REMARKS ON THE GIBBS MEASURES FOR NONLINEAR DISPERSIVE EQUATIONS

by

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Abstract. — We show, by the means of several examples, how we can use Gibbs measures to construct global solutions to dispersive equations at low regularity. The construction relies on the Prokhorov compactness theorem combined with the Skorohod convergence theorem. To begin with, we consider the non linear Schrödinger equation on the tri-dimensional sphere. Then we focus on the Benjamin-Ono equation and on the derivative nonlinear Schrödinger equation on the circle. Finally, we construct a Gibbs measure and global solutions to the so-called periodic half-wave equation.

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1. Introduction and main results

1.1. General introduction. — A Gibbs measure can be an interesting tool to show that local solutions to some dispersive PDEs are indeed global. Once we have a suitable local existence and uniqueness theory on the support of such a measure, we can expect to globalise these solutions; this measure in some sense compensates the lack of conservation law at some level of Sobolev regularity.

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See [**3, 4, 39, 36, 37, 13, 30, 31, 9**] where this approach has been fruitful.
Assume now that we have a Gibbs measure, but that we are not able to show that the equation is locally well-posed on its support. The aim of this paper is to show - through several examples - that in this case we can use some compactness methods to construct global (but non unique) solutions on the support of the measure.
Although this method of construction of solutions is well-known in other contexts, like for the Euler equation (see Albeverio-Cruzeiro [**1**]) or for the Navier-Stokes equation (see Da Prato-Debussche [**19**]), it seems to be not exploited in the context of dispersive equations.

In [**10**] we have constructed global rough solutions to the periodic wave equation in any dimension with stochastic tools. While in [**10**] we used the energy conservation and a regularisation property of the wave equation in the argument, here we use instead the invariance of the measure by the non linear flow. As a consequence we also obtain that the distribution of the solutions we construct is independent of time.

Our first example concerns the non linear Schrödinger equation on the sphere $\mathbb{S}^3$ restricted to zonal functions (the functions which only depend on the geodesic distance to the north pole). For sub-quintic nonlinearities, we are able to define a Gibbs measure with support in $H^\sigma(\mathbb{S}^3)$ for any $\sigma < 1/2$, and to construct global solutions in this space. This is the result of Theorem 1.1. Bourgain-Bulut [**5**] have announced a uniqueness result for a similar equation (the radial NLS on $\mathbb{R}^3$) in the case of the cubic nonlinearity. See also [**36**, Section 10] for a discussion on this topic.

In a second time we deal with the Benjamin-Ono equation on the circle $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. This model arises in the study of one-dimensional internal long waves. In [**26, 27**] L. Molinet has shown that the equation is globally well-posed in $L^2(\mathbb{S}^1)$ and that this result is sharp. For this problem, a Gibbs measure with support in $H^{-\sigma}(\mathbb{S}^1)$, for any $\sigma > 0$ has already been constructed by N. Tzvetkov in [**35**] (see also the recent work of Tzvetkov-Visciglia [**38**], where the authors construct a Gibbs measure associated to each conservation law of the equation). In this case, we also construct global solutions on the support of the measure and prove its invariance (Theorem 1.2). A uniqueness result of the dynamics on the support of the measure was recently proven in a remarkable paper by Y. Deng [**18**].

Our third example concerns the periodic derivative Schrödinger equation. Here we use the measure constructed by Thomann-Tzvetkov [**34**]. We construct a dynamics for which the measure is invariant (Theorem 1.3). This result may be seen as a consequence of a recent work by Nahmod, Oh, Rey-Bellet and Staffilani [**28**] and Nahmod, Ray-Bellet, Sheffield and Staffilani [**29**]. Their approach is based on the local deterministic theory of Grünrock-Herr [**21**] which gauges out (the worst part of) the nonlinearity, and the uniqueness is only proved in this gauged-out context.

Finally, we consider the so-called half-wave equation on the circle, which can be seen as a limit model of Schrödinger-like equations for which one has very few dispersion. This model has been studied by

Gérard-Grellier [**20**] who showed that it is well-posed in $H^{1/2}(\mathbb{S}^1)$ (see also O. Pocovnicu [**33**] and more recently Krieger-Lenzmann-Raphaël [**24**] for a study of the equation on the real line). Here a Gibbs measure with support in $H^{-\sigma}(\mathbb{S}^1)$, for any $\sigma > 0$ can be defined, and global solutions (see Theorem 1.6) can be constructed.

1.2. The Schrödinger equation on $\mathbb{S}^3$. — Let $\mathbb{S}^3$ be the unit sphere in $\mathbb{R}^4$. We then consider the non linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{S}^3} u = |u|^{r-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{S}^3, \\ u(0,x) = f(x) \in H^\sigma(\mathbb{S}^3), \end{cases} \tag{1.1}$$

for $1 \leq r < 5$. In [**6**] N. Burq, P. Gérard and N. Tzvetkov have shown that (1.1) is globally well-posed in the energy space $H^1(\mathbb{S}^3)$. In this paper we address the question of the existence of global solutions at regularity below the energy space. Denote by $Z(\mathbb{S}^3)$ the space of the zonal functions, *i.e.* the space of the functions which only depend on the geodesic distance to the north pole of $\mathbb{S}^3$. Set $H^\sigma_{rad}(\mathbb{S}^3) := H^\sigma(\mathbb{S}^3) \cap Z(\mathbb{S}^3)$, $L^2_{rad}(\mathbb{S}^3) = H^0_{rad}(\mathbb{S}^3)$ and

$$X^{1/2}_{rad} = X^{1/2}_{rad}(\mathbb{S}^3) = \bigcap_{\sigma<1/2} H^\sigma_{rad}(\mathbb{S}^3).$$

For $x \in \mathbb{S}^3$, denote by $\theta = \mathrm{dist}(x, N) \in [0,\pi]$ the geodesic distance of x to the north pole and define

$$P_n(x) = \sqrt{\frac{2}{\pi}}\frac{\sin n\theta}{\sin\theta}, \qquad n \geq 1. \tag{1.2}$$

Then, $(P_n)_{n\geq1}$ is a Hilbertian basis of $L^2_{rad}(\mathbb{S}^3)$, which will be used in the sequel. Next, in order to avoid the issue with the 0-frequency, we make the change of unknown $u \longmapsto \mathrm{e}^{-it}u$, so that we are reduced to consider the equation

$$\begin{cases} i\partial_t u + (\Delta_{\mathbb{S}^3} - 1)u = |u|^{r-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{S}^3, \\ u(0,x) = f(x) \in H^\sigma(\mathbb{S}^3). \end{cases} \tag{1.3}$$

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$, which means that g_n can be written

$$g_n(\omega) = \frac{1}{\sqrt{2}}\big(h_n(\omega) + i\ell_n(\omega)\big),$$

where$\big(h_n(\omega)\big)_{n\geq1}$, $\big(\ell_n(\omega)\big)_{n\geq1}$ are independent standard real Gaussians $\mathcal{N}_{\mathbb{R}}(0,1)$.

For $N \geq 1$ we define the random variable

$$\omega \mapsto \varphi_N(\omega, x) = \sum_{n=1}^N \frac{g_n(\omega)}{n} P_n(x),$$

and we can show that if $\sigma < \frac{1}{2}$, then (φ_N) is a Cauchy sequence in $L^2\big(\Omega;\, H^\sigma(\mathbb{S}^3)\big)$: this enables us to define its limit

$$\omega \mapsto \varphi(\omega, x) = \sum_{n\geq 1} \frac{g_n(\omega)}{n} P_n(x) \in L^2\big(\Omega;\, H^\sigma(\mathbb{S}^3)\big). \tag{1.4}$$

We then define the Gaussian probability measure μ on $X^{1/2}_{rad}(\mathbb{S}^3)$ by $\mu = \mathbf{p} \circ \varphi^{-1}$. In other words, μ is the image of the measure $\mathbf{p}$ under the map

$$\begin{array}{ccl} \Omega & \longrightarrow & X^{1/2}_{rad}(\mathbb{S}^3) \\ \omega & \longmapsto & \varphi(\omega, \cdot) = \displaystyle\sum_{n\geq 1} \frac{g_n(\omega)}{n} P_n. \end{array}$$

We now construct a Gibbs measure for the equation (1.3). For $u \in L^{r+1}(\mathbb{S}^3)$ and $\beta > 0$, define the density

$$G(u) = \beta \mathrm{e}^{-\frac{1}{r+1}\int_{\mathbb{S}^3} |u|^{r+1}}, \tag{1.5}$$

and with a suitable choice of $\beta > 0$, this enables to construct a probability measure ρ on $X^{1/2}_{rad}(\mathbb{S}^3)$ by

$$\mathrm{d}\rho(u) = G(u)\mathrm{d}\mu(u).$$

Then we can prove

Theorem 1.1. — *Let $1 \leq r < 5$. The measure ρ is invariant under a dynamics of* (1.1). *More precisely, there exists a set Σ of full ρ measure so that for every $f \in \Sigma$ the equation* (1.1) *with initial condition $u(0) = f$ has a solution*

$$u \in \mathcal{C}\big(\mathbb{R}\,; X^{1/2}_{rad}(\mathbb{S}^3)\big).$$

The distribution of the random variable $u(t)$ is equal to ρ (and thus independent of $t \in \mathbb{R}$):

$$\mathscr{L}_{X^{1/2}_{rad}}\big(u(t)\big) = \mathscr{L}_{X^{1/2}_{rad}}\big(u(0)\big) = \rho, \quad \forall\, t \in \mathbb{R}.$$

Here and after, we abuse notation and write

$$\mathcal{C}\big(\mathbb{R}\,; X^{1/2}_{rad}(\mathbb{S}^3)\big) = \bigcap_{\sigma<1/2} \mathcal{C}\big(\mathbb{R}\,; H^{\sigma}_{rad}(\mathbb{S}^3)\big).$$

In our work, the only point where we need to restrict to zonal functions is for the construction of the Gibbs measure. The other arguments do not need any radial assumption. The result of Theorem 1.1 can not be extended to the case $r = 5$. Indeed, it is shown in [**2**, Theorem 4] that $\|u\|_{L^6(\mathbb{S}^3)} = +\infty$, $\mu-$a.s.
Since $G(u) > 0$, $\mu-$a.s., both measures μ and ρ have same support. Indeed, $\mu(X^{1/2}_{rad}(\mathbb{S}^3)) = \rho(X^{1/2}_{rad}(\mathbb{S}^3)) = 1$, but we can check that $\mu(H^{1/2}_{rad}(\mathbb{S}^3)) = \rho(H^{1/2}_{rad}(\mathbb{S}^3)) = 0$ (see [**8**, Proposition C.1]).

Let us compare our result to the result given by the usual deterministic compactness methods. The energy of the equation (1.1) reads

$$H(u) = \frac{1}{2}\int_{\mathbb{S}^3} |\nabla u|^2 + \frac{1}{r+1}\int_{\mathbb{S}^3} |u|^{r+1}.$$

Then, one can prove (see *e.g.* [**14**]) that for all $f \in H^1(\mathbb{S}^3) \cap L^{r+1}(\mathbb{S}^3)$ there exists a solution to (1.1) so that

$$u \in \mathcal{C}_w\big(\mathbb{R}; H^1(\mathbb{S}^3)\big) \cap \mathcal{C}_w\big(\mathbb{R}; L^{r+1}(\mathbb{S}^3)\big), \tag{1.6}$$

(here $\mathcal{C}_w$ stands for weak continuity in time) and so that for all $t \in \mathbb{R}$, $H(u)(t) \leq H(f)$. Notice that in (1.6) we can replace the space H^1 with H^1_{rad} if $f \in H^1_{rad}$.
The advantage of this method is that there is no restriction on $r \geq 1$ and no radial assumption on the initial condition. However this strategy asks more regularity on f. We also point out that with the deterministic method one loses the conservation of the energy, while in Theorem 1.1 we obtain an invariant measure (see also Remark 2.3).

1.3. The Benjamin-Ono equation. — Recall that $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ and let us denote by

$$\|f\|^2_{L^2(\mathbb{S}^1)} = (2\pi)^{-1}\int_0^{2\pi} |f(x)|^2 \mathrm{d}x.$$

For $f(x) = \sum_{k\in\mathbb{Z}} \alpha_k \mathrm{e}^{ikx}$ and $N \geq 1$ we define the spectral projector Π_N by $\Pi_N f(x) = \sum_{|k|\leq N} \alpha_k \mathrm{e}^{ikx}$. We also define the space $X^0(\mathbb{S}^1) = \bigcap_{\sigma>0} H^{-\sigma}(\mathbb{S}^1)$.

Denote by $\mathcal{H}$ the Hilbert transform, which is defined by

$$\mathcal{H}u(x) = -i\sum_{n\in\mathbb{Z}^\star} \mathrm{sign}(n) c_n \mathrm{e}^{inx}, \quad \text{for} \quad u(x) = \sum_{n\in\mathbb{Z}^\star} c_n \mathrm{e}^{inx}.$$

In this section, we are interested in the periodic Benjamin-Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + \partial_x\big(u^2\big) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{S}^1, \\ u(0,x) = f(x). \end{cases} \tag{1.7}$$

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Set $g_{-n}(\omega) = \overline{g_n(\omega)}$. For any $\sigma > 0$, we can define the random variable

$$\omega \longmapsto \varphi(\omega, x) = \sum_{n\in\mathbb{Z}^*} \frac{g_n(\omega)}{2|n|^{\frac{1}{2}}} \mathrm{e}^{inx} \in L^2\big(\Omega;\, H^{-\sigma}(\mathbb{S}^1)\big), \tag{1.8}$$

and the measure μ on $X^0(\mathbb{S}^1)$ by $\mu = \mathbf{p}\circ\varphi^{-1}$. Next, as in [**35**] define the measure ρ_N on $X^0(\mathbb{S}^1)$ by

$$\mathrm{d}\rho_N(u) = \Psi_N(u)\mathrm{d}\mu(u), \tag{1.9}$$

where the weight Ψ is given by

$$\Psi_N(u) = \beta_N \chi\big(\|u_N\|^2_{L^2} - \alpha_N\big)\mathrm{e}^{-\frac{2}{3}\int_{\mathbb{S}^1} u_N^3(x)\mathrm{d}x}, \quad u_N = \Pi_N u,$$

with $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\alpha_N = \int_{X^0(\mathbb{S}^1)} \|u_N\|^2_{L^2(\mathbb{S}^1)} \mathrm{d}\mu(u) = \int_\Omega \|\varphi_N(\omega, .)\|^2_{L^2(\mathbb{S}^1)} \mathrm{d}\mathbf{p}(\omega) = \sum_{1 \leq n \leq N} \frac{1}{n},$$

and where the constant $\beta_N > 0$ is chosen so that ρ_N is a probability measure on $X^0(\mathbb{S}^1)$. Then the result of N. Tzvetkov [**35**] reads: There exists $\Psi(u)$ which satisfies for all $p \in [1, +\infty[$, $\Psi(u) \in L^p(\mathrm{d}\mu)$ and

$$\Psi_N(u) \longrightarrow \Psi(u) \quad \text{in} \quad L^p(\mathrm{d}\mu(u)). \tag{1.10}$$

As a consequence, we can define a probability measure ρ on $X^0(\mathbb{S}^1)$ by $\mathrm{d}\rho(u) = \Psi(u)\mathrm{d}\mu(u)$. Then our result is the following

Theorem 1.2. — *There exists a set Σ of full ρ measure so that for every $f \in \Sigma$ the equation* (1.7) *with initial condition $u(0) = f$ has a solution*

$$u \in \mathcal{C}\big(\mathbb{R}\,; X^0(\mathbb{S}^1)\big).$$

For all $t \in \mathbb{R}$, the distribution of the random variable $u(t)$ is ρ.

Some care has to be given for the definition of the non linear term in (1.7), since u has a negative Sobolev regularity. Here we can define $\partial_x(u^2)$ on the support of μ as a limit of a Cauchy sequence (see Lemma 5.3).

As in [**9**, Proposition 3.10] we can prove that

$$\bigcup_{\chi \in \mathcal{C}_0^\infty(\mathbb{R})} \operatorname{supp} \rho = \operatorname{supp} \mu.$$

Observe that φ in (1.8) has mean 0, thus μ and ρ are supported on 0-mean functions. This is not a restriction since the mean $\int_{\mathbb{S}^1} u$ is an invariant of (1.7).

1.4. The derivative non linear Schrödinger equation. — We consider the periodic DNLS equation.

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x\big(|u|^2 u\big), & (t,x) \in \mathbb{R} \times \mathbb{S}^1, \\ u(0,x) = u_0(x). \end{cases} \tag{1.11}$$

Here, for $\sigma < 1/2$ we define the random variable ($\langle n \rangle = (1+n^2)^{1/2}$)

$$\omega \longmapsto \varphi(\omega, x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} \mathrm{e}^{inx} \in L^2\big(\Omega;\, H^\sigma(\mathbb{S}^1)\big), \tag{1.12}$$

and the measure μ on $X^{1/2}(\mathbb{S}^1) = \bigcap_{\sigma < 1/2} H^\sigma(\mathbb{S}^1)$ by $\mu = \mathbf{p} \circ \varphi^{-1}$. Next, denote by

$$f_N(u) = \operatorname{Im} \int_{\mathbb{S}^1} \overline{u_N^2(x)}\, \partial_x(u_N^2(x)) \mathrm{d}x.$$

Let $\kappa > 0$, and let $\chi : \mathbb{R} \longrightarrow \mathbb{R}$, $0 \leq \chi \leq 1$ be a continuous function with support $\operatorname{supp} \chi \subset [-\kappa, \kappa]$ and so that $\chi = 1$ on $[-\kappa/2, \kappa/2]$. We define the density

$$\Psi_N(u) = \beta_N \chi\big(\|u_N\|_{L^2(\mathbb{S}^1)}\big) \mathrm{e}^{\frac{3}{4} f_N(u) - \frac{1}{2} \int_{\mathbb{S}^1} |u_N(x)|^6 \mathrm{d}x},$$

and the measure ρ_N on $X^{1/2}(\mathbb{S}^1)$ by

$$\mathrm{d}\rho_N(u) = \Psi_N(u) \mathrm{d}\mu(u), \tag{1.13}$$

and where $\beta_N > 0$ is chosen so that ρ_N is a probability measure on $X^{1/2}(\mathbb{S}^1)$. By Thomann-Tzvetkov [**34**, Theorem 1.1], ρ_N converges to a probability measure ρ so that $\mathrm{d}\rho(u) = \Psi(u)\mathrm{d}\mu(u)$. Moreover, for all $p \geq 2$, if $\kappa \leq \kappa_p$, then $\Psi(u) \in L^p(\mathrm{d}\mu)$. Then our result reads

Theorem 1.3. — *Assume that $\kappa \leq \kappa_2$. Then there exists a set Σ of full ρ measure so that for every $f \in \Sigma$ the equation* (1.11) *with initial condition $u(0) = f$ has a solution*

$$u \in \mathcal{C}\big(\mathbb{R}\,; X^{1/2}(\mathbb{S}^1)\big).$$

For all $t \in \mathbb{R}$, the distribution of the random variable $u(t)$ is ρ.

