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Strong Asymptotic Stability of a Nonlinear Non-Isotropic Elastodynamic System

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ABSTRACT. We propose a new approach to prove the strong asymptotic stability of a nonlinear and a non-isotropic elastodynamic system. Unlike the earlier works, our method can be applied in the case of feedbacks with no growth assumptions at the origin, and when LaSalle's invariance principle cannot be applied due to the lack of compactness.

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1. Introduction

Consider the nonlinear damped elastodynamic system

$$\begin{cases} u_i'' - \sigma_{ij,j} + g_i(u_i') = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u_i = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 \quad \text{in} \quad \Omega, \\ i = 1, \cdots, n \end{cases} \tag{P}$$

where Ω is an open set of finite measure in $\mathbb{R}^n$ ($n = 1, 2, \cdots$), having a boundary Γ of class C^2. The stress tensor σ_{ij} is related to the strain tensor

growth at the origin allows the construction of a standard Lyapounov functional, or the use of some specific integral inequalities which yield the desired decay rates.

As a consequence, our problem is now faced with some difficulties which require the development of a new approach in successfully solving the problem of global existence and asymptotic behavior. This approach was introduced by the first author in [1].

The paper in organized as follows. In section 2, we shall prove the global existence, and in section 3 we shall prove the strong asymptotic stability.

2. Global existence

Before the statement of our main result in this section, let us first introduce some notations. We denote by H and V the Hilbert spaces $L^2(\Omega)^n$ and $H_0^1(\Omega)^n$ endowed respectively by the norms

$$\|v\|_H^2 = \int_\Omega v_i v_i \, dx \quad \text{and} \quad \|v\|_V^2 = \int_\Omega \sigma_{ij}(v)\varepsilon_{ij}(v)\, dx.$$

Thanks to (1.1)-(1.2) and to the Korn's inequality, one can easily verify that $\|\cdot\|_V$ is a norm on V.

The problem (P) is well posed, in fact we have

THEOREM 2.1.
The problem (P) admits a unique global weak solution

$$u \in C(\mathbb{R}_+; V) \cap C^1(\mathbb{R}_+; H)$$

for all given initial data $(u^0, u^1) \in V \times H$.

PROOF.
It should be evident that problem (P) may be written abstractly as

$$\begin{cases} U' = \mathcal{A}U + F(U) \\ U(0) = U_0 =: (u^0, u^1) \end{cases}$$

where $U = (u, z)$, $z = u'$, $\mathcal{A}U = (z, Au)$, $A = [A_i]_{i=1,2,\ldots,n}$ and $A_i u_i = \sigma_{ij,j}(u)$.

The domain of $\mathcal{A}$ is $D(\mathcal{A}) = W \times V$ where $W =: (H^2(\Omega) \cap H_0^1(\Omega))^n$ endowed with the norm

$$\|v\|_W^2 = \int_\Omega (\Delta v_i \Delta v_i + \sigma_{ij}(v)\varepsilon_{ij}(v))\, dx.$$

The application $F : V \times H \to V \times H$ is defined by $F(u, z) = (0, -g(z))$. We will show that $\mathcal{A}$ is maximal monotone operator. In fact, for all $U = (u, z)$, $\tilde{U} = (\tilde{u}, \tilde{z}) \in D(\mathcal{A})$ we have

$$\langle \mathcal{A}U - \mathcal{A}\tilde{U}, U - \tilde{U}\rangle_{V\times H} = \langle z - \tilde{z}, u - \tilde{u}\rangle_V + \langle A(u - \tilde{u}), z - \tilde{z}\rangle_H$$

$$= \int_\Omega a_{ijkl}\varepsilon_{kl}(z - \tilde{z})\varepsilon_{ij}(u - \tilde{u})\,dx + \int_\Omega \sigma_{ij,j}(u - \tilde{u})(z_i - \tilde{z}_i)\,dx$$

$$= \int_\Gamma \sigma_{ij}(u - \tilde{u})(z_i - \tilde{z}_i)\nu_j\,d\Gamma = 0.$$

