ON RANDOM WEIGHTED SOBOLEV INEQUALITIES ON $\mathbb{R}^d$ AND APPLICATIONS

by

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Abstract. — Our aim in this paper is to give an overview of recent results concerning randomization methods applied to Schrödinger equations obtained in [**31, 30, 33**]. We focus here on properties of the harmonic oscillator and Hermite functions for simplicity.
The general idea is that several well known deterministic results like Sobolev type inequalities are improved in a large extent by introducing some randomness. This is very useful in particular for N.L.S with supercritical initial data.

1. Introduction

We present here the results we have obtained in [**30, 31**] in collaboration with Aurélien Poiret and the results of [**33**]. Using ideas of Shiffman-Zelditch [**35**] and Burq-Lebeau [**6**] who developed a randomisation method based on the Laplace operator on a compact Riemannian manifold, we give a randomisation method on $L^2(\mathbb{R}^d)$ associated with the Laplace operator with confining potentials. We are able to construct probability measures on $L^2(\mathbb{R}^d)$, on the support of which a typical function enjoys better Sobolev estimates than expected. These measures and estimates have an interest in themselves, but we can moreover use them to

- construct orthonormal bases of $L^2(\mathbb{R}^d)$ of Hermite functions which enjoy good L^∞ bounds. These bounds are shown to be optimal ;
- prove quantum ergodicity results ;
- show a.s. local and global well-posedness results for supercritical nonlinear Schrödinger equations with quadratic potential ;

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- show a.s. global well-posedness results for supercritical nonlinear Schrödinger equations (without potential).

To simplify the exposition, we will focus on estimates in the Sobolev spaces based on the harmonic oscillator in $L^2(\mathbb{R}^d)$

$$H = -\Delta + |x|^2 = \sum_{j=1}^{d}(-\partial_j^2 + x_j^2).$$

We get optimal stochastic weighted Sobolev estimates on $\mathbb{R}^d$ using the Burq-Lebeau method. In [**6**], the construction of the measures relied on Gaussian random variables but it is possible to consider general random variable which satisfy concentration of measure estimates (like Bernoulli random variables). However, the optimal estimates (including lower bounds) are obtained only for the Gaussian law.

Let us emphasis here that even if there exist many similarities between the spectral properties of the Laplace operators a compact Riemannian manifolds and Schrödinger Hamiltonians with a confining potential on $\mathbb{R}^d$, there exists a big difference due to the complex behaviour at high energy of the spectral function on a non-compact configuration space.

In [**33**] we prove that most of the results can be extended to more general Schrödinger Hamiltonians $-\triangle + V(x)$ with confining potentials V.

2. Spectral estimates and harmonic Sobolev spaces

Let us recall the well known spectral analysis of the quantum harmonic oscillator H on $\mathbb{R}^d$. This is explained in many text books on quantum mechanics. For mathematical details, we refer to Helffer [**18**] or to the course of Ramond [**32**].

The operator H is self-adjoint on the Hilbert space $L^2(\mathbb{R}^d)$, and has the discrete spectrum $\{2n+d\}_{n\in\mathbb{N}}$.

- For $d = 1$ each eigenvalue is simple, $H\mathfrak{h}_n = (2n+d)\mathfrak{h}_n$ where $\mathfrak{h}_n$ is the n^{th} Hermite function.
- For $d \geq 2$, an orthonormal basis is obtained by tensor products.
  Let $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$\mathfrak{h}_\alpha(x) = \mathfrak{h}_{\alpha_1}(x_1)\cdots\mathfrak{h}_{\alpha_d}(x_1), \quad \alpha_1 + \cdots + \alpha_d = n. \tag{2.1}$$

It is sometimes convenient to use different notations. Any orthonormal basis of $\ker(H - (2n+d))$ can be denoted by $\{\varphi_{n,k}\}_{1\leq k\leq m_n}$, so that $H\varphi_{n,k} = (2n+d)\varphi_{n,k}$. In the sequel, $\{\lambda_j\}_{j\in\mathbb{N}}$ denotes the non-decreasing sequence of eigenvalues where each eigenvalue $(2n+d)$ is repeated according its multiplicity, such that $H\varphi_j = \lambda_j\varphi_j$ where $\{\varphi_j\}_{j\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.
By (2.1) the multiplicity of the eigenvalue $2n+d$ is of order n^{d-1}, therefore λ_j is of order $j^{1/d}$.

The scale of harmonic Sobolev spaces is defined as follows: $s \geq 0$, $p \geq 1$.

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}^d) = \big\{u \in L^p(\mathbb{R}^d),\ H^{s/2}u \in L^p(\mathbb{R}^d)\big\}, \tag{2.2}$$

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d) = \mathcal{W}^{s,2}.$$

The natural norms are denoted by $\|u\|_{\mathcal{W}^{s,p}}$ and up to equivalence of norms we have (see [**41**, Lemma 2.4]) for $1 < p < +\infty$

$$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x\rangle^s u\|_{L^p}.$$

For $s \geq 0$, $\mathcal{H}^s$ is the domain of the self-adjoint operator $H^{s/2}$ and we have

$$\mathcal{H}^s = \big\{u \in L^2(\mathbb{R}^d),\ \sum_{j\geq 0} \lambda_j^s |\langle \varphi_j, u\rangle|^2 < +\infty\big\}$$

and

$$\|u\|^2_{\mathcal{H}^s} \approx \sum_{j\geq 0} \lambda_j^s |\langle \varphi_j, u\rangle|^2.$$

In the sequel we fix $d \geq 2$. Let us consider spectral windows I_h depending on a small parameter $h > 0$ (this notation is convenient to make use of semiclassical results here and in the more general setting considered in [**33**]).
Let us denote by $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$ and assume that a_h and b_h satisfy, for some $a, b, D > 0, \delta \in [0, 1]$,

$$\lim_{h\to 0} a_h = a, \quad \lim_{h\to 0} b_h = b, \quad 0 < a \leq b \quad \text{and} \quad b_h - a_h \geq Dh^\delta, \tag{2.3}$$

with any $D > 0$ if $\delta < 1$ and $D \geq 2$ in the case $\delta = 1$. This condition ensures that N_h, the number (with multiplicities) of eigenvalues of H in I_h tends to infinity when $h \to 0$. Indeed, we can check that $N_h \sim ch^{-d}(b_h - a_h)$, in particular $\lim_{h\to 0} N_h = +\infty$, since $d \geq 2$. In the sequel, we write $\Lambda_h = \{j \geq 1,\ \lambda_j \in I_h\}$ and $\mathcal{E}_h = \operatorname{span}\{\varphi_j,\ j \in \Lambda_h\}$, so that $N_h = \#\Lambda_h = \dim \mathcal{E}_h$. Finally, we denote by $\mathbf{S}_h = \big\{u \in \mathcal{E}_h\ :\ \|u\| = 1\big\}$ the unit sphere of $\mathcal{E}_h$ (which is a complex linear space of dimension N_h).

A very useful tool to get local and global estimates in the deterministic setting as well as in the probabilistic setting are L^p estimates of spectral projectors and of their kernels, the so-called spectral function. On compact manifolds this is obvious in [**36**] for example.