Here, for $\kappa \leq \kappa_2$, we have

$$\bigcup_{\chi \in \mathcal{C}_0^\infty([-\kappa,\kappa])} \operatorname{supp} \rho = \big\{\|u\|_{L^2} \leq \kappa\big\} \bigcap \operatorname{supp} \mu.$$

1.5. The half-wave equation. — The periodic cubic Schrödinger on the circle has been much studied and in particular rough solutions have been constructed. See Christ [**15**], Colliander-Oh [**17**], Kwon-Oh [**25**], and Bourgain [**4**] in the 2-dimensional case.
Here we investigate a related equation where one has no more dispersion: We replace the Laplacian with the operator $|D|$, *i.e.* the operator defined by $|D|\mathrm{e}^{inx} = |n|\mathrm{e}^{inx}$, and we consider the following half-wave Cauchy problem

$$\begin{cases} i\partial_t u - |D|u = |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{S}^1, \\ u(0,x) = f(x). \end{cases}$$

This model has been studied by P. Gérard and S. Grellier [**20**] who showed that it is well-posed in $H^{1/2}(\mathbb{S}^1)$. However, the Sobolev space which is invariant by scaling is $L^2(\mathbb{S}^1)$, hence it is natural to try to construct solutions which have low regularity. In the sequel, in order to avoid trouble with the 0-frequency, we make the change of unknown $u \longmapsto \mathrm{e}^{-it}u$, so that we are reduced to consider the equation

$$i\partial_t u - \Lambda u = |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{S}^1,$$

where $\Lambda := |D| + 1$.

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\in\mathbb{Z}}$ a sequence of independent complex normalised Gaussians. Here we define the random variable

$$\omega \mapsto \varphi(\omega, x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{(1+|n|)^{\frac{1}{2}}} \mathrm{e}^{inx} \in L^2\big(\Omega;\, H^{-\sigma}(\mathbb{S}^1)\big), \tag{1.14}$$

for any $\sigma > 0$, and we then define the measure μ on $X^0(\mathbb{S}^1)$ by $\mu = \mathbf{p} \circ \varphi^{-1}$.

We need to give a sense to $|u|^2 u$ on the support of μ. In order to avoid the worst interaction term, we rather consider a gauged version of the equation for which the nonlinearity is formally $|u|^2 u - 2\|u\|^2_{L^2(\mathbb{S}^1)} u$. More precisely, define the Hamiltonian

$$H_N(u) = \int_{\mathbb{S}^1} |\Lambda u|^2 + \frac{1}{2}\int_{\mathbb{S}^1} |\Pi_N u|^4 - \Big(\int_{\mathbb{S}^1} |\Pi_N u|^2\Big)^2,$$

and consider the equation

$$i\partial_t u = \frac{\delta H_N}{\delta \overline{u}},$$

which reads

$$\begin{cases} i\partial_t u - \Lambda u = G_N(u_N), \quad (t,x) \in \mathbb{R} \times \mathbb{S}^1, \\ u(0,x) = f(x), \end{cases} \tag{1.15}$$

with $u_N := \Pi_N u$ and where G_N stands for

$$G_N(u_N) = \Pi_N\big(|u_N|^2 u_N\big) - 2\|u_N\|^2_{L^2(\mathbb{S}^1)} u_N. \tag{1.16}$$

This modification of the nonlinearity is classical, and is the Wick ordered version of the usual cubic nonlinearity (see Bourgain [**4**], Oh-Sulem [**32**]). Recall, that since the L^2 norm of (1.15) is preserved by the flow, one can recover the standard cubic nonlinearity with the change of function $v_N(t) = u_N(t)\exp\big(-2\int_0^t \|u_N(\tau)\|^2_{L^2} \mathrm{d}\tau\big)$.

Here, the main interest for introducing the gauge transform in (1.16) is to define the limit equation, when $N \longrightarrow +\infty$.

Proposition 1.4. — *For all $p \geq 2$, the sequence $\big(G_N(u_N)\big)_{N\geq 1}$ is Cauchy in $L^p\big(X^0(\mathbb{S}^1), \mathcal{B}, d\mu; H^{-\sigma}(\mathbb{S}^1)\big)$. Namely, for all $p \geq 2$, there exist $\eta > 0$ and $C > 0$ so that for all $1 \leq M < N$,*

$$\int_{X^0(\mathbb{S}^1)} \|G_N(u_N) - G_M(u_M)\|^p_{H^{-\sigma}(\mathbb{S}^1)} d\mu(u) \leq \frac{C}{M^\eta}.$$

We denote by $G(u)$ the limit of this sequence.

It is then natural to consider the equation

$$\begin{cases} i\partial_t u - \Lambda u = G(u), \quad (t,x) \in \mathbb{R} \times \mathbb{S}^1, \\ u(0,x) = f(x). \end{cases} \tag{1.17}$$

We now define a Gibbs measure for (1.17) as a limit of Gibbs measures for (1.15). Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ so that $0 \leq \chi \leq 1$. Define

$$\alpha_N = \int_{X^0(\mathbb{S}^1)} \|u_N\|^2_{L^2(\mathbb{S}^1)} \mathrm{d}\mu(u) = \sum_{|n|\leq N} \frac{1}{1+|n|},$$

consider the density

$$\Theta_N(u) = \beta_N \chi\big(\|u_N\|^2_{L^2} - \alpha_N\big) \mathrm{e}^{-\big(\|u_N\|^4_{L^4} - 2\|u_N\|^4_{L^2}\big)}, \tag{1.18}$$

and define the measure

$$\mathrm{d}\rho_N(u) = \Theta_N(u)\mathrm{d}\mu(u),$$

where $\beta_N > 0$ is chosen so that ρ_N is a probability measure. In our next result, we define a weighted Wiener measure for the equation (1.17).

Theorem 1.5. — *The sequence $\Theta_N(u)$ defined in* (1.18) *converges in measure, as $N \to \infty$, with respect to the measure μ. Denote by $\Theta(u)$ the limit and define the probability measure*

$$d\rho(u) \equiv \Theta(u)d\mu(u). \tag{1.19}$$

Then for every $p \in [1, \infty[$, $\Theta(u) \in L^p(d\mu(u))$ and the sequence Θ_N converges to Θ in $L^p(d\mu(u))$, as N tends to infinity.

The sign of the nonlinearity in (1.17) (defocusing) plays a role. Indeed, Theorem 1.5 does not hold when $G(u)$ is replaced with $-G(u)$.

Again, with the arguments of [**9**, Proposition 3.10], we can prove that

$$\bigcup_{\chi\in\mathcal{C}_0^\infty(\mathbb{R})} \operatorname{supp}\rho = \operatorname{supp}\mu.$$

Consider the measure ρ defined in (1.19), then

Theorem 1.6. — *There exists a set Σ of full ρ measure so that for every $f \in \Sigma$ the equation* (1.17) *with initial condition $u(0) = f$ has a solution*

$$u \in \mathcal{C}\big(\mathbb{R}\,; X^0(\mathbb{S}^1)\big).$$

For all $t \in \mathbb{R}$, the distribution of the random variable $u(t)$ is ρ.

In equation (1.17) the dispersive effect is weak and it seems difficult to deal with the regularities on the support of the measure by deterministic methods.

Remark 1.7. — More generally, we can consider the equation

$$i\partial_t u - \Lambda^\alpha u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{S}^1,$$

with $\alpha > 1$ and $p \geq 1$. Define $X^\beta(\mathbb{S}^1) = \bigcap_{\tau<\beta} H^\tau(\mathbb{S}^1)$. In this case, the situation is better since the series

$$\omega \longmapsto \varphi_\alpha(\omega, x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{(1+|n|)^{\alpha/2}}\mathrm{e}^{inx},$$

are so that $\varphi_\alpha \in L^2\big(\Omega; H^\beta(\mathbb{S}^1)\big)$ for all $0 < \beta < (\alpha-1)/2$. Here we should be able to construct solutions

$$u \in \mathcal{C}\big(\mathbb{R}; X^{(\alpha-1)/2}(\mathbb{S}^1)\big).$$

1.6. Notations and structure of the paper. —

Notations. — *In this paper $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters. For $n \in \mathbb{Z}$, we write $\langle n \rangle = (1 + |n|^2)^{1/2}$ and $[n] = 1 + |n|$. We will sometimes use the notations $L^p_T = L^p(-T, T)$ for $T > 0$. For a manifiold M, we write $L^p_x = L^p(M)$ and for $s \in \mathbb{R}$ we define the Sobolev space $H^s_x = H^s(M)$ by the norm $\|u\|_{H^s_x} = \|\big(1-\Delta\big)^{s/2} u\|_{L^2(M)}$. If E is a Banach space and μ is a measure on E, we write $L^p_\mu = L^p(d\mu)$ and $\|u\|_{L^p_\mu E} = \big\|\|u\|_E\big\|_{L^p_\mu}$. For M a manifold, we define $X^\sigma(M) = \bigcap_{\tau<\sigma} H^\tau(M)$, and if $I \subset \mathbb{R}$ is an interval, $\mathcal{C}\big(I; X^\sigma(M)\big) = \bigcap_{\tau<\sigma} \mathcal{C}\big(I; H^\tau(M)\big)$. If X is a random variable, we denote by $\mathscr{L}(X)$ its law (its distribution).*

The rest of the paper is organised as follows. In Section 2 we recall the Prokhorov and the Skohorod theorems which are the crucial tools for the proof of our results. In Section 3 we present the general strategy for the construction of the weak stochastic solutions. Each of the remaining Sections is devoted to a different equation.

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2. The Prokhorov and Skorohod theorems

In this section, we state two basic results, concerning the convergence of random variables. To begin with, recall the following definition (see *e.g.* [**23**, page 114])

Definition 2.1. — Let S be a metric space and $(\rho_N)_{N\geq 1}$ a family of probability measures on the Borel $\sigma-$algebra $\mathcal{B}(S)$. The family (ρ_N) on $(S, \mathcal{B}(S))$ is said to be tight if for any $\varepsilon > 0$ one can find a compact set $K_\varepsilon \subset S$ such that $\rho_N(K_\varepsilon) \geq 1 - \varepsilon$ for all $N \geq 1$.

Then, we have the following compactness criterion (see *e.g.* [**23**, page 114] or [**22**, page 309])

***Theorem 2.2* (Prokhorov)**. — *Assume that the family $(\rho_N)_{N\geq 1}$ of probability measures on the metric space S is tight. Then it is weakly compact, i.e. there is a subsequence $(N_k)_{k\geq 1}$ and a limit measure ρ_∞ such that for every bounded continuous function $f : S \to \mathbb{R}$,*

$$\lim_{k\to\infty} \int_S f(x) d\rho_{N_k}(x) = \int_S f(x) d\rho_\infty(x).$$

In fact, the Prokhorov theorem is stronger: In the case where the space S is separable and complete, the converse of the previous statement holds true, but we will not use this here.

Remark 2.3. — Let us make a remark on the case $S = \mathbb{R}^n$. The measure given by the theorem allows mass concentration in a point and the tightness condition forbids the escape of mass to infinity.

The Prokhorov theorem is of different nature compared to the compactness theorems giving the deterministic weak solutions: In the latter case there can be a loss of energy (as mentioned below (1.6)).

A weak limit of L^2 functions may lose some mass whereas in the Prokhorov theorem a limit measure is a probability measure.

We now state the Skorohod theorem

***Theorem 2.4* (Skorohod).** — *Assume that S is a separable metric space. Let $(\rho_N)_{N\geq 1}$ and ρ_∞ be probability measures on S. Assume that $\rho_N \longrightarrow \rho_\infty$ weakly. Then there exists a probability space on which there are $S-$valued random variables $(X_N)_{N\geq 1}$, X_∞ such that $\mathcal{L}(X_N) = \rho_N$ for all $N \geq 1$, $\mathcal{L}(X_\infty) = \rho_\infty$ and $X_N \longrightarrow X_\infty$ a.s.*

For a proof, see *e.g.* [**22**, page 79]. We illustrate this result with two elementary but significant examples:

- Assume that $S = \mathbb{R}$. Let $(X_N)_{1\leq N\leq\infty}$ be standard Gaussians, *i.e.* $\mathcal{L}(X_N) = \mathcal{L}(X_\infty) = \mathcal{N}_{\mathbb{R}}(0,1)$. Then the convergence in law obviously holds, but in general we can not expect the almost sure convergence of the X_N to X_∞ (define for example $X_N = (-1)^N X_\infty$).
- Assume that $S = \mathbb{R}$. Let $(Y_N)_{1\leq N\leq\infty}$ be random variables. For any random variable Y on $\mathbb{R}$ we denote by $F_Y(t) = P(Y \leq t)$ its cumulative distribution function. Here we assume that for all $1 \leq N \leq \infty$, F_{Y_N} is bijective and continuous, and we prove the Skorohod theorem in this case. Let X be a r.v. so that $\mathcal{L}(X)$ is the uniform distribution on $[0,1]$ and define the r.v. $\widetilde{Y}_N = F_{Y_N}^{-1}(X)$. We now check that the $\widetilde{Y}_N$ satisfy the conclusion of the theorem. To begin with,
$$F_{\widetilde{Y}_N}(t) = P(\widetilde{Y}_N \leq t) = P(X \leq F_{Y_N}(t)) = F_{Y_N}(t),$$
therefore we have for $1 \leq N \leq \infty$, $\mathcal{L}(Y_N) = \mathcal{L}(\widetilde{Y}_N)$. Now if we assume that $Y_N \longrightarrow Y_\infty$ in law, we have for all $t \in \mathbb{R}$, $F_{Y_N}(t) \longrightarrow F_{Y_\infty}(t)$ and in particular $\widetilde{Y}_N \longrightarrow \widetilde{Y}_\infty$ almost surely.

3. General strategy and results

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Let M be a Riemanian compact manifold and let $(e_n)_{n\geq 1}$ be an Hilbertian basis of $L^2(M)$ (with obvious changes, we can allow $n \in \mathbb{Z}$). Consider one of the equations mentioned in the introduction. Denote by
$$X^\sigma = X^\sigma(M) = \bigcap_{\tau<\sigma} H^\tau(M).$$

3.1. General strategy of the proof.

— The general strategy for proving a global existence result is the following:

Step 1: The Gaussian measure μ: We define a measure μ on $X^\sigma(M)$ which is invariant by the flow of the linear part of the equation. The index $\sigma_c \in \mathbb{R}$ is determined by the equation and the manifold M. Indeed this measure can be defined as $\mu = \mathbf{p} \circ \varphi^{-1}$, where $\varphi \in L^2\big(\Omega; H^\sigma(M)\big)$ for all $\sigma < \sigma_c$ is a Gaussian random variable which takes the form

$$\varphi(\omega, x) = \sum_{n \geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(x).$$

Here the (λ_n) satisfy $\lambda_n \sim cn^\alpha$, $\alpha > 0$ and are given by the linear part and the Hamiltonian structure of the equation. Notice in particular that for all measurable $f : X^{\sigma_c}(M) \longrightarrow \mathbb{R}$

$$\int_{X^{\sigma_c}(M)} f(u) \mathrm{d}\mu(u) = \int_\Omega f\big(\varphi(\omega, \cdot)\big) \mathrm{d}\mathbf{p}(\omega). \tag{3.1}$$

Step 2: The invariant measure ρ_N: By working on the Hamiltonian formulation of the equation, we introduce an approximation of the initial problem which has a global flow Φ_N, and for which we can construct a measure ρ_N on $X^{\sigma_c}(M)$ which has the following properties

(i) The measure ρ_N is a probability measure which is absolutly continuous with respect to μ

$$\mathrm{d}\rho_N(u) = \Psi_N(u) \mathrm{d}\mu(u).$$

(ii) The measure ρ_N is invariant by the flow Φ_N by the Liouville theorem.

(iii) There exists $\Psi \not\equiv 0$ such that for all $p \geq 2$, $\Psi(u) \in L^p(\mathrm{d}\mu)$ and

$$\Psi_N(u) \longrightarrow \Psi(u), \quad \text{in} \quad L^p(\mathrm{d}\mu).$$

(In particular $\|\Psi_N(u)\|_{L^p_\mu} \leq C$ uniformly in $N \geq 1$.) This enables to define a probability measure on $X^{\sigma_c}(M)$ by

$$\mathrm{d}\rho(u) = \Psi(u) \mathrm{d}\mu(u),$$

which is formally invariant by the equation.

