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NEUMANN-BOUNDARY STABILIZATION OF THE WAVE EQUATION WITH DAMPING CONTROLS

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Abstract. *This note is devoted to boundary stabilization of a non-homogeneous n-dimensional wave equation subject to Neumann boundary conditions. Both linear and nonlinear feedback law involving only a damping term are dealt with. Using a new energy norm, asymptotic convergence to an equilibrium position depending on the initial data is proved for the solutions of the considered systems. The method presented can also be applied to a large class of distributed parameter systems such as Petrovsky system, coupled wave-wave equations and elasticity systems.*

1 Introduction

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. Given a partition (Γ_0, Γ_1) of Γ, consider the following wave equation

$$y_{tt}(x,t) - Ay(x,t) = 0, \qquad \text{in } \Omega \times (0,\infty) \tag{1.1}$$

with either static Neumann boundary conditions and initial conditions

$$\begin{cases} \partial_A y(x,t) = 0, & \text{on}\, \Gamma_0 \times (0,\infty) \\ \partial_A y(x,t) = U(t), & \text{on}\, \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega),\ y_t(x,0) = z_0(x) \in L^2(\Omega), & \end{cases} \tag{1.2}$$

or dynamical Neumann boundary conditions and initial conditions

$$\begin{cases} m(x)y_{tt}(x,t) + \partial_A y(x,t) = 0, & \text{on}\, \Gamma_0 \times (0,\infty) \\ M(x)y_{tt}(x,t) + \partial_A y(x,t) = U(t), & \text{on}\, \Gamma_1 \times (0,\infty) \\ y(x,0) = y^0(x) \in H^1(\Omega),\ y_t(x,0) = z^0(x) \in L^2(\Omega), & \\ y_t|_{\Gamma_0}(x,0) = w_0^0(x) \in L^2(\Gamma_0),\ y_t|_{\Gamma_1}(x,0) = w_1^0(x) \in L^2(\Gamma_1), & \end{cases} \tag{1.3}$$

where $A = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j)$, $\partial_A = \sum_{i,j=1}^n a_{ij}\nu_j\partial_j$, $\partial_k = \frac{\partial}{\partial x_k}$, $\nu = (\nu_1, \cdots, \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω and $a_{ij} \in C^1(\bar{\Omega})$ such that there exists $\alpha_0 > 0$ satisfying

$$a_{ij} = a_{ji}, \forall i, j = 1, \cdots, n, \qquad \sum_{i,j=1}^{n} a_{ij}\epsilon_i\epsilon_j \geq \alpha_0 \sum_{i=1}^{n} \epsilon_i^2, \forall\, (\epsilon_1, \cdots, \epsilon_n) \in \mathbb{R}^n. \tag{1.4}$$

Furthermore,

$$\begin{cases} m \in L^\infty(\Gamma_0);\ m(x) \geq m_0 > 0, \forall x \in \Gamma_0; \\ M \in L^\infty(\Gamma_1);\ M(x) \geq M_1 > 0, \forall x \in \Gamma_1. \end{cases} \tag{1.5}$$

and U is a feedback law depending only on a damping term, that is,

$$U(t) = -a(x)\, y_t(x,t), \qquad (x,t) \in \Gamma_1 \times (0, \infty); \tag{1.6}$$

where the function a satisfies: $a \in L^\infty(\Gamma_1);\ a(x) \geq a_0 > 0, \forall x \in \Gamma_1$. Note that Γ_1 is supposed to be nonempty whereas Γ_0 may be empty.

Then, it is proved that the solutions of each of the above closed-loop system tend asymptotically to a constant depending on the initial data y_0 and z_0. The nonlinear control is also treated in this note.

The stabilization problem of the wave equation has been extensively studied in the literature (see [1], [3]-[5], [15], [17]-[21], [23]-[26] and the references therein). In all references cited above, the displacement term y is present in the closed-loop system since the proposed energy-norm of the system is

$$E_0(t) = \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j y + |y_t|^2 \right) dx$$

which is only a semi-norm in our case. Note that J. Lagnese [16] has proved the energy decay of $E_0(t)$ in the case when $\Gamma_0 = \emptyset$ and $a_{ij} = \delta_{ij}$ for the system (1.1)-(1.2) and (1.6). Nevertheless, the proof of this result is very technical and requires a preliminary result (see Theorem 2 in [16]).

The main contribution of this paper is to provide an alternative proof of Lagnese's result [16] by means of a simple and direct method and extend the results of [8], where the one-dimensional equation is dealt with. The key idea of the proof is to introduce a new energy associated to each system.

