On spectral renormalization group and the theory of resonances in non-relativistic QED

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Spectral renormalization group: general strategy

Problem and general strategy

- Want to study the spectral properties of some given Hamiltonian H acting on a Hilbert space $\mathcal{H}$
- Construct an effective Hamiltonian H_{eff} acting in a Hilbert space with fewer degrees of freedom, such that H_{eff} has the same spectral properties as H
- Use a scaling transformation to map H_{eff} to a scaled Hamiltonian $H^{(0)}$ acting on some Hilbert space $\mathcal{H}_0$
- Iterate the procedure to obtain a family of effective Hamiltonians $H^{(n)}$ acting on $\mathcal{H}_0$
- Estimate the "renormalized" perturbation terms $W^{(n)}$ appearing in $H^{(n)}$ and show that $W^{(n)}$ vanishes in the limit $n \to \infty$
- Study the limit Hamiltonian $H^{(\infty)}$
- Go back to the original Hamiltonian H using isospectrality of the renormalization map

Contents of the talk

Part I

The model

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Physical system and model

Physical System

- Non-relativistic matter: atom, ion or molecule composed of non-relativistic quantum charged particles (electrons and nuclei)
- Interacting with the quantized electromagnetic field, i.e. the photon field

Model: Standard model of non-relativistic QED

- Obtained by quantizing the Newton equations (for the charged particles) minimally coupled to the Maxwell equations (for the electromagnetic field)
- Restriction: charges distribution are localized in small, compact sets. Corresponds to introducing an ultraviolet cutoff suppressing the interaction between the charged particles and the high-energy photons
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Description of the atomic system (I)

Simplest physical system

- Hydrogen atom with an infinitely heavy nucleus fixed at the orign
- Spin of the electron neglected
- Units such that $\hbar = c = 1$

Hilbert space and Hamiltonian for the electron

- Hilbert space

$$\mathcal{H}_{\mathrm{el}} = \mathrm{L}^2(\mathbb{R}^3)$$

- Hamiltonian

$$H_{\mathrm{el}} = \frac{p_{\mathrm{el}}^2}{2m_{\mathrm{el}}} + V_\alpha(x_{\mathrm{el}}), \quad V_\alpha(x_{\mathrm{el}}) = -\frac{\alpha}{|x_{\mathrm{el}}|},$$

where $p_{\mathrm{el}} = -i\nabla_{x_{\mathrm{el}}}$, m_{el} is the electron mass, and $\alpha = e^2$ is the fine-structure constant ($\alpha \approx 1/137$)

- H_{el} is a self-adjoint operator in $\mathrm{L}^2(\mathbb{R}^3)$ with domain

$$\mathcal{D}(H_{\mathrm{el}}) = \mathcal{D}(p_{\mathrm{el}}^2) = \mathrm{H}^2(\mathbb{R}^3)$$

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Description of the atomic system (II)

Spectrum of H_{el}

- An infinite increasing sequence of negative, isolated eigenvalues of finite multiplicities $\{E_j\}_{j\in\mathbb{N}}$
- The semi-axis $[0,\infty)$ of continuous spectrum

Bohr's condition

- According to the physical picture, the electron jumps from an initial state of energy E_i to a final state of lower energy E_f by emitting a photon of energy $E_i - E_f$
- To capture this image mathematically, we need to take into account the interaction between the electron and the photon field
- The ground state energy E_0 is expected to remain an eigenvalue (stability of the system)
- The excited eigenvalues E_j, $j \geq 1$, associated with bound states are expected to turn into resonances associated with metastable states of finite lifetime

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Description of the photon field: n-photons space

n-photons space

- Hilbert space for 1 photon

$$\mathfrak{h} = \mathrm{L}^2(\mathbb{R}^3 \times \{1,2\})$$

Notations: $\underline{\mathbb{R}}^3 = \mathbb{R}^3 \times \{1,2\}$, $K = (k,\lambda) \in \underline{\mathbb{R}}^3$,

$$\langle f, g \rangle = \int_{\underline{\mathbb{R}}^3} \bar{f}(K) g(K) \mathrm{d}K = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \bar{f}(k,\lambda) g(k,\lambda) \mathrm{d}k$$

- Hilbert space for n photons

$$\mathcal{F}_s^{(n)}(\mathfrak{h}) = S_n \otimes_{j=1}^n \mathfrak{h},$$

where S_n is the symmetrization operator. Hence a n-photons state is associated to a function

$$\Phi^{(n)}(K_1, \dots, K_n) \in \mathrm{L}^2((\underline{\mathbb{R}}^3)^n),$$

such that $\Phi^{(n)}(K_1, \dots, K_n)$ is symmetric with respect to $K_1, \dots K_n$

Description of the photon field: Fock space

Fock space

- Hilbert space for the photon field = symmetric Fock space over $\mathfrak{h}$,

$$\mathcal{H}_{\rm ph} = \mathcal{F}_s(\mathfrak{h}) = \bigoplus_{n=0}^{+\infty} \mathcal{F}_s^{(n)}(\mathfrak{h}), \quad \mathcal{F}_s^{(0)} = \mathbb{C}$$

- $\Phi \in \mathcal{H}_{\rm ph}$ can be written as

$$\Phi = (\underbrace{\Phi^{(0)}}_{\in\mathbb{C}}, \underbrace{\Phi^{(1)}(K_1)}_{\in \mathrm{L}^2(\mathbb{R}^3)}, \underbrace{\Phi^{(2)}(K_1, K_2)}_{\in \mathrm{L}^2((\mathbb{R}^3)^2)}, \dots)$$

- Scalar product

$$\langle \Phi, \Psi \rangle_{\mathcal{H}_{\rm ph}} = \sum_{n=0}^{+\infty} \langle \Phi^{(n)}, \Psi^{(n)} \rangle_{\mathcal{F}_s^{(n)}(\mathfrak{h})}$$

- Vacuum

$$\Omega = (1, 0, 0, \dots)$$

Description of the photon field: second quantization (I)

Second quantization of an operator

For b an operator acting on the 1-photon space $\mathfrak{h}$, the second quantization of b is the operator on $\mathcal{H}_{\rm ph}$ defined by

$$\mathrm{d}\Gamma(b)|_{\mathbb{C}} = 0,$$
$$\mathrm{d}\Gamma(b)|_{\mathcal{F}_s^{(n)}} = b \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes b \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes b$$

If b is self-adjoint, one verifies that $\mathrm{d}\Gamma(b)$ is essentially self-adjoint. The closure is then denoted by the same symbol

Examples

- Number of photons operator

$$N = \mathrm{d}\Gamma(\mathbb{1}), \quad \mathcal{D}(N) = \Big\{ \Phi \in \mathcal{H}_{\rm ph}, \sum_{n \in \mathbb{N}} n^2 \big\| \Phi^{(n)} \big\|^2_{\mathcal{F}_s^{(n)}} < +\infty \Big\},$$

For all $n \in \mathbb{N}$, $N\Phi^{(n)} = n\Phi^{(n)}$, and the spectrum is given by

$$\sigma(N) = \sigma_{\rm pp}(N) = \mathbb{N}$$

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Description of the photon field: second quantization (II)

