A TOPOLOGICAL OBSTRUCTION TO THE CONTROLLABILITY OF NONLINEAR WAVE EQUATIONS WITH BILINEAR CONTROL TERM

THOMAS CHAMBRION AND LAURENT THOMANN

Abstract. In this paper we prove that the Ball-Marsden-Slemrod controllability obstruction also holds for nonlinear equations, with L^1 bilinear controls. We first show an abstract result and then we apply it to nonlinear wave equations. The first application to the sine-Gordon equation directly follows from the abstract result, and the second application concerns the cubic wave/Klein-Gordon equation and needs some additional work.

Contents

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1. Introduction and main result

1.1. **Introduction.** Evolution equations with a bilinear control term are often used to model the dynamics of a system driven by an external field (for instance, a quantum system driven by an electric field). In view of their importance, very few satisfactory descriptions of the attainable sets of such systems are available (among the rare exceptions, see Beauchard [3] for the case of the linear Schrödinger equation on a 1D compact domain or [4] for the linear wave equation on a 1D compact domain). For an overview of controllability results of bilinear control systems, we refer to Khapalov [9].

Roughly speaking, the attainable set for such systems does not coincide with the natural functional space where the system is defined. An explanation was provided by a celebrated article by Ball, Marsden and Slemrod [2] who proved that the attainable set of the linear dynamics with a bounded bilinear control using L^r, $r > 1$ real valued controls, is contained in a countable union of compact sets. This result has been adapted to the case of the Schrödinger equation by Turinici [11]. For partial differential equations posed in an infinite dimensional Banach space, this represents a strong topological obstruction to controllability (since the attainable set has hence empty interior by the Baire theorem). The proof heavily relies on the reflectiveness of L^r, $r > 1$ and could not be directly extended to L^1 controls.

Boussaïd, Caponigro and Chambrion [6] recently extended this obstruction to the case of L^1 (and even Radon measures) controls by considering the Dyson expansion of the solution. We show here that this technique can be adapted to the case of some nonlinear wave equations. This shows in particular that the nonlinear term does not help to control the equation in its natural energy space.

We consider the following abstract control system

$$\begin{cases} \psi'(t) = A\psi(t) + u(t)B\psi(t) + K(\psi(t)), \\ \psi(0) = \psi_0 \in \mathcal{X}, \end{cases} \tag{1.1}$$

with real valued controls $u : \mathbf{R} \to \mathbf{R}$ and with the following assumptions.

Assumption 1.1. The element $(\mathcal{X}, A, B, K)$ satisfies

- (i) $\mathcal{X}$ is Banach space endowed with norm $\|\cdot\|_{\mathcal{X}}$.
- (ii) $A : D(A) \to \mathcal{X}$ is a linear operator with domain $D(A) \subset \mathcal{X}$ that generates a C^0 semi-group of bounded linear operators. We denote by $\omega \geq 0$ and $M > 0$ two numbers such that $\|e^{tA}\|_{L(\mathcal{X},\mathcal{X})} \leq Me^{\omega t}$ for every $t \geq 0$.
- (iii) $B : \mathcal{X} \to \mathcal{X}$ is a linear bounded operator.
- (iv) $K : \mathcal{X} \to \mathcal{X}$ is k-Lipschitz-continuous (not necessarily linear), with $k > 0$.

In the sequel, the equation (1.1) is interpreted in its mild form, namely, we say that a function $\psi : [0, T] \to \mathcal{X}$ is a solution of (1.1) if, for every t in $[0, T]$,

$$\psi(t) = e^{tA}\psi_0 + \int_0^t u(s)e^{(t-s)A}B\psi(s)\mathrm{d}s + \int_0^t e^{(t-s)A}K(\psi(s))\mathrm{d}s. \tag{1.2}$$

Equation (1.2) is often called Duhamel formula.

1.2. **Notations.** Throughout the paper, for the sake of readability, we omit the range in the notation of spaces of real-valued functions. For instance, if X is a space, $H^k(X)$ denotes the set of H^k regular real functions on X.