Step 3: The measure ν_N: We abuse notation and write

$$\mathcal{C}\big([-T, T]; X^{\sigma_c}(M)\big) = \bigcap_{\sigma < \sigma_c} \mathcal{C}\big([-T, T]; H^\sigma(M)\big).$$

We denote by $\nu_N = \rho_N \circ \Phi_N^{-1}$ the measure on $\mathcal{C}\big([-T, T]; X^{\sigma_c}(M)\big)$, defined as the image measure of ρ_N by the map

$$\begin{array}{ccc} X^{\sigma_c}(M) & \longrightarrow & \mathcal{C}\big([-T, T]; X^{\sigma_c}(M)\big) \\ v & \longmapsto & \Phi_N(t)(v). \end{array}$$

In particular, for any measurable $F : \mathcal{C}\big([-T, T]; X^{\sigma_c}(M)\big) \longrightarrow \mathbb{R}$

$$\int_{\mathcal{C}\left([-T,T]; X^{\sigma_c}\right)} F(u) \mathrm{d}\nu_N(u) = \int_{X^{\sigma_c}} F\big(\Phi_N(t)(v)\big) \mathrm{d}\rho_N(v). \tag{3.2}$$

For each model we consider, we show that the corresponding sequence (ν_N) is tight in $\mathcal{C}\big([-T, T]; H^\sigma(M)\big)$ for all $\sigma < \sigma_c$. Therefore, for all $\sigma < \sigma_c$, by the Prokhorov theorem, there exists a measure $\nu_\sigma = \nu$

on $\mathcal{C}\big([-T,T];H^{\sigma}(M)\big)$ so that the weak convergence holds (up to a sub-sequence): For all $\sigma<\sigma_c$ and all bounded continuous $F:\mathcal{C}\big([-T,T];H^{\sigma}(M)\big)\longrightarrow\mathbb{R}$

$$\lim_{N\to\infty}\int_{\mathcal{C}\big([-T,T];H^{\sigma}\big)}F(u)\mathrm{d}\nu_N(u)=\int_{\mathcal{C}\big([-T,T];H^{\sigma}\big)}F(u)\mathrm{d}\nu(u).$$

At this point, observe that if $\sigma_1<\sigma_2$, then $\nu_{\sigma_1}\equiv\nu_{\sigma_2}$ on $\mathcal{C}\big([-T,T];H^{\sigma_1}(M)\big)$. Moreover, by the standard diagonal argument, we can ensure that ν is a measure on $\mathcal{C}\big([-T,T];X^{\sigma_c}(M)\big)$.

Finally, with the Skorohod theorem, we can construct a sequence of random variables which converges to a solution of the initial problem.

We now state a result which will be useful in the sequel. Assume that ρ_N satisfies the properties mentioned in Step 2.

Proposition 3.1. — *Let $\sigma<\sigma_c$. Let $p\geq 2$ and $r>p$. Then for all $N\geq 1$*

$$\big\|\|u\|_{L^p_TH^\sigma_x}\big\|_{L^p_{\nu_N}}\leq CT^{1/p}\big\|\|v\|_{H^\sigma_x}\big\|_{L^r_\mu}. \tag{3.3}$$

Let $q\geq 1$, $p\geq 2$ and $r>p$. Then for all $N\geq 1$

$$\big\|\|u\|_{L^p_TL^q_x}\big\|_{L^p_{\nu_N}}\leq CT^{1/p}\big\|\|v\|_{L^q_x}\big\|_{L^r_\mu}. \tag{3.4}$$

In case $\Psi_N\leq C$, one can take $r=p$ in the previous inequalities.

Proof. — We apply (3.2) with the function $u\longmapsto F(u)=\|u\|^p_{L^p_TH^\sigma_x}$. Here and after, we make the abuse of notation

$$\big\|\|u\|_{L^p_TH^\sigma_x}\big\|_{L^p_{\nu_N}}=\|u\|_{L^p_{\nu_N}L^p_TH^\sigma_x}.$$

Then

$$\begin{aligned}\|u\|^p_{L^p_{\nu_N}L^p_TH^\sigma_x}&=\int_{\mathcal{C}\big([-T,T];X^{\sigma_c}\big)}\|u\|^p_{L^p_TH^\sigma_x}\mathrm{d}\nu_N(u)\\&=\int_{X^{\sigma_c}}\|\Phi_N(t)(v)\|^p_{L^p_TH^\sigma_x}\mathrm{d}\rho_N(v)\\&=\int_{X^{\sigma_c}}\Big[\int_{-T}^{T}\|\Phi_N(t)(v)\|^p_{H^\sigma_x}\mathrm{d}t\Big]\mathrm{d}\rho_N(v)\\&=\int_{-T}^{T}\Big[\int_{X^{\sigma_c}}\|\Phi_N(t)(v)\|^p_{H^\sigma_x}\mathrm{d}\rho_N(v)\Big]\mathrm{d}t,\end{aligned} \tag{3.5}$$

where in the last line we used Fubini. Now we use the invariance of ρ_N under Φ_N, and we deduce that for all $t\in[-T,T]$

$$\int_{X^{\sigma_c}}\|\Phi_N(t)(v)\|^p_{H^\sigma_x}\mathrm{d}\rho_N(v)=\int_{X^{\sigma_c}}\|v\|^p_{H^\sigma_x}\mathrm{d}\rho_N(v).$$

Therefore, from (3.5) and Hölder we obtain with $1/r_1 + 1/r_2 = 1$

$$\begin{aligned}\|u\|^p_{L^p_{\nu_N} L^p_T H^\sigma_x} &= 2T \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v) \\ &= 2T \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \Psi_N(v) \mathrm{d}\mu(v) \\ &\leq C \|v\|^p_{L^{pr_1}_\mu H^\sigma_x} \|\Psi_N(v)\|_{L^{r_2}_\mu}.\end{aligned}$$

Now, let $r > p$, take $r_1 = r/p$ and we can conclude since $\Psi_N \in L^{r_2}(\mathrm{d}\mu)$.
For the proof of (3.4), we proceed similarly. We take $F(u) = \|u\|^p_{L^p_T L^q_x}$ in (3.2), and use the same arguments as previously. $\square$

3.2. Some deterministic estimates. — We now state an interpolation result, which will be useful for the study of each model. Consider $(e_n)_{n\geq 1}$ a Hilbertian basis of $L^2 = L^2(M)$ of eigenfunctions of Δ:

$$-\Delta e_n = \lambda_n^2 e_n, \quad n \geq 1.$$

For $u = \sum_{n\geq 1} \alpha_n e_n$, we define the spectral projector

$$\Delta_j u = \sum_{n\geq 1\,:\, 2^j \leq \langle \lambda_n \rangle < 2^{j+1}} \alpha_n e_n,$$

so that we have $u = \sum_{j\geq 0} \Delta_j u$ and for $\sigma \in \mathbb{R}$

$$C_1 2^{j\sigma} \|\Delta_j u\|_{L^2} \leq \|\Delta_j u\|_{H^\sigma(M)} \leq C_2 2^{j\sigma} \|\Delta_j u\|_{L^2}.$$

Define the space $W^{1,p}_T$ by the norm $\|u\|_{W^{1,p}_T} = \|u\|_{L^p_T} + \|\partial_t u\|_{L^p_T}$. Then

Lemma 3.2. — *Let $T > 0$ and $p \in [1, +\infty]$. Assume that $u \in L^p\big([-T,T]; L^2\big)$ and $\partial_t u \in L^p\big([-T,T]; L^2\big)$. Then $u \in L^\infty\big([-T,T]; L^2\big)$ and*

$$\|u\|_{L^\infty_T L^2} \leq C \|u\|^{1-1/p}_{L^p_T L^2} \|u\|^{1/p}_{W^{1,p}_T L^2}.$$

Proof. — Let $\gamma \in L^2(M)$ be so that $\|\gamma\|_{L^2} = 1$, and define $v(t) = \langle u(t), \gamma\rangle$. Then we clearly have

$$\|v\|_{L^p_T} \leq \|u\|_{L^p_T L^2}, \qquad \|\partial_t v\|_{L^p_T} \leq \|\partial_t u\|_{L^p_T L^2},$$

and from the Gagliardo-Nirenberg inequality we deduce

$$\|v\|_{L^\infty_T} \leq C \|v\|^{1-1/p}_{L^p_T} \|v\|^{1/p}_{W^{1,p}_T} \leq C \|u\|^{1-1/p}_{L^p_T L^2} \|u\|^{1/p}_{W^{1,p}_T L^2}. \tag{3.6}$$

Now from (3.6) we get

$$\begin{aligned}\|u\|_{L^\infty_T L^2} &= \sup_{t\in[-T,T]} \|u(t)\|_{L^2}\\ &= \sup_{t\in[-T,T]}\ \sup_{\|\gamma\|_{L^2}=1} v(t)\\ &= \sup_{\|\gamma\|_{L^2}=1}\ \sup_{t\in[-T,T]} v(t) \leq C\|u\|^{1-1/p}_{L^p_T L^2}\|u\|^{1/p}_{W^{1,p}_T L^2}.\end{aligned}$$

This completes the proof of Lemma 3.2. □

Denote by $H^\sigma = H^\sigma(M)$. Using the previous result we can prove

Lemma 3.3. — *Let $T > 0$ and $p \in [1,+\infty]$. Let $-\infty < \sigma_2 \leq \sigma_1 < +\infty$ and assume that $u \in L^p\big([-T,T];H^{\sigma_1}\big)$ and $\partial_t u \in L^p\big([-T,T];H^{\sigma_2}\big)$. Then for all $\varepsilon > \sigma_1/p - \sigma_2/p$, $u \in L^\infty\big([-T,T];H^{\sigma_1-\varepsilon}\big)$ and*

$$\|u\|_{L^\infty_T H^{\sigma_1-\varepsilon}} \leq C\|u\|^{1-1/p}_{L^p_T H^{\sigma_1}}\|u\|^{1/p}_{W^{1,p}_T H^{\sigma_2}}. \tag{3.7}$$

Moreover, there exists $\eta > 0$ and $\theta \in [0,1]$ so that for all $t_1, t_2 \in [-T,T]$

$$\|u(t_1) - u(t_2)\|_{H^{\sigma_1-2\varepsilon}} \leq C|t_1-t_2|^\eta \|u\|^{1-\theta}_{L^p_T H^{\sigma_1}}\|u\|^{\theta}_{W^{1,p}_T H^{\sigma_2}}.$$

Proof. — We use the frequency decomposition as recalled at the beginning of the section, and apply Lemma 3.2 to $\Delta_j u$

$$\begin{aligned}\|\Delta_j u\|_{L^\infty_T H^{\sigma_1-\varepsilon}} &\leq C2^{j(\sigma_1-\varepsilon)}\|\Delta_j u\|_{L^\infty_T L^2}\\ &\leq C2^{j(\sigma_1-\varepsilon)}\|\Delta_j u\|^{1-1/p}_{L^p_T L^2}\big(\|\partial_t\Delta_j u\|_{L^p_T L^2} + \|\Delta_j u\|_{L^p_T L^2}\big)^{1/p}\\ &\leq C2^{j(\sigma_1-\varepsilon)}2^{-j\sigma_1(1-1/p)}2^{-j\sigma_2/p}\|\Delta_j u\|^{1-1/p}_{L^p_T H^{\sigma_1}}\|\Delta_j u\|^{1/p}_{W^{1,p}_T H^{\sigma_2}}\\ &\leq C2^{-j(\varepsilon-\sigma_1/p+\sigma_2/p)}\|u\|^{1-1/p}_{L^p_T H^{\sigma_1}}\|u\|^{1/p}_{W^{1,p}_T H^{\sigma_2}}.\end{aligned}$$

This inequality together with $\|u\|_{L^\infty_T H^{\sigma_1-\varepsilon}} \leq \sum_{j\geq 0}\|\Delta_j u\|_{L^\infty_T H^{\sigma_1-\varepsilon}}$ yields (3.7). By Hölder we get

$$\|u(t_1)-u(t_2)\|_{H^{\sigma_2}} = \|\int_{t_1}^{t_2}\partial_\tau u(\tau)\mathrm{d}\tau\|_{H^{\sigma_2}} \leq |t_1-t_2|^{1-1/p}\|\partial_t u\|_{L^p_T H^{\sigma_2}}. \tag{3.8}$$

Next by interpolation, there exists $\theta_0 \in (0,1)$ so that

$$\begin{aligned}\|u(t_1)-u(t_2)\|_{H^{\sigma_1-2\varepsilon}} &\leq \|u(t_1)-u(t_2)\|^{1-\theta_0}_{H^{\sigma_1-\varepsilon}}\|u(t_1)-u(t_2)\|^{\theta_0}_{H^{\sigma_2}}\\ &\leq C\|u\|^{1-\theta_0}_{L^\infty_T H^{\sigma_1-\varepsilon}}\|u(t_1)-u(t_2)\|^{\theta_0}_{H^{\sigma_2}},\end{aligned}$$

and the result follows from this latter inequality combined with (3.7) and (3.8). □

4. The Schrödinger equation

4.1. The setting. — Let $\mathbb{S}^3$ be the unit sphere in $\mathbb{R}^4$. Consider the non linear Schrödinger equation

$$\begin{cases} i\partial_t u + (\Delta - 1)u = |u|^{r-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{S}^3, \\ u(0,x) = f(x) \in H^\sigma(\mathbb{S}^3), \end{cases} \tag{4.1}$$

where $\Delta = \Delta_{\mathbb{S}^3}$ stands for the Laplace-Beltrami operator, and where $1 \leq r < 5$. In the sequel we consider functions which only depend on the geodesic distance to the north pole, these are called zonal functions. Denote by $Z(\mathbb{S}^3)$ this space. Roughly speaking, this is the same type of reduction as restricting to radial functions in $\mathbb{R}^3$. Denote by $L^2_{rad}(\mathbb{S}^3) = L^2(\mathbb{S}^3) \cap Z(\mathbb{S}^3)$. We endow this space with the natural norm

$$\|f\|_{L^2_{rad}(\mathbb{S}^3)} = \Big(\int_{\mathbb{S}^3} |f|^2\Big)^{\frac{1}{2}} = \Big(\int_0^\pi |f(x)|^2 (\sin x)^2 \mathrm{d}x\Big)^{\frac{1}{2}},$$

where $x \in [0,\pi]$ represents the geodesic distance to the north pole of $\mathbb{S}^3$. The operator Δ can be restricted to L^2_{rad}, and it reads

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{2}{\tan x}\frac{\partial}{\partial x}.$$

One of the main interests to restrict to zonal functions, is that the eigenvalues of Δ in $L^2_{rad}(\mathbb{S}^3)$ are simple. The family $(P_n)_{n\geq 1}$ defined in (1.2) is a Hilbertian basis of $L^2_{rad}(\mathbb{S}^3)$ of eigenfunction of the Laplacian: For all $n \geq 1$, $-\Delta P_n = (n^2-1)P_n$. We define the operator $\Lambda = \big(1-\Delta\big)^{\frac{1}{2}}$, in particular $\Lambda P_n = nP_n$.

Let us define the complex vector space $E_N = \operatorname{span}\Big((P_n)_{1\leq n\leq N}\Big)$. Then we introduce a smooth version of the usual spectral projector on E_N. Let $\chi \in \mathcal{C}_0^\infty(-1,1)$, so that $\chi \equiv 1$ on $(-1/2, 1/2)$. We then define

$$S_N\Big(\sum_{n\geq 1} c_n P_n\Big) = \chi(\frac{\Lambda}{N})\sum_{n\geq 1} c_n P_n = \sum_{n\geq 1}\chi(\frac{n}{N})c_n P_n.$$

One of the advantages of this operator compared with the usual spectral projector, is the following result. See Burq-Gérard-Tzvetkov [**7**] for a proof.

Lemma 4.1. — *Let $1 < p < \infty$. Then $S_N : L^p(\mathbb{S}^3) \longrightarrow L^p(\mathbb{S}^3)$ is continuous and there exists $C > 0$ so that for all $N \geq 1$,*

$$\|S_N\|_{L^p(\mathbb{S}^3)\to L^p(\mathbb{S}^3)} \leq C.$$

Moreover, for all $f \in L^p(\mathbb{S}^3)$, $S_N f \longrightarrow f$ in $L^p(\mathbb{S}^3)$, when $N \longrightarrow +\infty$.

4.2. Preliminaries: Some estimates. — In the sequel, we will need a particular case of Sogge's estimates.

Lemma 4.2. — *The following bounds hold true for $n \geq 1$*

$$\|P_n\|_{L^p(\mathbb{S}^3)} \leq \begin{cases} Cn^{1/2-1/p}, & \text{if } \ 2 \leq p \leq 4, \\ Cn^{1-3/p}, & \text{if } \ 4 \leq p \leq \infty. \end{cases} \tag{4.2}$$

Proof. — The bound for $p=\infty$ is clear by the definition (1.2). The case $p=4$ is proved in [**36**, Lemma 10.1] thanks to the formula

$$P_k P_\ell = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\min(k,\ell)} P_{|k-\ell|+2j-1}, \quad k,\ell \geq 1.$$

The general case follows by Hölder. □

The next Lemma (Khinchin inequality) shows a smoothing property of the random series in the L^p spaces. See *e.g.* [**12**, Lemma 4.2] for the proof.

