EXTREMALS FOR HARDY-SOBOLEV TYPE INEQUALITIES: THE INFLUENCE OF THE CURVATURE

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Abstract. We consider the optimal Hardy-Sobolev inequality on a smooth bounded domain of the Euclidean space. Roughly speaking, this inequality lies between the Hardy inequality and the Sobolev inequality. We address the questions of the value of the optimal constant and the existence of non-trivial extremals attached to this inequality. When the singularity of the Hardy part is located on the boundary of the domain, the geometry of the domain plays a crucial role: in particular, the convexity and the mean curvature are involved in these questions. The main difficulty to encounter is the possible bubbling phenomenon. We describe precisely this bubbling through refined concentration estimates. An offshot of these techniques allows us to provide general compactness properties for nonlinear equations, still under curvature conditions for the boundary of the domain.

Contents

1. The Hardy-Sobolev inequality and two questions

We consider the Euclidean space $\mathbb{R}^n$, $n \geq 3$. The famous Sobolev theorem asserts that there exists a constant $C_1(n) > 0$ such that

$$(1) \qquad \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}} \leq C_1(n) \int_{\mathbb{R}^n} |\nabla u|^2\, dx$$

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for all $u \in C_c^\infty(\mathbb{R}^n)$. Another very famous inequality is the Hardy inequality, which asserts that there exists $C_2(n) > 0$ such that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx \leq C_2(n) \int_{\mathbb{R}^n} |\nabla u|^2\, dx \tag{2}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. Interpolating these two inequalities, one gets the Hardy-Sobolev inequality: more precisely, let $s \in [0, 2]$, then there exists $C(s, n) > 0$ such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star}} \leq C(s, n) \int_{\mathbb{R}^n} |\nabla u|^2\, dx \tag{3}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$, where

$$2^\star(s) := \frac{2(n-s)}{n-2}.$$

Indeed, with $s = 0$, we recover the Sobolev inequality (1), and with $s = 2$, we recover the Hardy inequality (2). The Hardy-Sobolev inequality is a particular case of the family of functional inequalities obtained by Caffarelli-Kohn-Nirenberg [8]. When $s \in (0, 2)$, it is remarkable that the Hardy-Sobolev inequality inherites the singularity at 0 from the Hardy inequality and the superquadratic exponent from the Sobolev inequality. For completeness and density reasons, given Ω an open subset of $\mathbb{R}^n$, it is more convenient to work in the Sobolev space

$$H_{1,0}^2(\Omega) := \text{ Completion of } C_c^\infty(\Omega) \text{ for } \|\cdot\|$$

where $\|u\| := \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}$. Therefore, inequality (3) is valid for $u \in H_{1,0}^2(\Omega)$.

Following the programme developed for other functional inequalities, we saturate (3): given Ω an open subset of $\mathbb{R}^n$, we define

$$\mu_s(\Omega) := \inf_{u \in H_{1,0}^2(\Omega)\setminus\{0\}} I_\Omega(u), \text{ where } I_\Omega(u) := \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)}.$$

It follows fom the Hardy-Sobolev inequality that $\mu_s(\Omega) > 0$. We address the two following questions:

Question 1: What is the value of $\mu_s(\Omega)$?

Question 2: Are there extremals for $\mu_s(\Omega)$?
That is: is there some $u_\Omega \in H_{1,0}^2(\Omega) \setminus \{0\}$ such that $I_\Omega(u_\Omega) = \mu_s(\Omega)$?

The main difficulty here is due to the fact that $2^\star(s)$ is critical from the viewpoint of the Sobolev embeddings. More precisely, if Ω is bounded, then $H_{1,0}^2(\Omega)$ is embedded in the weighted space $L^p(\Omega, |x|^{-s})$ for $1 \leq p \leq 2^\star(s)$. And the embedding is compact iff $p < 2^\star(s)$ (in general, at least... see subsection 2.3 below). This lack of compactness defeats the classical minimization strategy to gets extremals for $\mu_s(\Omega)$. In fact, when $s = 0$, that is in the case of Sobolev inequalities, the same kind of difficulty occurs, and there have been some methods developed to bypass them. Concerning the same questions in the Riemannian context, we refer to Hebey-Vaugon [24] and Druet [10], and also to Aubin-Li [4].

2. A few answers in some specific cases

In this section, we collect a few facts and answers to questions 1 and 2: these results are essentially extensions of the methods developed in the case $s = 0$.

