ROUGH VOLTERRA EQUATIONS 2: CONVOLUTIONAL GENERALIZED INTEGRALS

AURÉLIEN DEYA AND SAMY TINDEL

ABSTRACT. We define and solve Volterra equations driven by a non-differentiable signal, by means of a variant of the rough path theory allowing to handle generalized integrals weighted by an exponential coefficient. The results are applied to a standard rough path $\mathbf{x} = (\mathbf{x^1}, \mathbf{x^2}) \in \mathcal{C}_2^\gamma(\mathbb{R}^m) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})$, with $\gamma > 1/3$, which includes the case of fractional Brownian motion with Hurst index $H > 1/3$.

1. Introduction

This paper is part of an ambitious ongoing project which aims at offering a new point of view on multidimensional stochastic calculus, via the semi-deterministic rough path approach initiated by Lyons [24]. We tackle the issue of the non-linear Volterra system

$$y_t^i = a^i + \int_0^t \sigma^{i0}(t,u,y_u)\,du + \sum_{j=1}^m \int_0^t \sigma^{ij}(t,u,y_u)\,dx_u^j, \quad i = 1,\ldots,d, \quad t \in [0,T], \tag{1}$$

where T stands for an arbitrary horizon, $x : [0,T] \to \mathbb{R}^m$ a multidimensional γ-Hölder path, $a \in \mathbb{R}^d$ an initial condition and $\sigma^{ij} : [0,T]^2 \times \mathbb{R}^d \to \mathbb{R}$ smooth enough functions.

The (ordinary) Volterra equation providing a relevant model in many biological or physical situations, it is not surprising that its noisy version already gave birth to a great amount of papers. A first analysis when x is a Brownian motion is contained in the pioneering works [6, 7], and then generalized to the case of a semimartingale in [31]. If the coefficients σ^{ij} are also seen as random functions, which often happens to be more appropriate, some anticipative stochastic calculus techniques are required in order to solve the system, and we refer to [1, 28, 30] for the main results in this direction. It should be mentioned at this point that the last of those references [30] is motivated by financial models of capital growth rate, which goes beyond the classical physical or biological applications of Volterra equations. Several authors also envisaged the possibility of a singularity for the application $u < t \to \sigma(t,u,.)$ as t tends to u [10, 11, 37], while examples of a so-called backward stochastic Volterra equations recently appeared in the literature [38, 40], stimulated (here again) by new financial applications [39]. Besides, one can find in [34, 21, 43] studies of infinite-dimensional versions of (1), often linked to the context of stochastic partial differential equations. It is finally worth noticing that the behaviour of the solutions to the Itô-Volterra equation is now deeply understood, through the consideration of numerical schemes [35, 42] or the existence of large deviations [17, 33, 27, 42] and Strassen's law [29] results.

Date: April 20, 2011.

2000 *Mathematics Subject Classification.* 60H05, 60H07, 60G15.

Key words and phrases. Rough paths theory; Stochastic Volterra equations; Fractional Brownian motion.

In this background, it seems quite natural to wonder if the interpretation and resolution of (1) can be extended to a non-semimartingale driving process x. The existence of a theoretical solution would for instance allow to study the influence of a more general gaussian noise in the asymptotic equilibria observed in [4, 2, 3, 5]. The interest in a generalization of the system has also been recently reinforced by the emergence, in the field of nanophysics, of a model involving a Volterra system perturbed by a fractional Brownian motion (fBm in the sequel) with Hurst index H different from 1/2 [22, 23]. In the latter references, the fractional process only intervenes through an additive noise: the resolution of the system (1) in its general form would here open the way to a sophistication of the model.

The particular case where x stands for a fBm with Hurst index $H > 1/2$ has been thoroughly treated in [16]: the integral is therein understood in the Young sense. Notice that in this situation, [8] provides a slighlty different approach to the equation, based on fractional calculus techniques. If one wishes to go one step further in the procedure and consider a γ-Hölder path with $\gamma \leq 1/2$, the rough paths methods must come into the picture. However, the classical rough path theory introduced by Terry Lyons [25] (see also the recent formulation in [18]) is mostly designed to handle the case of diffusion type equations, and there have been an intensive activity during the last couple of years in order to extend these semi-pathwise techniques to other systems, such as delay equations [26] or PDEs [9, 20]. The current article fits into this global project, and we shall see how to modify the original rough path setting in order to handle systems like (1). The method then leads to what appears to the authors as the first result of existence and uniqueness of a global solution ever shown for the rough Volterra equation (1), in case $\gamma < \frac{1}{2}$.

Our result more exactly applies to the *convolutional* Volterra equation:

$$y_t^i = a^i + \sum_{j=1}^{m} \int_0^t \phi(t-u)\sigma^{ij}(y_u)\, dx_u^j, \quad i = 1,\dots,d, \quad t \in [0,T], \tag{2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ and $\sigma^{ij} : \mathbb{R}^d \to \mathbb{R}$ are smooth enough applications. Notice that we have included the drift term in the sum, by assuming that the first component of x coincides with the identity function. In spite of its specificity, the formulation (2) covers most of the model aforementioned (it is in particular the model at stake in [22, 23]). The main result of this paper can be stated in the following way:

Theorem 1.1. *Assume that the path $x : [0,T] \to \mathbb{R}^m$ allows the construction of a geometric 2-rough path $\mathbf{x} = (\mathbf{x^1}, \mathbf{x^2}) \in \mathcal{C}_2^\gamma(\mathbb{R}^m) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})$ for some coefficient $\gamma > 1/3$. If $\phi \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$ and $\sigma^{ij} \in \mathcal{C}^{3,\mathbf{b}}(\mathbb{R}^d;\mathbb{R})$ for all $i = 1,\dots,d$, $j = 1,\dots,m$, then the system (2), interpreted thanks to Propositions 5.5 and 6.2, admits a unique global solution y in the space of controlled paths introduced in* [19] *(see Definition 2.5). Moreover, the Itô map associated to the system is locally Lipschitz continuous: if y (resp. $\hat{y}$) stands for the solution of the system driven by x (resp. $\hat{x}$) with initial condition a (resp. $\hat{a}$), then*

$$\mathcal{N}[y-\hat{y};\mathcal{C}_1^\gamma(\mathbb{R}^d)] \leq c_{\mathbf{x},\tilde{\mathbf{x}}} \left\{ |a-\hat{a}| + \mathcal{N}[x-\hat{x};\mathcal{C}_1^\gamma(\mathbb{R}^m)] + \mathcal{N}[\mathbf{x^2}-\hat{\mathbf{x}}^\mathbf{2};\mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})] \right\}, \tag{3}$$

where

$$c_{\mathbf{x},\tilde{\mathbf{x}}} = C\left(\mathcal{N}[x;\mathcal{C}_1^\gamma(\mathbb{R}^m)], \mathcal{N}[\hat{x};\mathcal{C}_1^\gamma(\mathbb{R}^m)], \mathcal{N}[\mathbf{x^2};\mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})], \mathcal{N}[\hat{\mathbf{x}}^\mathbf{2};\mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})]\right),$$

for some function $C : (\mathbb{R}^+)^* \to \mathbb{R}^+$ growing with its four arguments.

Beyond the interpretation and resolution of the fractional Volterra system, the continuity result (3) is likely to offer simplified proofs of the classical results (large deviations, support theorem) obtained in the (standard) Brownian case. For the sake of conciseness, we shall let the procedure in abeyance, though (this should follow the lines of Chapter 19 in [18]).

A first attempt to solve the deterministic system (2) has been initiated in [16] by resorting to the *standard* rough paths formalism. As evoked earlier, the method turns out to be successful in the Young case ($\gamma > 1/2$) with the existence of a unique global solution. Unfortunately, it incompletely answers the problem in the *rough* case ($\gamma \leq 1/2$), allowing a *local* resolution only. The difficulties raised by the extension of the path have been extensively commented in [16]. They are essentially due to the dependence of the system with respect to the past of the trajectory. To figure out this phenomenom, remember that the usual resolution framework in rough paths theory is a (well-chosen) space of Hölder paths (or paths with bounded p-variations). Here, the variations of the (potential) solution y between two times $s < t$ are given by

$$y_t^i - y_s^i = \int_s^t \phi(t-u)\,\sigma^{ij}(y_u)\,dx_u^j + \int_0^s \left[\phi(t-u) - \phi(s-u)\right]\,\sigma^{ij}(y_u)\,dx_u^j, \tag{4}$$

and through the latter integral pops out the problem in question: the variations of y between a time s (present) and a time t (future) are linked to the past ($[0,s]$) of the path. In the Young case, the right-hand-side of (4) can be estimated by an affine function of y, which allows to overcome the dependence to the past and settle a global fixed-point argument. The reasoning does not hold true anymore when $\gamma \leq 1/2$, the estimate giving this time rise to a (at least) quadratic term in y.

Let us say a few words about the strategy we have adopted in this paper in order to exhibit a *global* solution when $\gamma \in (1/3, 1/2]$:

(i) First, we will reformulate (2) (when x is differentiable) by writing ϕ as the Fourier transform of a function $\tilde{\phi} \in L^1(\mathbb{R})$, that is to say using the representation

$$\phi(v) = \int_{\mathbb{R}} d\xi\, S_v(\xi)\tilde{\phi}(\xi) \quad , \quad S_v(\xi) \equiv e^{-2\mathrm{i}\pi\xi v},\ v \in [0,T]. \tag{5}$$

Thanks to Fubini theorem, the system (2) can now be equivalently presented as: for all $i = 1, \ldots, d$,

$$y_t^i = a^i + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\tilde{y}_t^i(\xi) \quad , \quad \tilde{y}_t^i(\xi) = \int_0^t S_{t-u}(\xi)\,dx_u^j\,\sigma^{ij}(y_u)\ , \quad t \in [0,T]. \tag{6}$$

Owing to the additivity property $S_{t+t'}(\xi) = S_t(\xi)S_{t'}(\xi)$, it is easily seen that for any fixed $\xi \in \mathbb{R}$,

$$\tilde{y}_t^i(\xi) - \tilde{y}_s^i(\xi) = \int_s^t S_{t-u}(\xi)\,dx_u^i\,\sigma^{ij}(y_u) + A_{ts}(\xi)\tilde{y}_s^i(\xi), \tag{7}$$

with $A_{ts}(\xi) \equiv S_{t-s}(\xi) - 1$, and the dependence w.r.t the past ($[0,s]$) is here reduced to a dependence w.r.t the present (s) only, which makes it easier to control on successive patching intervals I_1, I_2,... Therefore, the system will first be solved under the form (7), before we go back to the original setting (2).

(ii) The transition from y to $\tilde{y}$ is however not priceless: we leave the Euclidian context of (2) to enter the framework of functional-valued paths. For instance, the definition of a

Hölder path will then be relative to a norm of functions to be precised (see (26)). Besides, observe that the expression

$$\tilde{y}_t^i(\xi) = \int_0^t S_{t-u}(\xi)\, dx_u^j\, \sigma^{ij}(y_u) \quad , \quad \tilde{y}_0^i = 0, \tag{8}$$

is quite close to the mild formulation of an evolution equation: in order to analyze this system, we have drown our inspiration from the method and formalism developped in [20] for a class of rough partial differential equations. In particular, the interpretation of the rough integral will involve an adaptation of the notion of 2-rough paths to the background at stake here: the standard path $(\mathbf{x^1}, \mathbf{x^2})$ will be replaced (in a first phase at least) by a convolutional path $(\tilde{X}^x, \tilde{X}^{ax}, \tilde{X}^{xx})$, given, when x is differentiable, by the three formulas $(i, j = 1, \ldots, m)$

$$\tilde{X}_{ts}^{x,i}(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dx_u^i \quad , \quad \tilde{X}_{ts}^{Ax,i}(\xi) \equiv \int_s^t A_{tu}(\xi)\, dx_u^i, \tag{9}$$

$$\tilde{X}_{ts}^{xx,ij}(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dx_u^i\, (x_u^j - x_s^i). \tag{10}$$

If x is a Hölder path, those three definitions are a priori only formal, but once we have admitted the existence of those integrals (see for instance Hypothesis 5 for a more precise statement), we can resort to an extension procedure for the integral $\int_s^t S_{t-u}(\xi)\, dx_u^j\, \sigma^{ij}(y_u)$ similar to the one used in the analysis of ordinary systems, and based on the intervention of an inverse operator $\tilde{\Lambda}$ (Proposition 3.8). The extension of the three expressions in (9) and (10) will be analyzed in the end of the paper (Section 6): for sake of conciseness, the question will actually be reduced to a loose integration by parts argument.

(iii) In the case $1/3 < \gamma \leq 1/2$, the reasoning that leads us to the existence of a global solution consists in a technical patching argument (Section 5) based on the following observation: in spite of the simplification suggested by (7), the system keeps some dependence w.r.t the past through the present. Consequently, if one wants to patch together local solutions $\tilde{y}^{(k)}$ on successive time intervals $I_k = [l_k, l_{k+1}]$, one must control the Hölder norm of $\tilde{y}^{(k)}$, but also the "initial condition" $\tilde{y}_{l_k}^{(k)}$. The general principle of the reasoning is contained in the proof of Theorem 5.10, but it actually leans on the controls obtained in Proposition 5.7, 5.8 and 5.9. It is worth noticing that the general scheme of the proof in question, as well as the scheme of the proof of Theorem 4.3, are refered to in [14] and [13] for the study of rough PDE models.

Here is how our article is organized: we recall some basic definitions of algebraic integration at Section 2, and we adapt those notions to the convolutional context at Section 3.2. Section 4 is devoted to the simpler case of Young equations, which allows to explain our method with less technical apparatus. Then at Section 5 we move to the rough case of our Volterra equation, and explain all the details of the method we have chosen in order to solve it. Finally, we apply our theory to (standard) rough paths at Section 6.

2. Algebraic integration

This section is devoted to recall the very basic elements of the algebraic integration theory introduced in [19], in order to fix notations for the remainder of the paper.

2.1. **Increments.** As explained in [19], the extension of the integral steming from the standard differential system $dy_t^i = dx_t^j\, \sigma^{ij}(y_t)$ is based on the notion of increment, together with an elementary operator δ acting on them. The notion of increment can be introduced in the following way: for two arbitrary real numbers $\ell_2 > \ell_1 \geq 0$, a vector space V, and an integer $k \geq 1$, we denote by $\mathcal{C}_k(V)$ the set of continuous functions $g : [\ell_1, \ell_2]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \cup_{k\geq 1} \mathcal{C}_k(V)$. The operator δ alluded to above can be seen as an operator acting on k-increments, and is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V) \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i+1} g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{11}$$

where $\hat{t}_i$ means that this particular argument is omitted. Then a fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $t, u, s \in [\ell_1, \ell_2]$, we have

$$(\delta g)_{ts} = g_t - g_s, \quad \text{and} \quad (\delta h)_{tus} = h_{ts} - h_{tu} - h_{us}. \tag{12}$$

The above-mentionned ordinary system is then of course equivalent to

$$y_0 = a \quad , \quad (\delta y^i)_{ts} = \int_s^t dx_u^j\, \sigma^{ij}(y_u). \tag{13}$$

Furthermore, it is readily checked that the complex $(\mathcal{C}_*, \delta)$ is *acyclic*, i.e. $\mathcal{ZC}_{k+1}(V) = \mathcal{BC}_k(V)$ for any $k \geq 1$. In particular, the following basic property, which we label for further use, holds true:

Lemma 2.1. *Let $k \geq 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (non unique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Observe that Lemma 2.1 implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an *exact* increment of a function (i.e. a finite difference).

Let us now introduce a convenient notation for the product of increments:

Definition 2.2. *Let V and W two normed spaces and I a subinterval of $[0,T]$. If $g \in \mathcal{C}_k(I; \mathcal{L}(V,W))$ and $h \in \mathcal{C}_l(I; W)$, for some $k, l \in \mathbb{N}^*$, we define the product gh as the $(k+l-2)$-increment (with values in W) given by the formula: for all $t_1 \leq t_2 \leq \ldots \leq t_{k+l-1}$,*

$$(gh)_{t_1 \ldots t_{k+l-1}} \equiv g_{t_1 \ldots t_k} h_{t_k t_{k+1} \ldots t_{k+l-1}}. \tag{14}$$

Notice again that our future discussions will mainly rely on k-increments with $k \leq 2$, for which we will use some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu \equiv \sup_{s,t \in [\ell_1, \ell_2]} \frac{\|f_{ts}\|_V}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_1^\mu(V) = \{ f \in \mathcal{C}_2(V);\, \|f\|_\mu < \infty \}.$$

In the same way, for $h \in \mathcal{C}_3(V)$, set

$$\|h\|_{\gamma,\rho} = \sup_{s,u,t\in[\ell_1,\ell_2]} \frac{\|h_{tus}\|_V}{|u-s|^\gamma |t-u|^\rho} \tag{15}$$

$$\|h\|_\mu \equiv \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\ h = \sum_i h_i,\ 0<\rho_i<\mu \right\},$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h=\sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,z)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) \equiv \{h \in \mathcal{C}_3(V);\ \|h\|_\mu < \infty\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \cup_{\mu>1}\mathcal{C}_3^\mu(V)$, and remark that the same kind of norms can be considered on the spaces $\mathcal{ZC}_3(V)$, leading to the definition of some spaces $\mathcal{ZC}_3^\mu(V)$ and $\mathcal{ZC}_3^{1+}(V)$. In order to avoid ambiguities, we shall denote by $\mathcal{N}[f;\,\mathcal{C}_j^\kappa]$ the κ-Hölder norm on the space $\mathcal{C}_j$, for $j=1,2,3$. For $\zeta \in \mathcal{C}_j(V)$, we also set $\mathcal{N}[\zeta;\mathcal{C}_j^0(V)] = \sup_{s\in[\ell_1;\ell_2]^j}\|\zeta_s\|_V$.

With these notations in mind, the following proposition is a basic result which is at the core of our approach to path-wise integration (see [19] for the original proof of the result, based on Stokes Theorem, and [20] for a simplified version):

Theorem 2.3 (The sewing map)**.** *Let $\mu>1$. For any $h \in \mathcal{ZC}_3^\mu([0,1];V)$, there exists a unique $\Lambda h \in \mathcal{C}_2^\mu([0,1];V)$ such that $\delta(\Lambda h) = h$. Furthermore,*

$$\|\Lambda h\|_\mu \le c_\mu\, \mathcal{N}[h;\,\mathcal{C}_3^\mu(V)], \tag{16}$$

with $c_\mu = 2+2^\mu\sum_{k=1}^\infty k^{-\mu}$. This gives rise to a linear continuous map $\Lambda : \mathcal{ZC}_3^\mu([0,1];V) \to \mathcal{C}_2^\mu([0,1];V)$ such that $\delta\Lambda = Id_{\mathcal{ZC}_3^\mu([0,1];V)}$.

The following corollary gives a first relation between the structures we have just introduced and generalized integrals, in the sense that it connects the operators δ and Λ with Riemann sums.

Corollary 2.4 (Integration of small increments)**.** *For any 1-increment $g \in \mathcal{C}_2(V)$, such that $\delta g \in \mathcal{C}_3^{1+}$, set $\delta f = (Id - \Lambda\delta)g$. Then*

$$(\delta f)_{ts} = \lim_{|\Pi_{ts}|\to 0} \sum_{i=0}^n g_{t_{i+1}\,t_i},$$

where the limit is over any partition $\Pi_{ts} = \{t_0=t,\dots,t_n=s\}$ of $[t,s]$ whose mesh tends to zero. The 1-increment δf is the indefinite integral of the 1-increment g.

Proof. For any partition $\Pi_t = \{s=t_0<t_1<...<t_n=t\}$ of $[s,t]$, write

$$(\delta f)_{ts} = \sum_{i=0}^n (\delta f)_{t_{i+1}t_i} = \sum_{i=0}^n g_{t_{i+1}t_i} - \sum_{i=0}^n \Lambda_{t_{i+1}t_i}(\delta g).$$

Observe now that for some $\mu>1$ such that $\delta g \in \mathcal{C}_3^\mu$,

$$\Big\| \sum_{i=0}^n \Lambda_{t_{i+1}t_i}(\delta g) \Big\|_V \le \sum_{i=0}^n \|\Lambda_{t_{i+1}t_i}(\delta g)\|_V \le \mathcal{N}[\Lambda(\delta g);\,\mathcal{C}_2^\mu(V)]\ |\Pi_{ts}|^{\mu-1}\ |t-s|,$$

and as a consequence, $\lim_{|\Pi_{ts}|\to 0}\sum_{i=0}^n \Lambda_{t_{i+1}t_i}(\delta g) = 0$. $\square$

2.2. **Dissection of a standard rough integral.** Let us say a few words about the way the tools introduced in the previous subsection interact with each other to lead to an interpretation of the rough integral $\int_s^t dx_u^i\, z_u^i$.