Lemma 4.3. — *There exists $C>0$ such that for all $p \geq 2$ and $(c_n) \in \ell^2(\mathbb{N})$*

$$\| \sum_{n\geq 1} g_n(\omega)\, c_n \|_{L^p_{\mathbf{p}}} \leq C\sqrt{p}\Big(\sum_{n\geq 1} |c_n|^2 \Big)^{\frac{1}{2}}. \tag{4.3}$$

Define $\mu = \mathbf{p} \circ \varphi^{\mathbf{-1}}$, with φ given in (1.4). Then we can state

Lemma 4.4. — *Let $\sigma < \frac{1}{2}$, then there exists $C>0$ so that for all $p \geq 2$*

$$\big\| \|v\|_{H^\sigma_x} \big\|_{L^p_\mu} \leq C\sqrt{p}. \tag{4.4}$$

Let $2 \leq q < 6$, then there exists $C>0$ so that for all $p \geq q$

$$\big\| \|v\|_{L^q_x} \big\|_{L^p_\mu} \leq C\sqrt{p}. \tag{4.5}$$

Proof. — We prove (4.4). Let $\sigma < 1/2$ and apply (4.3) to $(1-\Delta)^{\sigma/2}\varphi = \sum_{n\geq 1} \dfrac{g_n}{n^{1-\sigma}} P_n$. Then

$$\|(1-\Delta)^{\sigma/2}\varphi\|_{L^p_{\mathbf{p}}} \leq C\sqrt{p}\Big(\sum_{n\geq 1} \frac{|P_n|^2}{n^{2(1-\sigma)}} \Big)^{\frac{1}{2}}.$$

Take the $L^2(\mathbb{S}^3)$ norm of the previous inequality, and by the Minkowski inequality the claim follows. The proof of (4.5) is similar, using (4.2) and the Minkowski inequality. □

We will also need the next result. See [**9**, Lemma 3.3] for the proof.

Lemma 4.5. — *Let $2 \leq q < 6$. Then there exist $c, C > 0$ so that for all $N \geq 1$ and $\lambda > 0$*

$$\mu\big(u \in X^{1/2}(\mathbb{S}^3) \;:\; \|S_N u\|_{L^q(\mathbb{S}^3)} > \lambda \big) \leq C e^{-c\lambda^2}.$$

Moreover there exist $\alpha, c, C > 0$ so that for all $1 \leq M \leq N$ and $\lambda > 0$

$$\mu\big(u \in X^{1/2}(\mathbb{S}^3) \;:\; \|S_N u - S_M u\|_{L^q(\mathbb{S}^3)} > \lambda \big) \leq C e^{-cM^\alpha \lambda^2}. \tag{4.6}$$

4.3. A convergence result. — Let $1 \leq r < 5$ and recall the definition (1.5) of G. Let $N \geq 1$ and set $G_N = \beta_N G \circ S_N$, where $\beta_N > 0$ is chosen such that

$$\mathrm{d}\rho_N(u) = G_N(u)\mathrm{d}\mu(u),$$

defines a probability measure on $X^{1/2}(\mathbb{S}^3)$. The next statement shows that we can pass to the limit $N \longrightarrow +\infty$ in the previous expression.

Proposition 4.6. — *Let $p \in [1, \infty[$, then*

$$G_N(u) \longrightarrow G(u), \quad in \quad L^p(d\mu(u)),$$

when $N \longrightarrow +\infty$.

In particular, for any Borel set $A \subset X^{1/2}(\mathbb{S}^3)$, $\lim\limits_{N\to\infty} \rho_N(A) = \rho(A)$. Observe that for all $N \geq 1$, $\rho_N\big(X^{1/2} \backslash X^{1/2}_{rad}\big) = 0$, as well as $\rho\big(X^{1/2} \backslash X^{1/2}_{rad}\big) = 0$.

Proof. — Let $q < 6$. By (4.6), we deduce that $\|S_N u\|_{L^q_x} \longrightarrow \|u\|_{L^q_x}$ in mesure, w.r.t. μ, hence $G_N(u) = G(S_N u) \longrightarrow G(u)$. In other words, if for $\varepsilon > 0$ and $N \geq 1$ we denote by

$$A_{N,\varepsilon} = \big\{ u \in X^{1/2}(\mathbb{S}^3) \ : \ |G_N(u) - G(u)| \leq \varepsilon \big\},$$

then $\mu(A^c_{N,\varepsilon}) \longrightarrow 0$, when $N \longrightarrow +\infty$. Now use that $0 \leq G, G_N \leq 1$

$$\begin{aligned} \|G - G_N\|_{L^p_\mu} &\leq \|(G - G_N)\mathbf{1}_{A_{N,\varepsilon}}\|_{L^p_\mu} + \|(G - G_N)\mathbf{1}_{A^c_{N,\varepsilon}}\|_{L^p_\mu} \\ &\leq \varepsilon\big(\,\mu(A_{N,\varepsilon})\,\big)^{1/p} + 2\big(\,\mu(A^c_{N,\varepsilon})\,\big)^{1/p} \leq C\varepsilon, \end{aligned}$$

for N large enough. This ends the proof. □

4.4. Study of the measure ν_N. — Let $N \geq 1$. We then consider the following approximation of (4.1)

$$\begin{cases} i\partial_t u + (\Delta - 1)u = S_N\big(|S_N u|^{r-1} S_N u\big), \ \ (t,x) \in \mathbb{R} \times \mathbb{S}^3, \\ u(0,x) = v(x) \in X^{1/2}_{rad}(\mathbb{S}^3). \end{cases} \tag{4.7}$$

The main motivation to introduce this system is the following proposition, which is directly inspired from [**9**, Section 8]. Therefore we omit the proof.

Proposition 4.7. — *The equation (4.7) has a global flow Φ_N. Moreover, the measure ρ_N is invariant under Φ_N : For any Borel set $A \subset X^{1/2}_{rad}(\mathbb{S}^3)$ and for all $t \in \mathbb{R}$, $\rho_N\big(\Phi_N(t)(A)\big) = \rho_N\big(A\big)$.*

In particular if $\mathscr{L}_{X^{1/2}_{rad}}(v) = \rho_N$ then for all $t \in \mathbb{R}$, $\mathscr{L}_{X^{1/2}_{rad}}(\Phi_N(t)v) = \rho_N$.

Remark 4.8. — Observe that (4.7) is not a finite dimensional system of ODE, but its flow restricted to high frequencies is linear.

We denote by ν_N the measure on $\mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^3)\big)$, defined as the image measure of ρ_N by the map

$$\begin{array}{rcl} X^{1/2}(\mathbb{S}^3) & \longrightarrow & \mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^3)\big) \\ v & \longmapsto & \Phi_N(t)(v). \end{array}$$

Lemma 4.9. — *Let $\sigma<\frac{1}{2}$ and $p\geq 2$. Then for all $N\geq 1$*

$$\big\|\|u\|_{L^p_T H^\sigma_x}\big\|_{L^p_{\nu_N}}\leq C. \tag{4.8}$$

Let $2\leq q<6$ and $p\geq q$. Then for all $N\geq 1$

$$\big\|\|u\|_{L^p_T L^q_x}\big\|_{L^p_{\nu_N}}\leq C. \tag{4.9}$$

Proof. — By (3.3) and the fact that $G\leq C$ we already have

$$\|u\|_{L^p_{\nu_N}L^p_T H^\sigma_x}\leq C\|v\|_{L^p_\mu H^\sigma_x}=C\|\varphi\|_{L^p_{\mathbf{P}}H^\sigma_x},$$

where we used the transport property (3.1) with the map $f\,:\,u\longmapsto\|u\|^p_{H^\sigma_x}$. Finally we conclude with (4.4).
For the proof of (4.9), we use (3.4) and (4.5). □

Lemma 4.10. — *Let $\sigma>\frac{3}{2}$ and $p\geq 2$. Then there exists $C>0$ so that for all $N\geq 1$*

$$\big\|\|u\|_{W^{1,p}_T H^{-\sigma}_x}\big\|_{L^p_{\nu_N}}\leq C. \tag{4.10}$$

Proof. — By (4.8) it is enough to show that $\big\|\|\partial_t u\|_{L^p_T H^{-\sigma}_x}\big\|_{L^p_{\nu_N}}\leq C$. By definition

$$\begin{aligned} \|\partial_t u\|^p_{L^p_{\nu_N}L^p_T H^{-\sigma}_x} &= \int_{\mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^3)\big)}\|\partial_t u\|^p_{L^p_T H^{-\sigma}_x}\mathrm{d}\nu_N(u) \\ &= \int_{X^{1/2}(\mathbb{S}^3)}\|\partial_t\Phi_N(t)(v)\|^p_{L^p_T H^{-\sigma}_x}\mathrm{d}\rho_N(v). \end{aligned}$$

Now we use that $w_N:=\Phi_N(t)(v)$ satisfies (4.7) to get

$$\|\partial_t w_N\|_{L^p_{\rho_N}L^p_T H^{-\sigma}_x}\leq\|(\Delta-1)w_N\|_{L^p_{\rho_N}L^p_T H^{-\sigma}_x}+\|S_N\big(|S_N w_N|^{r-1}S_N w_N\big)\|_{L^p_{\rho_N}L^p_T H^{-\sigma}_x},$$

which in turn implies

$$\|\partial_t u\|_{L^p_{\nu_N}L^p_T H^{-\sigma}_x}\leq\|(\Delta-1)u\|_{L^p_{\nu_N}L^p_T H^{-\sigma}_x}+\|S_N\big(|S_N u|^{r-1}S_N u\big)\|_{L^p_{\nu_N}L^p_T H^{-\sigma}_x}. \tag{4.11}$$

Firstly, by (4.8) we get for $\sigma>1/2$

$$\|(\Delta-1)u\|_{L^p_{\nu_N}L^p_T H^{-\sigma}_x}=\|u\|_{L^p_{\nu_N}L^p_T H^{2-\sigma}_x}\leq C. \tag{4.12}$$

Then by Sobolev, since $\sigma>3/2$, we get $\|g\|_{H^{-\sigma}_x}\leq C\|g\|_{L^1_x}$. Therefore

$$\begin{aligned} \|S_N\big(|S_N u|^{r-1}S_N u\big)\|_{L^p_{\nu_N}L^p_T H^{-\sigma}_x} &\leq C\|S_N\big(|S_N u|^{r-1}S_N u\big)\|_{L^p_{\nu_N}L^p_T L^1_x} \\ &\leq C\|S_N u\|^r_{L^{rk}_{\nu_N}L^{rk}_T L^r_x} \\ &\leq C\|u\|^r_{L^{rk}_{\nu_N}L^{rk}_T L^r_x}, \end{aligned}$$

where we used twice the continuity of S_N on L^p_x spaces (see Lemma 4.1). Now, since $1 \leq r < 5$ we can apply (4.9) and this together with (4.12) implies the result.

□

4.5. The convergence argument. —

Proposition 4.11. — *Let $T > 0$ and $\sigma < \frac{1}{2}$. Then the family of measures*

$$\nu_N = \mathscr{L}_{\mathcal{C}_T H^\sigma}\big(u_N(t); t \in [-T, T]\big)_{N \geq 1}$$

is tight in $\mathcal{C}\big([-T, T]; H^\sigma(\mathbb{S}^3)\big)$.

Proof. — Let $\sigma < \frac{1}{2}$. Fix $\sigma < s' < s'' < \frac{1}{2}$ and $\alpha > 0$. We define the space $\mathcal{C}^\alpha_T H^{s'} = \mathcal{C}^\alpha\big([-T, T]; H^{s'}(\mathbb{S}^3)\big)$ by the norm

$$\|u\|_{\mathcal{C}^\alpha_T H^{s'}} = \sup_{t_1, t_2 \in [-T,T],\, t_1 \neq t_2} \frac{\|u(t_1) - u(t_2)\|_{H^{s'}_x}}{|t_1 - t_2|^\alpha} + \|u\|_{L^\infty_T H^{s'}_x},$$

and it is classical that the embedding $\mathcal{C}^\alpha_T H^{s'} \subset \mathcal{C}\big([-T, T]; H^\sigma(\mathbb{S}^3)\big)$ is compact.

We now claim that there exists $0 < \alpha \ll 1$ so that for all $p \geq 1$ we have the bound

$$\|u\|_{L^p_{\nu_N} \mathcal{C}^\alpha_T H^{s'}} \leq C. \tag{4.13}$$

Indeed apply Lemma 3.3 with $\sigma_1 = s''$ and $\sigma_2 = \sigma$. Then for p large enough we have

$$\|u\|_{\mathcal{C}^\alpha_T H^{s'}} \leq C \|u\|^{1-\theta}_{L^p_T H^{s''}} \|u\|^\theta_{W^{1,p}_T H^{-\sigma}} \leq C \|u\|_{L^p_T H^{s''}} + C \|u\|_{W^{1,p}_T H^{-\sigma}},$$

for some small $\alpha > 0$. By (4.8) and (4.10) we then deduce $\|u\|_{L^p_{\nu_N} \mathcal{C}^\alpha_T H^{s'}} \leq C$. (The fact that (4.13) is indeed true for any $p \geq 1$ is a consequence of Hölder.) Let $\delta > 0$ and define

$$K_\delta = \big\{u \in \mathcal{C}_T H^\sigma \ \ s.t.\ \ \|u\|_{\mathcal{C}^\alpha_T H^{s'}} \leq \delta^{-1}\big\}.$$

Thanks to the previous considerations, the set K_δ is compact. Finally, by Markov and (4.13) we get that

$$\nu_N(K^c_\delta) \leq \delta\, \|u\|_{L^1_{\nu_N} \mathcal{C}^\alpha_T H^{s'}} \leq \delta C,$$

which shows the tightness of (ν_N). □

The result of Proposition 4.11 enables us to use the Prokhorov theorem: For each $T > 0$ there exists a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T, T]; X^{1/2}(\mathbb{S}^3)\big)$ so that for all $\tau < 1/2$ and all bounded continuous function $F : \mathcal{C}\big([-T, T]; H^\tau(\mathbb{S}^3)\big) \longrightarrow \mathbb{R}$

$$\int_{\mathcal{C}([-T,T];H^\tau)} F(u) \mathrm{d}\nu_{N_k}(u) \longrightarrow \int_{\mathcal{C}([-T,T];H^\tau)} F(u) \mathrm{d}\nu(u).$$

By the Skohorod theorem, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{p}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable $\widetilde{u}$ with values in $\mathcal{C}\big([-T, T]; X^{1/2}(\mathbb{S}^3)\big)$ so that

$$\mathscr{L}\big(\widetilde{u}_{N_k}; t \in [-T, T]\big) = \mathscr{L}\big(u_{N_k}; t \in [-T, T]\big) = \nu_{N_k}, \quad \mathscr{L}\big(\widetilde{u}; t \in [-T, T]\big) = \nu, \tag{4.14}$$

and for all $\tau < 1/2$

$$\widetilde{u}_{N_k} \longrightarrow \widetilde{u}, \qquad \widetilde{\mathbf{p}} - \text{a.s. in } \mathcal{C}\big([-T,T]; H^\tau(\mathbb{S}^3)\big). \tag{4.15}$$

We now claim that $\mathscr{L}_{X^{1/2}}(u_{N_k}(t)) = \mathscr{L}_{X^{1/2}}(\widetilde{u}_{N_k}(t)) = \rho_{N_k}$, for all $t \in [-T,T]$ and $k \geq 1$. For all $t \in [-T,T]$, the evaluation map

$$\begin{array}{rcl} R_t\,:\,\mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^3)\big) & \longrightarrow & X^{1/2}(\mathbb{S}^3) \\ u & \longmapsto & u(t,.), \end{array}$$

is well defined and continuous. Then, for all $t \in [-T,T]$, $u_{N_k}(t)$ and $\widetilde{u}_{N_k}(t)$ have same distribution. Let us now determine the distribution of $u_{N_k}(t)$ which we denote by $\nu^t_{N_k}$. By definition of $\nu^t_{N_k}$ and ν_{N_k} we have for all measurable $F : X^{1/2}(\mathbb{S}^3) \longrightarrow \mathbb{R}$

$$\begin{aligned} \int_{X^{1/2}(\mathbb{S}^3)} F(v)\mathrm{d}\nu^t_{N_k}(v) &= \int_{\mathcal{C}\left([-T,T];X^{1/2}(\mathbb{S}^3)\right)} F(R_t u)\mathrm{d}\nu_{N_k}(u) \\ &= \int_{X^{1/2}(\mathbb{S}^3)} F\big(R_t\Phi_{N_k}(\cdot)w\big)\mathrm{d}\rho_{N_k}(w) \\ &= \int_{X^{1/2}(\mathbb{S}^3)} F\big(\Phi_{N_k}(t)(w)\big)\mathrm{d}\rho_{N_k}(w). \end{aligned}$$

From the invariance of ρ_{N_k} under Φ_{N_k} we get $\nu^t_{N_k} = \rho_{N_k}$.

Thus from (4.15) and the convergence property of Proposition 4.6, we deduce that

$$\mathscr{L}_{X^{1/2}}(\widetilde{u}(t)) = \rho, \quad \forall\, t \in [-T,T]. \tag{4.16}$$

Let $k \geq 1$ and $t \in \mathbb{R}$ and consider the r.v. X_k given by

$$X_k = i\partial_t u_{N_k} + (\Delta - 1)u_{N_k} - S_{N_k}\Big(|S_{N_k}u_{N_k}|^{r-1}S_{N_k}u_{N_k}\Big).$$

Define $\widetilde{X}_k$ similarly to X_k with u_{N_k} replaced with $\widetilde{u}_{N_k}$. Then by (4.14), $\mathscr{L}_{\mathcal{C}_T X^{1/2}}(\widetilde{X}_{N_k}) = \mathscr{L}_{\mathcal{C}_T X^{1/2}}(X_{N_k}) = \delta_0$, in other words, $\widetilde{X}_k = 0$ $\widetilde{\mathbf{p}}-$a.s. and $\widetilde{u}_{N_k}$ satisfies the following equation $\widetilde{\mathbf{p}}-$a.s.

$$i\partial_t\widetilde{u}_{N_k} + (\Delta - 1)\widetilde{u}_{N_k} = S_{N_k}\Big(|S_{N_k}\widetilde{u}_{N_k}|^{r-1}S_{N_k}\widetilde{u}_{N_k}\Big). \tag{4.17}$$

We now show that we can pass to the limit $k \longrightarrow +\infty$ in (4.17) in order to show that $\widetilde{u}$ is $\widetilde{\mathbf{p}}-$a.s. a solution to (4.1). Firstly, from (4.15) we deduce the convergence of the linear terms of the equation. Indeed, $\widetilde{\mathbf{p}}-$a.s. , when $k \longrightarrow +\infty$

$$i\partial_t\widetilde{u}_{N_k} + (\Delta - 1)\widetilde{u}_{N_k} \longrightarrow i\partial_t\widetilde{u} + (\Delta - 1)\widetilde{u} \quad \text{in} \quad \mathcal{D}'\big([-T,T]\times\mathbb{S}^3\big).$$

To handle the nonlinear term, we apply the next lemma.