2.1. **The case $s = 0$.** In this context, the situation is well understood. In particular,

$$\mu_0(\mathbb{R}^n) = n(n-2)\left(\frac{\omega_{n-1}}{2}\cdot\frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}\right)^{\frac{2}{n}} = \frac{n(n-2)\omega_n^{2/n}}{4}$$

where ω_k is the volume of the standard $k-$sphere of $\mathbb{R}^{k+1}$. The extremals exist and are known: indeed, $u \in H_{1,0}^2(\mathbb{R}^n)\setminus\{0\}$ is an extremal for $\mu_0(\mathbb{R}^n)$ if and only if there exist $x_0 \in \mathbb{R}^n$, $\lambda \in \mathbb{R}\setminus\{0\}$, $\alpha > 0$ such that

$$u(x) = \lambda\left(\frac{\alpha}{\alpha^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}} \text{ for all } x \in \mathbb{R}^n. \tag{4}$$

These results are due to Rodemich [29], Aubin [2] and Talenti [31]. We also refer to Lieb [25] and Lions [26] for other nice points of view.

Concerning general open subsets of $\mathbb{R}^n$, one can show that

$$\mu_0(\Omega) = \mu_0(\mathbb{R}^n) = \frac{n(n-2)\omega_n^{2/n}}{4}$$

for all Ω open subset of $\mathbb{R}^n$. Moreover, if there is an extremal for $\mu_s(\Omega)$, then it is also an extremal for $\mu_0(\mathbb{R}^n)$ and it is of the form of (4). In particular, there is no extremal for $\mu_s(\Omega)$ if Ω is bounded (more general conditions involving the capacity are available).

From now on, we concentrate on the case $s \in (0,2)$. Here, due to the singularity at 0, the situation will depend drastically on the location of 0 with respect to Ω

2.2. **The case $0 \in \Omega$, $s \in (0,2)$.** Here again, when $\Omega = \mathbb{R}^n$, the constant $\mu_s(\Omega)$ is explicit, and we know what the extremals are (see Ghoussoub-Yuan [20], Lieb [25], we refer also to Catrina-Wang [9]). More precisely,

$$\mu_s(\mathbb{R}^n) = (n-2)(n-s)\left(\frac{\omega_{n-1}}{2-s}\cdot\frac{\Gamma^2\left(\frac{n-s}{2-s}\right)}{\Gamma\left(\frac{2n-2s}{2-s}\right)}\right)^{\frac{2-s}{n-s}}$$

and given $\alpha > 0$, the functions

$$u_\alpha(x) := \left(\frac{\alpha}{\alpha^2 + |x|^{2-s}}\right)^{\frac{n-2}{2-s}}$$

are extremals for $\mu_s(\mathbb{R}^n)$, and $u \in H_{1,0}^2(\mathbb{R}^n)\setminus\{0\}$ is an extremal for $\mu_s(\mathbb{R}^n)$ iff there exists $\lambda \in \mathbb{R}\setminus\{0\}$ and $\alpha > 0$ such that $u = \lambda\cdot u_\alpha$. when $s = 0$, we recover some of the extremals for the standard Sobolev inequality. Here, it is important to note the following asymptotics for u_α when $\alpha \to 0$:

$$\lim_{\alpha\to 0} u_\alpha(0) = +\infty \text{ and } \lim_{\alpha\to 0} u_\alpha(x) = 0 \text{ for all } x \neq 0.$$

In other words, the function u_α concentrates at 0 when $\alpha \to 0$.

When dealing with an open subset Ω of $\mathbb{R}^n$ such that $0 \in \Omega$, one can follow the approach developed for $s = 0$. Indeed, it follows from the definition of $\mu_s(\Omega)$ that

$$\mu_s(\Omega) \geq \mu_s(\mathbb{R}^n).$$

The reverse inequality is obtained via the estimate of I_Ω at a suitable test-function. Let $\eta \in C_c^\infty(\Omega)$ such that $\eta(x) \equiv 1$ in a neighborhood of 0. Then $\eta u_\alpha \in C_c^\infty(\Omega)$. Simple computations then yield

$$I_\Omega(\eta u_\alpha) = \mu_s(\mathbb{R}^n) + o(1)$$

where $\lim_{\alpha\to 0} o(1) = 0$. It then follows that $\mu_s(\Omega) \leq \mu_s(\mathbb{R}^n)$, and then

$$\mu_s(\Omega) = \mu_s(\mathbb{R}^n).$$

Indeed, this is exactly the standard proof in the case $s = 0$. Concerning the extremals, the same argument as for $s = 0$ proves that there is no extremal for $\mu_s(\Omega)$ if Ω is bounded. To conclude, one can say that the case $s \in (0,2)$ when $0 \in \Omega$ is quite similar to the case $s = 0$.

2.3. **The case $0 \notin \overline{\Omega}$, $s \in (0,2)$.** This case is not the most interessant. Indeed, when $0 \notin \overline{\Omega}$ and Ω is bounded, then $L^{2^\star(s)}(\Omega, |x|^{-s}) = L^{2^\star(s)}(\Omega)$ and the embedding $H_{1,0}^2(\Omega) \hookrightarrow L^{2^\star(s)}(\Omega)$ is compact since $1 \leq 2^\star(s) < \frac{2n}{n-2}$. Therefore, the standard minimization methods work and there are extremals for $\mu_s(\Omega)$. However, finding the explicit value of $\mu_s(\Omega)$ is almost impossible in general.