The spectral function is then defined as

$$\pi_H(\lambda; x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)}$$

(recall that this definition does not depend on the choice of $\{\varphi_j, j \in \mathbb{N}\}$). When the energy λ is localized in $I \subseteq \mathbb{R}^+$ we denote by $\Pi_H(I)$ the spectral projector of H on I. The range $\mathcal{E}_H(I)$ of $\Pi_H(I)$ is spanned by $\{\varphi_j; \lambda_j \in I\}$ and $\Pi_H(I)$ has an integral kernel given by

$$\pi_H(I; x, y) = \sum_{[j\,:\,\lambda_j \in I]} \varphi_j(x)\overline{\varphi_j(y)}.$$

We will also use the notation $\mathcal{E}_H(\lambda) = \mathcal{E}_H([0, \lambda])$, $N_H(\lambda) = \dim[\mathcal{E}_H(\lambda)]$.
The relationship between Sobolev type estimates and the spectral function is illustrated by the following elementary result:

$$|u(x)| \leq (\pi_H(I; x, x))^{1/2} \|u\|_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

We have the following uniform estimate:

$$\pi_H(\lambda; x, x) \le C_\theta \lambda^{(d+\theta)/2} \langle x \rangle^{-\theta},$$

where C_θ depends only on $\theta > 0$. Now using (2.4) we get (with the semiclassical parameter $h = \lambda^{-1}$)

$$\langle x \rangle^{\theta/2} h^{(d+\theta)/4} |u(x)| \le C_\theta \|u\|_{L^2(\mathbb{R}^d)}, \quad \forall u \in \mathcal{E}_H(h^{-1}).$$

For smaller spectral windows like I_h we need more accurate estimates on the spectral function

$$e_x := \pi_H(\frac{b_h}{h}; x, x) - \pi_H(\frac{a_h}{h}; x, x).$$

For any $\theta \ge 0$ there exists $C_\theta > 0$ such that

$$\langle x \rangle^{\theta} e_x \le C_\theta N_h h^{(d-\theta)/2}. \tag{2.5}$$

where C_θ depends only on $\theta \ge 0$. Using (2.4) and interpolation inequalities we get Sobolev type inequalities for $u \in \mathcal{E}_h$, $\theta \ge 0$, $p \ge 2$

$$\|u\|_{L^{\infty,\theta/2}(\mathbb{R}^d)} \le C \left(N_h h^{(d-\theta)/2} \right)^{1/2} \|u\|_{L^2(\mathbb{R}^d)}, \tag{2.6}$$

which in turn implies using interpolation

$$\|u\|_{L^{p,\theta(p/2-1)}(\mathbb{R}^d)} \le C \left(N_h h^{(d-\theta)/2} \right)^{\frac{1}{2}-\frac{1}{p}} \|u\|_{L^2(\mathbb{R}^d)}. \tag{2.7}$$

The previous inequality can be written as

$$\|u\|_{L^{p,\theta(p/2-1)}(\mathbb{R}^d)} \le C (b_h - a_h)^{\frac{1}{2}-\frac{1}{p}} h^{-(\frac{d+\theta}{2})(\frac{1}{2}-\frac{1}{p})} \|u\|_{L^2(\mathbb{R}^d)}, \quad \forall p \in [2, +\infty], \ \forall \theta \in [0, d].$$

We shall see that these Sobolev type inequalities in a large extent can be improved in a random sense that we shall explain now.

3. Probabilistic weighted estimates for frequency localized functions

3.1. Probabilistic setting. — Let us introduce now our probabilistic setting. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{X_n, \ n \ge 1\}$ be a sequence of i.i.d random variables, centered and normalized following a law ν. We assume for simplicity in all this paper that ν **is either the standard complex Gaussian** $\mathcal{N}_{\mathbb{C}}(0, 1)$ **or the Bernoulli law** $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ (see [**31**] for more general laws satisfying a concentration of measure property).
Let $\gamma := (\gamma_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers and define the random vector in $\mathcal{E}_h$

$$v_\gamma(\omega) := \sum_{j \in \Lambda_h} \gamma_j X_j(\omega) \varphi_j. \tag{3.1}$$

The probability law of v_γ is denoted by ν_γ. We define a probability measure $\mathbf{P}_\gamma$ on the sphere $\mathbf{S}_h$ by: for all measurable and bounded function $f : \mathbf{S}_h \longrightarrow \mathbb{R}$,

$$\int_{\mathbf{S}_h} f(u) \mathrm{d}\mathbf{P}_\gamma(u) = \int_\Omega f\left(\frac{v_\gamma(\omega)}{\|v_\gamma(\omega)\|_{L^2(\mathbb{R}^d)}} \right) \mathrm{d}\mathbb{P}(\omega).$$

It is not difficult to see that in the isotropic case ($\gamma_j = \frac{1}{\sqrt{N_h}}$ for all $j \in \Lambda_h$) and when $X_j \sim \mathcal{N}_{\mathbb{C}}(0,1)$, then $\mathbf{P}_\gamma$ is the uniform probability on the sphere $\mathbf{S}_h$.

For more general sequences γ we need to assume the following squeezing conditions. There exists $K_0 > 0$, $K_1 > 0$ such that

$$\|\gamma\|_{\ell^\infty(\Lambda_h)} \le \frac{K_0}{\sqrt{N_h}} \|\gamma\|_{\ell^2(\Lambda_h)}, \quad \forall h \in]0,1]. \tag{3.2}$$

We also need the stronger condition

$$\frac{K_1}{\sqrt{N_h}} \|\gamma\|_{\ell^2(\Lambda_h)} \le |\gamma_j| \le \frac{K_0}{\sqrt{N_h}} \|\gamma\|_{\ell^2(\Lambda_h)}, \quad \forall h \in]0,1], \forall j \in \Lambda_h. \tag{3.3}$$

Notice that

$$\mathbb{E}(\|v_\gamma\|^2_{L^2(\mathbb{R}^d)}) = \sum_{j\in\Lambda_h} |\gamma_j|^2 = |\gamma|^2_{\ell^2(\Lambda_h)}.$$

Let us comment now our assumptions.

Remark 3.1. — (i) Assumptions (3.2) and (3.3) mean that in the index set Λ_h the sequence γ is almost constant. In applications it is useful to be able to modify the γ_j. By considering two sequences γ, β and using the Kakutani theorem we can construct mutually singular probability measures μ_γ, μ_β on the Sobolev space $\mathcal{H}^s$ (see [**31**, Section 2.2]). In particular this gives larger sets of initial data for solutions of NLS (see Section 6.2 of this paper).

(ii) Assumptions (3.2) and (3.3) are very useful to be able to use the accurate spectral estimate (2.5) as we have shown in [**31**, Section 3]. This condition allows to normalize the $\mathcal{H}^s$-norm of v_γ. Moreover it is compatible with the Lévy contraction principle (see [**20**] for details).

(iii) It is possible to avoid assumption (3.2) at the price of introducing new weighted Sobolev spaces (see the definition of the space $\mathcal{Z}$ in [**20**]).

(iv) The Gaussian complex law is natural because it gives the uniform probability measure on the sphere $\mathbf{S}_h$, but in this case the $\mathcal{H}^s$-norm of v_γ (3.1) depends on ω. This is not the case if ν is a centred Bernoulli law. Moreover the probabilistic information we get clearly depends on ν and it may be interesting to consider also discrete probability laws. Actually, our results are proven under the assumption that the family $(\nu^{\otimes N}, \mathbb{R}^N)_{N\ge 1}$ satisfies the concentration property of measures. Finally, we point out that the results of Section 6 hold true under the weaker assumption that ν is subgaussian: there exists $\sigma \ge 0$ such that

$$\int_{\mathbb{R}} \mathrm{e}^{sx} d\nu(x) \le \mathrm{e}^{\frac{\sigma^2}{2}s^2}, \ \ \forall s \in \mathbb{R}.$$

For a discussion about these conditions we refer to [**31**, Section 2.1].

A very useful tool here is a measure concentration property satisfied by the probability $\mathbf{P}_\gamma$, proved by P. Lévy for the uniform law. For a study of this notion, we refer to the book [**23**].

Proposition 3.2. — *Suppose that Assumption* (3.3) *is satisfied. Then there exist constants* $K > 0$, $\kappa > 0$ *such that for every Lipschitz function* $F : \mathbf{S}_h \longrightarrow \mathbb{R}$ *satisfying*

$$|F(u) - F(v)| \leq \|F\|_{Lip} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h,$$

we have

$$\mathbf{P}_\gamma \Big[u \in \mathbf{S}_h : |F - \mathcal{M}_F| > r \Big] \leq K \mathrm{e}^{-\frac{\kappa N_h r^2}{\|F\|^2_{Lip}}}, \quad \forall r > 0, \;\; h \in]0,1], \tag{3.4}$$

where $\mathcal{M}_F$ *is a median for* F.

Recall that a median $\mathcal{M}_F$ for F is defined by

$$\mathbf{P}_\gamma \big[u \in \mathbf{S}_h : F \geq \mathcal{M}_F \big] \geq \frac{1}{2}, \quad \mathbf{P}_\gamma \big[u \in \mathbf{S}_h : F \leq \mathcal{M}_F \big] \geq \frac{1}{2}.$$

The factor N_h in the exponential of r.h.s of (3.4) is crucial in applications to get large deviation estimates.

