ON THE CONTINUOUS RESONANT EQUATION FOR NLS
II. STATISTICAL STUDY

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ABSTRACT. We consider the continuous resonant (CR) system of the 2D cubic nonlinear Schrödinger (NLS) equation. This system arises in numerous instances and regimes as an effective equation for the long-time dynamics of NLS in the presence of a confinement (like on a compact domain or with a trapping potential). The system was derived and studied from a deterministic viewpoint in several earlier works [10, 13, 12], which uncovered many of its striking properties. This manuscript is devoted to a probabilistic study of this system. Most notably, we construct global solutions in negative Sobolev spaces, which leave Gibbs and white noise measures invariant. Invariance of white noise measure seems particularly interesting in view of the absence of similar results for NLS.

1. INTRODUCTION

The purpose of this manuscript is to construct some invariant measures for the so-called continuous resonant (CR) system of the cubic nonlinear Schrödinger equation. This system can be written as

$$(CR) \qquad \begin{cases} i\partial_t u = \mathcal{T}(u, u, u), & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0, x) = f(x), \end{cases}$$

where the operator $\mathcal{T}$ defining the nonlinearity has several equivalent formulations corresponding to different interpretations/origins of this system. In its original formulation [10] as the large-box limit[1] of the resonant cubic NLS[2], $\mathcal{T}$ can be written as follows: For $z \in \mathbb{R}^2$ and $(x_1, x_2) \in \mathbb{R}^2$, denoting by $x^\perp = (-x_2, x_2)$, we have

$$\mathcal{T}(f_1, f_2, f_3)(z) \stackrel{def}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^2} f_1(x + z) f_2(\lambda x^\perp + z) \overline{f_3(x + \lambda x^\perp + z)} \, dx \, d\lambda.$$

[1]Starting with the equation on a torus of size L and letting $L \to \infty$.

[2]This is NLS with only the resonant interactions retained (a.k.a. first Birkhoff normal form). It gives an approximate NLS for sufficiently small initial data.

This integral can be understood as an integral over all rectangles having z as a vertex. It has the following equivalent formulation [12]:

$$\mathcal{T}(f_1, f_2, f_3) = 2\pi \int_{\mathbb{R}} e^{-i\tau\Delta}\Big[(e^{i\tau\Delta}f_1)(e^{i\tau\Delta}f_2)(\overline{e^{i\tau\Delta}f_3})\Big]d\tau.$$

It was shown in [10] that the dynamics of (CR) approximate that of the cubic NLS equation on a torus of size L (with L large enough) over time scales $\sim L^2/\varepsilon^2$ (up to logarithmic loss in L), where ε denotes the size of the initial data.

Another formulation of (CR) comes from the fact that it is also the resonant system for the cubic nonlinear Schrödinger equation with harmonic potential given by:

$$(1.1) \qquad\qquad i\partial_t u - \Delta u + |x|^2 u = \mu|u|^2 u, \qquad \mu = cst \in \mathbb{R}.$$

In this picture, $\mathcal{T}$ can be written as follows: Denoting by $H := -\Delta + |x|^2 = -\partial_{x_1}^2 - \partial_{x_2}^2 + x_1^2 + x_2^2$ the harmonic oscillator on $\mathbb{R}^2$, then

$$\mathcal{T}(f_1, f_2, f_3) = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{i\tau H}\Big[(e^{-i\tau H}f_1)(e^{-i\tau H}f_2)(\overline{e^{-i\tau H}f_3})\Big]d\tau.$$

As a result, the dynamics of (CR) approximate the dynamics of (1.1) over long nonlinear time scales for small enough initial data.

The equation (CR) is Hamiltonian: Indeed, introducing the functional

$$\mathcal{E}(u_1, u_2, u_3, u_4) \stackrel{def}{=} \langle \mathcal{T}(u_1, u_2, u_3), u_4 \rangle_{L^2}$$
$$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} (e^{-itH}u_1)(e^{-itH}u_2)(\overline{e^{-itH}u_3})(\overline{e^{-itH}u_4})dx\,dt,$$

and setting

$$\mathcal{E}(u) := \mathcal{E}(u, u, u, u),$$

then (CR) derives from the Hamiltonian $\mathcal{E}$ given the symplectic form $\omega(f, g) = -4\Im\langle f, g\rangle_{L^2(\mathbb{R}^2)}$ on $L^2(\mathbb{R}^2)$, so that (CR) is equivalent to

$$i\partial_t f = \frac{1}{2}\frac{\partial\mathcal{E}(f)}{\partial\bar{f}}.$$

In addition to the two instances mentioned above in which (CR) appears to describe the long-time dynamics of the cubic NLS equation – with or without potential – we mention the following:

- The equation (CR) appears as a modified scattering limit of the cubic NLS on $\mathbb{R}^3$ with harmonic tapping in two directions. Here, (CR) appears as an asymptotic system and any information on the asymptotic dynamics of (CR) directly gives the corresponding behavior for NLS with partial harmonic trapping. We refer to Hani-Thomann [13] for more details.
- When restricted to the Bargmann-Fock space (see below), the equation (CR) turns out to be the Lowest-Landau-Level equation, which describes fast rotating Bose-Einstein condensates (see [1, 16, 11]).
- The equation (CR) can also be interpreted as describing the effective dynamics of high frequency envelopes for NLS on the unit torus $\mathbb{T}^2$. This means that if the initial data $\varphi(0)$ for

NLS has its Fourier transform given by[3] $\{\widehat{\varphi}(0,k) \sim g_0(\frac{k}{N})\}_{k \in \mathbb{Z}^2}$, and if $g(t)$ evolves according to (CR) with initial data g_0 and $\varphi(t)$ evolves according to NLS with initial data $\varphi(0)$, then $g(t, \frac{k}{N})$ approximates the dynamics of $\widehat{\varphi}(t, k)$ in the limit of large N (see [10, Theorem 2.6]).

1.1. **Some properties and invariant spaces.** We review some of the properties of the (CR) equation that will be useful for us in this paper. For a more detailed study of the equation we refer to [10, 12].

First, (CR) is globally well-posed in $L^2(\mathbb{R}^2)$. Amongst its conserved quantities, we note

$$\int_{\mathbb{R}^2} |u|^2 \, dx, \quad \int_{\mathbb{R}^2} (|x|^2|u|^2 + |\nabla u|^2) \, dx = \int_{\mathbb{R}^2} \overline{u} H u \, dx,$$

(recall that H denotes the harmonic oscillator $H = -\Delta + |x|^2$). Many subspaces of $L^2(\mathbb{R}^2)$ are left invariant by the flow of (CR), in particular:

- The eigenspaces $(E_N)_{N \geq 0}$ of the harmonic oscillator are stable (see [10, 12]). This is a manifestation of the fact that (CR) is the resonant equation associated to (1.1). Recall that H admits a complete basis of eigenvectors for $L^2(\mathbb{R}^2)$; each eigenspace E_N ($N = 0, 1, 2, ...$) has dimension $N + 1$.
- The set of radial functions is stable, as follows from the invariance of H under rotations (see [12]). Global dynamics on $L^2_{rad}(\mathbb{R}^2)$, the radial functions of $L^2(\mathbb{R}^2)$, can be defined. A basis of normalized eigenfunctions of H for $L^2_{rad}(\mathbb{R}^2)$ is given by,

  $$\text{if } n \in \mathbb{N}, \quad \varphi_n^{rad}(x) = \frac{1}{\sqrt{\pi}} L_n^{(0)}(|x|^2) e^{-\frac{|x|^2}{2}} \quad \text{with} \quad L_k^{(0)}(x) = e^x \frac{1}{n!} \left(\frac{d}{dx} \right)^k (e^{-x} x^k).$$

  We record that $H\varphi_n^{rad} = (4n+2)\varphi_k^{rad}$.

- If $\mathcal{O}(\mathbb{C})$ stands for the set of entire functions on $\mathbb{C}$ (with the identification $z = x_1 + ix_2$), the Bargmann-Fock space $L^2_{hol}(\mathbb{R}^2) = L^2(\mathbb{R}^2) \cap (\mathcal{O}(\mathbb{C})e^{-|z|^2/2})$ is invariant by the flow of (CR). Global dynamics on $L^2_{hol}(\mathbb{R}^2)$ can be defined. A basis of normalized eigenfunctions of H for $L^2_{hol}(\mathbb{R}^2)$ is given by the "holomorphic" Hermite functions, also known as the "special Hermite functions", namely

  $$\text{if } n \in \mathbb{N}, \quad \varphi_n^{hol}(x) = \frac{1}{\sqrt{\pi n!}} (x_1 + ix_2)^n e^{-|x|^2/2}.$$

  Notice that $H\varphi_n = 2(n+1)\varphi_n$. It is proved in [12] that

  (1.2) $$\mathcal{T}(\varphi_{n_1}, \varphi_{n_2}, \varphi_{n_3}) = \alpha_{n_1, n_2, n_3, n_4} \varphi_{n_4}, \qquad n_4 = n_1 + n_2 - n_3,$$

  with

  $$\alpha_{n_1, n_2, n_3, n_4} = \mathcal{H}(\varphi_{n_1}, \varphi_{n_2}, \varphi_{n_3}, \varphi_{n_4}) = \frac{\pi}{8} \frac{(n_1 + n_2)!}{2^{n_1 + n_2} \sqrt{n_1! n_2! n_3! n_4!}} \mathbf{1}_{n_1 + n_2 = n_3 + n_4}.$$

  As a result, the (CR) system reduces to the following infinite system of ODE when restricted to $\text{Span}\{\varphi_n\}_{n \in \mathbb{N}}$:

  $$i\partial_t c_n(t) = \sum_{\substack{n_1, n_2, n_3 \in \mathbb{N} \\ n_1 + n_2 - n_3 = n}} \alpha_{n_1, n_2, n_3, n} c_{n_1}(t) c_{n_2}(t) \overline{c_{n_3}}(t).$$

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[3]Up to a normalizing factor in H^s, $s > 1$.

1.2. **Statistical solutions.** In this paper we construct global probabilistic solutions on each of these spaces which leave invariant either Gibbs or white noise measures. More precisely, our main results can be summarized as follows:

- We construct global strong flows on
$$X^0_{rad}(\mathbb{R}^2) = \cap_{\sigma>0} \mathcal{H}^{-\sigma}_{rad}(\mathbb{R}^2),$$
  and on
$$X^0_{hol}(\mathbb{R}^2) := \big(\cap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R}^2) \big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}),$$
  which leave Gibbs measures invariant (see Theorem 2.5).