Examples

- Energy of the free photon field

$$H_f = \mathrm{d}\Gamma(\omega),$$

where ω is the operator of multiplication by the relativistic dispersion relation

$$\omega(k) = |k|$$

For all $n \in \mathbb{N}$,

$$(H_f \Phi)^{(n)}(K_1, \dots, K_n) = \Big(\sum_{j=1}^{n} |k_j| \Big) \Phi^{(n)}(K_1, \dots, K_n)$$

Spectrum

$$\sigma(H_f) = [0, \infty), \quad \sigma_{\mathrm{pp}}(H_f) = \{0\}$$

Description of the photon field: creation and annihilation operators (I)

Creation and annihilation operators

- For $h \in \mathfrak{h}$, the creation operator $a^*(h) : \mathcal{H}_{\mathrm{ph}} \to \mathcal{H}_{\mathrm{ph}}$ is defined for $\Phi \in \mathcal{F}_s^{(n)}$ by

$$a^*(h)\Phi = \sqrt{n+1} S_{n+1} h \otimes \Phi$$

- The annihilation operator $a(h)$ is defined as the adjoint of $a^*(h)$
- $a^*(h)$ and $a(h)$ are closable, their closures are denoted by the same symbols
- Other expressions for $a^*(h)$ and $a(h)$ are

$$(a(h)\Phi)^{(n)}(K_1, \dots, K_n) = \sqrt{n+1} \int_{\underline{\mathbb{R}}^3} \bar{h}(K)\Phi^{(n+1)}(K, K_1, \dots, K_n) \mathrm{d}K,$$

$$(a^*(h)\Phi)^{(n)}(K_1, \dots, K_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(K_i)\Phi^{(n-1)}(K_1, \dots, \hat{K}_i, \dots, K_n),$$

where $\hat{K}_i$ means that the variable K_i is removed

Description of the photon field: creation and annihilation operators (II)

Canonical commutation relations

$$[a^*(f), a^*(g)] = [a(f), a(g)] = 0,$$
$$[a(f), a^*(g)] = \langle f, g \rangle_{\mathfrak{h}}$$

Physical notations

- We will use the following *notations*

$$a^*(f) = \int_{\mathbb{R}^3} f(K) a^*(K) \mathrm{d}K, \quad a(f) = \int_{\mathbb{R}^3} \bar{f}(K) a(K) \mathrm{d}K$$

(where $a^*(K)$ and $a(K)$ can be defined as operator-valued distributions)

- Likewise, we can write, for instance

$$N = \int_{\mathbb{R}^3} a^*(K) a(K) \mathrm{d}K, \quad H_f = \int_{\mathbb{R}^3} \omega(k) a^*(K) a(K) \mathrm{d}K$$

Description of the photon field: creation and annihilation operators (II)

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Description of the photon field: field operators

Field operators

For $h \in \mathfrak{h}$, the field operator $\Phi(h)$ is defined by

$$\Phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h))$$

One verifies that $\Phi(h)$ is essentially auto-adjoint, its closure is denoted by the same symbol

Standard model of non-relativistic QED: the Hamiltonian

Hilbert space for the electron and the photon field

$$\mathcal{H} = \mathcal{H}_{\rm el} \otimes \mathcal{H}_{\rm ph} = \mathrm{L}^2(\mathbb{R}^3; \mathcal{H}_{\rm ph})$$

Pauli-Fierz Hamiltonian

$$H_\alpha = \frac{1}{2m_{\rm el}}(p_{\rm el} - \alpha^{\frac{1}{2}} A(x_{\rm el}))^2 + V_\alpha(x_{\rm el}) + H_f$$

where, for all $x \in \mathbb{R}^3$,

$$A(x) = \int_{\mathbb{R}^3} \frac{\chi_{\alpha\Lambda}(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \left(a^*(K) e^{-ik\cdot x} + a(K) e^{ik\cdot x} \right) \mathrm{d}K$$

In other words, for all $x \in \mathbb{R}^3$, $A(x) = (A_1(x), A_2(x), A_3(x))$ where $A_j(x)$ is the field operator given by

$$A_j(x) = \Phi(h_j(x)), \quad h_j(x, K) = \frac{\chi_{\alpha\Lambda}(k)}{\sqrt{|k|}} \varepsilon_{\lambda,j}(k) e^{-ik\cdot x}$$

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Standard model of non-relativistic QED: coupling functions

Polarization vectors

The vectors $\varepsilon_\lambda(k) = (\varepsilon_{\lambda,1}(k), \varepsilon_{\lambda,2}(k), \varepsilon_{\lambda,3}(k))$, for $\lambda \in \{1,2\}$, are polarization vectors that can be chosen, for instance, as

$$\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k) = \frac{(-k_1 k_3, -k_2 k_3, k_1^2 + k_2^2)}{\sqrt{k_1^2 + k_2^2}\sqrt{k_1^2 + k_2^2 + k_3^2}}$$

(The family $(k/|k|, \varepsilon_1(k), \varepsilon_2(k))$ is an orthonormal basis of $\mathbb{R}^3$ for all $k \neq 0$)

Ultraviolet cutoff

The function $\chi_{\alpha\Lambda}$ is an ultraviolet cutoff at energy scale $\alpha\Lambda$ that can be chosen for instance as

$$\chi_{\alpha\Lambda}(k) = e^{-\frac{k^2}{\alpha^2\Lambda^2}},$$

where $\Lambda > 0$ is arbitrary large. Thanks to $\chi_{\alpha\Lambda}$, the coupling functions $h_j(x)$ belong to $\mathfrak{h}$ and hence the Hamiltonian is well-defined

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Standard model of non-relativistic QED: small coupling regime

Scaling transformation

- Fine-structure constant α treated as a small coupling parameter
- To treat the interaction (electron)-(transverse photons) as a perturbation, useful to apply a certain scaling transformation (corresponds to conjugating the Hamiltonian H_α with a unitary transformation). One then arrives at the new Hamiltonian (still denoted by H_α)

$$H_\alpha = \frac{1}{2m_{\rm el}}(p_{\rm el} - \alpha^{\frac{3}{2}} A(\alpha x_{\rm el}))^2 + V(x_{\rm el}) + H_f$$

where, for all $x \in \mathbb{R}^3$,

$$A(x) = \int_{\underline{\mathbb{R}}^3} \frac{\chi_\Lambda(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \left(a^*(K) e^{-ik\cdot x} + a(K) e^{ik\cdot x} \right) \mathrm{d}K,$$

and

$$V(x_{\rm el}) = -\frac{1}{|x_{\rm el}|}$$

Standard model of non-relativistic QED: spectral problems (I)

The non-interacting Hamiltonian H_0

- For $\alpha = 0$, we obtain

$$H_0 = \frac{p_{\mathrm{el}}^2}{2m_{\mathrm{el}}} + V(x_{\mathrm{el}}) + H_f = H_{\mathrm{el}} \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{ph}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{el}}} \otimes H_f$$