In a metric space X endowed with distance d_X, we define the ball centered in $x \in X$ with radius $r > 0$ by $B_X(x, r) = \{y \in X | d_X(x, y) < r\}$. If X is a vector space endowed with norm $\|\cdot\|_X$, the distance associated with the norm is denoted d_X: $d_X(x, y) = \|x - y\|_X$, for every x, y in X.

1.3. **Main result.** Under Assumption 1.1, one can show that equation (1.1) admits a global flow Φ^u (see Propositions 2.2 and 2.3). Our main result concerning the control of (1.1) gives a description of the attainable set and reads as follows

Theorem 1.2. *Let $(\mathcal{X}, A, B, K)$ satisfy Assumption 1.1. Then, for every ψ_0 in $\mathcal{X}$, the attainable set from ψ_0 of (1.1) with controls u in $L^1([0,+\infty))$, $\bigcup_{t\geq 0}\bigcup_{u\in L^1([0,t])}\left\{\Phi^u(t)\psi_0\right\}$, is contained in a countable union of compact subsets of $\mathcal{X}$.*

This result gives a definite obstruction to the controllability of (1.1) in a general setting, since it shows that the attainable set is meager in the sense of Baire. However, as noted by Beauchard and Laurent in [5, Section 1.4.1], this result does not exclude exact controllability in a smaller space, endowed with a stronger norm (for which the operator B is no longer continuous). In this sense, this obstruction to controllability may be seen as an unfortunate choice of the ambient space.

The proof of Theorem 1.2 relies on the description of the solutions of (1.1) by series, called Dyson expansion (see Section 2). This strategy has been successfully carried out for the case $K = 0$ (linear dynamics) in [6, Section 5.1], and we show here that it can also be applied to nonlinear problems. For more details on Dyson expansions, we refer to [10, Theorem X.69 and equation (X.129)].

In the assumptions of the Theorem 1.2, the fact that K is Lipschitz is needed in order to ensure the existence of a global flow of (1.1), but in the core of the proof of our result we only need that K is continuous (see Proposition 2.6).

We provide two explicit applications of Theorem 1.2 to nonlinear wave equations. We first give the example of the sine-Gordon equation, which exactly matches Assumption 1.1 and to which Theorem 1.2 directly applies. Then, by means of the 3-dimensional cubic Klein-Gordon equation, we show that the hypothesis "K is Lipschitz" can be relaxed. Actually, for the nonlinear wave equation (see Section 3.2), the gain of derivative in the Duhamel formula allows one to bound the nonlinearity using Sobolev estimates, and the global existence of a flow can be obtained by energy estimates.

We are also able to obtain negative controllability results for the nonlinear Schrödinger equation, and this will be treated in our forthcoming paper [7].

Remark 1.3. By rather simple modifications, the result of Theorem 1.2 can be extended to the case of the equation

$$\psi'(t) = A\psi(t) + \sum_{j=1}^{n} u_j(t)B_j\psi(t) + \alpha(t)K(\psi(t)),$$

with the same assumptions on the controls $u_j \in L^1([0,+\infty))$ and with $\alpha \in L^1([0,+\infty))$ being given. Such models are relevant in some physical contexts (e.g. the Schrödinger equation with electric and magnetic fields combined with coupling to the environment in the spirit of [8]), but we omit the details to simplify the presentation.

2. Ball-Marsden-Slemrod obstructions for nonlinear equations

2.1. **Dyson expansion of the solutions.** Let $T > 0$ and u be given in $L^1([0,T])$. Define by induction on $p \geq 0$,

$$\begin{cases} Y^u_{0,t}\psi_0 = 0 \\ Y^u_{p+1,t}\psi_0 = e^{tA}\psi_0 + \displaystyle\int_0^t e^{(t-s)A}\left[u(s)BY^u_{p,s}\psi_0 + K(Y^u_{p,s}\psi_0)\right]\mathrm{d}s \end{cases} \tag{2.1}$$

and $Z^u_{p,t}\psi_0 = Y^u_{p+1,t}\psi_0 - Y^u_{p,t}\psi_0$.