Lemma 4.12. — *Let $1 \leq r < 5$. Up to a sub-sequence, the following convergence holds true*

$$\widetilde{u}_{N_k} \longrightarrow \widetilde{u}, \qquad \widetilde{\mathbf{p}} - \textit{a.s. in } L^r\big([-T,T]\times\mathbb{S}^3\big).$$

Proof. — In order to simplify the notations in the proof, we drop all the tildes and write $N_k \equiv k$ and $L^p_{t,x} = L^p([-T,T]\times\mathbb{S}^3)$. If $1 \leq r \leq 2$, the result immediately follows from (4.15). For $2 < r < 5$, by the Hölder inequality,

$$\|u_k - u\|_{L^r_{t,x}} \leq \|u_k - u\|^{\theta}_{L^2_{t,x}} \|u_k - u\|^{1-\theta}_{L^{r+1}_{t,x}}, \tag{4.18}$$

with $\theta = \frac{2}{r(r-1)}$. By (4.15), a.s. in $\omega \in \Omega$

$$\|u_k - u\|_{L^2_{t,x}} \longrightarrow 0. \tag{4.19}$$

Let $\varepsilon > 0$ and $\lambda > 0$. By the inclusion

$$\forall\ X, Y \geq 0 \qquad \{XY > \lambda\} \subset \{X > \varepsilon^{\theta}\lambda\} \cup \{Y > \varepsilon^{-\theta}\},$$

together with (4.18) and the Markov inequality we have

$$\begin{aligned} &\mathbf{p}\big(\|u_k - u\|_{L^r_{t,x}} > \lambda\big) \\ &\qquad \leq \mathbf{p}\big(\|u_k - u\|^{\theta}_{L^2_{t,x}} > \varepsilon^{\theta}\lambda\big) + \mathbf{p}\big(\|u_k - u\|^{1-\theta}_{L^{r+1}_{t,x}} > \varepsilon^{-\theta}\big) \\ &\qquad\qquad \leq \mathbf{p}\big(\|u_k - u\|_{L^2_{t,x}} > \varepsilon\lambda^{1/\theta}\big) + \varepsilon^{2/(r-2)} \int_{\Omega} \|u_k - u\|^{r+1}_{L^{r+1}_{t,x}} \mathrm{d}\mathbf{p}. \end{aligned} \tag{4.20}$$

By (4.9) and the definition of ν_k

$$\int_{\Omega} \|u_k\|^{r+1}_{L^{r+1}_{t,x}} \mathrm{d}\mathbf{p} = \int \|w\|^{r+1}_{L^{r+1}_{t,x}} \mathrm{d}\nu_k(w) \leq C_T.$$

Similarly, $\displaystyle\int_{\Omega} \|u\|^{r+1}_{L^{r+1}_{t,x}} \mathrm{d}\mathbf{p} \leq C_T$. Therefore $\displaystyle\int_{\Omega} \|u_k - u\|^{r+1}_{L^{r+1}_{t,x}} \mathrm{d}\mathbf{p}$ is bounded uniformly in k. Thus, thanks to (4.19) and (4.20), we get the following convergence in probability

$$\forall\, \lambda > 0, \qquad \mathbf{p}\big(\|u_k - u\|_{L^r_{t,x}} > \lambda\big) \longrightarrow 0, \quad \text{when} \quad k \longrightarrow +\infty,$$

and after passing to a sub-sequence, we obtain the announced almost sure convergence. $\square$

4.6. Conclusion of the proof of Theorem 1.1. — Define $\widetilde{f} = \widetilde{u}(0)$. Then by (4.16), $\mathscr{L}_{X^{1/2}}(\widetilde{f}) = \rho$ and by the previous arguments, there exists $\widetilde{\Omega}' \subset \widetilde{\Omega}$ such that $\widetilde{\mathbf{p}}(\widetilde{\Omega}') = 1$. Set $\Sigma = \widetilde{f}(\Omega')$, then $\rho(\Sigma) = \widetilde{\mathbf{p}}(\widetilde{\Omega}') = 1$. Moreover, for $\omega' \in \widetilde{\Omega}'$, the r.v. $\widetilde{u}$ satisfies the equation

$$\begin{cases} i\partial_t \widetilde{u} + (\Delta - 1)\widetilde{u} = |\widetilde{u}|^{r-1}\widetilde{u}, \quad (t,x) \in \mathbb{R}\times\mathbb{S}^3, \\ \widetilde{u}(0,x) = \widetilde{f}(x) \in X^{1/2}_{rad}(\mathbb{S}^3). \end{cases} \tag{4.21}$$

It remains to check that we can construct a global dynamics. Take a sequence $T_N \to +\infty$, and perform the previous argument for $T = T_N$. For all $N \geq 1$, let Σ_N be the corresponding set of initial conditions and set $\Sigma = \cap_{N\in\mathbb{N}}\Sigma_N$. Then $\rho(\Sigma) = 1$ and for all $\widetilde{f} \in \Sigma$, there exists

$$\widetilde{u} \in \mathcal{C}\big(\mathbb{R}\,; X^{1/2}_{rad}(\mathbb{S}^3)\big),$$

which solves (4.21).

This completes the proof of Theorem 1.1.

5. The Benjamin-Ono equation

5.1. Preliminaries. — As in [**35**], consider the following approximation of (1.7)

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + \Pi_N \partial_x\big((\Pi_N u)^2\big) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{S}^1, \\ u(0,x) = f(x). \end{cases} \tag{5.1}$$

This equation is a linear PDE for the high frequencies (modes larger than $2N$) and an ODE for the low frequencies. It is staightforward to check that the quantity $\|u\|_{L^2(\mathbb{S}^1)}$ is preserved by the equation, thus (5.1) admits a global flow $\Phi_N(t)$. The motivation for introducing (5.1), is that it is given by the Hamiltonian

$$H_N(u) = -\frac{1}{2}\int_{\mathbb{S}^1} \big(|D_x|^{1/2}u\big)^2 - \frac{1}{3}\int_{\mathbb{S}^1} \big(\Pi_N u\big)^3.$$

As a consequence, we can check that the measure ρ_N as defined in (1.9) is invariant by Φ_N. See [**35**] for more details.

We now state a technical result which we will need in the sequel.

Lemma 5.1. — *Let $\alpha > 1/2$, then there exists $C_\beta > 0$ so that for all $N \in \mathbb{Z}$*

$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^\alpha} \leq \frac{C_\beta}{\langle N\rangle^\beta}, \tag{5.2}$$

for all $\beta < 2\alpha - 1$ when $1/2 < \alpha \leq 1$ and $\beta = \alpha$ when $\alpha > 1$.

Proof. — Cut the sum in two parts

$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^\alpha} \leq \sum_{|n|\leq N/2} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^\alpha} + \sum_{|n|> N/2} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^\alpha}. \tag{5.3}$$

Assume that $\alpha > 1$. Then by (5.3)

$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^\alpha} \leq \frac{C}{\langle N\rangle^\alpha}\sum_{|n|\leq N} \frac{1}{\langle n\rangle^\alpha} \leq \frac{C}{\langle N\rangle^\alpha}.$$

Assume that $1/2 < \alpha \leq 1$ and fix $\beta < 2\alpha - 1$. Then by (5.3)

$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^\alpha} \leq \frac{C}{\langle N\rangle^\beta}\sum_{|n|\leq N/2} \frac{1}{\langle n\rangle^\alpha \langle n-N\rangle^{\alpha-\beta}} + \frac{C}{\langle N\rangle^\beta}\sum_{|n|> N/2} \frac{1}{\langle n\rangle^{\alpha-\beta} \langle n-N\rangle^\alpha} \leq \frac{C}{\langle N\rangle^\beta}.$$

□

5.2. Definition of the nonlinear term in (1.7). — To begin with, we have

Lemma 5.2. — *Let $\sigma > 0$. Then there exists $C > 0$ so that for all $p \geq 2$*

$$\big\| \|v\|_{H_x^{-\sigma}} \big\|_{L^p_\mu} \leq C\sqrt{p}.$$

The proof is analogous to (4.4) and is omitted here.

We define the term $\partial_x(u^2)$ in (1.7) on the support of μ as the limit of a Cauchy sequence. Recall the notation $u_N = \Pi_N u$ and set $\Pi^0 = 1 - \Pi_0$ the orthogonal projection on 0-mean functions. The next result is inspired from [**35**, Lemma 5.1]

Lemma 5.3. — *For all* $p \geq 2$, *the sequence* $\big(\Pi^0(u_N^2)\big)_{N\geq 1}$ *is Cauchy in* $L^p\big(X^0(\mathbb{S}^1), \mathcal{B}, d\mu; H^{-\sigma}(\mathbb{S}^1)\big)$. *Namely, for all* $p \geq 2$, *there exist* $\eta > 0$ *and* $C > 0$ *so that for all* $1 \leq M < N$,

$$\int_{X^0(\mathbb{S}^1)} \|\Pi^0(u_N^2) - \Pi^0(u_M^2)\|^p_{H^{-\sigma}(\mathbb{S}^1)} d\mu(u) \leq \frac{C}{M^\eta}.$$

We denote by $\Pi^0(u^2)$ *its limit. This enables to define*

$$\partial_x\big(u^2\big) := \partial_x\big(\,\Pi^0(u^2)\,\big).$$

Proof. — By the result [**34**, Proposition 2.4] on the Wiener chaos, we only have to prove the statement for $p = 2$.

Firstly, by definition of the measure μ

$$\int_{X^0(\mathbb{S}^1)} \|\Pi^0(u_N^2) - \Pi^0(u_M^2)\|^2_{H^{-\sigma}(\mathbb{S}^1)} \mathrm{d}\mu(u) = \int_\Omega \|\Pi^0(\varphi_N^2) - \Pi^0(\varphi_M^2)\|^2_{H^{-\sigma}(\mathbb{S}^1)} \mathrm{d}\mathbf{p}.$$

Therefore, it is enough to prove that $\big(\Pi^0(\varphi_N^2)\big)_{N\geq 1}$ is a Cauchy sequence in $L^2\big(\Omega; H^{-\sigma}(\mathbb{S}^1)\big)$. Let $1 \leq M < N$, let $k \in \mathbb{Z}$ and denote by $\mathrm{e}_k(x) = \mathrm{e}^{ikx}$. Then, by definition of φ_N,

$$\Pi^0(\varphi_N^2) = \sum_{\substack{0<|n_1|,|n_2|\leq N \\ n_1 \neq -n_2}} \frac{g_{n_1} g_{n_2}}{|n_1|^{\frac12}|n_2|^{\frac12}} \mathrm{e}^{i(n_1+n_2)x},$$

and thus we get

$$\langle\, \Pi^0(\varphi_N^2 - \varphi_M^2)\,|\, \mathrm{e}_k\,\rangle = \sum_{B^{(k)}_{M,N}} \frac{g_{n_1} g_{n_2}}{|n_1|^{\frac12}|n_2|^{\frac12}},$$

where $B^{(k)}_{M,N}$ is the set defined by

$$B^{(k)}_{M,N} = \big\{(n_1,n_2) \in \mathbb{Z}^2 \;\text{ s.t. }\; 0 < |n_1|, |n_2| \leq N,\; n_1 \neq -n_2, \\ \big(|n_1| > M \;\text{ or }\; |n_2| > M\big) \;\text{ and }\; n_1 + n_2 = k\big\}.$$

Therefore we obtain

$$\big\|\langle\, \Pi^0(\varphi_N^2 - \varphi_M^2)\,|\, \mathrm{e}_k\,\rangle\big\|^2_{L^2(\Omega)} = \int_\Omega \sum_{\substack{(n_1,n_2)\in B^{(k)}_{M,N} \\ (m_1,m_2)\in B^{(k)}_{M,N}}} \frac{g_{n_1} g_{n_2} \overline{g}_{m_1} \overline{g}_{m_2}}{|n_1|^{\frac12}|n_2|^{\frac12}|m_1|^{\frac12}|m_2|^{\frac12}} \mathrm{d}\mathbf{p}.$$

Since $(g_n)_{n\in\mathbb{Z}^*}$ are independent and centred Gaussians, we deduce that each term in the r.h.s. vanishes, unless $(n_1,n_2) = (m_1,m_2)$ or $(n_1,n_2) = (m_2,m_1)$. Thus by interpolation between (5.2) and the inequality

$$\sum_{|n|>M} \frac{1}{|n||n-k|} \leq \frac{1}{M^\theta} \sum_{n\neq 0} \frac{1}{|n|^{1-\theta}|n-k|} \leq \frac{C_\theta}{M^\theta},$$

we obtain that for all $0<\eta<1$ there exists $C>0$ so that for all $1<M<N$

$$\begin{aligned}\big\|\langle\, \Pi^0(\varphi_N^2-\varphi_M^2)\,|\, \mathrm{e}_k\,\rangle\big\|^2_{L^2(\Omega)} &\leq C \sum_{(n_1,n_2)\in B^{(k)}_{M,N}} \frac{1}{|n_1||n_2|} \\ &\leq C\sum_{|n|>M}\frac{1}{|n||n-k|} \leq \frac{C}{M^\eta \langle k\rangle^{1-\eta}}.\end{aligned}$$

As a consequence we get

$$\begin{aligned}\big\|\Pi^0(\varphi_N^2-\varphi_M^2)\big\|^2_{L^2(\Omega;H^{-\sigma}(\mathbb{S}^1))} &= \sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{2\sigma}} \big\|\langle\, \Pi^0(\varphi_N^2-\varphi_M^2)\,|\, \mathrm{e}_k\,\rangle\big\|^2_{L^2(\Omega)} \\ &\leq \frac{C}{M^\eta}\sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{1+2\sigma-\eta}} \leq \frac{C}{M^\eta},\end{aligned}$$

whenever we choose $\eta<2\sigma$.

□

5.3. Study of the measure ν_N. — Consider the probability measure ρ_N defined by (1.9). Define the measure ν_N on $\mathcal{C}\big([-T,T];X^0(\mathbb{S}^1)\big)$ as the image of ρ_N by the map

$$\begin{array}{ccc} X^0(\mathbb{S}^1) & \longrightarrow & \mathcal{C}\big([-T,T];X^0(\mathbb{S}^1)\big) \\ v & \longmapsto & \Phi_N(t)(v),\end{array}$$

where Φ_N is the flow of (5.1). Then, we are able to prove the following bounds

Lemma 5.4. — *Let $\sigma>0$ and $p\geq 2$. Then there exists $C>0$ such that for all $N\geq 1$*

$$\big\|\|u\|_{L^p_T H^{-\sigma}_x}\big\|_{L^p_{\nu_N}} \leq C, \tag{5.4}$$

and

$$\big\|\|\partial_t u\|_{L^p_T H^{-\sigma-2}_x}\big\|_{L^p_{\nu_N}} \leq C. \tag{5.5}$$

Proof. — The bound (5.4) is obtained thanks to (1.10), (3.3) and Lemma 5.2. We now turn to (5.5). From the equation

$$\partial_t u = -\mathcal{H}\partial_x^2 u - \Pi_N \partial_x\big((\Pi_N u)^2\big),$$

similarly to (4.11), we deduce

$$\|\partial_t u\|_{L^p_{\nu_N} L^p_T H^{-\sigma-2}_x} \leq \|u\|_{L^p_{\nu_N} L^p_T H^{-\sigma}_x} + \|\Pi^0(\Pi_N u)^2\|_{L^p_{\nu_N} L^p_T H^{-\sigma}_x}.$$

By the invariance of the measure ρ_N by Φ_N we get

$$\begin{aligned}\big\| \Pi^0\big[(\Pi_N u)^2\big] \big\|^p_{L^p_{\nu_N}L^p_T H^{-\sigma}_x} &= \int_{\mathcal{C}([-T,T];X^0)} \big\| \Pi^0\big[(\Pi_N u)^2\big] \big\|^p_{L^p_T H^{-\sigma}_x} \mathrm{d}\nu_N(u) \\ &= \int_{X^0(\mathbb{S}^1)} \Big\| \Pi^0\big[\big(\Pi_N\big[\Phi_N(t)(v)\big]\,\big)^2\big] \Big\|^p_{L^p_T H^{-\sigma}_x} \mathrm{d}\rho_N(v) \\ &= \int_{X^0(\mathbb{S}^1)} \big\| \Pi^0\big[(\Pi_N v)^2\big] \big\|^p_{L^p_T H^{-\sigma}_x} \mathrm{d}\rho_N(v) \\ &= 2T\int_{X^0(\mathbb{S}^1)} \big\| \Pi^0\big[(\Pi_N v)^2\big] \big\|^p_{H^{-\sigma}_x} \Psi_N(v)\mathrm{d}\mu(v), \end{aligned} \tag{5.6}$$

and by Cauchy-Schwarz and Lemma 5.3

$$\big\| \Pi^0\big[(\Pi_N u)^2\big] \big\|^p_{L^p_{\nu_N}L^p_T H^{-\sigma}_x} \leq C_T \big\| \Pi^0\big[(\Pi_N v)^2\big] \big\|^p_{L^{2p}_\mu H^{-\sigma}_x} \|\Psi_N(v)\|_{L^2_\mu} \leq C,$$

which concludes the proof. □

Proposition 5.5. — *Let $T>0$ and $\sigma>0$. Then the family of measures*

$$\nu_N = \mathscr{L}_{\mathcal{C}_T H^{-\sigma}}\big(u_N(t); t\in[-T,T]\big)_{N\geq 1}$$

is tight in $\mathcal{C}\big([-T,T];H^{-\sigma}(\mathbb{S}^1)\big)$.