- Spectrum: $\sigma(H_0) = \sigma(H_{\mathrm{el}}) + \sigma(H_f)$

Main problems concerning the spectrum of H_α

- The full Hamiltonian H_α is decomposed as

$$H_\alpha = H_0 + W_\alpha$$

- Aim: behavior of the unperturbed eigenvalues E_j as the perturbation W_α is added. One expects that
[1] The lowest eigenvalue E_0 remains an eigenvalue, giving the existence of a (stable) ground state for H_α
[2] Excited eigenvalues E_j turn into resonances associated to metastable states

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Standard model of non-relativistic QED: spectral problems (II)

Results

• Problem [1] can be solved in various ways [Bach-Frohlich-Sigal CMP'99], [Griesemer-Lieb-Loss Inventiones'01], [Bach-Frohlich-Pizzo CMP'07]. In fact one can show that for arbitrary α,

$$E_\alpha = \inf \sigma(H_\alpha),$$

is an eigenvalue of H_α [Griesemer-Lieb-Loss'01]
• Up to now, Problem [2] (existence of resonances) is only solved using the Bach-Fröhlich-Sigal spectral renormalization group [Bach-Fröhlich-Sigal Adv.Math.'98], [Sigal JSP'09]

In these talks

• We describe the BFS spectral renormalization group, applying it to obtain the existence of a ground state (Problem [1])
• We explain the modifications used to prove the existence of resonances (Problem [2])

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Part II

Spectral renormalization group

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General strategy

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The Feshbach-Schur map (I)

Abstract setting

- $\mathcal{H}$ complex, separable Hilbert space
- H, H_0 closed operators on $\mathcal{H}$ such that $H = H_0 + W$, $\mathcal{D}(H) = \mathcal{D}(H_0)$
- Assumptions:

  a) ("Projections") χ, $\bar{\chi}$ bounded operators on $\mathcal{H}$ such that

$$[\chi, \bar{\chi}] = 0 = [\chi, H_0] = [\bar{\chi}, H_0], \quad \chi^2 + \bar{\chi}^2 = \mathbb{1}$$

(Typically, $\chi, \bar{\chi}$ are spectral projections of H_0)

  b) (Invertibility assumptions) Let

$$H_{\bar{\chi}} = H_0 + \bar{\chi} W \bar{\chi}$$

The operators $H_0, H_{\bar{\chi}} : \mathcal{D}(H_0) \cap \mathrm{Ran}\bar{\chi} \to \mathrm{Ran}\bar{\chi}$ are bijections with bounded inverses. Moreover, the operator

$$\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi : \mathcal{D}(H_0) \to \mathcal{H}$$

is bounded

The Feshbach-Schur map (II)

Main properties

• Under the previous hypotheses, H is invertible with bounded inverse iff the Feshbach-Schur operator $F_\chi(H, H_0) : \mathcal{D}(H_0) \cap \overline{\mathrm{Ran}\chi} \to \overline{\mathrm{Ran}\chi}$ defined by

$$F_\chi(H, H_0) = H_0 + \chi W \chi - \chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$$

is invertible with bounded inverse. In this case,

$$H^{-1} = Q_\chi F_\chi(H, H_0)^{-1} Q_\chi^\# + \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi},$$
$$F_\chi(H, H_0)^{-1} = \chi H^{-1} \chi + \bar{\chi} H_0^{-1} \bar{\chi},$$

where

$$Q_\chi : \chi - \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi, \quad Q_\chi^\# = \chi - \chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi}$$

• The maps

$$\chi : \mathrm{Ker}\, H \to \mathrm{Ker}\, F_\chi(H, H_0), \quad Q_\chi : \mathrm{Ker}\, F_\chi(H, H_0) \to \mathrm{Ker}\, H$$

are linear isomorphisms and inverse to each other

The Feshbach-Schur map (III)

Consequences

• Under the previous hypotheses,

$$\lambda \in \sigma(H) \iff 0 \in \sigma(H-\lambda) \iff 0 \in \sigma\big(F_\chi(H-\lambda, H_0-\lambda)\big)$$

• Likewise,

$$\lambda \in \sigma_{\mathrm{pp}}(H) \iff 0 \in \sigma_{\mathrm{pp}}(H-\lambda) \iff 0 \in \sigma_{\mathrm{pp}}\big(F_\chi(H-\lambda, H_0-\lambda)\big),$$

and if ψ is an eigenstate of $F_\chi(H-\lambda, H_0-\lambda)$ associated to the eigenvalue 0, then $Q_\chi \psi$ is an eigenstate of H associated to the eigenvalue λ

• The Feshbach-Schur operator $F_\chi(H-\lambda, H_0-\lambda)$ is viewed as an effective Hamiltonian acting in the Hilbert space $\overline{\mathrm{Ran}\chi}$.

Application to non-relativistic QED (I)

The "projections"

- Recall $H_0 = H_{\rm el} + H_f$, $H_\alpha = H_0 + W_\alpha$. Choose $\chi = \Pi_{\rm el} \otimes \chi_{H_f \leq \rho}$, where $\Pi_{\rm el}$ is the projection onto the (non-degenerate) ground state of $H_{\rm el}$, and $\chi_{\cdot \leq \rho}$ is a "smoothed out" characteristic function of the interval $[0, \rho]$
- Let

$$\bar{\chi} = \Pi_{\rm el}^{\perp} \otimes \mathbb{1} + \Pi_{\rm el} \otimes \sqrt{\mathbb{1} - \chi^2_{H_f \leq \rho}}.$$

Hence $[\chi, \bar{\chi}] = 0 = [\chi, H_0] = [\bar{\chi}, H_0]$ and $\chi^2 + \bar{\chi}^2 = \mathbb{1}$

The invertibility assumptions

- By definition of $\bar{\chi}$, for $\lambda \leq E_0 + \rho/2$, $H_0 - \lambda : \mathcal{D}(H_0) \cap \mathrm{Ran}(\bar{\chi}) \to \mathrm{Ran}(\bar{\chi})$ is invertible with bounded inverse
- Using the Neumann series decomposition

$$(H_\alpha - \lambda)^{-1}_{\bar{\chi}} = (H_0 - \lambda)^{-1} \sum_{n \geq 0} \Big(-\bar{\chi} W_\alpha \bar{\chi} (H_0 - \lambda)^{-1} \Big)^n,$$

we see that $(H_\alpha - \lambda)_{\bar{\chi}}$ is invertible with bounded inverse for $\alpha \ll \rho$ and $\lambda \leq E_0 + \rho/2$. Likewise, $\bar{\chi}(H_\alpha - \lambda)^{-1}_{\bar{\chi}} \bar{\chi} W_\alpha \chi$ is bounded

Application to non-relativistic QED (I)

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Application to non-relativistic QED (II)

Feshbach-Schur operator

With the previous notations, the operator

$$F_\chi(H_\alpha - \lambda, H_0 - \lambda) = H_0 - \lambda + \chi W_\alpha \chi - \chi W_\alpha \bar{\chi}(H_\alpha - \lambda)_{\bar{\chi}}^{-1} \bar{\chi} W_\alpha \chi$$
$$= E_0 - \lambda + H_f + \chi W_\alpha \chi - \chi W_\alpha \bar{\chi}(H_\alpha - \lambda)_{\bar{\chi}}^{-1} \bar{\chi} W_\alpha \chi$$

acting on $\overline{\mathrm{Ran}\,\chi} \equiv \mathrm{Ran}\,\mathbb{1}_{H_f \leq \rho}$ is isospectral to H_α in the sense that

$$\lambda \in \sigma_\#(H_\alpha) \iff 0 \in \sigma_\#\big(F_\chi(H_\alpha - \lambda, H_0 - \lambda)\big),$$

where $\sigma_\#$ stands for σ or σ_{pp}

Effective Hamiltonian

The effective Hamiltonian acting on $\mathrm{Ran}\,\mathbb{1}_{H_f \leq \rho}$ is thus

$$H_{\mathrm{eff}}(\lambda) = F_\chi(H_\alpha - \lambda, H_0 - \lambda)$$

Application to non-relativistic QED (II)

