ONSAGER-MACHLUP FUNCTIONAL FOR UNIFORMLY ELLIPTIC TIME-INHOMOGENEOUS DIFFUSION

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Abstract. In this paper, we will compute the Onsager-Machlup functional of an inhomogeneous uniformly elliptic diffusion process. This functional is very similar to the corresponding functional for homogeneous diffusions; indeed, the only difference come from the infinitesimal variation of the volume. For the Ricci flow, for instance, the functional is close to the $\mathcal{L}_0$ distance used by Lott in [6]. We will also use the Onsager-Machlup functional to study small ball probability for weighted sup-norm of some inhomogeneous diffusion.

1. Introduction

Let M be a n-dimensional Riemannian manifold, and L_t be an inhomogeneous uniformly elliptic second order operator over M, without constant term. It is always possible to endow M with a time-dependent family of metrics $g(t)$ such that

$$L_t = \frac{1}{2}\Delta_t + Z(t), \tag{1.1}$$

where Δ_t is a Laplace Beltrami operator for the metric $g(t)$ and $Z(t,.)$ is a time-dependent vector field on M. Let $X_t(x_0)$ be a L_t-diffusion process on M, starting at the point x_0. An example of such a diffusion is the $g(t)$-Brownian motion, introduced in [2], where the family of metrics $(g(t))_{t\in[0,T]}$ comes from the Ricci flow on M.

Let $d(t,x,y)$ be the Riemannian distance on M according to the metric $g(t)$. Consider a smooth curve $\varphi : [0,T] \to M$, such that $\varphi(0) = x_0$. We are now interested in the asymptotic equivalent as ϵ goes to zero of the following probability

$$\mathbb{P}_{x_0}[\forall t \in [0,T] \quad d(t, X_t, \varphi(t)) \leq \epsilon].$$

This asymptotic will be expressed as the product of two terms. The first one is a decreasing function of ϵ that does not depend on the curve nor geometries (except the dimension), while the second term depends on the geometries along the curve φ. This second term is expressed as a Lagrangian. So maximizing this term reduces to finding the most probable path of the diffusion. This term is usually called the Onsager-Machlup functional of the diffusion X_t.

To compute the O.M. functional, we will use both the technics introduced by Takahashi and Watanabe in [7], and the non-singular drift introduced by

Hara. Using this drift, Hara and Takahashi made in [4] a substantial simplification of the latter proof (of O.M. functional) of Takahashi and Watanabe.

We propose here to introduce a time-dependent parallel transport along a curve, according to a family of metrics. It will allow us to compute the Onsager-Machlup functional in the time-inhomogeneous case.

Let $\operatorname{div}_{g(t)}$ and $R_{g(t)}$ be respectively the divergence operator and the scalar curvature with respect to the metric $g(t)$. Let H be a time-dependent function on the tangent bundle defined for $v \in T_xM$ as:

$$\begin{aligned} H(t,x,v) = \frac{1}{2}\|Z(t,x)-v\|^2_{g(t)} + \frac{1}{2}\operatorname{div}_{g(t)}(Z)(t,x) - \frac{1}{12}R_{g(t)}(x) \\ + \frac{1}{4}\operatorname{trace}_{g(t)}(\dot g(t)). \end{aligned}$$

The main result of this paper is the following:

Theorem 1.1. *Let $X_t(x_0)$ be a L_t diffusion process starting at point x_0, where $L_t = \frac{1}{2}\Delta_t + Z(t,.)$. Then we have the following asymptotic:*

$$\begin{aligned} &\mathbb{P}_{x_0}[\forall t \in [0,T], d(t,X_t,\varphi(t)) \leq \epsilon] \\ &\sim_{\epsilon\downarrow 0} C\exp\{-\frac{\lambda_1 T}{\epsilon^2}\}\exp\{-\int_0^T H(t,\varphi(t),\dot\varphi(t))\,dt\}. \end{aligned}$$

Here C and λ_1 are explicit constants.

A similar result was obtained by [7],[4], and [1] in the homogeneous case. Our contribution comes from the time-inhomogeneity of the diffusion.

The paper is organized as follows: first, we will define in section 2 a parallel transport along a curve according to a family of metrics $(g(t)_t)$. This parallel transport will enable us to obtain a Fermi coordinates in a neighborhood of a smooth curve φ. We will also give a (local) development of a tensor that will be used in the following.

Then, we will introduce some useful tools in section 3. They are not new, and clearly exposed by Capitaine in [1]. So we will keep the same notation as in [1] in this paper. In [1], the author has investigated the case of different norms, in the homogeneous case. In the literature, non smooth functions φ are also considered, but this will not be discussed here. In the second part of section 3, we will establish the proof of Theorem 1.1. Finally, section 4 is devoted to some applications. First, we will describe the most probable path of an inhomogeneous diffusion. Then, we will obtain a small ball estimate (for the weighted sup-norm) for inhomogeneous diffusions.

2. Parallel transport along a curve, and Fermi coordinate

Let $\varphi : [0,T] \longrightarrow M$ be a smooth curve. Suppose that the manifold M is endowed with a family of metrics $g(t)_{t\in[0,T]}$ which is C^1 in time and C^2 in space . This family of metrics induces a time dependent family of Levi-Civita connexions, denoted by ∇^t.

Suppose that A is a bilinear form on a given vector space E. Let v, w be in E. Suppose that there exists scalar product $\langle ., . \rangle_{g(t)}$ on E. Then define $A^{\#g(t)}(v) \in E$ as the element of E such that $\langle A^{\#g(t)}v, w\rangle_{g(t)} = A(v, w)$.

Let v be a vector on $T_{\varphi(0)}M$ and define $\tau_t v$ as the solution of the following ODE :

$$\begin{cases} \nabla^t_{\dot\varphi(t)}(\tau_t v) = -\frac{1}{2}\dot g(t)^{\#g(t)}(\tau_t v) \\ \tau_0 v = v. \end{cases}$$

The map $\tau_t v$ is called a parallel transport of v along the curve φ according to the family of metrics $g(t)$.