Proof. — The proof is similar to the proof of Proposition 4.11. Here we use the estimates (5.4) and (5.5). □

5.4. Proof of Theorem 1.2. — By Proposition 5.5 we can use the Prokhorov theorem: For each $T>0$ there exists a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T,T];X^0(\mathbb{S}^1)\big)$ so that $\nu_{N_k} \longrightarrow \nu$ weakly on $\mathcal{C}\big([-T,T];H^{-\sigma}(\mathbb{S}^1)\big)$, for all $\sigma>0$. By the Skohorod theorem, there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbf{p}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable $\widetilde{u}$ with values in $\mathcal{C}\big([-T,T];X^0(\mathbb{S}^1)\big)$ so that

$$\mathscr{L}\big(\widetilde{u}_{N_k}; t\in[-T,T]\big) = \mathscr{L}\big(u_{N_k}; t\in[-T,T]\big) = \nu_{N_k}, \quad \mathscr{L}\big(\widetilde{u}; t\in[-T,T]\big) = \nu,$$

and for all $\sigma>0$

$$\widetilde{u}_{N_k} \longrightarrow \widetilde{u}, \qquad \widetilde{\mathbf{p}} - \text{a.s. in } \mathcal{C}\big([-T,T];H^{-\sigma}(\mathbb{S}^1)\big). \tag{5.7}$$

We have that $\mathscr{L}_{X^0(\mathbb{S}^1)}(u_{N_k}(t)) = \mathscr{L}_{X^0(\mathbb{S}^1)}(\widetilde{u}_{N_k}(t)) = \rho_{N_k}$, for all $t\in[-T,T]$ and $k\geq 1$. Therefore, for all $t\in[-T,T]$, $\mathscr{L}_{X^0(\mathbb{S}^1)}(u(t)) = \rho$. Next, $\widetilde{u}_{N_k}$ satisfies the following equation $\widetilde{\mathbf{p}}-$a.s.

$$\partial_t \widetilde{u}_{N_k} + \mathcal{H}\,\partial_x^2 \widetilde{u}_{N_k} + \Pi_{N_k}\partial_x\big((\Pi_{N_k}\widetilde{u}_{N_k})^2\big) = 0.$$

We now show that we can pass to the limit $k\longrightarrow+\infty$ in the previous equation. Firstly, from (5.7) we deduce the convergence of the linear terms of the equation. Indeed, $\widetilde{\mathbf{p}}-$a.s. , when $k\longrightarrow+\infty$

$$\partial_t \widetilde{u}_{N_k} + \mathcal{H}\,\partial_x^2 \widetilde{u}_{N_k} \longrightarrow \partial_t\widetilde{u} + \mathcal{H}\,\partial_x^2\widetilde{u} \quad \text{in} \quad \mathcal{D}'\big([-T,T]\times\mathbb{S}^1\big).$$

The only difficulty is to pass to the limit in the non linear term. Here we can proceed as in **[19]**.

Lemma 5.6. — *Let $\sigma > 0$. Up to a sub-sequence, the following convergence holds true*

$$\Pi^0\big[(\Pi_{N_k}\widetilde{u}_{N_k})^2\big] \longrightarrow \Pi^0\big[\widetilde{u}^2\big], \qquad \widetilde{\mathbf{p}} - a.s. \ in \ L^2\big([-T,T]; H^{-\sigma}(\mathbb{S}^1)\big).$$

Proof. — In order to simplify the notations, in this proof we drop the tildes and write $N_k = k$. Let $M \geq 1$ and write

$$\Pi^0\Big[(\Pi_k u_k)^2 - u^2\Big] = \Pi^0\Big[\big((\Pi_k u_k)^2 - u_k^2\big) + \big(u_k^2 - (\Pi_M u_k)^2\big) + \big((\Pi_M u_k)^2 - (\Pi_M u)^2\big) + \big((\Pi_M u)^2 - u^2\big)\Big].$$

To begin with, by continuity of the square in finite dimension, when $k \longrightarrow +\infty$

$$\Pi^0\big[(\Pi_M u_k)^2\big] \longrightarrow \Pi^0\big[(\Pi_M u)^2\big], \qquad \widetilde{\mathbf{p}} - \text{a.s. in } \ L^2\big([-T,T]; H^{-\sigma}(\mathbb{S}^1)\big).$$

We now deal with the other terms. It is sufficient to show the convergence in the space $X := L^2\big(\Omega \times [-T,T]; H^{-\sigma}(\mathbb{S}^1)\big)$, since the almost sure convergence follows after exaction of a sub-sequence.
With the same arguments as in (5.6) we obtain

$$\begin{aligned}
\big\| \Pi^0\big[(\Pi_M u_k)^2 - u_k^2\big] \big\|_X^2 &= \int_{\mathcal{C}([-T,T];X^0)} \big\| \Pi^0\big[(\Pi_M v)^2 - v^2\big] \big\|^2_{L^2_T H^{-\sigma}_x} \mathrm{d}\nu_k(v) \\
&= \int_{X^0(\mathbb{S}^1)} \Big\| \Pi^0\Big[\big[\Pi_M \Phi_k(t)(f)\big]^2 - \big[\Phi_k(t)(f)\big]^2\Big] \Big\|^2_{L^2_T H^{-\sigma}_x} \mathrm{d}\rho_k(f) \\
&= \int_{X^0(\mathbb{S}^1)} \big\| \Pi^0\big[\big(\Pi_M f\big)^2 - f^2\big] \big\|^2_{L^2_T H^{-\sigma}_x} \mathrm{d}\rho_k(f) \\
&= 2T \int_{X^0(\mathbb{S}^1)} \big\| \Pi^0\big[\big(\Pi_M f\big)^2 - f^2\big] \big\|^2_{H^{-\sigma}_x} \Psi_k(f) \mathrm{d}\mu(f),
\end{aligned}$$

and by Cauchy-Schwarz and (1.10),

$$\big\| \Pi^0\big[(\Pi_M u_k)^2 - u_k^2\big] \big\|_X \leq C \big\| \Pi^0\big[\big(\Pi_M f\big)^2 - f^2\big] \big\|_{L^4_\mu H^{-\sigma}_x}.$$

This latter term tends to 0 uniformly in $k \geq 1$ when $M \longrightarrow +\infty$, according to Lemma 5.3. The term $\big\| \Pi^0\big[(\Pi_M u)^2 - u^2\big] \big\|_X$ is treated similarly.
Finally, with the same argument we show

$$\big\| \Pi^0\big[(\Pi_k u_k)^2 - u_k^2\big] \big\|_X \leq C \big\| \Pi^0\big[\big(\Pi_k f\big)^2 - f^2\big] \big\|_{L^4_\mu H^{-\sigma}_x},$$

which tends to 0 when $k \longrightarrow +\infty$. This completes the proof. □

The conclusion of the proof of Theorem 1.2 is similar to the argument in Subsection 4.6.

6. The derivative nonlinear Schrödinger equation

6.1. Hamiltonian formalism of DNLS. — To begin with, we recall some facts which are explained in the appendix of [**34**]. We define the operator ∂^{-1} by

$$\partial^{-1} \,:\, f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{inx} \longmapsto \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\alpha_n}{in} \mathrm{e}^{inx},$$

and the skew symmetric operator $(K(u,v)^* = -K(u,v))$

$$K(u,v) = \begin{pmatrix} -u\partial^{-1}u\cdot & -i + u\partial^{-1}v\cdot \\ i + v\partial^{-1}u\cdot & -v\partial^{-1}v\cdot \end{pmatrix}. \tag{6.1}$$

Define H by

$$H(u(t)) = \int_{\mathbb{S}^1} |\partial_x u|^2 \mathrm{d}x + \frac{3}{4} i \int_{\mathbb{S}^1} \overline{u}^2\, \partial_x(u^2)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}^1} |u|^6 \mathrm{d}x,$$

and introduce the Hamiltonian system

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = K(u,v) \begin{pmatrix} \frac{\delta H}{\delta u}(u,v) \\ \frac{\delta H}{\delta v}(u,v) \end{pmatrix}. \tag{6.2}$$

Denote by

$$T_u(t) = 2\,\mathrm{Im} \int_{\mathbb{S}^1} u\partial_x \overline{u} + \frac{3}{2}\int_{\mathbb{S}^1} |u|^4, \tag{6.3}$$

then the system (6.2) is a Hamiltonian formulation of the equation

$$i\partial_t u + \partial_x^2 u = i\partial_x\big(|u|^2 u\big) + T_u(t)u, \tag{6.4}$$

in the coordinates $(u,v) = (u,\overline{u})$ (see [**34**, Proposition A.2]). Now, if we set

$$v(t,x) = \mathrm{e}^{i\int_0^t T_u(s)\mathrm{d}s} u(t,x), \tag{6.5}$$

then v is the solution of the equation

$$\begin{cases} i\partial_t v + \partial_x^2 v = i\partial_x\big(|v|^2 v\big), \quad (t,x) \in \mathbb{R}\times\mathbb{S}^1, \\ v(0,x) = u_0(x). \end{cases}$$

Moreover, if u and v are linked by (6.5), we have $T_u = T_v$.

Thanks to these observations, we can focus on the equation (6.4). We introduce a natural truncation for which we can construct an invariant Gibbs measure. Namely, let K be given by (6.1), and consider the following system

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \Pi_N K(u_N, v_N)\Pi_N \begin{pmatrix} \frac{\delta H}{\delta u}(u_N,v_N) \\ \frac{\delta H}{\delta v}(u_N,v_N) \end{pmatrix}. \tag{6.6}$$

This an Hamiltonian system with Hamiltonian $H(\Pi_N u, \Pi_N v)$. Now we assume that $v = \overline{u}$ and we compute the equation satisfied by u_N: this will be a finite dimensional approximation of (6.4). Denote by $\Pi_N^{\perp} = 1 - \Pi_N$, then we have

In the coordinates $v_N = \overline{u_N}$, the system (6.6) reads

$$i\partial_t u + \partial_x^2 u_N = i\Pi_N\Big(\partial_x(|u_N|^2 u_N)\Big) + u_N T_{u_N} + R_N(u_N), \quad (t,x) \in \mathbb{R}\times\mathbb{S}^1, \tag{6.7}$$

where

$$\begin{aligned} R_N(u_N) &= \frac{3}{2}\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(u_N\partial_x(\overline{u_N}^2)\big)+\overline{u_N}\Pi_N^{\perp}\big(\overline{u_N}\partial_x(u_N^2)\big)\Big]\Big)\\ &\quad+\frac{3}{2}i\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(|u_N|^4\overline{u_N}\big)-\overline{u_N}\Pi_N^{\perp}\big(|u_N|^4u_N\big)\Big]\Big)\\ &:= R_N^1(u_N)+R_N^2(u_N). \end{aligned} \tag{6.8}$$

For all $N\geq 1$, this equation is globally well-posed in $L^2(\mathbb{S}^1)$ and denote by Φ_N the flowmap. Moreover, the measure ρ_N defined in (1.13) is invariant by Φ_N (see [**34**, Proposition A.4]).

Recall that $\mu=\mathbf{p}\circ\varphi^{-\mathbf{1}}$ with φ as in (1.12). We need to give a sense to the expression T_u in (6.3) on the support of μ.

Lemma 6.1. — *For all $p\geq 2$, the sequence $\big(T_{u_N}\big)_{N\geq 1}$ is a Cauchy sequence in $L^p\big(X^{1/2}(\mathbb{S}^1),\mathcal{B},d\mu;\mathbb{R}\big)$. Namely, for all $p\geq 2$, there exists $C>0$ so that for all $1\leq M<N$,*

$$\int_{X^{1/2}(\mathbb{S}^1)}|T_{u_N}-T_{u_M}|^p\,d\mu(u)\leq\frac{C}{M}.$$

We denote by T_u the limit of this sequence which is formally given by (6.3).

Proof. — Denote by $J(u)=\operatorname{Im}\int_{\mathbb{S}^1}u\partial_x\overline{u}$. Let $1\leq M<N$. Then for $\varphi_N(\omega,x)=\sum_{|n|\leq N}\frac{g_n(\omega)}{\langle n\rangle}\mathrm{e}^{inx}$ we compute

$$J(\varphi_N)-J(\varphi_M)=-\sum_{M<|n|\leq N}\frac{n|g_n|^2}{\langle n\rangle^2}=-\sum_{M<|n|\leq N}\frac{n(|g_n|^2-1)}{\langle n\rangle^2},$$

where we used that $\sum_{M<|n|\leq N}\frac{n}{\langle n\rangle^2}=0$. Define the r.v. $G_n(\omega)=|g_n(\omega)|^2-1$, hence

$$|J(\varphi_N)-J(\varphi_M)|^2=\sum_{M<|n_1|,|n_2|\leq N}\frac{n_1n_2G_{n_1}G_{n_2}}{\langle n_1\rangle^2\langle n_2\rangle^2}. \tag{6.9}$$

By independence of the g_n, $\mathbb{E}[G_nG_m]=C\delta_{n,m}$. Thus by integration of (6.9)

$$\int_\Omega|J(\varphi_N)-J(\varphi_M)|^2\mathrm{d}\mathbf{p}=\sum_{M<|n|\leq N}\frac{n^2}{\langle n\rangle^4}\leq\frac{C}{M}.$$

By definition of μ we have proved the result for $p=2$. The general case $p\geq 2$ follows from the Wiener chaos estimates (see *e.g.* [**34**, Proposition 2.4]). □

6.2. Study of the measure ν_N. — Now define the measure $\nu_N=\rho_N\circ\Phi_N^{-1}$ on $\mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^1)\big)$ and we have

Lemma 6.2. — *Let $\sigma < \frac{1}{2}$ and $p \geq 2$. Then for all $N \geq 1$*

$$\big\| \|u\|_{L^p_T H^\sigma_x} \big\|_{L^p_{\nu_N}} \leq C. \tag{6.10}$$

$$\big\| \|\partial_t u\|_{L^p_T H^{\sigma-2}_x} \big\|_{L^p_{\nu_N}} \leq C. \tag{6.11}$$

Proof. — The estimate (6.10) is obtained with Proposition 3.1 and the definition (1.12) of φ. Similarly, we also have that for all $2 \leq q \leq p$

$$\|u\|_{L^p_{\nu_N} L^p_T L^q_x} \leq C. \tag{6.12}$$

We turn to (6.11). From the equation (6.7) we get (similarly to (4.11))

$$\begin{aligned}
&\|\partial_t u\|_{L^p_{\nu_N} L^p_T H^{\sigma-2}_x} \leq \\
&\quad \leq \|\partial_x^2 u\|_{L^p_{\nu_N} L^p_T H^{\sigma-2}_x} + \|\partial_x(|u_N|^2 u_N)\|_{L^p_{\nu_N} L^p_T H^{\sigma-2}_x} + \|u_N T_{u_N}\|_{L^p_{\nu_N} L^p_T H^{\sigma-2}_x} + \|R_N(u_N)\|_{L^p_{\nu_N} L^p_T H^{\sigma-2}_x} \\
&\quad \leq \|u\|_{L^p_{\nu_N} L^p_T H^\sigma_x} + \|u_N\|^3_{L^p_{\nu_N} L^p_T L^6_x} + \|u_N T_{u_N}\|_{L^p_{\nu_N} L^p_T L^2_x} + \|R_N(u_N)\|_{L^p_{\nu_N} L^p_T H^{\sigma-2}_x}.
\end{aligned}$$

We estimate each term of the r.h.s. By (6.10) and (6.12) we only have to consider the two last ones. By Cauchy-Schwarz (recall that T_u does not depend on x)

$$\|u_N T_{u_N}\|_{L^p_{\nu_N} L^p_T L^2_x} \leq \|u_N\|_{L^{2p}_{\nu_N} L^{2p}_T L^2_x} \|T_{u_N}\|_{L^{2p}_{\nu_N} L^{2p}_T}. \tag{6.13}$$

Then using the invariance of ρ_N (see the proof of Proposition 3.1) and Lemma 6.1 we have

$$\begin{aligned}
\|T_{u_N}\|^{2p}_{L^{2p}_{\nu_N} L^{2p}_T} &= 2T \int_{X^{1/2}(\mathbb{S}^1)} |T_{v_N}|^{2p} \Psi_N(v) \mathrm{d}\mu(v) \\
&\leq C \|T_{v_N}\|^{2p}_{L^{4p}_\mu} \|\Psi_N(v)\|_{L^2_\mu} \leq C,
\end{aligned}$$

which by (6.13) implies

$$\|u_N T_{u_N}\|_{L^p_{\nu_N} L^p_T L^2_x} \leq C.$$

The conclusion of the proof is given by the next result. □

Lemma 6.3. — *Let $\sigma > 1/2$ and $p \geq 2$. Then*

$$\big\| \|R_N(u_N)\|_{L^p_T H^{-\sigma}_x} \big\|_{L^p_{\nu_N}} \longrightarrow 0 \quad \textit{when} \quad N \longrightarrow +\infty.$$

Proof. — To begin with, using the same arguments as in the proof of Proposition 3.1 with $F(u) = \|R_N(\Pi_N u)\|^p_{L^p_T H^{-\sigma}_x}$ we have,

$$\|R_N(u_N)\|_{L^p_{\nu_N} L^p_T H^{-\sigma}_x} \leq C \|R_N(v_N)\|_{L^{2p}_\mu H^{-\sigma}_x},$$

where we used that $\|\Psi_N\|_{L^2_\mu} \leq C$. We estimate each contribution in the r.h.s. of (6.8).