2 The Main Results

Consider the state space $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$ equipped with the inner product

$$\langle (y,z), (\tilde{y}, \tilde{z}) \rangle_{\mathcal{H}} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \tilde{y} + z\tilde{z} \right) dx + \epsilon \left(\int_\Omega z\, dx + \int_{\Gamma_1} ay\, d\sigma \right) \left(\int_\Omega \tilde{z}\, dx + \int_{\Gamma_1} a\tilde{y}\, d\sigma \right),$$

where ϵ is a positive constant. Then, one can prove that $\mathcal{H}$ endoweded with this inner product is a Hilbert space provided that ϵ is small enough. Next, the system (1.1)-(1.2) and (1.6) can be written as follows

$$\begin{cases} \Phi_t(t) = \mathcal{A}\Phi(t), \\ \Phi(0) = \Phi_0 = (y_0, z_0), \end{cases} \tag{2.1}$$

where $\mathcal{A}$ is an unbounded linear operator such that $\mathcal{A}(y,z) = (z, Ay)$ for any (y,z) in its domain

$$\mathcal{D}(\mathcal{A}) = \Big\{ (y,z) \in H^1(\Omega) \times H^1(\Omega); Ay \in L^2(\Omega);\ \partial_A y = 0 \,\text{on}\, \Gamma_0;\ \partial_A y + az = 0 \,\text{on}\, \Gamma_1 \Big\}.$$

The well-posedness result for the closed-loop system (2.1) can be easily proved by applying semigroups theory of linear operators [22]. Moreover, one can establish that the canonical embedding $i : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is compact, where $\mathcal{D}(\mathcal{A})$ is equipped with the graph norm. The first main result of this note is:

Theorem 2.1. *For any initial data $\Phi_0 = (y_0, z_0) \in \mathcal{H}$, the solution $\Phi(t) = (y(t), y_t(t))$ of (2.1) tends in $\mathcal{H}$ to $(\mathcal{C}, 0)$ as $t \longrightarrow +\infty$, where*

$$\mathcal{C} = \left(\int_{\Gamma_1} a \, d\sigma \right)^{-1} \left(\int_{\Omega} z_0 \, dx + \int_{\Gamma_1} a y_0 \, d\sigma \right). \tag{2.2}$$

Proof of Theorem 2.1. By a standard argument, it suffices to prove Theorem 2.1 for a smooth initial data $\Phi_0 = (y_0, z_0) \in \mathcal{D}(\mathcal{A}^2)$. Then, let $\Phi(t) = (y(t), y_t(t)) = S(t)\Phi_0$ be the solution of (2.1). Since the trajectory of solution $\{\Phi(t)\}_{t \geq 0}$ is a precompact set, one can apply LaSalle's principle to deduce that $\omega(\Phi_0)$ is non empty, compact, invariant under the semigroup $S(t)$ and $S(t)\Phi_0 \longrightarrow \omega(\Phi_0)$, as $t \to +\infty$ [13]. Let $\tilde{\Phi}_0 = (\tilde{y}_0, \tilde{z}_0) \in \omega(\Phi_0) \subset \mathcal{D}(\mathcal{A})$ and $\tilde{\Phi}(t) = (\tilde{y}(t), \tilde{y}_t(t)) = S(t)\tilde{\Phi}_0 \in \mathcal{D}(\mathcal{A})$ the unique strong solution of (2.1). Using the fact that $\|\tilde{\Phi}(t)\|_{\mathcal{H}}$ is constant [13] and hence $< \mathcal{A}\tilde{\Phi}, \tilde{\Phi} >_{\mathcal{H}} = -\int_{\Gamma_1} a|z|^2 \, d\sigma = 0$, it implies that $\tilde{z} = \tilde{y}_t = 0$ on Γ_1. This yields $\tilde{y} \equiv$ constant and the desired result follows. ⋄

Now, we treat the case when the boundary conditions are dynamical. First, the well-posedness of the problem (1.1)-(1.3) and (1.6) can be proved on the state space $\mathcal{H}_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1)$ equipped with the inner product

$$\begin{aligned} \Big\langle (y, z, w_0, w_1), (\tilde{y}, \tilde{z}, \tilde{w}_0, \tilde{w}_1) \Big\rangle_{\mathcal{H}_d} &= \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij} \partial_i y \partial_j \tilde{y} + z\tilde{z} \right) dx + \int_{\Gamma_0} m w_0 \tilde{w}_0 \, d\sigma + \int_{\Gamma_1} M w_1 \tilde{w}_1 \, d\sigma \\ +\epsilon \left(\int_{\Omega} z \, dx + \int_{\Gamma_0} m w_0 \, d\sigma + \int_{\Gamma_1} (M w_1 + a y) \, d\sigma \right) &\left(\int_{\Omega} \tilde{z} \, dx + \int_{\Gamma_0} m \tilde{w}_0 \, d\sigma + \int_{\Gamma_1} (M \tilde{w}_1 + a \tilde{y}) \, d\sigma \right), \end{aligned} \tag{2.3}$$

where $\epsilon > 0$ is a small enough constant. Indeed, setting $z = y_t, w_0 = z|_{\Gamma_0}, w_1 = z|_{\Gamma_1}$ and $\Phi(t) = (y(t), z(t), w_0(t), w_1(t))$, the closed loop system can be written into the following form:

$$\left\{ \begin{array}{l} \Phi_t(t) = \mathcal{A}_d \Phi(t), \\ \Phi(0) = \Phi_0 = (y^0, z^0, w_0^0, w_1^0). \end{array} \right. \tag{2.4}$$