2.4. **The case $0 \in \partial\Omega$, $s \in (0,2)$: first results.** This case is much more intricate. If we want to mimick the arguments above, one is stuck by the fact that $\eta u_\alpha \notin H_{1,0}^2(\Omega)$ when $0 \in \partial\Omega$. Indeed, around 0, the set Ω looks like $\mathbb{R}^n_- := \{x \in \mathbb{R}^n / \, x_1 < 0\}$ (and not like $\mathbb{R}^n$ in the case $0 \in \Omega$): therefore, we are going to compare $\mu_s(\Omega)$ with $\mu_s(\mathbb{R}^n_-)$.

Since Ω is smooth, there exists U, V open subsets of $\mathbb{R}^n$ such that $0 \in U$, $0 \in V$ and there exists $\varphi : U \to V$ a $C^\infty-$diffeomorphism such that $\varphi(0) = 0$ and

$$\varphi(U \cap \{x_1 < 0\}) = \varphi(U) \cap \Omega, \quad \varphi(U \cap \{x_1 = 0\}) = \varphi(U) \cap \partial\Omega.$$

Up to an affine transformation, we can assume that the differential of φ at 0 is the identity map. Let $u \in H_{1,0}^2(\mathbb{R}^n_-) \setminus \{0\}$ and let a sequence $(\mu_i)_{i\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $\lim_{i\to+\infty} \mu_i = 0$ and $\eta \in C_c^\infty(U)$ such that $\eta(x) \equiv 1$ in a neighborhood of 0. We define

$$v_i(x) := \eta(x)\mu_i^{-\frac{n-2}{2}} u\left(\mu_i^{-1}\varphi^{-1}(x)\right)$$

for $x \in U \cap \mathbb{R}^n_-$ and 0 elsewhere. One easily gets that $v_i \in H_{1,0}^2(\Omega)$ for all $i \in \mathbb{N}$. Straightforward computations yield

$$I_\Omega(v_i) = \frac{\int_{\mathbb{R}^n_-} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n_-} \frac{|u|^{2^\star(s)}}{|x|^s} \, dx\right)} + o(1)$$

where $\lim_{i\to+\infty} o(1) = 0$. Therefore, taking the infimum for all u, we get that

$$\mu_s(\Omega) \leq \mu_s(\mathbb{R}^n_-). \tag{5}$$

Indeed, aguing as in the case $0 \in \Omega$, one gets that when $\Omega \subset \mathbb{R}^n_-$, then the reverse inequality holds, and then

$$\mu_s(\Omega) = \mu_s(\mathbb{R}^n_-) \text{ when } \Omega \subset \mathbb{R}^n_-.$$

Moreover, if $\Omega \subset \mathbb{R}^n_-$ and Ω is bounded, then there is no extremal for $\mu_s(\Omega)$.

Actually, in case $0 \in \partial\Omega$, the method for $s = 0$ can be extended only when $\Omega \subset \mathbb{R}^n_-$, which is an hypothesis of convexity at 0. In particular, this hypothesis is satisfied for balls. In the sequel, we are going to tackle our problem when $0 \in \partial\Omega$

without convexity assumptions: what is interesting here is that the geometry of the boundary will be concerned.

3. The case $0 \in \partial\Omega$: statement of the results

In this context, one of the first contributions is due to Egnell:

Theorem 1 (Egnell [15]). *Let D be a nonempty connected domain of $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$. Let $C := \{r\theta / r > 0,\, \theta \in D\}$ be the cone based at 0 induced by D. Then there are extremals for $\mu_s(C)$.*

Indeed, in the spirit of Lions [26], Egnell takes advantage of the invariance of the problem after rescaling in the directions of D to prove relative compactness of minimizers of $\mu_s(C)$ after rescaling. An important point here is that the domain C is not necesseraly smooth at 0. Moreover, Theorem 1 proves that there are extremals for $\mu_s(\mathbb{R}^n_-)$. But we do not know the value of $\mu_s(\mathbb{R}^n_-)$.

More recently, Ghoussoub and Kang came back to this problem when the domain Ω is bounded and smooth at 0:

Theorem 2 (Ghoussoub-Kang [16]). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Assume that*

$$\mu_s(\Omega) < \mu_s(\mathbb{R}^n_-). \tag{6}$$

Then there are extremals for $\mu_s(\Omega)$.

This kind of condition is very classical in best constant problems, see Aubin [3], Brézis-Nirenberg [5].