In a first phase, those tools enable a real dissection of the ordinary version of the integral (when x and possibly z are differentiable). For instance, by combining the elementary decomposition $\int_s^t dx_u^i\, z_u^i = (\delta x^i)_{ts}\, z_s^i + \int_s^t dx_u^i\, (\delta z^i)_{us}$ with the relation $\delta\left(\int dx^i\,(\delta z^i)\right) = (\delta x^i)(\delta z^i)$, one deduces from Theorem 2.3 the expression

$$\int_s^t dx_u^i\, z_u^i = (\delta x^i)_{ts}\, z_s^i + \Lambda_{ts}\left((\delta x^i)(\delta z^i)\right).$$

It is now readily checked that if $x, z \in \mathcal{C}_1^\gamma$, with $\gamma > 1/2$ (Young case), the right-hand-side of the latter equality still makes sense: the development is then legitimately chosen as a definition for the rough integral.

When $\gamma \leq 1/2$, a deeper analysis of the ordinary integral is required. In order to bring the procedure to a successful result, the class of potential integrands z has to be restricted to a particular set of pre-integrated paths, that will be met again at Section 5:

Definition 2.5. *Let I a subinterval of $[0,T]$ and $x \in \mathcal{C}_1^\gamma(I;\mathbb{R}^m)$ with $\gamma > 1/3$. For any $l \in \mathbb{N}^*$, a path $y \in \mathcal{C}_1(I;\mathbb{R}^l)$ is said to be γ-controlled (by x) on I, with values in $\mathbb{R}^l$, if its increments δy can be decomposed in the following way: for all $s < t \in I$,*

$$(\delta y^i)_{ts} = (\delta x^j)_{ts} y_s^{x,ji} + y_{ts}^{\sharp,i}, \quad \text{avec } y^x \in \mathcal{C}_1^\gamma(I;\mathbb{R}^{l,m}) \text{ et } y^\sharp \in \mathcal{C}_2^{2\gamma}(I;\mathbb{R}^l). \tag{17}$$

The set of γ-controlled paths will be denoted by $\mathcal{Q}_x^\gamma(I;\mathbb{R}^l)$ and provided with the seminorm

$$\mathcal{N}[y;\mathcal{Q}_x^\gamma(I;\mathbb{R}^l)] \equiv \mathcal{N}[y;\mathcal{C}_1^\gamma(I;\mathbb{R}^l)] + \mathcal{N}[y^x;\mathcal{C}_1^{0,\gamma}(I;\mathbb{R}^{l,m})] + \mathcal{N}[y^\sharp;\mathcal{C}_2^{2\gamma}(I;\mathbb{R}^l)], \tag{18}$$

Then we define $\mathcal{Q}_x^\gamma(I;\mathbb{R}^{k,l})$ ($k \in \mathbb{N}^*$) as the set of paths $y \in \mathcal{C}_1(I;\mathbb{R}^{k,l})$ such that $y^i = y^{i\cdot} \in \mathcal{Q}_x^\gamma(I;\mathbb{R}^l)$ for all $i = 1,\ldots,k$, and we associate to the elements of this set the quantity $\mathcal{N}[y;\mathcal{Q}_x^\gamma(I;\mathbb{R}^{k,l})] \equiv \sum_{i=1}^k \mathcal{N}[y^i;\mathcal{Q}_x^\gamma(I;\mathbb{R}^l)]$.

If x is differentiable and $z \in \mathcal{Q}_x^\gamma$, a quick algebraic computation shows that, by setting $\mathbf{x}_{ts}^{\mathbf{2},ij} \equiv \int_s^t dx_u^i\,(\delta x^j)_{us}$, we get $\int_s^t dx_u^i\, z_u^i = (\delta x^i)_{ts}\, z_s^i + \mathbf{x}_{ts}^{\mathbf{2},ij}\, z_s^{x,ji} + r_{ts}$, with $\delta r = (\delta x^i)\, z^{\sharp,i} + \mathbf{x}^{\mathbf{2},ij}\,\delta z^{x,ji}$, and so

$$\int_s^t dx_u^i\, z_u^i = (\delta x^i)_{ts}\, z_s^i + \mathbf{x}_{ts}^{\mathbf{2},ij}\, z_s^{x,ji} + \Lambda_{ts}\left((\delta x^i)\, z^{\sharp,i} + \mathbf{x}^{\mathbf{2},ij}\,\delta z^{x,ji}\right). \tag{19}$$

The right-hand-side of the latter equality can now be extended to any 2-rough path $\mathbf{x} = (\delta x, \mathbf{x}^{\mathbf{2}}) \in \mathcal{C}_2^\gamma \times \mathcal{C}_2^{2\gamma}$ with $\gamma > 1/3$, that is to say to any γ-Hölder path x allowing the construction of a Lévy area $\mathbf{x}_{ts}^{\mathbf{2},ij} \equiv \int_s^t dx_u^i\,(\delta x^j)_{us}$ (see [25] for a thorough definition), a hypothesis which is for instance known to be satisfied by a fractional Brownian motion with Hurst index $H > 1/3$ (see [12] or [36]).

In fact, if one permits to restrict the class of integrands to $\mathcal{Q}_x^\gamma$, it is because the latter space is large and stable enough to make possible the interpretation and resolution of the ordinary rough system $(\delta y^i)_{ts} = \int_s^t dx_u^j\, \sigma^{ij}(y_u)$ therein, for a sufficiently smooth vector field σ . It is indeed not difficult to see that if $y \in \mathcal{Q}_x^\gamma$ and $\sigma \in \mathcal{C}^{2,\mathbf{b}}$, then $z \equiv \sigma(y) \in \mathcal{Q}_x^\gamma$, while (19) immediately shows that $\int dx\, z \in \mathcal{Q}_x^\gamma$.

All of those considerations will be kept in mind when analyzing the system (2).

3. Algebraic convolutional integration

We already announced it in the introduction: in order to reduce the dependence of equation (2) with respect to the past, we will appeal to a preliminary rewriting of the system, based on the representation of ϕ as the Fourier transform of a function $\tilde{\phi}$. The resulting formulation will be close to the model studied in [20]: just as in the latter reference, it suggests a natural adaptation of the standard algebraic formalism presented in the previous section.

3.1. **Transformation of the ordinary system.** Assume in this subsection that x is differentiable. Let us go back for a short while on the transformation sketched out in the introduction, and which started from the assumption that ϕ could be written as in (5). Note here and now that this hypothesis is actually not very restricting. Indeed, insofar as we are working with finite fixed horizon T, only the behaviour of ϕ on $[0,T]$ matters, and it is possible to replace, in (2), ϕ with a compactly supported function ϕ_T such that $\phi_{|[0,T]} = \phi_{T|[0,T]}$. If ϕ is assumed to be continuous on $\mathbb{R}$, then ϕ_T can be picked in $L^2(\mathbb{R})$, and in this case

$$\phi_T = \mathcal{F}\tilde{\phi}, \quad \text{with } \tilde{\phi} = \tilde{\phi}_T = \mathcal{F}^{-1}\phi_T \in L^2(\mathbb{R}),$$

where $\mathcal{F}$ stands for the Fourier transform. In fact, under the hypotheses of Theorem 1.1 ($\phi \in \mathcal{C}^3(\mathbb{R})$), it is easy to show that $\tilde{\phi}$ is integrable (see Proposition 6.6). Nevertheless, for the time being, we record this condition in the following hypothesis:

Hypothesis 1. *We assume, in this section and the two following, that the function ϕ admits the representation (5), for some function $\tilde{\phi} \in L^1(\mathbb{R})$.*

We are then allowed to apply Fubini Theorem and assert that the system (2) is equivalent to

$$\begin{cases} y_t^i & = a^i + \int_{\mathbb{R}} \tilde{y}_t^i(\xi)\, \tilde{\phi}(\xi)\, d\xi \\ \tilde{y}_t^i(\xi) & = \int_0^t S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v). \end{cases} \tag{20}$$

Besides, as we also evoked in the introduction, the increments $(\delta\tilde{y}^i)_{ts}(\xi) \equiv \tilde{y}_t^i(\xi) - \tilde{y}_s^i(\xi)$ are governed by the equation

$$\begin{aligned} (\delta\tilde{y}^i(\xi))_{ts} &= \int_s^t S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v) + A_{ts}(\xi) \int_0^s S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v) \\ &= \int_s^t S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v) + A_{ts}(\xi) \tilde{y}_s^j(\xi), \end{aligned}$$

where we have set

$$A_{ts}(\xi) \equiv S_{t-s}(\xi) - 1. \tag{21}$$

Notice now that the first term $\int_s^t S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v)$ above is really similar to what one obtains in the diffusion case, namely an integral of the form $\int_s^t$ (see (13)). However, the second term $A_{ts}(\xi)\tilde{y}_s(\xi)$ is a little clumsy for further expansions. Hence, a straightforward idea is to make it disappear by just setting

$$(\tilde{\delta}\tilde{y}^i)_{ts}(\xi) \equiv (\delta\tilde{y}^i)_{ts}(\xi) - A_{ts}(\xi)\tilde{y}_s^i(\xi) \tag{22}$$

Then the last equation can be read as $(\tilde{\delta}\tilde{y}^i)_{ts}(\xi) = \int_s^t S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v)$, and the system (20) becomes

$$\begin{cases} y_t^i & = a^i + \int_{\mathbb{R}} \tilde{y}_t^i(\xi)\, \tilde{\phi}(\xi)\, d\xi \\ (\tilde{\delta}\tilde{y}^i)_{ts}(\xi) & = \int_s^t S_{t-v}(\xi) dx_v^j\, \sigma^{ij}(y_v), \end{cases} \tag{23}$$

with the initial condition $\tilde{y}_0 \equiv 0$. In the sequel, we shall essentially focus on the path $\tilde{y}$, by merging the two equations of the last system into a single one:

$$\tilde{y}_0 = 0 \quad , \quad (\tilde{\delta}\tilde{y}^i)_{ts}(\xi) = \int_s^t S_{t-v}(\xi)\, dx_v^j \left[\sigma^{ij} \circ T_{a,\tilde{\phi}}\right](\tilde{y}_v), \tag{24}$$

where the operator $T_{a,\phi}$ is defined by

$$T_{a,\tilde{\phi}}(\varphi) \equiv a + \int_{\mathbb{R}} d\eta\, \tilde{\phi}(\eta)\varphi(\eta). \tag{25}$$

The original solution path y can then be recovered in an obvious way, so that it will be sufficient to solve the Volterra equation under the more suitable form (24), with a right-hand-side written as an integral from s to t against x (compare with (13)).

Actually, if we take the liberty of focusing on $\tilde{\delta}$ rather than on the standard increment δ, it is because the former operator also makes possible the building of an integration theory, by means of an inversion mapping similar to Λ, and that will be denoted by $\tilde{\Lambda}$ (see Proposition 3.8). This is what we mean to elaborate on in the following subsections.

3.2. **Convolutional increments.** Notice that, due to the fact that $S_{t_1-t_2}(\xi)$ is studied only for $t_1 > t_2$, our integration domains will be of the form $\mathcal{S}_n = \mathcal{S}_n([\ell_1, \ell_2])$, where $\mathcal{S}_n$ stands for the n-simplex

$$\mathcal{S}_n = \{(t_1, \dots, t_n) : \ell_2 \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq \ell_1\}.$$

For any Banach space E, the notation $\mathcal{C}_n([\ell_1, \ell_2]; E)$ will henceforth refer to the set of paths h which are continuous on $\mathcal{S}_n$, with values in E, and such that $h_{t_1 \dots t_k} = 0$ if there exist $i \neq j$ for which $t_i = t_j$.

According to the (first) definition (22), $\tilde{\delta}$ is supposed to act on functional-valued paths. Let us anticipate here the next sections by introducing the spaces of functions that will spontaneously arise during the study of (24) (see for instance Proposition 4.2). Those are the $\mathcal{L}^1$-type spaces induced by the norm

$$\mathcal{N}[\tilde{g}; \mathcal{L}_\beta(V)] = \mathcal{N}[\tilde{g}; \mathcal{L}_{\beta,\tilde{\phi}}(V)] \equiv \int_{\mathbb{R}} d\xi\, |\tilde{\phi}(\xi)|(1 + |\xi|^\beta)\|\tilde{g}(\xi)\|_V, \tag{26}$$

where $\beta > 0$ is a fixed parameter and V a Euclidian space. Then we define

$$\tilde{\mathcal{C}}_{k,\beta}(I; V) \equiv \mathcal{C}_k(I; \mathcal{L}_\beta(V)). \tag{27}$$

The standard incremental operator δ acts on those spaces through the obvious formula:

$$\text{If } \tilde{h} \in \tilde{\mathcal{C}}_{k,\beta}(I; V), \quad (\delta\tilde{h})_{t_1 \dots t_{k+1}}(\xi) \equiv \delta(\tilde{h}(\xi))_{t_1 \dots t_{k+1}}, \quad \xi \in \mathbb{R}. \tag{28}$$

As for $\tilde{\delta}$, it can be naturally extended to any $\tilde{\mathcal{C}}_{k,\beta}(I; V)$ ($k \in \mathbb{N}^*$):

Definition 3.1. *Let I an interval of $\mathbb{R}^+$ and V a Euclidian space. For any $\beta > 0$, we define the sequence of operators $\tilde{\delta}_k : \tilde{\mathcal{C}}_{k,\beta}(I; V) \to \tilde{\mathcal{C}}_{k+1,\beta}(I; V)$ by the formula: if $\tilde{h} \in \tilde{\mathcal{C}}_{k,\beta}(I; V)$, then for all $\xi \in \mathbb{R}$,*

$$(\tilde{\delta}_k \tilde{h})_{t_1 \dots t_{k+1}}(\xi) \equiv (\delta_k \tilde{h})_{t_1 \dots t_{k+1}}(\xi) - A_{t_1 t_2}(\xi)\, \tilde{h}_{t_2 \dots t_{k+1}}(\xi), \quad (t_1, \dots t_{k+1}) \in \mathcal{S}_{k+1}(I). \tag{29}$$

In particular, if $s < u < t \in I$,

$$(\tilde{\delta}_1 \tilde{h})_{ts}(\xi) = \tilde{h}_t(\xi) - S_{t-s}(\xi)\, \tilde{h}_s(\xi) \quad , \quad (\tilde{\delta}_2 \tilde{h})_{tus}(\xi) = \tilde{h}_{ts}(\xi) - \tilde{h}_{tu}(\xi) - S_{t-u}(\xi)\, \tilde{h}_{us}(\xi).$$

For sake of clarity, we shall use the same notation $\tilde{\delta}$ for the operators $\tilde{\delta}_k$, $k \in \mathbb{N}^*$.

Remark 3.2. In the rest of the paper, we will explicitly write down the "space" variable ξ only when there might be a confusion. Thus, we will for instance simply write $\tilde{\delta}\tilde{h} = \delta\tilde{h} - a\tilde{h}$.

The convention given by (14) for products of increments can be translated in this context as:

Lemma 3.3. *If $\tilde{M} \in \tilde{\mathcal{C}}_{n,\beta}(I;\mathbb{R}^{k,l})$ and $L \in \mathcal{C}_m(I;\mathbb{R}^l)$, then the product $\tilde{M}L$, defined by the relation*

$$(\tilde{M}L)_{t_1\dots t_{m+n-1}}(\xi) \equiv \tilde{M}_{t_1\dots t_n}(\xi)\, L_{t_n\dots t_{m+n-1}},$$

belongs to $\tilde{\mathcal{C}}_{m+n-1,\beta}(I;\mathbb{R}^k)$. Moreover, when $n=2$, the following algebraic relations are satisfied:

$$\delta(\tilde{M}L) = \delta\tilde{M}\, L - \tilde{M}\,\delta L, \quad et \quad \tilde{\delta}(\tilde{M}L) = \tilde{\delta}\tilde{M}\, L - \tilde{M}\,\delta L. \tag{30}$$

Proof. The first part of the assertion is obvious. As for the algebraic relations when $n=2$, the first one is immediate, while for the second one, it suffices to notice that

$$\begin{aligned}
\tilde{\delta}(\tilde{M}L)_{t_1\dots t_{m+2}} &= \delta(\tilde{M}L)_{t_1\dots t_{m+2}} - A_{t_1t_2}\tilde{M}_{t_2t_3}L_{t_3\dots t_{m+2}} \\
&= (\delta\tilde{M})_{t_1t_2t_3}L_{t_3\dots t_{m+2}} - \tilde{M}_{t_1t_2}(\delta L)_{t_2\dots t_{m+2}} - A_{t_1t_2}\tilde{M}_{t_2t_3}L_{t_3\dots t_{m+2}} \\
&= [(\delta\tilde{M})_{t_1t_2t_3} - A_{t_1t_2}\tilde{M}_{t_2t_3}]L_{t_3\dots t_{m+2}} - (\tilde{M}\,\delta L)_{t_1\dots t_{m+2}}.
\end{aligned}$$

□

With those notations and preliminary results in hand, we are in position to prove that the starting property of standard algebraic integration (summed up in Section 2), namely the cohomological relation $\delta\delta = 0$, remains true for $\tilde{\delta}$:

Proposition 3.4. $\tilde{\delta}\tilde{\delta} = 0$. *More precisely, for any $\beta > 0$ and any $k \in \mathbb{N}^*$, $Im\,\tilde{\delta}_{|\tilde{\mathcal{C}}_{k,\beta}(I;V)} = Ker\,\tilde{\delta}_{|\tilde{\mathcal{C}}_{k+1,\beta}(I;V)}$.*

Proof. If $\tilde{F} \in \tilde{\mathcal{C}}_{k,\beta}(I;V)$, then using the relation $\delta\delta = 0$ and the result of Lemma 3.3, we deduce

$$\begin{aligned}
\tilde{\delta}\tilde{\delta}\tilde{F} &= (\delta - A)\,[(\delta - A)\,\tilde{F}] = \delta\delta\tilde{F} - \delta(A\,\tilde{F}) - A\,\delta\tilde{F} + A\,A\,\tilde{F} \\
&= -\delta A\,\tilde{F} + A\,\delta\tilde{F} - A\,\delta\tilde{F} + A\,A\,\tilde{F} = A\,A\,\tilde{F} - \delta A\,\tilde{F}.
\end{aligned}$$

It is then readily checked, owing to the additivity $S_t \cdot S_{t'} = S_{t+t'}$, that

$$(\delta A)_{tus} = A_{tu}\,A_{us}, \qquad (t,u,s) \in \mathcal{S}_3(I),$$

which gives $\tilde{\delta}\tilde{\delta}\tilde{F} = 0$.

Now, if $\tilde{C} \in \tilde{\mathcal{C}}_{k+1,\beta}(I;V)$ is such that $\tilde{\delta}\tilde{C} = 0$, we set $\tilde{B}_{t_1\dots t_n} \equiv \tilde{C}_{t_1\dots t_n s}$, for some arbitrary time $s \in I$. Then

$$\begin{aligned}
[\tilde{\delta}\tilde{B}]_{t_1\dots t_{n+1}} &= [\delta\tilde{C}]_{t_1\dots t_{n+1}s} + (-1)^{n+1}\tilde{C}_{t_1\dots t_{n+1}} - A_{t_1t_2}\tilde{C}_{t_2\dots t_n s} \\
&= [\tilde{\delta}\tilde{C}]_{t_1\dots t_{n+1}s} + (-1)^{n+1}\tilde{C}_{t_1\dots t_{n+1}} = (-1)^{n+1}\tilde{C}_{t_1\dots t_{n+1}}.
\end{aligned}$$

Therefore, by setting $\tilde{D} \equiv (-1)^{n+1}\tilde{B}$, we get $\tilde{\delta}\tilde{D} = \tilde{C}$.

