

Chapter 4

Frictional versus viscoelastic damping for Timoshenko-type systems

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Abstract

In this paper we consider the following Timoshenko system

$$\varphi_{tt} - (\varphi_x + \psi)_x = 0, \quad (0, 1) \times (0, +\infty)$$

$$\psi_{tt} - \psi_{xx} + \int_0^t g(t - \tau)(a(x)\psi_x(\tau))_x d\tau + \varphi_x + \psi + b(x)h(\psi_t) = 0, \quad (0, 1) \times (0, +\infty)$$

with Dirichlet boundary conditions where a , b , g , and h are specific functions. We establish an exponential and polynomial decay results. This result improves and generalizes some existing results in the literature.

Keywords and phrases: exponential decay, frictional damping, polynomial decay, relaxation function, Timoshenko, viscoelastic.

4.1 Introduction

A simple model describing the transverse vibration of a beam, which was developed in [23], is given by the following system of coupled hyperbolic equations

$$\begin{aligned}\rho u_{tt} &= (K(u_x - \varphi))_x, \quad \text{in } (0, L) \times (0, +\infty) \\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_x - \varphi), \quad \text{in } (0, L) \times (0, +\infty),\end{aligned}\tag{4.1}$$

where t denotes the time variable and x is the space variable along the beam of length L , in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

Kim and Renardy [9] considered (1.1) together with two boundary control of the form

$$\begin{aligned}K\varphi(L, t) - K\frac{\partial u}{\partial x}(L, t) &= \alpha\frac{\partial u}{\partial t}(L, t), \quad \forall t \geq 0 \\ EI\frac{\partial \varphi}{\partial x}(L, t) &= -\beta\frac{\partial \varphi}{\partial t}(L, t), \quad \forall t \geq 0\end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1). An analogous result was also established by Feng *et al.* [7], where the stabilization of vibrations in a Timoshenko system was studied. Raposo *et al.* [15] studied (1.1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings. Precisely, they looked into the following system

$$\begin{aligned}\rho_1 u_{tt} - K(u_x - \varphi)_x + u_t &= 0, \quad \text{in } (0, L) \times (0, +\infty) \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + K(u_x - \varphi) + \varphi_t &= 0, \quad \text{in } (0, L) \times (0, +\infty) \\ u(0, L) = u(L, t) = \varphi(0, t) = \varphi(L, t) &= 0, \quad \forall t > 0\end{aligned}\tag{4.2}$$

and proved that the energy associated with (1.2) decays exponentially. This result is similar to the one by Taylor *et al.* [22] but, as they mentioned, the originality in their work lies in the method they used, which was developed by Liu and Zheng [12]. This method is different from the usual ones such as the classical energy method. It mainly uses the semigroup theory. Soufyane and Wehbe [20] showed that it is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback. So, they considered

$$\begin{aligned}\rho u_{tt} &= (K(u_x - \varphi))_x, \quad \text{in } (0, L) \times (0, +\infty) \\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, \quad \text{in } (0, L) \times (0, +\infty) \\ u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) &= 0, \quad \forall t > 0,\end{aligned}\tag{4.3}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_p}\right)$; otherwise only the asymptotic stability has been proved. This result improves earlier ones by Soufyane [21] and Shi and Feng [17], where an exponential decay of the solution energy of (1.1) together, with two locally distributed feedbacks, had been proved. Xu and Yung [24] studied a system of Timoshenko beams with pointwise feedback controls, sought information about the eigenvalues and eigenfunctions of the system, and used this information to examine the stability of the system. Muñoz Rivera and Racke [14] treated a system of the form

$$\begin{aligned}\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x &= 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma\theta_x &= 0 \\ \rho_3\theta_t - K\theta_{xx} + \gamma\psi_{xt} &= 0,\end{aligned}$$

where $\varphi, \psi,$ and θ are functions of (x, t) model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of $\sigma, \rho_i, b, K, \gamma,$ they proved several exponential decay results for the linearized system and non exponential stability result for the case of different wave speeds. Ammar-Khodja *et al.* [1] considered a linear Timoshenko-type system with memory of the form

$$\begin{aligned}\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x &= 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) &= 0\end{aligned}\tag{4.4}$$

in $(0, L) \times (0, +\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. The feedback of memory type has also been used by De Lima Santos [6]. He considered a Timoshenko system and showed that the presence of two feedback of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. Shi and Feng [19] investigated a nonuniform Timoshenko beam and showed that, under some locally distributed controls, the vibration of the beam decays exponentially. To achieve their goal, the authors used the frequency multiplier method. For more results concerning well-posedness and controllability of Timoshenko systems, we refer the reader to [8], [10], [11], [16], [18], [24], and [25].

In the present work we are concerned with

$$\left\{\begin{array}{l} \varphi_{tt} - (\varphi_x + \psi)_x = 0, \quad (0, 1) \times \mathbb{R}_+ \\ \psi_{tt} - \psi_{xx} + \varphi_x + \psi + \int_0^t g(t-\tau)(a(x)\psi_x(\tau))_x d\tau \\ + b(x)h(\psi_t) = 0, \quad (0, 1) \times \mathbb{R}_+ \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \quad t \geq 0 \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad x \in (0, 1) \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad x \in (0, 1). \end{array}\right.\tag{4.5}$$

Our aim in this work is to investigate the effect of both frictional and viscoelastic dampings, where each one of them can vanish on the whole domain or in a part of it. In addition, we would

like to see the influence of these dissipations on the rate of decay of solutions. Of course, the most interesting case occurs when we have simultaneous and complementary damping mechanisms. This result generalizes the one in [1] and improves it. Precisely, we obtain an exponential or polynomial decay result under weaker conditions on the relaxation function g (see remark 3.1 by the end). Our proof combines arguments from [1-5]. In particular, the use of a functional similar to the one in [2,3] played an essential role in weakening the requirements on g . We should note here that we do not lose the generality by taking ρ_1, ρ_2, K, b , appeared in (1.4), to be equal to one and our argument also works for $\rho_1/\rho_2 = K/b$. The paper is organized as follows. In Section 2, We present some notations and material needed for our work and state our main result. The proof will be given in section 3.