- We construct global weak probabilistic solutions on
$$X^{-1}_{hol}(\mathbb{R}^2) := \big(\cap_{\sigma>1} \mathcal{H}^{-\sigma}(\mathbb{R}^2) \big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}),$$
  and this dynamics leaves invariant the white noise measure (see Theorem 2.6).

Since the 90's, there have been many works on the construction of Gibbs measures for dispersive equations, and more recently, much attention has been paid to the well-posedness of these equations with random initial conditions. We refer to the introduction of [20] for references on the subject. In particular, concerning the construction of strong solutions for the nonlinear harmonic oscillator (which is related to (CR)), we refer to [26, 3, 8, 17, 18, 20].

Construction of flows invariant by white noise measure is much trickier due to the low regularity of the support of such measures, and there seems to be no results of this sort for NLS equations. We construct weak solutions on the support of the white noise measure on $X^{-1}_{hol}(\mathbb{R}^2)$ using a method based on a compactness argument in the space of measures (the Prokhorov theorem) combined with a representation theorem of random variables (the Skorohod theorem). This approach has been first applied to the Navier-Stokes and Euler equations in Albeverio-Cruzeiro [2] and Da Prato-Debussche [7] and extended to dispersive equations by Burq-Thomann-Tzvetkov [4]. We refer to [4] for a self-contained presentation of the method.

1.3. **Notations.** Define the harmonic Sobolev spaces for $s \in \mathbb{R}$, $p \geq 1$ by
$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}^2) = \big\{ u \in L^p(\mathbb{R}^2),\ H^{s/2}u \in L^p(\mathbb{R}^2) \big\}, \qquad \mathcal{H}^s = \mathcal{W}^{s,2}.$$
They are endowed with the natural norms $\|u\|_{\mathcal{W}^{s,p}}$. Up to equivalence of norms we have for $s \geq 0$, $1 < p < +\infty$ (see [28, Lemma 2.4])

(1.3) $$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x \rangle^s u\|_{L^p}.$$

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{p})$. In all the paper, $\{g_n,\ n \geq 0\}$ and $\{g_{n,k},\ n \geq 0,\ 0 \leq k \leq n\}$ are independent standard complex Gaussians $\mathcal{N}_{\mathbb{C}}(0,1)$ (their probability density function reads thus $\frac{1}{\pi}e^{-|z|^2}dz$, dz being Lebesgue measure on $\mathbb{C}$). If X is a random variable, we denote by $\mathscr{L}(X)$ its law (or distribution).

We will sometimes use the notations $L^p_T = L^p(-T, T)$ for $T > 0$. If E is a Banach space and μ is a measure on E, we write $L^p_\mu = L^p(\mathrm{d}\mu)$ and $\|u\|_{L^p_\mu E} = \big\| \|u\|_E \big\|_{L^p_\mu}$. We define $X^\sigma(\mathbb{R}^2) = \bigcap_{\tau<\sigma} \mathcal{H}^\tau(\mathbb{R}^2)$, and if $I \subset \mathbb{R}$ is an interval, with an abuse of notation, we write $\mathcal{C}\big(I; X^\sigma(\mathbb{R}^2)\big) = \bigcap_{\tau<\sigma} \mathcal{C}\big(I; \mathcal{H}^\tau(\mathbb{R}^2)\big)$.

Finally, $\mathbb{N}$ denotes the set of natural integers including 0; $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded

with respect to the other parameters. For two quantities A and B, we denote $A \lesssim B$ if $A \leq CB$, and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

2. STATEMENT OF THE RESULTS

As mentioned above, we will construct strong solutions on the support of Gibbs measures and prove the invariance of such measures. For white noise measures, solutions are weak and belong to the space $C_T X^{-1}$. We start by discussing the former case.

2.1. Global strong solutions invariant by Gibbs measure.

2.1.1. **Measures and dynamics on the space E_N.** The operator H is self-adjoint on $L^2(\mathbb{R}^2)$, and has the discrete spectrum $\{2N + 2, N \in \mathbb{N}\}$. For $N \geq 0$, denote by E_N the eigenspace associated to the eigenvalue $2N + 2$. This space has dimension $N + 1$. Consider $(\varphi_{N,k})_{0 \leq k \leq N}$ any orthonormal basis of E_N. Define $\gamma_N \in L^2(\Omega; E_N)$ by

$$\gamma_N(\omega, x) = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} g_{N,k}(\omega)\varphi_{N,k}(x).$$

The distribution of the random variable γ_N does not depend on the choice of the basis, and observe that the law of large numbers gives

$$\|\gamma_N\|_{L^2(\mathbb{R}^2)}^2 = \frac{1}{N+1} \sum_{k=0}^{N} |g_{N,k}(\omega)|^2 \longrightarrow 1 \quad \text{a.s. when} \quad N \longrightarrow +\infty.$$

Then we define the probability measure $\mu_N = \gamma_\# \mathbf{p} := \mathbf{p} \circ \gamma_N^{-1}$ on E_N.

The L^p properties of the measures μ_N have been studied in [19] with an improvement in [21]. We mention in particular the following result

Theorem 2.1 ([19, 21]). *There exist $c, C_1, C_2 > 0$ such that for all $N \geq N_0$*

$$\mu_N \left[u \in E_N : C_1 N^{-1/2}(\log N)^{1/2}\|u\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^\infty(\mathbb{R}^2)} \leq C_2 N^{-1/2}(\log N)^{1/2}\|u\|_{L^2(\mathbb{R}^2)} \right] \geq 1 - N^{-c}.$$

This proposition is a direct application of [21, Theorem 3.8] with $h = N^{-1}$ and $d = 2$. Notice that for all $u \in E_N$, we have $\|u\|_{\mathcal{H}^s} = (2N + 2)^{s/2}\|u\|_{L^2}$. The best (deterministic) L^∞-bound for an eigenfunction $u \in E_N$ is given by Koch-Tataru [15]

$$(2.1) \qquad\qquad\qquad\qquad \|u\|_{L^\infty(\mathbb{R}^2)} \leq C\|u\|_{L^2(\mathbb{R}^2)},$$

and the previous estimate is optimal, since it is saturated by the radial Hermite functions. Therefore the result of Theorem 2.1 shows that there is almost a gain of one derivative compared to the deterministic estimate (2.1).

It turns out that the measures μ_N are invariant under the flow of (CR), and we have the following result.

Theorem 2.2. *For all $N \geq 1$, the measure μ_N is invariant under the flow Φ of (CR) restricted to E_N. Therefore, by the Poincaré theorem, μ_N-almost all $u \in E_N$ is recurrent in the following sense: for μ_N-almost all $u_0 \in E_N$ there exists a sequence of times $t_n \longrightarrow +\infty$ so that*

$$\lim_{n \to +\infty} \left\| \Phi(t_n)u_0 - u_0 \right\|_{L^2(\mathbb{R}^2)} = 0.$$

In the previous result, one only uses the invariance of the probability measure μ_N by the flow, and no additional property of the equation (CR).

2.1.2. **Gibbs Measure on the space $X^0(\mathbb{R}^2)$ and a well-posedness result.** In the sequel we either consider the family $(\varphi_n^{rad})_{n \geq 0}$ of the radial Hermite functions, which are eigenfunctions of H associated to the eigenvalue $\lambda_n^{rad} = 4n + 2$; or the family $(\varphi_n^{hol})_{n \geq 0}$ of the holomorphic Hermite functions, which are eigenvalues of H associated to the eigenvalue $\lambda_n^{hol} = 2n + 2$. Set

$$X_{rad}^0(\mathbb{R}^2) = \cap_{\sigma>0} \mathcal{H}_{rad}^{-\sigma}(\mathbb{R}^2),$$

$$X_{hol}^0(\mathbb{R}^2) := \left(\cap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R}^2) \right) \cap (\mathcal{O}(\mathbb{C}) \mathrm{e}^{-|z|^2/2}).$$

In the following, we denote $X_\star^0(\mathbb{R}^2)$ for $X_{rad}^0(\mathbb{R}^2)$ or $X_{hol}^0(\mathbb{R}^2)$, $\varphi_n^\star$ for φ_n^{rad} or φ_n^{hol}, etc...

Define now $\gamma_\star \in L^2(\Omega; X_\star^0(\mathbb{R}^2))$ by

$$\gamma_\star(\omega, x) = \sum_{n=0}^{+\infty} \frac{g_n(\omega)}{\sqrt{\lambda_n^\star}} \varphi_n^\star(x),$$

and consider the Gaussian probability measure $\mu_\star = (\gamma_\star)_\# \mathbf{p} := \mathbf{p} \circ \gamma_\star^{-1}$.

Lemma 2.3. *In each of the previous cases, the measure $\mu_\star$ is a probability measure on $X_\star^0(\mathbb{R}^2)$.*

Notice that since (CR) conserves the $\mathcal{H}^1$ norm, $\mu_\star$ is formally invariant by its flow. More generally, we can define a family $(\rho_{\star,\beta})_{\beta \geq 0}$ of probability measures on $X_\star^0(\mathbb{R}^2)$ which are formally invariant by (CR) in the following way: define for $\beta \geq 0$ the measure $\rho_\star = \rho_{\star,\beta}$ by

$$(2.2) \qquad\qquad d\rho_\star(u) = C_\beta \mathrm{e}^{-\beta \mathcal{E}(u)} d\mu_\star(u),$$

where $C_\beta > 0$ is a normalising constant. In Lemma 3.2, we will show that $\mathcal{E}(u) < +\infty$, $\mu_\star$ a.s., which enables us to define this probability measure.

For all $\beta \geq 0$, $\rho_\star(X_\star^0(\mathbb{R}^2)) = 1$ and $\rho_\star(L_\star^2(\mathbb{R}^2)) = 0$.

Remark 2.4. *Observe that we could also give a sense to a generalised version of (2.2) when $\beta < 0$ using the renormalizing method of Lebowitz-Rose-Speer. We do not give the details and refer to [3] for such a construction.*

We are now able to state the following global existence result.