Proposition 2.1. *The parallel transport τ_t is an isometry between the tangent space $(T_{\varphi(0)}M, g(0))$ and the tangent space $(T_{\varphi(t)}M, g(t))$. In particular, if $(e_1, e_2, ..., e_n)$ is an orthonormal basis of $T_{\varphi(0)}M$ for the metric $g(0)$, then $(\tau_t e_1, \tau_t e_2, ..., \tau_t e_n)$ is an orthonormal basis of $T_{\varphi(t)}M$ for the metric $g(t)$.*

Proof. Let v, w be in $T_{\varphi(0)}M$, we have:

$$\begin{aligned} \frac{d}{dt}\langle \tau_t v, \tau_t w\rangle_{g(t)} &= \nabla^t g(t)(\tau_t v, \tau_t w) + \langle \nabla^t \tau_t v, \tau_t w\rangle_{g(t)} + \langle \tau_t v, \nabla^t \tau_t w\rangle_{g(t)} \\ &+ \dot g(t)(\tau_t v, \tau_t w) \\ &= -\frac{1}{2}\langle \dot g(t)^{\#g(t)}(\tau_t v), \tau_t w\rangle_{g(t)} - \frac{1}{2}\langle \tau_t v, \dot g(t)^{\#g(t)}(\tau_t w)\rangle_{g(t)} \\ &+ \dot g(t)(\tau_t v, \tau_t w) \\ &= 0. \end{aligned}$$

□

We are now able to write the Fermi coordinates in a neighborhood of a curve. Let $\varphi : [0, T] \longrightarrow M$ be a smooth curve and let τ be the parallel transport above φ in the sense of proposition 2.1, where we have fixed a $g(0)$-orthonormal basis $(e_1, ..., e_n)$ of $T_{\varphi(0)}M$. Consider the map

$$\begin{aligned} \Psi : [0, T] \times \mathbb{R}^n &\longrightarrow [0, T] \times M \\ (t, v_1, ..., v_n) &\longmapsto (t, \exp^{g(t)}_{\varphi(t)}(\tau_t \sum_1^n v_i e_i)), \end{aligned}$$

where $\exp^{g(t)}_x$ means the exponential map for the metric $g(t)$. The map Ψ is clearly a diffeomorphism on some neighborhood U of $[0, T] \times 0$. Define now $V = \Psi(U)$. Remark that, for each fixed t, the map $\Psi(t, .)$ is the normal coordinates for the metric $g(t)$ in a neighborhood of the point $\varphi(t)$.

Let $X_t(x_0)$ be an L_t-diffusion starting at the point x_0, where L_t is a time-dependent operator as in (1.1). Using these Fermi coordinates, the time-dependent norm of Theorem 1.1 can be translated in terms of Euclidean norm, while the generator will be the pull back operator of L_t by Ψ. By assumption, the generator of (t, X_t) is $\partial_t + \frac{1}{2}\Delta_t + Z(t,.)$. We will now compute the generator of $\Psi^{-1}(t, X_t)$, or more precisely its local development:

$$\Psi^*(\partial_t + \frac{1}{2}\Delta_t + Z(t,.)) = \frac{\tilde{\partial}}{\partial_t} + \frac{1}{2}\tilde{\Delta}_t + \tilde{Z}(t,.). \tag{2.1}$$

The second term in the right hand side is computed in [2] as :

$$\tilde{\Delta}_t = g^{ij}(\Psi(t,.))\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - g^{kl}(\Psi(t,.))\Gamma^i_{kl}(\Psi(t,.))\frac{\partial}{\partial x_i},$$

where $(x_1^t, ..., x_n^t)$, $g_{ij}(\Psi(t,.))$, $g^{ij}(\Psi(t,.))$, and $\Gamma^i_{kl}(\Psi(t,.))$ are respectively the normal coordinates at the point $\varphi(t)$ for the metric $g(t)$ with respect to the vector basis $(\tau_t e_1, ..., \tau_t e_1)$, the coefficient of metric $g(t)$ in this basis, its inverse, and the Christoffel symbols of the Levi-Civita connexion of the metric $g(t)$ in this basis.

Clearly we have

$$\tilde{Z}(t,.) = \sum_{i=1}^n Z^i(t,.)\frac{\partial}{\partial x_i}$$

where $Z^i(t,.) = \langle Z(\Psi(t,.)), \frac{\partial}{\partial x_i^t}|_{\Psi(t,.)}\rangle_{g(t)}$.

Recall that V is a neighborhood of $\{(t, \varphi(t)), t \in [0,T]\}$ and $V = \Psi(U)$. For a point $(t, x) \in V$ such that Ψ induces a diffeomorphism in (t, x), we will write $\Psi^{-1}(t, x) = (t, x_1^t, ..., x_n^t) \in [0,T] \times \mathbb{R}^n$.

To study (2.1), we have to compute $\frac{\tilde{\partial}}{\partial_t}|_{(t,x)} = \sum_{i=1}^n a_i(t,x)\frac{\partial}{\partial x_i} + a_0(t,x)\frac{\partial}{\partial t}$. For any $1 \le i \le n$, we easily see that $a_i(t_0, x) = \frac{\partial}{\partial_t}_{|t_0}(x_i^t)|_{\Psi(t_0,x)}$. For any fixed $x \in M$, we have the equality:

$$\frac{\partial}{\partial t}(\exp^{g(t)}(\varphi(t), \sum_{i=1}^n \tau_t e_i x_i^t)) = 0,$$

where for any $v \in T_x M$, $\exp^{g(t)}(x, v)$ is the exponential map for the metric $g(t)$ at the point x. The next technical result will be useful to compute the term $a_i(t,x) = \frac{\partial}{\partial_t}(x_i^t)|_{\Psi(t,x)}$.

Lemma 2.2. *Let $v \in T_x M$. Then*

$$\frac{\partial}{\partial t}_{|t_0} \exp^{g(t)}(x, v) = O(\|v\|^2_{g(t_0)}).$$

Proof. Let $x_i(t, s)$ be the i-th coordinate of the geodesic $\exp^{g(t)}(x, s.v)$ in the normal coordinates system centered at $\varphi(t_0)$ with respect to the metric $g(t_0)$. In the following, we will shorten the notation an write $\dot{x}(t, s)$ for $\frac{\partial}{\partial s}x(t, s)$.

The usual equation of geodesics shows that:

$$\frac{\partial}{\partial t}_{|t_0} x_i(t,s) = -\sum_{jk} \frac{\partial}{\partial t}_{|t_0} \Big[\int_0^s du \int_0^u dl\, \Gamma^i_{jk}(t, x(t,l))\dot{x}_j(t,l)\dot{x}_k(t,l) \Big]$$

$$= \sum_{jk} - \int_0^s du \int_0^u dl \left(\frac{\partial}{\partial t}_{|t_0} \Gamma^i_{jk}(t, x(t_0,l))\dot{x}_j(t_0,l)\dot{x}_k(t_0,l) \right.$$

$$+ \langle d\Gamma^i_{jk}(t_0,.), \frac{\partial}{\partial t}_{|t_0} x(t,l)\rangle \dot{x}_j(t_0,l)\dot{x}_k(t_0,l)$$

$$\left. + 2\Gamma^i_{jk}(t_0, x(t_0,l)) \frac{\partial}{\partial t}_{|t_0} (\dot{x}_j(t,l))\dot{x}_k(t_0,l) \right).$$