4.2 Preliminaries

In order to state our main result we make the following hypotheses.

(H1) $a, b : [0, 1] \rightarrow \mathbb{R}_+$ are such that

$$a \in C^1([0, 1]), \quad b \in L^\infty([0, 1]),$$

$$a(0) + a(1) > 0, \quad \inf_{x \in [0, 1]} \{a(x) + b(x)\} > 0.$$

(H2) $h : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable nondecreasing function such that there exist constants $c_1, c_2 > 0$ and $q \geq 1$ for which

$$c_1 \min\{|s|, |s|^q\} \leq |h(s)| \leq c_2 \max\{|s|, |s|^{\frac{1}{q}}\}, \quad s \in \mathbb{R}.$$

(H3) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \quad 1 - \|a\|_\infty \int_0^\infty g(s) ds = l > 0.$$

(H4) There exist constants $\xi > 0$ and $1 \leq p < 3/2$ such that

$$g'(s) \leq -\xi g^p(s), \quad s \geq 0.$$

Remark 2.1. We note that, by hypothesis (H1), we have either $a(0) > 0$ or $a(1) > 0$. So, without loss of generality we take $a(0) > 0$ in the whole paper.

Remark 2.2. Hypothesis (H4) implies that

$$\int_0^{+\infty} g^{2-p}(s) ds < +\infty.$$

For completeness we state, without proof, an existence and regularity result.

Proposition 2.1. Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, 1) \times L^2(0, 1)$ be given. Assume that (H1)-(H3) are satisfied, then problem (1.5) has a unique global (weak) solution

$$\varphi, \psi \in C(\mathbb{R}_+; H_0^1(0, 1)) \cap C^1(\mathbb{R}_+; L^2(0, 1)). \quad (4.1)$$

Moreover, if

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1)$$

then the solution satisfies

$$\varphi, \psi \in L^\infty(\mathbb{R}_+; H^2(0, 1) \cap H_0^1(0, 1)) \cap W^{1,+\infty}(\mathbb{R}_+; H_0^1(0, 1)) \cap W^{2,+\infty}(\mathbb{R}_+; L^2(0, 1)). \quad (4.2)$$

Remark 2.3. If h is linear and $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1)$ then

$$\varphi, \psi \in C(\mathbb{R}_+; H^2(0, 1) \cap H_0^1(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+; H_0^1(0, 1)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(0, 1)).$$

Remark 2.4. This result can be proved using standard arguments such as the nonlinear semi-group method or the Galerkin method.

Now, we introduce the energy functional

$$E(t) := \frac{1}{2} \int_0^1 [\varphi_t^2 + \psi_t^2 + (1 - a(x) \int_0^t g(s) ds) \psi_x^2 + (\varphi_x + \psi)^2] dx + \frac{1}{2} (g \circ \psi_x), \quad (4.3)$$

where, for all $v \in L^2(0, 1)$ and for all $1 \leq p < 3/2$,

$$(g^p \circ v)(t) = \int_0^1 a^p(x) \int_0^t g^p(t-s)(v(t) - v(s))^2 ds dx. \quad (4.4)$$

We are now ready to state our main stability result.

Theorem 2.2. Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, 1) \times L^2(0, 1)$ be given. Assume that (H1)-(H4) are satisfied, then there exist two positive constants c and ω , for which the solution of problem (1.5) satisfies

$$E(t) \leq ce^{-\omega t}, \quad \forall t \geq 0 \quad \text{if } p = q = 1, \quad (4.5)$$

$$E(t) \leq c(1+t)^{-\frac{2}{p(q+1)-2}}, \quad \forall t \geq 0 \quad \text{if } (p, q) \neq (1, 1) \text{ and } (2p-1)(q+1) < 4, \quad (4.6)$$

and

$$E(t) \leq c(1+t)^{-\frac{2}{(2p-1)(q+1)-2}}, \quad \forall t \geq 0 \quad \text{if } (p, q) \neq (1, 1) \text{ and } (2p-1)(q+1) \geq 4. \quad (4.7)$$

4.3 Proof of the main result

In this section we prove our main result. For this purpose we shall establish several lemmas.

Lemma 3.1. Let (φ, ψ) be the solution of (1.5). Then the energy functional satisfies

$$E'(t) = -\frac{1}{2} g(t) \int_0^1 a(x) \psi_x^2 dx - \int_0^1 b(x) \psi_t h(\psi_t) dx + \frac{1}{2} (g' \circ \psi_x) \leq 0. \quad (4.1)$$

Proof. By multiplying equations in (1.5) by φ_t and ψ_t respectively and integrating over $(0, 1)$, using integration by parts, hypotheses (H1)-(H4) and some manipulations as in [13], we obtain (3.1) for any regular solution. This equality remains valid for weak solutions by simple density argument.