□

Remark 3.5. A straightforward iteration of the relation $\tilde{\delta}\tilde{\delta} = 0$ leads to the formula: for any partition $\{s = t_0 < t_1 < \ldots < t_n = t\}$ of $[s,t]$, for any $\tilde{f} \in \tilde{\mathcal{C}}_{1,\beta}([s,t];V)$,

$$(\tilde{\delta}\tilde{f})_{ts} = \sum_{i=0}^{n-1} S_{t-t_{i+1}} \cdot (\tilde{\delta}\tilde{f})_{t_{i+1}t_i}. \tag{31}$$

This kind of decomposition will be appealed to several times in the sequel, especially in the proofs of Lemma 3.7 and Corollary 3.9. In some way, this is the convolutional analog of the usual telescopic sum $(\delta f)_{ts} = \sum_{i=0}^{n-1} (\delta f)_{t_{i+1}t_i}$.

The cochain complex $(\tilde{\mathcal{C}}_{k,\beta}(I;V), \tilde{\delta})$ will stand for the structure at the base of all the constructions in this paper. Let us try to give an idea of the relevance of this structure in the context of equation (24). To this end, we set, for two smooth paths $f : [0,T] \to W$, $g : [0,T] \to \mathcal{L}(W,V)$,

$$\mathcal{J}_{ts}(\tilde{d}g\, f)(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dg_u\, f_u, \quad \xi \in \mathcal{A}, \tag{32}$$

and for any smooth $h : [0,T]^2 \to W$,

$$\mathcal{J}_{ts}(\tilde{d}g\, h)(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dg_u\, h_{us}. \tag{33}$$

The usual Chasles relation $\delta\left(\int dg\, f\right) = 0$ becomes here:

Proposition 3.6. *With the notations (32) and (33), one has, if $f : [0,T] \to W$ and $g : [0,T] \to \mathcal{L}(W,V)$ stand for two differentiable paths,*

$$\tilde{\delta}\left(\mathcal{J}(\tilde{d}g\, f)\right) = 0 \quad , \quad \tilde{\delta}\left(\mathcal{J}(\tilde{d}g\, \delta f)\right) = \mathcal{J}(\tilde{d}g)\, \delta f. \tag{34}$$

Proof. This is a matter of straightforward computations: if $s < u < t$,

$$\tilde{\delta}\left(\mathcal{J}(\tilde{d}g\, f)\right)_{tus} = \mathcal{J}_{ts}(\tilde{d}g\, f) - \mathcal{J}_{tu}(\tilde{d}g\, f) - S_{t-u} \cdot \mathcal{J}_{us}(\tilde{d}g\, f),$$

and $S_{t-u} \cdot \mathcal{J}_{us}(\tilde{d}g\, f) = \int_s^u S_{t-v}\, dg_v\, f_v$, which easily yields $\tilde{\delta}\left(\mathcal{J}(\tilde{d}g\, f)\right) = 0$. In the same way,

$$\tilde{\delta}\left(\mathcal{J}(\tilde{d}g\, \delta f)\right) = \int_u^t S_{t-v}\, dg_v\, (\delta f)_{vs} - \int_u^t S_{t-v}\, dg_v\, (\delta f)_{vu} = \left(\int_u^t S_{t-v}\, dg_v\right) (\delta f)_{us}.$$

□

3.3. **Convolutional Hölder spaces and $\tilde{\Lambda}$ map.** In order to cope with (24), the notion of generalized Hölder path presented in the previous section has to be adapted to the convolutional formalism we have just introduced. We first define, for all (fixed) parameters $\mu, \beta, \gamma > 0$, any interval I of $\mathbb{R}^+$ and any Euclidian space V,

$$\tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V) \equiv \{\tilde{y} \in \tilde{\mathcal{C}}_{2,\beta}(I;V) :\ \mathcal{N}[\tilde{y}; \tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V)] \equiv \sup_{s<t\in I} \frac{\mathcal{N}[\tilde{y}_{ts}; \mathcal{L}_\beta(V)]}{|t-s|^{\mu}} < \infty\},$$

$$\tilde{\mathcal{C}}^{\mu}_{1,\beta}(I;V) \equiv \{\tilde{y} \in \tilde{\mathcal{C}}_{1,\beta}(I;V) :\ \tilde{\delta}\tilde{y} \in \tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V)\}. \tag{35}$$

As for paths with three variables, we define, as in the standard case, the intermediate space $\tilde{\mathcal{C}}^{(\gamma,\beta)}_{3,\beta}(I;V)$ induced by the norm

$$\mathcal{N}[\tilde{h}; \tilde{\mathcal{C}}^{(\gamma,\rho)}_{3,\beta}(I;V)] \equiv \sup_{s<u<t\in I} \frac{\mathcal{N}[\tilde{h}_{tus}; \mathcal{L}_\beta(V)]}{|t-u|^{\gamma}\, |u-s|^{\rho}},$$

and then set $\tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V) \equiv \oplus_{0\le\alpha\le\mu} \tilde{\mathcal{C}}^{\alpha,\mu-\alpha}_{3,\beta}(I;V)$. We also provide the latter space with the norm

$$\mathcal{N}[\tilde{h}; \tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V)] \equiv \inf\left\{ \sum_i \mathcal{N}[h_i; \tilde{\mathcal{C}}^{(\rho_i,\mu-\rho_i)}_{3,\beta}(I;V)];\ h = \sum_i h_i,\ 0 < \rho_i < \mu \right\}.$$

It is worth noticing that the elementary results asserting that $\operatorname{Im}\delta_1 \cap \mathcal{C}_2^{\mu}(V) = \{0\}$ if $\mu > 1$, admits a direct analog:

Lemma 3.7. *Fix $\beta > 0$. If $\mu > 1$, then $\operatorname{Im}\tilde{\delta}_{|\tilde{\mathcal{C}}_{1,\beta}(I;V)} \cap \tilde{\mathcal{C}}^{\mu}_{2,\beta}(V) = \{0\}$.*

Proof. Let $\tilde{M} = \tilde{\delta}\tilde{f} \in \operatorname{Im}\tilde{\delta}_{|\tilde{\mathcal{C}}_{1,\beta}(I;V)} \cap \tilde{\mathcal{C}}^{\mu}_{2,\beta}(V)$. According to (31), we can write, for all $s < t$, $\tilde{M}_{ts} = \sum_{i=0}^{n-1} S_{t-t_{i+1}} \cdot \tilde{M}_{t_{i+1}t_i}$, for any partition $\Pi_{ts} = \{s = t_0 < t_1 < \ldots < t_n = t\}$ of $[s,t]$. Since $|S_t(\xi)| = 1$, this entails

$$\mathcal{N}[\tilde{M}_{ts}; \mathcal{L}_\beta(V)] \le \sum_{i=0}^{n-1} \mathcal{N}[\tilde{M}_{t_{i+1}t_i}; \mathcal{L}_\beta(V)] \le \mathcal{N}[\tilde{M}; \tilde{\mathcal{C}}^{\mu}_{2,\beta}(V)]\, |t-s|\, |\Pi_{ts}|^{\mu-1},$$

and the latter estimate tends to 0 as mesh $|\Pi_{ts}|$ tends to 0. □

With all of those results in hand, it is now easy to follow the same lines as in the proof of Theorem 2.3 in order to establish the existence of an inverse operator for $\tilde{\delta}$ (see [20] for a similar adaptation):

Proposition 3.8. *Let $\mu > 1, \beta > 0$, I an interval of $\mathbb{R}^+$ and V a Euclidian space. For all $\tilde{h} \in \operatorname{Ker}\delta_{|\tilde{\mathcal{C}}_{3,\beta}(I;V)} \cap \tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V)$, there exists a unique path $\tilde{\Lambda}\tilde{h} \in \tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V)$ such that $\tilde{\delta}(\tilde{\Lambda}\tilde{h}) = \tilde{h}$. Moreover, the following contraction property holds true:*

$$\mathcal{N}[\tilde{\Lambda}\tilde{h}; \tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V)] \le c_\mu\, \mathcal{N}[\tilde{h}; \tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V)], \tag{36}$$

with c_μ a constant that only depends on μ. This statement gives birth to a continuous linear mapping

$$\tilde{\Lambda} : \operatorname{Ker}\delta_{|\tilde{\mathcal{C}}_{3,\beta}(I;V)} \cap \tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V) \to \tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V)$$

such that

$$\tilde{\delta}\tilde{\Lambda} = Id_{\operatorname{Ker}\delta_{|\tilde{\mathcal{C}}_{3,\beta}(I;V)} \cap \tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V)} \quad \text{and} \quad \tilde{\Lambda}\tilde{\delta} = Id_{\tilde{\mathcal{C}}^{\mu}_{2,\beta}(I;V)}. \tag{37}$$

We also have the following equivalent of Corollary 2.4 at our disposal:

Corollary 3.9. *Let $\tilde{g} \in \tilde{\mathcal{C}}_{2,\beta}(I;V)$ such that $\tilde{\delta}\tilde{g} \in \tilde{\mathcal{C}}^{\mu}_{3,\beta}(I;V)$, for some coefficient $\mu > 1$. If $\tilde{\delta}\tilde{f} \equiv (Id - \tilde{\Lambda}\tilde{\delta})\tilde{g}$, then*

$$(\tilde{\delta}\tilde{f})_{ts} = \lim_{|\Pi_{ts}| \to 0} \sum_{i=0}^{n} S_{t-t_{i+1}} \cdot \tilde{g}_{t_{i+1}t_i} \quad \text{in } \mathcal{L}_\beta,$$

where the limit is over any partition $\Pi_{ts} = \{t_0 = t, \ldots, t_n = s\}$ of $[t,s]$ whose mesh tends to zero.

Proof. Here again, it suffices to use the same arguments as in the standard case (Corollary 2.4), starting from the decomposition (31). □

4. The Young case

Remember that we first wish to solve the system in the form (24), which can also be written, with the notation (32), as

$$\tilde{y}_0 \equiv 0, \qquad \tilde{\delta}\tilde{y}^i = \mathcal{J}\left(\tilde{d}x^j\, \sigma^{ij}(y)\right), \quad y_u = T_{a,\tilde{\phi}}(\tilde{y}_u) = a + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\tilde{y}_u(\xi). \tag{38}$$

For the time being, the right-hand-side of the latter equality only makes sense for a differentiable path x. The aim of this section is to extend the definition of the equation to a γ-Hölder path x with $\gamma > 1/2$, and then solve it with the resulting interpretation.

To this end, we will follow the same general strategy as in the standard case (Subsection 2.2), which begins with a dissection of the ordinary integral.

4.1. **Heuristic considerations and interpretation of the system.** Let us assume for the moment that x and $\tilde{y}$ are differentiable (in time) and let us successively set $y \equiv T_{a,\tilde{\phi}}(\tilde{y})$, $z^{ij} \equiv \sigma^{ij}(y)$, so that the integral at stake here is given by $\mathcal{J}(\tilde{d}x^j\, z^{ij})$.

Before we turn to the dissection procedure for this integral, it is important to ponder about the regularity one can expect for z, or equivalently for y (we will suppose that σ is smooth enough), when x and $\tilde{y}$ become non-differentiable. To answer the question, observe the decomposition

$$(\delta y^i)_{ts} = \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, (\delta \tilde{y})_{ts}(\xi) = \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, (\tilde{\delta} \tilde{y}^i)_{ts}(\xi) + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, A_{ts}(\xi)\, \tilde{y}^i_s(\xi). \tag{39}$$

As $\tilde{y}$ stands for the (potential) solution of (38) and $|S_t(\xi)| = 1$, $\tilde{\delta}\tilde{y}$ is expected to inherit the regularity of x, or otherwise stated $|(\tilde{\delta}\tilde{y})_{ts}(\xi)| \le c_x\, |t-s|^{\gamma}$ (uniformly in ξ), which would lead, as we have assumed $\int_{\mathbb{R}} d\xi\, |\tilde{\phi}(\xi)| < \infty$ (Hypothesis 1), to an estimate such that: $|\int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, (\tilde{\delta}\tilde{y}^i)_{ts}(\xi)| \le c_x\, |t-s|^{\gamma}$.
To retrieve $|t-s|$-increments from the term $\int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, A_{ts}(\xi)\, \tilde{y}^i_s(\xi)$, we shall lean on the elementary estimate

$$|A_{ts}(\xi)| = |S_{t-s}(\xi) - 1| \le c_{\gamma}\, |t-s|^{\gamma}\, |\xi|^{\gamma}\,. \tag{40}$$

This is where the spaces $\mathcal{L}_{\beta}(V)$ defined by (26) occurs. Indeed, from (40), one has

$$\left| \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, A_{ts}(\xi) \tilde{y}_s(\xi) \right| \le c_{\gamma}\, |t-s|^{\gamma}\, \mathcal{N}[\tilde{y}_s; \mathcal{L}_{\gamma}(\mathbb{R}^d)]. \tag{41}$$

Going back to decomposition (39), we see that, by starting with a path $\tilde{y}$ that takes values in $\mathcal{L}_{\gamma}(\mathbb{R}^d)$, we should retrieve a path y, and then a path z, both Hölder-continuous *in the classical sense.*

Those considerations (that will be precised through Proposition 4.2) will help us in the dissection procedure of the integral $\mathcal{J}(\tilde{d}x^j\, z^{ij})$. Indeed, we will not hesitate anymore to let the *standard* increment δz come (back) into the picture, and we will thus start, just as in the diffusion case, with the decomposition (x is still assumed to be differentiable)

$$\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij}) = \mathcal{J}(\tilde{d}\tilde{x}^j)\, z^{ij}_s + \mathcal{J}_{ts}(\tilde{d}\tilde{x}^j\, \delta z^{ij}), \tag{42}$$

where $\mathcal{J}_{ts}(\tilde{d}x^j) \equiv \mathcal{J}_{ts}(\tilde{d}x^j\, 1) = \int_s^t S_{t-u}\, dx^j_u$. When x becomes rough (that is to say γ-Hölder with $0 < \gamma < 1$), the integral $\int_s^t S_{t-u}\, dx^j_u$ can still be understood as a Young integral ([41]). In the spirit of the rough paths methodology and by anticipating the computations of Proposition 4.1 and Theorem 4.3 below, we will make the following more precise hypothesis:

Hypothesis 2. *Let $x \in \mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^m)$, with $\gamma > 1/2$. We admit the existence of a sequence x^{ε} of differentiable paths that satisfies*

$$\mathcal{N}[x^{\varepsilon} - x; \mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^m)] \xrightarrow{\varepsilon \to 0} 0,$$

and such that the associated sequence of paths

$$\tilde{X}^{x^{\varepsilon},i}_{ts}(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dx^{\varepsilon,i}_u$$

converges to $\tilde{X}^{x,i}_{ts}(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dx^i_u$ *(understood as a Young integral) w.r.t the topology of the space* $\tilde{\mathcal{C}}^{\gamma}_{2,\gamma}([0,T];\mathbb{R}^m)$*. In particular,*

$$\tilde{X}^x \in \mathcal{C}^{\gamma}_{2,\gamma}([0,T];\mathbb{R}^m) \quad \text{and} \quad \tilde{\delta}\tilde{X}^x = 0.$$

If x *is differentiable, we assume that this result holds true for* $x^{\varepsilon} \equiv x$.

Proposition 4.1. *Let* $x : [0,T] \to \mathbb{R}^m$ *a path that satisfies Hypothesis 2,* I *a subinterval of* $[0,T]$*. For any* $z \in \mathcal{C}^{\gamma}_1(I;\mathbb{R}^{d,m})$ *and* $\xi \in \mathbb{R}$*, set*

$$\mathcal{J}(\tilde{d}x^j\, z^{ij})(\xi) \equiv \tilde{X}^{x,j}(\xi)\, z^{ij} + \tilde{\Lambda}(\tilde{X}^{x,j}\, \delta z^{ij})(\xi) = (Id - \tilde{\Lambda}\tilde{\delta})(\tilde{X}^{x,j}\, z^{ij})(\xi). \tag{43}$$

Then

(1) $\mathcal{J}(\tilde{d}x^j\, z^{ij})$ *is well-defined as an element of* $\tilde{\mathcal{C}}^{\gamma}_{2,\gamma}(I;\mathbb{R}^d)$*, and it coincides with the usual Riemann integral* $\int_s^t S_{t-v}(\xi)\, dx_v\, z_v$ *when* x *is differentiable.*

(2) *The following estimate holds true (remember that we have set* $\mathcal{N}[z;\mathcal{C}^0_1(I;\mathbb{R}^{d,m})] \equiv \sup_{s\in I} |z_s|$*):*

$$\mathcal{N}[\mathcal{J}(\tilde{d}x\, z);\tilde{\mathcal{C}}^{\gamma}_{2,\gamma}(I;\mathbb{R}^d)] \le c_x \left\{ \mathcal{N}[z;\mathcal{C}^0_1(I;\mathbb{R}^{d,m})] + |I|^{\gamma}\, \mathcal{N}[z;\mathcal{C}^{\gamma}_1(I;\mathbb{R}^{d,m})] \right\}. \tag{44}$$

(3) *For all* $s < t \in I$*,*

$$\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij}) = \lim_{|\Pi_{ts}|\to 0} \sum_{k=0}^{n-1} S_{t-t_{k+1}} \cdot \tilde{X}^{x,j}_{t_{k+1},t_k}\, z^{ij}_{t_k} \quad \text{in } \mathcal{L}_{\gamma}, \tag{45}$$

where the limit is taken over any partition $\Pi_{ts} = \{t_0 = t, \dots, t_n = s\}$ *of* $[s,t]$ *whose mesh tends to* 0.

Proof. To show that the increment defined by (43) coincides with the Riemann integral $\int_s^t S_{t-u}(\xi)\, dx^j_u\, z^{ij}_u$ in case x is differentiable, let us go back to the decomposition (42), that can also be written as

$$\mathcal{J}_{ts}(\tilde{d}x^j\, \delta z^{ij}) = \mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij}) - \tilde{X}^{x,j}_{ts}\, z^{ij}_s.$$

By applying $\tilde{\delta}$ to the two sides of the relation, and then using (34) and (30), we get

$$\tilde{\delta}\left(\mathcal{J}(\tilde{d}x^j\, z^{ij})\right) = -\tilde{\delta}\tilde{X}^{x,j}\, z^{ij} + \tilde{X}^{x,j}\delta z^{ij} = \tilde{X}^{x,j}\, \delta z^{ij},$$

and so, via (37),

$$\mathcal{J}(\tilde{d}x^j\, \delta z^{ij}) = \tilde{\Lambda}\left(\tilde{X}^{x,j}\, \delta z^{ij}\right),$$

which enables to recover (43). The fact that formula (43) is well-defined in $\tilde{\mathcal{C}}^{\gamma}_{2,\gamma}$ is a straightforward consequence of Hypothesis 2. Indeed, owing to the latter hypothesis, we know that $\tilde{X}^x\, \delta z \in \mathcal{C}^{2\gamma}_{3,\gamma}(I;\mathbb{R}^d) \cap \operatorname{Ker} \delta_{|\mathcal{C}_{2,\gamma}(I;\mathbb{R}^d)}$, and we are thus in position to apply $\tilde{\Lambda}$. The estimate (44) is then due to the contraction property (36). As for the expression (45), it stems from Corollary 3.9.