Proof. Let us briefly sketch the proof of this result. First, given $\epsilon \in (0, 2^\star(s) - 2)$, consider the approximate minimization:

$$\mu_s^\epsilon(\Omega) := \inf_{u \in H^2_{1,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)-\epsilon}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)-\epsilon}}}.$$

Since the exponent is subcritical, there is compactness of the embedding $H^2_{1,0}(\Omega) \hookrightarrow L^{2^\star(s)-\epsilon}(\Omega, |x|^{-s})$ and we get that there is a minimizer $u_\epsilon \in H^2_{1,0}(\Omega) \setminus \{0\}$ of $\mu_s^\epsilon(\Omega)$. Moreover, regularity theory yields that $u_\epsilon \in C^\infty(\overline{\Omega} \setminus \{0\}) \cap C^1(\overline{\Omega})$ and we can assume that u_ϵ verifies the system

$$\begin{cases} \Delta u_\epsilon = \frac{u_\epsilon^{2^\star(s)-1-\epsilon}}{|x|^s} & \text{in } \Omega \\ u_\epsilon > 0 & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

Concerning the energy, we have that

$$\int_\Omega \frac{u_\epsilon^{2^\star(s)-\epsilon}}{|x|^s}\, dx = \mu_s^\epsilon(\Omega)^{\frac{2^\star(s)-\epsilon}{2^\star(s)-\epsilon-2}}. \tag{7}$$

The standard strategy is then to let $\epsilon \to 0$: this is not straightforward since the embedding $H^2_{1,0}(\Omega) \rightharpoonup L^{2^\star(s)}(\Omega; |x|^{-s})$ is not compact. In the case $s = 0$, Struwe [30] gave a very nice decomposition describing precisely this lack of compactness for Palais-Smale sequence. Struwe's result was extended to our situation by Ghoussoub-Kang. We need to define a bubble:

Definition 1. *A family $(B_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$ is a bubble if there exists a family $(\mu_\epsilon)_{\epsilon>0} \in \mathbb{R}_{>0}$ such that $\lim_{\epsilon\to 0}\mu_\epsilon = 0$, there exists $u \in H_{1,0}^2(\mathbb{R}^n_-)\setminus\{0\}$ such that*

$$\Delta u = \frac{|u|^{2^\star(s)-2}u}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-)$$

and

$$B_\epsilon(x) = \eta(x)\mu_\epsilon^{-\frac{n-2}{2}} u\left(k_\epsilon^{-1}\varphi^{-1}(x)\right)$$

for $x \in U \cap \mathbb{R}^n_-$ and 0 elsewhere, where

$$k_\epsilon = \mu_\epsilon^{1-\frac{\epsilon}{2^\star(s)-2}} \text{ and } \lim_{\epsilon\to 0} k_\epsilon^\epsilon = c \in (0,1].$$

In the definition, η and φ are as in Subsection 2.4.

An important remark is that a consequence of the definition of $\mu_s(\mathbb{R}^n_-)$ is that the same computations as in Subsection 2.4 yield

$$\int_\Omega \frac{|B_\epsilon|^{2^\star(s)-\epsilon}}{|x|^s}\,dx + o(1) \geq \mu_s(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}} + o(1)$$

for any bubble. Then, in the spirit of Struwe and following the proofs of Ghoussoub-Kang [16] and Robert [28], we get that for any family $(u_\epsilon)_{\epsilon>0}$ of solutions to (3) such that there exists $\Lambda > 0$ such that $\|u_\epsilon\|_{H_{1,0}^2(\Omega)} \leq \Lambda$ for all $\epsilon > 0$, there exists $u_0 \in H_{1,0}^2(\Omega)$, there exists $N \in \mathbb{N}$ and there exists N positive bubbles $(B_{i,\epsilon})_{\epsilon>0}$, $i \in \{1,...,N\}$ such that

$$u_\epsilon = u_0 + \sum_{i=1}^N B_{i,\epsilon} + R_\epsilon \tag{8}$$

where $\lim_{\epsilon\to 0} R_\epsilon = 0$ strongly in $H_{1,0}^2(\Omega)$.

We apply (8) to the function u_ϵ in (3). Assume that there is a bubble in the decomposition, then one gets that

$$\int_\Omega \frac{u_\epsilon^{2^\star(s)-\epsilon}}{|x|^s}\,dx \geq \int_\Omega \frac{B_{i,\epsilon}^{2^\star(s)-\epsilon}}{|x|^s}\,dx + o(1) \geq \mu_s(\mathbb{R}^n_-)^{\frac{2^\star(s)}{2^\star(s)-2}} + o(1)$$

where $\lim_{\epsilon\to 0} o(1) = 0$. Since $\lim_{\epsilon\to 0}\mu_s^\epsilon(\Omega) = \mu_s(\Omega)$, we get with (7) that

$$\mu_s(\Omega) \geq \mu_s(\mathbb{R}^n_-),$$

a contradiction with the initial hypothesis. Therefore there is not bubble and $\lim_{\epsilon\to 0} u_\epsilon = u_0$ in $H_{1,0}^2(\Omega)$, and u_0 is an extremal for $\mu_s(\Omega)$. □

But when is inequality (6) fulfilled? For this type of problems, the traditional method (see Aubin [3]) is to compute the functional I_Ω at bubbles modelized on extremals for $\mu_s(\mathbb{R}^n_-)$ and to make a Taylor expansion, hoping that one succeeds in getting below the energy threshold. But at this stage, a difficulty occurs: the extremals for $\mu_s(\mathbb{R}^n_-)$ are not explicit, and therefore, the coefficients that appear in the estimate of I_Ω at the bubbles are not explicit, and we do not have informations on their sign in general. Then, it is not possible to prove that one goes below the energy threshold with this method.