Next, we introduce a function α which helps in establishing some needed estimates. By using Remark 2.1 and the fact that a is continuous, then there exists $\varepsilon_0 > 0$ such that $\inf_{x \in [0, \varepsilon_0]} a(x) \geq \varepsilon_0$. Set

$$d = \min\{\varepsilon_0, \inf_{x \in [0, 1]} \{a(x) + b(x)\}\} > 0$$

and let $\alpha \in C^1([0, 1])$ be such that $0 \leq \alpha \leq a$ and

$$\begin{aligned} \alpha(x) &= 0 & \text{if } a(x) &\leq \frac{d}{4}, \\ \alpha(x) &= a(x) & \text{if } a(x) &\geq \frac{d}{2}. \end{aligned}$$

Lemma 3.2. *The function α is not identically zero and satisfies*

$$\inf_{x \in [0, 1]} \{\alpha(x) + b(x)\} \geq \frac{d}{2}.$$

Proof. For all $x \in [0, \varepsilon_0]$, we have $a(x) \geq \varepsilon_0 \geq d > \frac{d}{2}$, so, by definition, $\alpha(x) = a(x) \geq \varepsilon_0$, hence, α is not identically zero over $[0, 1]$.

In the other hand, if $a(x) \geq \frac{d}{2}$, then $\alpha(x) \geq \frac{d}{2}$, which implies that $\alpha(x) + b(x) \geq \frac{d}{2}$. If $a(x) < \frac{d}{2}$, then, by (H1) and the definition of d we have $b(x) > \frac{d}{2}$. Consequently $\alpha(x) + b(x) \geq \frac{d}{2}$. Therefore $\inf_{x \in [0, 1]} \{\alpha(x) + b(x)\} \geq \frac{d}{2}$.

The key point to show the exponential and the polynomial decay is to construct a Lyapunov functional \mathcal{L} equivalent to E and satisfying, for positive constants λ_1 and λ_2 ,

$$\mathcal{L}'(t) \leq -\lambda_2 \mathcal{L}^{\lambda_1}(t), \quad \forall t \geq 0.$$

For this, we define several functionals which allow us to obtain the needed estimates. To simplify the computations we set

$$g \odot v = \int_0^1 \alpha(x) \int_0^t g(t-s)(v(t) - v(s)) ds dx$$

for all $v \in L^2(0, 1)$ and use c , throughout this paper, to denote a generic positive constant.

Lemma 3.3. *There exists a positive constant c such that*

$$(g \odot v)^2 \leq cg^p \circ v_x$$

for all $v \in H^1(0, 1)$ with $v(0) = 0$.

Proof. Let $S_a = \{x \in [0, 1]: a(x) > \frac{d}{4}\}$. We should note that, by definition of d , $0 \in S_a$, hence $\partial S_a \cap \partial(0, 1) \neq \emptyset$ and $\text{supp } \alpha \subset S_a$.

$$(g \odot v)^2 = \left(\int_{\text{supp } \alpha} \alpha(x) \int_0^t g^{1-\frac{p}{2}}(t-\tau) g^{\frac{p}{2}}(t-s)(v(t) - v(s)) ds dx \right)^2.$$

By using Hölder's inequality, a variant of Poincaré's inequality (see [5]) and Remark 2.2, we get

$$\begin{aligned} (g \odot v)^2 &\leq c \left(\int_0^t g^{2-p}(s) ds \right) \left(\int_{\text{supp } \alpha} \int_0^t g^p(t-s)(v(t) - v(s))^2 ds dx \right) \\ &\leq c \int_{S_a} \int_0^t g^p(t-s)(v_x(t) - v_x(s))^2 ds dx. \end{aligned}$$

Recalling the definition of S_a , we arrive at

$$(g \odot v)^2 \leq c \int_{S_a} a^p(x) \int_0^t g^p(t-s)(v_x(t) - v_x(s))^2 ds dx \leq c g^p \circ v_x.$$

Lemma 3.4. Under the assumptions (H1)-(H4), the functional I defined by

$$I(t) := - \int_0^1 \alpha(x) \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} I'(t) &\leq - \left(\int_0^t g(s) ds - \delta \right) \int_0^1 \alpha(x) \psi_t^2 dx + \delta \int_0^1 (\varphi_x + \psi)^2 dx + c \delta \int_0^1 \psi_x^2 dx \quad (4.2) \\ &\quad - \frac{c}{\delta} g' \circ \psi_x + c \left(\delta + \frac{1}{\delta} \right) g^p \circ \psi_x + \frac{c}{\delta} \int_0^1 b(x) h^2(\psi_t(t)) dx, \end{aligned}$$

for all $\delta > 0$.

Proof. By using equations in (1.5), we get

$$\begin{aligned} I'(t) &= - \int_0^1 \alpha \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s)) ds dx - \int_0^1 \alpha \psi_t^2 \left(\int_0^t g(s) ds \right) dx \\ &\quad - \int_0^1 \alpha [\psi_{xx} - \int_0^t g(t-s)(a(x) \psi_x(s))_x ds - \varphi_x - \psi - b(x) h(\psi_t)] \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx \\ &= - \int_0^1 \alpha \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s)) ds dx - \int_0^1 \alpha \psi_t^2 \left(\int_0^t g(s) ds \right) dx \\ &\quad + \int_0^1 \alpha \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx + \int_0^1 \alpha (\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 \alpha a \left(\int_0^t g(t-s) \psi_x(s) ds \right) \left(\int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds \right) dx \\
& + \int_0^1 \alpha' (\psi_x - a) \int_0^t g(t-s) \psi_x(s) ds \left(\int_0^t g(t-s) (\psi(t) - \psi(s)) ds \right) dx \\
& \quad + \int_0^1 \alpha b(x) h(\psi_t) \int_0^t g(t-s) (\psi(t) - \psi(s)) ds dx.
\end{aligned}$$

We now estimate the terms in the right side of the above equality as follows.