Note that we have $\| \dot{x}(t_0,s) \|^2_{g(t_0)} = \| v \|^2_{g(t_0)}$ and $\Gamma^i_{jk}(t_0,x) = O(\| x \|_{g(t_0)})$. In a neighborhood V of $\{(t,\varphi(t)), t \in [0,T]\}$, the quantities $| \frac{\partial}{\partial t}\Gamma^i_{jk}(t,.) |$ and $\| d\Gamma^i_{jk}(t,.) \|$ are bounded by some constant C, hence we get the equality:

$$\frac{\partial}{\partial t}_{|t_0} x(t,s) := \frac{\partial}{\partial t}_{|t_0} (x_1(t,s),...,x_n(t,s))$$

$$= O(\|v\|^2_{g(t_0)}) + \int_0^s dl\, O(\|v\|^2_{g(t_0)}) \frac{\partial}{\partial t}_{|t_0} x(t,l)$$

$$+ \int_0^s du \int_0^u dl\, O(\|v\|^2_{g(t_0)}) \frac{\partial}{\partial t}_{|t_0} x(t,l).$$

By Gronwall's lemma we deduce that :

$$\| \frac{\partial}{\partial t}_{|t_0} x(t,1) \| = O(\|v\|^2_{g(t_0)}).$$

$\square$

Lemma 2.3. *Let $(x_1(t),...,x_n(t))$ be the coordinates of*

$$\exp^{g(t_0)}(\varphi(t), \sum_{i=1}^n \tau_t e_i x_i^t)$$

in the normal coordinates system at the point $\varphi(t_0)$ for the metric $g(t_0)$ with reference basis $(\tau_{t_0} e_i)_{i=1,...,n}$ and $\partial_i = \frac{\partial}{\partial_{x_i^{t_0}}}$ be the associated vector field. Then we have:

$$\frac{\partial}{\partial t}_{|t_0} x_i(t) = \frac{\partial}{\partial t}_{|t_0} x_i^t - \frac{1}{2}\frac{\partial}{\partial t}_{|t_0} (g(t))_{\varphi(t_0)}(\partial_i, \sum_{j=1}^n x_j^{t_0}\partial_j) + \langle \frac{\partial}{\partial t}_{|t_0} \varphi(t), \partial_i \rangle_{g(t_0)}$$

$$+ O(\| x^{t_0} \|^2).$$

Proof. As in the previous proof, we write the geodesic

$$\exp^{g(t_0)}(\varphi(t), s.\sum_{i=1}^n \tau_t e_i x_i^t)$$

in normal coordinate. It satisfies the system :

$$\begin{cases} \ddot{x}_i(t,s) = -\sum_{jk} \Gamma^i_{jk}(t_0, x(t,s))\dot{x}_j(t,s)\dot{x}_k(t,s), \\ \dot{x}_i(t,0) = \langle \sum_{l=1}^n \tau_t e_l x^t_l, \partial_{i|_{\varphi(t)}} \rangle_{g(t_0)}, \\ x_i(t,0) = \varphi(t)^i. \end{cases}$$

Moreover, we have :

$$\begin{aligned} \frac{\partial}{\partial t}_{|t_0} x_i(t,s) = &-\int_0^s du \int_0^u dl \sum_{jk} \frac{\partial}{\partial t}_{|t_0} \Big[\Gamma^i_{jk}(t_0, x(t,l))\dot{x}_j(t,l)\dot{x}_k(t,l) \Big] \\ &+ s\frac{\partial}{\partial t}_{|t_0} \dot{x}_i(t,0) + \frac{\partial}{\partial t}_{|t_0} x_i(t,0). \end{aligned} \tag{2.2}$$

Similarly to the latter proof, the equality (2.2) can be rewritten in a matrix. Then using again Gronwall's lemma, we see that $\frac{\partial}{\partial t}_{|t_0} x_i(t,s)$ is bounded for any $s \in [0,1]$. So the integral term of (2.2) is an $O(\|x^{t_0}\|^2_{g(t_0)})$. Hence, we deduce that

$$\begin{aligned} \frac{\partial}{\partial t}_{|t_0} x_i(t,1) = O(\|x^{t_0}\|^2_{g(t_0)}) &+ \frac{\partial}{\partial t}_{|t_0} \langle \sum_{l=1}^n \tau_t e_l x^t_l, \partial_{i_{|_{\varphi(t)}}} \rangle_{g(t_0)} \\ &+ \langle \frac{\partial}{\partial t}_{|t_0} \varphi(t), \partial_{i_{|_{\varphi(t_0)}}} \rangle_{g(t_o)}. \end{aligned}$$

Remark that for $t = t_0$, we have $\partial_{i_{|_{\varphi(t_0)}}} = \tau_{t_0} e_i$. So, we obtain the equality:

$$\begin{aligned} \frac{\partial}{\partial t}_{|t_0} x_i(t,1) = O(\|x^{t_0}\|^2_{g(t_0)}) &+ \frac{\partial}{\partial t}_{|t_0} x^t_l \delta^l_i \\ &+ \sum_{l=1}^n x^{t_0}_l \frac{\partial}{\partial t}_{|t_0} \langle \tau_t e_l, \partial_{i_{|_{\varphi(t)}}} \rangle_{g(t_0)} + \langle \frac{\partial}{\partial t}_{|t_0} \varphi(t), \partial_{i_{|_{\varphi(t_0)}}} \rangle_{g(t_o)}. \end{aligned}$$

By construction of the parallel transport τ, we have :

$$\begin{aligned} \frac{\partial}{\partial t}_{|t_0} \langle \tau_t e_l, \partial_{i_{|_{\varphi(t)}}} \rangle_{g(t_0)} &= \langle \nabla^{t_0} \tau_t e_l, \partial_{i_{|_{\varphi(t_0)}}} \rangle_{g(t_0)} + \langle \tau_{t_0} e_l, \nabla^{t_0} \partial_i \rangle_{g(t_0)} \\ &= -\frac{1}{2}\dot{g}(t_0)(\tau_{t_0} e_l, \partial_{i_{|_{\varphi(t_0)}}}) \\ &= -\frac{1}{2}\dot{g}(t_0)(\partial_{l_{|_{\varphi(t_0)}}}, \partial_{i_{|_{\varphi(t_0)}}}). \end{aligned}$$

The last term of the right hand side has vanished because ∂_i comes from normal coordinates for the metric $g(t_0)$. Putting all pieces together leads to the result.