However, Ghoussoub and Kang were able to prove an existence result:

Theorem 3 (Ghoussoub-Kang [16]). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Assume that the principal curvatures at 0 are all negative and that $n \geq 4$. Then there are extremals for $\mu_s(\Omega)$.*

Concerning terminology, the principal curvatures are the eigenvalues of the second fundamental form of the hypersurface $\partial\Omega$ oriented by the outward normal vector. The second fundamental form being

$$II_0(\vec{X}, \vec{Y}) = (dn_0(\vec{X}), \vec{Y}) \text{ for } \vec{X}, \vec{Y} \in T_0\partial\Omega$$

where dn_0 is the differential of the outward normal vector at 0 and $(\cdot, \cdot)$ is the Euclidean scalar product. Concerning the proof, Ghoussoub and Kang are able to exhibit a family $(w_i)_{i\in\mathbb{N}} \in H_{1,0}^2(\Omega) \setminus \{0\}$ such that $I_\Omega(w_i) < \mu_s(\mathbb{R}^n_-)$ for i large and under the assumptions of the theorem: this family is not constructed via the bubbles and the construction is quite intricate.

The condition in Theorem 3 means that the domain is locally concave at 0: a condition that is consistant with the non-existence of extremals when $\Omega \subset \mathbb{R}^n_-$. However, these two cases do not cover all situations, and dimension 3 is not treated in Theorem 3. In fact, in the proof of Ghoussoub-Kang, the bubbling phenomenon is ruled out at the beginning of the argument for energy considerations. To get more general results, the strategy is to describe precisely the potential bubbling and then to get a contradiction: techniques different from the standard minimization ones are required to go any further.

The suitable quantity to consider is the mean curvature (that is the trace of the second fondamental form). In a joint work with N.Ghoussoub, we use blow-up techniques to prove the following:

Theorem 4 (Ghoussoub-Robert [17]). *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Assume that the mean curvature of $\partial\Omega$ at 0 is negative and that $n \geq 3$. Then there are extremals for $\mu_s(\Omega)$.*

This results clearly includes Theorem 3. Qualitatively, Theorem 4 tells us that there are extremals for $\mu_s(\Omega)$ when the domain is "more" concave than convex at 0 in the sense that the negative principal directions dominate quantitatively the positive principal directions. This allows us to exhibit new examples neither convex or concave for which the extremals exist. Note that this results does not tell anything about the value of the best constant.

4. Sketch of the proof of Theorem 4

As in the proof of Theorem 2, we consider the subcritical problem. Indeed, given $\epsilon \in (0, 2^\star(s) - 2)$, there exists $u_\epsilon \in H_{1,0}^2(\Omega) \cap C^\infty(\overline{\Omega} \setminus \{0\}) \cap C^1(\overline{\Omega})$ such that

$$\left\{ \begin{array}{ll} \Delta u_\epsilon = \frac{u_\epsilon^{2^\star(s)-1-\epsilon}}{|x|^s} & \text{in } \Omega \\ u_\epsilon > 0 & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{array} \right. \tag{9}$$

and

$$\lim_{\epsilon\to 0} \int_\Omega \frac{u_\epsilon^{2^\star(s)-\epsilon}}{|x|^s}\, dx = \mu_s^\epsilon(\Omega)^{\frac{2^\star(s)}{2^\star(s)-2}}. \tag{10}$$

With (5) and Theorem 2, we can assume that $\mu_s(\Omega) = \mu_s(\mathbb{R}^n_-)$. With the decomposition (8) above, we get that we are in one and only one of the following situations:

a. either there exists $u_0 \in H_{1,0}^2(\Omega) \setminus \{0\}$ such that $\lim_{\epsilon\to 0} u_\epsilon = u_0$ in $H_{1,0}^2(\Omega)$,

b. or there exists a bubble $(B_\epsilon)_{\epsilon>0}$ such that

$$u_\epsilon = B_\epsilon + o(1) \tag{11}$$

where $\lim_{\epsilon\to 0} o(1) = 0$ in $H_{1,0}^2(\Omega)$. Moreover, the function $u \in H_{1,0}^2(\Omega)$ defining the bubble in Definition 1 is positive: in particular, $u \in H_{1,0}^2(\mathbb{R}^n_-) \cap C^\infty(\overline{\mathbb{R}^n_-} \setminus \{0\}) \cap C^1(\overline{\mathbb{R}^n_-})$ and satisfies

$$\Delta u = \frac{u^{2^\star(s)-1}}{|x|^s} \text{ in } \mathcal{D}'(\mathbb{R}^n_-),\; u > 0 \text{ in } \mathbb{R}^n_-,\; u = 0 \text{ on } \partial\mathbb{R}^n_-. \tag{12}$$

We are going to prove that **b.** does not hold when the mean curvature is negative at 0. Indeed, if **b.** does not hold, then situation **a.** holds and u_0 is an extremal for $\mu_s(\Omega)$, and Theorem 4 is proved.