□

In order to give sense to the system (38) through the definition (43), we will rely on the following proposition, which actually summarizes the above considerations:

Proposition 4.2. *Let $I=[l_1,l_2]$ a subinterval of $[0,T]$ and $\sigma\in\mathcal{C}^{2,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$. For any $\tilde{y}\in\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}(I;\mathbb{R}^d)$, we set $y\equiv T_{a,\tilde{\phi}}(\tilde{y})$ and define*

$$\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}(I;\mathbb{R}^d)]\equiv\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0}_{1,\gamma}(I;\mathbb{R}^d)]+\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}(I;\mathbb{R}^d)],$$

with $\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0}_{1,\gamma}(I;\mathbb{R}^d)]\equiv\sup_{s\in I}\mathcal{N}[\tilde{y}_s;\mathcal{L}_\gamma(\mathbb{R}^d)]$. Then $\sigma(y)\in\mathcal{C}^{\gamma}_1(I;\mathbb{R}^{d,m})$ and

$$\mathcal{N}[\sigma(y);\mathcal{C}^{\gamma}_1(I;\mathbb{R}^{d,m})]\le c_\sigma\,\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}(I;\mathbb{R}^d)].\tag{46}$$

Moreover, if $\tilde{y}^{(1)},\tilde{y}^{(2)}\in\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}(I;\mathbb{R}^d)$ are such that $\tilde{y}^{(1)}_{l_1}=\tilde{y}^{(2)}_{l_1}$, then

$$\mathcal{N}[\sigma(y^{(1)})-\sigma(y^{(2)});\mathcal{C}^{0}_1(I;\mathbb{R}^{d,m})]\le c_\sigma\ |I|^\gamma\ \mathcal{N}[\tilde{y}^{(1)}-\tilde{y}^{(2)};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}(I;\mathbb{R}^d)],\tag{47}$$

$$\mathcal{N}[\sigma(y^{(1)})-\sigma(y^{(2)});\mathcal{C}^{\gamma}_1(I;\mathbb{R}^{d,m})]\le c_\sigma\left\{1+\mathcal{N}[\tilde{y}^{(2)};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}(I;\mathbb{R}^d)]\right\}\mathcal{N}[\tilde{y}^{(1)}-\tilde{y}^{(2)};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}(I;\mathbb{R}^d)].\tag{48}$$

Proof. By using (40), we get

$$\begin{aligned}|\delta(\sigma(y))_{ts}| &\le \|D\sigma\|_\infty\int_{\mathbb{R}}d\xi\,|\tilde{\phi}(\xi)|\,|(\delta\tilde{y})_{ts}(\xi)|\\ &\le \|D\sigma\|_\infty\left\{\int_{\mathbb{R}}d\xi\,|\tilde{\phi}(\xi)||(\tilde{\delta}\tilde{y})_{ts}(\xi)|+\int_{\mathbb{R}}d\xi\,|\tilde{\phi}(\xi)|\,|A_{ts}(\xi)|\,|\tilde{y}_s(\xi)|\right\}\\ &\le c_\gamma\|D\sigma\|_\infty\,|t-s|^\gamma\left\{\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}]+\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0}_{1,\gamma}]\right\},\end{aligned}$$

which corresponds to (46). The inequality (47) can be obtained in the same way, after noticing that, for any $s\in I$,

$$\left|\sigma(y^{(1)}_s)-\sigma(y^{(2)}_s)\right|\le\|D\sigma\|_\infty\int_{\mathbb{R}}d\xi\,|\tilde{\phi}(\xi)|\left|\delta(\tilde{y}^{(1)}-\tilde{y}^{(2)})_{s\ell_1}(\xi)\right|.$$

As for (48), this is a consequence of the classical estimate

$$\begin{aligned}\left|\delta(\sigma(y^{(1)})-\sigma(y^{(2)}))_{ts}\right|\le\|D\sigma\|_\infty\left|\delta(y^{(1)}-y^{(2)})_{ts}\right|+\|D^2\sigma\|_\infty\left|\delta(y^{(2)})_{ts}\right|\\ \left(\left|y^{(1)}_t-y^{(2)}_t\right|+\left|y^{(1)}_s-y^{(2)}_s\right|\right).\end{aligned}$$

□

4.2. **Solving the equation.** Proposition 4.1, together with Proposition 4.2, provides a reasonable interpretation of (38). We can now state the main result of this section:

Theorem 4.3. *Let x a path that satisfies Hypothesis 2. If $\sigma\in\mathcal{C}^{2,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$, then the equation (38), interpreted with Propositions 4.1 and 4.2, admits a unique solution in the space $\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,T];\mathbb{R}^d)$ defined by (35).*

Proof. Consider a constant $\varepsilon>0$, $l\in\mathbb{N}$, and assume that we have already constructed a solution $\tilde{y}^{(l)}\in\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon])$. If $l=0$, then $\tilde{y}^{(0)}=\tilde{y}^{(0)}_0=0$. The proof will consist in showing that one can extend $\tilde{y}^{(l)}$ into a solution $\tilde{y}^{(l+1)}\in\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,(l+1)\varepsilon])$, by means of a fixed-point argument.

Step 1: Existence of invariant balls. Let $\tilde{y}\in\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,(l+1)\varepsilon])$ such that $\tilde{y}_{|[0,l\varepsilon]}=\tilde{y}^{(l)}$, and denote by $\tilde{z}=\Gamma(\tilde{y})$ the element of $\tilde{\mathcal{C}}_{1,\gamma}([0,(l+1)\varepsilon])$ characterized by $\tilde{z}_{|[0,l\varepsilon]}=\tilde{y}^{(l)}$ and for all $s,t\in[0,(l+1)\varepsilon]$, $(\tilde{\delta}\tilde{z})_{ts}=\mathcal{J}_{ts}\left(\tilde{d}x\,\sigma(y)\right)$, where, as in Proposition 4.2, $y\equiv T_{a,\tilde{\phi}}(\tilde{y})$ (remember the notation (25)).

First, the estimate (44) provides

$$\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([l\varepsilon,(l+1)\varepsilon])] \le c_x \left\{ \mathcal{N}[\sigma(y);\mathcal{C}^0_1([0,(l+1)\varepsilon])] + \varepsilon^{\gamma}\,\mathcal{N}[\sigma(y);\mathcal{C}^{\gamma}_1([0,(l+1)\varepsilon])] \right\},$$

which, together with (46), gives

$$\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([l\varepsilon,(l+1)\varepsilon])] \le c^1_{x,\sigma} \left\{ 1 + \varepsilon^{\gamma}\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])] \right\}.$$

If $0 \le s \le l\varepsilon \le t \le (l+1)\varepsilon$, we use (31) to deduce

$$\begin{aligned} \mathcal{N}[(\tilde{\delta}\tilde{z})_{ts};\mathcal{L}_{\gamma}] &\le \mathcal{N}[(\tilde{\delta}\tilde{z})_{t,l\varepsilon};\mathcal{L}_{\gamma}] + \mathcal{N}[(\tilde{\delta}\tilde{z})_{l\varepsilon,s};\mathcal{L}_{\gamma}] \\ &\le 2\max\left(\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([l\varepsilon,(l+1)\varepsilon]), \mathcal{N}[\tilde{y}^{(l)};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon])] \right) |t-s|^{\gamma}. \end{aligned} \tag{49}$$

Besides, for any $s \in [0,(l+1)\varepsilon]$, $\tilde{z}_s = (\tilde{\delta}\tilde{z})_{s0}$, and so

$$\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^0_{1,\gamma}([0,(l+1)\varepsilon])] \le \mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])]T^{\gamma}. \tag{50}$$

We are thus led to set

$$\begin{aligned} \varepsilon &\equiv \left(4c^1_{x,\sigma}(1+T^{\gamma})\right)^{-1/\gamma} \\ N_{l+1} &\equiv \max\left(2(1+T^{\gamma})\mathcal{N}[\tilde{y}^{(l)};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon])], 4c^1_{x,\sigma}(1+T^{\gamma}) \right). \end{aligned}$$

Indeed, for such values, it is readily checked from (49) and (50) that if $\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])] \le N_{l+1}$, then $\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,(l+1)\varepsilon])] \le \frac{N_{l+1}}{1+T^{\gamma}}$ and $\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^0_{1,\gamma}([0,(l+1)\varepsilon])] \le \frac{N_{l+1}}{1+T^{\gamma}}T^{\gamma}$, hence $\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])] \le N_{l+1}$. In other words, the ball

$$B^{N_{l+1}}_{\tilde{y}^{(l)},(l+1)\varepsilon} = \{\tilde{y} \in \tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon]) :\ \tilde{y}_{|[0,l\varepsilon]} = \tilde{y}^{(l)},\ \mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])] \le N_{l+1}\}$$

is invariant by Γ.

The independence of ε with respect to $\tilde{y}^{(l)}$ will allow us to repeat the procedure (with the same ε) and thus to get a sequence of radii $(N_k)_{k\ge1}$ such that the sets $B^{N_k}_{\tilde{y}^{(k)},k\varepsilon}$ are invariant by Γ. Of course, the definition of the latter application has to be adapted to each of those sets.

Step 2: Contraction property. We are now going to look for a splitting of $[l\varepsilon,(l+1)\varepsilon]$ into subintervals $[l\varepsilon,l\varepsilon+\eta],[l\varepsilon+\eta,l\varepsilon+2\eta],\ldots$ of the same length η (that could depend on ε and l), on which Γ is a contraction mapping.

Let $\tilde{y}^{\mathbf{a}},\tilde{y}^{\mathbf{b}} \in \tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon+\eta])$ such that $\tilde{y}^{\mathbf{a}}_{|[0,l\varepsilon]} = \tilde{y}^{\mathbf{b}}_{|[0,l\varepsilon]} = \tilde{y}^{(l)}$, $\mathcal{N}[\tilde{y}^{\mathbf{a}};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])] \le N_{l+1}$, $\mathcal{N}[\tilde{y}^{\mathbf{b}};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])] \le N_{l+1}$, and set $\tilde{z}^{\mathbf{a}} \equiv \Gamma(\tilde{y}^{\mathbf{a}})$, $\tilde{z}^{\mathbf{b}} \equiv \Gamma(\tilde{y}^{\mathbf{b}})$, where Γ is defined just as in Step 1, but restricted to $\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon+\eta])$. By using (44) again, we deduce

$$\begin{aligned} \mathcal{N}[\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([l\varepsilon,l\varepsilon+\eta]) \le c_{\gamma,x}\big\{&\mathcal{N}[\sigma(y^{\mathbf{a}})-\sigma(y^{\mathbf{b}});\mathcal{C}^0_1([l\varepsilon,l\varepsilon+\eta])] \\ &+ \eta^{\gamma}\mathcal{N}[\sigma(y^{\mathbf{a}})-\sigma(y^{\mathbf{b}});\mathcal{C}^{\gamma}_1([l\varepsilon,l\varepsilon+\eta])]\big\}, \end{aligned}$$

and then, according to (47) and (48),

$$\mathcal{N}[\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([l\varepsilon,l\varepsilon+\eta])] \le c^2_{x,\sigma}\left\{1+N_{l+1}\right\}\eta^{\gamma}\mathcal{N}[\tilde{y}^{\mathbf{a}}-\tilde{y}^{\mathbf{b}};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([l\varepsilon,l\varepsilon+\eta])].$$

Since the paths $\tilde{y}^{\mathbf{a}}-\tilde{y}^{\mathbf{b}}$, $\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}}$ vanish on $[0,l\varepsilon]$, the latter estimate implies

$$\mathcal{N}[\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon+\eta])] \le c^2_{x,\sigma}\left\{1+N_{l+1}\right\}\eta^{\gamma}\mathcal{N}[\tilde{y}^{\mathbf{a}}-\tilde{y}^{\mathbf{b}};\tilde{\mathcal{C}}^{\gamma}_{1,\gamma}([0,l\varepsilon+\eta])].$$

Besides, $(\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}})_s=\tilde{\delta}(\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}})_{s,l\varepsilon}$, so that $\mathcal{N}[\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}};\tilde{\mathcal{C}}^0_{1,\gamma}([0,l\varepsilon+\eta])]\leq\mathcal{N}[\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]\eta^\gamma$. Therefore,

$$\mathcal{N}[\tilde{z}^{\mathbf{a}}-\tilde{z}^{\mathbf{b}};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]\leq c^2_{x,\sigma}\left\{1+N_{l+1}\right\}(1+T^\gamma)\eta^\gamma\mathcal{N}[\tilde{y}^{\mathbf{a}}-\tilde{y}^{\mathbf{b}};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]. \quad (51)$$

Fix $\eta\equiv\inf\left(\varepsilon,(2c^2_{x,\sigma}\left\{1+N_{l+1}\right\}(1+T^\gamma))^{-1/\gamma}\right)$ so as to make Γ a strict contraction of the set

$$\{\tilde{y}\in\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta]):\ \tilde{y}_{|[0,l\varepsilon]}=\tilde{y}^{(l)},\ \mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]\leq N_{l+1}\}.$$

Using the invariance of $B^{N_{l+1}}_{\tilde{y}^{(l)},(l+1)\varepsilon}$, it is easily seen that the latter set is invariant by Γ too (see Lemma 4.4 below). Consequently, there exists a unique fixed-point in this set, that we denote by $\tilde{y}^{(l),\eta}$. Insofar as η does not depend on $\tilde{y}^{(l)}$, the reasoning remains true on the (invariant) set

$$\{\tilde{y}\in\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+2\eta]):\ \tilde{y}_{|[0,l\varepsilon+\eta]}=\tilde{y}^{(l),\eta},\ \mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+2\eta])]\leq N_{l+1}\}.$$

Thus, $\tilde{y}^{(l),\eta}$ can be extended into a solution $\tilde{y}^{(l),2\eta}$ defined on $[0,l\varepsilon+2\eta]$ and by iterating the procedure until the interval $[l\varepsilon,(l+1)\varepsilon]$ is covered, we get the expected extension $\tilde{y}^{(l+1)}$.

The uniqueness of the solution can be easily shown with the arguments of Step 2 (replace $\tilde{z}^{\mathbf{a}},\tilde{z}^{\mathbf{b}}$ with $\tilde{y}^{\mathbf{a}},\tilde{y}^{\mathbf{b}}$ in (51)). The details are left to the reader.

□

Lemma 4.4. *With the notations of the previous proof, the set*

$$\{\tilde{y}\in\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta]):\ \tilde{y}_{|[0,l\varepsilon]}=\tilde{y}^{(l)},\ \mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]\leq N_{l+1}\}$$

is invariant by Γ.

Proof. Consider an element $\tilde{y}$ in the set in question and denote $\tilde{z}\equiv\Gamma(\tilde{y})$. Then define

$$\hat{y}_t=\begin{cases}\tilde{y}_t & \text{if } t\leq l\varepsilon+\eta\\ S_{t-(l\varepsilon+\eta)}\cdot\tilde{y}_{l\varepsilon+\eta} & \text{if } t\in[l\varepsilon+\eta,(l+1)\varepsilon].\end{cases}$$

The path $\hat{y}$ is clearly continuous and accordingly belongs to $\tilde{\mathcal{C}}_{1,\gamma}([0,(l+1)\varepsilon])$. Moreover, if $s,t\in[l\varepsilon+\eta,(l+1)\varepsilon]$, $(\tilde{\delta}\hat{y})_{ts}=0$, while if $s\leq l\varepsilon+\eta\leq t$, $(\tilde{\delta}\hat{y})_{ts}=S_{t-(l\varepsilon+\eta)}\cdot(\tilde{\delta}\tilde{y})_{l\varepsilon+\eta,s}$, so that $\mathcal{N}[\hat{y};\tilde{\mathcal{C}}^\gamma_{1,\gamma}([0,(l+1)\varepsilon])]\leq\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^\gamma_{1,\gamma}([0,l\varepsilon+\eta])]$. Since $\mathcal{N}[\hat{y};\tilde{\mathcal{C}}^0_{1,\gamma}([0,(l+1)\varepsilon])]\leq\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^0_{1,\gamma}([0,l\varepsilon+\eta])]$, we deduce $\mathcal{N}[\hat{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])]\leq\mathcal{N}[\tilde{y};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]\leq N_{l+1}$, which means that $\hat{y}\in B^{N_{l+1}}_{\tilde{y}^l,(l+1)\varepsilon}$. According to Step 1 of the previous proof, $B^{N_{l+1}}_{\tilde{y}^l,(l+1)\varepsilon}$ is invariant by Γ, and so, if $\hat{z}\equiv\Gamma(\hat{y})$, then $\mathcal{N}[\hat{z};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])]\leq N_{l+1}$. It is now obvious that $\tilde{z}=\hat{z}_{|[0,l\varepsilon+\eta]}$, which finally leads to $\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,l\varepsilon+\eta])]\leq\mathcal{N}[\hat{z};\tilde{\mathcal{C}}^{0,\gamma}_{1,\gamma}([0,(l+1)\varepsilon])]\leq N_{l+1}$.

□

To conclude with this section, let us go back to the original setting of the equation:

Corollary 4.5. *Under Hypothesis 2, and assuming that* $\sigma\in\mathcal{C}^{2,b}(\mathbb{R}^d;\mathbb{R}^{d,m})$*, the system (2), interpreted with Proposition 4.1, admits a unique solution* y *in* $\mathcal{C}^\gamma_1([0,T];\mathbb{R}^d)$.

Proof. If $\tilde{y}$ stands for the solution of (38) given by Theorem 4.3, it suffices to set, for any $t\in[0,T]$, $y_t\equiv T_{a,\tilde{\phi}}(\tilde{y}_t)$. The details are left to the reader. □

5. The rough case

Our aim still consists in studying the system (38), but we will suppose in this section that the Hölder coefficient γ of x belongs to $(1/3, 1/2]$. Definition (43) does not make sense anymore, and some developments at order 2 are required. To this end, we will resort to the same strategy as in the diffusion case (see Subsection 2.2), divided into two steps:

(1) Identifying the algebraic structure of the potential solution $\tilde{y}$, which will lead to the introduction of a space $\tilde{\mathcal{Q}}$ of controlled paths.
(2) Extending the integral of the system above $x \in \mathcal{C}_1^\gamma$ when $\tilde{y} \in \tilde{\mathcal{Q}}$.

5.1. Convolutional controlled paths.

Let us start with some heuristic considerations. As in the Young case, the system will be analyzed in the form (remember the notation (32))

$$\tilde{y}_0 \equiv 0, \qquad \tilde{\delta}\tilde{y}^i = \mathcal{J}\left(\tilde{d}x^j\,\sigma^{ij}(y)\right), \quad y_u = T_{a,\tilde{\phi}}(\tilde{y}_u) := a + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{y}_u(\xi). \tag{52}$$

Assume for the moment that x is a differentiable path. Equation (52) admits in this case a unique solution $\tilde{y}$, whose (convolutional) increments can be expanded into

$$(\tilde{\delta}\tilde{y}^i)_{ts}(\xi) = \int_s^t S_{t-u}(\xi)\,dx_u^j\,\sigma^{ij}(y_u) = \tilde{X}_{ts}^{x,j}(\xi)\sigma^{ij}(y_s) + \tilde{r}_{ts}^i(\xi), \tag{53}$$

with

$$\tilde{X}_{ts}^{x,j}(\xi) = \int_s^t S_{t-u}(\xi)\,dx_u^j \quad , \quad \tilde{r}_{ts}^i(\xi) = \int_s^t S_{t-u}(\xi)\,dx_u^j\,(\delta\sigma^{ij}(y))_{us}. \tag{54}$$

This elementary decomposition lets already emerge the structure likely to replace $\mathcal{Q}_x^\gamma$ (Definition 2.5) in the convolutional setting. Let us go a little bit deeper into the analysis of (53): if x and y are γ-Hölder ($\gamma \in (1/3, 1/2]$), it is natural to expect that, on the one side, $\tilde{X}^x$ belongs to a space such that $\tilde{\mathcal{C}}_{2,\beta}^\gamma([0,T];\mathbb{R}^m)$, for some coefficient $\beta > 0$, and on the other side, $\tilde{r} \in \tilde{\mathcal{C}}_{2,\beta}^{2\gamma}([0,T];\mathbb{R}^d)$. For some technical reasons that will pop out in the proof of Theorem 5.10, we shall actually be prompted to take $\beta = 1$ in order to exhibit a *global* solution for (52).

Notations: For sake of clarity, we henceforth use the shortcut

$$\tilde{\mathcal{C}}_k^\gamma(I;V) \equiv \tilde{\mathcal{C}}_{k,1}^\gamma(I;V), \quad k \in \{1,2,3\}. \tag{55}$$

As in the previous section, let us label the appropriate regularity assumptions relative to the path $\tilde{X}^x$:

Hypothesis 3. *Let $x \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^m)$, with $\gamma \in (1/3, 1/2]$. We assume that there exists a sequence x^ε of differentiable paths that satisfies*

$$\mathcal{N}[x^\varepsilon - x; \mathcal{C}_1^\gamma([0,T];\mathbb{R}^m)] \xrightarrow{\varepsilon\to 0} 0,$$

and such that the sequence of paths defined by

$$\tilde{X}_{ts}^{x^\varepsilon,i}(\xi) \equiv \int_s^t S_{t-u}(\xi)\,dx_u^{\varepsilon,i}$$

converges to $\tilde{X}_{ts}^{x,i}(\xi) \equiv \int_s^t S_{t-u}(\xi)\,dx_v^i$ (understood as a Young integral) w.r.t the topology of $\tilde{\mathcal{C}}_2^\gamma([0,T];\mathbb{R}^m)$. In particular,

$$\tilde{X}^x \in \tilde{\mathcal{C}}_2^\gamma([0,T];\mathbb{R}^m) \quad \textit{and} \quad \tilde{\delta}\tilde{X}^x = 0.$$

If x is a differentiable path, we assume that this result holds true for $x^\varepsilon \equiv x$.