□

Proposition 2.4. *We have*

$$\frac{\partial}{\partial t_{|t_0}} x_i^t = \frac{1}{2}\frac{\partial}{\partial t_{|t_0}}(g(t))_{\varphi(t_0)}(\partial_i, \sum_{j=1}^n x_j^{t_0}\partial_j) - \langle \frac{\partial}{\partial t_{|t_0}}\varphi(t), \partial_i\rangle_{g(t_0)} + O(\| x^{t_0} \|^2_{g(t_0)}).$$

Proof. Recall that:

$$\frac{\partial}{\partial t}(\exp^{g(t)}(\varphi(t), \sum_{i=1}^n \tau_t e_i x_i^t)) = 0.$$

The expected equation follows from the previous two lemmas. □

We can now conclude that the Taylor series of the generator is:

$$\begin{aligned}
\frac{\tilde{\partial}}{\partial_t} + \tilde{L}_t &:= \Psi^*(\partial_t + \frac{1}{2}\Delta_t + Z(t,.))_{|(t,x)} \\
&= \frac{\tilde{\partial}}{\partial_t} + \frac{1}{2}\sum_{i,j=1}^n g^{ij}(t,x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^n \tilde{b}^i(t,x)\frac{\partial}{\partial x_i} \\
&= \frac{\partial}{\partial_t} + \sum_{i,j=1}^n \Big(\frac{1}{2}\dot{g}(t)(\frac{\partial}{\partial x_i^t}, \frac{\partial}{\partial x_j^t})x_j - \dot{\varphi}(t)^i\Big)\frac{\partial}{\partial x_i} \\
&- \frac{1}{2}\sum_{k,l,i=1}^n g^{kl}(t,x)\Gamma_{kl}^i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n g^{ij}(t,x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
&+ \sum_{i=1}^n Z^i(t,x)\frac{\partial}{\partial x_i} + O(\|x\|^2),
\end{aligned}$$

where $g^{ij}(t,x)$ corresponds to the inverse of the metric $g(t)$ in the normal coordinates $(x_1^t, ..., x_n^t)$ evaluated at the point $\Psi(t,x)$, $\Gamma_{ij}^k(t,x)$ are the Christoffel symbols in these coordinates at the point $\Psi(t,x)$, $\dot{\varphi}^i(t)$ and $Z^i(t,x)$ are the coordinates of the corresponding vector in these normal coordinates.

Remark 2.5. We have no time-dependence term such as $O(\|.\|_{g(t)})$ because all metrics are equivalent on U.

3. Proof of the main result

3.1. **A useful tool : Besselizing drift.** Let $X(t)$ be a L_t-diffusion, and $\tilde{T} = inf\{t \in [0,T], s.t.\, (t, X(t)) \notin V\}$. Let us define $\tilde{X}(t)$ a $\mathbb{R}^n$-valued process such that $(t \wedge \tilde{T}, \tilde{X}(t)) = \Psi^{-1}(t \wedge \tilde{T}, X(t \wedge \tilde{T}))$. Then for any small enough ϵ, we have :

$$\mathbb{P}_{x_0}[\sup_{t\in[0,T]} d(t, X(t), \varphi(t)) \leq \epsilon] = \mathbb{P}_0[\sup_{t\in[0,T]} \|\tilde{X}(t)\| \leq \epsilon].$$

It is obvious that $(t, \tilde{X}(t))$ is a $\frac{\tilde{\partial}}{\partial t} + \tilde{L}_t$ diffusion. So there exists a $\mathbb{R}^n$-valued Brownian motion $\tilde{B}$, such that $\tilde{X}(t)$ is a solution of the following Itô stochastic differential equation :

$$\begin{cases} d\tilde{X}^i(t) = \sum_{j=1}^n \sqrt{g}^{ij}(t, \tilde{X}(t))\, d\tilde{B}_t^j + \tilde{b}^i(t, \tilde{X}(t))\, dt, \\ \tilde{X}(0) = 0. \end{cases}$$

Here $\sqrt{g}^{ij}(t,x)$ is the square root of the inverse metric $g(t)$ in the coordinates $(x_1^t, ..., x_n^t)$ at the point $\Psi(t,x)$ and the drift term is defined by:

$$\tilde{b}^i(t,x) = -\dot{\varphi}^i(t) - \frac{1}{2}\sum_{kl} g^{kl}(t,x)\Gamma_{kl}^i(t,x) + \frac{1}{2}\dot{g}(t)_{|\varphi(t)}\Big(\frac{\partial}{\partial x_i^t}, \sum_{j=1}^n x^j \frac{\partial}{\partial x_j^t}\Big) + Z^i(t,x) + O(\|x\|^2).$$

The Onsager Machlup functional for the sup-norm was studied by Takahashi and Watanabe. They introduced a drift, which is singular at the origin. The smooth Besselizing drift we will use here has been found by Hara. Let us describe shortly the Hara Besselizing drift. Since the coordinates are normal, we use Gauss' Lemma to find that for any $i \in [1..n]$:

$$\sum_{j=1}^m g^{ij}(t,x)x_j = x_i, \text{ and } \sum_{j=1}^m \sqrt{g}^{ij}(t,x)x_j = x_i.$$

The Hara drift γ is then defined by:

$$\gamma^i(t,x) = \frac{1}{2}\sum_{j=1}^n \frac{\partial g^{ij}}{\partial x_j}(t,x).$$

It satisfies the following equation :

$$\sum_{i=1}^n (1 - g^{ii}(t,x)) = 2\sum_j \gamma^j(t,x)x_j.$$

Let us denote $\tilde{\sigma}_{ij}(t,.) = \sqrt{g}^{ij}(t,x)$. We remind the reader that the process $\tilde{X}(t)$ satisfies the equation :

$$\begin{cases} d\tilde{X}(t) = \tilde{\sigma}(t, \tilde{X}(t))\, d\tilde{B}_t + \tilde{b}(t, \tilde{X}(t))\, dt \\ \tilde{X}(0) = 0. \end{cases} \tag{3.1}$$

Define the $\mathbb{R}^n$-valued process $Y(t)$ as the solution of the following Itô equation:

$$\begin{cases} dY(t) = \tilde{\sigma}(t, Y(t))\, d\tilde{B}_t + \gamma(t, Y(t))\, dt \\ Y(0) = 0. \end{cases} \tag{3.2}$$

By definition of the vector field γ, we get by Itô's formula:

$$d \parallel Y(t) \parallel^2 = 2\sum_{k=1}^{n} Y^k(t) d\tilde{B}_t^k + n\, dt.$$

By Lévy's Theorem, we see that $B(t) = \sum_{k=1}^{n} \int_0^t \frac{Y^k(s)}{\|Y(s)\|} d\tilde{B}_s^k$ is a one dimensional Brownian motion in the filtration generated by $\tilde{B}$ and

$$d \parallel Y(t) \parallel^2 = 2 \parallel Y(t) \parallel dB_t + n\, dt.$$

It proves that $\parallel Y(t) \parallel$ is a n-dimensional Bessel process.

