_1) \Big) dx \tag{7.3}$$

$$+ \int_\Omega e^{\varphi_2(x)} \Big( |u_2'|^2 + |\nabla u_2|^2 + 2l_2(x)F_2(u_2) \Big) dx + 2\int_\Omega e^{\varphi_1(x)} a_1(x)u_1u_2 dx,$$

which is nonincreasing,

$$E'(t) = -2\int_\Omega \Big( e^{\varphi_1(x)} q_3(x)u_1'g_1(u_1') + e^{\varphi_2(x)} l_3(x)u_2'g_2(u_2') \Big) \leq 0.$$

In the general case, we assume that $a_1(x) = a_2(x)$ and we define the energy by (7.3) with $\varphi_1 \equiv \varphi_2 \equiv 0$.

**7.3. Elasticity systems.** We are interested in the precise decay property of the solution for elasticity systems:

$$(7.4)\qquad \begin{cases} u_i'' - \sigma_{ij,j} + q_{1,i}(x)h_i(\sigma_{i1}, \dots, \sigma_{in}) \\ \qquad\qquad + q_{2,i}(x)f_i(u_i) + q_{3,i}(x)g_i(u_i') = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u_i = 0 & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u_i(x,0) = u_i^0(x) \quad \text{and} \quad u_i'(x,0) = u_i^1(x) & \text{in} \quad \Omega, \\ i = 1, \dots, n, \end{cases}$$

with the same notations as before. Here for $i = 1, \dots, n$, $h_i$, $f_i$, and $g_i$ satisfy the same hypothesis as $h$, $f$, and $g$ in section 2, respectively, and $q_{1,i}$, $q_{2,i}$, and $q_{3,i}$ are defined as $q_1$, $q_2$, and $q_3$ in section 7.1, respectively.

We define the equivalent energy of (7.4) by the formula

$$E(t) = \int_\Omega \sum_{i=1}^{i=n} e^{\varphi_i(x)} \left( |u_i'|^2 + \sum_{j=1}^{j=n} \sigma_{ij}\varepsilon_{ij} + 2q_{2,i}(x)F_i(u_i) \right) dx,$$

where $\varphi_i \equiv 0$ if $h_i$ is nonlinear, and if $h_i$ is linear, $h_i(\zeta) = -\nabla\phi_i \cdot \zeta$ for all $\zeta \in \mathbb{R}^n$ with $\phi_i \in W^{1,\infty}(\Omega)$, then we take $\varphi_i \in W^{1,\infty}(\Omega)$ such that $\nabla\varphi_i = q_{1,i}(x)\nabla\phi_i$. In the case where all the functions $h_i$ are linear, our system is dissipative:

$$E'(t) = -2\int_\Omega \sum_{i=1}^{i=n} e^{\varphi_i(x)} q_{3,i}(x) u_i' g_i(u_i') dx \le 0.$$

We obtain the results of Theorem 2.1 and Theorem 2.2.

Under some geometric condition as in [3], the results of Theorem 2.3 and Theorem 2.4 can be easily proved in the case of boundary feedback; that is, we consider the homogenous Dirichlet condition on $\Gamma_0$, and we consider the following one on $\Gamma_1$ (see [3]):

$$\sum_{j=1}^{j=n} \sigma_{ij}\nu_j + q_{4,i}(x)u_i + q_{3,i}(x)g_i(u_i') = 0.$$

*Remark.* The method developed in this paper is direct and very flexible; it can be applied to various nondissipative problems (elasticity, thermoelasticity, Kirchoff, von Karman, coupled systems, ...) with an internal or a boundary feedback, and it can generalize the decay estimates (known in the dissipative case) to the nondissipative one.

**Open questions.** The main restrictive assumptions under which the stability results are valid are the smallness conditions on $\beta$ (defined by (2.6)) assumed in Theorems 2.1, 2.3, and 2.4. In the case of nonlinear function $h$, these assumptions are required to obtain the inequalities $(*)$ (given in the introduction). In Theorem 2.4 (stability of $(P')$ with $h(\nabla u) = -\nabla\phi \cdot \nabla u$), the smallness assumption on $\beta$ is required to absorb some terms caused by the use of the second multiplier $m \cdot \nabla u$. It would be interesting to know if the stability estimates still hold true under weaker assumption on $\beta$, using more sophisticated tools, for example, general multipliers. And if it is not the case, it would be interesting to know if other weaker stability estimates can be obtained.

Another important aspect of the case of nonlinear function $h$ is assumption (2.5) imposed on the damping $g$. It would be interesting to prove the same polynomial stability (obtained in the case of linear function $h$) under the weaker assumption (2.12). With this perspective, it would be interesting to look at what we can conclude at $\infty$ on a positive function satisfying the following inequalities more general than $(*)$:

$$\begin{cases} \int_S^T E^{a_0}(t)dt \leq a_1(E(S)+E(T)) + a_2(E(S)-E(T)) \qquad \forall 0 \leq S \leq T < \infty, \\ E'(t) \leq a_3 E(t) \qquad \forall t \geq 0, \end{cases}$$

where $a_i$, $i = 0, 1, 2, 3$, are nonnegative constants.

It would also be very interesting (particularly from the point of view of applications) to explore a more general class of hyperbolic equations based on the equation

$$K(x,t)u'' - Au + F(x,t,u,u',\nabla u) = 0,$$

where $K$ and $F$ are given functions and $Au = \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}u)$ is a second-order elliptic differential operator with smooth coefficients $a_{ij}$.

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