By using Young's inequality and Lemma 3.3 (for g' and $p = 1$) we obtain, for all $\delta > 0$,

$$- \int_0^1 \alpha \psi_t \int_0^t g'(t-s) (\psi(t) - \psi(s)) ds dx \leq \delta \int_0^1 \alpha(x) \psi_t^2 dx - \frac{c}{\delta} g' \circ \psi_x.$$

Similarly, we have

$$\begin{aligned}
& - \int_0^1 \alpha \psi_x \int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds dx \leq \delta \int_0^1 \psi_x^2 dx + \frac{c}{\delta} g^p \circ \psi_x, \\
& - \int_0^1 \alpha (\varphi_x + \psi) \int_0^t g(t-s) (\psi(t) - \psi(s)) ds dx \leq \delta \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\delta} g^p \circ \psi_x, \\
& - \int_0^1 \alpha a \left(\int_0^t g(t-s) \psi_x(s) ds \right) \left(\int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds \right) dx \\
\leq & \delta' \int_0^1 a \left(\int_0^t g(t-s) (\psi_x(s) - \psi_x(t) + \psi_x(t)) ds \right)^2 dx + \frac{c}{\delta'} \int_0^1 a \left(\int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\
\leq & 2\delta' \int_0^1 a \psi_x^2 \left(\int_0^t g(s) ds \right)^2 dx + (2\delta' + \frac{c}{\delta'}) \int_0^1 a \left(\int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\
\leq & c\delta' \int_0^1 \psi_x^2 dx + c(\delta' + \frac{1}{\delta'}) g^p \circ \psi_x \leq \delta \int_0^1 \psi_x^2 dx + c(\delta + \frac{1}{\delta}) g^p \circ \psi_x, \\
& \int_0^1 \alpha' (\psi_x - a) \int_0^t g(t-s) \psi_x(s) ds \left(\int_0^t g(t-s) (\psi(t) - \psi(s)) ds \right) dx \leq \delta \int_0^1 \psi_x^2 dx + c(\delta + \frac{1}{\delta}) g^p \circ \psi_x,
\end{aligned}$$

and

$$\int_0^1 \alpha b(x) h(\psi_t) \int_0^t g(t-s) (\psi(t) - \psi(s)) ds dx \leq \delta \int_0^1 b(x) h^2(\psi_t) dx + c(\delta + \frac{1}{\delta}) g^p \circ \psi_x.$$

By combining all the above estimates, the assertion of Lemma 3.4 is proved.

Lemma 3.5. Under the assumptions (H1)-(H4), the functional J defined by

$$J(t) := - \int_0^1 (\psi \psi_t + \varphi \varphi_t) dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} J'(t) &\leq - \int_0^1 (\psi_t^2 + \varphi_t^2) dx + \int_0^1 (\psi + \varphi_x)^2 dx \\ &+ c \int_0^1 \psi_x^2 dx + cg^p \circ \psi_x + c \int_0^1 b(x)h^2(\psi_t) dx. \end{aligned} \quad (4.3)$$

Proof. By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned} J'(t) &= - \int_0^1 (\psi_t^2 + \varphi_t^2) dx - \int_0^1 \varphi(\psi_x + \varphi_{xx}) dx \\ &- \int_0^1 \psi[\psi_{xx} - \int_0^t g(t-s)(a(x)\psi_x(s))_x ds - \varphi_x - \psi - b(x)h(\psi_t)] dx \\ &= - \int_0^1 (\psi_t^2 + \varphi_t^2) dx + \int_0^1 \psi_x^2 dx - \int_0^1 a(x)\psi_x \left(\int_0^t g(t-s)\psi_x(s) ds \right) dx \\ &\quad + \int_0^1 (\psi + \varphi_x)^2 dx + \int_0^1 b(x)\psi h(\psi_t) dx \\ &\leq - \int_0^1 (\psi_t^2 + \varphi_t^2) dx + \int_0^1 (\psi + \varphi_x)^2 + c \int_0^1 \psi_x^2 dx + cg^p \circ \psi_x + c \int_0^1 b(x)h^2(\psi_t) dx. \end{aligned}$$

This completes the proof of Lemma 3.5.

Lemma 3.6. Assume that (H1)-(H4) hold. Then, the functional K defined by

$$K(t) := \int_0^1 \psi_t(\psi + \varphi_x) dx + \int_0^1 \psi_x \varphi_t dx - \int_0^1 a(x) \varphi_t \int_0^t g(t-s) \psi_x(s) ds dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} K'(t) &\leq [(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s) ds) \varphi_x]_{x=0}^{x=1} - (1-\varepsilon) \int_0^1 (\psi + \varphi_x)^2 dx \\ &+ \varepsilon \int_0^1 \varphi_t^2 dx - \frac{c}{\varepsilon} g' \circ \psi_x + \frac{c}{\varepsilon} \int_0^1 \psi_x^2 dx + \int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon} \int_0^1 b(x)h^2(\psi_t) dx \end{aligned} \quad (4.4)$$

for any $0 < \varepsilon < 1$.