We aim to show that the series $(\sum_p Z^u_{p,t}\psi_0)$ converges. To this end, we need some quantitative bounds, which are stated in the next result.

Proposition 2.1. *For every j in $\mathbf{N}$, every $t>0$ and every u in $L^1([0,+\infty))$,*

$$\|Z^u_{j,t}\psi\|_{\mathcal{X}} \leq \frac{e^{\omega t}M^{j+1}\left(kt+\|B\|_{L(\mathcal{X},\mathcal{X})}\int_0^t|u(s)|\mathrm{d}s\right)^j}{j!}\|\psi\|_{\mathcal{X}}. \tag{2.2}$$

Proof. We proceed by induction on $j\geq 0$. Inequality (2.2) for $j=0$ follows from Assumption 1.1(*ii*). Assume now that we have proved (2.2) for a given j. Then, since

$$\begin{aligned}
\|Z^u_{j+1,t}\psi\|_{\mathcal{X}} &\leq \int_0^t Me^{\omega(t-s)}\left(k+|u(s)|\|B\|_{L(\mathcal{X},\mathcal{X})}\right)\|Z^u_{j,s}\psi\|_{\mathcal{X}}\mathrm{d}s\\
&\leq \frac{M^{j+2}}{j!}e^{\omega t}\left[\int_0^t\left(k+|u(s)|\|B\|_{L(\mathcal{X},\mathcal{X})}\right)\left(ks+\|B\|_{L(\mathcal{X},\mathcal{X})}\int_0^s|u(\tau)|\mathrm{d}\tau\right)^j\mathrm{d}s\right]\|\psi\|_{\mathcal{X}}\\
&\leq \frac{M^{j+2}}{(j+1)!}e^{\omega t}\left(kt+\|B\|_{L(\mathcal{X},\mathcal{X})}\int_0^t|u(s)|\mathrm{d}s\right)^{j+1}\|\psi\|_{\mathcal{X}},
\end{aligned}$$

which concludes the proof. □

From Proposition 2.1, for every t in $[0,T]$ and every ψ in $\mathcal{X}$, the sum $\sum_j Z^u_{j,t}\psi$ converges in $\mathcal{X}$. We denote this sum by $Y^u_{\infty,t}\psi$:

$$Y^u_{\infty,t}\psi=\sum_{j=0}^{+\infty}Z^u_{j,t}\psi$$

Proposition 2.2. *For every ψ in $\mathcal{X}$, every $T>0$ and every u in $L^1([0,+\infty),\mathbf{R})$, the function $(t,\psi)\mapsto Y^u_{\infty,t}\psi$ is continuous from $\mathbf{R}\times\mathcal{X}$ to $\mathcal{X}$.*

Proof. This follows from the continuity of the functions $(t,\psi)\mapsto Z^u_{j,t}\psi$ for every $j\geq 0$ and from the convergence of $\sum_j Z^u_{j,t}\psi$ (locally uniform in t and ψ) from Proposition 2.1. □

Proposition 2.3. *For every $T\in[0,+\infty)$, every u in $L^1([0,T],\mathbf{R})$ and every ψ_0 in $\mathcal{X}$, $t\mapsto Y^u_{\infty,t}\psi_0$ is the unique mild solution on $[0,T]$ of (1.1) taking value ψ_0 at 0.*

Proof. The mapping

$$\begin{array}{rccc}
F: & C^0([0,T],\mathcal{X}) & \longrightarrow & C^0([0,T],\mathcal{X})\\
 & \left(t\mapsto\psi(t)\right) & \longmapsto & \left(t\mapsto e^{tA}\psi_0+\int_0^t e^{(t-s)A}\left[u(s)B\psi+K(\psi)\right]\mathrm{d}s\right)
\end{array}$$

is continuous for the norm $L^\infty([0,T],\mathcal{X})$. By (2.1), $t\mapsto Y^u_{\infty,t}\psi_0$ is a fixed point of F, hence a mild solution on $[0,T]$ of (1.1) taking value ψ_0 at 0.