Feshbach-Schur operator

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$$\begin{aligned} F_\chi(H_\alpha-\lambda, H_0-\lambda) &= H_0-\lambda+\chi W_\alpha \chi - \chi W_\alpha \bar\chi (H_\alpha-\lambda)^{-1}_{\bar\chi} \bar\chi W_\alpha \chi \\ &= E_0-\lambda+H_f+\chi W_\alpha \chi - \chi W_\alpha \bar\chi (H_\alpha-\lambda)^{-1}_{\bar\chi} \bar\chi W_\alpha \chi \end{aligned}$$

acting on $\overline{\mathrm{Ran}\,\chi} \equiv \mathrm{Ran}\,\mathbb{1}_{H_f\le\rho}$ is isospectral to H_α in the sense that

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$$H_{\mathrm{eff}}(\lambda) = F_\chi(H_\alpha-\lambda, H_0-\lambda)$$

Expression of the interaction Hamiltonian (I)

Interaction Hamiltonian

Recall that

$$H_\alpha = \frac{1}{2m_{\mathrm{el}}}\left(p_{\mathrm{el}} - \alpha^{\frac{3}{2}} A(\alpha x_{\mathrm{el}})\right)^2 + V(x_{\mathrm{el}}) + H_f = H_0 + W_\alpha,$$

with

$$W_\alpha = \frac{1}{2m_{\mathrm{el}}}\left(- 2\alpha^{\frac{3}{2}} p_{\mathrm{el}} \cdot A(\alpha x_{\mathrm{el}}) + \alpha^3 A(\alpha x_{\mathrm{el}})^2\right),$$

and

$$A(x) = \int_{\mathbb{R}^3} \frac{\chi_\Lambda(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \left(a^*(K) e^{-ik\cdot x} + a(K) e^{ik\cdot x}\right) \mathrm{d}K$$

Expression of the interaction Hamiltonian (II)

Interaction Hamiltonian

The interaction Hamiltonian W_α can be written under the form

$$W_\alpha = W_1 + W_2,$$

with

$$W_1 = \int_{\underline{\mathbb{R}}^3} \big(G_{1,0}(K) \otimes a^*(K) + G_{0,1}(K) \otimes a(K) \big) \mathrm{d}K,$$

$$W_2 = \int_{\underline{\mathbb{R}}^3 \times \underline{\mathbb{R}}^3} \big(G_{2,0}(K,K') \otimes a^*(K) a^*(K') + G_{0,2}(K,K') \otimes a(K) a(K')$$

$$G_{1,1}(K,K') \otimes a^*(K) a(K') \big) \mathrm{d}K \mathrm{d}K'$$

where $G_{i,j}(K)$, $G_{i,j}(K,K')$ are operators acting on $\mathcal{H}_{\mathrm{el}}$

Generalized Wick normal form (I)

Normal form

- Use the previous Neumann series decomposition

$$H_{\text{eff}}(\lambda) = E_0 - \lambda + H_f$$
$$+ \chi W_\alpha \chi - \chi W_\alpha \bar{\chi}(H_0 - \lambda)^{-1} \sum_{n \geq 0} \left(-\bar{\chi} W_\alpha \bar{\chi}(H_0 - \lambda)^{-1} \right)^n \bar{\chi} W_\alpha \chi,$$

- Use the CCR

$$[a(K), a(K')] = 0 = [a^*(K), a^*(K')], \quad [a(K), a^*(K')] = \delta(K - K'),$$

and the "pull-through" formula $a(K)f(H_f) = f(H_f + |k|)a(K)$, to rewrite $H_{\text{eff}}(\lambda)$ under the (generalized) Wick ordered form

$$H_{\text{eff}}(\lambda) = w_{0,0}(\lambda, H_f) + \sum_{m+n \geq 1} \chi_{H_f \leq \rho} \int_{B_\rho^{m+n}} \Big(\prod_{j=1}^{m} a^*(K_j) \Big)$$
$$w_{m,n}(\lambda, H_f; K_1, \dots, K_{m+n}) \Big(\prod_{j=m+1}^{m+n} a(K_j) \Big) \chi_{H_f \leq \rho} \mathrm{d}K_1 \dots \mathrm{d}K_{m+n}$$

Generalized Wick normal form (II)

Normal form

$$H_{\mathrm{eff}}(\lambda) = w_{0,0}(\lambda, H_f) + \sum_{m+n\geq 1} \chi_{H_f \leq \rho} \int_{B_\rho^{m+n}} \Big(\prod_{j=1}^{m} a^*(K_j) \Big)$$

$$w_{m,n}(\lambda, H_f; K_1, \dots, K_{m+n}) \Big(\prod_{j=m+1}^{m+n} a(K_j) \Big) \chi_{H_f \leq \rho} \mathrm{d}K_1 \dots \mathrm{d}K_{m+n},$$

where $B_\rho = \{K = (k, \lambda) \in \mathbb{R}^3, |k| \leq \rho\}$, and

$$w_{m,n}(\lambda, \cdot) : [0, \rho] \times B_\rho^{m+n} \to \mathbb{C}$$

For instance,

$$w_{0,0}(\lambda, H_f) = E_0 - \lambda + H_f + \alpha^3(\cdots)$$

Generalized Wick normal form (III)

Example

Consider the term coming from $\chi W_\alpha \bar\chi (H_0-\lambda)^{-1}\bar\chi W\chi$ given by

$$\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)a(K_1)\bar\chi(H_0)(H_0-\lambda)^{-1}\bar\chi(H_0)$$

$$G_{1,0}(K_2)a^*(K_2)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

$$=\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)\bar\chi(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar\chi(H_0+|k_1|)$$

$$G_{1,0}(K_2)a(K_1)a^*(K_2)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

$$=\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)\bar\chi(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar\chi(H_0+|k_1|)$$

$$G_{1,0}(K_2)\big(\delta(K_1-K_2)+a^*(K_2)a(K_1)\big)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

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Generalized Wick normal form (IV)

Example

$$\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)\bar\chi(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar\chi(H_0+|k_1|)$$

$$G_{1,0}(K_2)\big(\delta(K_1-K_2)+a^*(K_2)a(K_1)\big)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

$$=\chi(H_0)\int_{\mathbb{R}^3} G_{0,1}(K_1)\bar\chi(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar\chi(H_0+|k_1|)$$

$$G_{1,0}(K_1)\mathrm{d}K_1\chi(H_0)$$

$$+\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)a^*(K_2)\bar\chi(H_0+|k_1|+|k_2|)(H_0+|k_1|+|k_2|-\lambda)^{-1}$$

$$\bar\chi(H_0+|k_1|+|k_2|)G_{1,0}(K_2)a(K_1)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