3.2. L^p bounds, $2 \leq p < +\infty$. — The following result shows that for $\theta = d$ the Sobolev estimate (2.7) is improved by the factor $N_h^{\frac{1}{p} - \frac{1}{2}}$ for u in a set of probability close to 1.

***Theorem 3.3* ([31], Theorem 4.7)**. — *Let* $p \geq 2$. *Denote by* $\mathcal{M}_p$ *a median of* $\|u\|_{L^{p,d(p/2-1)}}$. *Assume condition* (3.2) *and let* $\delta \in [0,1]$. *Then there exist* $0 < C_0 < C_1$, $K > 0$, $c_1 > 0$, $h_0 > 0$ *such that for all* $r \in [2, K|\log h|]$ *and* $h \in]0, h_0]$ *such that*

$$\mathbf{P}_{\gamma,h} \Big[u \in \mathbf{S}_h : \Big| \|u\|_{L^{p,d(p/2-1)}} - \mathcal{M}_p \Big| > \Lambda \Big] \leq 2 \exp\big(- c_2 N_h^{2/p} \Lambda^2 \big). \tag{3.5}$$

Moreover if condition (3.3) *is satisfied and if* $\delta \in [0, 2/3[$, *then we have*

$$C_0 \sqrt{p} \leq \mathcal{M}_p \leq C_1 \sqrt{p}, \quad \forall p \in [2, K \log N_h].$$

This result shows that $\|u\|_{L^{p,d(p/2-1)}}$ has a Gaussian concentration around its median. The first part of the theorem is a direct application of Proposition 3.2 to $F(u) = \|u\|_{L^{p,d(p/2-1)}}$. The estimate of $\mathcal{M}_p$ is more involved and needs the following estimate for the spectral function.

***Lemma 3.4* ([31], Lemma 4.9)**. — *Let* $\delta \in [0, 2/3[$ *and* $-d/(p-1) < \theta \leq 1$. *There exist* $0 < C_0 < C_1$ *and* $h_0 > 0$ *such that*

$$C_0 N_h h^{\frac{d-\theta}{2}(1-\frac{1}{p})} \leq \Big(\int_{\mathbb{R}^d} \langle x \rangle^{\theta(p-1)} e_x^p \, dx \Big)^{1/p} \leq C_1 N_h h^{\frac{d-\theta}{2}(1-\frac{1}{p})},$$

for every $p \in [1, \infty[$ *and* $h \in]0, h_0]$.

Let $0 < \eta < 1$, then we are able to precise the concentration estimate (3.5) in the regime $p \sim |\ln h|^{1-\eta}$. Assume that for all $j \in \Lambda_h$, $\gamma_j = N_h^{-1/2}$ and that $X_j \sim \mathcal{N}_{\mathbb{C}}(0,1)$, so that $\mathbf{P} := \mathbf{P}_\gamma$ is the uniform probability on $\mathbf{S}_h$.

Theorem 3.5. — *Let $\delta \in [0, 2/3[$, $0 < \eta < 1$ and set $p_h = |\ln h|^{1-\eta}$. Then there exists constants $C_\star = C_\star(d)$ and $c > 0$ such that for all $\varepsilon > 0$ there exists $h_0 > 0$ so that for all $h \in]0, h_0]$*

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : \Big|\|u\|_{L^{p_h, d(p_h/2-1)}} - C_\star\sqrt{p_h}\Big| > \varepsilon\sqrt{p_h}\Big] \le 2e^{-c\varepsilon^2|\ln h|}.$$

We give the main lines of the proof in the appendix.

3.3. L^∞ bounds. — The next result shows that for $\theta = d$ the Sobolev estimate (2.6) is improved by the factor $N_h^{-\frac{1}{2}}|\log h|$ for u in a set of probability close to 1. We suppose that (3.2) and (2.3) with $0 \le \delta \le 1$ are satisfied.

***Theorem 3.6* ([31], Theorem 4.1).** — *There exist $h_0 \in]0, 1]$, $c_2 > 0$ and $C > 0$ such that if $c_1 = d(1 + d/4)$, we have*

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h :\ h^{-\frac{d-\theta}{4}}\|u\|_{L^{\infty,\theta/2}(\mathbb{R}^d)} > \Lambda\Big] \le Ch^{-c_1}\mathrm{e}^{-c_2\Lambda^2},\ \forall \Lambda > 0,\ \forall h \in]0, h_0].$$

We can deduce probabilistic estimates for the derivatives as well. Recall that the Sobolev spaces $\mathcal{W}^{s,p}(\mathbb{R}^d)$ are defined in (2.2). The following result say that Theorem 3.6 is sharp for large enough spectral windows.

***Theorem 3.7* ([31], Theorem 1.1).** — *Let $d \ge 2$. Assume that $0 \le \delta < 2/3$ in (2.3) and that condition (3.3) holds. Then there exist $0 < C_0 < C_1$, $c_1 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$.*

$$\mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : C_0|\log h|^{1/2} \le \|u\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \le C_1|\log h|^{1/2}\right] \ge 1 - h^{c_1}.$$

It is clear that under condition (3.3), there exist $0 < C_2 < C_3$, so that for all $u \in \mathbf{S}_h$, and $s \ge 0$

$$C_2h^{-s/2} \le \|u\|_{\mathcal{H}^s(\mathbb{R}^d)} \le C_3h^{-s/2},$$

since all elements of $\mathbf{S}_h$ oscillate with frequency $h^{-1/2}$. Thus Theorem 3.7 shows a gain of $d/2$ derivatives in L^∞, and this induces a gain of d derivatives compared to the usual deterministic Sobolev embeddings. This can be compared with the results of [6] where the authors obtain a gain of $d/2$ derivatives on compact manifolds: this comes from different behaviours of the spectral function, see Section 2. Notice that the bounds in Theorem 3.7 (and in the results of [6] as well) do not depend on the length of the interval of the frequency localisation I_h (see (2.3)), but only on the size of the frequencies. This is a consequence of the randomisation, and from the bound (2.5).

In our works we give estimates for the eigenfunctions in the configuration space, while Feng, Shiffman and Zelditch [35, 17] give similar estimates in the Bargmann representation for holomorphic fields.

We now state a result which gives optimal L^∞ bounds in the general case $0 \le \delta \le 1$

Theorem 3.8. — *Let $d \ge 2$. Assume that $0 \le \delta \le 1$ in (2.3) and that condition (3.3) holds. Then there exist $0 < C_0 < C_1$, $c_1 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$.*

$$\mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : C_0|\log h|^{1/2} \le h^{-d/4}\|u\|_{L^\infty(\mathbb{R}^d)} \le C_1|\log h|^{1/2}\right] \ge 1 - h^{c_1}.$$

We will give the main lines of the proof of this result in the Appendix B. The key obervation is that the estimates of the L^p norm of the spectral function (which are not optimal when δ is close to 1), become optimal in the regime $p = p_h \sim c|\ln h|$.

4. Hermite functions estimates

In this section the previous results are applied to obtain L^∞ estimates for Hermite functions.

Theorem 4.1 **([31], Theorem 1.3)**. — *Let $d \geq 2$. Then there exists an orthonormal basis of $L^2(\mathbb{R}^d)$ of eigenfunctions of the harmonic oscillator H denoted by $\{\varphi_n^\sharp\}_{n\geq 1}$ such that $\|\varphi_n^\sharp\|_{L^2(\mathbb{R}^d)} = 1$ and so that for some $M > 0$ and all $n \geq 1$,*

$$\|\varphi_n^\sharp\|_{L^\infty(\mathbb{R}^d)} \leq M\lambda_n^{-\frac{d}{4}}(1+\log\lambda_n)^{1/2}. \tag{4.1}$$

In other words, all elements of this basis are decreasing in $L^\infty(\mathbb{R}^d)$ norm.
Let us compare (4.1) with the general known bounds on Hermite functions. We have $H\varphi_n = \lambda_n\varphi_n$, with $\lambda_n \sim cn^{1/d}$, therefore (4.1) can be rewritten

$$\|\varphi_n^\sharp\|_{L^\infty(\mathbb{R}^d)} \leq Mn^{-1/4}(1+\log n)^{1/2},\ \forall n \geq 0., \tag{4.2}$$

For a general orthonormal basis of Hermite functions, with $d \geq 2$, Koch and Tataru [**21**] (see also [**22**]) prove that

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq C\lambda_n^{\frac{d}{4}-\frac{1}{2}},$$

which shows that (4.2) induces a gain of $d-1$ derivatives compared to the general case.
Notice that for the tensorial Hermite basis $\mathfrak{h}_\alpha(x)$, using the bound for $d = 1$ [**40, 21**] we have

$$\|\mathfrak{h}_\alpha\|_{L^\infty(\mathbb{R}^d)} \leq C\lambda_n^{-\frac{1}{12}},\ \text{ for } \alpha_1 + \cdots + \alpha_d = \lambda_n.$$

Observe also that the basis of radial Hermite functions does not satisfy (4.1) in dimension $d \geq 2$, as we have recently proved in [**20**]. We will see in Theorem 4.5 that the log term in (4.1) can not be avoided.