Theorem 2.5. *Let $\beta \geq 0$. There exists a set $\Sigma \subset X_\star^0(\mathbb{R}^2)$ of full $\rho_\star$ measure so that for every $f \in \Sigma$ the equation (CR) with initial condition $u(0) = f$ has a unique global solution $u(t) = \Phi(t)f$ such that for any $0 < s < 1/2$*

$$u(t) - f \in \mathcal{C}(\mathbb{R}; \mathcal{H}^s(\mathbb{R}^2)).$$

Moreover, for all $\sigma > 0$ and $t \in \mathbb{R}$

$$\|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)} \leq C \left(\Lambda(f, \sigma) + \ln^{\frac{1}{2}} \left(1 + |t| \right) \right),$$

and the constant $\Lambda(f,\sigma)$ satisfies the bound $\mu_\star\big(f:\Lambda(f,\sigma)>\lambda\big)\leq Ce^{-c\lambda^2}$.
Furthermore, the measure $\rho_\star$ is invariant by Φ: For any $\rho_\star$ measurable set $A\subset\Sigma$, for any $t\in\mathbb{R}$, $\rho_\star(A)=\rho_\star(\Phi(t)(A))$.

2.2. White noise measure on the space $X_{hol}^{-1}(\mathbb{R}^2)$ and weak solutions.

Our aim is now to construct weak solutions on the support of the white noise measure. Consider the Gaussian random variable

$$(2.3)\qquad \gamma(\omega,x)=\sum_{n=0}^{+\infty}g_n(\omega)\varphi_n^{hol}(x)=\frac{1}{\sqrt{\pi}}\Big(\sum_{n=0}^{+\infty}\frac{(x_1+ix_2)^n g_n(\omega)}{\sqrt{n!}}\Big)e^{-|x|^2/2},$$

and the measure $\mu=\mathbf{p}\circ\gamma^{-1}$. As in Lemma 2.3 we can show that the measure μ is a probability measure on

$$X_{hol}^{-1}(\mathbb{R}^2):=\big(\cap_{\sigma>1}\mathcal{H}^{-\sigma}(\mathbb{R}^2)\big)\cap(\mathcal{O}(\mathbb{C})e^{-|z|^2/2}).$$

Since $\|u\|_{L^2(\mathbb{R}^2)}$ is preserved by (CR), μ is formally invariant under (CR). We are not able to define a flow at this level of regularity, however using compactness arguments combined with probabilistic methods, we will construct weak solutions.

Theorem 2.6. *There exists a set $\Sigma\subset X_{hol}^{-1}(\mathbb{R}^2)$ of full μ measure so that for every $f\in\Sigma$ the equation (CR) with initial condition $u(0)=f$ has a solution*

$$u\in\bigcap_{\sigma>1}\mathcal{C}\big(\mathbb{R};\mathcal{H}^{-\sigma}(\mathbb{R}^2)\big).$$

The distribution of the random variable $u(t)$ is equal to μ (and thus independent of $t\in\mathbb{R}$):

$$\mathscr{L}_{X^{-1}(\mathbb{R}^2)}\big(u(t)\big)=\mathscr{L}_{X^{-1}(\mathbb{R}^2)}\big(u(0)\big)=\mu,\quad\forall t\in\mathbb{R}.$$

Remark 2.7. *One can also define the Gaussian measure $\mu=\mathbf{p}\circ\gamma^{-1}$ on $X^{-1}(\mathbb{R}^2)=\cap_{\sigma>1}\mathcal{H}^{-\sigma}(\mathbb{R}^2)$ by*

$$\gamma(\omega,x)=\sum_{n=0}^{+\infty}\frac{1}{\sqrt{\lambda_n}}\sum_{k=-n}^{n}g_{n,k}(\omega)\varphi_{n,k}(x),\quad\lambda_n=2n+2$$

(where the $\varphi_{n,k}$ are an orthonormal basis of eigenfunctions of the harmonic oscillator and the angular momentum operator). Since $\|u\|_{\mathcal{H}^1(\mathbb{R}^2)}$ is preserved by (CR), μ is formally invariant under (CR), but we are not able to obtain an analogous result in this case.

The same comment holds for the white noise measure $\mu=\mathbf{p}\circ\gamma^{-1}$ on $X_{rad}^{-1}(\mathbb{R}^2)=\cap_{\sigma>1}\mathcal{H}_{rad}^{-\sigma}(\mathbb{R}^2)$ with

$$\gamma(\omega,x)=\sum_{n=0}^{+\infty}g_n(\omega)\varphi_n^{rad}(x),$$

which is also formally invariant under (CR).

2.3. Plan of the paper.

The rest of the paper is organised as follows. In Section 3 we prove the results concerning the strong solutions, and in Section 4 we construct the weak solutions.

3. Strong solutions

3.1. **Proof of Theorem 2.2.** The proof of Theorem 2.2 is an application of the Liouville theorem. Indeed, write $u_N = \sum_{k=0}^{N} c_{N,k}\varphi_{N,k} \in E_N$, then

$$d\mu_N = \frac{(N+1)^{N+1}}{\pi^{N+1}} \exp\left(-(N+1)\sum_{k=0}^{N}|c_{N,k}|^2\right) \prod_{k=0}^{N} da_{N,k}\, db_{N,k},$$

where $c_{N,k} = a_{N,k} + ib_{N,k}$.

The Lebesgue measure $\prod_{k=0}^{N} da_{N,k}db_{N,k}$ is preserved since (CR) is Hamiltonian and $\sum_{k=0}^{N}|c_{N,k}|^2 = \|u_N\|_{L^2}^2$ is a constant of motion.

3.2. **Proof of Theorem 2.5.** We start with the proof of Lemma 2.3.

Proof of Lemma 2.3. We only consider the case $X_\star^0(\mathbb{R}^2) = X_{hol}^0(\mathbb{R}^2)$. It is enough to show that $\gamma_{hol} \in X_{hol}^0(\mathbb{R}^2)$, $\mathbf{p}$-a.s. First, for all $\sigma > 0$ we have

$$(3.1) \qquad \int_\Omega \|\gamma_{hol}\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}^2 d\mathbf{p}(\omega) = \int_\Omega \sum_{n=0}^{+\infty} \frac{|g_n|^2}{(\lambda_n^{hol})^{\sigma+1}} d\mathbf{p}(\omega) = C \sum_{n=0}^{+\infty} \frac{1}{(n+1)^{\sigma+1}} < +\infty,$$

therefore $\gamma_{hol} \in \bigcap_{\sigma>0} L^2\big(\Omega\,;\mathcal{H}^{-\sigma}(\mathbb{R}^2)\big)$. Next, by [6, Lemma 3.4], for all $A \geq 1$ there exists a set $\Omega_A \subset \Omega$ such that $\mathbf{p}(\Omega_A^c) \leq \exp(-A^\delta)$ and for all $\omega \in \Omega_A$, $\varepsilon > 0$, $n \geq 0$

$$|g_n(\omega)| \leq CA(n+1)^\varepsilon.$$

Then for $\omega \in \bigcup_{A\geq 1}\Omega_A$, $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{\lambda_n^{hol}\, n!}} \in \mathcal{O}(\mathbb{C})$. $\qquad\qquad\square$

We first define a smooth version of the usual spectral projector. Let $\chi \in \mathcal{C}_0^\infty(-1,1)$, so that $0 \leq \chi \leq 1$, with $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. We define the operators $S_N = \chi\big(\frac{H}{\lambda_N}\big)$ as

$$S_N\Big(\sum_{n=0}^{\infty} c_n\varphi_n^\star\Big) = \sum_{n=0}^{\infty} \chi\big(\frac{\lambda_n^\star}{\lambda_N^\star}\big) c_n\varphi_n^\star.$$

Then for all $1 < p < +\infty$, the operator S_N is bounded in $L^p(\mathbb{R}^2)$ (see [8, Proposition 2.1] for a proof).

3.2.1. **Local existence.** It will be useful to work with an approximation of (CR). We consider the dynamical system given by the Hamiltonian $\mathcal{H}_N(u) := \mathcal{H}(S_N u)$. This system reads

$$(3.2) \qquad \begin{cases} i\partial_t u_N = \mathcal{T}_N(u_N), & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u_N(0,x) = f, \end{cases}$$

and $\mathcal{T}_N(u_N) := S_N\mathcal{T}(S_N u, S_N u, S_N u)$. Observe that (3.2) is a finite dimensional dynamical system on $\bigoplus_{k=0}^{N} E_k$ and that the projection of $u_N(t)$ on its complement is constant. For $\beta \geq 0$ and $N \geq 0$ we define the measures $\rho_\star^N$ by

$$d\rho_\star^N(u) = C_\beta^N e^{-\beta\mathcal{H}_N(u)} d\mu_\star(u),$$

where $C_\beta^N > 0$ is a normalising constant. We have the following result

Lemma 3.1. *The system* (3.2) *is globally well-posed in* $L^2(\mathbb{R}^2)$. *Moreover, the measures* $\rho_\star^N$ *are invariant by its flow denoted by* Φ_N.

Proof. The global existence follows from the conservation of $\|u_N\|_{L^2(\mathbb{R}^2)}$. The invariance of the measures is a consequence of the Liouville theorem and the conservation of $\sum_{k=0}^\infty \lambda_k |c_k|^2$ by the flow of (CR) (see [10]). We refer to [3, Lemma 8.1 and Proposition 8.2] for the details. $\square$

We now state a result concerning dispersive bounds of Hermite functions

Lemma 3.2. *For all* $2 \le p \le +\infty$,

$$(3.3) \qquad \|\varphi_n^{hol}\|_{L^p(\mathbb{R}^d)} \le C n^{\frac{1}{2p}-\frac{1}{4}},$$

$$(3.4) \qquad \|\varphi_n^{rad}\|_{L^4(\mathbb{R}^d)} \le C n^{-\frac{1}{4}} (\ln n)^{\frac{1}{4}}.$$

Proof. By Stirling, we easily get that $\|\varphi_n^{hol}\|_{L^\infty(\mathbb{R}^d)} \le C n^{-\frac{1}{4}}$, and the general case is obtained by interpolation. For the the proof of (3.4), we refer to [14, Proposition 2.4]. $\square$

Lemma 3.3. *(i) We have*

(3.5) $\exists C > 0, \exists c > 0, \forall \lambda \ge 1, \forall N \ge 1,$

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH}S_N u\|_{L^4([-\frac{\pi}{4},\frac{\pi}{4}]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}.$$

(ii) There exists $\beta > 0$ *such that*

(3.6) $\exists C > 0, \exists c > 0, \forall \lambda \ge 1, \forall N \ge N_0 \ge 1,$

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH}(S_N - S_{N_0})u\|_{L^4([-\frac{\pi}{4},\frac{\pi}{4}]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-cN_0^\beta \lambda^2}.$$