Generalized Wick normal form (IV)

Example

$$
\begin{aligned}
&\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)\bar\chi(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar\chi(H_0+|k_1|)\\
&\qquad G_{1,0}(K_2)\big(\delta(K_1-K_2)+a^*(K_2)a(K_1)\big)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)\\
=&\chi(H_0)\int_{\mathbb{R}^3} G_{0,1}(K_1)\bar\chi(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar\chi(H_0+|k_1|)\\
&\qquad G_{1,0}(K_1)\mathrm{d}K_1\chi(H_0)\\
+&\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)a^*(K_2)\bar\chi(H_0+|k_1|+|k_2|)(H_0+|k_1|+|k_2|-\lambda)^{-1}\\
&\qquad \bar\chi(H_0+|k_1|+|k_2|)G_{1,0}(K_2)a(K_1)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)
\end{aligned}
$$

Generalized Wick normal form (IV)

Example

$$\chi(H_0)\int_{\mathbb{R}^3\times\mathbb{R}^3} G_{0,1}(K_1)\bar{\chi}(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar{\chi}(H_0+|k_1|)$$

$$G_{1,0}(K_2)\big(\delta(K_1-K_2)+a^*(K_2)a(K_1)\big)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

$$=\chi(H_0)\int_{\mathbb{R}^3} G_{0,1}(K_1)\bar{\chi}(H_0+|k_1|)(H_0+|k_1|-\lambda)^{-1}\bar{\chi}(H_0+|k_1|)$$

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$$\bar{\chi}(H_0+|k_1|+|k_2|)G_{1,0}(K_2)a(K_1)\mathrm{d}K_1\mathrm{d}K_2\chi(H_0)$$

Scaling transformation (I)

Scaling transformation

- Effective Hamiltonian $H_{\text{eff}}(\lambda)$ acts on the Hilbert space $\mathrm{Ran}\,\mathbb{1}_{H_f \le \rho}$ at energy scale ρ. To obtain an Hamiltonian at energy scale 1 we use the unitary scaling transformation

$$U_\rho : \mathrm{Ran}\,\mathbb{1}_{H_f \le \rho} \to \mathrm{Ran}\,\mathbb{1}_{H_f \le 1} =: \mathcal{H}_0,$$

$$(U_\rho \Phi)^{(n)}(K_1, \dots, K_n) = \rho^{\frac{3n}{2}} \Phi^{(n)}((\rho k_1, \lambda_1), \dots, (\rho k_n, \lambda_n))$$

- Note that the free photon field Hamiltonian is scaled as

$$U_\rho H_f U_\rho^* = \rho H_f$$

- Define the new Hamiltonian $\tilde{H}_{\text{eff}}(\lambda)$ acting on $\mathcal{H}_0$ by

$$\tilde{H}_{\text{eff}}(\lambda) = \frac{1}{\rho}\left(U_\rho H_{\text{eff}}(\lambda) U_\rho^* + E_0 - \lambda\right)$$

Scaling transformation (II)

Scaling transformation

- In generalized Wick ordered form,

$$\tilde{H}_{\text{eff}}(\lambda) = \tilde{w}_{0,0}(\lambda, H_f) + \sum_{m+n\geq 1} \chi_{H_f \leq 1} \int_{B_1^{m+n}} \Big(\prod_{j=1}^{m} a^*(K_j)\Big)$$

$$\tilde{w}_{m,n}(\lambda, H_f; K_1, \dots, K_{m+n}) \Big(\prod_{j=m+1}^{m+n} a(K_j) \Big) \chi_{H_f \leq 1} \mathrm{d}K_1 \dots \mathrm{d}K_{m+n},$$

where $\tilde{w}_{0,0}(\lambda, H_f) = H_f + \alpha^3(\cdots)$ and for $m+n \geq 1$,

$$\tilde{w}_{m,n}(\lambda, \cdot) : [0,1] \times B_1^{m+n} \to \mathbb{C}$$

$$\tilde{w}_{m,n}(\lambda, H_f; K_1, \dots, K_n) = \rho^{\frac{3}{2}(m+n)-1} w_{m,n}(\lambda, \rho H_f; \rho K_1, \dots, \rho K_n)$$

Remark: Infrared singularity

Consider a (coupling) function of the form $f(K) = \chi_\Lambda(k)/|k|^{\frac{1}{2}-\mu}$. Then

$$\rho^{-1} U_\rho a(f) U_\rho^* = \rho^\mu a\Big(\frac{\chi_{\rho^{-1}\Lambda}}{|\cdot|^{\frac{1}{2}-\mu}}\Big)$$

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$$\tilde{w}_{m,n}(\lambda, H_f; K_1, \dots, K_{m+n})\Big(\prod_{j=m+1}^{m+n} a(K_j)\Big)\chi_{H_f\leq 1}\mathrm{d}K_1\dots\mathrm{d}K_{m+n},$$

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Scaling transformation of the spectral parameter

Scaling transformation of the spectral parameter

• Effective Hamiltonian $\tilde{H}_{\mathrm{eff}}(\lambda)$ acting on $\mathcal{H}_0$ is defined for $\lambda \le E_0 + \rho/2$. To obtain a family of operators defined on $[-1/2, 1/2]$, we consider the map

$$Z_{(0)} : \Big[E_0 - \frac{\rho}{2}, E_0 + \frac{\rho}{2} \Big] \to \Big[-\frac{1}{2}, \frac{1}{2} \Big]$$
$$\lambda \mapsto \frac{1}{\rho}(\lambda - E_0)$$

• For $\lambda \in [-1/2, 1/2]$, define the new Hamiltonian $H_{(0)}(\lambda)$ acting on $\mathcal{H}_0$ by

$$H_{(0)}(\lambda) = \tilde{H}_{\mathrm{eff}}(Z_{(0)}^{-1}(\lambda))$$

Isospectrality

Using isospectrality of the Feshbach-Schur map, we obtain

$$\lambda \in \sigma(H_{(0)}(\lambda)) \cap \Big[-\frac{1}{2}, \frac{1}{2} \Big] \iff Z_{(0)}^{-1}(\lambda) \in \sigma(H_\alpha) \cap \Big[E_0 - \frac{\rho}{2}, E_0 + \frac{\rho}{2} \Big]$$

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Banach space of operators (I)

The function space $\mathcal{W}^{\#}_{0,0}$ (relevant and marginal parts)

- Let

$$\mathcal{W}^{\#}_{0,0} = \mathrm{C}^1([0,1];\mathbb{C}), \quad \|w_{0,0}\| = |w_{0,0}(0)| + \|w'_{0,0}\|_\infty$$

- Can be decomposed into $\mathcal{W}^{\#}_{0,0} = \mathbb{C} \oplus \mathcal{T}$, $\quad \mathcal{T} = \{w_{0,0} \in \mathcal{W}^{\#}_{0,0}, w_{0,0}(0) = 0\}$

The function space $\mathcal{W}^{\#}_{m,n}$, $m+n \geq 1$ (irrelevant part)