Theorem 4.1 is a consequence of a more powerful result, obtained following an idea in [**43**]. Here we follow the main lines of [**6**, Section 3]. In this part the upper bounds estimates of Section 3 are used in their full strength.

Firstly, we assume that for all $j \in \Lambda_h$, $\gamma_j = N_h^{-1/2}$ and that $X_j \sim \mathcal{N}_{\mathbb{C}}(0,1)$, so that $\mathbf{P} := \mathbf{P}_\gamma$ is the uniform probability on $\mathbf{S}_h$. We set $h_k = 1/k$ with $k \in \mathbb{N}^*$, and

$$a_{h_k} = 2 + dh_k, \quad b_{h_k} = 2 + (2+d)h_k.$$

Then (2.3) is satisfied with $\delta = 1$ and $D = 2$. In particular, each interval

$$I_{h_k} = \Big[\frac{a_{h_k}}{h_k}, \frac{b_{h_k}}{h_k}\Big[= [2k+d, 2k+d+2[$$

only contains the eigenvalue $2k+d$ with multiplicity $N_{h_k} \sim ck^{d-1}$, and $\mathcal{E}_{h_k}$ is the corresponding eigenspace of the harmonic oscillator H. We can identify the space of the orthonormal basis of $\mathcal{E}_{h_k}$ with the unitary group $U(N_{h_k})$ and we endow $U(N_{h_k})$ with its Haar probability measure ρ_k. Then the space $\mathcal{B}$ of the Hilbertian bases of eigenfunctions of H in $L^2(\mathbb{R}^d)$ can be identified with

$$\mathcal{B} = \times_{k\in\mathbb{N}} U(N_{h_k}), \tag{4.3}$$

which can be endowed with the measure

$$\mathrm{d}\rho = \otimes_{k\in\mathbb{N}}\, \mathrm{d}\rho_k. \tag{4.4}$$

Denote by $B = (\varphi_{k,\ell})_{k\in\mathbb{N},\,\ell\in[\![1,N_{h_k}]\!]} \in \mathcal{B}$ a typical orthonormal basis of $L^2(\mathbb{R}^d)$ so that for all $k \in \mathbb{N}$, $(\varphi_{k,\ell})_{\ell\in[\![1,N_{h_k}]\!]} \in U(N_{h_k})$ is an orthonormal basis of $\mathcal{E}_{h_k}$.

Then the main result of the section which implies Theorem 4.1, is the following.

Theorem 4.2 **([31], Theorem 5.1)**. — *Let $d \geq 2$. Then, if $M > 0$ is large enough, there exist $c, C > 0$ so that for all $r > 0$*

$$\rho\Big[\, B = (\varphi_{k,\ell})_{k\in\mathbb{N},\,\ell\in[\![1,N_{h_k}]\!]} \in \mathcal{B} : \exists k,\ell;\ \|\varphi_{k,\ell}\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \,\Big] \leq C\mathrm{e}^{-cr^2}.$$

The previous result relies on the following proposition

Proposition 4.3 **([31], Proposition 5.2)**. — *Let $d \geq 2$. Then, if $M > 0$ is large enough, there exist $c, C > 0$ so that for all $r > 0$ and $k \geq 1$*

$$\rho_k\Big[\, B_k = (\psi_\ell)_{\ell\in[\![1,N_{h_k}]\!]} \in U(N_{h_k}) : \exists \ell \in [\![1,N_{h_k}]\!];\ \|\psi_\ell\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \,\Big] \\ \leq Ck^{-2}\mathrm{e}^{-cr^2}.$$

Let us show how this result implies Theorem 4.2: We set

$$\mathcal{F}_{k,r} = \big\{\, B_k = (\psi_\ell)_{\ell\in[\![1,N_{h_k}]\!]} \in U(N_{h_k}) : \forall \ell \in [\![1,N_{h_k}]\!];\ \ \|\psi_\ell\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \leq M(\log k)^{1/2} + r \big\},$$

and $\mathcal{F}_r = \cap_{k\geq 1}\mathcal{F}_{k,r}$. Then for all $r > 0$

$$\rho(\mathcal{F}_r^c) \leq \sum_{k\geq 1} \rho_k(\mathcal{F}_{k,r}^c) \leq C\sum_{k\geq 1} k^{-2}\mathrm{e}^{-cr^2} = C'\mathrm{e}^{-cr^2},$$

and this completes the proof.

Then from the Borel-Cantelli Lemma we get

Corollary 4.4 **([31], Corolllary 5.3)**. — *For ρ-almost all orthonormal basis $(\varphi_{k,\ell})_{k\in\mathbb{N},\,\ell\in[\![1,N_{h_k}]\!]}$ of eigenfunctions of H we have*

(i) For $2 \leq p < +\infty$

$$\|\varphi_{k,\ell}\|_{L^p(\mathbb{R}^d)} \leq C_p k^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}, \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in [\![1,N_{h_k}]\!].$$

(ii) For $p=+\infty$

$$\|\varphi_{k,\ell}\|_{L^\infty(\mathbb{R}^d)}\leq (M+1)k^{-d/4}(1+\log k)^{1/2},\qquad \forall\, k\in\mathbb{N},\ \forall\,\ell\in[\![1,N_{h_k}]\!].$$

We now state a result we shows that the log factor in the previous bounds is optimal

Theorem 4.5. — *Let* $d\geq 2$. *Then for all* $M>0$

$$\rho\Big[\,B=(\varphi_{k,\ell})_{k\in\mathbb{N},\,\ell\in[\![1,N_{h_k}]\!]}\in\mathcal{B}:\ \liminf_{k\to+\infty}\ \inf_{\ell=1,\dots,N_k} k^{d/4}\|\varphi_{k,\ell}\|_{L^\infty(\mathbb{R}^d)}\leq M\,\Big]=0.$$

This means that, from an orthonormal basis, one cannot extract a subsequence which satisfies better bounds than in Corollary 4.4.

This result is a consequence of Theorem 3.8. For the proof, we can follow the main lines of [**6**, Théorème 8] where an analogous result for the spherical harmonics is proven.

5. Application to Quantum ergodicity

Recall that "quantum ergodicity" for a quantum Hamiltonian H usually means that in some semi-classical regime almost all eigenfunctions of H are distributed according to an invariant measure with a support equal to the classical energy shell.

Assume that $I_h=]a_h,b_h]$ is such that

$$\lim_{h\to 0}a_h=\lim_{h\to 0}b_h=\eta>0\qquad\text{and}\qquad \lim_{h\to 0}\frac{b_h-a_h}{h}=+\infty.$$

The Liouville measure L_η associated with the classical Hamiltonian $H_{c\ell}(x,\xi)=|x|^2+|\xi|^2$ is here the uniform probability on the sphere $\sqrt{\eta}\,\mathbb{S}^{2d-1}$. We define the class of symbols

$$S(1)=\big\{A\in C^\infty(\mathbb{R}^{2d}),\ \forall\alpha,\forall\beta,\ \sup_{(x,\xi)}|\partial_x^\alpha\partial_\xi^\beta A(x,\xi)|<+\infty\big\},$$

and $S(1,1)\subset S(1)$ the class of symbols such that $A\in C^\infty(\mathbb{R}^{2d})$ and A is homogeneous of degree 0 outside a small neighbourhood of $(0,0)$ in $\mathbb{R}^d_x\times\mathbb{R}^d_\xi$: $A(\lambda x,\lambda\xi)=A(x,\xi)$ for every $\lambda\geq 1$ and $|(x,\xi)|\geq\varepsilon$. For $A\in S(1)$ let us denote by $\hat{A}$ the Weyl quantization of A (here $h=1$). Notice that if $\|u_h\|_{L^2(\mathbb{R}^d)}=1$ then $A\mapsto\langle u_h,\hat{A}u_h\rangle$ defines a semiclassical measure on $\sqrt{\eta}\,\mathbb{S}^{2d-1}$ (see e.g. [**5**]).