(iii) In the holomorphic case: for all $2 \le p < +\infty$ *and* $s < \frac{1}{2} - \frac{1}{p}$

$$\exists C > 0, \exists c > 0, \forall \lambda \ge 1, \forall N \ge 1,$$

$$(3.7) \qquad \mu_{hol}\big(u \in X_{hol}^0(\mathbb{R}^2) : \|e^{-itH}u\|_{L^p_{[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,p}(\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2},$$

$$\mu_{hol}\big(u \in X_{hol}^0(\mathbb{R}^2) : \|e^{-itH}u\|_{L^{8/3}([-\frac{\pi}{4},\frac{\pi}{4}]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}.$$

(iv) In the radial case: for all $s < 1/2$

(3.8) $\exists C > 0, \exists c > 0, \forall \lambda \ge 1, \forall N \ge 1,$

$$\mu_{rad}\big(u \in X_{rad}^0(\mathbb{R}^2) : \|e^{-itH}u\|_{L^4_{[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,4}(\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}.$$

Proof. We have that

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH}S_N u\|_{L^4([-\frac{\pi}{4},\frac{\pi}{4}]\times\mathbb{R}^2)} > \lambda\big)$$

$$= \mathbf{p}\Big(\Big\|\sum_{n=0}^\infty e^{-it\lambda_n} \chi\big(\frac{\lambda_n}{\lambda_N}\big) \frac{g_n(\omega)}{\sqrt{\lambda_n}} \varphi_n^\star(x)\Big\|_{L^4([-\frac{\pi}{4},\frac{\pi}{4}]\times\mathbb{R}^2)} > \lambda\Big).$$

Set

$$F(\omega, t, x) \equiv \sum_{n=0}^{\infty} e^{-it\lambda_n^\star} \chi\left(\frac{\lambda_n^\star}{\lambda_N^\star}\right) \frac{g_n(\omega)}{\sqrt{\lambda_n^\star}} \varphi_n^\star(x) \,.$$

Let $q \geq p \geq 2$ and $s \geq 0$. Recall here the Khintchine inequality (see e.g. [5, Lemma 3.1] for a proof): there exists $C > 0$ such that for all real $k \geq 2$ and $(a_n) \in \ell^2(\mathbb{N})$

$$(3.9) \qquad \left\| \sum_{n \geq 0} g_n(\omega) \, a_n \right\|_{L_\mathbf{p}^k} \leq C\sqrt{k} \left(\sum_{n \geq 0} |a_n|^2 \right)^{\frac{1}{2}},$$

if the g_n are iid normalized Gaussians. Applying it to (3.9) we get

$$\|H^{s/2} F(\omega, t, x)\|_{L_\omega^q} \leq C\sqrt{q} \left(\sum_{n=0}^{\infty} \chi^2\left(\frac{\lambda_n^\star}{\lambda_N^\star}\right) \frac{|\varphi_n^\star(x)|^2}{\lambda_n^{\star\,1-s}} \right)^{1/2} \leq C\sqrt{q} \left(\sum_{n=0}^{\infty} \frac{|\varphi_n^\star(x)|^2}{\langle n \rangle^{1-s}} \right)^{1/2},$$

and using twice the Minkowski inequality for $q \geq p$ gives

$$(3.10) \qquad \|H^{s/2} F(\omega, t, x)\|_{L_\omega^q L_{t,x}^p} \leq \|H^{s/2} F(\omega, t, x)\|_{L_{t,x}^p L_\omega^q} \leq C\sqrt{q} \left(\sum_{n=0}^{\infty} \frac{\|\varphi_n^\star(x)\|_{L^p(\mathbb{R}^2)}^2}{\langle n \rangle^{1-s}} \right)^{1/2}.$$

We are now ready to prove (3.5). Set $p = 4$ and $s = 0$. Since by Lemma 3.2 we have $\|\varphi_n^\star\|_{L^4(\mathbb{R}^2)} \leq Cn^{-1/8}$, we get from (3.10)

$$\|F(\omega, t, x)\|_{L_\omega^q L_{t,x}^4} \leq C\sqrt{q} \,.$$

The Bienaymé-Tchebichev inequality gives then

$$\mathbf{p}\left(\|F(\omega, t, x)\|_{L_{t,x}^4} > \lambda \right) \leq (\lambda^{-1} \|F(\omega, t, x)\|_{L_\omega^q L_{t,x}^4})^q \leq (C\lambda^{-1}\sqrt{q})^q \,.$$

Thus by choosing $q = \delta\lambda^2 \geq 4$, for δ small enough, we get the bound

$$\mathbf{p}\left(\|F(\omega, t, x)\|_{L_{t,x}^4} > \lambda \right) \leq Ce^{-c\lambda^2} \,,$$

which is (3.5).

For the proof of (3.6), we analyze the function

$$G(\omega, t, x) \equiv \sum_{n=0}^{\infty} e^{-it\lambda_n^\star} \left(\chi\left(\frac{\lambda_n^\star}{\lambda_N^\star}\right) - \chi\left(\frac{\lambda_n^\star}{\lambda_{N_0}^\star}\right) \right) \frac{g_n(\omega)}{\sqrt{\lambda_n^\star}} \varphi_n^\star(x),$$

and we use that a negative power of N_0 can be gained in the estimate. Namely, there is $\gamma > 0$ such that

$$\|F_{N_0}(\omega, t, x)\|_{L_\omega^q L_{t,x}^4} \leq C\sqrt{q} N_0^{-\gamma} \,,$$

which implies (3.6).

To prove (3.7)-(3.8), we come back to (3.10) and argue similarly. This completes the proof of Lemma 3.3. $\qquad\square$

Lemma 3.4. *Let $\beta \geq 0$. Let $p \in [1, \infty[$, then when $N \longrightarrow +\infty$.*

$$C_\beta^N e^{-\beta\mathcal{H}_N(u)} \longrightarrow C_\beta e^{-\beta\mathcal{H}(u)} \quad in \quad L^p(d\mu_\star(u)).$$

In particular, for all measurable sets $A \subset X_\star^0(\mathbb{R}^2)$,

$$\rho_\star^N(A) \longrightarrow \rho_\star(A).$$

Proof. Denote by $G_\beta^N(u) = e^{-\beta \mathcal{H}_N(u)}$ and $G_\beta(u) = e^{-\beta \mathcal{H}(u)}$. By (3.6), we deduce that $\mathcal{H}_N(u) \longrightarrow \mathcal{H}(u)$ in measure, w.r.t. $\mu_\star$. In other words, for $\varepsilon > 0$ and $N \geq 1$ we denote by

$$A_{N,\varepsilon} = \left\{ u \in X_\star^0(\mathbb{R}^2) \ : \ |G_\beta^N(u) - G_\beta(u)| \leq \varepsilon \right\},$$

then $\mu_\star(A_{N,\varepsilon}^c) \longrightarrow 0$, when $N \longrightarrow +\infty$. Since $0 \leq G, G_N \leq 1$,

$$
\begin{aligned}
\|G_\beta - G_\beta^N\|_{L_{\mu_\star}^p} &\leq \|(G_\beta - G_\beta^N)\mathbf{1}_{A_{N,\varepsilon}}\|_{L_{\mu_\star}^p} + \|(G_\beta - G_\beta^N)\mathbf{1}_{A_{N,\varepsilon}^c}\|_{L_{\mu_\star}^p} \\
&\leq \varepsilon\big(\mu_\star(A_{N,\varepsilon})\big)^{1/p} + 2\big(\mu_\star(A_{N,\varepsilon}^c)\big)^{1/p} \leq C\varepsilon,
\end{aligned}
$$

for N large enough. Finally, we have when $N \longrightarrow +\infty$

$$C_\beta^N = \Big(\int e^{-\beta \mathcal{H}_N(u)} d\mu_\star(u)\Big)^{-1} \longrightarrow \Big(\int e^{-\beta \mathcal{H}(u)} d\mu_\star(u)\Big)^{-1} = C_\beta,$$

and this ends the proof. $\qquad\square$

We look for a solution to (CR) of the form $u = f + v$, thus v has to satisfy

(3.11)
$$\begin{cases} i\partial_t v = \mathcal{T}(f + v), & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ v(0,x) = 0, \end{cases}$$

with $\mathcal{T}(u) = \mathcal{T}(u,u,u)$. Similarly, we introduce

(3.12)
$$\begin{cases} i\partial_t v_N = \mathcal{T}_N(f + v_N), & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ v(0,x) = 0. \end{cases}$$

Recall that $X_\star^0(\mathbb{R}^2) = X_{hol}^0(\mathbb{R}^2)$ or $X_{rad}^0(\mathbb{R}^2)$. Define the sets, for $s < \frac{1}{2}$,

$$A_{rad}^s(D) = \left\{ f \in X_{rad}^0(\mathbb{R}^2) \ : \ \|e^{-itH} f\|_{L_{[-\frac{\pi}{4}, \frac{\pi}{4}]}^4 \mathcal{W}^{s,4}(\mathbb{R}^2)} \leq D \right\},$$

and choosing $p(s) = \frac{4}{1-2s}$ so that $s < \frac{1}{2} - \frac{1}{p}$,

$$A_{hol}^s(D) = \left\{ f \in X_{hol}^0(\mathbb{R}^2) \ : \ \|e^{-itH} f\|_{L_{[-\frac{\pi}{4}, \frac{\pi}{4}]}^{8/3} L^{8/3}(\mathbb{R}^2)} + \|e^{-itH} f\|_{L_{[-\frac{\pi}{4}, \frac{\pi}{4}]}^{p(s)} \mathcal{W}^{s,p(s)}(\mathbb{R}^2)} \leq D \right\}.$$

In the sequel we write $A_\star(D) = A_{hol}(D)$ or $A_{rad}(D)$. Then we have the following result

Lemma 3.5. *Let $\beta \geq 0$. There exist $c, C > 0$ so that for all $N \geq 0$*

$$\rho_\star^N\big(A_\star(D)^c\big) \leq Ce^{-cD^2}, \quad \rho_\star\big(A_\star(D)^c\big) \leq Ce^{-cD^2}, \quad \mu_\star\big(A_\star(D)^c\big) \leq Ce^{-cD^2}.$$

Proof. Since $\beta \geq 0$, we have $\rho_\star^N\big(A_\star(D)^c\big), \rho_\star\big(A_\star(D)^c\big) \leq C\mu_\star\big(A_\star(D)^c\big)$. The result is therefore given by (3.7) and (3.8). $\qquad\square$