Now we will show that $\mathcal{A}$ is maximal, to do this let us take $U^0 = (u^0, z^0) \in V \times H$, we will prove that there exists $U = (u, z) \in D\mathcal{A})$ satisfying

$$U + \mathcal{A}U = U^0,$$

which is equivalent to prove the existence of a solution in W to the following equation

$$u_i - \sigma_{ij,j}(u) = u^0 + z^0 \in H. \tag{2.1}$$

Thanks to (1.2) and the well known Lax-Milgram's lemma, the equation (2.1) can be easily solved, we omit the details here.

Hence $\mathcal{A}$ generats a C_0-group of isometries on $V \times H$. Furthermore since F is locally Lipschitz and by $(H1) - (H2)$, we have

$$\langle F(U), U\rangle_{V\times H} = -\int_\Omega z \cdot g(z)\,dx \le 0,$$

we deduce the global existence of a weak solution to (P) by using the following general result of Ball [3]:

THEOREM 2.2.

Let $\mathcal{A}$ be the infinitesimal generator of a linear C_0-semigroup $e^{\mathcal{A}t}$ on a real Hilbert space H, and $F : H \to H$ satisfying

$$F \text{ is locally Lipschitz,} \tag{i}$$

$$\langle F(U), U\rangle_H \le 0 \quad \text{for all} \quad U \in H. \tag{ii}$$

Then the problem

$$\begin{cases} U' = \mathcal{A}U + F(U) \\ U(0) = U_0 \end{cases}$$

possesses a unique weak solution $U(t)$ on $\mathbb{R}_+$ for each $U_0 \in H$.

3. Strong asymptotic stabilization

We define the energy of the solution u to the problem (P) by

$$E(t) = \frac{1}{2}\int_\Omega (u'_i u'_i + \sigma_{ij}\varepsilon_{ij})\,dx.$$

A simple computation shows that

$$E'(t) = -\int_\Omega u'_i g(u'_i)\,dx \leq 0 \quad \text{by}\,(H2);$$

hence the energy is non-increasing, and our main result in this section is

THEOREM 3.1.
We have

$$E(t) \to 0 \quad \textit{as} \quad t \to +\infty$$

for every weak solution of (P).

For the proof, we need the two following lemmas

LEMMA 3.2.
We have

$$\int_0^t \int_\Omega u_i g_i(u'_i)\,dx\,ds = o(t), \quad t \to +\infty.$$

LEMMA 3.3.
We have

$$\int_0^t \int_\Omega u'_i u'_i\,dx\,ds = o(t), \quad t \to +\infty.$$

PROOF OF LEMMA 3.2.
From now on we denote by c various positive constants which may be different at different steps.

By $(H1)$, we see that $|g_i(x)| \leq c|x|$ for $|x| \leq 1$, then we have

$$\int_{|u'_i|\leq 1} |u_i g_i(u'_i)|\,dx \leq c\int_{|u'_i|\leq 1} (u'_i g_i(u'_i))^{\frac{1}{2}} |u_i|\,dx$$

$$\leq c\Big(\int_\Omega u'_i g_i(u'_i)\,dx\Big)^{\frac{1}{2}} \|u\|_H.$$

Similarly by (H3) we have

$$\int_{|u_i'|>1} |u_i g_i(u_i')|\, dx \le c\Big(\int_\Omega u_i' g_i(u_i')\, dx\Big)^{\frac{1}{q'}} \|u\|_{(L^q(\Omega))^n}$$

where $q' = \frac{q}{q-1}$ is the Hölder conjugate of q. Then from the Hölder's inequality we obtain

$$\int_0^t \int_\Omega u_i g_i(u_i')\, dx\, ds \le c\Big(\int_0^t \int_\Omega u_i' g_i(u_i')\, dx\, ds\Big)^{\frac{1}{2}} \sqrt{t} \sup_{[0,t]} \|u(s)\|_H$$

$$+c\Big(\int_0^t \int_\Omega u_i' g_i(u_i')\, dx ds\Big)^{\frac{1}{q'}} t^{\frac{1}{q}} \sup_{[0,t]} \|u(s)\|_{(L^q(\Omega))^n}.$$