Let us define $\hat{Y}_t := (t, Y(t))$. The next step consists in finding a well-suited probability measure such that, under this measure, the process $Y(t)$ has the same distribution as the process $\tilde{X}(t)$. Let us define:

$$N_t = \int_0^t \langle \tilde{\sigma}^{-1}(\hat{Y}_t)(\tilde{b}(\hat{Y}_t) - \gamma(\hat{Y}_t)),\, d\tilde{B}_t \rangle,$$

$$M_t = \exp(N_t - \frac{1}{2}\langle N \rangle_t)$$

$$\mathbb{Q} = M_T.\mathbb{P}.$$

Girsanov's Theorem ensures that $(Y, \mathbb{Q})$ is a solution of (3.1). The uniqueness in law of such a solution then implies that:

$$\begin{aligned}
&\mathbb{P}_0[\sup_{t\in[0,T]} \quad \|\tilde{X}(t)\| \le \epsilon] = \mathbb{Q}[\sup_{t\in[0,T]} \quad \|Y(t)\| \le \epsilon] \\
&= \mathbb{E}_{\mathbb{P}}[M_T; \sup_{t\in[0,T]} \quad \|Y(t)\| \le \epsilon] \\
&= \mathbb{E}_{\mathbb{P}}[M_T \mid \sup_{t\in[0,T]} \quad \|Y(t)\| \le \epsilon]\mathbb{P}[\sup_{t\in[0,T]} \quad \|Y(t)\| \le \epsilon].
\end{aligned} \tag{3.3}$$

The term $\mathbb{P}[\sup_{t\in[0,T]} \quad \|Y(t)\| \le \epsilon]$ is easily controlled by a stopping time argument. So finding the Onsager Machlup functional reduces to the study of the behavior of a conditioned exponential martingale, as in the paper [7]. We will study the behavior of:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Bigg[\exp\Big(\sum_{i,j=1}^{n} \int_0^T \sqrt{g}_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t) d\tilde{B}_t^i \\
&- \frac{1}{2}\sum_{i,j=1}^{n} \int_0^T g_{ij}(\hat{Y}_t)\delta^i(\hat{Y}_t)\delta^j(\hat{Y}_t) dt\Big) \Bigg| \sup_{t\in[0,T]} \quad \|Y(t)\| \le \epsilon\Bigg].
\end{aligned} \tag{3.4}$$

Where $\delta^i(t,x) = \tilde{b}^i(t,x) - \gamma^i(t,x)$.

Remark 3.1. From Lemma 1 in [1] it is sufficient to control the exponential moments one by one in the following sense.

Let us recall briefly this lemma :

Lemma 3.2 ([5],[1]). *Let $I_1,...,I_n$ be n random variables, $\{A_\epsilon\}_{0<\epsilon}$ a family of events, and $a_1,...,a_n$ some real numbers. If, for every real number c and every $1\leq i\leq n$, we have*

$$\limsup_{\epsilon\to 0}\mathbb{E}[\exp(cI_i)\mid A_\epsilon]\leq \exp(ca_i),$$

then,

$$\lim_{\epsilon\to 0}\mathbb{E}(\exp(\sum_{i=1}^n I_i)\mid A_\epsilon)=\exp(\sum_{i=1}^n a_i).$$

Note that, in the case studied here, all the metrics $g(t)$ are equivalent. Recall Cartan's Theorem dealing with Taylor series of metric and curvature in normal coordinates. We have :

$$g_{ij}(t,x)=\delta_i^j-\frac{1}{3}\sum_{kl}R_{iklj}(t,0)x_kx_l+O(\|x\|^3),$$

where $R_{iklj}(t,0)$ are the components of the Riemannian curvature tensor, for the metric $g(t)$ in normal coordinates centered at the point $\varphi(t)$. We thus deduce the following equalities :

$$g^{ij}(t,x)=\delta_i^j+O(\|x\|^2),$$
$$\gamma^i(t,x)=-\frac{1}{6}\sum_{j=1}^n R_{ij}(t,0)x_j+O(\|x\|^2),$$

where $R_{ij}(t,0)$ are the component of the Ricci curvature tensor for the metric $g(t)$ in normal coordinates, at the point $\varphi(t)$. By definition of the Christoffel symbol, we have,

$$\begin{aligned}\Gamma_{ij}^k(t,x)&=\frac{1}{2}(\frac{\partial}{\partial_{x_i}}g_{jk}(t,x)+\frac{\partial}{\partial_{x_j}}g_{ik}(t,x)-\frac{\partial}{\partial_{x_k}}g_{ij}(t,x))\\&=-\frac{1}{3}\sum_{l=1}^n(R_{jlik}(t,0)+R_{iljk}(t,0))x_l+O(\|x\|^2).\end{aligned}\tag{3.5}$$

So we obtain,

$$-\frac{1}{2}\sum_{i,j=1}^n g^{ij}(t,x)\Gamma_{ij}^k(t,x)=-\frac{1}{3}\sum_{l=1}^n R_{lk}(t,0)x_l+O(\|x\|^2),$$

and thus,

(3.6)

$$\begin{aligned}\delta^i(t,x)&=-\dot\varphi^i(t)+\sum_{j=1}^n\big(\frac{1}{2}\dot g_{ij}(t,0)-\frac{1}{6}R_{ij}(t,0)\big)x_j+Z^i(t,x)+O(\|x\|^2),\\&=-\dot\varphi^i(t)+Z^i(t,0)+\sum_{j=1}^n\big(\frac{1}{2}\dot g_{ij}(t,0)-\frac{1}{6}R_{ij}(t,0)+\frac{\partial}{\partial x_j}Z^i(t,0)\big)x_j\\&\quad+O(\|x\|^2),\end{aligned}$$

where $\dot{g}_{ij}(t,0) = \dot{g}(t)\big(\frac{\partial}{\partial x_i^t}\mid_{\varphi(t)}, \frac{\partial}{\partial x_j^t}\mid_{\varphi(t)}\big)$.

3.2. **Proof of the theorem 1.1.** According to Lemma 3.2 we will separately estimate the terms of (3.4). The easiest one is the drift term. Namely, we have :

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \mathbb{E}[\exp\{-\frac{c}{2}\int_0^T g_{ij}(\hat{Y}_t)\delta^i(\hat{Y}_t)\delta^j(\hat{Y}_t)dt\} \mid \sup_{t\in[0,T]} \|Y(t)\| \leq \epsilon] \\
(3.7)\qquad &\leq \lim_{\epsilon\to 0} \exp[-\frac{c}{2}\int_0^T \delta_i^j(-\dot{\varphi}^i(t) + Z^i(t,0))^2 + O(\epsilon)\,dt] \\
&\leq \exp(-\frac{c}{2}\int_0^T \delta_i^j(-\dot{\varphi}^i(t) + Z^i(t,0))^2\,dt),
\end{aligned}$$

where we have used in the second inequality the fact that $O(\epsilon)$ is uniform in t according to the uniform equivalence of the family of metrics $\{g(t)\}_{t\in[0,T]}$. So, it remains to control the first term in (3.4). To this aim, we will use the following Theorem established in [4].