Proposition 3.6. *Let $s < 1/2$. There exists $c > 0$ such that, for any $D \geq 0$, setting $\tau(D) = cD^{-2}$, for any $f \in A(D)$ there exists a unique solution $v \in L^\infty([-\tau, \tau]; L^2(\mathbb{R}^2))$ to the equation (3.11) and a unique solution $v_N \in L^\infty([-\tau, \tau]; L^2(\mathbb{R}^2))$ to the equation (3.12) which furthermore satisfy*

$$\|v\|_{L^\infty([-\tau,\tau]; \mathcal{H}^s(\mathbb{R}^2))}, \quad \|v_N\|_{L^\infty([-\tau,\tau]; \mathcal{H}^s(\mathbb{R}^2))} \leq D.$$

The key ingredient in the proof of this result, is the following trilinear estimate

Lemma 3.7. *Assume that for $1 \leq j \leq 3$ and $1 \leq k \leq 4$, $(p_{jk}, q_{jk}) \in [2, +\infty[^2$ are Strichartz admissible pairs, or in other words satisfy*

$$\frac{1}{q_{jk}} + \frac{1}{p_{jk}} = \frac{1}{2},$$

and are such that for $1 \leq j \leq 4$,

$$\frac{1}{p_{j1}} + \frac{1}{p_{j2}} + \frac{1}{p_{j3}} + \frac{1}{p_{j4}} = \frac{1}{q_{j1}} + \frac{1}{q_{j2}} + \frac{1}{q_{j3}} + \frac{1}{q_{j4}} = 1.$$

Then for all $s \geq 0$, there exists $C > 0$ such that

$$\begin{aligned}
\|\mathcal{T}(u_1, u_2, u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} \leq & C \|e^{-itH}u_1\|_{L^{p_{11}}\mathcal{W}^{s,q_{11}}} \|e^{-itH}u_2\|_{L^{p_{12}}L^{q_{12}}} \|e^{-itH}u_3\|_{L^{p_{13}}L^{q_{13}}} \\
& + C \|e^{-itH}u_1\|_{L^{p_{21}}L^{q_{21}}} \|e^{-itH}u_2\|_{L^{p_{22}}\mathcal{W}^{s,q_{22}}} \|e^{-itH}u_3\|_{L^{p_{23}}L^{q_{23}}} \\
& + C \|e^{-itH}u_1\|_{L^{p_{31}}L^{q_{31}}} \|e^{-itH}u_2\|_{L^{p_{32}}L^{q_{32}}} \|e^{-itH}u_3\|_{L^{p_{33}}\mathcal{W}^{s,q_{33}}},
\end{aligned}$$

with the notation $L^p\mathcal{W}^{s,q} = L^p\big([-\frac{\pi}{4}, \frac{\pi}{4}]; \mathcal{W}^{s,q}(\mathbb{R}^2)\big)$.

Proof. By duality

$$\begin{aligned}
\|\mathcal{T}(u_1, u_2, u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} & = \sup_{\|u\|_{L^2(\mathbb{R}^2)}=1} \langle H^{s/2}\mathcal{T}(u_1, u_2, u_3), u \rangle_{L^2(\mathbb{R}^2)} \\
& = 2\pi \sup_{\|u\|_{L^2(\mathbb{R}^2)}=1} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} H^{s/2}\Big((e^{-itH}u_1)(e^{-itH}u_2)(\overline{e^{-itH}u_3})\Big)(\overline{e^{-itH}u})\, dx\, dt.
\end{aligned}$$

Then by Strichartz for all u of unit norm in L^2 and for any admissible pair (p_4, q_4)

$$\begin{aligned}
\|\mathcal{T}(u_1, u_2, u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} & \leq C \|(e^{-itH}u_1)(e^{-itH}u_2)(\overline{e^{-itH}u_3})\|_{L^{p'_4}\mathcal{W}^{s,q'_4}} \|e^{-itH}u\|_{L^{p_4}L^{q_4}} \\
& \leq C \|(e^{-itH}u_1)(e^{-itH}u_2)(\overline{e^{-itH}u_3})\|_{L^{p'_4}\mathcal{W}^{s,q'_4}}.
\end{aligned}$$

We then conclude using (1.3) and applying twice the following lemma. $\qquad\qquad\square$

We have the following product rule

Lemma 3.8. *Let $s \geq 0$, then the following estimates hold*

$$\|u\,v\|_{\mathcal{W}^{s,q}} \leq C\|u\|_{L^{q_1}}\|v\|_{\mathcal{W}^{s,q'_1}} + C\|v\|_{L^{q_2}}\|u\|_{\mathcal{W}^{s,q'_2}},$$

with $1 < q < \infty$, $1 < q_1, q_2 < \infty$ and $1 \leq q'_1, q'_2 < \infty$ so that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q'_1} = \frac{1}{q_2} + \frac{1}{q'_2}.$$

For the proof with the usual Sobolev spaces, we refer to [24, Proposition 1.1, p. 105]. The result in our context follows by using (1.3).

Proof of Proposition 3.6. We only consider the equation (3.11), the other case being similar by the boundedness of S_N on $L^p(\mathbb{R}^2)$. For $s < 1/2$, we define the space

$$Z^s(\tau) = \big\{ v \in \mathcal{C}\big([-\tau, \tau]; \mathcal{H}^s(\mathbb{R}^2)\big) \text{ s.t. } v(0) = 0 \text{ and } \|v\|_{Z^s(\tau)} \leq D \big\},$$

with $\|v\|_{Z^s(\tau)} = \|v\|_{L^\infty_{[-\tau,\tau]} \mathcal{H}^s(\mathbb{R}^2)}$, and for $f \in A_\star(D)$ we define the operator

$$K(v) = -i \int_0^t \mathcal{T}(f + v) ds.$$

We will show that K has a unique fixed point $v \in Z^s(\tau)$.

<u>The case of radial Hermite functions.</u> By Lemma 3.7 with $(p_{jk}, q_{jk}) = (4, 4)$, we have for all $v \in Z^s(\tau)$

$$\|K(v)\|_{Z^s(\tau)} \leq \tau \big\| \mathcal{T}(f + v) \big\|_{Z^s(\tau)}$$

(3.13)
$$\leq C\tau \Big\| \|\mathrm{e}^{-isH}(f + v)(t)\|^3_{L^4_{s\in[-\frac{\pi}{4}, \frac{\pi}{4}]} \mathcal{W}^{s,4}(\mathbb{R}^2)} \Big\|_{L^\infty_{t\in[-\tau,\tau]}}.$$

Next, by Strichartz and since $v \in Z^s(\tau)$

$$\big\| \mathrm{e}^{-isH}(f + v)(t) \big\|_{L^4_{s\in[-\frac{\pi}{4}, \frac{\pi}{4}]} \mathcal{W}^{s,4}(\mathbb{R}^2)} \leq \big\| \mathrm{e}^{-isH} f \big\|_{L^4_{s\in[-\frac{\pi}{4}, \frac{\pi}{4}]} \mathcal{W}^{s,4}(\mathbb{R}^2)} + \big\| \mathrm{e}^{-isH} v(t) \big\|_{L^4_{s\in[-\frac{\pi}{4}, \frac{\pi}{4}]} \mathcal{W}^{s,4}(\mathbb{R}^2)}$$

$$\leq C\big(D + \|v(t)\|_{\mathcal{H}^s(\mathbb{R}^2)}\big) \leq 2CD.$$

Therefore, from (3.13) we deduce

$$\|K(v)\|_{Z^s(\tau)} \leq C\tau D^3,$$

which implies that K maps $Z^s(\tau)$ into itself when $\tau \leq cD^{-2}$, for $c > 0$ small enough.

Similarly, for $v_1, v_2 \in Z^s(\tau)$, we have the bound

(3.14)
$$\|K(v_2) - K(v_1)\|_{Z^s(\tau)} \leq C\tau D^2 \|v_2 - v_1\|_{Z^s(\tau)},$$

which shows that if $\tau \leq cD^{-2}$, K is a contraction of $Z^s(\tau)$. The Picard fixed point theorem gives the desired result.

<u>The case of holomorphic Hermite functions.</u> For $s < \frac{1}{2}$, recall that we set $p = p(s) = \frac{4}{1-2s}$ so that $s < \frac{1}{2} - \frac{1}{p}$.

$$\|K(v)\|_{Z^s(\tau)} \leq \tau \big\| \mathcal{T}(f + v) \big\|_{Z^s(\tau)}$$

$$\leq C\tau \big(\|\mathcal{T}(f, f, f)\|_{Z^s} + \|\mathcal{T}(f, f, v)\|_{Z^s} + \|\mathcal{T}(f, v, v)\|_{Z^s} + \|\mathcal{T}(v, v, v)\|_{Z^s} \big).$$

We estimate each term thanks to Lemma 3.7 and Strichartz. The conjugation plays no role, so we forget it.

Estimate of the trilinear term in v:

$$\|\mathcal{T}(v, v, v)\|_{\mathcal{H}^s} \leq C\|\mathrm{e}^{-it'H} v\|^3_{L^4_{t'\in[-\frac{\pi}{4}, \frac{\pi}{4}]} \mathcal{W}^{s,4}(\mathbb{R}^2)} \leq C\|v\|^3_{\mathcal{H}^s(\mathbb{R}^2)}.$$

Estimate of the quadratic term in v: for $\delta > 0$ such that $\frac{2}{\frac{8}{3}+\delta} + \frac{1}{p} + \frac{1}{4} = 1$,

$$\|\mathcal{T}(v,v,f)\|_{\mathcal{H}^s} \leq C\|e^{-it'H}v\|^2_{L^{8/3+\delta}_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^{8/3+\delta}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^p_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,p}(\mathbb{R}^2)}$$

$$+ \|e^{-it'H}v\|_{L^4_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,4}(\mathbb{R}^2)}\|e^{-it'H}v\|_{L^4_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^4(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^4_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^4(\mathbb{R}^2)}$$

$$\leq CD\|v\|^2_{\mathcal{H}^s(\mathbb{R}^2)}.$$

Estimate of the linear term in v: with the same δ as above,

$$\|\mathcal{T}(v,f,f)\|_{\mathcal{H}^s} \leq C\|e^{-it'H}v\|_{L^{8/3+}_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^{8/3+}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^{8/3+}_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^{8/3+}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^p_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,p}(\mathbb{R}^2)}$$

$$+ \|e^{-it'H}v\|_{L^4_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,4}(\mathbb{R}^2)}\|e^{-it'H}f\|^2_{L^4_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^4(\mathbb{R}^2)}$$

$$\leq CD^2\|v\|_{\mathcal{H}^s(\mathbb{R}^2)}.$$

Estimate of the constant term in v:

$$\|\mathcal{T}(v,f,f)\|_{\mathcal{H}^s} \leq C\|e^{-it'H}f\|^2_{L^{8/3+}_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}L^{8/3+}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^p_{t'\in[-\frac{\pi}{4},\frac{\pi}{4}]}\mathcal{W}^{s,p}(\mathbb{R}^2)} \leq CD^3.$$

With these estimates at hand, the result follows by the Picard fixed point theorem. $\qquad\square$

3.2.2. **Approximation and invariance of the measure.**

Lemma 3.9. *Fix $D \geq 0$ and $s < 1/2$. Then for all $\varepsilon > 0$, there exists $N_0 \geq 0$ such that for all $f \in A_\star(D)$ and $N \geq N_0$*

$$\big\|\Phi(t)f - \Phi_N(t)f\big\|_{L^\infty([-\tau_1,\tau_1];\mathcal{H}^s(\mathbb{R}^2))} \leq \varepsilon,$$

where $\tau_1 = cD^{-2}$ for some $c > 0$.