Assume that $t\mapsto\psi_1(t)$ and $t\mapsto\psi_2(t)$ are two mild solutions on $[0,T]$ of (1.1) taking value ψ_0 at 0. Define $T^*=\sup_{t\in[0,T]}\left\{t\mid\psi_1(s)=\psi_2(s),\text{ for almost every }s\leq t\right\}$. We will prove by contradiction that $T^*=T$, that is, $\psi_1=\psi_2$ almost everywhere. Assume that $T^*<T$. We chose $t_1\in(T^*,T]$ such that

$$Me^{(t_1-T^*)\omega}\left(k(t_1-T^*)+\|u\|_{L^1([T^*,t_1],\mathbf{R})}\|B\|_{L(\mathcal{X},\mathcal{X})}\right):=C_0<1.$$

Then, for all $T^* \leq t_2 \leq t_1$

$$\begin{aligned}
&\|\psi_2(t_2) - \psi_1(t_2)\|_{\mathcal{X}} \\
&= \left\| \int_{T^*}^{t_2} u(s) e^{(t_2-s)A} B(\psi_2(s) - \psi_1(s)) \mathrm{d}s + \int_{T^*}^{t_2} e^{(t_2-s)A} \Big(K(\psi_2(s)) - K(\psi_1(s)) \Big) \mathrm{d}s \right\|_{\mathcal{X}} \\
&\leq \int_{T^*}^{t_2} |u(s)| M e^{(t_2-s)\omega} \|B\|_{L(\mathcal{X},\mathcal{X})} \|\psi_2(s) - \psi_1(s)\|_{\mathcal{X}} \mathrm{d}s + \int_{T^*}^{t_2} M e^{(t_2-s)\omega} k \|\psi_2(s) - \psi_1(s)\|_{\mathcal{X}} \mathrm{d}s \\
&\leq \|\psi_2 - \psi_1\|_{L^\infty([T^*,t_1),\mathcal{X})} \int_{T^*}^{t_1} \left(k + |u(s)| \|B\|_{L(\mathcal{X},\mathcal{X})} \right) M e^{(t_1-s)\omega} \mathrm{d}s \\
&\leq C_0 \|\psi_2 - \psi_1\|_{L^\infty([T^*,t_1),\mathcal{X})},
\end{aligned}$$

therefore we deduce that

$$\|\psi_2 - \psi_1\|_{L^\infty([T^*,t_1),\mathcal{X})} \leq C_0 \|\psi_2 - \psi_1\|_{L^\infty([T^*,t_1),\mathcal{X})},$$

which gives the desired contradiction. To conclude the proof, it remains to show that any mild solution is continuous (since two continuous functions coincide as soon as they are equal almost everywhere). And indeed, any mild solution of (1.1) is equal almost everywhere to Y^u_∞, which is continuous (Proposition 2.2), hence any mild solution of (1.1) is essentially bounded and then is continuous by its definition (1.2). □

Definition 2.4. Let $T > 0$, u in $L^1([0,+\infty), \mathbf{R})$ and ψ_0 in $\mathcal{X}$. In the following, we denote by $t \mapsto \Phi^u(t)\psi_0$ the mild solution of system (1.1) associated with the initial condition ψ_0 and the control u in $L^1([0,T))$.

We sum up the above results in the following

Proposition 2.5 (Dyson expansion of the solutions of (1.2))**.** *Let $t > 0$, u in $L^1([0,+\infty), \mathbf{R})$ and ψ_0 in $\mathcal{X}$. Then*

$$\Phi^u(t)\psi_0 = \sum_{j=0}^{\infty} Z^u_{j,t}(\psi_0). \tag{2.3}$$

2.2. **A compactness result.** Recall that $Y^u_{j,t}\psi_0$ is defined in (2.1) and that $Z^u_{j,t}\psi_0 = Y^u_{j+1,t}\psi_0 - Y^u_{j,t}\psi_0$.