• Denote by $Q_N(v_N) = v_N \Pi_N^{\perp}\big(v_N \partial_x(\overline{v_N}^2)\big)$. Then by Sobolev and Cauchy-Schwarz

$$\begin{aligned}
\|R^1_N(v_N)\|_{L^r_\mu H^{-\sigma}_x} &\leq C\|R^1_N(v_N)\|_{L^r_\mu L^1_x} \\
&\leq C\|v_N \partial^{-1} Q_N(v_N)\|_{L^r_\mu L^1_x} \\
&\leq \|v_N\|_{L^{2r}_\mu L^2_x}\|Q_N(u_N)\|_{L^{2r}_\mu H^{-1}_x} \\
&\leq C\|Q_N(u_N)\|_{L^{2r}_\mu H^{-1}_x}. \qquad (6.14)
\end{aligned}$$

Next, by the definition of μ and the Wiener chaos estimates

$$\begin{aligned}
\|R^1_N(v_N)\|_{L^r_\mu H^{-\sigma}_x} &\leq C\|Q_N(\varphi_N)\|_{L^{2r}_{\mathbf{p}} H^{-1}_x} \\
&\leq C\|Q_N(\varphi_N)\|_{L^2_{\mathbf{p}} H^{-1}_x}. \qquad (6.15)
\end{aligned}$$

We now compute the term $\|Q_N(\varphi_N)\|_{L^2_{\mathbf{p}} H^{-1}_x}$. We have

$$\varphi_N \partial_x\big(\overline{\varphi_N^2}\big) = -i \sum_{|n_1|,|n_2|,|n_3|\leq N} \frac{(n_1+n_2)\overline{g_{n_1}}\,\overline{g_{n_2}}\,g_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} \mathrm{e}^{i(n_3-n_2-n_1)x},$$

so that

$$\partial^{-1} Q_N(\varphi_N) = -\sum_{n\in A_N} \frac{(n_1+n_2)\overline{g_{n_1}}\,\overline{g_{n_2}}\,g_{n_3}\,g_{n_4}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle\langle n_4\rangle(n_4+n_3-n_2-n_1)} \mathrm{e}^{i(n_4+n_3-n_2-n_1)x},$$

where the set A_N is given by

$$A_N := \Big\{ n=(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4 \;\text{ s.t. }\; |n_1|,|n_2|,|n_3|,|n_4|\leq N, \\ |n_1+n_2-n_3|>N \;\text{ and }\; n_4+n_3-n_2-n_1\neq 0\Big\}.$$

As a consequence we obtain the following expression

$$\|Q_N(\varphi_N)\|^2_{H^{-1}_x} = \sum_{n,m\in B_N} \frac{(n_1+n_2)(m_1+m_2)\overline{g_{n_1}}\,\overline{g_{n_2}}\,g_{n_3}\,g_{n_4}\,g_{m_1}\,g_{m_2}\,\overline{g_{m_3}}\,\overline{g_{m_4}}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle\langle n_4\rangle\langle m_1\rangle\langle m_2\rangle\langle m_3\rangle\langle m_4\rangle(n_4+n_3-n_2-n_1)^2}, \qquad (6.16)$$

with

$$B_N := \Big\{ n,m\in A_N \;\text{ s.t. }\; m_4+m_3-m_2-m_1 = n_4+n_3-n_2-n_1\Big\}.$$

We take the expectation of (6.16). By independence of the g_n and since they are centered, each contribution in the r.h.s. is zero, unless $\{n_1,n_2,m_3,m_4\} = \{m_1,m_2,n_3,n_4\}$. But coming back to the definition of A_N, the condition $|n_1+n_2-n_3|>N$ implies that $n_3 \notin \{n_1,n_2\}$. Similarly,

$m_3 \notin \{m_1, m_2\}$. Therefore, up to permutation we have $n = m$ and by (5.2) with $\alpha = 2$

$$\begin{aligned}\int_\Omega \|Q_N(\varphi_N)\|^2_{H_x^{-1}} \mathrm{d}\mathbf{p} &\leq C \sum_{n \in A_N} \frac{(n_1+n_2)^2}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2(n_4+n_3-n_2-n_1)^2} \\ &\leq CN^2 \sum_{n \in A_N} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_3-n_2-n_1\rangle^2} \\ &\leq C \sum_{n \in A_N} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}.\end{aligned}$$

Next, use that on A_N, $\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle \geq CN$ to get that

$$\int_\Omega \|Q_N(\varphi_N)\|^2_{H_x^{-1}} \mathrm{d}\mathbf{p} \leq \frac{C}{N^{1/2}} \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n_1\rangle^{3/2}\langle n_2\rangle^{3/2}\langle n_3\rangle^{3/2}} \leq \frac{C}{N^{1/2}}. \tag{6.17}$$

Finally, from (6.14), (6.15) and (6.17) we conclude that

$$\|R_N^1(u_N)\|_{L^p_{\nu_N} L^p_T H_x^{-\sigma}} \longrightarrow 0.$$

• We now consider the contribution of R_N^2. With the same arguments as previously,

$$\begin{aligned}\|R_N^2(v_N)\|_{L^r_\mu H_x^{-\sigma}} &\leq C\|R_N^2(v_N)\|_{L^r_\mu L^1_x} \\ &\leq C\Big\| v_N \partial^{-1}\Big[v_N \Pi_N^\perp\big(|v_N|^4\overline{v_N}\big)\Big]\Big\|_{L^r_\mu L^1_x} \\ &\leq C\|v_N\|_{L^{2r}_\mu L^2_x} \|v_N \Pi_N^\perp\big(|v_N|^4\overline{v_N}\big)\|_{L^{2r}_\mu H_x^{-1}} \\ &\leq C\|v_N \Pi_N^\perp\big(|v_N|^4\overline{v_N}\big)\|_{L^{2r}_\mu L^1_x} \\ &\leq C\|\Pi_N^\perp\big(|v_N|^4\overline{v_N}\big)\|_{L^{4r}_\mu L^2_x}.\end{aligned}$$

Denote by $V_N = |v_N|^4\overline{v_N}$. Then by [**34**, Lemma 2.2], $(V_N)_{N\geq 1}$ is a Cauchy sequence in $L^{4r}_\mu L^2_x$, and denote by V its limit. Write

$$\begin{aligned}\|\Pi_N^\perp V_N\|_{L^{4r}_\mu L^2_x} &\leq \|\Pi_N^\perp(V_N - V)\|_{L^{4r}_\mu L^2_x} + \|\Pi_N^\perp V\|_{L^{4r}_\mu L^2_x} \\ &\leq \|V_N - V\|_{L^{4r}_\mu L^2_x} + \|\Pi_N^\perp V\|_{L^{4r}_\mu L^2_x},\end{aligned}$$

which tends to 0 when $N \longrightarrow +\infty$. □

Proposition 6.4. — *Let $T > 0$ and $\sigma < 1/2$. Then the family of measures*

$$\nu_N = \mathscr{L}_{\mathcal{C}_T H^\sigma}\big(u_N(t); t \in [-T,T]\big)_{N\geq 1}$$

is tight in $\mathcal{C}\big([-T,T]; H^\sigma(\mathbb{S}^1)\big)$.

Proof. — The proof is similar to the proof of Proposition 4.11. Here we use the estimates (6.10) and (6.11). □

6.3. Proof of Theorem 1.3. — We can proceed as in the proofs of Theorems 1.1 and 1.2. By Proposition 6.4 and the Prokhorov theorem we can extract a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^1)\big)$ so that $\nu_{N_k}\longrightarrow\nu$ weakly on $\mathcal{C}\big([-T,T];H^{\sigma}(\mathbb{S}^1)\big)$ for all $\sigma<1/2$. Thanks to the Skohorod theorem, there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbf{p}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable $\widetilde{u}$ with values in $\mathcal{C}\big([-T,T];X^{1/2}(\mathbb{S}^1)\big)$ so that

$$\mathscr{L}\big(\widetilde{u}_{N_k};t\in[-T,T]\big)=\mathscr{L}\big(u_{N_k};t\in[-T,T]\big)=\nu_{N_k},\quad \mathscr{L}\big(\widetilde{u};t\in[-T,T]\big)=\nu,$$

and for all $\sigma<1/2$

$$\widetilde{u}_{N_k}\longrightarrow\widetilde{u},\qquad \widetilde{\mathbf{p}}-\text{a.s. in }\ \mathcal{C}\big([-T,T];H^{\sigma}(\mathbb{S}^1)\big).$$

Moreover, $\widetilde{u}_{N_k}$ satisfies $\widetilde{\mathbf{p}}$-a.s. the equation (6.7). Passing to the limit in the linear terms makes no difficulty, we only have to take care on the nonlinear terms. Denote by

$$\mathcal{G}_N(u)=i\Pi_N\Big(\partial_x(|u_N|^2u_N)\Big)+u_NT_{u_N}+R_N(u_N).$$

The next result completes the proof of Theorem 1.3 (the conclusion of the proof is similar to the argument in Subsection 4.6).

Lemma 6.5. — *Up to a sub-sequence, the following convergence holds true. For any $\sigma>0$*

$$\mathcal{G}_{N_k}(\widetilde{u}_{N_k})\longrightarrow i\partial_x(|\,\widetilde{u}\,|^2\,\widetilde{u})+\widetilde{u}T_{\widetilde{u}},\qquad \widetilde{\mathbf{p}}-a.s.\ in\ L^2\big([-T,T];H^{-\sigma}(\mathbb{S}^1)\big).$$

Proof. — We drop the tildes and write $N_k\equiv N$. Since $\mathscr{L}(u_N)=\nu_N$, we can apply Lemma 6.3

$$\|R_N(u_N)\|_{L^2_{\mathbf{p}}L^2_TH^{-\sigma}_x}=\|R_N(u_N)\|_{L^2_{\nu_N}L^2_TH^{-\sigma}_x}\longrightarrow0,$$

when $N\longrightarrow+\infty$. The convergence of the two other terms is obtained as in Lemma 5.6. □

Remark 6.6. — Observe that in all the proof, we only used the fact that $\Psi_N\in L^2(\mathrm{d}\mu)$ uniformly in N (and not higher order integrability). Therefore the result of Theorem 1.3 holds for $\kappa\leq\kappa_2$, and the support of ρ is not empty.

7. The half-wave equation

7.1. Justification of the equation. —

Proof of Proposition 1.4. — We prove the result when $p=2$. The general case follows by the Wiener chaos estimates.

To begin with, use that

$$\int_{X^0(\mathbb{S}^1)}\|G_N(u_N)-G_M(u_M)\|^2_{H^{-\sigma}(\mathbb{S}^1)}\mathrm{d}\mu(u)=\int_{\Omega}\|G_N(\varphi_N)-G_M(\varphi_M)\|^2_{H^{-\sigma}(\mathbb{S}^1)}\mathrm{d}\mathbf{p}.$$

Therefore, we are reduced to prove that $\big(G_N(\varphi_N)\big)_{N\geq1}$ is a Cauchy sequence in $L^2\big(\Omega;H^{-\sigma}(\mathbb{S}^1)\big)$. Denote by

$$\chi_N=|\varphi_N|^2\varphi_N-2\|\varphi_N\|^2_{L^2(\mathbb{S}^1)}\varphi_N.$$

It is enough to show the result for (χ_N), because once we know that $\chi_N \longrightarrow \chi$ in $L^2\big(\Omega; H^{-\sigma}(\mathbb{S}^1)\big)$, we deduce that $G_N(\varphi_N) = \Pi_N \chi_N \longrightarrow \chi$ in $L^2\big(\Omega; H^{-\sigma}(\mathbb{S}^1)\big)$. In the sequel, we will use the notation $[n] = 1 + |n|$. Then, by definition of φ_N we can compute

$$\begin{aligned}\chi_N &= \sum_{|n_1|,|n_2|,|n_3|\leq N} \frac{g_{n_1}\overline{g}_{n_2}g_{n_3}}{[n_1]^{\frac12}[n_2]^{\frac12}[n_3]^{\frac12}}\mathrm{e}^{i(n_1-n_2+n_3)x} - 2\sum_{|n_1|,|n_3|\leq N}\frac{|g_{n_1}|^2 g_{n_3}}{[n_1][n_3]^{\frac12}}\mathrm{e}^{in_3x}\\ &= \sum_{\substack{|n_1|,|n_2|,|n_3|\leq N,\\ n_1\neq n_2, n_3\neq n_2}} \frac{g_{n_1}\overline{g}_{n_2}g_{n_3}}{[n_1]^{\frac12}[n_2]^{\frac12}[n_3]^{\frac12}}\mathrm{e}^{i(n_1-n_2+n_3)x}.\end{aligned}$$

Next, denote by $\mathrm{e}_k(x) = \mathrm{e}^{ikx}$. Then for all $1 \leq M \leq N$

$$\langle \chi_N - \chi_M \,|\, \mathrm{e}_k\rangle = \sum_{B^{(k)}_{M,N}} \frac{g_{n_1}\overline{g}_{n_2}g_{n_3}}{[n_1]^{\frac12}[n_2]^{\frac12}[n_3]^{\frac12}}, \tag{7.1}$$

where the set $B^{(k)}_{M,N}$ is defined by

$$\begin{aligned}B^{(k)}_{M,N} = \Big\{&(n_1,n_2,n_3)\in\mathbb{Z}^3 \;\text{ s.t. }\; 0<|n_1|,|n_2|,|n_3|\leq N,\;\; n_1\neq n_2,\;\; n_3\neq n_2,\\ &\text{and }\; \big(|n_1|>M \;\text{ or }\; |n_2|>M \;\text{ or }\; |n_3|>M\big) \;\text{ and }\; n_1-n_2+n_3=k\Big\}.\end{aligned}$$

From (7.1) we obtain

$$\big\|\langle \chi_N - \chi_M \,|\, \mathrm{e}_k\rangle\big\|^2_{L^2(\Omega)} = \int_\Omega \sum_{\substack{(n_1,n_2,n_3)\in B^{(k)}_{M,N}\\ (m_1,m_2,m_3)\in B^{(k)}_{M,N}}} \frac{g_{n_1}\overline{g}_{n_2}g_{n_3}\overline{g}_{m_1}g_{m_2}\overline{g}_{m_3}}{[n_1]^{\frac12}[n_2]^{\frac12}[n_3]^{\frac12}[m_1]^{\frac12}[m_2]^{\frac12}[m_3]^{\frac12}}\mathrm{d}\mathbf{p}.$$

Since the (g_n) are independent and centered, we deduce that each term in the r.h.s. vanishes, unless $n_2 = m_2$ and $(n_1,n_3) = (m_1,m_3)$ or $(n_1,n_3) = (m_3,m_1)$. Thus

$$\big\|\langle \chi_N - \chi_M \,|\, \mathrm{e}_k\rangle\big\|^2_{L^2(\Omega)} \leq C \sum_{(n_1,n_2,n_3)\in B^{(k)}_{M,N}} \frac{1}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle}.$$

By symmetry in the previous sum, we can assume that $M < |n_1| \leq N$, $0 < |n_2| \leq N$ and write $n_3 = k + n_2 - n_1$. Then by (5.2) for some small $\varepsilon > 0$

$$\begin{aligned}\big\|\langle \chi_N - \chi_M \,|\, \mathrm{e}_k\rangle\big\|^2_{L^2(\Omega)} &\leq C \sum_{M<|n_1|\leq N}\frac{1}{\langle n_1\rangle}\sum_{n_2\in\mathbb{Z}}\frac{1}{\langle n_2\rangle\langle n_2-(n_1-k)\rangle}\\ &\leq C\sum_{M<|n_1|\leq N}\frac{1}{\langle n_1\rangle\langle n_1-k\rangle^{1-\varepsilon}} \leq \frac{C}{M^\varepsilon\langle k\rangle^{1-2\varepsilon}}.\end{aligned} \tag{7.2}$$

Now, by (7.2) we get

$$\begin{aligned}\big\|\chi_N-\chi_M\big\|^2_{L^2(\Omega;H^{-\sigma}(\mathbb{S}^1))} &= \sum_{k\in\mathbb{Z}}\frac{1}{\langle k\rangle^{2\sigma}}\big\|\langle \chi_N-\chi_M\,|\,\mathrm{e}_k\rangle\big\|^2_{L^2(\Omega)}\\ &\leq \frac{C}{M^{\varepsilon}}\sum_{k\in\mathbb{Z}}\frac{1}{\langle k\rangle^{1+2\sigma-2\varepsilon}}\leq \frac{C}{M^{\varepsilon}},\end{aligned}$$

if we choose $\varepsilon<\sigma$, and this concludes the proof.