Proof. Denoting for simplicity $\mathcal{T}(f) = \mathcal{T}(f,f,f)$,

$$v - v_N = -i\int_0^t \big[S_N\big(\mathcal{T}(f+v) - \mathcal{T}(f+v_N)\big) + (1-S_N)\mathcal{T}(f+v)\big]ds.$$

As in (3.14) we get

$$\|v - v_N\|_{Z^s(\tau)} \leq C\tau D^2\|v - v_N\|_{Z^s(\tau)} + \int_{-\tau}^{\tau} \|(1-S_N)\mathcal{T}(f+v)\|_{\mathcal{H}^s(\mathbb{R}^2)}ds,$$

which in turn implies when $C\tau D^2 \leq 1/2$

$$\|v - v_N\|_{Z^s(\tau)} \leq 2\int_{-\tau}^{\tau} \|(1-S_N)\mathcal{T}(f+v)\|_{\mathcal{H}^s(\mathbb{R}^2)}ds.$$

Let $\eta > 0$ so that $s + \eta < 1/2$. Then by the proof of Proposition 3.6, $\|\mathcal{T}(f+v)\|_{L^\infty_{[-\tau,\tau]}\mathcal{H}^{s+\eta}(\mathbb{R}^2)} \leq CD^3$ if $\tau \leq c_0 D^{-2}$ and therefore there exists $N_0 = N_0(\varepsilon, D)$ which satisfies the claim. $\qquad\square$

In the next result, we summarize the results obtained by Suzzoni in [9, Sections 3.3 and 4]. Since the proofs are very similar in our context, we skip them.

Let $D_{i,j} = (i + j^{1/2})^{1/2}$, with $i, j \in \mathbb{N}$ and set $T_{i,j} = \sum_{\ell=1}^j \tau_1(D_{i,\ell})$. Let

$$\Sigma_{N,i} := \big\{f : \forall j \in \mathbb{N},\ \Phi_N(\pm T_{i,j})f \in A(D_{i,j+1})\big\},$$

and

$$\Sigma_i := \limsup_{N \to +\infty} \Sigma_{N,i}, \qquad \Sigma := \bigcup_{i \in \mathbb{N}} \Sigma_i.$$

Proposition 3.10. *Let $\beta \geq 0$, then*

(i) *The set Σ is of full $\rho_\star$ measure.*

(ii) *For all $f \in \Sigma$, there exists a unique global solution $u = f + v$ to (CR), and for all $t \in \mathbb{R}$, $u(t) \in \Sigma$. This define a global flow Φ on Σ*

(iii) *For all measurable set $A \subset \Sigma$, and all $t \in \mathbb{R}$,*

$$\rho_\star(A) = \rho_\star(\Phi(t)(A)).$$

4. Weak solutions: proof of Theorem 2.6

4.1. **Definition of $\mathcal{T}(u, u, u)$ on the support of μ.** For $N \geq 0$, denote by Π_N the orthogonal projector on the space $\bigoplus_{k=0}^N E_k$ (in this section, we do not need the smooth cut-offs S_N). In the sequel, we denote by $\mathcal{T}(u) = \mathcal{T}(u, u, u)$ and $\mathcal{T}_N(u) = \Pi_N \mathcal{T}(\Pi_N u, \Pi_N u, \Pi_N u)$

Proposition 4.1. *For all $p \geq 2$ and $\sigma > 1$, the sequence $(\mathcal{T}_N(u))_{N \geq 1}$ is a Cauchy sequence in $L^p\big(X^{-1}(\mathbb{R}^2), \mathcal{B}, d\mu; \mathcal{H}^{-\sigma}(\mathbb{R}^2)\big)$. Namely, for all $p \geq 2$, there exist $\delta > 0$ and $C > 0$ so that for all $1 \leq M < N$,*

$$\int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}^p \, d\mu(u) \leq C M^{-\delta}.$$

We denote by $\mathcal{T}(u) = \mathcal{T}(u, u, u)$ the limit of this sequence and we have for all $p \geq 2$

$$(4.1) \qquad \|\mathcal{T}(u)\|_{L_\mu^p \mathcal{H}^{-\sigma}(\mathbb{R}^2)} \leq C_p.$$

Before we turn to the proof of Proposition 4.1, let us state two elementary results which will be needed in the sequel

Lemma 4.2. *For all $n \in \mathbb{N}$,*

$$\sum_{k=n}^{+\infty} \frac{1}{2^k} \binom{k}{n} = \sum_{k=n}^{+\infty} \frac{k!}{2^k \, n! \, (k-n)!} = 2.$$

Proof. For $|z| < 1$ we have $\dfrac{1}{1-z} = \displaystyle\sum_{k=0}^{+\infty} z^k$. If one differentiates n times this formula we get

$$\frac{n!}{(1-z)^{n+1}} = \sum_{k=n}^{+\infty} \frac{k!}{(k-n)!} z^{k-n},$$

which implies the result, taking $z = 1/2$. $\square$

Lemma 4.3. *Let $0 < \varepsilon < 1$ and $p, L \geq 1$ so that $p \leq L^\varepsilon$. Then*

$$\frac{L!}{2^L \, (L-p)!} \leq C 2^{-L/2}.$$

Proof. The proof is straightforward. By the assumption $p \leq L^\varepsilon$

$$\frac{L!}{(L-p)!} \leq L^p \leq C2^{L/2},$$

which was the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 4.1. By the result [27, Proposition 2.4] on the Wiener chaos, we only have to prove the statement for $p = 2$.

Firstly, by definition of the measure μ

$$\int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}^2 \mathrm{d}\mu(u) = \int_\Omega \|\mathcal{T}_N(\gamma(\omega)) - \mathcal{T}_M(\gamma(\omega))\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}^2 \mathrm{d}\mathbf{p}(\omega).$$

Therefore, it is enough to prove that $(\mathcal{T}_N(\gamma))_{N \geq 1}$ is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$. Let $1 \leq M < N$ and fix $\alpha > 1/2$. By (1.2) we get

$$
\begin{aligned}
H^{-\alpha} \mathcal{T}_N(\gamma) &= \frac{1}{2^\alpha} \sum_{A_N} \frac{g_{n_1} g_{n_2} \overline{g_{n_3}}}{(n_1 + n_2 - n_3 + 1)^\alpha} \mathcal{T}(\varphi_{n_1}, \varphi_{n_2}, \varphi_{n_3}) \\
&= \frac{\pi}{8 \cdot 2^\alpha} \sum_{A_N} \frac{(n_1 + n_2)!}{2^{n_1+n_2} \sqrt{n_1! \, n_2! \, n_3! \, (n_1 + n_2 - n_3)!}} \frac{g_{n_1} g_{n_2} \overline{g_{n_3}}}{(n_1 + n_2 - n_3 + 1)^\alpha} \varphi_{n_1 + n_2 - n_3} \\
&= \frac{\pi}{8 \cdot 2^\alpha} \sum_{p=0}^N \frac{1}{(p+1)^\alpha} \Big(\sum_{A_N^{(p)}} \frac{(n_1 + n_2)!}{2^{n_1+n_2} \sqrt{n_1! \, n_2! \, n_3! \, p!}} g_{n_1} g_{n_2} \overline{g_{n_3}} \Big) \varphi_p,
\end{aligned}
$$

with

$$A_N = \big\{ n \in \mathbb{N}^3 \text{ s.t. } 0 \leq n_j \leq N, \ 0 \leq n_1 + n_2 - n_3 \leq N \big\},$$

$$A_N^{(p)} = \big\{ n \in \mathbb{N}^3 \text{ s.t. } 0 \leq n_j \leq N, \ n_1 + n_2 - n_3 = p \big\} \quad \text{if } 0 \leq p \leq N.$$

Therefore,

$$\|\mathcal{T}_N(\gamma) - \mathcal{T}_M(\gamma)\|_{\mathcal{H}^{-\alpha}(\mathbb{R}^2)}^2 =$$

$$= \frac{\pi^2}{64 \cdot 2^{2\alpha}} \sum_{p=0}^N \frac{1}{(p+1)^{2\alpha}} \sum_{(n,m) \in A_{M,N}^{(p)} \times A_{M,N}^{(p)}} \frac{(n_1 + n_2)! \, (m_1 + m_2)! \, g_{n_1} g_{n_2} \overline{g_{n_3} g_{m_1} g_{m_2}} g_{m_3}}{2^{n_1+n_2} 2^{m_1+m_2} p! \, \sqrt{n_1! \, n_2! \, n_3!} \, \sqrt{m_1! \, m_2! \, m_3!}}$$

where $A_{M,N}^{(p)}$ is the set defined by

$$A_{M,N}^{(p)} = \Big\{ n \in \mathbb{N}^3 \text{ s.t. } 0 \leq n_j \leq N, \ n_1 + n_2 - n_3 = p \in \{0 \dots N\},$$

$$\big(n_1 > M \ \text{ or } \ n_2 > M \ \text{ or } \ n_3 > M \ \text{ or } \ p > M \big) \Big\}.$$

Now we take the integral over Ω. Since $(g_n)_{n \geq 0}$ are independent and centred Gaussians, we deduce that each term in the r.h.s. vanishes, unless

Case 1: $(n_1, n_2, n_3) = (m_1, m_2, m_3)$ or $(n_1, n_2, n_3) = (m_2, m_1, m_3)$

or

Case 2: $(n_1, n_2, m_1) = (n_3, m_2, m_3)$ or $(n_1, n_2, m_2) = (n_3, m_1, m_3)$ or $(n_1, n_2, m_3) = (m_1, n_3, m_2)$ or $(n_1, n_2, m_3) = (m_2, n_3, m_1)$.