- Let $\mathcal{W}^{\#}_{m,n}$ be the set of functions $w_{m,n} : [0,1] \times B_1^{m+n} \to \mathbb{C}$ such that
  * For all $\omega \in [0,1]$, $(K_1, \dots K_{m+n}) \mapsto w_{m,n}(\omega, K_1, \dots, K_{m+n})$ is bounded and symmetric w.r.t. $(K_1, \dots, K_m)$ and $(K_{m+1}, \dots, K_n)$
  * For all $(K_1, \dots, K_{m+n}) \in B_1^{m+n}$, $\omega \mapsto w_{m,n}(\omega, K_1, \dots, K_{m+n})$ belongs to $\mathrm{C}^1([0,1];\mathbb{C})$
- $\mathcal{W}^{\#}_{m,n}$ is equipped with the norm (where $\mu > 0$ is related to the infrared singularity of the model)

$$\|w_{m,n}\| = \|w_{m,n}\|_\mu + \|\partial_\omega w_{m,n}\|_\mu,$$

$$\|w_{m,n}\|_\mu = \sup_{[0,1]\times B_1^{m+n}} \left| w_{m,n}(\omega, K_1, \dots, K_{m+n}) \right| \prod_{j=1}^{m+n} |k_j|^{\frac{1}{2}-\mu}$$

Banach space of operators (I)

The function space $\mathcal{W}_{0,0}^{\#}$ (relevant and marginal parts)

- Let
$$\mathcal{W}_{0,0}^{\#} = \mathrm{C}^1([0,1];\mathbb{C}), \quad \|w_{0,0}\| = |w_{0,0}(0)| + \|w_{0,0}'\|_\infty$$
- Can be decomposed into $\mathcal{W}_{0,0}^{\#} = \mathbb{C} \oplus \mathcal{T}$, $\quad \mathcal{T} = \{w_{0,0} \in \mathcal{W}_{0,0}^{\#}, w_{0,0}(0) = 0\}$

The function space $\mathcal{W}_{m,n}^{\#}$, $m+n \geq 1$ (irrelevant part)

- Let $\mathcal{W}_{m,n}^{\#}$ be the set of functions $w_{m,n} : [0,1] \times B_1^{m+n} \to \mathbb{C}$ such that
  ∗ For all $\omega \in [0,1]$, $(K_1, \dots K_{m+n}) \mapsto w_{m,n}(\omega, K_1, \dots, K_{m+n})$ is bounded and symmetric w.r.t. $(K_1, \dots, K_m)$ and $(K_{m+1}, \dots, K_n)$
  ∗ For all $(K_1, \dots, K_{m+n}) \in B_1^{m+n}$, $\omega \mapsto w_{m,n}(\omega, K_1, \dots, K_{m+n})$ belongs to $\mathrm{C}^1([0,1];\mathbb{C})$
- $\mathcal{W}_{m,n}^{\#}$ is equipped with the norm (where $\mu > 0$ is related to the infrared singularity of the model)

$$\|w_{m,n}\| = \|w_{m,n}\|_\mu + \|\partial_\omega w_{m,n}\|_\mu,$$

$$\|w_{m,n}\|_\mu = \sup_{[0,1]\times B_1^{m+n}} \left| w_{m,n}(\omega, K_1, \dots, K_{m+n}) \right| \prod_{j=1}^{m+n} |k_j|^{\frac{1}{2}-\mu}$$

Banach space of operators (II)

The Banach space $\mathcal{W}^\#$

Let

$$\mathcal{W}^\# = \bigoplus_{m+n\geq 0} \mathcal{W}^\#_{m,n}, \quad \|\underline{w}\| = \sum_{m+n\geq 0} \xi^{-(m+n)} \|w_{m,n}\|,$$

with the notation $\underline{w} = (w_{0,0}, w_{1,0}, w_{0,1}, \dots) \in \mathcal{W}^\#$ and where $0 < \xi < 1$ is a suitably chosen parameter

Operators associated to elements of $\mathcal{W}^\#$

- To $\underline{w} \in \mathcal{W}^\#$ we associate a bounded operator on $\mathcal{H}_0$ by letting

$$H(\underline{w}) = w_{0,0}(H_f) + \sum_{m+n\geq 1} \chi_{H_f\leq 1} \int_{B_1^{m+n}} \Big(\prod_{j=1}^{m} a^*(K_j)\Big) w_{m,n}(\lambda, H_f; K_1, \dots, K_{m+n}) \Big(\prod_{j=m+1}^{m+n} a(K_j)\Big) \chi_{H_f\leq 1} \mathrm{d}K_1 \dots \mathrm{d}K_{m+n}$$

- For all $\mu \geq 0$ and $0 < \xi < 1$, the map $H : \underline{w} \to H(\underline{w})$ is injective and continuous with $\|H(\underline{w})\| \leq \|\underline{w}\|$

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Banach space of operators (III)

Dependence on the spectral parameter

Let

$$\mathcal{W} = \mathrm{C}^1\Big(\Big[-\frac{1}{2},\frac{1}{2}\Big];\mathcal{W}^{\#}\Big), \quad \|\underline{w}(\cdot)\| = \sup_{\lambda\in[-\frac{1}{2},\frac{1}{2}]} \|\underline{w}(\lambda)\|_{\mathcal{W}^{\#}}$$

The Banach space $H(\mathcal{W})$

The Banach space in which the renormalization map will be defined is

$$H(\mathcal{W}) = \Big\{H(\underline{w}(\cdot)) \in \mathrm{C}^1\Big(\Big[-\frac{1}{2},\frac{1}{2}\Big];H(\mathcal{W}^{\#})\Big)\Big\},$$

equipped with the norm

$$\big\|H(\underline{w}(\cdot))\big\| = \sup_{\lambda\in[-\frac{1}{2},\frac{1}{2}]} \big\|H(\underline{w}(\lambda))\big\|_{\mathcal{B}(\mathcal{H}_0)}$$

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Banach space of operators (IV)

A polydisc in $\mathcal{W}$

Let

$$\mathcal{D}(\beta,\varepsilon) = \Big\{ \underline{w}(\cdot) = \big(E(\cdot), T(\cdot), (w_{m,n}(\cdot))_{m+n\geq 1}\big) \in \mathcal{W},$$

$$\sup_{\lambda\in[-\frac12,\frac12]} |E(\lambda)| \leq \varepsilon,$$

$$\sup_{\lambda\in[-\frac12,\frac12]} \sup_{\omega\in[0,1]} \big|\partial_\omega T(\lambda,\omega) - 1\big| \leq \beta,$$

$$\sup_{\lambda\in[-\frac12,\frac12]} \big\|(w_{m,n}(\lambda))_{m+n\geq 1}\big\|_{\mathcal{W}^\#} \leq \varepsilon \Big\}$$

The initial Hamiltonian

Let $\beta,\varepsilon > 0$. Let $\alpha^{\frac12} \ll \rho \leq \xi < 1$. Then $H_{(0)}(\cdot) \in H(\mathcal{W})$, and, with $H_{(0)}(\cdot) = H(\underline{w}_{(0)}(\cdot))$,

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Renormalization map (I)