Then we have

Theorem 5.1 **([33], Theorem 1.4)**. — *Assume that we are in the isotropic case* ($\gamma_j=\frac{1}{\sqrt{N_h}}$ *for all* $j\in\Lambda_h$). *Then there exist* $c,C>0$ *so that for all* $r\geq 1$ *and* $A\in S(1,1)$,

$$\mathbf{P}_h\Big[\,u\in\mathbf{S}_h:|\langle u,\hat{A}u\rangle-L_\eta(A)|>r\Big]\leq Ce^{-cN_hr^2},\ \ \forall h\in]0,1]. \tag{5.1}$$

This result can be related with quantum ergodicity (see [**42, 39, 19, 24**] and the book [**46**, Chapter 15] for an introduction to this subject) which concerns the semi-classical behavior of $\langle\varphi_j,\hat{A}\varphi_j\rangle$ when the classical flow is ergodic on the energy hyper surface $H_{c\ell}^{-1}(\eta)$. Then, for "almost all" eigenfunctions φ_j, we have $\langle\varphi_j,\hat{A}\varphi_j\rangle\overset{j\to+\infty}{\longrightarrow}L_\eta(A)$. The meaning of Theorem 5.1 is that we have $\langle u,\hat{A}u\rangle\overset{h\to 0}{\longrightarrow}L_\eta(A)$

for almost all $u \in \mathbf{S}_h$ such that all modes $(\varphi_j)_{j\in\Lambda_h}$ are "almost uniformly distributed " in u. For related results on compact manifolds see Zelditch [**44**].

As a consequence of Theorem 5.1 we get easily that almost all bases of Hermite functions is Quantum Uniquely Ergodic (see Theorem 5.2 for a precise statement). In [**43**] the author proved that on the standard sphere a random orthonormal basis of eigenfunctions of the Laplace operator is ergodic.

Burq-Lebeau [**6**, Théorème 3] obtained a similar result for the Laplacian on a compact manifold. A modification of their proof allows to consider more general random variables satisfying the Gaussian concentration assumption instead of the uniform law.

From (5.1) we directly deduce that there exists $C > 0$ so that for all $p \geq 2$ and $h \in]0,1]$,

$$\big\| \langle u, \hat{A}u\rangle - L_\eta(A) \big\|_{L^p_{\mathbf{P}_h}} \leq C N_h^{-1/2}\sqrt{p}.$$

Therefore, if one denotes by

$$u_h^\omega = \frac{1}{\sqrt{N_h}} \sum_{j\in\Lambda_h} X_j(\omega)\varphi_j,$$

then we have $\langle u_h^\omega, \hat{A}u_h^\omega\rangle \xrightarrow{h\to 0} L_\eta(A)$ in $L^p(\Omega)$-norm with a remainder estimate.
For example taking the Bernoulli law for the X_j then $u_h^\omega = \dfrac{1}{\sqrt{N_h}} \displaystyle\sum_{j\in\Lambda_h} (-1)^{\varepsilon_j(\omega)}\varphi_j$ where $\{\varepsilon_j\}_{j\geq 1}$ are i.i.d Bernoulli variables.

We are also able to prove that a random orthonormal basis of eigenfunctions of the Harmonic oscillator H is Quantum Uniquely Ergodic (QUE, according the terminology used in [**44**] and introduced in [**34**]).

Recall the definitions (4.3) and (4.4) of $\mathcal{B}$ and ρ. Denote by $B = (\varphi_{j,\ell})_{j\in\mathbb{N},\,\ell\in[\![1,N_{h_j}]\!]} \in \mathcal{B}$ a typical orthonormal basis of $L^2(\mathbb{R}^d)$ so that for all $j\in\mathbb{N}$, $(\varphi_{j,\ell})_{\ell\in[\![1,N_{h_j}]\!]} \in U(N_{h_j})$ is an orthonormal basis of $\mathcal{E}_{h_j}$. Then

Theorem 5.2 **([33], Theorem 1.6)**. — *For $B\in\mathcal{B}$ and $A\in S(1,1)$ let us denote by*

$$D_j(B) = \max_{1\leq \ell\leq N_{h_j}} \big| \langle \varphi_{j,\ell}, \hat{A}\varphi_{j,\ell}\rangle - L_\eta(A)\big|.$$

Then we have

$$\lim_{j\to+\infty} D_j(B) = 0, \ \ \rho-\text{a.s on } \ \mathcal{B}.$$

In other words, ρ-almost all orthonormal basis of Hermite functions is QUE.

Using estimates proved in [**6**], an analogous result to Theorem 5.2 can be proved for the Laplace operator on Riemannian compact manifolds with the same method. This holds true in particular for the sphere in any dimension $d \geq 2$ and more generally for Zoll manifolds (in this last setting a random orthonormal basis of quasi-modes is obtained).

For Schrödinger operators with super-quadratic potentials a similar result can be obtained (see [**33**]), considering orthonormal basis of quasi-modes (approximated eigenfunctions) satisfying the conclusion of Theorem 5.1.

6. Application to supercritical nonlinear Schrödinger equations

6.1. Construction of measures on $L^2(\mathbb{R}^d)$ and Strichartz inequalities. — For $j \geq 1$ denote by

$$I(j) = \big\{n \in \mathbb{N},\ 2j \leq \lambda_n < 2(j+1)\big\}.$$

Observe that for all $j \geq d/2$, $I(j) \neq \emptyset$ and that $\#I(j) \sim c_d j^{d-1}$ when $j \longrightarrow +\infty$. It might also be noticed that $I(j)$ corresponds to exactly one eigenvalue λ_n of H, with multiplicity $\#I(j)$, where $\lambda_n = 2j$ if d is even and $\lambda_n = 2j+1$ if d is odd.
Let us consider any fixed orthonormal basis $(\varphi_n)_{n\geq 0}$ of eigenfunctions for the harmonic oscillator H. Let $s \in \mathbb{R}$, then any $u \in \mathcal{H}^s(\mathbb{R}^d)$ can be written in a unique fashion

$$u = \sum_{j=1}^{+\infty} \sum_{n\in I(j)} c_n \varphi_n, \tag{6.1}$$

and we make the following condition on the coefficients (this is actually condition (3.2))

$$|c_k|^2 \leq \frac{C}{\#I(j)} \sum_{n\in I(j)} |c_n|^2, \quad \forall k \in I(j), \quad \forall j \geq 1. \tag{6.2}$$

We then define the random variable u^ω by

$$u^\omega = \sum_{j=1}^{+\infty} \sum_{n\in I(j)} X_n(\omega) c_n \varphi_n,$$

where $\{X_n\}_{n\geq 0}$ is a sequence of i.i.d random variables, following either the standard complex Gaussian law $\mathcal{N}_{\mathbb{C}}(0,1)$ or the Bernoulli law $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

Remark 6.1. — Here condition (6.2) allows us to use the optimal estimates on the spectral function e_x (see Lemma 3.4 and inequality (6.3) below). This will induce a gain of Sobolev regularity for the random series u^ω as it is stated in Proposition 6.2, compared to the deterministic Strichartz estimates (6.4).

We then have the following probabilistic improvement of the Strichartz estimates.