Proposition 2.6. *For every j in $\mathbf{N}$, $T \geq 0$ and $L \geq 0$, and ψ_0 in $\mathcal{X}$, the sets*

$$\mathcal{Z}_j^{T,L} = \left\{ Z^u_{j,t}\psi_0 \mid 0 \leq t \leq T, \|u\|_{L^1(0,T)} \leq L \right\} \text{ and } \mathcal{Y}_j^{T,L} = \left\{ Y^u_{j,t}\psi_0 \mid 0 \leq t \leq T, \|u\|_{L^1(0,T)} \leq L \right\}$$

are relatively compact in $\mathcal{X}$.

Proof. We adapt the proof of [6] (valid for $K = 0$) to the general case of a continuous function K.

Since a finite sum of relatively compact sets is still relatively compact, it is enough to prove the result for $\mathcal{Y}_j^{T,L}$. We prove this by induction on $j \geq 0$.

For $j = 0$, the result is clear.

Assume that $\mathcal{Y}_j^{T,L}$ is relatively compact in $\mathcal{X}$ for some $j \geq 0$. We aim to prove that $\mathcal{Y}_{j+1}^{T,L}$ is relatively compact in $\mathcal{X}$ as well. For this, we chose $\varepsilon > 0$ and we try to exhibit an ε-net of $\mathcal{Y}_{j+1}^{T,L}$.

Since the mappings

$$\begin{array}{rccc} G_1 : & [0,T] \times \mathcal{X} & \longrightarrow & \mathcal{X} \\ & (s,\psi) & \longmapsto & e^{(T-s)A} B\psi \end{array} \quad \text{and} \quad \begin{array}{rccc} G_2 : & [0,T] \times \mathcal{X} & \longrightarrow & \mathcal{X} \\ & (s,\psi) & \longmapsto & e^{(T-s)A} K(\psi) \end{array}$$

are continuous, the sets $G_1([0,T]\times\mathcal{Y}_j^{T,L})$ and $G_2([0,T]\times\mathcal{Y}_j^{T,L})$ are relatively compact. Hence, there exists a finite family $(x_i)_{1\leq i\leq N}$ such that, for $\ell=1,2$,

$$G_\ell([0,T]\times\mathcal{Y}_j^{T,L})\subset\bigcup_{i=1}^N B_{\mathcal{X}}\left(x_i,\frac{\varepsilon}{4(L+T)}\right).$$

Let $(\varphi_i)_{1\leq i\leq N}$ be a partition of unity associated with the covering of $G_\ell([0,T]\times\mathcal{Y}_j^{T,L})$, $\ell=1,2$. That is, the functions φ_i satisfy $0\leq\varphi_i\leq 1$ and, for every x in $G_\ell([0,T]\times\mathcal{Y}_j^{T,L})$, $\sum_{i=1}^N\varphi_i(x)=1$ and $\Big\|x-\sum_{i=1}^N\varphi_i(x)x_i\Big\|_{\mathcal{X}}<\frac{\varepsilon}{2(L+T)}$.

Then, for every u in $L^1([0,T],\mathbf{R})$ such that $\|u\|_{L^1(0,T)}\leq L$,

$$\Big\|\int_0^t u(s)e^{(t-s)A}(BY^u_{j,s}\psi_0)\mathrm{d}s-\sum_{i=1}^N\int_0^t u(s)\varphi_i\big(e^{(t-s)A}(BY^u_{j,s}\psi_0\big)x_i\mathrm{d}s\Big\|_{\mathcal{X}}\leq\frac{L\varepsilon}{2(L+T)},$$

and

$$\Big\|\int_0^t e^{(t-s)A}K(Y^u_{j,s}\psi_0)\mathrm{d}s-\sum_{i=1}^N\int_0^t\varphi_i(e^{(t-s)A}K(Y^u_{j,s}\psi_0))x_i\mathrm{d}s\Big\|_{\mathcal{X}}\leq\frac{T\varepsilon}{2(L+T)}.$$