Proof. By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned} K'(t) &= \int_0^1 (\varphi_x + \psi) [\psi_{xx} - \int_0^t g(t-s)(a(x)\psi_x(s))_x ds - \varphi_x - \psi - b(x)h(\psi_t)] dx \\ &\quad + \int_0^1 (\varphi_{xt} + \psi_t) \psi_t dx + \int_0^1 \psi_{xt} \varphi_t dx + \int_0^1 \psi_x (\varphi_x + \psi)_x dx \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 a(x)(\varphi_x + \psi)_x \int_0^t g(t-s)\psi_x(s)ds dx - \int_0^1 a(x)\varphi_t(g(0)\psi_x + \int_0^t g'(t-s)\psi_x(s)ds)dx \\
& = [(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s)ds)\varphi_x]_{x=0}^{x=1} \\
& \quad - \int_0^1 (\psi + \varphi_x)^2 dx - \int_0^1 b(x)(\psi + \varphi_x)h(\psi_t)dx + \int_0^1 \psi_t^2 dx \\
& \quad + g(t) \int_0^1 a(x)\psi_x\varphi_t dx - \int_0^1 a(x)\varphi_t \int_0^t g'(t-s)(\psi_x(s) - \psi_x(t))ds dx.
\end{aligned}$$

By using Young's inequality, (3.4) is established.

Lemma 3.7. Assume that (H1)-(H4) hold. Let $m \in C^1([0, 1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that for any $\varepsilon > 0$ we have, along the solution,

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 m(x)\psi_t(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s)ds)dx \\
& \leq - \left((\psi_x(1, t) - a(1) \int_0^t g(t-s)\psi_x(1, s)ds)^2 + (\psi_x(0, t) - a(0) \int_0^t g(t-s)\psi_x(0, s)ds)^2 \right) \\
& \quad + \varepsilon \int_0^1 (\psi + \varphi_x)^2 dx + \frac{c}{\varepsilon} \left(\int_0^1 \psi_x^2 dx + g^p \circ \psi_x \right) + c \left(\int_0^1 (\psi_t^2 + b(x)h^2(\psi_t))dx - g' \circ \psi_x \right)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 m(x)\varphi_t\varphi_x dx \leq -(\varphi_x^2(1, t) + \varphi_x^2(0, t)) \\
& \quad + c \int_0^1 (\varphi_t^2 + \varphi_x^2 + \psi_x^2)dx.
\end{aligned}$$

Proof. By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 m(x)\psi_t(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s)ds)dx \\
& = \int_0^1 m(x)(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s)ds)_x(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s)ds)dx \\
& \quad - \int_0^1 m(x)(\psi_x - a(x) \int_0^t g(t-s)\psi_x(s)ds)(\varphi_x + \psi + b(x)h(\psi_t))dx \\
& \quad + \int_0^1 m(x)\psi_t(\psi_{xt} - a(x)g(0)\psi_x - a(x) \int_0^t g'(t-s)\psi_x(s)ds)dx \\
& = - \left((\psi_x(1, t) - a(1) \int_0^t g(t-s)\psi_x(1, s)ds)^2 + (\psi_x(0, t) - a(0) \int_0^t g(t-s)\psi_x(0, s)ds)^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \int_0^1 m'(x) (\psi_x - a(x) \int_0^t g(t-s) \psi_x(s) ds)^2 dx \\
& - \int_0^1 m(x) (\psi_x - a(x) \int_0^t g(t-s) \psi_x(s) ds) (\varphi_x + \psi + b(x)h(\psi_t)) dx - \frac{1}{2} \int_0^1 m'(x) \psi_t^2 dx \\
& + \int_0^1 m(x) a(x) \psi_t (\int_0^t g'(t-s) (\psi_x(t) - \psi_x(s)) ds) dx + g(t) \int_0^1 m(x) a(x) \psi_x \psi_t dx.
\end{aligned}$$

By using Young's inequality and Lemma 3.3, the first estimate of Lemma 3.7 is established. Similarly, we can prove the second estimate of Lemma 3.7.

Lemma 3.8. Assume that (H1)-(H4) hold. Then, the functional L defined by

$$L(t) := K(t) + \frac{1}{4\varepsilon} \int_0^1 m(x) \psi_t (\psi_x - a(x) \int_0^t g(t-s) \psi_x(s) ds) dx + \varepsilon \int_0^1 m(x) \varphi_t \varphi_x dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}
L'(t) & \leq -\left(\frac{3}{4} - c\varepsilon\right) \int_0^1 (\varphi_x + \psi)^2 dx + c\varepsilon \int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon} \int_0^1 \psi_t^2 dx \\
& + \frac{c}{\varepsilon} \int_0^1 b(x) h^2(\psi_t) dx + \frac{c}{\varepsilon^2} \int_0^1 \psi_x^2 dx - \frac{c}{\varepsilon} g' \circ \psi_x + \frac{c}{\varepsilon^2} g^p \circ \psi_x
\end{aligned} \tag{4.5}$$

for any $0 < \varepsilon < 1$.

Proof. By using Lemma 3.6, Lemma 3.7, Young's and Poincaré's inequalities, and the fact that

$$\varphi_x^2 \leq 2(\psi + \varphi_x)^2 + 2\psi^2$$

and

$$(\psi_x - a(x) \int_0^t g(t-s) \psi_x(s) ds) \varphi_x \leq \varepsilon \varphi_x^2 + \frac{1}{4\varepsilon} (\psi_x - a(x) \int_0^t g(t-s) \psi_x(s) ds)^2,$$

we obtain (3.5).