The renormalization map

- The renormalization map $\mathcal{R}_\rho : H(\mathcal{W}) \to H(\mathcal{W})$ is defined by

$$\mathcal{R}_\rho\big(H(\underline{w}(\lambda))\big) = \frac{1}{\rho} U_\rho F_{\chi_{H_f \le \rho}}\Big(H\big(\underline{w}(Z^{-1}(\lambda))\big) - Z^{-1}(\lambda), E(Z^{-1}(\lambda)) + T(Z^{-1}(\lambda)) - Z^{-1}(\lambda)\Big) U_\rho^* + \lambda$$

- Decimation of the degrees of freedom. One verifies that for suitably chosen ρ's, the Feshbach-Schur operator above is well-defined (use the C^1 property "with respect to H_f")
- U_ρ is a scaling transformation
- Z is a scaling transformation of the spectral parameter (use the C^1 property with respect to λ)

$$Z : \Big\{\lambda \in \Big[-\frac{1}{2}, \frac{1}{2}\Big], |\lambda - E(\lambda)| \le \frac{\rho}{2}\Big\} \ni \lambda \to \frac{1}{\rho}(\lambda - E(\lambda)) \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$$

- Using Neumann series decomposition and generalized Wick ordered form, $\mathcal{R}_\rho\big(H(\underline{w}(\cdot))\big)$ is written as an element of $H(\mathcal{W})$

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Renormalization map (II)

Perturbation decreases with application of $\mathcal{R}_\rho$

Let $\alpha \ll \rho < 1$, $\mu > 0$, $\xi = \rho^{1/2}$. For all $0 < \beta, \varepsilon \leq \rho$,

$$\mathcal{R}_\rho : H\big(\mathcal{D}(\beta, \varepsilon)\big) \to H\big(\mathcal{D}(\beta + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})\big)$$

Iteration

- Let
$$H_{(l)}(\cdot) = \mathcal{R}_\rho^l\big(H_{(0)}(\cdot)\big) = H\big(E_{(l)}(\cdot), T_{(l)}(\cdot), (w_{m,n}^{(l)}(\cdot))_{m+n\geq 1}\big)$$

- Let $Z_{(l)}$ be the scaling transformation of the spectral parameter appearing in the l^{th} application of $\mathcal{R}_\rho$

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Existence of a ground state

Existence of a ground state

The sequence $Z_{(0)}^{-1} \circ Z_{(1)}^{-1} \circ \cdots \circ Z_{(l)}^{-1}(0)$ converges as $l \to \infty$. The limit

$$E_{(\infty)} = \lim_{l \to \infty} Z_{(0)}^{-1} \circ Z_{(1)}^{-1} \circ \cdots \circ Z_{(l)}^{-1}(0)$$

is an eigenvalue of H_α and

$$\sigma(H_\alpha) \cap \left[E_0 - \frac{\rho}{2}, E_0 + \frac{\rho}{2} \right] \subset E_{(\infty)} + [0, 1].$$

In particular H_α has a ground state associated to the eigenvalue $E_{(\infty)}$

Algorithm to compute $E_{(\infty)}$

- The method provides an algorithm to compute $E_{(\infty)}$ up to any order in α
- One can show [Halser-Herbst JFA'12] that $E_{(\infty)}$ is an analytic function of α

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Part III

Resonances and lifetime of metastable states

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Complex dilatations

Unitary scaling transformation of electron position and photon momenta

Recall $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3; \mathcal{H}_{\mathrm{ph}})$. For $\theta \in \mathbb{R}$, let U_θ be the unitary dilatations operator that implements the transformations

$$x_{\mathrm{el}} \mapsto e^{\theta} x_{\mathrm{el}}, \quad k \mapsto e^{-\theta} k$$

More precisely, for $\Phi \in \mathcal{H}$,

$$(U_\theta \Phi)^{(n)}(x_{\mathrm{el}}, K_1, \dots, K_n) = e^{-\frac{3}{2}(n-1)\theta} \Phi^{(n)}(e^{\theta} x_{\mathrm{el}}, (e^{-\theta} k_1, \lambda_1), \dots, (e^{-\theta} k_n, \lambda_n))$$

The dilated Hamiltonian

- For $\theta \in \mathbb{R}$, let $H_\alpha(\theta) = U_\theta H_\alpha U_\theta^{-1}$, which gives

$$H_\alpha(\theta) = H_{\mathrm{el}}(\theta) + e^{-\theta} H_f + W_\alpha(\theta), \quad H_{\mathrm{el}}(\theta) = e^{-2\theta} \frac{p_{\mathrm{el}}^2}{2m_{\mathrm{el}}} + V(e^{\theta} x_{\mathrm{el}})$$

- Using assumptions on the coupling function, we can define $H_\alpha(\theta)$ by the same expression, for $\theta \in \mathcal{D}(0, \theta_0) \subset \mathbb{C}$, θ_0 sufficiently small. The family $\theta \mapsto H_\alpha(\theta)$ is then analytic of type (A) in the sense of Kato

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Existence of resonances

Existence of resonances ([Bach-Fröhlich-Sigal Adv.Math.'98], [F. AHP'08], [Sigal JSP'09])

Let $E_j < 0$ be a simple eigenvalue of H_{el}. There exists $\alpha_c > 0$ such that for all $0 < \alpha \leq \alpha_c$, there exists a non-degenerate eigenvalue $E_{j,\alpha}$ of $H_\alpha(\theta)$ such that $E_{j,\alpha}$ does not depend on θ (for θ suitably chosen) and

$$E_{j,\alpha} \underset{\alpha \to 0}{\longrightarrow} E_j$$

The eigenvalue $E_{j,\alpha}$ of $H_\alpha(\theta)$ is called a resonance of H_α

Perturbative expansion in α

Expansion in α can be computed up to any order; first terms:

$$E_{j,\alpha} = E_j + \alpha^3 c_0 + \mathcal{O}(\alpha^4),$$

where $\mathrm{Im}\, c_0 < 0$ (given by Fermi's Golden Rule)

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Lifetime of metastable states

Estimation of the lifetime of metastable states ([Hasler-Herbst-Huber AHP'08], [Abou Salem-F-Fröhlich-Sigal Adv.Appl.Math.'09])

- Let φ_j be a normalized eigenstate of $H_{\rm el}$ associated to E_j
- Then $\varphi_j \otimes \Omega$ (with Ω the Fock vacuum) is a normalized eigenstate of H_0 associated to E_j
- There exists $\alpha_c > 0$ such that for all $0 < \alpha \leq \alpha_c$ and $t \geq 0$,

$$\left\langle \varphi_j \otimes \Omega, e^{-itH_\alpha} \varphi_j \otimes \Omega \right\rangle = e^{-itE_{j,\alpha}} + \mathcal{O}(\alpha)$$

- Consequence: for $t \ll \alpha^{-3}$,

$$\left| \left\langle \varphi_j \otimes \Omega, e^{-itH_\alpha} \varphi_j \otimes \Omega \right\rangle \right| = e^{t \operatorname{Im} c_0} + \mathcal{O}(\alpha)$$