***Proposition 6.2* ([30], Proposition 2.1)**. — *Let $s \in \mathbb{R}$ and let $u \in \mathcal{H}^s(\mathbb{R}^d)$ as in* (6.1)*. Assume that $(c_n)_{n\in\mathbb{N}}$ satisfies* (6.2)*. Let $1 \leq q < +\infty$, $2 \leq p \leq +\infty$, and set $\alpha = d(1/2 - 1/p)$ if $p < +\infty$ and $\alpha < d/2$ if $p = +\infty$. Then there exist $c, C > 0$ so that for all $\tau \in \mathbb{R}$*

$$\mathbb{P}\Big[\omega :\ \big\|e^{-i(t+\tau)H} u^\omega\big\|_{L^q_{[0,T]}\mathcal{W}^{s+\alpha,p}(\mathbb{R}^d)} > K\Big] \leq C e^{-\frac{cK^2}{T^{2/q}\|u\|^2_{\mathcal{H}^s(\mathbb{R}^d)}}}.$$

The first key ingredient in the proof is the Khinchin inequality (see *e.g.* [**9**, Lemma 4.2] for a proof): There exists $C > 0$ such that for all real $k \geq 2$ and $(a_n) \in \ell^2(\mathbb{N})$

$$\| \sum_{n\geq 0} X_n(\omega)\, a_n \|_{L^k_{\mathbb{P}}} \leq C\sqrt{k} \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{\frac{1}{2}}.$$

The second key ingredient in the proof is the L^∞ estimate of the spectral function given by Thangavelu/Karadzhov (see [**31**, Lemma 3.5]) which reads for $d \geq 2$

$$\sup_{x\in\mathbb{R}} \sum_{n\in I(j)} \big|\varphi_n(x)\big|^2 \leq C j^{d/2-1}, \tag{6.3}$$

and which does not depend on the choice of the $(\varphi_n)_{n\geq 1}$.

Let us recall the deterministic Strichartz estimates for the harmonic oscillator, which can be established using the Mehler formula (see [**8**, Lemma 5.1] for the argument in 1D which can be extended). We say that a couple $(q,p) \in [2,+\infty]^2$ is admissible if

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2} \quad \text{and} \quad (d,q,p) \neq (2,2,+\infty).$$

Then for all $T > 0$ there exists $C_T > 0$ so that for all $u_0 \in \mathcal{H}^s(\mathbb{R}^d)$ we have

$$\|\mathrm{e}^{-itH} f\|_{X^s_T} \leq C_T \|f\|_{\mathcal{H}^s(\mathbb{R}^d)}, \tag{6.4}$$

where

$$X^s_T := \bigcap_{(q,p)\ admissible} L^q\big([-T,T]\,; \mathcal{W}^{s,p}(\mathbb{R}^d)\big).$$

When p is close to $+\infty$, the result of Proposition 6.2 expresses a gain $\mathbb{P}-$a.s. of almost $d/2$ derivatives in space compared to the bound (6.4).

Before we state our well-posedness results, we need to define the measures on the space of initial conditions. We define the set $\mathcal{A}_s \subset \mathcal{H}^s(\mathbb{R}^d)$ by

$$\mathcal{A}_s = \big\{ u = \sum_{j=1}^{+\infty} \sum_{n\in I(j)} c_n\varphi_n \in \mathcal{H}^s(\mathbb{R}^d) \ \text{ s.t. condition (6.2) holds for some } C > 0 \big\}.$$

Let $\gamma \in \mathcal{A}_s$. We define the probability measure μ_γ on $\mathcal{H}^s$ via the map

$$\begin{array}{rcl} \Omega & \longrightarrow & \mathcal{H}^s(\mathbb{R}^d) \\ \omega & \longmapsto & \gamma^\omega = \displaystyle\sum_{j=1}^{+\infty} \sum_{n\in I(j)} c_n X_n(\omega) \varphi_n, \end{array}$$

in other words, μ_γ is defined by: for all measurable $F : \mathcal{H}^s \longrightarrow \mathbb{R}$

$$\int_{\mathcal{H}^s(\mathbb{R}^d)} F(v)\mathrm{d}\mu_\gamma(v) = \int_\Omega F(\gamma^\omega)\mathrm{d}\mathbb{P}(\omega).$$

In particular, we can check that μ_γ satisfies

- If $\gamma \in \mathcal{H}^s \backslash \mathcal{H}^{s+\varepsilon}$, then $\mu_\gamma(\mathcal{H}^{s+\varepsilon}) = 0$.
- Assume that for all $j \geq 1$ such that $I(j) \neq \emptyset$ we have $c_j \neq 0$. Then for all nonempty open subset $B \subset \mathcal{H}^s$, $\mu_\gamma(B) > 0$.

Finally, we denote by $\mathcal{M}^s$ the set of all such measures

$$\mathcal{M}^s = \bigcup_{\gamma \in \mathcal{A}_s} \{\mu_\gamma\}.$$

For more properties of these measures, we refer to the introduction of [**30**].

Each element $\gamma \in \mathcal{A}_s$ defines a probability measure μ_γ. We will see in the next section that for a typical $u_0 \in \mathcal{H}^s$ in the support of μ_γ, the nonlinear Schrödinger equation with initial condition u_0 is well-posed.

By the Kakutani theorem (see also Remark 3.1), the space $\mathcal{M}^s$ contains mutually singular measures, and this extends the set of initial conditions for which we are able to solve NLS.

6.2. Application to NLS. — We now consider the Cauchy problem for nonlinear Schrödinger equations, where the initial condition is random. We apply the estimates developed in the previous sections, in particular Proposition 6.2, in order to show local and global well-posedness results, for problems with Sobolev supercritical regularity.

Much work has been done on dispersive PDEs with random initial conditions since the papers of Burq-Tzvetkov [**9, 10**]. In these articles, the authors showed that thanks to a randomisation of the initial condition one can prove well-posedness results even for data with supercritical Sobolev regularity. We also refer to [**11, 38, 8, 29, 28, 37, 7, 27, 25**] and references therein for further developments.

Let us consider nonlinear Schrödinger equation with harmonic potential

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u - |x|^2 u = \pm |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \\ u(0) = u_0, \end{cases} \tag{6.5}$$

with $d \geq 2$, $p \geq 3$ an odd integer. This is a model used in the description of the Bose-Einstein condensates.

Before we state our results, let us recall some facts concerning the deterministic study of the nonlinear Schrödinger equation (6.5). We say that (6.5) is locally well-posed in $\mathcal{H}^s(\mathbb{R}^d)$, if for all initial condition $u_0 \in \mathcal{H}^s(\mathbb{R}^d)$, there exists a unique local in time solution $u \in \mathcal{C}([-T,T];\mathcal{H}^s(\mathbb{R}^d))$, and if the flow-map is uniformly continuous. We denote by

$$s_c = \frac{d}{2} - \frac{2}{p-1},$$

the critical Sobolev index. Then one can show that NLS is well-posed in $\mathcal{H}^s(\mathbb{R}^d)$ when $s > \max(s_c, 0)$, and ill-posed when $s < s_c$. We refer to the introduction of [**38**] for more details on this topic.

6.2.1. **Local existence results.** — We are now able to state our first result on the local well-posedness of (6.5).

Theorem 6.3 **([30], Theorem 1.1).** — *Let $d \geq 2$, $p \geq 3$ an odd integer and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^d)$ with $\mu(\Sigma) = 1$ and so that:*

(i) For all $u_0 \in \Sigma$ there exist $T > 0$ and a unique local solution u to (6.5) with initial data u_0 satisfying

$$u(t) - e^{-itH} u_0 \in \mathcal{C}\big([-T,T]; \mathcal{H}^s(\mathbb{R}^d)\big), \tag{6.6}$$

for some $\frac{d}{2} - \frac{2}{p-1} < s < \frac{d}{2}$.

(ii) More precisely, for all $T > 0$, there exists $\Sigma_T \subset \Sigma$ with

$$\mu(\Sigma_T) \geq 1 - C \exp\big(- cT^{-\delta} \|\gamma\|^{-2}_{L^2(\mathbb{R}^2)}\big), \quad C, c, \delta > 0,$$

and such that for all $u_0 \in \Sigma_T$ the lifespan of u is larger than T.

Denote by $\gamma = \displaystyle\sum_{n=0}^{+\infty} c_n \varphi_n(x)$, then $u_0^\omega := \displaystyle\sum_{n=0}^{+\infty} g_n(\omega) c_n \varphi_n(x)$ is a typical element in the support of μ_γ. Another way to state Theorem 6.3 is : for any $T > 0$, there exists an event $\Omega_T \subset \Omega$ so that

$$\mathbb{P}(\Omega_T) \geq 1 - C \exp\big(- cT^{-\delta} \|\gamma\|^{-2}_{L^2(\mathbb{R}^d)}\big), \quad C, c, \delta > 0,$$

and so that for all $\omega \in \Omega_T$, there exists a unique solution of the form (6.6) to (6.5) with initial data u_0^ω.