Let $L_1(t) := L(t) + 2c\varepsilon J(t)$. By using Lemma 3.5 and Lemma 3.8, and fixing ε small enough, we obtain

$$\begin{aligned}
L_1'(t) & \leq -\frac{1}{2} \int_0^1 (\psi + \varphi_x)^2 dx - \tau \int_0^1 \varphi_t^2 dx + c \int_0^1 \psi_t^2 dx \\
& + c \int_0^1 \psi_x^2 dx + c \int_0^1 b(x) h^2(\psi_t) dx + cg^p \circ \psi_x - cg' \circ \psi_x
\end{aligned} \tag{4.6}$$

where $\tau = c\varepsilon$.

As in [1], we use the multiplier w given by the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0. \tag{4.7}$$

Lemma 3.9. The solution of (3.7) satisfies

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. We multiply equation (3.7) by w , integrate by parts, and use the Cauchy-Schwarz inequality, to get

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (3.7) with respect to t to obtain, by similar calculations,

$$\int_0^1 w_{xt}^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Poincaré's inequality, then yields

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma 3.9.

Lemma 3.10. Under the assumptions (H1)-(H4), the functional J_1 defined by

$$J_1(t) := \int_0^1 (\psi\psi_t + w\varphi_t) dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} J_1'(t) &\leq -\frac{l}{2} \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_1} \int_0^1 \psi_t^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx \\ &\quad + c \left(\int_0^1 b(x)h^2(\psi_t) dx + g^p \circ \psi_x \right) \end{aligned} \quad (4.8)$$

for any $0 < \varepsilon_1 < 1$ (l is defined in (H3)).

Proof. By exploiting equations (1.5) and integrating by parts, we have

$$\begin{aligned} J_1'(t) &= \int_0^1 (\psi_t^2 - \psi_x^2) dx + \int_0^1 a(x)\psi_x \int_0^t g(t-s)\psi_x(s) ds dx \\ &\quad - \int_0^1 \psi(\psi + \varphi_x + b(x)h(\psi_t)) dx - \int_0^1 w_x(\psi + \varphi_x) dx + \int_0^1 w_t\varphi_t dx \\ &\leq \int_0^1 \psi_t^2 dx - \frac{l}{2} \int_0^1 \psi_x^2 dx + c \left(\int_0^1 b(x)h^2(\psi_t) dx + g^p \circ \psi_x \right) \\ &\quad + \int_0^1 (w_x^2 - \psi^2) dx + \frac{c}{\varepsilon_1} \int_0^1 \varphi_t^2 dx + \varepsilon_1 \int_0^1 w_t^2 dx. \end{aligned}$$

Lemma 3.9 gives the desired result.

For $N_1, N_2, N_3 > 1$, let

$$\mathcal{L}(t) := N_1 E(t) + N_2 I(t) + N_3 J_1 + L_1(t)$$

and $g_0 = \int_0^{t_0} g(s) ds > 0$ for some fixed $t_0 > 0$. By combining (3.1), (3.2), (3.6), (3.8), and taking $\delta = \frac{1}{4N_2}$, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq & -(N_2 g_0 - \frac{1}{4}) \int_0^1 \alpha(x) \psi_t^2 dx + c \frac{N_3}{\varepsilon_1} \int_0^1 \psi_t^2 dx \\ & - N_1 \int_0^1 b(x) \psi_t h(\psi_t) dx + c(N_2^2 + N_3) \int_0^1 b(x) h^2(\psi_t) dx \\ & - (\frac{N_3}{2} - c - \frac{c}{N_2}) \int_0^1 \psi_x^2 dx - (c - N_3 \varepsilon_1) \int_0^1 \varphi_t^2 dx - \frac{1}{4} \int_0^1 (\psi + \varphi_x)^2 dx \\ & + (\frac{N_1}{2} - cN_2^2) g' \circ \psi_x + c(N_2^2 + N_3) g^p \circ \psi_x \end{aligned} \quad (4.9)$$

for all $t \geq t_0$.

We distinguish different cases:

Case 1: $p = q = 1$. In this case, we choose N_3 large enough so that

$$\frac{N_3}{2} > c,$$

then ε_1 small enough so that

$$\varepsilon_1 < \frac{c}{N_3}.$$

Next, we choose N_2 large enough so that

$$N_2 g_0 - \frac{1}{4} > \frac{2cN_3}{d\varepsilon_1}, \quad \frac{N_3}{2} - c - \frac{c}{N_2} > 0.$$

Finally, we choose N_1 large enough so that

$$N_1 c_1 - c(N_2^2 + N_3) > N_2 g_0 - \frac{1}{4}, \quad \xi(\frac{N_1}{2} - cN_2^2) > c(N_2^2 + N_3).$$

By using (H2) and (H3), we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq & -(N_2 g_0 - \frac{1}{4}) \int_0^1 (\alpha(x) + b(x)) \psi_t^2 dx + c \frac{N_3}{\varepsilon_1} \int_0^1 \psi_t^2 dx \\ & - (\frac{N_3}{2} - c - \frac{c}{N_2}) \int_0^1 \psi_x^2 dx - (c - N_3 \varepsilon_1) \int_0^1 \varphi_t^2 dx - \frac{1}{4} \int_0^1 (\psi + \varphi_x)^2 dx - c g \circ \psi_x. \end{aligned}$$

Lemma 3.2, then, gives

$$\mathcal{L}'(t) \leq -c \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx + g \circ \psi_x \right) \leq -cE(t)$$

for all $t \geq t_0$.