We argue by contradiction and assume that **b.** holds. The idea is to prove that the family $(u_\epsilon)_{\epsilon>0}$ behaves more or less like the bubble $(B_\epsilon)_{\epsilon>0}$. In fact (11) indicates that these two families are equal up to the addition of a term vanishing in $H_{1,0}^2(\Omega)$. We need something more precise, indeed a pointwise description, not a description in Sobolev spaces. This requires a good knowledge of the bubbles: a difficult question since bubbles are not explicit here.

4.1. **Strong pointwise estimate.** When $u \in H_{1,0}^2(\mathbb{R}^n_-) \cap C^1(\overline{\mathbb{R}^n_-})$ is a positive weak solution to (12), we prove that there exists a constant $C > 0$ such that

$$\frac{1}{C} \cdot \frac{|x_1|}{(1+|x|^2)^{n/2}} \le u(x) \le C \frac{|x_1|}{(1+|x|^2)^{n/2}}$$

for all $x \in \mathbb{R}^n_-$. Coming back to the definition of the bubble, and letting $(\mu_\epsilon)_{\epsilon>0} \in \mathbb{R}_{>0}$ the parameter in Definitin 1, we get that

$$B_\epsilon(x) \le C \frac{\mu_\epsilon^{n/2} d(x, \partial\Omega)}{(\mu_\epsilon^2 + |x|^2)^{n/2}}$$

for all $x \in \Omega$. Instead of comparing directly with the bubble, we are going to prove the following claim:

Claim: there exists $C_1 > 0$ such that

$$u_\epsilon(x) \le C_1 \frac{\mu_\epsilon^{n/2} d(x, \partial\Omega)}{(\mu_\epsilon^2 + |x|^2)^{n/2}} \tag{13}$$

for all $x \in \Omega$ and all $\epsilon > 0$.

This type of optimal pointwise estimates have their origin in Atkinson-Peletier [1] and Brézis-Peletier [6]. In the general case when $s = 0$, such an estimate was obtained by Han [22] with the use of the Kelvin transform and in Hebey [23] and in Robert [27]. In the Riemannian context, such pointwise estimates are in Hebey-Vaugon [24], Druet [10] and Druet-Robert [14]. These techniques were used by Druet [11] to solve the three-dimensional conjecture of Brézis. In the context of high energy, that is with arbitrary many bubbles, we refer to the monography Druet-Hebey-Robert [13] and to Druet [12].

The proof we present here uses the machinery developed in Druet-Hebey-Robert [13] for equations of Yamabe-type on manifolds: in particular, this allows to tackle problems with arbitrary high energy. These techniques can be extended to our context where there is a singularity at 0, a point on the boundary. The proof of (13) proceeds in three steps:

Step 1: We have that

$$\lim_{\epsilon\to 0}\mu_\epsilon^{\frac{n-2}{2}}u_\epsilon(\varphi(\mu_\epsilon x)) = u \text{ in } C^1_{loc}(\overline{\mathbb{R}^n_-}).$$

Indeed, rescaling (11) yields that the convergence above holds locally in $H^2_{1,0}(\mathbb{R}^n_-)$. The convergence in C^1 is a consequence of elliptic regularity.

Step 2: For all $\nu \in (0, 2^\star(s)-2)$, there exists $R_\nu > 0$ and $C_\nu > 0$ such that

$$u_\epsilon(x) \le C_\nu \mu_\epsilon^{\frac{n}{2}-\nu(n-1)}\frac{d(x,\partial\Omega)^{1-\nu}}{|x|^{n(1-\nu)}} \tag{14}$$

for all $\epsilon > 0$ small enough and all $x \in \Omega\setminus\varphi(B_{R_\nu\mu_\epsilon}(0))$.