The key argument in the proof is the use of Proposition 6.2 which yields a gain of $d/2$ derivatives compared to the deterministic theory. To prove Theorem 6.3 we only have to gain $s_c = d/2 - 2/(p-1)$ derivatives. The solution is constructed by a fixed point argument in a Strichartz space $X_T^s \subset \mathcal{C}\big([-T,T]; \mathcal{H}^s(\mathbb{R}^d)\big)$ with continuous embedding, and uniqueness holds in the class X_T^s.

The deterministic Cauchy problem for (6.5) was studied by Oh [**26**] (see also Cazenave [**14**, Chapter 9] for more references). In [**38**], Thomann has proven an almost sure local existence result for (6.5) in the supercritical regime (with a gain of 1/4 of derivative), for any $d \geq 1$. This local existence result was improved by Burq-Thomann-Tzvetkov [**8**] when $d = 1$ (gain of 1/2 derivatives), by Deng [**16**] when $d = 2$, and by Poiret [**29, 28**] in any dimension.

6.2.2. **Global existence and scattering results for NLS.** — As an application of the results of the previous part, we are able to construct global solutions to the non-linear Schrödinger equation without potential, which scatter when $t \to \pm\infty$. Consider the following equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = \pm |u|^{p-1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \\ u(0) = u_0. \end{cases} \tag{6.7}$$

The well-posedness indexes for this equation are the same as for equation (6.5). Namely, (6.7) is well-posed in $H^s(\mathbb{R}^d)$ when $s > \max(s_c, 0)$, and ill-posed when $s < s_c$.

Then we can prove

Theorem 6.4 **([30], Theorem 1.3)**. — *Let $d \geq 2$, $p \geq 3$ an odd integer and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^d)$ with $\mu(\Sigma) > 0$ and so that:*

(i) For all $u_0 \in \Sigma$ there exists a unique global solution u to (6.7) with initial data u_0 satisfying
$$u(t) - e^{it\Delta}u_0 \in \mathcal{C}\big(\mathbb{R}; \mathcal{H}^s(\mathbb{R}^d)\big),$$
for some $\frac{d}{2} - \frac{2}{p-1} < s < \frac{d}{2}$.

(ii) For all $u_0 \in \Sigma$ there exist states $f_+, f_- \in \mathcal{H}^s(\mathbb{R}^d)$ so that when $t \longrightarrow \pm\infty$
$$\big\| u(t) - e^{it\Delta}(u_0 + f_\pm) \big\|_{H^s(\mathbb{R}^d)} \longrightarrow 0.$$

(iii) If we assume that the distribution of ν is symmetric, then
$$\mu\Big(u_0 \in L^2(\mathbb{R}^d) : \textit{the assertion (ii) holds true} \;\Big|\; \|u_0\|_{L^2(\mathbb{R}^d)} \leq \eta\Big) \longrightarrow 1,$$
when $\eta \longrightarrow 0$.

We can show [**29**, Théorème 20], that for all $s > 0$, if $u_0 \notin \mathcal{H}^\sigma(\mathbb{R}^d)$ then $\mu(H^\sigma(\mathbb{R}^d)) = 0$. This shows that the randomisation does not yield a gain of derivative in the Sobolev scale; thus Theorem 6.4 gives results for initial conditions which are not covered by the deterministic theory.

There is a large literature for the deterministic local and global theory with scattering for (6.7). We refer to [**2, 12**] for such results and more references.

One of the key points in the proof is to use the lens transform $\mathscr{L}$ defined as
$$u(t,x) \longmapsto \mathscr{L}u(t,x) = \Big(\frac{1}{\sqrt{1+4t^2}}\Big)^{d/2} u\Big(\frac{\arctan(2t)}{2}, \frac{x}{\sqrt{1+4t^2}}\Big) \mathrm{e}^{\frac{ix^2 t}{1+4t^2}},$$
which permits link equation (6.7) to equation (6.5). In particular, local in time results for (6.5) on the time interval $[-\pi/4, \pi/4]$ imply global in time results for (6.7).

In Theorem 6.4 we assumed that $d \geq 2$ and $p \geq 3$ was an odd integer, so we had $p \geq 1 + 4/d$, or in other words we were in a L^2-supercritical setting. Our approach also allows to get global in time results in an L^2-subcritical context, i.e. when $1 + 2/d < p < 1 + 4/d$.

Theorem 6.5 **([30], Theorem 1.4)**. — *Let $d = 2$ and $2 < p < 3$ and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^2)$ with $\mu(\Sigma) > 0$ and so that for all $0 < \varepsilon < 1$*

(i) For all $u_0 \in \Sigma$ there exists a unique global solution u to (6.7) with initial data u_0 satisfying
$$u(t) - e^{it\Delta}u_0 \in \mathcal{C}\big(\mathbb{R}; \mathcal{H}^{1-\varepsilon}(\mathbb{R}^2)\big).$$

(ii) For all $u_0 \in \Sigma$ there exist states $f_+, f_- \in \mathcal{H}^{1-\varepsilon}(\mathbb{R}^2)$ so that when $t \longrightarrow \pm\infty$
$$\big\| u(t) - e^{it\Delta}(u_0 + f_\pm) \big\|_{H^{1-\varepsilon}(\mathbb{R}^2)} \longrightarrow 0.$$

(iii) If we assume that the distribution of ν is symmetric, then

$$\mu\Big(u_0 \in L^2(\mathbb{R}^2) : \textit{the assertion (ii) holds true} \;\Big|\; \|u_0\|_{L^2(\mathbb{R}^2)} \leq \eta\Big) \longrightarrow 1,$$

when $\eta \longrightarrow 0$.

In the case $p \leq 1 + 2/d$, Barab [**3**] showed that a non trivial solution to (6.7) never scatters, therefore even with a stochastic approach one can not have scattering in this case. When $d = 2$, the condition $p > 2$ in Theorem 6.5 is therefore optimal. Usually, deterministic scattering results in L^2-subcritical contexts are obtained in the space $H^1 \cap \mathcal{F}(H^1)$. Here we assume $u_0 \in L^2$, and thus we relax both the regularity and the decay assumptions (this latter point is the most striking in this context). Again we refer to [**2, 1**] for an overview of scattering theory for NLS.

6.2.3. **Global existence results for NLS with quadratic potential.** — We also get global existence results for defocusing Schrödinger equation with harmonic potential. For $d = 2$ or $d = 3$, consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} - Hu = |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{6.8}$$

and denote by E the energy of (6.8), namely

$$E(u) = \|u\|^2_{\mathcal{H}^1(\mathbb{R}^d)} + \frac{1}{2}\|u\|^4_{L^4(\mathbb{R}^d)}.$$

Deterministic global existence for (6.8) has been studied by Zhang [**45**] and by Carles [**13**] in the case of time-dependent potentials.

When $d = 3$, our global existence result for (6.8) is the following

Theorem 6.6 **([30], Theorem 1.5)**. — *Let $d = 3$, $1/6 < s < 1$ and fix $\mu = \mu_\gamma \in \mathcal{M}^s$. Then there exists a set $\Sigma \subset \mathcal{H}^s(\mathbb{R}^3)$ so that $\mu(\Sigma) = 1$ and so that the following holds true*

(i) For all $u_0 \in \Sigma$, there exists a unique global solution to (6.8) which reads

$$u(t) = e^{-itH}u_0 + w(t), \quad w \in \mathcal{C}\big(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)\big).$$

(ii) The previous line defines a global flow Φ, which leaves the set Σ invariant

$$\Phi(t)(\Sigma) = \Sigma, \quad \textit{for all } t \in \mathbb{R}.$$

(iii) There exist $C, c_s > 0$ so that for all $t \in \mathbb{R}$,

$$E\big(w(t)\big) \leq C(M + |t|)^{c_s +},$$

where M is a positive random variable so that

$$\mu(u_0 \in \mathcal{H}^s(\mathbb{R}^3) \,: M > K) \leq C e^{-\frac{cK^\delta}{\|\gamma\|^2_{\mathcal{H}^s(\mathbb{R}^3)}}}.$$

Here the critical Sobolev space is $\mathcal{H}^{1/2}(\mathbb{R}^3)$, thus the local deterministic theory combined with the conservation of the energy, immediately gives global well-posedness in $\mathcal{H}^1(\mathbb{R}^3)$. To prove Theorem 6.6, we use the high/low frequency decomposition method of Bourgain [**4**, page 84]), which relies on the almost well-posedness result of Theorem 6.3 and the global well-posedness in $\mathcal{H}^1(\mathbb{R}^3)$. As a result, we get a global well-posedness result in a supercritical context. This strategy has been successful in different contexts, and has been first used together with probabilistic arguments by Colliander-Oh [**15**] for the cubic Schrödinger below $L^2(\mathbb{S}^1)$ and later on by Burq-Tzvetkov [**11**] for the wave equation.