As a conclusion, we are able to define a limit $G(u)$ so that for all $p\geq 2$

$$\|G(u)\|_{L^p_\mu H^{-\sigma}(\mathbb{S}^1)}\leq C_p. \tag{7.3}$$

□

7.2. Construction of the measure ρ. — In this section φ is given by (1.14). Denote by $[n]=1+|n|$, then define $\alpha_N=\displaystyle\sum_{|n|\leq N}\frac{1}{[n]}$ and

$$g_N(u)=\|\Pi_N u\|^2_{L^2}-\alpha_N.$$

7.2.1. Preliminar results. — We begin with the following result due to N. Tzvetkov. See [**35**, Lemma 4.8] for a proof.

Lemma 7.1. — *The sequence $\big(g_N(u)\big)_{N\geq 1}$ is Cauchy in $L^2\big(X^0(\mathbb{S}^1),\mathcal{B},d\mu\big)$. Moreover there exists $c>0$ so that for all $\lambda>0$ and $N>M\geq 1$*

$$\mu\Big(u\in X^0(\mathbb{S}^1)\ :\ |g_N(u)-g_M(u)|>\lambda\Big)\leq Ce^{-c\lambda M^{1/2}}.$$

Define the sequence

$$f_N(u)=-\int_{\mathbb{S}^1}|u_N|^4+2\Big(\int_{\mathbb{S}^1}|u_N|^2\Big)^2=-\|u_N\|^4_{L^4}+2\|u_N\|^4_{L^2}. \tag{7.4}$$

Proposition 7.2. — *The sequence $(f_N)_{N\geq 1}$ is Cauchy in $L^2\big(X^0(\mathbb{S}^1),\mathcal{B},d\mu\big)$. More precisely, there exists $C>0$ so that for all $N>M\geq 1$*

$$\|f_N(u)-f_M(u)\|_{L^2\left(X^0(\mathbb{S}^1),\mathcal{B},d\mu\right)}\leq \frac{C}{M^{\frac12}}. \tag{7.5}$$

Moreover, for all $p\geq 2$ and $N>M\geq 1$

$$\|f_N(u)-f_M(u)\|_{L^p\left(X^0(\mathbb{S}^1),\mathcal{B},d\mu\right)}\leq \frac{C\,(p-1)^2}{M^{\frac12}}. \tag{7.6}$$

Corollary 7.3. — *There exists $c>0$ so that for all $\lambda>0$ and $N>M\geq 1$*

$$\mu\Big(u\in X^0(\mathbb{S}^1)\ :\ |f_N(u)-f_M(u)|>\lambda\Big)\leq Ce^{-c\lambda^{1/2}M^{1/4}}.$$

Proof of Corollary 7.3. — By Markov and (7.6) we have that for all $p \geq 2$

$$\mu\Big(u \in X^0(\mathbb{S}^1) \ : \ |f_N(u) - f_M(u)| > \lambda\Big) \leq \frac{1}{\lambda^p} \|f_N(u) - f_M(u)\|^p_{L^p\big(X^0(\mathbb{S}^1),\mathcal{B},\mathrm{d}\mu\big)} \leq \big(\frac{Cp^2}{\lambda M^{1/2}}\big)^p.$$

Then choose $p = c_0 \lambda^{1/2} M^{1/4}$ for $c_0 > 0$ small enough.

□

Proof of Proposition 7.2. — We prove (7.5). The estimate (7.6) immediately follows from [**34**, Proposition 2.4]. Firstly, we have $\displaystyle\int_{\mathbb{S}^1} |\varphi_N|^2 = \sum_{|n| \leq N} \frac{|g_n|^2}{[n]}$, with the notation $[n] = 1 + |n|$. Thus

$$\big(\int_{\mathbb{S}^1} |\varphi_N|^2\big)^2 = \sum_{|n|,|m| \leq N} \frac{|g_n|^2 |g_m|^2}{[n][m]}. \tag{7.7}$$

Similarly, we explicitly obtain

$$\int_{\mathbb{S}^1} |\varphi_N|^4 = \sum_{\substack{|n_1|,|n_2|,|n_3|,|n_4| \leq N \\ n_1 - n_2 + n_3 - n_4 = 0}} \frac{g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_{n_4}}}{[n_1]^{\frac{1}{2}} [n_2]^{\frac{1}{2}} [n_3]^{\frac{1}{2}} [n_4]^{\frac{1}{2}}}. \tag{7.8}$$

We introduce the set

$$A_N = \{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \ \text{ s.t. } \ |n_1|, |n_2|, |n_3|, |n_4| \leq N \ \text{ and } \ n_1 - n_2 + n_3 - n_4 = 0\}.$$

We now split the sum (7.8) in two parts, by distinguishing the cases $n_3 = n_1$ and $n_3 \neq n_1$ in A_N and write

$$\int_{\mathbb{S}^1} |\varphi_N|^4 = X_N + Y_N, \tag{7.9}$$

with

$$X_N = \sum_{B_N} \frac{g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_{n_4}}}{[n_1]^{\frac{1}{2}} [n_2]^{\frac{1}{2}} [n_3]^{\frac{1}{2}} [n_4]^{\frac{1}{2}}},$$

where $B_N = A_N \cap \{\, n_1 = n_2 \ \text{ or } \ n_1 = n_4 \,\}$, and

$$Y_N = \sum_{\substack{A_N, n_1 \neq n_2 \\ n_1 \neq n_4}} \frac{g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_{n_4}}}{[n_1]^{\frac{1}{2}} [n_2]^{\frac{1}{2}} [n_3]^{\frac{1}{2}} [n_4]^{\frac{1}{2}}}. \tag{7.10}$$

We observe that if $(n_1, n_2, n_3, n_4) \in B_N$, then either $(n_1, n_3) = (n_2, n_4)$ or $(n_1, n_3) = (n_4, n_2)$. Thus

$$\begin{aligned} X_N &= \sum_{|n_1|,|n_3| \leq N} \frac{|g_{n_1}|^2 |g_{n_3}|^2}{[n_1][n_3]} + \sum_{\substack{|n_1|,|n_3| \leq N \\ n_1 \neq n_3}} \frac{|g_{n_1}|^2 |g_{n_3}|^2}{[n_1][n_3]} \\ &= 2\big(\int_{\mathbb{S}^1} |\varphi_N|^2\big)^2 - \sum_{|n| \leq N} \frac{|g_n|^4}{[n]^2}, \end{aligned}$$

where in the last line we used (7.7). Thus, with (7.9) we obtain

$$f_N(\varphi_N) = -\int_{\mathbb{S}^1} |\varphi_N|^4 + 2\big(\int_{\mathbb{S}^1} |\varphi_N|^2\big)^2 = \sum_{|n|\leq N} \frac{|g_n|^4}{[n]^2} - Y_N.$$

We now show that $(Y_N)_{N\geq 1}$ is Cauchy in $L^2(\Omega, \mathcal{F}, \mathbf{p})$. Let $1 \leq N < M$, then we define

$$\begin{aligned} A_{M,N} = \big\{(n_1,n_2,n_3,n_4) \in \mathbb{Z}^4 \;\text{ s.t. }\; M < |n_1|, |n_2|, |n_3|, |n_4| \leq N,\\ n_1 - n_2 + n_3 - n_4 = 0 \;\text{ and s.t. }\; |n_j| > M \;\text{ for some }\; 1 \leq j \leq 4\big\}. \end{aligned}$$

Thus, thanks to (7.10) we have

$$(Y_M - Y_N)^2 = \sum_{\substack{A_{M,N},\\ n_1\neq n_2\\ n_1\neq n_4}} \sum_{\substack{A_{M,N},\\ m_1\neq m_2\\ m_1\neq m_4}} \frac{g_{n_1}\overline{g_{n_2}}g_{n_3}\overline{g_{n_4}}}{[n_1]^{\frac12}[n_2]^{\frac12}[n_3]^{\frac12}[n_4]^{\frac12}} \frac{\overline{g_{m_1}}g_{m_2}\overline{g_{m_3}}g_{m_4}}{[m_1]^{\frac12}[m_2]^{\frac12}[m_3]^{\frac12}[m_4]^{\frac12}}.$$

We take the integral over Ω of the previous sum. By the independence of the Gaussians each term vanishes unless $\{n_1,n_2,n_3,n_4\} = \{m_1,m_2,m_3,m_4\}$. Thus

$$\|Y_M - Y_N\|^2_{L^2(\Omega)} \leq C \sum_{A_{M,N}} \frac{1}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle\langle n_4\rangle}.$$

By symmetry of the sum, we can assume that $|n_1| \geq M$ and we replace $n_4 = n_1 - n_2 + n_3$. Then by (5.2)

$$\begin{aligned} \|Y_M - Y_N\|^2_{L^2(\Omega)} &\leq C \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}\\ |n_1|>M}} \frac{1}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle\langle n_1-n_2+n_3\rangle} \\ &\leq C \sum_{\substack{n_1,n_2\in\mathbb{Z}\\ |n_1|>M}} \frac{1}{\langle n_1\rangle\langle n_2\rangle\langle n_1-n_2\rangle^{1-\varepsilon}} \\ &\leq C \sum_{|n_1|\geq M} \frac{1}{\langle n_1\rangle^{2-2\varepsilon}} \leq \frac{C}{M^{1-2\varepsilon}}, \end{aligned}$$

which was the claim.

□

7.2.2. The crucial estimate. — We now have all the ingredients to prove the following proposition, which is the key point in the proof of Theorem 1.5. Recall the definition (7.4).

Proposition 7.4. — *Let $\chi \in \mathcal{C}_0^\infty\big([-R,R]\big)$. Then for all $1 \leq p < \infty$ there exists $C > 0$ such that for every $N \geq 1$,*

$$\Big\|\chi\Big(\|\Pi_N u\|^2_{L^2(\mathbb{S}^1)} - \alpha_N\Big)\, e^{f_N(u)}\Big\|_{L^p(d\mu(u))} \leq C\,.$$

Proof. — Our aim is to show that the integral $\int_0^\infty \lambda^{p-1}\mu(A_{\lambda,N})d\lambda$ is convergent uniformly with respect to N, where

$$A_{\lambda,N}=\Big\{u\in X^0(\mathbb{S}^1)\,:\,\chi\Big(\|\Pi_N u\|^2_{L^2(\mathbb{S}^1)}-\alpha_N\Big)e^{f_N(u)}>\lambda\Big\}.$$

Proposition 7.4 is a straightforward consequence of the following lemma. □

Lemma 7.5. — *For any $L>0$, there exists $C>0$ such that for every N and every $\lambda\geq 1$,*

$$\mu(A_{\lambda,N})\leq C\lambda^{-L}.$$

Proof. — Fistly, observe that we can assume that $\lambda\geq C_R$ for any constant $C_R>0$. Let $c_0>0$ a small number which will be fixed later and set

$$M=\mathrm{e}^{c_0(\ln\lambda)^{1/2}}.$$

To begin with $\mu(A_{\lambda,N})\leq\mu(\widetilde{A}_{\lambda,N})$, where

$$\widetilde{A}_{\lambda,N}=\Big\{u\in X^0(\mathbb{S}^1)\,:\,f_N(u)>\ln\lambda,\quad |g_N(u)|\leq R\Big\}.$$

• Assume that $N\leq M$. On the set $\{\,|g_N(u)|\leq R+1\,\}$ we have

$$f_N(u)\leq 2\|\Pi_N u\|^4_{L^2(\mathbb{S}^1)}\leq 2(C\ln N+R)^2\leq 2(C\ln M+R)^2=Cc_0^2\ln\lambda,$$

if $\lambda\geq C_R$ large enough. We fix $c_0>0$ so that $Cc_0^2<1/4$. In particular $\mu(A_{\lambda,N})\leq\mu(\widetilde{A}_{\lambda,N})=0$.
• Assume that $N\geq M$. First observe that if we define

$$B_{\lambda,N}=\Big\{u\in X^0(\mathbb{S}^1)\,:\,\Big|g_N(u)-g_M(u)\Big|>1\Big\},$$

by Lemma 7.1 and the definition of M, we get for any $L\geq 1$

$$\mu(B_{\lambda,N})\leq C\exp(-cM^{1/2})\leq C_L\lambda^{-L}\,.$$

Similarly, set

$$C_{\lambda,N}=\Big\{u\in X^0(\mathbb{S}^1)\,:\,\Big|f_N(u)-f_M(u)\Big|>1\Big\},$$

then by Corollary 7.3, for any $L\geq 1$ we have

$$\mu(C_{\lambda,N})\leq C\exp(-cM^{1/4})\leq C_L\lambda^{-L}\,.$$

We have $\widetilde{A}_{\lambda,N}\subset C_{\lambda,N}\cup D_{\lambda,N}$ where

$$D_{\lambda,N}=\Big\{u\in X^0(\mathbb{S}^1)\,:\,f_M(u)>\frac{1}{2}\ln\lambda,\quad |g_N(u)|\leq R\Big\}.$$

Then observe that $\{\,|g_N(u)|\leq R\,\}\cap\{\,|g_N(u)-g_M(u)|\leq 1\,\}\subset\big\{|g_M(u)|\leq R+1\big\}$, therefore we can write $D_{\lambda,N}\subset B_{\lambda,N}\cup E_{\lambda,N}$ where

$$E_\lambda=\Big\{u\in X^0(\mathbb{S}^1)\,:\,f_M(u)>\frac{1}{2}\ln\lambda,\quad |g_M(u)|\leq R+1\Big\}.$$

In the first part of the proof, we have already shown that $\mu(E_\lambda)=0$. Finally, we put all the estimates together and obtain $\mu(A_{\lambda,N})\leq C_L\lambda^{-L}$. □

7.2.3. Convergence to the mesure ρ. — We now have all the ingredients to complete the proof of Theorem 1.5.

First we define the density $\Theta : X^0(\mathbb{S}^1) \longrightarrow \mathbb{R}$ with respect to the measure μ of the measure ρ. By Lemma 7.1 and Proposition 7.2, we have the following convergences in the μ measure: $g_N(u)$ converges to $g(u)$ and $f_N(u)$ to $f(u)$. Then, by composition and multiplication of continuous functions, we obtain

$$\Theta_N(u) \longrightarrow \beta\chi\big(g(u)\big)\mathrm{e}^{f(u)} \equiv \Theta(u),$$

in measure, with respect to the measure μ, and where $\beta > 0$ is so that $\mathrm{d}\rho(u) = \Theta(u)\mathrm{d}\mu(u)$ is a probability measure on $X^0(\mathbb{S}^1)$. By this construction, Θ is measurable from $\big(X^0(\mathbb{S}^1), \mathcal{B}\big)$ to $\mathbb{R}$.

Then, we can extract a sub-sequence $\Theta_{N_k}(u)$ so that $\Theta_{N_k}(u) \longrightarrow \Theta(u)$, μ a.s. and by Proposition 7.4 and the Fatou lemma, for all $p \in [1, +\infty)$,

$$\int_{X^0(\mathbb{S}^1)} |\Theta(u)|^p \mathrm{d}\mu(u) \leq \liminf_{k\to\infty} \int_{X^0(\mathbb{S}^1)} |\Theta_{N_k}(u)|^p \mathrm{d}\mu(u) \leq C,$$

thus $\Theta(u) \in L^p(\mathrm{d}\mu(u))$.

It remains to prove the convergence of $\Theta_N(u)$ in $L^p(\mathrm{d}\mu(u))$: Here we can follow the proof of Proposition 4.6. We do not write de details.

7.3. Study of the measure ν_N. — Let $N \geq 1$ and consider the equation (1.15). Observe that $u_N = \Pi_N u$ satisfies an ODE, while $u_N^{\perp} = (1 - \Pi_N)u$ is solution to the linear problem $(i\partial_t - \Lambda)u_N^{\perp} = 0$. Since the $L^2(\mathbb{S}^1)$-norm of a solution u to (1.15) is preserved, it follows that the equation is globally well-posed in $L^2(\mathbb{S}^1)$. We denote by Φ_N the flowmap. Moreover, because of the Hamiltonian structure and the Liouville theorem, the measure ρ_N is invariant by Φ_N.

Similarly to the previous section, for $T > 0$ we define the measure ν_N on $\mathcal{C}\big([-T, T]; X^0(\mathbb{S}^1)\big)$ as the image of ρ_N by the flowmap

$$\begin{array}{rcl} X^0(\mathbb{S}^1) & \longrightarrow & \mathcal{C}\big([-T, T]; X^0(\mathbb{S}^1)\big) \\ v & \longmapsto & \Phi_N(t)(v). \end{array}$$

Using this definition, we can prove

Lemma 7.6. — *Let $\sigma > 0$, then for all $p \geq 2$*

$$\big\| \|G(u)\|_{L^p_T H^{-\sigma}_x} \big\|_{L^p_{\nu_N}} \leq C. \tag{7.11}$$

Proof. — By definition, invariance of ρ_N and Cauchy-Schwarz

$$\begin{aligned} \|G(u)\|^p_{L^p_{\nu_N} L^p_T H^{-\sigma}_x} &= \int_{\mathcal{C}\big([-T,T];X^0\big)} \|G(u)\|^p_{L^p_T H^{-\sigma}_x} \mathrm{d}\nu_N(u) \\ &= \int_{X^0} \|G\big(\Phi_N(t)(v)\big)\|^p_{L^p_T H^{-\sigma}_x} \mathrm{d}\rho_N(v) \\ &= 2T \int_{X^0} \|G(v)\|^p_{H^{-\sigma}_x} \theta_N(v) \mathrm{d}\mu(v) \\ &\leq 2T \|G(v)\|^p_{L^{2p}_\mu H^{-\sigma}_x} \|\theta_N(v)\|_{L^2_\mu}. \end{aligned}$$

We conclude with (7.3) and Proposition 7.4. □

Lemma 7.7. — *Let $\sigma > 0$, then for all $p \geq 2$*

$$\big\| \|u\|_{L^p_T H^{-\sigma}_x} \big\|_{L^p_{\nu_N}} \leq C, \tag{7.12}$$

$$\big\| \|u\|_{W^{1,p}_T H^{-\sigma-1}_x} \big\|_{L^p_{\nu_N}} \leq C. \tag{7.13}$$

Proof. — The proof of (7.12) is a consequence of (3.3) and Lemma 5.2. The estimate (7.13) is obtained from (7.11) and (7.12): The proof is similar to (5.5) and we do not write the details. □

As a consequence we can show

Proposition 7.8. — *Let $T > 0$ and $\sigma > 0$. Then the family of measures*

$$\nu_N = \mathscr{L}_{\mathcal{C}_T H^{-\sigma}}\big(u_N(t); t \in [-T,T]\big)_{N\geq 1}$$

is tight in $\mathcal{C}\big([-T,T]; H^{-\sigma}(\mathbb{S}^1)\big)$.

7.4. Proof of Theorem 1.6. — The proof is similar to the Benjamin-Ono case. The only difficulty lies in the limit of the nonlinear term. Here we can proceed as in Lemma 5.6 to obtain the next result. Recall the definition (1.16). Then

Lemma 7.9. — *Up to a sub-sequence, the following convergence holds true*

$$G_{N_k}(\widetilde{u}_{N_k}) \longrightarrow G(\widetilde{u}), \qquad \widetilde{\mathbf{p}} - a.s. \text{ in } L^2\big([-T,T]; H^{-\sigma}(\mathbb{S}^1)\big),$$

where G is defined by Proposition 1.4.

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