Infrared cutoff

Introduction of an infrared cutoff

Define the infrared cutoff Hamiltonian

$$H_{\alpha,\sigma}(\theta) = H_0(\theta) + W_{\alpha,\sigma}(\theta)$$

where the interaction between the electron and the photons of energies $\leq \sigma$ has been suppressed in the interaction Hamiltonian $W_\alpha(\theta)$. For $\theta = 0$, this corresponds to replacing the electromagnetic vector potential $A(x)$ by

$$A_\sigma(x) = \int_{\mathbb{R}^3} \mathbb{1}_{|k|\geq\sigma} \frac{\chi_\Lambda(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \left(a^*(K) e^{-ik\cdot x} + a(K) e^{ik\cdot x} \right) \mathrm{d}K$$

Spectrum of the infrared cutoff Hamiltonian

- There exists a complex eigenvalue $E_{j,\alpha}^{\geqslant\sigma}$ of $H_{\alpha,\sigma}(\theta)$ arising from E_j, but $E_{j,\alpha}^{\geqslant\sigma}$ depends on θ
- When restricted to the Fock space of photons of energies $\geq \sigma$, there is a gap of order $\mathcal{O}(\sigma)$ around $E_{j,\alpha}^{\geqslant\sigma}$ in the spectrum of $H_{\alpha,\sigma}(\theta)$

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Hunziker's method (I)

Relation between propagator and resolvent, Combes' formula

- Let $\Psi_j = \varphi_j \otimes \Omega$. Let $f \in \mathrm{C}_0^\infty(\mathbb{R})$ be supported into a neighborhood of order $\mathcal{O}(\sigma)$ of E_j, $f = 1$ near E_j
- Stone's formula

$$\left\langle \Psi_j, e^{-itH_\alpha} f(H_\alpha)\Psi_j \right\rangle$$
$$= \lim_{\varepsilon \searrow 0} \frac{1}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \left\langle \Psi_j, \left[(H_\alpha - z - i\varepsilon)^{-1} - (H_\alpha - z + i\varepsilon)^{-1} \right] \Psi_j \right\rangle dz$$

- Combes' formula (first for $\theta \in \mathbb{R}$, then for $\theta \in \mathbb{C}$ using analyticity)

$$\left\langle \Psi_j, e^{-itH_\alpha} f(H_\alpha)\Psi_j \right\rangle$$
$$= \frac{1}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \Big[\left(\Psi_j(\theta), (H_\alpha(\bar\theta) - z)^{-1} \Psi_j(\bar\theta) \right)$$
$$- \left\langle \Psi_j(\bar\theta), (H_\alpha(\theta) - z)^{-1} \Psi_j(\theta) \right\rangle \Big] dz$$

Hunziker's method (I)

Relation between propagator and resolvent, Combes' formula

- Let $\Psi_j = \varphi_j \otimes \Omega$. Let $f \in \mathrm{C}_0^\infty(\mathbb{R})$ be supported into a neighborhood of order $\mathcal{O}(\sigma)$ of E_j, $f = 1$ near E_j
- Stone's formula

$$\left\langle \Psi_j, e^{-itH_\alpha} f(H_\alpha)\Psi_j \right\rangle$$
$$= \lim_{\varepsilon \searrow 0} \frac{1}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \left\langle \Psi_j, \left[(H_\alpha - z - i\varepsilon)^{-1} - (H_\alpha - z + i\varepsilon)^{-1} \right] \Psi_j \right\rangle dz$$

- Combes' formula (first for $\theta \in \mathbb{R}$, then for $\theta \in \mathbb{C}$ using analyticity)

$$\left\langle \Psi_j, e^{-itH_\alpha} f(H_\alpha)\Psi_j \right\rangle$$
$$= \frac{1}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \Big[\left\langle \Psi_j(\theta), (H_\alpha(\bar\theta) - z)^{-1} \Psi_j(\bar\theta) \right\rangle - \left\langle \Psi_j(\bar\theta), (H_\alpha(\theta) - z)^{-1} \Psi_j(\theta) \right\rangle \Big] dz$$

Hunziker's method (II)

Infrared cutoff Hamiltonian

Approximate the resolvent of $H_\alpha(\theta)$ by the resolvent of $H_{\alpha,\sigma}(\theta)$

$$\left\langle \Psi_j, e^{-itH_\alpha} f(H_\alpha)\Psi_j \right\rangle$$

$$= \frac{1}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \Big[\left\langle \Psi_j(\theta), (H_{\alpha,\sigma}(\bar\theta) - z)^{-1} \Psi_j(\bar\theta) \right\rangle$$

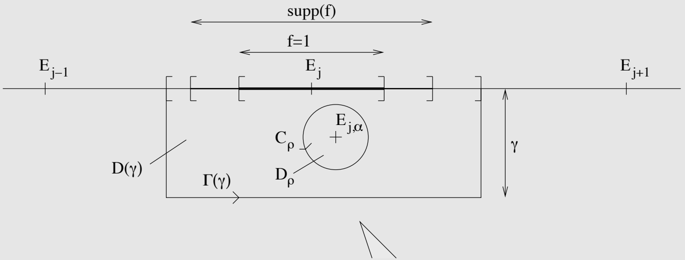
$$- \left\langle \Psi_j(\bar\theta), (H_{\alpha,\sigma}(\theta) - z)^{-1} \Psi_j(\theta) \right\rangle \Big] dz + \mathrm{Rem}(\alpha,\sigma)$$

Hunziker's method (III)

Deformation of the path of integration

• Using the gap property for $H_{\alpha,\sigma}(\theta)$, deform the path of integration (with $\alpha^3 \ll \gamma \leq \mathrm{C}\sigma$ and $\tilde{f}$ a suitable almost analytic extension of f)

$$\int_{\mathbb{R}} e^{-itz} f(z)[\dots]dz = \int_{\Gamma(\gamma)} e^{-itz}\tilde{f}(z)[\dots]dz + \int_{\mathcal{C}_\rho} e^{-itz}\tilde{f}(z)[\dots]dz$$
$$+ \iint_{\mathrm{D}(\gamma)\setminus D_\rho} e^{-itz}(\partial_{\bar{z}}\tilde{f})(z)[\dots]d\mathrm{Re}(z)d\mathrm{Im}(z)$$



• Use Cauchy's formula and estimates of the resolvent of $H_{\alpha,\sigma}(\theta)$

Continuation of the resolvent

Pole of an analytic continuation of the resolvent? ([Abou Salem-F-Fröhlich-Sigal Adv.Appl.Math.'09])

There exists $\alpha_c > 0$ and a dense domain $\mathcal{D}$ such that for all $0 < \alpha \leq \alpha_c$ and $\Psi \in \mathcal{D}$, the map

$$z \mapsto F_\Psi(z) = \langle \Psi, (H_\alpha - z)^{-1} \Psi \rangle$$

has an analytic continuation from $\mathbb{C}^+$ to a domain $\mathcal{W}_{j,\alpha}$ related to $E_{j,\alpha}$, such that

$$F_\Psi(z) = \frac{p(\Psi)}{E_{j,\alpha} - z} + r(z, \Psi), \quad |r(z,\Psi)| \leq \frac{C(\Psi)}{|E_{j,\alpha} - z|^\beta},$$

with $\beta < 1$, and where $p(\cdot)$, $C(\cdot)$ are bounded quadratic forms

Thank you!