With a similar approach, in dimension $d = 2$, we can prove the following result

Theorem 6.7 **([30], Theorem 1.6)**. — *Let $d = 2$, $0 < s < 1$ and fix $\mu = \mu_\gamma \in \mathcal{M}^s$. Then there exists a set $\Sigma \subset \mathcal{H}^s(\mathbb{R}^2)$ so that $\mu(\Sigma) = 1$ and so that for all $u_0 \in \Sigma$, there exists a unique global solution to* (6.8) *which reads*

$$u(t) = e^{-itH}u_0 + w(t), \quad w \in \mathcal{C}\big(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^2)\big).$$

In addition, statements (ii) and (iii) of Theorem 6.6 are also satisfied with $c_s = \frac{1-s}{s}$.

Here the critical Sobolev space is $L^2(\mathbb{R}^2)$, thus Theorem 6.7 shows global well-posedness for any subcritical cubic non linear Schrödinger equations in dimension two.

Appendix A
Proof of Theorem 3.5

Lemma A.1. — *Let $0 < \eta < 1$ and set $p_h = |\ln h|^{1-\eta}$. Then there exists a constant $C_\star = C_\star(d)$ such that*

$$(C_\star - \varepsilon/2)\sqrt{p_h} \leq \mathcal{M}_{p_h} \leq (C_\star + \varepsilon/2)\sqrt{p_h}, \tag{A.1}$$

for $h > 0$ small enough.

Proof. — In the sequel, write $p = p_h$. As in the proof of [**31**, Theorem 4.7], we denote by $\mathcal{A}_p = \mathbf{E}_h(\|u\|_{L^{p,d(p/2-1)}})^{1/p}$. Then from [**31**, estimate (4.14)] we have

$$|\mathcal{A}_p - \mathcal{M}_p| \leq CN^{-1/p}\sqrt{p} \leq \mathrm{e}^{-c|\ln h|^\eta}\sqrt{p},$$

hence it is enough to prove estimate (A.1) for $\mathcal{A}_p$. The proof then consists in tracking all the constants in [**31**] and to show that they are optimal. We do not write all the details, but we give the main steps. Let us recall [**31**, estimate (4.12)]

$$\begin{aligned} C_1\, p(c_1N)^{-p/2}\left(\int_{\mathbb{R}^d}\langle x\rangle^{d(\frac{p}{2}-1)}e_x^{p/2}\mathrm{d}x\right)\int_0^{c_0N} t^{p/2-1}\mathrm{e}^{-t}\,\mathrm{d}t \leq \mathcal{A}_p^p \leq \\ \leq C_2\, p(c_2N)^{-p/2}\left(\int_{\mathbb{R}^d}\langle x\rangle^{d(\frac{p}{2}-1)}e_x^{p/2}\mathrm{d}x\right)\Gamma(p/2). \end{aligned} \tag{A.2}$$

Then for $\varepsilon > 0$

- With an inspection of the proof of [**31**, Theorem 2.6, (2.11)] and in [**31**, Lemma 2.11] we can show that we can take in the previous line
$$1/2-\varepsilon/8<c_2<c_1<1/2+\varepsilon/8.$$
- We can construct the parametrix in [**31**, Lemma 4.9] in such a way that
$$(\sqrt{C^\star}-\varepsilon/8)\sqrt{N}\leq\Big(\int_{\mathbb{R}^d}\langle x\rangle^{d(\frac{p}{2}-1)}e_x^{p/2}\,\mathrm{d}x\Big)^{1/p}\leq(\sqrt{C^\star}+\varepsilon/8)\sqrt{N},$$
where $C^\star$ is a constant which only depends on the dimension.
- With the Laplace method, we can show that for $p\geq 2$ large enough
$$(1-\varepsilon/8)\sqrt{\frac{p}{2\mathrm{e}}}\leq\Big(\int_0^{c_0N}t^{p/2-1}\mathrm{e}^{-t}\,\mathrm{d}t\Big)^{1/p}\leq\big(\Gamma(p/2)\big)^{1/p}\leq(1+\varepsilon/8)\sqrt{\frac{p}{2\mathrm{e}}}.$$

Putting the previous estimates together with (A.2), we get
$$(C_\star-\varepsilon/2)\sqrt{p}\leq\mathcal{A}_p\leq(C_\star+\varepsilon/2)\sqrt{p},$$
with $C_\star=(C^\star/(2\mathrm{e}))^{1/2}$, which was the claim. □

We are now able to complete the proof of Theorem 3.5. By Lemma A.1, for $p\geq 2$ large enough
$$\Big\{\big|\|u\|_{L^{p,d(p/2-1)}}-C_\star\sqrt{p}\big|>\varepsilon\sqrt{p}\Big\}\subset\Big\{\Big|\|u\|_{L^{p,d(p/2-1)}}-\mathcal{M}_p\Big|>\frac{\varepsilon}{2}\sqrt{p}\Big\},$$
and an application of Theorem 3.3 gives the result.

Appendix B
Proof of Theorem 3.8

Assume here that $\delta=1$ (the general case $0\leq\delta\leq1$ can be treated in the same way). To begin with, we state an estimate of $\|u\|_{L^r}$.

Theorem B.1. — *Let $2d/(d-1)<r\leq\infty$ and denote by $\mathcal{M}_r$ a median of $\|u\|_{L^r}$. Then there exist $0<C_0<C_1$, $K>0$, $c_1>0$ and $h_0>0$ such that for all $r\in[2,K|\log h|]$ and $h\in]0,h_0]$ such that*
$$\mathbf{P}_{\gamma,h}\Big[u\in\mathbf{S}_h:\Big|\|u\|_{L^r}-\mathcal{M}_r\Big|>\Lambda\Big]\leq 2\exp\big(-c_2N_h^{2/r}h^{-\frac{d}{4}(1+\frac{2}{r})}\Lambda^2\big).$$
and where
$$C_0\sqrt{r}h^{\frac{d}{4}(1+\frac{2}{r})}\leq\mathcal{M}_r\leq C_1\sqrt{r}h^{\frac{d}{4}(1-\frac{2}{r})},\quad\forall r\in[2,K|\log h|\,].\tag{B.1}$$

This is a result similar to [**31**, Theorem 4.7] in which we made the restriction $\delta<2/3$. However, the price to pay, is that the estimate (B.1) is no more optimal. The reason is that, in this case we prove the following estimate on the spectral function when $p>d/(d-1)$ (here $r=2p$)
$$C_0h^{1-\frac{d}{2}(1-\frac{1}{p})}\leq\Big(\int_{\mathbb{R}^d}e_x^p\,\mathrm{d}x\Big)^{1/p}\leq C_1h^{1-\frac{d}{2}(1+\frac{1}{p})}.\tag{B.2}$$

Let us prove (B.2). The upper bound is the same as in Lemma 3.4 and is proved using (2.5) (see also [**33**, Appendix A.5] for more general results). To get the lower bound, we oberve that among the family $(\varphi_j)_{j\in\Lambda_h}$ there exists a radial function $\varphi_{rad,h}$ and which satisfies $H\varphi_{rad,h} \sim \frac{1}{h}\varphi_{rad,h}$. Therefore

$$e_x := \sum_{j\in\Lambda_h} |\varphi_j|^2 \geq |\varphi_{rad,h}|^2.$$

Now we invoke the sharp L^p bounds of radial Hermite functions, proved in [**20**, Proposition 2.4] which imply the result. Finally, to prove Theorem B.1, we proceed as in the proof of [**31**, Theorem 4.7], using the estimate (B.2).

The proof of Theorem 3.8 is analogous to [**31**, Corollary 4.8]. Roughly speaking, the L^∞ norm is reached in the regime $r = r_h = c|\ln h|$ and $h \ll 1$. In this regime, the estimate (B.1) becomes optimal, since $h^{c_1/r_h} \sim c_2$.

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