With the decomposition (53) in mind, the most natural and consistent framework to study the system (52) is the following:

Definition 5.1. *Assume that Hypothesis 3 is satisfied. For any interval I of $[0,T]$, we call convolutional controlled path (by $\tilde{X}^x$) on I, with values in $\mathbb{R}^d$, any element $\tilde{y}$ in $\tilde{\mathcal{C}}_1^\gamma(I;\mathbb{R}^d)$ whose convolutional increments can be written as*

$$(\tilde{\delta}\tilde{y}^i)_{ts} = \tilde{X}^{x,j}_{ts}\,\tilde{y}^{x,ij}_s + \tilde{y}^{\sharp,i}_{ts}, \quad \text{with} \quad \tilde{y}^x \in \mathcal{C}_1^\gamma(I;\mathbb{R}^{d,m}) \text{ and } \tilde{y}^\sharp \in \tilde{\mathcal{C}}_2^{2\gamma}(I;\mathbb{R}^d), \tag{56}$$

The set of convolutional controlled paths on I will be denoted by $\tilde{\mathcal{Q}}_x^\gamma(I;\mathbb{R}^d)$ and we provide the latter space with the seminorm

$$\mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}_x^\gamma(I;\mathbb{R}^d)] \equiv \mathcal{N}[\tilde{y};\tilde{\mathcal{C}}_1^\gamma(I;\mathbb{R}^d)] + \mathcal{N}[\tilde{y}^x;\mathcal{C}_1^0(I;\mathbb{R}^{d,m})] + \mathcal{N}[\tilde{y}^x;\mathcal{C}_1^\gamma(I;\mathbb{R}^{d,m})] + \mathcal{N}[\tilde{y}^\sharp;\tilde{\mathcal{C}}_2^{2\kappa}(I;\mathbb{R}^d]. \tag{57}$$

Remark 5.2. It may be worth noticing that in spite of its notation, the path $\tilde{y}^x$ defined through (56) takes values in a Euclidian space, and not in a functional space.

In order to give sense to the system (52) when $\tilde{y} \in \tilde{\mathcal{Q}}_x^\gamma(I;\mathbb{R}^k)$, it is now important to identify the algebraic structure of the integrand $\sigma(y_u)$, where $y_u \equiv T_{a,\tilde{\phi}}(\tilde{y}_u)$. To begin with, observe that if $\tilde{\delta}\tilde{y}$ admits the decomposition (56), then the increments of y can be written as:

$$\begin{aligned}
(\delta y^i)_{ts} &= \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)(\delta\tilde{y}^i)_{ts}(\xi) \\
&= \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)(\tilde{\delta}\tilde{y}^i)_{ts}(\xi) + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)A_{ts}(\xi)\tilde{y}^i_s(\xi) \\
&= \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{X}^{x,j}_{ts}(\xi)\tilde{y}^{x,ij}_s + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{y}^{\sharp,i}_{ts}(\xi) + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)A_{ts}(\xi)\tilde{y}^i_s(\xi) \\
&= X^{x,j}_{ts}\tilde{y}^{x,ij}_s + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{y}^{\sharp,i}_{ts}(\xi) + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)A_{ts}(\xi)\tilde{y}^i_s(\xi),
\end{aligned} \tag{58}$$

where $X^{x,j}_{ts} \equiv \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{X}^{x,j}_{ts}(\xi)$ is well-defined as an element of $\mathcal{C}_2^\gamma([0,T];\mathbb{R}^m)$, thanks to Hypothesis 3. Let us analyze (58) as far as Hölder-continuity is concerned. For the last term of the composition, remember the obvious estimate $|A_{ts}(\xi)| \le c\,|\xi|\,|t-s|$, which entails here

$$\left|\int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)A_{ts}(\xi)\tilde{y}^i_s(\xi)\right| \le |t-s|\,\mathcal{N}[\tilde{y}_s;\mathcal{L}_1],$$

and consequently suggests that the path at stake is quite smooth. Besides, the regularity assumption on $\tilde{y}^{\sharp,i}$ immediately gives

$$\left|\int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{y}^{\sharp,i}_{ts}(\xi)\right| \le |t-s|^{2\gamma}\,\mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}_x^\gamma].$$

With those two controls in hand, it would be tempting to envisage an algebraic structure such that

$$\{y:\ (\delta y^i)_{ts} = X^{x,j}_{ts}y^{x,ij}_s + y^{\sharp,i}_{ts},\ \text{with } y^x \in \mathcal{C}_1^\gamma(\mathbb{R}^{m,l}) \text{ and } y^\sharp \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^k)\}.$$

It is indeed possible to show that the latter set is invariant when composing the path with a smooth enough mapping, which would ensure the transition between y and $\sigma(y)$.

Nevertheless, a little bit more subtle analysis of (58) leads to more convenient algebraic handlings. It actually suffices to observe that the path $\tilde{X}^x$ can be decomposed as

$$\tilde{X}^x_{ts}(\xi) = \int_s^t S_{t-u}(\xi)\, dx_u = (\delta x)_{ts} + \int_s^t A_{tu}(\xi)\, dx_u.$$

When $x \in \mathcal{C}_1^\gamma(\mathbb{R}^m)$, the latter transformation is at this point purely formal. Let us record it through the following theoretical hypothesis, that will examined into details at Section 6:

Hypothesis 4. *Under Hypothesis 3, we admit that the sequence of paths defined by*

$$\tilde{X}^{Ax^\varepsilon,i}_{ts}(\xi) \equiv \int_s^t A_{tu}(\xi)\, dx^{\varepsilon,i}_u,$$

converges w.r.t to the topology of the space $\tilde{\mathcal{C}}_{2,0}^{1+\gamma}(\mathbb{R}^m)$ (we recall that this space has been defined at Subsection 3.3). In particular,

$$\tilde{X}^{Ax} \in \tilde{\mathcal{C}}_{2,0}^{1+\gamma}(\mathbb{R}^m) \quad \text{and} \quad \tilde{X}^x_{ts}(\xi) = \mathbf{x}^{\mathbf{1}}_{ts} + \tilde{X}^{Ax}_{ts}(\xi), \tag{59}$$

where we have denoted, according to [25], $\mathbf{x}^{\mathbf{1}} \equiv \delta x$.

If x is a differentiable path, we assume that this result holds true for $x^\varepsilon \equiv x$.

Remark 5.3. The regularity assumption contained in (59) is of course suggested by the estimate $|A_{ts}(\xi)| \leq c|\xi|\,|t-s|$, having also in mind the fact that we are working with the underlying functional space $\mathcal{L}_1$ (Notation (55)).

Going back to (58), the increments of y can now be expanded into

$$(\delta y^i)_{ts} = \mathbf{x}^{\mathbf{1},j}_{ts}\,(L_{\tilde{\phi}}\, \tilde{y}^{x,ij}_s) + \left[X^{Ax,j}_{ts} \tilde{y}^{x,ij}_s + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) \tilde{y}^{\sharp,i}_{ts}(\xi) + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) A_{ts}(\xi) \tilde{y}^i_s(\xi) \right], \tag{60}$$

where we have set

$$L_{\tilde{\phi}} \equiv \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) \quad \text{and} \quad X^{Ax}_{ts} \equiv \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) \tilde{X}^{Ax}_{ts}(\xi). \tag{61}$$

Therefore, owing to the regularity assumption (59), we recover here the same structure of controlled paths as in the analysis of standard systems (see Definition 2.5), and we have established the following transition:

Proposition 5.4. *Under Hypotheses 3 and 4, if $\tilde{y} \in \tilde{\mathcal{Q}}^\gamma_x(I;\mathbb{R}^d)$ admits the decomposition $\tilde{\delta}\tilde{y}^i = \tilde{X}^{x,j}\tilde{y}^{x,ij} + \tilde{y}^{\sharp,i}$, then the path $y \equiv T_{a,\tilde{\phi}}(\tilde{y})$ belongs to $\mathcal{Q}^\gamma_x(I;\mathbb{R}^d)$ and admits the decomposition $\delta y^i = \mathbf{x}^{\mathbf{1},j}\, y^{x,ij} + y^{\sharp,i}$, with*

$$y^{x,ij}_t = L_{\tilde{\phi}}\, \tilde{y}^{x,ij}_t \quad , \quad y^{\sharp,i}_{ts} = X^{Ax,j}_{ts} \tilde{y}^{x,i}_s + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) \tilde{y}^{\sharp,i}_{ts}(\xi) + \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) A_{ts}(\xi) \tilde{y}^i_s(\xi). \tag{62}$$

The expected structure for the integrand $\sigma(y)$ immediately arises from this result. Indeed, we have already recalled the invariance property: if $y \in \mathcal{Q}^\gamma_x(I;\mathbb{R}^d)$ and $\sigma \in \mathcal{C}^{2,\mathbf{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$, then $z \equiv \sigma(y) \in \mathcal{Q}^\gamma_x(I;\mathbb{R}^{d,m})$, where the space $\mathcal{Q}^\gamma_x(I;\mathbb{R}^{d,m})$ has also been introduced in Definition 2.5.

5.2. **Convolutional integration of controlled paths.** Taking the above considerations into account, the interpretation of the system (52) is now reduced to the problem of extending the integral $\mathcal{J}(\tilde{d}x^j\, z^{ij})$ to the case x is γ-Hölder ($\gamma \in (1/3, 1/2]$) and $z \in \mathcal{Q}_x^\gamma(I;\mathbb{R}^{d,m})$. Observe that with a view to settling a fixed-point argument, it also matters that the extension gives birth to an element in $\tilde{\mathcal{Q}}_x^\gamma(I;\mathbb{R}^d)$.

In order to construct the integral in question, we will rely, as in the standard case, on the a priori existence of a *convolutional Levy area* adapted to the context:

Hypothesis 5. *Under Hypothesis 3, we admit that the sequence of paths defined by*

$$\tilde{X}_{ts}^{x^\varepsilon x^\varepsilon, ij}(\xi) \equiv \int_s^t S_{t-u}(\xi)\, dx_u^{\varepsilon,i}\, (\delta x^{\varepsilon,j})_{us}$$

converges to a path $\tilde{X}^{xx}$ w.r.t the topology of $\tilde{\mathcal{C}}_2^{2\gamma}(\mathbb{R}^{m,m})$. In particular,

$$\tilde{X}^{xx} \in \tilde{\mathcal{C}}_2^{2\gamma}(I;\mathbb{R}^{m,m}) \quad \text{and} \quad (\tilde{\delta}\tilde{X}^{xx})_{tus} = \tilde{X}_{tu}^x \otimes (\delta x)_{us}. \tag{63}$$

If x is a differentiable path, we assume that this result holds true for $x^\varepsilon \equiv x$.

Once endowed with this second-order path, here is the natural way to integrate a controlled path:

Proposition 5.5. *We assume that both Hypotheses 3 and 5 are satisfied, and let $I = [l_1, l_2]$ a fixed subinterval of $[0,T]$. For any path $z \in \mathcal{Q}_x^\gamma(I;\mathbb{R}^{d,m})$ with decomposition*

$$\delta z^{ij} = \mathbf{x}^{\mathbf{1},k}\, z^{x,ijk} + z^{\sharp,ij}, \tag{64}$$

we set, for any $s < t \in I$,

$$\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij}) \equiv \tilde{X}_{ts}^{x,j}\, z_s^{ij} + \tilde{X}_{ts}^{xx,jk}\, z_s^{x,ijk} + \tilde{\Lambda}_{ts}\left(\tilde{X}^{x,j}\, z^{\sharp,ij} + \tilde{X}^{xx,jk}\, \delta z^{x,ijk}\right). \tag{65}$$

Then:

1. *$\mathcal{J}(\tilde{d}x^j\, z^{ij})$ is well-defined as an element of $\tilde{\mathcal{C}}_2^\gamma(I;\mathbb{R}^d)$ and for any $\xi \in \mathbb{R}$, $\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij})(\xi)$ coincides with the usual Riemann integral $\int_s^t S_{t-u}(\xi)\, dx_u^j\, z_u^{ij}$ when x is a differentiable path.*
2. *For any $\tilde{h} \in \mathcal{L}_1$, there exists a unique path $\tilde{z} \in \tilde{\mathcal{Q}}_x^\gamma(I;\mathbb{R}^d)$ such that $\tilde{z}_{l_1} = \tilde{h}$ and $\tilde{\delta}\tilde{z}^i = \mathcal{J}(\tilde{d}x^j\, z^{ij})$.*
3. *For any $s < t \in I$, $\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij})$ can be described by the formula:*

$$\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij}) = \lim_{|\Pi_{ts}|\to 0} \sum_{l=0}^n \left[\tilde{X}_{t_{l+1},t_l}^{x,j}\, z_{t_l}^{ij} + \tilde{X}_{t_{l+1},t_l}^{xx,jk}\, z_{t_l}^{x,ijk}\right] \quad \text{in } \mathcal{L}_1, \tag{66}$$

where the limit is taken over any partition $\Pi_{ts} = \{t_0 = t, \dots, t_n = s\}$ of $[s,t]$ whose mesh tends to 0.

Proof. (1) If x is a differentiable path, then, as in the Young case, we first write

$$\mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij})(\xi) = \int_s^t S_{t-v}(\xi)\, dx_v^j\, z_v^{ij} = \tilde{X}_{ts}^{x,j}(\xi)\, z_s^{ij} + \int_s^t S_{t-v}(\xi)\, dx_v^j\, (\delta z^{ij})_{vs}.$$

By injecting the decomposition (64) of $(\delta z^{ij})_{vs}$ in the latter relation, we get

$$\begin{aligned} \mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij})(\xi) &= \tilde{X}_{ts}^{x,j}(\xi)\, z_s^{ij} + \int_s^t S_{t-v}(\xi)\, dx_v^j\, \left[\mathbf{x}_{vs}^{\mathbf{1},k}\, z_s^{x,ijk} + z_{vs}^{\sharp,ij}\right] \\ &= \tilde{X}_{ts}^{x,j}(\xi)\, z_s^{ij} + \tilde{X}_{ts}^{xx,jk}(\xi)\, z_s^{x,ijk} + \int_s^t S_{t-v}(\xi) dx_v^j\, z_{vs}^{\sharp,ij}, \end{aligned}$$

and so, with the notation (33),

$$\mathcal{J}_{ts}(\tilde{d}x^j\, z^{\sharp,ij}) = \mathcal{J}_{ts}(\tilde{d}x^j\, z^{ij}) - \tilde{X}^{x,j}_{ts}\, z^{ij}_s - \tilde{X}^{xx,jk}_{ts}\, z^{x,ijk}_s. \tag{67}$$

Let us now apply the operator $\tilde{\delta}$ to the two sides of this equality: thanks to (34), (30), (64) and (63), we successively deduce

$$\begin{aligned}
\tilde{\delta}\left(\mathcal{J}(\tilde{d}x^j\, z^{\sharp,ij})\right) &= \tilde{X}^{x,j}\,\delta z^{ij} - \tilde{\delta}\tilde{X}^{xx,jk}\, z^{x,ijk} + \tilde{X}^{xx,jk}\,\delta z^{x,ijk}\\
&= \tilde{X}^{x,j}\,(\mathbf{x}^{\mathbf{1},k}\, z^{x,ijk}) + \tilde{X}^{x,j}\, z^{\sharp,ij} - (\tilde{X}^{x,j}\,\mathbf{x}^{\mathbf{1},k})\, z^{x,ijk} + \tilde{X}^{xx,jk}\,\delta z^{x,ijk}\\
&= \tilde{X}^{x,j}\, z^{\sharp,ij} + \tilde{X}^{xx,jk}\,\delta z^{x,ijk},
\end{aligned}$$

and we are therefore allowed to write, via (37),

$$\mathcal{J}(\tilde{d}x^j\, z^{\sharp,ij}) = \tilde{\Lambda}\left(\tilde{X}^{x,j}\, z^{\sharp,ij} + \tilde{X}^{xx,jk}\,\delta z^{x,ijk}\right).$$

Going back to (67), we recover (65). The validity of the latter formula for a Hölder path x is then a straightforward consequence of the algebraic and analytic assumptions contained in Hypotheses 3 and 5, which also accounts for *(2)*. As for *(3)*, it stems from Corollary 3.9, after noticing that

$$\tilde{\delta}\tilde{z}^i = (\mathrm{Id} - \tilde{\Lambda}\tilde{\delta})(\tilde{X}^{x,j}\, z^{ij} + \tilde{X}^{xx,jk}\, z^{x,ijk}).$$

□

5.3. **Localized controlled paths.** At this point, we are able to interpret the system (52) under Hypotheses 3, 4 and 5, as the following loop summarizes it:

$$\begin{array}{ccccccc}
\tilde{\mathcal{Q}}^\gamma_x & \overset{\text{Prop. 5.4}}{\longrightarrow} & \mathcal{Q}^\gamma_x & \longrightarrow & \mathcal{Q}^\gamma_x & \overset{\text{Prop. 5.5}}{\longrightarrow} & \tilde{\mathcal{Q}}^\gamma_x\\
\tilde{y}^i & \longmapsto & y = T_{a,\tilde{\phi}}(\tilde{y}) & \longmapsto & \sigma^{ij}(y) & \longmapsto & (\tilde{\delta}\tilde{z}^i) = \mathcal{J}(\tilde{d}x^j\,\sigma^{ij}(y)).
\end{array}$$

The proof of existence (and uniqueness) of a *global* solution to the system will stem from successive fixed point arguments in the spaces $\tilde{\mathcal{Q}}^\gamma_x(I_n)$, for a particular sequence I_n of intervals that covers $[0,T]$. Patching those local solutions together will require a simultaneous control on both the norms of $\tilde{y}$ and the initial condition $\tilde{h}^n = \tilde{y}_{l_n}$ on each interval $I_n = [l_n, l_{n+1}]$, when applying the 3-step procedure described by the above loop.

The most natural idea to do so consists in splitting up the reasoning into three successive estimates, each of them corresponding to a particular step, and when the intermediate space $\mathcal{Q}^\gamma_x(I)$ is provided with its usual norm $\mathcal{N}[.;\mathcal{Q}^\gamma_x(I)]$, defined by (18).

Unfortunately, using the latter norm turns out not to be sufficient in order to get a sharp enough final estimate expressed in terms of $\mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}^\gamma_x(I_n)]$ and $\mathcal{N}[\tilde{y}_{l_n};\mathcal{L}_1]$, and an additional technical argument has to be settled here. It involves the introduction of a specific (affine) subspace of $\mathcal{Q}^\gamma_x(I_n)$, intended to isolate the terms that depend only on the initial condition $\tilde{y}_{l_n}$.

We assume in this subsection that x satisfies the three hypotheses 3, 4 and 5, and we fix an arbitrary subinterval $I = [l_1, l_2]$ of $[0,T]$.

Definition 5.6. *Let $k \in \mathbb{N}^*$, $f \in \mathcal{C}^1_2(I;\mathbb{R}^k)$. A path $y \in \mathcal{C}^\gamma_1(I;\mathbb{R}^k)$ will be said γ-controlled (by x) around f on I if its increments admit the following decomposition: for all $s < t \in I$,*

$$(\delta y^i)_{ts} - f^i_{ts} = \mathbf{x}^{\mathbf{1},j}_{ts}\, y^{x,ij}_s + y^{\flat,i}_{ts} \quad \text{with } y^x \in \mathcal{C}^\gamma_1(I;\mathbb{R}^{m,k}) \text{ and } y^\flat \in \mathcal{C}^{2\gamma}_2(I;\mathbb{R}^k). \tag{68}$$

The set of such paths will be denoted by $\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)$*, and to any* $y \in \mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)$*, we associate the quantity*

$$\begin{aligned}&\mathcal{M}[y;\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)]\\ &\qquad \equiv \mathcal{N}[y^x;\mathcal{C}_1^0(I;\mathbb{R}^{k,m})] + \mathcal{N}[y^x;\mathcal{C}_1^{\gamma}(I;\mathbb{R}^{k,m})] + \mathcal{N}[y^{\flat};\mathcal{C}_2^{2\gamma}(I;\mathbb{R}^{k,m})] + \mathcal{N}[y;\mathcal{C}_1^{\gamma}(I;\mathbb{R}^k)].\end{aligned}$$

As with the controlled paths, we then define, for any $f \in \mathcal{C}_2^1(I;\mathbb{R}^{k,l})$*,* $\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^{k,l})$ *as the set of paths* $y \in \mathcal{C}_1^{\gamma}(I;\mathbb{R}^{k,l})$ *such that, for any* $j = 1,\dots,l$*,* $y^{\cdot j} \in \mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)$*, and we associate to those elements the quantity*

$$\mathcal{M}[y;\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^{k,l})] \equiv \sum_{j=1}^{l} \mathcal{N}[y^{\cdot j};\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)].$$

Obviously, $\mathcal{A}^{\gamma}_{x,0}(I) = \mathcal{Q}^{\gamma}_x(I)$ and more generally: for any $f \in \mathcal{C}_2^1(I;\mathbb{R}^k)$, $\mathcal{A}^{\gamma}_{x,f}(I) \subset \mathcal{Q}^{\gamma}_x(I)$. The crucial point in our localization around f is precisely that this latter increment does not (directly) intervene in the computation of $\mathcal{M}[y;\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)]$.