Let us denote by $*d$ the Stratonovich differential.

Theorem 3.3 ([4]). *Let α be a one form on $[0,T]\times\mathbb{R}^n$, which does not depend on dt and $Y(t)$ be a diffusion process in $\mathbb{R}^n$ whose radial part is a Bessel process, and such that for any $1\leq i,j\leq n$,*

$$\langle \int_0^{\cdot} Y^i dY^j - Y^j dY^i, \|Y\|_{\cdot}\rangle = 0.$$

Then the following estimate holds for the stochastic line integral $\int_{*d(t,Y_t)}\alpha$ (in the sense of Stratonovich integration of a one form along a process):

$$\mathbb{E}[\exp(\int_{*d(t,Y_t)}\alpha) \mid \sup_{t\in[0,T]} \|Y(t)\| \leq \epsilon] = \exp(O(\epsilon)).$$

The proof of this Theorem is based on the stochastic Stokes theorem which is deduced from Stokes' theorem by using Stratonovich integrals and the Kunita-Watanabe theorem for orthogonal martingales.

To use the above Theorem we first have to write the first term of (3.4) in terms of Stratonovich integral of a one form along a Bessel radial part process. Using the definition of y, (see (3.2)):

$$d\tilde{B}_t^i = \sum_{j=1}^n \tilde{\sigma}_{ij}^{-1}(\hat{Y}_t)\,dY_t^j - \sum_{j=1}^n \tilde{\sigma}_{ij}^{-1}(\hat{Y}_t)\gamma^j(\hat{Y}_t)\,dt,$$

so

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\int_0^T \sqrt{g}_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)d\tilde{B}_t^i \\
&= \sum_{i,j=1}^{n}\int_0^T g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)dY_t^i - \int_0^T g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)\gamma^i(\hat{Y}_t)\,dt \\
&= \sum_{i,j=1}^{n}\int_0^T g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t) * dY_t^i \\
&- \frac{1}{2}\sum_{i,j=1}^{n}\int_0^T \langle d(g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)), dY_t^i\rangle - \int_0^T g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)\gamma^i(\hat{Y}_t)\,dt.
\end{aligned}
\tag{3.8}
$$

Proposition 3.4. *Denote by A_ϵ the event $\{\sup_{t\in[0,T]} \quad \|Y(t)\| \leq \epsilon\}$. Then, the following equalities hold for any $1 \leq i,j \leq n$ and $c \in \mathbb{R}$:*

i) $\mathbb{E}[\exp(c\int_0^T \sum_{i,j=1}^n g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t) * dY_t^i) \mid A_\epsilon] = \exp(O(\epsilon))$.

ii)
$$
\begin{aligned}
&\limsup_{\epsilon\to 0}\mathbb{E}[\exp(-\frac{c}{2}\int_0^T \langle d(g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)), dY_t^i\rangle) \mid A_\epsilon] \\
&\leq \exp(-\frac{c}{2}\int_0^T \delta_i^j\{\frac{1}{2}\dot{g}_{ij}(t,0) - \frac{1}{6}R_{ij}(t,0) + \frac{\partial}{\partial x_j}Z^i(t,0)\}\,dt).
\end{aligned}
$$

iii) $\limsup_{\epsilon\to 0}\mathbb{E}[\exp(-c\int_0^T g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)\gamma^i(\hat{Y}_t)\,dt) \mid A_\epsilon] = 1$.

Proof. i) Let $\alpha = c\sum_{i,j=1}^n g_{ij}(t,x)\delta^j(t,x)dx^i$ be defined in the neighborhood $U \subset [0,T]\times\mathbb{R}^n$, and extend it to the whole space. The expected asymptotic expansion is straightforward corollary of Theorem 3.3.

ii) Using Itô's formula, the definition of Y leads to, for any $1 \leq i \leq n$:

$$
\begin{aligned}
&\limsup_{\epsilon\to 0}\mathbb{E}[\exp(-\frac{c}{2}\int_0^T \langle d(g_{ij}(\hat{Y}_t)\delta^j(\hat{Y}_t)), dY_t^i\rangle) \mid A_\epsilon] \\
&= \limsup_{\epsilon\to 0}\mathbb{E}[\exp(-\frac{c}{2}\int_0^T \sum_{l=1}^n \frac{\partial}{\partial x_l}(g_{ij}(t,\cdot)\delta^j(t,\cdot))(Y_t)dY_t^l dY_t^i) \mid A_\epsilon] \\
&= \limsup_{\epsilon\to 0}\mathbb{E}[\exp(-\frac{c}{2}\int_0^T \sum_{l=1}^n \frac{\partial}{\partial x_l}(g_{ij}(t,.)\delta^j(t,.))(Y_t)g_{il}(\hat{Y}_t)\,dt) \mid A_\epsilon] \\
&\leq \exp(-\frac{c}{2}\int_0^T (\frac{1}{2}\dot{g}_{ii}(t,0) - \frac{1}{6}R_{ii}(t,0) + \frac{\partial}{\partial x_i}Z^i(t,0))\,dt).
\end{aligned}
$$

For the latter inequality, we have used the Taylor expansion computed in the last section.

iii) We have :

$$
\begin{aligned}
(3.9)\qquad &\limsup_{\epsilon\to 0}\mathbb{E}[\exp(-c\int_0^T g_{ij}(t,Y(t)\delta^j(\hat{Y}_t)\gamma^i(t,Y(t))\,dt)\mid A_\epsilon]\\
&=\limsup_{\epsilon\to 0}\mathbb{E}[\exp(-c\int_0^T O(\|Y(t)\|)\,dt)\mid A_\epsilon]=1.
\end{aligned}
$$

□

We are now ready to prove Theorem 1.1

Proof. Theorem 1.1
Using Lemma 3.2,formula (3.3), (3.4), (3.7) and Proposition 3.4, we obtain :

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\mathbb{E}_{\mathbb{P}}[M_T \mid \sup_{t\in[0,T]}\|Y(t)\|\leq\epsilon]\\
(3.10)\qquad &=\exp\Big(\int_0^T\{-\frac{1}{2}\|Z(t,\varphi(t))-\dot{\varphi}(t)\|^2_{g(t)}-\frac{1}{4}(\mathrm{Tr}_{g(t)}(\dot{g}(t)))_{\varphi(t)}\\
&+\frac{1}{12}R(t,\varphi(t))-\frac{1}{2}\operatorname{div}_{g(t)}Z(t,\varphi(t))\}dt\Big)\\
&=\exp\Big(-\int_0^T H(t,\varphi(t),\dot{\varphi}(t))\,dt\Big).
\end{aligned}
$$