Here $\mathcal{A}_d$ is a linear operator, which generates a C_0 semigroup of contractions on $\mathcal{H}_d$, and defined by

$$\mathcal{D}(\mathcal{A}_d) = \left\{ (y, z, w_0, w_1) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1);\ Ay \in L^2(\Omega),\ w_0 = z|_{\Gamma_0},\ w_1 = z|_{\Gamma_1} \right\},$$

$$\mathcal{A}_d(y, z, w_0, w_1) = \left(z, Ay, -\frac{1}{m}\partial_A y, -\frac{1}{M}(a w_1 + \partial_A y) \right), \text{ for any } (y, z, w_0, w_1) \in \mathcal{D}(\mathcal{A}_d).$$

Then, using the same arguments as for Theorem 2.1, one can prove the second main result:

Theorem 2.2. *For any initial data $\Phi_0 = (y_0, z_0, w_0^0, w_1^0) \in \mathcal{H}_d$, the solution $\Phi(t) = (y(t), y_t(t), w_0(t), w_1(t))$ of (2.4) goes in $\mathcal{H}_d$ to $(\mathcal{C}, 0, 0, 0)$ as $t \longrightarrow +\infty$, where $\mathcal{C}$ is given by (2.2).*

We turn now to the case of dynamical boundary conditions with a nonlinear damping control. For sake of simplicity and without loss of generality, we shall consider the system with constant coefficients

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ m y_{tt}(x,t) + \dfrac{\partial y}{\partial \nu}(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ M y_{tt}(x,t) + \dfrac{\partial y}{\partial \nu}(x,t) = -f(y_t(x,t)), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega),\ y_t(x,0) = z_0(x) \in L^2(\Omega), & \\ y_t|_{\Gamma_0}(x,0) = w_0^0(x) \in L^2(\Gamma_0),\ y_t|_{\Gamma_1}(x,0) = w_1^0(x) \in L^2(\Gamma_1), & \end{cases} \tag{2.5}$$

where f satisfies the classical assumptions, namely, f is a non-decreasing continuous function such that $f(0) = 0$. Moreover, suppose that there exists a positive constant K such that $|f'(0)s - f(s)| \leq K s f(s)$, for any s in some neighborhood of 0.

Let the state space $\mathcal{H}_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1)$ equipped with the new inner product

$$\Big\langle (y, z, w_0, w_1), (\tilde{y}, \tilde{z}, \tilde{w}_0, \tilde{w}_1) \Big\rangle_{\mathcal{H}_d} = \int_\Omega (\nabla y \nabla \tilde{y} + z\tilde{z})\, dx + \int_{\Gamma_0} m w_0 \tilde{w}_0\, d\sigma + M \int_{\Gamma_1} w_1 \tilde{w}_1\, d\sigma + \rho \int_{\Gamma_1} y\tilde{y}\, d\sigma, \tag{2.6}$$

where ρ is any positive constant. Then, it is shown that the system is well-posed in the sense of semigroups of nonlinear operators [2] and as $t \to \infty$, the solution $(y(t), y_t(t), y_t|_{\Gamma_0}(t), y_t|_{\Gamma_1}(t))$ of the system tends to $\left(\tilde{\mathcal{C}}, 0, 0, 0\right)$, where $\tilde{\mathcal{C}}$ is given by

$$(f'(0)\,|\Gamma_1|)^{-1} \left(\int_\Omega z^0 dx \right) + m \left(\int_{\Gamma_0} w_0^0 d\sigma \right) + \int_{\Gamma_1} \left(M w_1^0 + f'(0) y^0 \right) d\sigma + \int_0^\infty \int_{\Gamma_1} (f'(0) w_1(s) - f(w_1(s)))\, d\sigma.$$

Remark 2.1. The method presented in this note can be applied for a large class of distributed systems (where the classical energy defines only a semi-norm in the state space) to prove the convergence, as the time goes to infinity, of solutions to an equilibrium point which can be determined. Indeed, applications to Petrovsky system, coupled wave-wave equations and elasticity systems can be carried out.

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