We write

$$\int_\Omega \|\mathcal{T}_N(\gamma) - \mathcal{T}_M(\gamma)\|^2_{\mathcal{H}^{-2\alpha}(\mathbb{R}^2)} d\mathbf{p} = J_1 + J_2,$$

where J_1 and J_2 correspond to the contribution in the sum of each of the previous cases.

Contribution in case 1: By symmetry, we can assume that $(n_1, n_2, n_3) = (m_1, m_2, m_3)$. Define

$$B^{(p)}_{M,N} = \left\{ n \in \mathbb{N}^2 \text{ s.t. } 0 \le n_j \le N, \text{ and } \left(n_1 > M \text{ or } n_2 > M \text{ or } n_1 + n_2 - p > M \text{ or } p > M \right) \right\}.$$

Then

$$J_1 \le C \sum_{p \ge 0} \frac{1}{(1+p)^{2\alpha}} \sum_{B^{(p)}_{M,N}} \frac{\left((n_1 + n_2)! \right)^2}{2^{2(n_1+n_2)} p! \, n_1! \, n_2! \, (n_1 + n_2 - p)!}.$$

In the previous sum, we make the change of variables $L = n_1 + n_2$, and we observe that on $B^{(p)}_{M,N}$, we have $L \ge M$, then

$$\begin{aligned}
J_1 &\le C \sum_{p \ge 0} \frac{1}{(1+p)^{2\alpha}} \sum_{L \ge p+M} \sum_{n_1=0}^{L} \frac{(L!)^2}{2^{2L} p! \, n_1! \, (L - n_1)! \, (L - p)!} \\
&= C \sum_{p \ge 0} \frac{1}{(1+p)^{2\alpha}} \sum_{L \ge p+M} \frac{L!}{2^L p! \, (L - p)!},
\end{aligned}$$

where we used the fact that $\sum_{n_1=0}^{L} \binom{L}{n_1} = 2^L$. Let $\varepsilon > 0$ and split the previous sum in two parts

$$\begin{aligned}
J_1 &\le C \sum_{p=0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha}} \sum_{L=M}^{+\infty} \frac{L!}{2^L p! \, (L - p)!} + C \sum_{p=M^\varepsilon+1}^{+\infty} \frac{1}{(1+p)^{2\alpha}} \sum_{L=p}^{+\infty} \frac{L!}{2^L p! \, (L - p)!} \\
&\le C \sum_{p=0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha}} \sum_{L=M}^{+\infty} \frac{L!}{2^L p! \, (L - p)!} + 2C \sum_{p=M^\varepsilon+1}^{+\infty} \frac{1}{(1+p)^{2\alpha}} \\
&:= J_{11} + J_{12},
\end{aligned}$$

by Lemma 4.2. For the first sum, we can use Lemma 4.3, since $p \le M^\varepsilon \le L^\varepsilon$, thus

$$J_{11} \le C \sum_{p=0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha} p!} \sum_{L=M}^{+\infty} \frac{1}{2^{L/2}} \le C \sum_{L=M}^{+\infty} \frac{1}{2^{L/2}} \le C M^{-\delta}.$$

Next, clearly $J_{12} \le C M^{-\delta}$ because $\alpha > 1/2$, and this gives $J_1 \le C M^{-\delta}$.

Contribution in case 2: We can assume that $(n_1, n_2, m_1) = (n_3, m_2, m_3)$. Then for $n, m \in A_{M,N}^{(p)}$ we have $n_2 = m_2 = p$. Moreover, by symmetry, we can assume that $n_1 > M$ or $p > M$. Thus

$$J_2 \leq C \sum_{p \geq 0} \frac{1}{(1+p)^{2\alpha}} \sum_{n_1 = M+1}^{+\infty} \sum_{m_1 = 0}^{+\infty} \frac{(n_1 + p)!\,(m_1 + p)!}{2^{n_1 + p} 2^{m_1 + p} n_1!\, m_1!\, (p!\,)^2}$$

$$+ C \sum_{p \geq M+1} \frac{1}{(1+p)^{2\alpha}} \sum_{n_1 = 0}^{+\infty} \sum_{m_1 = 0}^{+\infty} \frac{(n_1 + p)!\,(m_1 + p)!}{2^{n_1 + p} 2^{m_1 + p} n_1!\, m_1!\, (p!\,)^2} := J_{21} + J_{22}.$$

To begin with, by Lemma 4.2, we have

$$
\begin{aligned}
J_{22} &= C \sum_{p \geq M+1} \frac{1}{(1+p)^{2\alpha}} \Big(\sum_{n_1 = 0}^{+\infty} \frac{(n_1 + p)!}{2^{n_1 + p} n_1!\, p!} \Big) \Big(\sum_{m_1 = 0}^{+\infty} \frac{(m_1 + p)!}{2^{m_1 + p} m_1!\, p!} \Big) \\
&= 4C \sum_{p \geq M+1} \frac{1}{(1+p)^{2\alpha}} \leq c M^{-\delta}.
\end{aligned}
$$

Then by Lemma 4.2 again

$$
\begin{aligned}
J_{21} &= C \sum_{p \geq 0} \frac{1}{(1+p)^{2\alpha}} \Big(\sum_{n_1 = M+1}^{+\infty} \frac{(n_1 + p)!}{2^{n_1 + p} n_1!\, p!} \Big) \Big(\sum_{m_1 = 0}^{+\infty} \frac{(m_1 + p)!}{2^{m_1 + p} m_1!\, p!} \Big) \\
&= 2C \sum_{p \geq 0} \frac{1}{(1+p)^{2\alpha}} \Big(\sum_{n_1 = M+1}^{+\infty} \frac{(n_1 + p)!}{2^{n_1 + p} n_1!\, p!} \Big) \\
&= 2C \sum_{p = 0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha}} \Big(\sum_{n_1 = M+1}^{+\infty} \frac{(n_1 + p)!}{2^{n_1 + p} n_1!\, p!} \Big) + 2C \sum_{p = M^\varepsilon + 1}^{+\infty} \frac{1}{(1+p)^{2\alpha}} \Big(\sum_{n_1 = M+1}^{+\infty} \frac{(n_1 + p)!}{2^{n_1 + p} n_1!\, p!} \Big) \\
&:= K_1 + K_2.
\end{aligned}
$$

One the one hand, by Lemma 4.3

$$K_1 \leq C \Big(\sum_{p=0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha} p!} \Big) \Big(\sum_{n_1 = M+1}^{+\infty} 2^{-n_1/2} \Big) \leq C M^{-\delta},$$

and one the other hand, by Lemma 4.2, since $\alpha > 1/2$

$$K_2 \leq C \sum_{p = M^\varepsilon + 1}^{+\infty} \frac{1}{(1+p)^{2\alpha}} \leq C M^{-\delta}.$$

Putting all the estimates together, we get $J_2 \leq C M^{-\delta}$, which concludes the proof. $\square$

4.2. **Study of the measure ν_N.** Let $N \geq 1$. We then consider the following approximation of (CR)

$$(4.2) \qquad \begin{cases} i\partial_t u = \mathcal{T}_N(u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0,x) = f(x) \in X^{-1}(\mathbb{R}^2). \end{cases}$$

The equation (4.2) is an ODE in the frequencies less than N, $(1 - \Pi_N)u(t) = (1 - \Pi_N)f$ and for all $t \in \mathbb{R}$.

The main motivation to introduce this system is the following proposition, whose proof we omit.

Proposition 4.4. *The equation* (4.2) *has a global flow* Φ_N. *Moreover, the measure* μ *is invariant under* Φ_N : *For any Borel set* $A \subset X^{-1}(\mathbb{R}^2)$ *and for all* $t \in \mathbb{R}$, $\mu\big(\Phi_N(t)(A)\big) = \mu\big(A\big)$.

In particular if $\mathscr{L}_{X^{-1}}(v) = \mu$ then for all $t \in \mathbb{R}$, $\mathscr{L}_{X^{-1}}(\Phi_N(t)v) = \mu$.

We denote by ν_N the measure on $\mathcal{C}\big([-T, T]; X^{-1}(\mathbb{R}^2)\big)$, defined as the image measure of μ by the map

$$
\begin{aligned}
X^{-1}(\mathbb{R}^2) &\longrightarrow \mathcal{C}\big([-T, T]; X^{-1}(\mathbb{R}^2)\big) \\
v &\longmapsto \Phi_N(t)(v).
\end{aligned}
$$

Lemma 4.5. *Let* $\sigma > 1$ *and* $p \geq 2$. *Then there exists* $C > 0$ *so that for all* $N \geq 1$

$$
\big\| \|u\|_{W_T^{1,p} \mathcal{H}_x^{-\sigma}} \big\|_{L_{\nu_N}^p} \leq C.
$$

Proof. Firstly, we have that for $\sigma > 1$, $p \geq 2$ and $N \geq 1$

$$
\big\| \|u\|_{L_T^p \mathcal{H}_x^{-\sigma}} \big\|_{L_{\nu_N}^p} \leq C.
$$

Indeed, by the definition of ν_N and the invariance of μ by Φ_N we have

$$
\|u\|_{L_{\nu_N}^p L_T^p \mathcal{H}_x^{-\sigma}} = (2T)^{1/p} \|v\|_{L_\mu^p \mathcal{H}_x^{-\sigma}} = (2T)^{1/p} \|\gamma\|_{L_{\mathbf{p}}^p \mathcal{H}_x^{-\sigma}}.
$$

Then, by the Khintchine inequality (3.9) and (3.1), for all $p \geq 2$

$$
\|\gamma\|_{L_{\mathbf{p}}^p \mathcal{H}_x^{-\sigma}} \leq C \sqrt{p} \|\gamma\|_{L_{\mathbf{p}}^2 \mathcal{H}_x^{-\sigma}} \leq C.
$$

We refer to [4, Proposition 3.1] for the details.