Let us now see how the sets $\mathcal{A}^{\gamma}_{x,f}(I)$ pop out naturally when one integrates a convolutional controlled path with respect to ξ.

Proposition 5.7. *We assume that both Hypotheses 3 and 4 are satisfied. Let* $\tilde{y} \in \tilde{\mathcal{Q}}^{\gamma}_x(I;\mathbb{R}^k)$ *such that* $\tilde{y}_{l_1} = \tilde{h} \in \mathcal{L}_1$ *and* $\tilde{\delta}\tilde{y} = \tilde{X}^{x,j}\,\tilde{y}^{x,ij} + \tilde{y}^{\sharp,i}$*, and set* $y \equiv T_{a,\tilde{\phi}}(\tilde{y})$*. Then* $y \in \mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^k)$*, with* $f_{ts} \equiv \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi)\, A_{ts}(\xi) S_{s-l_1}(\xi)\tilde{h}(\xi)$*. Moreover,*

$$\mathcal{M}[y;\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^d)] \le c_x \left\{ \mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}^{\gamma}_x(I;\mathbb{R}^d)] + |I|^{1-\gamma}\,\mathcal{N}[\tilde{h};\mathcal{L}_1] \right\}. \tag{69}$$

Proof. From (60), we can write, for all $s < t \in I$,

$$\begin{aligned}&(\delta y^i)_{ts}\\ &\quad= \mathbf{x}^{\mathbf{1},j}_{ts}\,(L_{\tilde{\phi}}\,\tilde{y}^{x,ij}_s) + \tilde{X}^{Ax,j}_{ts}\,\tilde{y}^{x,ij}_s + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\,\tilde{y}^{\sharp,i}_{ts}(\xi) + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi) A_{ts}(\xi)\tilde{y}^i_s(\xi)\\ &\quad= \mathbf{x}^{\mathbf{1},j}_{ts}\,(L_{\tilde{\phi}}\,\tilde{y}^{x,ij}_s) + \tilde{X}^{Ax,j}_{ts}\,\tilde{y}^{x,ij}_s + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\,\tilde{y}^{\sharp,i}_{ts}(\xi) + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi) A_{ts}(\xi)(\tilde{\delta}\tilde{y}^i)_{sl_1}(\xi) + f^i_{ts}.\end{aligned}$$

Now set $y^{x,ij}_s \equiv L_{\tilde{\phi}}\,\tilde{y}^{x,ij}_s$, $y^{\flat,i}_{ts} \equiv \tilde{X}^{Ax,j}_{ts}\,\tilde{y}^{x,ij}_s + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\left\{\tilde{y}^{\sharp,i}_{ts}(\xi) + A_{ts}(\xi)(\tilde{\delta}\tilde{y}^i)_{sl_1}(\xi)\right\}$. Clearly,

$$\mathcal{N}[y^{\flat};\mathcal{C}_2^{2\gamma}] \le c_x \left\{ \mathcal{N}[\tilde{y}^x;\mathcal{C}_1^0] + \mathcal{N}[\tilde{y}^{\sharp};\tilde{\mathcal{C}}_2^{2\gamma}] + \mathcal{N}[\tilde{y};\tilde{\mathcal{C}}_1^{\gamma}] \right\} \le c_x\,\mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}^{\gamma}_x],$$

and $|(\delta y)_{ts}| \le |f_{ts}| + |t-s|^{\gamma}\,\mathcal{N}[X^x;\mathcal{C}_2^{\gamma}]\mathcal{N}[\tilde{y}^x;\mathcal{C}_1^0] + |t-s|^{2\gamma}\,\mathcal{N}[y^{\flat};\mathcal{C}_2^{2\gamma}]$.
As $|f_{ts}| \le |t-s|\,\mathcal{N}[\tilde{h};\mathcal{L}_1]$, we get $\mathcal{N}[y;\mathcal{C}_1^{\gamma}] \le |I|^{1-\gamma}\,\mathcal{N}[\tilde{h};\mathcal{L}_1] + c_x\mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}^{\gamma}_x]$, and (69) is thus proved.

□

The following result is the analog of [19, Proposition 4] in the context of localized controlled paths:

Proposition 5.8. *Let* $y \in \mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^d)$ *with* $y_{l_1} = h$ *and* $\delta y^i - f^i = \mathbf{x}^{\mathbf{1},j}\,y^{x,ij} + y^{\flat,i}$*, and consider a mapping* $\sigma \in \mathcal{C}^{3,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$*. Then* $\sigma(y) \in \mathcal{A}^{\gamma}_{x,D\sigma(h)f}(I;\mathbb{R}^{d,m})$ *and the following* *estimate holds true:*

$$\mathcal{M}[\sigma(y);\mathcal{A}^{\gamma}_{x,D\sigma(h)f}(I)]$$
$$\leq c_{x,\sigma}\left\{1+\mathcal{M}[y;\mathcal{A}^{\gamma}_{x,f}(I)]^2+|I|^{1-\gamma}\,\mathcal{M}[y;\mathcal{A}^{\gamma}_{x,f}(I)]\mathcal{N}[f;\mathcal{C}^1_2(I)]+|I|^{1-\gamma}\,\mathcal{N}[f;\mathcal{C}^1_2(I)]\right\}. \tag{70}$$

Moreover, if $y^{(1)},y^{(2)}\in\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^{d,m})$ *are such that* $y^{(1)}_{l_1}=y^{(2)}_{l_1}$*, then*

$$\mathcal{N}[\sigma(y^{(1)})-\sigma(y^{(2)});\mathcal{Q}^{\gamma}_x(I)]$$
$$\leq c_{x,\sigma}\mathcal{N}[y^{(1)}-y^{(2)};\mathcal{Q}^{\gamma}_x(I)]\big\{1+\mathcal{M}[y^{(1)};\mathcal{A}^{\gamma}_{x,f}(I)]^2+\mathcal{M}[y^{(2)};\mathcal{A}^{\gamma}_{x,f}(I)]^2$$
$$+|I|^{1-\gamma}\,\mathcal{N}[f;\mathcal{C}^1_2(I)](1+\mathcal{N}[y^{(1)};\mathcal{C}^{\gamma}_1(I)]+\mathcal{N}[y^{(2)};\mathcal{C}^{\gamma}_1(I)])\big\}. \tag{71}$$

Proof. This is a matter of standard differential calculus, which mostly appeals to the same arguments as in the proofs of [19, Proposition 4] and [32, Lemma 3.1]. For sake of conciseness, we refer the reader to the latter articles for further details. □

Let us again point out the fact that $\mathcal{A}^{\gamma}_{x,f}$ is a subset of $\mathcal{Q}^{\gamma}_x$. This means in particular that for any element $z\in\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^{d,m})$, the integral $\mathcal{J}(\tilde{d}x^j\,z^{ij})$ can be defined using Proposition 5.5. For those particular paths, we have the following control at our disposal:

Proposition 5.9. *Assume that both Hypotheses 3 and 5 are satisfied. If* $z\in\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^{d,m})$*, then the seminorm of the path* $\tilde{z}\in\tilde{\mathcal{Q}}^{\gamma}(I;\mathbb{R}^d)$ *defined by* $\tilde{z}_{l_1}=\tilde{h}\in\mathcal{L}_1$ *and* $\tilde{\delta}\tilde{z}^i=\mathcal{J}(\tilde{d}x^j\,z^{ij})$ *can be estimated by*

$$\mathcal{N}[\tilde{z};\tilde{\mathcal{Q}}^{\gamma}_x(I;\mathbb{R}^d)]$$
$$\leq c_x\left\{\mathcal{N}[z;\mathcal{C}^0_1(I;\mathbb{R}^{d,m})]+\left|z^x_{l_1}\right|+|I|^{\gamma}\,\mathcal{M}[z;\mathcal{A}^{\gamma}_{x,f}(I;\mathbb{R}^{d,m})]+|I|^{1-\gamma}\,\mathcal{N}[f;\mathcal{C}^1_2(I;\mathbb{R}^{d,m})]\right\}. \tag{72}$$

Proof. According to Proposition 5.5, the decomposition of $\tilde{z}$ as a convolutional controlled path is given by $\tilde{\delta}\tilde{z}^i=\tilde{X}^{x,j}\,\tilde{z}^{x,ij}+\tilde{z}^{\sharp,i}$, with $\tilde{z}^x=z$ and $\tilde{z}^{\sharp}=\tilde{z}^{\sharp,1}+\tilde{z}^{\sharp,2}$, where

$$\tilde{z}^{\sharp,1,i}\equiv\tilde{X}^{xx,jk}\,z^{x,ijk}\quad\text{and}\quad\tilde{z}^{\sharp,2,i}\equiv\tilde{\Lambda}(\tilde{X}^{x,j}\,(z^{\flat,ij}+f^{ij})+\tilde{X}^{xx,jk}\,\delta z^{x,ijk}).$$

Since $(\delta z^{ij})_{ts}=f^{ij}_{ts}+\mathbf{x}^{\mathbf{1},k}_{ts}\,z^{x,ijk}_s+z^{\flat,ij}_{ts}=f^{ij}_{ts}+\mathbf{x}^{\mathbf{1},k}_{ts}\,z^{x,ijk}_{l_1}+\mathbf{x}^{\mathbf{1},k}_{ts}\,(\delta z^{x,ijk})_{sl_1}+z^{\flat,ij}_{ts}$,

$$\mathcal{N}[\zeta^{\tilde{z}};\mathcal{C}^{\gamma}_1(I)]=\mathcal{N}[z;\mathcal{C}^{\gamma}_1(I)]\leq c_x\left\{|I|^{1-\gamma}\,\mathcal{N}[f;\mathcal{C}^1_2(I)]+\left|z^x_{l_1}\right|+|I|^{\gamma}\,\mathcal{M}[z;\mathcal{A}^{\gamma}_{f,h}(I)]\right\}.$$

As for the residual term, we first have, by writing $\tilde{z}^{\sharp,1,i}_{ts}=\tilde{X}^{xx,jk}_{ts}\,z^{x,ijk}_{l_1}+\tilde{X}^{xx,jk}_{ts}\,(\delta z^{x,ijk})_{sl_1}$, $\mathcal{N}[\tilde{z}^{\sharp,1};\tilde{\mathcal{C}}^{2\gamma}_2]\leq c_x\left\{|z^x_{l_1}|+|I|^{\gamma}\,\mathcal{M}[z;\mathcal{A}^{\gamma}_{x,f}(I)]\right\}$, while, due to the contraction property (16),

$$\mathcal{N}[\tilde{z}^{\sharp,2};\tilde{\mathcal{C}}^{2\gamma}_2(I)]\leq c_x\left\{|I|^{\gamma}\,\mathcal{M}[z;\mathcal{A}^{\gamma}_{x,f}(I)]+|I|^{1-\gamma}\,\mathcal{N}[f;\mathcal{C}^1_2(I)]\right\}.$$

Finally, as $\tilde{\delta}\tilde{z}^i=\tilde{X}^{x,j}\,\tilde{z}^{x,ij}+\tilde{z}^{\sharp,i}$,

$$\mathcal{N}[\tilde{z};\tilde{\mathcal{C}}^{\gamma}_1(I)]\leq c_x\left\{\mathcal{N}[z;\mathcal{C}^0_1(I)]+\left|z^x_{l_1}\right|+|I|^{\gamma}\,\mathcal{M}[z;\mathcal{A}^{\gamma}_{x,f}(I)]+|I|^{1-\gamma}\,\mathcal{N}[f;\mathcal{C}^1_2(I)]\right\},$$

which achieves the proof of (72). □

5.4. **Solving the equation.** We are now in position to solve the system:

Theorem 5.10. *Assume that the three hypotheses 3, 4 and 5 are satisfied. If* $\sigma \in \mathcal{C}^{3,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$*, then the system (52), interpreted with Proposition 5.5, admits a unique solution* $\tilde{y}$ *in* $\tilde{\mathcal{Q}}_x^\gamma([0,T];\mathbb{R}^d)$*. Moreover, there exists a function* $C:(\mathbb{R}^+)^3 \to \mathbb{R}^+$ *growing with each of its three arguments, such that*

$$\mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}_x^\gamma([0,T];\mathbb{R}^d)] \leq C(\mathcal{N}[X^x;\tilde{\mathcal{C}}_2^\gamma],\mathcal{N}[\tilde{X}^{Ax};\tilde{\mathcal{C}}_{2,0}^{1+\gamma}],\mathcal{N}[\tilde{X}^{xx};\tilde{\mathcal{C}}_2^{2\gamma}]). \tag{73}$$

Proof. As we announced it in the introduction, the proof will consist in successive fixed-point arguments on a sequence of intervals $(I_n)_n$ that covers $[0,T]$. We shall more precisely consider the sequence given by:

$$I_n^N = [l_n^N, l_{n+1}^N] \quad \text{with } l_0^N = 0 \text{ and } \varepsilon_n = \varepsilon_n^N = l_{n+1}^N - l_n^N = \frac{1}{N+n}, \tag{74}$$

where N is a positive integer that will be determined in the course of the proof.

On each of those intervals, the procedure will (as usual) be divided into two steps: we first establish the existence of invariant subsets for the mapping Γ associated to the system, and then show that the restriction of Γ to some of those subsets is a strict contraction.

The results of Subsection 5.3 show that in order to control the image $\tilde{z} \equiv \Gamma(\tilde{y})$ of a path $\tilde{y} \in \tilde{\mathcal{Q}}_x^\gamma(I_n^N)$, it is important to have an estimate of the norm of $\tilde{y}$ in $\tilde{\mathcal{Q}}_x^\gamma(I_n^N)$, but also of the norm of the initial condition $\tilde{h}_n \equiv \tilde{y}_{l_n^N}$. This general observation will be at the core of our reasoning.

Step 1: Invariance of balls. Let us temporarily fix the parameter N in (74), and introduce two additional parameters $\alpha_1,\alpha_2>0$, the value of which will also be determined during the proof. We consider the sets

$$B_n^{\tilde{h}_n} \equiv \{\tilde{y} \in \tilde{\mathcal{Q}}_x^\gamma(I_n^N):\ \tilde{y}_{l_n^N} = \tilde{h}_n,\, \tilde{y}_{l_n^N}^x = \sigma(h_n),\, \mathcal{N}[\tilde{y};\tilde{\mathcal{Q}}_x^\gamma(I_n^N)] \leq (N+n)^{\alpha_2}\},$$

where $\tilde{h}_n \in \mathcal{L}_1$ is such that $\mathcal{N}[\tilde{h}_n;\mathcal{L}_1] \leq (N+n)^{\alpha_1}$. As in the proof of Theorem 4.3, if $\tilde{y} \in B_n^{\tilde{h}_n}$, $\tilde{z} \equiv \Gamma(\tilde{y})$ stands for the path in $\tilde{\mathcal{Q}}_x^\gamma(I_n^N)$ defined by the two conditions: $\tilde{z}_{l_n^N} = \tilde{h}_n$ and for all $s,t \in I_n^N$, $(\tilde{\delta}\tilde{z})_{ts} = \mathcal{J}_{ts}(\tilde{d}x^j\,\sigma^{ij}(y))$, where $y \equiv T_{a,\tilde{\phi}}(\tilde{y})$.

With those notations, we are going to prove that α_1 and α_2 can be picked in such a way that, on the one hand, the sets $B_n^{\tilde{h}_n}$ are invariant by Γ, and, on the other hand, the following property holds true:

$$\text{(H)} \qquad \text{If } \tilde{y} \in B_n^{\tilde{h}_n}, \text{ then } \mathcal{N}[\tilde{y}_{l_{n+1}^N};\mathcal{L}_1] \leq (N+n+1)^{\alpha_1}.$$

The latter condition will allow us to patch successive fixed points together at Step 3.

Let $\tilde{y} \in B_n^{\tilde{h}_n}$, $\tilde{z} \equiv \Gamma(\tilde{y})$. In order to apply the results of Subsection 5.3, denote, for all $s<t \in I_n^N$,

$$y_t \equiv T_{a,\tilde{\phi}}(\tilde{y}_t)\ ,\ f_{ts}^n \equiv \int_{\mathbb{R}} d\xi\, \tilde{\phi}(\xi) A_{ts}(\xi) S_{s-l_n^N}(\xi) \tilde{h}_n(\xi)\ ,\ g_{ts}^n \equiv D\sigma(y_{l_n^N})\, f_{ts}^n. \tag{75}$$

Estimate (72) first gives

$$\begin{aligned}&\mathcal{N}[\tilde{z};\tilde{\mathcal{Q}}_x^\gamma(I_n^N)]\\ &\leq c_x \Big\{\mathcal{N}[\sigma(y);\mathcal{C}_1^0(I_n^N)] + |\sigma(y)_{l_n^N}^x| + \varepsilon_n^\gamma\,\mathcal{M}[\sigma(y);\mathcal{A}_{x,g^n}^\gamma(I_n^N)] + \varepsilon_n^{1-\gamma}\,\mathcal{N}[g^n;\mathcal{C}_2^1(I_n^N)]\Big\}. \end{aligned}\tag{76}$$

According to Propositions 5.8 and 5.7, we know that

$$\sigma(y)^{x,ijk}_{l_n^N}=\partial_p\sigma^{ij}(y_{l_n^N})\,y^{x,pk}_{l_n^N}=\partial_p\sigma^{ij}(y_{l_n^N})\,(L_{\tilde\phi}\,\tilde y^{x,pk}_{l_n^N})=L_{\tilde\phi}\,\partial_p\sigma^{ij}(y_{l_n^N})\,\sigma^{pk}(\tilde h_n),$$

so that $|\sigma(y)^x_{l_n^N}|\le c_\sigma$. Besides, we obviously have

$$\mathcal{N}[g^n;\mathcal{C}_2^1(I_n^N)]\le\|D\sigma\|_\infty\,\mathcal{N}[f^n;\mathcal{C}_2^1(I_n^N)]\le c_\sigma\,\mathcal{N}[\tilde h_n;\mathcal{L}_1].$$

Going back to (76), we deduce

$$\mathcal{N}[\tilde z;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\le c_{x,\sigma}\left\{1+\varepsilon_n^\gamma\,\mathcal{M}[\sigma(y);\mathcal{A}^\gamma_{x,g^n}(I_n^N)]+\varepsilon_n^{1-\gamma}\,\mathcal{N}[\tilde h_n;\mathcal{L}_1]\right\}.$$

The association of estimates (70) and (69) then entails

$$\begin{aligned}\mathcal{N}[\tilde z;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\le c^1_{x,\sigma}\big\{&1+\varepsilon_n^{1-\gamma}\,\mathcal{N}[\tilde h_n;\mathcal{L}_1]\\&+\varepsilon_n^\gamma\,\mathcal{N}[\tilde y;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]^2+\varepsilon_n\,\mathcal{N}[\tilde y;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\,\mathcal{N}[\tilde h_n;\mathcal{L}_1]+\varepsilon_n^{2-\gamma}\,\mathcal{N}[\tilde h_n;\mathcal{L}_1]^2\big\}.\end{aligned}\tag{77}$$

In order to establish the invariance of $B_n^{\tilde h_n}$, or in other words to prove that $\mathcal{N}[\tilde z;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\le(N+n)^{\alpha_2}$ (for N large enough), a first series of conditions naturally arises from (77):

$$\begin{cases}\alpha_1-(1-\gamma)<\alpha_2\\2\alpha_2-\gamma<\alpha_2\\\alpha_1+\alpha_2-1<\alpha_2\\2\alpha_1-2+\gamma<\alpha_2,\end{cases}\tag{78}$$

and it is easily seen that this system reduces to

$$\begin{cases}\alpha_2<\gamma\\\alpha_1<1-\gamma+\alpha_2.\end{cases}\tag{79}$$

If $\alpha_1,\alpha_2>0$ are assumed to satisfy those two conditions, then we can pick N large enough so that the expected stability property is checked. Indeed, from (77), we get $\mathcal{N}[\tilde z;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\le 6c^1_{x,\sigma}\,(N+n)^{\alpha_3}$, where α_3 stands for the largest left-hand side of system (78). As $\alpha_3<\alpha_2$, we can pick N such that for any $n\ge 0$, $(N+n)^{\alpha_2-\alpha_3}\ge 6c^1_{x,\sigma}$, and so $\mathcal{N}[\tilde z;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\le(N+n)^{\alpha_2}$.