Proof. This is one of the most difficult steps: we only briefly outline the proof. Thanks to Step 1, proving (13) amounts to proving that

$$u_\epsilon(x) \le C_1\frac{\mu_\epsilon^{n/2}d(x,\partial\Omega)}{|x|^n}$$

for $\Omega\setminus\varphi(B_{R_0\mu_\epsilon}(0))$ for some $R_0 > 0$. We denote by G the Green's function for $\Delta - \epsilon_0$ with $\epsilon_0 > 0$ small, that is

$$\Delta G(x,\cdot) - \epsilon_0 G(x,\cdot) = \delta_x \text{ in } \mathcal{D}'(\Omega) \text{ and } G(x,\cdot) = 0 \text{ in } \partial\Omega$$

for all $x \in \Omega$. In particular, denoting by $\partial/\partial_1\vec{\nu}$ the exterior normal derivative with respect to the first variable, one proves that there exists $\delta > 0$ such that

$$0 < -\frac{\partial G(0,x)}{\partial_1\vec{\nu}} \le C_2\frac{d(x,\partial\Omega)}{|x|^n}$$

for all $x \in \Omega\cap B_\delta(0)$, and, up to multiplication by a constant, the right-hand-side is exactly what we want to compare u_ϵ with. Given $\nu > 0$ small enough, with the use of a comparison principle and some refined estimates, we are able to compare u_ϵ and

$$C_\epsilon\cdot\left(-\frac{\partial G(0,x)}{\partial_1\vec{\nu}}\right)^{1-\nu}$$

on $\Omega\setminus\varphi(B_{R_\nu\mu_\epsilon}(0))$ for R_ν large enough and a suitable constant C_ϵ depending on ϵ. Then we get (14). We refer to the articles [17, 18] for the proof of this assertion. □

Step 3: We plug the above estimates of Steps 1 and 2 into Green's representation fomula

$$u_\epsilon(x) = \int_\Omega H(x,y)u_\epsilon^{2^\star(s)-1-\epsilon}(y)\,dy$$

for all $x \in \Omega$, where H is the Green's function for Δ with Dirichlet boundary conditions. Then, it is necessary to divide the domain Ω in various subdomains, and on each of these subdomains, we use different estimates for u_ϵ. At the end, we get (13). This proves the claim.

4.2. **Pohozaev identity.** The final contradiction comes from the Pohozaev identity. Indeed, integrating by parts, we get that

$$\int_\Omega x^i\partial_i u_\epsilon \Delta u_\epsilon\, dx + \frac{n-2}{2}\int_\Omega u_\epsilon\Delta u_\epsilon\, dx = -\frac{1}{2}\int_{\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\, d\sigma$$

and then, with the system (9), we get that

$$\left(\frac{n-2}{2}-\frac{n-s}{2^\star(s)-\epsilon}\right)\int_\Omega \frac{u_\epsilon^{2^\star(s)-\epsilon}}{|x|^s}\, dx = -\frac{1}{2}\int_{\partial\Omega}(x,\nu)|\nabla u_\epsilon|^2\, d\sigma.$$

The left-hand-side is easy to estimate with (10). For the right-hand-side, we need to use the optimal estimate (13), and we get that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\mu_\epsilon} = \frac{(n-s)\int_{\partial\mathbb{R}^n_-} II_0(x,x)|\nabla u|^2\, dx}{(n-2)^2\int_{\mathbb{R}^n_-}|\nabla u|^2\, dx}$$

where II_0 is the second fondamental form at 0 defined on the tangent space of $\partial\Omega$ at 0 that we assimilate to $\partial\mathbb{R}^n_-$.

In addition, in the spirit of Caffarelli-Gidas-Spruck [7] and Gidas-Ni-Nirenberg [21], we prove that the positive function u satisfying (12) enjoys the best symmetry possible: indeed, writing $x=(x_1,\bar{x})\in\mathbb{R}^n$ with $x_1\in\mathbb{R}$, we get that $u(x_1,\bar{x}) = \tilde{u}(x_1,|x|)$ where $\tilde{u}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Therefore, the limit above rewrites as

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\mu_\epsilon} = \frac{(n-s)\int_{\partial\mathbb{R}^n_-}|x|^2\cdot|\nabla u|^2\, dx}{n(n-2)^2\int_{\mathbb{R}^n_-}|\nabla u|^2\, dx}\cdot H(0),$$

where $H(0)$ is the mean curvature at 0. Since the left-hand-side is nonnegative, we get that $H(0)\geq 0$: a contradiction with our initial assumption. Then **b.** does not hold and we have extremals for $\mu_s(\Omega)$. This proves Theorem 4.

4.3. **General compactness.** The proof that we have sketched here involved functions developing one bubble in the Struwe decomposition. As in Druet-Hebey-Robert [13], this analysis can be extended to functions developing arbitrary many bubbles, that is when the energy is arbitrary. The new difficulty here is that many bubbles accumulate at 0. The following result holds:

Theorem 5 (Ghoussoub-Robert [18])**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, with $0\in\partial\Omega$. Let $(u_\epsilon)_{\epsilon>0}\in H_{1,0}^2(\Omega)$ and $(a_\epsilon)_{\epsilon>0}\in C^1(U)$ (with $\Omega\subset\subset U$) be a family of solutions to the equation*

$$\Delta u_\epsilon + a_\epsilon u_\epsilon = \frac{|u_\epsilon|^{2^\star(s)-2-\epsilon}u_\epsilon}{|x|^s}\ \text{ in } \mathcal{D}'(\Omega).$$

Assume that there exists $\Lambda>0$ such that $\|u_\epsilon\|_{H_{1,0}^2(\Omega)}\leq\Lambda$ and that $\lim_{\epsilon\to 0}a_\epsilon = a_\infty$ in $C^1_{loc}(U)$. Assume that the principal curvatures at 0 are nonpositive, but not all null. Then there exists $u\in C^1(\overline{\Omega})$ such that, up to a subsequence, $\lim_{\epsilon\to 0}u_\epsilon = u$ in $C^1(\overline{\Omega})$.