In the other hand, we can choose N_1 even larger (if needed) so that

$$\mathcal{L}(t) \sim E(t). \quad (4.10)$$

Therefore, by combining the last two inequalities, we obtain, for a positive constant ω ,

$$\mathcal{L}'(t) \leq -\omega \mathcal{L}(t), \quad t \geq t_0.$$

A simple integration over (t_0, t) , leads to

$$\mathcal{L}(t) \leq ce^{-\omega t}, \quad t \geq t_0.$$

Consequently, (2.5) is established by virtue of (3.10) and boundedness of E

Case 2: $q = 1$, $p > 1$. With the same choice of constants as in Case 1, we deduce, from (3.9),

$$\mathcal{L}'(t) \leq -c \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx + g^p \circ \psi_x \right). \quad (4.11)$$

But using (H3) and (H4), we easily see that

$$\int_0^\infty g^{1-\theta}(s) ds < \infty, \quad \theta < 2 - p,$$

so lemma 3.3 [5] (see also [4]) yields

$$g \circ \psi_x \leq c \left\{ \left(\int_0^\infty g^{1-\theta}(s) ds \right) E(0) \right\}^{(p-1)/(p-1+\theta)} \{g^p \circ \psi_x\}^{\theta/(p-1+\theta)}.$$

Therefore we get, for $\gamma \geq 1$,

$$\begin{aligned} E^\gamma(t) &\leq cE^{\gamma-1}(0) \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx \right) + (g \circ \psi_x)^\gamma \quad (4.12) \\ &\leq cE^{\gamma-1}(0) \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx \right) \\ &\quad + c \left\{ \left(\int_0^\infty g^{1-\theta}(s) ds \right) E(0) \right\}^{\gamma(p-1)/(p-1+\theta)} \{g^p \circ \psi_x\}^{\theta\gamma/(p-1+\theta)} \end{aligned}$$

By choosing $\theta = \frac{1}{2}$ and $\gamma = 2p - 1$ (hence $\gamma\theta/(p-1+\theta) = 1$), estimate (3.12) gives

$$E^\gamma(t) \leq c \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx + g^p \circ \psi_x \right). \quad (4.13)$$

By combining (3.9), (3.10) and (3.12), we arrive at

$$\mathcal{L}'(t) \leq -c\mathcal{L}^\gamma(t), \quad t \geq t_0.$$

By integration, we get

$$\mathcal{L}(t) \leq -c(1+t)^{-\frac{1}{\gamma-1}}(t), \quad t \geq t_0. \quad (4.14)$$

Since $p < 3/2$ then $\frac{1}{\gamma-1} = \frac{1}{2(p-1)} > 1$. As a consequence of (3.14), we have

$$\int_0^\infty \mathcal{L}(t)dt + \sup_{t \geq 0} t\mathcal{L}(t) < +\infty.$$

Therefore, by using again Lemma 3.3 of [5] (see also [4]), we have

$$\begin{aligned} g \circ \psi_x &\leq c \left(\int_0^t \|\psi(s)\|_{H^1(0,1)} ds + t \|\psi(t)\|_{H^1(0,1)} \right)^{\frac{p-1}{p}} (g^p \circ \psi_x)^{\frac{1}{p}} \\ &\leq c \left(\int_0^t \mathcal{L}(s)dt + t\mathcal{L}(t) \right)^{\frac{p-1}{p}} (g^p \circ \psi_x)^{\frac{1}{p}} \leq c(g^p \circ \psi_x)^{\frac{1}{p}}, \end{aligned}$$

which implies that

$$g^p \circ \psi_x \geq (g \circ \psi_x)^p.$$

So

$$\mathcal{L}'(t) \leq -c \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx + (g \circ \psi_x)^p \right)$$

and, for (3.12) with $\gamma = p$,

$$E^p(t) \leq C \left(\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 (\psi + \varphi_x)^2 dx + (g \circ \psi_x)^p \right).$$

Combining the last two inequalities and (3.10), we obtain

$$\mathcal{L}'(t) \leq -c\mathcal{L}^p(t), \quad t \geq t_0.$$

A simple integration over (t_0, t) and by virtue of boundedness of \mathcal{L} , we arrive at

$$\mathcal{L}(t) \leq c(1+t)^{-\frac{1}{p-1}}, \quad t \geq 0.$$

Therefore (2.6) is satisfied using (3.10). Note that, in this case, $(2p-1)(q+1) = 2(2p-1) < 4$ thanks to (H4).

Case 3: $q > 1, p = 1$. In this case we consider the following partition of $(0, 1)$:

$$\Omega^+ = \{x \in (0, 1) : |\psi_t| > 1\} \text{ and } \Omega^- = \{x \in (0, 1) : |\psi_t| \leq 1\}.$$

From hypothesis (H2) and Holder's inequality, we easily show that

$$\int_{\Omega^+} b(x)(\psi_t^2 + h^2(\psi_t))dx \leq c \int_{\Omega^+} b(x)\psi_t h(\psi_t)dx \leq c \int_0^1 b(x)\psi_t h(\psi_t)dx \quad (4.15)$$

$$\int_{\Omega^-} b(x)(\psi_t^2 + h^2(\psi_t))dx \leq c \int_{\Omega^-} b(x)(\psi_t h(\psi_t))^{\frac{2}{q+1}} dx \leq c \left(\int_0^1 b(x)\psi_t h(\psi_t)dx \right)^{\frac{2}{q+1}}. \quad (4.16)$$