It now remains to analyze condition (H). To this end, write

$$\tilde y^i_{l^N_{n+1}}=S_{\varepsilon_n}\,\tilde y^i_{l_n^N}+(\delta\tilde y^i)_{l^N_{n+1}l^N_n}=S_{\varepsilon_n}\,\tilde h^i_n+\tilde X^{x,j}_{l^N_{n+1}l^N_n}\,\sigma^{ij}(\tilde h_n)+\tilde y^{\sharp,i}_{l^N_{n+1}l^N_n},$$

which leads to

$$|\tilde y_{l^N_{n+1}}|\le|\tilde h_n|+c_{x,\sigma}\,\varepsilon_n^\gamma+\varepsilon_n^{2\gamma}\,\mathcal{N}[\tilde y;\tilde{\mathcal{Q}}^\gamma_x(I_n^N)]\le(N+n)^{\alpha_1}+c_{x,\sigma}\,(N+n)^{-\gamma}+(N+n)^{\alpha_2-2\gamma}.\tag{80}$$

Then observe the asymptotic equivalent $\frac{c_{x,\sigma}m^{-\gamma}+m^{\alpha_2-2\gamma}}{(m+1)^{\alpha_1}-m^{\alpha_1}}\sim_{m\to\infty}\frac{c_{x,\sigma}m^{-\gamma}+m^{\alpha_2-2\gamma}}{\alpha_1\,m^{\alpha_1-1}}$: thus, by adding to (79) the (compatible) condition

$$\alpha_1>1-\gamma,\tag{81}$$

there exists an integer N large enough such that, for any $n\in\mathbb{N}^*$,

$$(N+n)^{\alpha_1}+c_{x,\sigma}\,(N+n)^{-\gamma}+(N+n)^{\alpha_2-\gamma}\le(N+n+1)^{\alpha_1}.$$

We pick N in this way to retrieve, from (80), property (H).

Step 2: Contraction property. Let $\tilde y^{(1)},\tilde y^{(2)}\in B_n^{\tilde h_n}$, $\tilde z^{(1)}\equiv\Gamma(\tilde y^{(1)}),\tilde z^{(2)}\equiv\Gamma(\tilde y^{(2)})$, and set $y^{(1)}\equiv T_{a,\tilde\phi}(\tilde y^{(1)}),y^{(2)}\equiv T_{a,\tilde\phi}(\tilde y^{(2)})$. Here again, the expected property will stem from the estimates of subsection 5.3. It is first worth noticing that if $y^{(1)}, y^{(2)} \in \mathcal{A}^\gamma_{x,f}(I)$, then $y^{(1)}-y^{(2)} \in \mathcal{A}^\gamma_{x,0}(I)$ and $\mathcal{N}[y^{(1)}-y^{(2)}; \mathcal{A}^\gamma_{x,0}(I)] = \mathcal{N}[y^{(1)}-y^{(2)}; \mathcal{Q}^\gamma_x(I)]$. Therefore, according to (72),

$$\begin{aligned}&\mathcal{N}[\tilde{z}^{(1)} - \tilde{z}^{(2)}; \tilde{\mathcal{Q}}^\gamma_x(I^N_n)] \\ &\leq c\left\{\mathcal{N}[\sigma(y^{(1)}) - \sigma(y^{(2)}); \mathcal{C}^0_1(I^N_n)] + |\sigma(y^{(1)})^x_{l^N_n} - \sigma(y^{(2)})^x_{l^N_n}| + \varepsilon^\gamma_n \mathcal{N}[\sigma(y^{(1)}) - \sigma(y^{(2)}); \mathcal{Q}^\gamma_x(I^N_n)]\right\}.\end{aligned}$$

Of course, $\sigma(y^{(1)})^x_{l^N_n} = \sigma(y^{(2)})^x_{l^N_n}$ and

$$\mathcal{N}[\sigma(y^{(1)}) - \sigma(y^{(2)}); \mathcal{C}^0_1(I^N_n)] \leq \varepsilon^\gamma_n \mathcal{N}[\sigma(y^{(1)}) - \sigma(y^{(2)}); \mathcal{Q}^\gamma_x(I^N_n)],$$

which, together with estimates (71) and (69), easily gives

$$\mathcal{N}[\tilde{z}^{(1)} - \tilde{z}^{(2)}; \tilde{\mathcal{Q}}^\gamma_x(I^N_n)] \leq c_{x,\sigma}\, J_{N+n}\, \mathcal{N}[\tilde{y}^{(1)} - \tilde{y}^{(2)}; \tilde{\mathcal{Q}}^\gamma_x(I^N_n)],$$

with

$$J_n = n^{-\gamma} + n^{-\gamma+2\alpha_2} + n^{2\alpha_1-(2-\gamma)} + n^{\alpha_1-1} + n^{\alpha_1+\alpha_2-1},$$

In order to ensure that $\lim_{N\to\infty} J_N = 0$, we are this time led to the system

$$\begin{cases} 2\alpha_2 - \gamma < 0 \\ 2\alpha_1 - 2 + \gamma < 0 \\ \alpha_1 - 1 < 0 \\ \alpha_1 + \alpha_2 - 1 < 0, \end{cases} \tag{82}$$

which, intersected with both conditions (79) and (81), provides the final assumption

$$\begin{cases} 0 < \alpha_2 < \frac{\gamma}{2} \\ 1 - \gamma < \alpha_1 < 1 - \gamma + \alpha_2. \end{cases}$$

Once such coefficients fixed, we can choose N large enough so that both the contraction property and the property (H) are satisfied on the invariant balls $B^{\tilde{h}_n}_n$, $n \geq 0$.

Step 3: Patching the solutions. The construction of the expected global solution $\tilde{y} \in \tilde{\mathcal{Q}}^\gamma_x([0,T])$ is now reduced to a patching argument.

First, we define the sequence $(\tilde{y}^{(n)}, \tilde{y}^{(n),x})_{n\geq 0}$ according to the following iterative procedure: $(\tilde{y}^{(0)}, \tilde{y}^{(0),x}) \in \tilde{\mathcal{Q}}^\gamma_x(I^N_0)$ is the fixed point of Γ in B^0_0 and for any $n \geq 1$, $(\tilde{y}^{(n)}, \tilde{y}^{(n),x}) \in \tilde{\mathcal{Q}}^\gamma_x(I^N_n)$ is the fixed point of Γ in $B_n^{\tilde{y}^{(n-1)}_{l^N_n}}$. The latter construction is made possible by the two previous steps. Then we define, for any $t \in [0,T]$,

$$\tilde{y}_t \equiv \sum_{n=0}^{N_T} \tilde{y}^{(n)}_t\, \mathbf{1}_{I^N_n}(t) \quad , \quad \tilde{y}^x_t \equiv \sum_{n=0}^{N_T} \tilde{y}^{(n),x}_t\, \mathbf{1}_{I^N_n}(t),$$

where N_T stands for the smallest integer such that $\sum_{n=0}^{N_T} |I^N_n| \geq T$.

If $l^N_{k-1} < s \leq l^N_k < \ldots < l^N_{k'} \leq t < l^N_{k'+1}$, one can appeal to the decomposition

$$(\tilde{\delta}\tilde{y})_{ts} = S_{t-l^N_k} \cdot (\tilde{\delta}\tilde{y})_{l^N_k s} + (\tilde{\delta}\tilde{y})_{t l^N_{k'}} + \sum_{i=k}^{k'-1} S_{t-l^N_{i+1}} \cdot (\tilde{\delta}\tilde{y})_{l^N_{i+1} l^N_i}, \tag{83}$$

together with the relation $\tilde{\delta}\tilde{X}^x = 0$, to deduce

$$(\tilde{\delta}\tilde{y}^i)_{ts} = \tilde{X}^{x,j}_{ts}\, \tilde{y}^{x,ij}_s + \tilde{y}^{\sharp,i}_{ts}, \tag{84}$$

with $\tilde{y}^{\sharp,i}_{ts} = \tilde{y}^{\sharp,1,i}_{ts} + \tilde{y}^{\sharp,2,i}_{ts}$,

$$\tilde{y}^{\sharp,1,i}_{ts} \equiv \tilde{X}^{x,j}_{tl^N_k}\left[\tilde{y}^{(k),x,ij}_{l^N_k} - \tilde{y}^{(k-1),x,ij}_{s}\right] + \sum_{p=k+1}^{k'} \tilde{X}^{x,j}_{tl^N_p}\left[\tilde{y}^{(p),x,ij}_{l^N_p} - \tilde{y}^{(p-1),x,ij}_{l^N_{p-1}}\right],$$

$$\tilde{y}^{\sharp,2}_{ts} \equiv S_{t-l^N_k}\cdot \tilde{y}^{(k-1),\sharp,i}_{l^N_k s} + \tilde{y}^{(k'),\sharp,i}_{tl^N_{k'}} + \sum_{p=k}^{k'-1} S_{t-l^N_{p+1}}\cdot \tilde{y}^{(p),\sharp,i}_{l^n_{p+1}l^N_p}.$$

From those expressions, and owing to the regularity of each $\tilde{y}^{(k),x}$, it is easily seen that $(\tilde{y},\tilde{y}^x)$ defines an element of $\tilde{\mathcal{Q}}^{\gamma}_x([0,T])$.

Let us finally go back to (83), which can also be written as

$$(\tilde{\delta}\tilde{y}^i)_{ts} = S_{t-l^N_k}\cdot \mathcal{J}_{l^N_k s}(\tilde{d}x^j\,\sigma^{ij}(y)) + \mathcal{J}_{tl^N_{k'}}(\tilde{d}x^j\,\sigma^{ij}(y)) + \sum_{p=k}^{k'-1} S_{t-l^N_{p+1}}\cdot \mathcal{J}_{l^N_{p+1}l^N_p}(\tilde{d}x^j\,\sigma^{ij}(y)).$$

By invoking the relation $\tilde{\delta}\left(\mathcal{J}(\tilde{d}x^j\,z^{ij})\right) = 0$, we get

$$\mathcal{J}_{tl^N_{k'-1}}(\tilde{d}x^j\,\sigma^{ij}(y)) = \mathcal{J}_{tl^N_{k'}}(\tilde{d}x^j\,\sigma^{ij}(y)) + S_{t-l^N_{k'}}\cdot \mathcal{J}_{l^N_{k'}l^N_{k'-1}}(\tilde{d}x^j\,\sigma^{ij}(y)),$$

hence

$$(\tilde{\delta}\tilde{y}^i)_{ts} = S_{t-l^N_k}\cdot \mathcal{J}_{l^N_k s}(\tilde{d}x^j\,\sigma^{ij}(y)) + \mathcal{J}_{tl^N_{k'-1}}(\tilde{d}x^j\,\sigma^{ij}(y)) + \sum_{p=k}^{k'-2} S_{t-l^N_{p+1}}\cdot \mathcal{J}_{l^N_{p+1}l^N_p}(\tilde{d}x^j\,\sigma^{ij}(y)).$$

The iteration of this simplification procedure leads to $(\tilde{\delta}\tilde{y}^i)_{ts} = \mathcal{J}_{ts}(\tilde{d}x^j\,\sigma^{ij}(y))$ for all $s,t\in[0,T]$.

The uniqueness of the solution is easy to establish with the estimates of Step 2, just as in the diffusion case (see for instance the proof of [15, Theorem 2.6]). As for the control result (73), it is a consequence of decomposition (84), having in mind the local controls induced by the balls $B^{\tilde{h}_n}_n$. □

Once endowed with the control result (73), the continuity of the Itô map associated to (38) can be proved along the same lines as in the case of ordinary systems. The reader is (here again) refered to the proof of [15, Theorem 2.6] for a detailed analysis of the method. For the statement of this result, we call 'initial' condition of (52) the constant a that appears in the system. This actually corresponds to the initial condition of the original equation (2).

Corollary 5.11. *Assume that the three hypotheses 3, 4 and 5 are satisfied for two distinct paths $x^{(1)}$ and $x^{(2)}$, and let $\sigma\in\mathcal{C}^{3,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$. If $\tilde{y}^{(1)}$ (resp. $\tilde{y}^{(2)}$) denotes the solution of the system (52) driven by $x^{(1)}$ (resp. $x^{(2)}$) in the sense of Proposition 5.5, with 'initial' condition $a^{(1)}$ (resp. $a^{(2)}$), then*

$$\mathcal{N}[\tilde{y}^{(1)}-\tilde{y}^{(2)};\tilde{\mathcal{C}}^{\gamma}_1([0,T];\mathbb{R}^m)] \le c_{x^{(1)},x^{(2)}}\Big\{ \left|a^{(1)}-a^{(2)}\right|$$
$$+\mathcal{N}[\tilde{X}^{x^{(1)}}-\tilde{X}^{x^{(2)}};\tilde{\mathcal{C}}^{\gamma}_2] + \mathcal{N}[\tilde{X}^{Ax^{(1)}}-\tilde{X}^{Ax^{(2)}};\tilde{\mathcal{C}}^{1+\gamma}_{2,0}] + \mathcal{N}[\tilde{X}^{x^{(1)}x^{(1)}}-\tilde{X}^{x^{(2)}x^{(2)}};\tilde{\mathcal{C}}^{2\gamma}_2]\Big\}, \quad (85)$$

with

$$c_{x^{(1)},x^{(2)}} \equiv C\left(\tilde{X}^{x^{(1)}},\tilde{X}^{x^{(2)}},\tilde{X}^{Ax^{(1)}},\tilde{X}^{Ax^{(2)}},\tilde{X}^{x^{(1)}x^{(1)}},\tilde{X}^{x^{(2)}x^{(2)}}\right),$$

where C is a function that grows with its arguments.

Let us conclude with a transposition of this result into the original setting of (2):

Corollary 5.12. *Under Hypotheses 3, 4 and 5, and assuming that $\sigma \in \mathcal{C}^{3,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$, the system (2), interpreted with Propositions 5.5 and 5.7, admits a unique solution y in $\mathcal{Q}_x^\gamma([0,T];\mathbb{R}^d)$. Moreover, the continuity result (85) remains true for y, w.r.t the (classical) Hölder norm $\mathcal{N}[.;\mathcal{C}_1^\gamma]$ in the left-hand-side.*

Proof. As in the Young case, it suffices to set, for any $t \in [0,T]$, $y_t \equiv T_{a,\tilde{\phi}}(\tilde{y}_t)$, where $\tilde{y}$ is the path given by Theorem 5.10. □

6. Application to rough paths

The aim now consists in proving that the hypotheses we have raised all through the previous two sections can actually be checked for a large class of Hölder paths x. If we put those different hypotheses (Hypotheses 2, 3, 4 and 5) together, we have to show the existence of three paths $(\tilde{X}^x, \tilde{X}^{Ax}, \tilde{X}^{xx})$ that would extend the three definitions (valid when x is differentiable)

$$\tilde{X}_{ts}^x(\xi) = \int_s^t S_{t-u}(\xi)\,dx_u \quad , \quad \tilde{X}_{ts}^{Ax}(\xi) = \int_s^t A_{t-u}(\xi)\,dx_u, \tag{86}$$

$$\tilde{X}_{ts}^{xx}(\xi) = \int_s^t S_{t-u}(\xi)\,dx_u \otimes \mathbf{x}_{us}^{\mathbf{1}}, \tag{87}$$

above a γ-Hölder x, with $\gamma > 1/3$ (remember that $\mathbf{x}^{\mathbf{1}} \equiv \delta x$).

6.1. An integration by parts argument.
We propose here to extend (86)-(87) via elementary integrations by parts, following the general scheme:

$$\int_s^t S_{t-u}(\xi)\,dx_u = \int_s^t S_{t-u}(\xi)\,d(x_u - x_s) = \mathbf{x}_{ts}^{\mathbf{1}} - \int_s^t \frac{d}{du}(S_{t-u}(\xi))\mathbf{x}_{us}^{\mathbf{1}}\,du. \tag{88}$$

Let us first evoke the Young case ($\gamma > 1/2$), for which only $\tilde{X}^x$ comes into the picture:

Proposition 6.1. *Let $x \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^m)$, with $\gamma > 1/2$. If $\int_{\mathbb{R}} d\xi\,|\tilde{\phi}(\xi)|(1+|\xi|^{1+\gamma}) < \infty$, then any sequence of differentiable paths x^ε such that*

$$\mathcal{N}[x^\varepsilon - x;\mathcal{C}_1^\gamma([0,T];\mathbb{R}^m)] \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

satisfies Hypothesis 2.

Proof. For any differentiable path $\tilde{x}$, one has, thanks to (88),

$$|\tilde{X}_{ts}^{\tilde{x}}(\xi)| \le |\tilde{\mathbf{x}}_{ts}^{\mathbf{1}}| + |\xi| \int_s^t |\tilde{\mathbf{x}}_{us}^{\mathbf{1}}|\,du \le c\,\mathcal{N}[\tilde{x};\mathcal{C}_1^\gamma]\,|t-s|^\gamma\,\{1+|\xi|\}\,. \tag{89}$$

Since $\int_{\mathbb{R}} d\xi\,|\tilde{\phi}(\xi)|(1+|\xi|^{1+\gamma}) < \infty$, it is then easily seen that $(\tilde{X}^{x^\varepsilon})_{\varepsilon>0}$ is a Cauchy sequence in $\mathcal{C}_{2,\gamma}^\gamma(\mathbb{R}^m)$. □

The extension of the two paths $\tilde{X}^{Ax}$ et $\tilde{X}^{xx}$, which is needed in order to apply the results of Section 5, that is to say when $\gamma \in (1/3,1/2]$, will stem from the same kind of argument. It suffices to notice that, if x is a differentiable path,

$$\tilde{X}_{ts}^{Ax}(\xi) = \int_s^t \frac{d}{du}(S_{t-u}(\xi))\mathbf{x}_{us}^{\mathbf{1}}\,du, \tag{90}$$

and if we denote by $\mathbf{x^2}$ the *standard* Lévy area of x ($\mathbf{x}^{\mathbf{2}}_{ts} \equiv \int_s^t dx_v \otimes (\delta x)_{vs}$), which is at the core of the rough paths methods, one has

$$\tilde{X}^{xx}_{ts}(\xi) = \int_s^t S_{t-u}(\xi) \frac{d}{du}(\mathbf{x}^{\mathbf{2}}_{us})\, du = \mathbf{x}^{\mathbf{2}}_{ts} + \int_s^t \frac{d}{du}(S_{t-u}(\xi))\, \mathbf{x}^{\mathbf{2}}_{us}\, du. \tag{91}$$

With the same argument as in the previous proof, those transformations lead to the assertion:

Proposition 6.2. *Let x a path allowing the construction of a 2-rough path $\mathbf{x} = (\mathbf{x^1}, \mathbf{x^2}) \in \mathcal{C}_2^\gamma(\mathbb{R}^m) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})$, for some coefficient $\gamma > 1/3$. If $\int_{\mathbb{R}} d\xi\, |\tilde{\phi}(\xi)|(1+|\xi|^2) < \infty$, then any sequence x^ε of differentiable paths such that*

$$\mathcal{N}[x^\varepsilon - x; \mathcal{C}_1^\gamma([0,T];\mathbb{R}^m)] + \mathcal{N}[\mathbf{x}^{\varepsilon,\mathbf{2}} - \mathbf{x^2}; \mathcal{C}_2^{2\gamma}([0,T];\mathbb{R}^{m,m})] \xrightarrow{\varepsilon \to 0} 0 \tag{92}$$

satisfies the three hypotheses 3, 4 and 5.