In other words, there is no bubble under the assumption on the curvature at 0. Here, as in the proof of Theorem 4, we prove that the u_ϵ's are controled pointwisely by a sum of bubbles. Then, plugging u_ϵ in the Pohozaev identity, we get that, in case there is at least one bubble, there exists $v \in H_{1,0}^2(\mathbb{R}_-^n) \setminus \{0\}$, $C > 0$ and $(\mu_\epsilon)_{\epsilon>0} \in \mathbb{R}_{>0}$ such that $\lim_{\epsilon\to 0} \mu_\epsilon = 0$ and

$$\lim_{\epsilon\to 0} \frac{\epsilon}{\mu_\epsilon} = C \cdot \int_{\partial\mathbb{R}_-^n} II_0(x,x)|\nabla v|^2\, dx.$$

Under the assumptions of the theorem, the right-hand-side is negative. A contradiction. Then there is no bubble and one recovers compactness. Note that since we have no information on the sign of v, we cannot prove symmetry as in the proof of Theorem 4.

5. About low dimensions

A remarkable point here is that there is not low-dimensional phenomenon in Theorems 4 and 5. Moreover, there is no condition on the function a to recover compactness: the geometry of $\partial\Omega$ dominates the linear perturbation a.

This is quite surprising in view of some existing results for Yamabe-type equations. Here is an example: consider the functional

$$J_\Omega(u) := \frac{\int_\Omega (|\nabla u|^2 + au^2)\, dx}{\left(\int_\Omega |u|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}}$$

for $u \in H_{1,0}^2(\Omega) \setminus \{0\}$, where $\Delta + a$ is coercive and $a \in C^\infty(\overline{\Omega})$. We let G_a be the Green's function for $\Delta + a$ with Dirichlet boundary conditions on $\partial\Omega$ and when $n = 3$, we define $g_a(x,y)$ by

$$G_a(x,y) = \frac{1}{\omega_2|x-y|} + g_a(x,y).$$

In particular, one gets that $g_a \in C^0(\Omega \times \Omega)$. Then the following theorem holds:

Theorem 6. **i.** *if $n \geq 4$, $\inf_{u\in H_{1,0}^2(\Omega)\setminus\{0\}} J_\Omega(u)$ is achieved iff there exists $x \in \Omega$ such that $a(x) < 0$ (Brézis-Nirenberg [5]).*

ii. *if $n = 3$, $\inf_{u\in H_{1,0}^2(\Omega)\setminus\{0\}} J_\Omega(u)$ is achieved iff there exists $x \in \Omega$ such that $g_a(x,x) > 0$ (Druet [11]).*

Therefore, in dimension $n \geq 4$, the geometry of Ω is not to be taken into account; but in dimension $n = 3$, the condition relies on both a and Ω (the Green's function depends on the geometry).

Another example arises from Yamabe-type equations on manifolds. We denote by R_g the scalar curvature of a metric g. O.Druet proved the following:

Theorem 7 (Druet [12])**.** *Let (M,g) be a compact manifold of dimension $n \geq 3$. Let $(h_\epsilon)_{\epsilon>0} \in C^2(M)$ such that $\lim_{\epsilon\to+\infty} h_\epsilon = h_0$ in $C^2(M)$ with $\Delta_g + h_0$ coercive. Let $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ such that*

$$\Delta_g u_\epsilon + h_\epsilon u_\epsilon = u_\epsilon^{2^\star(0)-1} \text{ in } M.$$

Assume that there exists $\Lambda > 0$ such that $\|u_\epsilon\|_{2^\star(0)} \leq \Lambda$ for all $\epsilon > 0$. Moreover, assume that

i. *$h_0(x) \neq \frac{n-2}{4(n-1)} R_g(x)$ for all $x \in M$ if $n \geq 4$, $n \neq 6$,*

ii. *$h_\epsilon(x) \leq \frac{n-2}{4(n-1)} R_g(x)$ for all $x \in M$ for all $x \in M$ and all $\epsilon > 0$ and (M,g) is not conformally diffeomorphic to the $n-$sphere in case $h_0 \equiv \frac{n-2}{4(n-1)} R_g$.*

Then, up to a subsequence, there exists $u_0 \in C^2(M)$ *such that* $\lim_{\epsilon \to 0} u_\epsilon = u_0$.

Here again, there is a difference depending of the dimension and on the linear term h. In this context, dimension six is a quite intriguing dimension.

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