Since the second term of (3.3) is given by the scaling property of the Brownian motion, we see that

$$\mathbb{P}_0[\sup_{t\in[0,T]}\|Y(t)\|\leq\epsilon]=\mathbb{P}_0[\tau_1^n(B)>\frac{T}{\epsilon^2}],$$

where $\tau_1^n(B)$ is the first hitting time of the ball of radius 1 by the n-dimensional Brownian motion. Thus, using arguments of stopping time, Dirichlet problem and spectral Theorem, we get the following :

$$\mathbb{P}_0[\sup_{t\in[0,T]}\|Y(t)\|\leq\epsilon]\sim_{\epsilon\to 0}C\exp(-\lambda_1\frac{T}{\epsilon^2}),$$

where λ_1 is the first eigenvalue of the Laplace operator $(-\frac{1}{2}\Delta_{\mathbb{R}^n})$ in the unit ball in $\mathbb{R}^n$ with Dirichlet's boundary conditions, and C is an explicit constant that only depends on the dimension, (see lemme 8.1 [5]). □

4. Applications

4.1. **The most probable path.** In this section, we will deduce from Theorem 1.1 the equation of the "most likely"curve. Namely, we will find a second order differential equation for the critical curve of the Onsager Machlup functional $E[\varphi]=\int_0^T H(t,\varphi(t),\dot{\varphi}(t))\,dt$. This will be done below, in proposition 4.1.

Let φ and ψ be two smooth curves in M such that $\varphi(0)=\psi(0)$ and $\varphi(T)=\psi(T)$. Using the notation of Theorem 1.1, it implies that, for any given initial x_0, we have :

$$\lim_{\epsilon\to 0}\frac{\mathbb{P}_{x_0}[\sup_{t\in[0,T]} \quad d(t,X(t),\varphi(t))\le\epsilon]}{\mathbb{P}_{x_0}[\sup_{t\in[0,T]} \quad d(t,X(t),\psi(t))\le\epsilon]}$$
$$=\frac{\exp\left(-\int_0^T H(t,\varphi(t),\dot\varphi(t))\,dt\right)}{\exp\left(-\int_0^T H(t,\psi(t),\dot\psi(t))\,dt\right)}.$$

Our goal consists in computing the critical curve of the functional :

$$E[\varphi]=\int_0^T H(t,\varphi(t),\dot\varphi(t))\,dt,$$

when both the initial and ending points are fixed. In the next result, we determine the equation of this curve in the case of $g(t)$-Brownian motion (see [2]). The general case could be deduced by the same computation.

Proposition 4.1. *Let X_t be a $L_t:=\frac{1}{2}\Delta_t$ diffusion, where Δ_t is the Laplace operator with respect to a family of metrics $g(t)$ coming from the Ricci flow $\partial_t g(t)=\alpha\,\mathrm{Ric}_{g(t)}$, (as in [2]). Then the critical curve φ for the functional E satisfies the following second order differential equation:*

$$\nabla^t_{\partial_t}\dot\varphi(t)+\alpha\,\mathrm{Ric}^{\#g(t)}(\dot\varphi(t))+\frac{1-3\alpha}{12}\nabla^t R_t(\varphi(t))=0.$$

Proof. Let φ be a critical curve for E and let exp be the exponential map according to some fixed metric. Then for all vector fields V over φ such that $V(0)=V(1)=0$, we have:

$$\frac{\partial}{\partial_s}_{|s=0}E[t\mapsto\exp_{\varphi(t)}(sV(t))]=0.$$

Let us recall that the generator of (t,X_t) is given by $\partial_t+\frac{1}{2}\Delta_t+Z(t,\cdot)$. So when $L_t=\frac{1}{2}\Delta_t$, we have $Z(t,.)=0$, and hence

$$H(t,x,v)=\frac{1}{2}\|v\|^2_{g(t)}-\frac{1-3\alpha}{12}R_{g(t)}(x).$$

Let us now denote the variation of the curve φ by $\varphi_V(t,s):=\exp_{\varphi(t)}(sV(t))$, and $\dot{\varphi_V}(t,s):=\frac{\partial}{\partial_t}\varphi_V(t,s)$. The preceding equation of E becomes:

$$\begin{aligned}
0=&\frac{\partial}{\partial_s}_{|s=0}\int_0^T\frac{1}{2}\|\dot{\varphi_V}(t,s)\|^2_{g(t)}-\frac{1-3\alpha}{12}R_{g(t)}(\varphi_V(t,s))dt\\
=&\int_0^T\langle\dot{\varphi_V}(t,0),\nabla^t_{\partial_s}\dot{\varphi_V}(t,0)\rangle_{g(t)}\\
&-\frac{1-3\alpha}{12}\langle\nabla^t R_{g(t)}(\varphi_V(t,0)),\frac{\partial}{\partial_s}_{|s=0}\varphi_V(t,s)\rangle_{g(t)}\,dt\\
=&\int_0^T\langle\dot\varphi(t),\nabla^t_{\partial_s}\dot{\varphi_V}(t,s)\rangle_{g(t)}-\frac{1-3\alpha}{12}\langle\nabla^t R_{g(t)}(\varphi(t)),V(t)\rangle_{g(t)}\,dt.
\end{aligned}$$

Since ∂_t and ∂_s commute, and since the connection ∇^t is torsion free, we have $\nabla^t_{\partial_s}\dot{\varphi_V}(t,s) = \nabla^t_{\partial_t}\frac{\partial}{\partial_s}\varphi_V(t,s)$. So, for any vector field V such that $V(0) = V(T) = 0$, the critical curve satisfies:

$$\int_0^T \langle \dot\varphi(t), \nabla^t_{\partial_t} V(t)\rangle_{g(t)} - \frac{1-3\alpha}{12}\langle \nabla^t R_{g(t)}(\varphi(t)), V(t)\rangle_{g(t)}\, dt = 0. \tag{4.1}$$

Moreover, a straightforward computation shows that,

$$\begin{aligned}\partial_t \langle \dot\varphi(t), V(t)\rangle_{g(t)} = \langle \nabla^t_{\partial_t}\dot\varphi(t), V(t)\rangle_{g(t)} + \langle \dot\varphi(t), \nabla^t_{\partial_t} V(t)\rangle_{g(t)} \\ + \dot g(t)\Big(\dot\varphi(t), V(t)\Big).\end{aligned}$$

The final condition of the vector field $V(0) = V(T) = 0$, gives:

$$\int_0^T \partial_t\Big(\langle \dot\varphi(t), V(t)\rangle_{g(t)}\Big)\, dt = 0.$$