Therefore, with the same choice of constants N_3, ε_1, N_2 , and N_1 as in Case 1 we deduce, from (3.9), Lemma 3.2, hypothesis (H3) and the definition (2.3) of energy,

$$\mathcal{L}'(t) \leq -cE(t) + c \left(\int_0^1 b(x)\psi_t h(\psi_t)dx \right)^{\frac{2}{q+1}}, \quad \forall t \geq t_0.$$

We multiply this last inequality by $E^{\frac{q-1}{2}}$ and use (3.1) to arrive at

$$\mathcal{L}'(t)E(t)^{\frac{q-1}{2}} \leq -cE(t)^{\frac{q+1}{2}} + c(-E'(t))^{\frac{2}{q+1}}E(t)^{\frac{q-1}{2}}, \quad \forall t \geq t_0.$$

By using Young's inequality and (3.10), we obtain, for all $\varepsilon > 0$,

$$\begin{aligned} \left(\mathcal{L}(t)E(t)^{\frac{q-1}{2}} \right)' &\leq \frac{q-1}{2}\mathcal{L}(t)E^{\frac{q-3}{2}}(t)E'(t) - cE^{\frac{q+1}{2}}(t) + \varepsilon E^{\frac{q+1}{2}}(t) - c_\varepsilon E'(t) \\ &\leq -(c-\varepsilon)E^{\frac{q+1}{2}}(t) - c_\varepsilon E'(t), \quad \forall t \geq t_0. \end{aligned}$$

By choosing ε small enough, we obtain

$$(\mathcal{L}(t)E^{\frac{q-1}{2}}(t) + cE(t))' \leq -cE^{\frac{q+1}{2}}(t), \quad \forall t \geq t_0.$$

We put

$$F(t) = \mathcal{L}(t)E^{\frac{q-1}{2}}(t) + cE(t) \quad (4.17)$$

and we use (3.10), to deduce that

$$F(t) \sim E(t), \quad (4.18)$$

hence

$$F'(t) \leq -cF^{\frac{q+1}{2}}(t), \quad \forall t \geq t_0$$

A simple integration then leads to

$$F(t) \leq c(1+t)^{\frac{-2}{q-1}}, \quad \forall t \geq t_0,$$

consequently, the use of (3.18) yields

$$E(t) \leq c(1+t)^{\frac{-2}{q-1}}, \quad \forall t \geq t_0,$$

which gives (2.6) and (2.7). Note that, if $p = 1$ then (2.6) and (2.7) are the same.

Case 4: $q > 1, p > 1$

With the same choice of constants N_3, ε_1, N_2 and N_1 as in Case 1 and the use of (3.15), (3.16), and (3.1), we get

$$\mathcal{L}'(t) \leq -c \int_0^1 (\varphi_t^2 + \psi_t^2 + \psi_x^2 + (\psi + \varphi_x)^2) - cg^p \circ \psi_x + c(-E')^{\frac{2}{q+1}}, \quad \forall t \geq t_0.$$

Therefore, using (3.13), we have

$$\mathcal{L}'(t) \leq -c(E(t))^{2p-1} + c(-E'(t))^{\frac{2}{q+1}}, \quad \forall t \geq t_0. \quad (4.19)$$

As in Case 3 we deduce, from (3.19), Young's inequality, and the fact that E is bounded, that

$$(\mathcal{L}(t)E^{\frac{q-1}{2}}(t) + cE(t))' \leq -cE(t)^{2p-1+B}, \quad \forall t \geq t_0,$$

where $B = \frac{(2p-1)(q-1)}{2}$. By setting

$$F(t) = \mathcal{L}(t)E^{\frac{q-1}{2}}(t) + cE(t),$$

we easily see, using (3.10), that

$$F(t) \sim E(t) \quad (4.20)$$

Consequently we have

$$F'(t) \leq -cE^{2p-1+B}(t), \quad \forall t \geq t_0.$$

A simple integration then gives

$$F(t) \leq c(1+t)^{\frac{-1}{2p-2+B}}, \quad \forall t \geq t_0. \quad (4.21)$$

Hence,

$$E(t) \leq c(1+t)^{-\frac{1}{(2p-1)(q+1)-1}}, \quad \forall t \geq t_0$$

which gives (2.7) by virtue of boundedness and continuity of E . If $(2p-1)(q+1) < 4$ (hence $\frac{1}{2p-2+B} > 1$) then, using (3.21), we obtain

$$\int_0^\infty F(t)dt + \sup_{t \geq 0} tF(t) < +\infty$$

and consequently, as in Case 2, we have

$$g^p \circ \psi_x \geq c(g \circ \psi_x)^p,$$

which gives, by virtue of (3.13) with $\gamma = p$,

$$\mathcal{L}'(t) \leq -c(E(t))^p + c(-E'(t))^{\frac{2}{q+1}}, \quad \forall t \geq t_0. \quad (4.22)$$

Using (3.22) instead of (3.19) and repeating the same calculations, we conclude that

$$E(t) \leq c(1+t)^{-\frac{2}{p(q+1)-2}}, \quad \forall t \geq t_0. \quad (4.23)$$

Estimate (2.6) is obtained by virtue of (3.23) and the boundedness of E . This completes the proof of Theorem 2.2.

Remark 3.1. By taking $a \equiv 1$ and $b \equiv 0$, (1.5) reduces to the system studied in [1]. In this case our result is established under weaker conditions on g . Precisely, we do not require anything on g'' as in (1.6) and (1.7) of [1] and g''' as in (2.4) of [5]. We only need g to be differentiable satisfying (H3) and (H4).

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