Using the Hölder, Sobolev and Poincaré inequalities we have

$$\|u(s)\|_H \le c\|u(s)\|_{(L^q(\Omega))^n} \le c\|u\|_V \le cE(s)^{\frac{1}{2}} \le cE(0)^{\frac{1}{2}} \quad \forall s \ge 0.$$

From these estimates, it follows that

$$\int_0^t \int_\Omega u_i g_i(u_i')\, dx\, ds \le ct^{\frac{1}{2}} + ct^{\frac{1}{q}} = o(t), \quad t \to +\infty.$$

PROOF OF LEMMA 3.3.

Let ϵ be an arbitrarily small real number and set

$$M_i(\epsilon) =: \sup\{\frac{x}{g_i(x)}, |x| \ge \sqrt{\frac{\epsilon}{|\Omega|}}\} \quad \text{and} \quad M(\epsilon) =: \max_{i=1}^{i=n} M_i(\epsilon),$$

by hypotheses $(H1)-(H3)$, we have $M(\epsilon) < +\infty$. Clearly,

$$\int_{|u_i'|<\sqrt{\frac{\epsilon}{|\Omega|}}} u_i' u_i'\, dx \le n\epsilon.$$

On the other hand

$$\int_{|u_i'|\ge\sqrt{\frac{\epsilon}{|\Omega|}}} u_i' u_i'\, dx = \int_{|u_i'|\ge\sqrt{\frac{\epsilon}{|\Omega|}}} \frac{u_i'}{g_i(u_i')} u_i' g_i(u_i')\, dx$$

$$\le M(\epsilon) \int_\Omega u_i' g_i(u_i')\, dx.$$

As

$$\int_{|u_i'|\ge\sqrt{\frac{\epsilon}{|\Omega|}}} u_i' u_i'\, dx \le \sqrt{2E(0)}\Big(\int_{|u_i'|\ge\sqrt{\frac{\epsilon}{|\Omega|}}} u_i' u_i'\, dx\Big)^{\frac{1}{2}}$$

we deduce that

$$\int_\Omega u_i' u_i' \, dx \le n\epsilon + \sqrt{2E(0)M(\epsilon)}\Big(\int_\Omega u_i' g_i(u_i') \, dx\Big)^{\frac{1}{2}}.$$

and then by the Hölder inequality, we have

$$\int_0^t \int_\Omega u_i' u_i' \, dx \, ds \le c\epsilon t + c\sqrt{2E(0)M(\epsilon)}\Big(\int_0^t \int_\Omega u_i' g_i(u_i') \, dx \, ds\Big)^{\frac{1}{2}} \sqrt{t}$$

$$\le c\epsilon t + c\sqrt{2M(\epsilon)}E(0)\sqrt{t} = o(t), \quad t \to +\infty.$$

PROOF OF THEOREM 2.1.
Assume on the contrary that $l =: \lim_{t\to+\infty} E(t) > 0$, then we have

$$\phi(t) - \phi(0) = 2\int_0^t \int_\Omega u_i' u_i' \, dx \, ds - 2\int_0^t E(s) \, ds - \int_0^t \int_\Omega u_i g_i(u_i') \, dx \, ds$$

$$\le -2lt + o(t), \quad t \to +\infty$$

where we set $\phi(t) = \int_\Omega u_i u_i' \, dx$, we used the two lemmas in the last step. It follows that $\phi(t) \to -\infty$ as $t \to +\infty$, but this is impossible because

$$|\phi(t)| = |\int_\Omega u_i u_i' \, dx| \le \frac{1}{2}\int_\Omega (u_i' u_i' + u_i u_i) \, dx \le cE(t) \le cE(0).$$

Hence $\lim_{t\to+\infty} E(t) = 0$.

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