Hence, for any vector field V such that $V(0) = V(T) = 0$, the preceding equation (4.1) becomes

$$\begin{aligned}\int_0^T \Big(\langle \nabla^t_{\partial_t}\dot\varphi(t), V(t)\rangle_{g(t)} + \langle \alpha \operatorname{Ric}^{\#g(t)}(\dot\varphi(t)), V(t)\rangle \\ + \frac{1-3\alpha}{12}\langle \nabla^t R_{g(t)}(\varphi(t)), V(t)\rangle_{g(t)}\Big)\, dt = 0.\end{aligned}$$

Thus we conclude that φ is a critical value of E if and only if it satisfies:

$$\nabla^t_{\partial_t}\dot\varphi(t) + \alpha \operatorname{Ric}^{\#g(t)}(\dot\varphi(t)) + \frac{1-3\alpha}{12}\nabla^t R_{g(t)}(\varphi(t)) = 0.$$

□

Remark 4.2. The choice $\alpha = \frac{1}{3}$ for the speed of the backward Ricci flow produces a simplification in the above expressions and makes the functional E positive for all time, for any fixed metric $g(0)$ (when the backward Ricci flow exists).

Remark 4.3. The more general case of $g(t)$-BM can be easily deduced by the same proof. Let X_t be a $L_t := \frac{1}{2}\Delta_t$ diffusion, where Δ_t is the Laplace operator with respect to a family of metric $g(t)$, then the E-critical curve φ satisfies:

$$\nabla^t_{\partial_t}\dot\varphi(t) + \dot g(t)^{\#g(t)}(\dot\varphi(t)) + \frac{1}{12}\nabla^t R_{g(t)}(\varphi(t)) - \frac{1}{4}\nabla^t(\operatorname{Tr}_{g(t)}\dot g(t))(\varphi(t)) = 0.$$

We can also use this formula for the Brownian motion induced by the mean curvature flow as in [3], and compute the most probable path for this inhomogeneous diffusion. This result can be used to compute the most probable path for the degenerated diffusion $Z(t)$ (see Remark 2.9 of [3]).

4.2. **Small ball properties of inhomogeneous diffusion for weighted sup norm.** Let $X_t(x)$ be a $L_t = \frac{1}{2}\Delta_t + Z(t)$ diffusion. Let $f \in C^1([0,T])$ be a positive function on $[0,T]$. In this paragraph, we wish to estimate the following probability

$$\mathbb{P}_{x_0}[\forall t \in [0,T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)],$$

when ϵ is positive and close to 0. We deduce the following small ball estimate:

Proposition 4.4. *There exists an explicit positive constant $C > 0$ such that:*

$$\mathbb{P}_{x_0}[\forall t \in [0,T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)] \sim_{\epsilon\downarrow 0}$$
$$C \exp\{-\frac{\lambda_1 \int_0^T \frac{1}{f^2(s)}\,ds}{\epsilon^2}\} \exp\{-\int_0^T \tilde{H}(t, \varphi(t), \dot{\varphi}(t))\,dt\}$$

where

$$\begin{aligned}\tilde{H}(t,x,v) = \|Z(t,x) - v\|^2_{g(t)} &+ \frac{1}{2}\operatorname{div}_{g(t)}(Z)(t,x) - \frac{1}{12}R_{g(t)}(x) \\ &+ \frac{1}{4}f^{-2}(t)\operatorname{trace}_{g(t)}(\dot{g}(t)) - \frac{1}{2}n(f'(t)f^{-3}(t)).\end{aligned}$$

Proof. Let $\tilde{g}(t) = \frac{1}{f^2(t)}g(t)$, and let $\tilde{d}(t,.,.)$ be the associated distance. Then the probability we wish to estimate is

$$\mathbb{P}_{x_0}[\forall t \in [0,T] \quad \tilde{d}(t, X_t, \varphi(t)) \leq \epsilon].$$

Now after a change of time we will turn the L_t diffusion X into a $\tilde{L}_t$ diffusion, in order to apply Theorem 1.1. Let us define

$$\delta(t) = \left(\int_0^{\cdot} \frac{1}{f^2(s)}\,ds\right)^{-1}(t),$$

and let $\tilde{X}(t) := X_{\delta(t)}$. Then $\tilde{X}$ is a $\tilde{L}_t$ diffusion, where

$$\tilde{L}_t := \frac{1}{2}\Delta_{\tilde{g}(\delta(t))} + f^2(\delta(t))Z(\delta(t),.).$$

We deduce that:

$$\begin{aligned}&\mathbb{P}_{x_0}[\forall t \in [0,T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)] \\ &= \mathbb{P}_{x_0}[\forall t \in [0,T] \quad \tilde{d}(t, X_t, \varphi(t)) \leq \epsilon] \\ &= \mathbb{P}_{x_0}[\forall t \in [0,\delta^{-1}(T)] \quad \tilde{d}(\delta(t), \tilde{X}_t, \varphi(\delta(t))) \leq \epsilon] \\ &\sim_{\epsilon\downarrow 0} C\exp\{-\frac{\lambda_1\delta^{-1}(T)}{\epsilon^2}\}\exp\{-\int_0^{\delta^{-1}(T)} H(\delta(t), \varphi(\delta(t)), \dot{\delta}(t)\dot{\varphi}(\delta(t))\,dt\}.\end{aligned}$$

In the last line, we have used Theorem 1.1, and the Lagrangian H related to the diffusion $\tilde{X}$. After a change of variables we get the result. $\square$

Corollary 4.5.

$$\epsilon^2 \log\{\mathbb{P}_{x_0}[\forall t \in [0,T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)]\} \to_{\epsilon\to 0} -\lambda_1 \int_0^T \frac{1}{f^2(s)}\, ds.$$

References

[1] Mireille Capitaine. On the Onsager-Machlup functional for elliptic diffusion processes. In *Séminaire de Probabilités, XXXIV*, volume 1729 of *Lecture Notes in Math.*, pages 313–328. Springer, Berlin, 2000.

[2] A. K. Coulibaly-Pasquier. Brownian motion with respect to time-changing riemannian metrics, applications to Ricci flow. Ann. Inst. Henri Poincaré Probab. Stat. 47 (2011), no. 2, 515–538.

[3] A. K. Coulibaly-Pasquier. Some stochastic process without birth, linked to the mean curvature flow. Ann. Probab. 39 (2011), no. 4, 1305–1331.

[4] Keisuke Hara and Yoichiro Takahashi. Lagrangian for pinned diffusion process. In *Itô's stochastic calculus and probability theory*, pages 117–128. Springer, Tokyo, 1996.

[5] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.

[6] John Lott. Optimal transport and Perelman's reduced volume. arXiv:0804.0343v2.

[7] Y. Takahashi and S. Watanabe. The probability functionals (Onsager-Machlup functions) of diffusion processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 433–463. Springer, Berlin, 1981.

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