We are thus in position to provide a more explicit formulation of Corollary 5.12:

Theorem 6.3. *Let $x : [0,T] \to \mathbb{R}^m$ a γ-Hölder path ($\gamma > 1/3$) allowing the construction of a geometric 2-rough path $\mathbf{x} = (\mathbf{x^1}, \mathbf{x^2}) \in \mathcal{C}_2^\gamma(\mathbb{R}^m) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{m,m})$. Assume that ϕ can be represented as (5) on $[0,T]$, for some function $\tilde{\phi}$ such that the integrability condition $\int_{\mathbb{R}} d\xi\, |\tilde{\phi}(\xi)|(1+|\xi|^2) < \infty$ is satisfied. Then, if $\sigma \in \mathcal{C}^{3,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$, the system (2), interpreted with Propositions 5.5 and 6.2, admits a unique solution in the space $\mathcal{Q}_x^\gamma([0,T];\mathbb{R}^d)$ of controlled paths. Moreover, the continuity statement (3) holds true.*

Remark 6.4. In retrospect, with the help of the continuity result (3), we can provide another (equivalent) interpretation of the rough system (2). Remember first that when x is a differentiable path, the interpretation given in Section 4 or in Section 5 coincides with the ordinary Volterra equation, understood in the Riemann-Lebesgue sense: this is the content of points (1) in Proposition 4.1 and Proposition 5.5, and one of the main principles of our approach. Consequently, due to (3), our understanding of the rough Volterra equation can also be summed up as follows: *for any sequence x^ε of differentiable paths that converges to x in the sense of (92), the sequence y^ε of ordinary solutions to (2) associated to x^ε converges to a path y with respect to the γ-Hölder topology.*

Remark 6.5. With the interpretation exhibited in Remark 6.4, it is easily seen that the solution y given by Theorem 6.3 does not depend on the particular representative $\tilde{\phi}$ in (5), provided the integrability condition is satisfied. Assume indeed that $\tilde{\phi}^1, \tilde{\phi}^2$ are such that $\int_{\mathbb{R}} d\xi\, |\tilde{\phi}^i(\xi)|(1+|\xi|^2) < \infty$ and $\phi^1_{|[0,T]} = \phi^2_{[0,T]} = \phi_{[0,T]}$, where $\phi^i(t) \equiv \int_{\mathbb{R}} d\xi\, S_t(\xi)\, \tilde{\phi}^i(\xi)$. If x is a differentiable path, the path y^1 (resp. y^2) associated to $\tilde{\phi}^1$ (resp. $\tilde{\phi}^2$) through Theorem 6.3 is known to be solution of the ordinary equation

$$y^i_t = a^i + \int_0^t \phi^i(t-u)\sigma(y^i_u)\, dx_u = a^i + \int_0^t \phi(t-u)\sigma(y^i_u)\, dx_u,$$

hence, by uniqueness, $y^1 = y^2$. The result in the general rough case can then be deduced by passing to the limit.

Keeping Remark 6.5 in mind, Theorem 1.1 is now obtained via the following elementary result:

Proposition 6.6. *If $\phi \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$, then there exists a function $\tilde{\phi}_T$ satisfying*

$$\int_{\mathbb{R}} d\xi\, |\tilde{\phi}_T(\xi)|(1+|\xi|^2) < \infty$$

and such that ϕ admits the representation (5) on $[0,T]$.

Proof. As we announced it in Subsection 3.1, it suffices to extend the restriction $\phi_{|[0,T]}$ into a compactly supported function $\phi_T \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$. Then

$$\phi_T = \mathcal{F}\tilde{\phi}_T, \quad \text{with } \tilde{\phi}_T(\xi) \equiv (\mathcal{F}^{-1}\phi_T)(\xi) = c\int_{\mathbb{R}} e^{2i\pi t\xi}\phi_T(t)\,dt.$$

Since $\tilde{\phi}_T \in L^2(\mathbb{R})$, one has

$$\begin{aligned}\int_{\mathbb{R}} d\xi\,|\tilde{\phi}_T(\xi)|(1+|\xi|^2) &\leq 2\int_{|\xi|\leq 1} d\xi\,|\tilde{\phi}_T(\xi)| + c\int_{|\xi|\geq 1}\frac{|\mathcal{F}^{-1}(\phi_T''')(\xi)|}{|\xi|}\\ &\leq c\left\{\|\tilde{\phi}_T\|_{L^2} + \|\mathcal{F}^{-1}(\phi_T''')\|_{L^2}\right\} < \infty.\end{aligned}$$

□

6.2. **The (fractional) Brownian motion case.** Owing to the results of [18] or [36], we know that the existence of a geometric 2-rough path holds true for a fractional Brownian motion with Hurst index $H > 1/3$. This means that Theorem 1.1 can be applied in this situation, giving birth to the first result of existence and uniqueness of a global solution for (2) when $1/3 < H < 1/2$. In the standard Brownian case ($H = 1/2$), this solution can be shown to (almost surely) coincide with the Stratonovich one (see for instance [18, Section 17.2] for a similar statement).

The Itô interpretation of (2) in presence of a standard Brownian motion $x = B$ can also be recovered from the considerations of Section 5, by defining the convolutional 2-rough path $(\tilde{X}^B, \tilde{X}^{AB}, \tilde{X}^{BB})$ as Itô integrals, ie $\tilde{X}^{B,i}_{ts}(\xi) \equiv \int_s^t S_{t-u}(\xi)\,dB^i_u$, $\tilde{X}^{AB,i}_{ts}(\xi) \equiv \int_s^t A_{t-u}(\xi)\,dB^i_u$, $\tilde{X}^{BB,ij}_{ts}(\xi) \equiv \int_s^t S_{t-u}(\xi)\,dB^i_u\,(\delta B^j)_{us}$. Let us sketch out the two steps of this identification, which essentially follows the lines of [14, Section 6.2].

First of all, remember that the Itô-Volterra equation

$$Y^i_t = a^i + \int_0^t \phi(t-u)\sigma^{ij}(Y_u)\,dB^j_u \quad , \quad t\in[0,T], \tag{93}$$

is known to have a unique solution under the assumptions of our study, namely ϕ, σ (at least) differentiable (see for instance [6]). Then, assuming that $\tilde{\phi} \in L^1(\mathbb{R})$, one can see with the help of the stochastic Fubbini theorem that (93) is equivalent to

$$(\tilde{\delta}\tilde{Y}^i)_{ts}(\xi) = \int_s^t S_{t-u}(\xi)\,dB^j_u\,\sigma^{ij}(Y_u) \quad , \quad Y^i_u = a^i + \int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\tilde{Y}^i_u(\xi). \tag{94}$$

The latter formulation allows to make the link with the formalism of Section 5:

Lemma 6.7. *Assume that* $\sigma \in \mathcal{C}^{1,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$ *and that* $\tilde{\phi}$ *satisfies* $\int_{\mathbb{R}} d\xi\,|\tilde{\phi}(\xi)|(1+|\xi|) < \infty$. *Then, with the notations of Section 5, the Itô solution* $\tilde{Y}$ *of (94) almost surely belongs to* $\tilde{\mathcal{Q}}^\gamma_B([0,T];\mathbb{R}^d)$ *for any* $0 < \gamma < 1/2$.

Proof. The decomposition of $\tilde{Y}$ as an element of $\tilde{\mathcal{Q}}^\gamma_B([0,T];\mathbb{R}^d)$ is naturally given by

$$(\tilde{\delta}\tilde{Y}^i)_{ts} = \tilde{X}^{B,j}_{ts}\sigma^{ij}(Y_s) + \tilde{Y}^{\sharp,i}_{ts}, \quad \text{with } \tilde{Y}^{\sharp,i}_{ts}(\xi) \equiv \int_s^t S_{t-u}(\xi)\,dB^j_u\,(\delta\sigma^{ij}(Y))_{us}.$$

In order to see that $\sigma(Y)$ (resp. $\tilde{Y}^\sharp$) almost surely belongs to $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^{d,m})$ (resp. $\tilde{\mathcal{C}}_2^{2\gamma}([0,T];\mathbb{R}^d)$), one can rely on a ($\tilde{\delta}$-)adapted version of the classical Garsia-Rodemich-Rumsey lemma, which reduces the problem to (easy) moments estimates. The reader is refered to [20, Lemma 3.8] for the statement of such a result in a convolutional context. Some additional details about this standard reasoning can also be found in [14, Proposition 6.8]. □

Once $\tilde{Y}$ has been identified as an element of $\tilde{\mathcal{Q}}_B^\gamma([0,T];\mathbb{R}^d)$, Proposition 5.5 provides us with a pathwise definition of the integral $\mathcal{J}\left(\tilde{d}B^j\,\sigma^{ij}(Y)\right)$ based on the Itô 2-rough paths $(\tilde{X}^B,\tilde{X}^{AB},\tilde{X}^{BB})$. The second step towards the expected identification can now be expressed as follows:

Proposition 6.8. *Assume that $\sigma\in\mathcal{C}^{3,\boldsymbol{b}}(\mathbb{R}^d;\mathbb{R}^{d,m})$ and that $\tilde{\phi}$ satisfies $\int_{\mathbb{R}} d\xi\,|\tilde{\phi}(\xi)|(1+|\xi|)<\infty$. Then, for any $\xi\in\mathbb{R}$, the integral $\mathcal{J}\left(\tilde{d}B^j\,\sigma^{ij}(Y)\right)(\xi)$ constructed in Proposition 5.5 almost surely coincides with the Itô integral $\int_s^t S_{t-u}(\xi)\,dB_u^j\,\sigma^{ij}(Y_u)$. Consequently, the solution given by Theorem 5.10 is (a.s.) equal to the Itô solution of (93) and the following continuity property holds: if Y (resp. $\hat{Y}$) stands for the solution of (93) with initial condition a (resp. $\hat{a}$), one has*

$$\mathcal{N}[Y-\hat{Y};\mathcal{C}_1^\gamma(\mathbb{R}^d)]\leq C_B\,|a-\hat{a}|\,, \tag{95}$$

for some (a.s.) finite random variable C_B.

Proof. Similarly to (60), one can decompose the Itô integral as

$$\int_s^t S_{t-u}(\xi)\,dB_u^j\,\sigma^{ij}(Y_u)=\tilde{X}_{ts}^{B,j}(\xi)\sigma^{ij}(Y_s)+\tilde{X}_{ts}^{BB,jl}(\xi)\,L_{\tilde{\phi}}\,\partial_k\sigma^{ij}(Y_s)\sigma^{kl}(Y_s)+\tilde{R}_{ts}^i(\xi),$$

where $L_{\tilde{\phi}}\equiv\int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)$ and $\tilde{R}_{ts}^i(\xi)\equiv\int_s^t S_{t-v}(\xi)\,dB_v^j\,M_{vs}^{ij}$ with

$$\begin{aligned}M_{vs}^{ij}\equiv{}&\left[(\delta\sigma^{ij}(Y))_{us}-(\delta Y^k)_{us}\partial_k\sigma^{ij}(Y_s)\right]\\&+\int_{\mathbb{R}} d\xi\,\tilde{\phi}(\xi)\left\{\tilde{X}^{AB,l}(\xi)\sigma^{kl}(Y_s)+\tilde{Y}_{us}^{\sharp,k}(\xi)+A_{us}(\xi)\tilde{Y}_s^k(\xi)\right\}\cdot\partial_k\sigma^{ij}(Y_s).\end{aligned}$$

From this expression, one can apply the ($\tilde{\delta}$-)G-R-R lemma we have already evoked in the proof of Lemma 6.7 and assert that $\tilde{R}\in\tilde{\mathcal{C}}_2^\mu([0,T];\mathbb{R}^d)$ a.s., for some coefficient $\mu>1$ (this actually follows the lines of [14, Proposition 6.11]). Consequently, by setting $(\tilde{\delta}\tilde{Z}^i)_{ts}\equiv\mathcal{J}_{ts}\left(\tilde{d}B^j\,\sigma^{ij}(Y)\right)$, one gets $\tilde{\delta}(\tilde{Y}-\tilde{Z})\in\operatorname{Im}\tilde{\delta}\cap\tilde{\mathcal{C}}_2^{\tilde{\mu}}([0,T];\mathbb{R}^d)$ with $\tilde{\mu}>1$, which, according to Lemma 3.7, leads to $\tilde{\delta}\tilde{Y}=\tilde{\delta}\tilde{Z}$, so that the two integrals indeed coincide.

The identification of the solutions now follows from the uniqueness property contained in Theorem 5.10, while (95) is deduced from Corollary 5.12. □

Remark 6.9. The above integrability assumption $\int_{\mathbb{R}} d\xi\,|\tilde{\phi}(\xi)|(1+|\xi|)<\infty$ (possibly translated into $\phi\in\mathcal{C}^2(\mathbb{R};\mathbb{R})$ as in Proposition 6.6) is here weaker than the hypothesis of Theorem 6.3, namely $\int_{\mathbb{R}} d\xi\,|\tilde{\phi}(\xi)|(1+|\xi|^2)<\infty$. This is due to the relative crudeness of the integration by parts argument used in Subsection 6.1, which entails a loss of "spatial" regularity through the derivative $\frac{d}{du}S_u(\xi)=c\,\xi\,S_u(\xi)$. The more direct definition of the convolutional Brownian rough paths as Itô integrals allows to avoid this issue.

Remark 6.10. Of course, the interest of our study in the Brownian case does not lie in the exhibition of a solution for (93), which has been known for a long time. On the other hand, the continuity property of the flow, which appears as a typical consequence of the rough paths strategy, is new to our knowledge. Similarly to [13, 15], it is likely to offer new perspectives as far as the discretization of stochastic Volterra systems is concerned (to be compared with [35]). For sake of conciseness, we prefer to leave this task in abeyance, though.

Acknowledgements

We are very grateful to an anonymous referee whose comments and suggestions have led to significant improvements of the paper, both in the content and in the presentation of the results.

References

[1] E. Alòs and D. Nualart. Anticipating stochastic Volterra equations. *Stochastic Process. Appl.*, 72(1):73–95, 1997.

[2] J. A. D. Appleby. Exponential asymptotic stability of nonlinear Itô-Volterra equations with damped stochastic perturbations. *Funct. Differ. Equ.*, 12(1-2):7–34, 2005.

[3] John A. D. Appleby, Siobhán Devin, and David W. Reynolds. Almost sure convergence of solutions of linear stochastic Volterra equations to nonequilibria limits. *J. Integral Equations Appl.*, 19(4):405–437, 2007.

[4] John A. D. Appleby and Alan Freeman. Exponential asymptotic stability of linear Itô-Volterra equations with damped stochastic perturbations. *Electron. J. Probab.*, 8:no. 22, 22 pp. (electronic), 2003.

[5] John A. D. Appleby and David W. Reynolds. Decay rates of solutions of linear stochastic Volterra equations. *Electron. J. Probab.*, 13:no. 30, 922–943, 2008.

[6] M. A. Berger and V. J. Mizel. Volterra equations with Itô integrals. I. *J. Integral Equations*, 2(3):187–245, 1980.

[7] M. A. Berger and V. J. Mizel. Volterra equations with Itô integrals. II. *J. Integral Equations*, 2(4):319–337, 1980.

[8] M. Besalú and C. Rovira. Stochastic Volterra equations driven by fractional Brownian motion with Hurst parameter H $> 1/2$, 2010.

[9] M. Caruana and P. Friz. Partial differential equations driven by rough paths. *J. Differential Equations*, 247(1):140–173, 2009.

[10] W. G. Cochran, J. Lee, and J. Potthoff. Stochastic Volterra equations with singular kernels. *Stochastic Process. Appl.*, 56(2):337–349, 1995.

[11] L. Coutin and L. Decreusefond. Stochastic Volterra equations with singular kernels. In *Stochastic analysis and mathematical physics*, volume 50 of *Progr. Probab.*, pages 39–50. Birkhäuser Boston, Boston, MA, 2001.

[12] L. Coutin and Z. Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields*, 122(1):108–140, 2002.

[13] A. Deya. Numerical schemes for the rough heat equation, 2010.

[14] A. Deya, M. Gubinelli, and S. Tindel. Non-linear rough heat equations. *To appear in Probab. Theory Related Fields*, 2011.

[15] A. Deya, A. Neuenkirch, and S. Tindel. A Milstein-type scheme without Levy area terms for SDEs driven by fractional Brownian motion. *To appear in Ann. Inst. H. Poincaré*, 2011.

[16] A. Deya and S. Tindel. Rough Volterra equations. I. The algebraic integration setting. *Stoch. Dyn.*, 9(3):437–477, 2009.

[17] Boualem Djehiche and M'hamed Eddahbi. Large deviations for a stochastic Volterra-type equation in the Besov-Orlicz space. *Stochastic Process. Appl.*, 81(1):39–72, 1999.

[18] P. Friz and N. Victoir. *Multidimensional dimensional processes seen as rough paths.* Cambridge University Press, 2010.
[19] M. Gubinelli. Controlling rough paths. *J. Funct. Anal.*, 216(1):86–140, 2004.
[20] Massimiliano Gubinelli and Samy Tindel. Rough evolution equations. *Ann. Probab.*, 38(1):1–75, 2010.
[21] David N. Keck and Mark A. McKibben. Abstract semilinear stochastic Itô-Volterra integrodifferential equations. *J. Appl. Math. Stoch. Anal.*, pages Art. ID 45253, 22, 2006.
[22] S. C. Kou. Stochastic modeling in nanoscale biophysics: subdiffusion within proteins. *Ann. Appl. Stat.*, 2(2):501–535, 2008.
[23] S. C. Kou. A selective view of stochastic inference and modeling problems in nanoscale biophysics. *Sci. China Ser. A*, 52(6):1181–1211, 2009.
[24] T. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.
[25] T. Lyons and Z. Qian. *System control and rough paths.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002. Oxford Science Publications.
[26] A. Neuenkirch, I. Nourdin, and S. Tindel. Delay equations driven by rough paths. *Electron. J. Probab.*, 13:no. 67, 2031–2068, 2008.
[27] David Nualart and Carles Rovira. Large deviations for stochastic Volterra equations. *Bernoulli*, 6(2):339–355, 2000.
[28] B. Øksendal and T. S. Zhang. The stochastic Volterra equation. In *Barcelona Seminar on Stochastic Analysis (St. Feliu de Guíxols, 1991)*, volume 32 of *Progr. Probab.*, pages 168–202. Birkhäuser, Basel, 1993.
[29] M. Ait Ouahra and M. Mellouk. Strassen's law of the iterated logarithm for stochastic Volterra equations and applications. *Stochastics*, 77(2):191–203, 2005.
[30] E. Pardoux and P. Protter. Stochastic Volterra equations with anticipating coefficients. *Ann. Probab.*, 18(4):1635–1655, 1990.
[31] P. Protter. Volterra equations driven by semimartingales. *Ann. Probab.*, 13(2):519–530, 1985.
[32] L. Quer-Sardanyons and S. Tindel. The 1-d stochastic wave equation driven by a fractional Brownian sheet. *Stochastic Process. Appl.*, 117(10):1448–1472, 2007.
[33] Carles Rovira and Marta Sanz-Solé. Large deviations for stochastic Volterra equations in the plane. *Potential Anal.*, 12(4):359–383, 2000.
[34] Constantin Tudor. On Volterra stochastic equations. *Boll. Un. Mat. Ital. A (6)*, 5(3):335–344, 1986.
[35] Constantin Tudor and Maria Tudor. Approximation schemes for Itô-Volterra stochastic equations. *Bol. Soc. Mat. Mexicana (3)*, 1(1):73–85, 1995.
[36] J. Unterberger. Stochastic calculus for fractional Brownian motion with Hurst exponent $H > \frac{1}{4}$: a rough path method by analytic extension. *Ann. Probab.*, 37(2):565–614, 2009.
[37] Z. Wang. Existence and uniqueness of solutions to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients. *Statist. Probab. Lett.*, 78(9):1062–1071, 2008.
[38] Jiongmin Yong. Backward stochastic Volterra integral equations and some related problems. *Stochastic Process. Appl.*, 116(5):779–795, 2006.
[39] Jiongmin Yong. Continuous-time dynamic risk measures by backward stochastic Volterra integral equations. *Appl. Anal.*, 86(11):1429–1442, 2007.
[40] Jiongmin Yong. Well-posedness and regularity of backward stochastic Volterra integral equations. *Probab. Theory Related Fields*, 142(1-2):21–77, 2008.
[41] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67(1):251–282, 1936.
[42] Xicheng Zhang. Euler schemes and large deviations for stochastic Volterra equations with singular kernels. *J. Differential Equations*, 244(9):2226–2250, 2008.
[43] Xicheng Zhang. Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J. Funct. Anal.*, 258(4):1361–1425, 2010.

Aurélien Deya and Samy Tindel: Institut Élie Cartan Nancy, Nancy-Université, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France. *Email:* `deya@iecn.u-nancy.fr`, `tindel@iecn.u-nancy.fr`