Next, we show that $\big\| \|\partial_t u\|_{L_T^p \mathcal{H}_x^{-\sigma}} \big\|_{L_{\nu_N}^p} \leq C$. By definition of ν_N

$$
\begin{aligned}
\|\partial_t u\|_{L_{\nu_N}^p L_T^p \mathcal{H}_x^{-\sigma}}^p &= \int_{\mathcal{C}\big([-T,T]; X^{-1}(\mathbb{R}^2)\big)} \|\partial_t u\|_{L_T^p \mathcal{H}_x^{-\sigma}}^p \mathrm{d}\nu_N(u) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \|\partial_t \Phi_N(t)(v)\|_{L_T^p \mathcal{H}_x^{-\sigma}}^p \mathrm{d}\mu(v).
\end{aligned}
$$

Now, since $\Phi_N(t)(v)$ satisfies (4.2) and by the invariance of μ, we have

$$
\begin{aligned}
\|\partial_t u\|_{L_{\nu_N}^p L_T^p \mathcal{H}_x^{-\sigma}}^p &= \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(\Phi_N(t)(v))\|_{L_T^p \mathcal{H}_x^{-\sigma}}^p \mathrm{d}\mu(v) \\
&= 2T \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(v)\|_{\mathcal{H}_x^{-\sigma}}^p \mathrm{d}\mu(v),
\end{aligned}
$$

and conclude with (4.1) and Proposition 4.1. $\qquad\square$

4.3. **The convergence argument.**   The importance of Proposition 4.5 above comes from the fact that it allows to establish the following tightness result for the measures ν_N. We refer to [4, Proposition 4.11] for the proof.

Proposition 4.6. *Let $T > 0$ and $\sigma > 1$. Then the family of measures*

$$(\nu_N)_{N \geq 1} \quad with \quad \nu_N = \mathscr{L}_{\mathcal{C}_T \mathcal{H}^{-\sigma}}\big(u_N(t); t \in [-T, T]\big)$$

is tight in $\mathcal{C}\big([-T, T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)\big)$.

The result of Proposition 4.6 enables us to use the Prokhorov theorem: For each $T > 0$ there exists a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T, T]; X^{-1}(\mathbb{R}^2)\big)$ so that for all $\tau > 1$ and all bounded continuous function $F : \mathcal{C}\big([-T, T]; \mathcal{H}^{-\tau}(\mathbb{R}^2)\big) \longrightarrow \mathbb{R}$

$$\int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{R}^2)\big)} F(u)\mathrm{d}\nu_{N_k}(u) \longrightarrow \int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{R}^2)\big)} F(u)\mathrm{d}\nu(u).$$

By the Skohorod theorem, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{p}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable $\widetilde{u}$ with values in $\mathcal{C}\big([-T, T]; X^{-1}(\mathbb{R}^2)\big)$ so that

$$(4.3) \qquad \mathscr{L}\big(\widetilde{u}_{N_k}; t \in [-T, T]\big) = \mathscr{L}\big(u_{N_k}; t \in [-T, T]\big) = \nu_{N_k}, \quad \mathscr{L}\big(\widetilde{u}; t \in [-T, T]\big) = \nu,$$

and for all $\tau > 1$

$$(4.4) \qquad\qquad \widetilde{u}_{N_k} \longrightarrow \widetilde{u}, \quad \widetilde{\mathbf{p}} - \text{a.s. in } \mathcal{C}\big([-T, T]; \mathcal{H}^{-\tau}(\mathbb{R}^2)\big).$$

We now claim that $\mathscr{L}_{X^{-1}}(u_{N_k}(t)) = \mathscr{L}_{X^{-1}}(\widetilde{u}_{N_k}(t)) = \mu$, for all $t \in [-T, T]$ and $k \geq 1$. Indeed, for all $t \in [-T, T]$, the evaluation map

$$\begin{aligned} R_t \,:\, \mathcal{C}\big([-T, T]; X^{-1}(\mathbb{R}^2)\big) &\longrightarrow X^{-1}(\mathbb{R}^2) \\ u &\longmapsto u(t, .), \end{aligned}$$

is well defined and continuous.

Thus, for all $t \in [-T, T]$, $u_{N_k}(t)$ and $\widetilde{u}_{N_k}(t)$ have same distribution $(R_t)_\# \nu_{N_k}$. But by Proposition 4.4, the distribution of $u_{N_k}(t)$ is μ, which justifies our claim.

Thus from (4.4) we deduce that

$$(4.5) \qquad\qquad \mathscr{L}_{X^{-1}}(\widetilde{u}(t)) = \mu, \quad \forall\, t \in [-T, T].$$

Let $k \geq 1$ and $t \in \mathbb{R}$ and consider the r.v. X_k given by

$$X_k = u_{N_k}(t) - R_0(u_{N_k}(t)) + i \int_0^t \mathcal{T}_{N_k}(u_{N_k})ds.$$

Define $\widetilde{X}_k$ similarly to X_k with u_{N_k} replaced with $\widetilde{u}_{N_k}$. Then by (4.3),

$$\mathscr{L}_{\mathcal{C}_T X^{-1}}(\widetilde{X}_{N_k}) = \mathscr{L}_{\mathcal{C}_T X^{-1}}(X_{N_k}) = \delta_0.$$

In other words, $\widetilde{X}_k = 0 \ \widetilde{\mathbf{p}} - \text{a.s.}$ and $\widetilde{u}_{N_k}$ satisfies the following equation $\widetilde{\mathbf{p}} - \text{a.s.}$

$$(4.6) \qquad\qquad \widetilde{u}_{N_k}(t) = R_0(\widetilde{u}_{N_k}(t)) - i \int_0^t \mathcal{T}_{N_k}(\widetilde{u}_{N_k})ds.$$

We now show that we can pass to the limit $k \longrightarrow +\infty$ in (4.6) in order to show that $\widetilde{u}$ is $\widetilde{\mathbf{p}}$−a.s. a solution to (CR) written in integral form as:

$$(4.7) \qquad \widetilde{u}(t) = R_0(\widetilde{u}(t)) - i \int_0^t \mathcal{T}(\widetilde{u})ds.$$

Firstly, from (4.4) we deduce the convergence of the linear terms in equation (4.6) to those in (4.7). The following lemma gives the convergence of the nonlinear term.

Lemma 4.7. *Up to a sub-sequence, the following convergence holds true*

$$\mathcal{T}_{N_k}(\widetilde{u}_{N_k}) \longrightarrow \mathcal{T}(\widetilde{u}), \qquad \widetilde{\mathbf{p}} - a.s. \ in \ L^2\big([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)\big).$$

Proof. In order to simplify the notations, in this proof we drop the tildes and write $N_k = k$. Let $M \geq 1$ and write

$$\mathcal{T}_k(u_k) - \mathcal{T}(u) = \big(\mathcal{T}_k(u_k) - \mathcal{T}(u_k)\big) + \big(\mathcal{T}(u_k) - \mathcal{T}_M(u_k)\big) + \big(\mathcal{T}_M(u_k) - \mathcal{T}_M(u)\big) + \big(\mathcal{T}_M(u) - \mathcal{T}(u)\big).$$

To begin with, by continuity of the product in finite dimension, when $k \longrightarrow +\infty$

$$\mathcal{T}_M(u_k) \longrightarrow \mathcal{T}_M(u), \qquad \widetilde{\mathbf{p}} - \text{a.s. in } L^2\big([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)\big).$$

We now deal with the other terms. It is sufficient to show the convergence in the space $X := L^2\big(\Omega \times [-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)\big)$, since the almost sure convergence follows after exaction of a sub-sequence. By definition and the invariance of μ we obtain

$$\begin{aligned}
\big\| \mathcal{T}_M(u_k) - \mathcal{T}(u_k) \big\|_X^2 &= \int_{\mathcal{C}([-T,T];X^{-1})} \big\| \mathcal{T}_M(v) - \mathcal{T}(v) \big\|_{L_T^2 \mathcal{H}_x^{-\sigma}}^2 \mathrm{d}\nu_k(v) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \big\| \mathcal{T}_M\big(\Phi_k(t)(f)\big) - \mathcal{T}\big(\Phi_k(t)(f)\big) \big\|_{L_T^2 \mathcal{H}_x^{-\sigma}}^2 \mathrm{d}\mu(f) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \big\| \mathcal{T}_M(f) - \mathcal{T}(f)) \big\|_{L_T^2 \mathcal{H}_x^{-\sigma}}^2 \mathrm{d}\mu(f) \\
&= 2T \int_{X^{-1}(\mathbb{R}^2)} \big\| \mathcal{T}_M(f) - \mathcal{T}(f) \big\|_{\mathcal{H}_x^{-\sigma}}^2 \mathrm{d}\mu(f),
\end{aligned}$$

which tends to 0 uniformly in $k \geq 1$ when $M \longrightarrow +\infty$, according to Proposition 4.1.

The term $\big\| \mathcal{T}_M(u) - \mathcal{T}(u) \big\|_X$ is treated similarly. Finally, with the same argument we show

$$\big\| \mathcal{T}_k(u_k) - \mathcal{T}(u_k) \big\|_X \leq C \big\| \mathcal{T}_k(f) - \mathcal{T}(f) \big\|_{L_\mu^2 \mathcal{H}_x^{-\sigma}},$$

which tends to 0 when $k \longrightarrow +\infty$. This completes the proof. $\qquad \square$

4.4. **Conclusion of the proof of Theorem 2.6.** Define $\widetilde{f} = \widetilde{u}(0) := R_0(\widetilde{u})$. Then by (4.5), $\mathscr{L}_{X^{-1}}(\widetilde{f}) = \mu$ and by the previous arguments, there exists $\widetilde{\Omega}' \subset \widetilde{\Omega}$ such that $\widetilde{\mathbf{p}}(\widetilde{\Omega}') = 1$ and for each $\omega' \in \widetilde{\Omega}'$, the random variable $\widetilde{u}$ satisfies the equation

$$(4.8) \qquad \widetilde{u} = \widetilde{f} - i \int_0^t \mathcal{T}(\widetilde{u})dt, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2.$$

Set $\Sigma = \widetilde{f}(\Omega')$, then $\mu(\Sigma) = \widetilde{\mathbf{p}}(\widetilde{\Omega}') = 1$. It remains to check that we can construct a global dynamics. Take a sequence $T_N \to +\infty$, and perform the previous argument for $T = T_N$. For all $N \geq 1$, let Σ_N

be the corresponding set of initial conditions and set $\Sigma = \cap_{N \in \mathbb{N}} \Sigma_N$. Then $\mu(\Sigma) = 1$ and for all $\widetilde{f} \in \Sigma$, there exists

$$\widetilde{u} \in \mathcal{C}\big(\mathbb{R}\,;X^{-1}(\mathbb{R}^2)\big),$$

which solves (4.8). This completes the proof of Theorem 2.6.

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