Now using that the compact sets $\sum_{i=1}^N[0,L]x_i$ and $\sum_{i=1}^N[0,T]x_i$ admit a $\varepsilon/4$-net $(y_i)_{1\leq i\leq N_2}$, and the previous estimates, we get $\mathcal{Y}_{j+1}^{T,L}\subset\bigcup_{i=1}^{N_2}B_{\mathcal{X}}(y_i,\varepsilon)$, which concludes the proof. □

Remark 2.7. In the proof of Proposition 2.6, we only used the continuity of K. In this paper, we assume that K is Lipschitz continuous in order to ensure the global existence of a flow of (1.2) and the Dyson expansion (2.3). In Section 3.2, we will show that our approach applies to more general nonlinearities, which are only locally (not globally) Lipschitz continuous.

2.3. **Proof of the nonlinear Ball-Marsden-Slemrod obstructions.** We are now able to complete the proof of Theorem 1.2.

For $T>0$ and $L>0$ define

$$\mathcal{V}^{T,L}=\big\{\Phi^u(t)\psi_0\mid u\in L^1([0,T]),\|u\|_{L^1([0,T])}\leq L,0\leq t\leq T\big\},$$

and notice that

$$\bigcup_{t\geq0}\bigcup_{u\in L^1}\{\Phi^u(t)\psi_0\}=\bigcup_{T\in\mathbf{N}}\bigcup_{L\in\mathbf{N}}\mathcal{V}^{T,L}.$$

Thus it is enough to prove that, for every $T>0$ and every $L>0$, the set $\mathcal{V}^{T,L}$ is relatively compact.

Let $\delta>0$ be given. We aim to find a δ-net of $\mathcal{V}^{T,L}$.

From Propositions 2.1 and 2.5, since $\Big\|\sum_{j=N}^\infty Z^u_{j,T}(\psi_0)\Big\|_{\mathcal{X}}$ tends to zero as N tends to infinity uniformly with respect to u in $B_{L^1([0,T],\mathbf{R})}(0,L)$, there exists N_1 large enough such that, for every u in $B_{L^1([0,T],\mathbf{R})}(0,L)$,

$$\Big\|\sum_{j=N_1}^\infty Z^u_{j,T}(\psi_0)\Big\|_{\mathcal{X}}<\frac{\delta}{2}.$$

The set $\mathcal{Y}_{N_1}^{T,L}$ is relatively compact (Proposition 2.6); hence it admits a $\delta/2$-net.

Thus
$$\mathcal{V}^{T,L} \subset \left\{ x \in \mathcal{X} \,|\, d_{\mathcal{X}}\left(x, \mathcal{Y}^{T,L}_{N_1}\right) \le \frac{\delta}{2} \right\}$$
admits a δ-net, which finishes the proof. □

3. Applications

3.1. The sine-Gordon equation. We consider the sine-Gordon equation which reads

$$\begin{cases} \partial_t^2 \psi - \partial_x^2 \psi = u(t)B(x)\psi - \sin\psi, & (t,x) \in \mathbf{R}\times\mathbf{R}, \\ \psi(0,.) = \psi_0 \in H^1(\mathbf{R}), \\ \partial_t\psi(0,.) = \psi_1 \in L^2(\mathbf{R}), \end{cases} \tag{3.1}$$

where B is a given function, and $u \in L^1_{loc}(\mathbf{R})$ is the control. In the case $B \equiv 0$, this equation appears in relativistic field theory or in the study of mechanical transmission lines. We rewrite this equation as a first order (in time) system, so that it fits the framework of our study. Equation (3.1) is equivalent to

$$\begin{cases} \partial_t \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} + u(t)B(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ -\sin\psi \end{pmatrix}, & (t,x) \in \mathbf{R}\times\mathbf{R}, \\ (\psi(0,.), \varphi(0,.)) = (\psi_0, \psi_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R}). \end{cases}$$

Then Theorem 1.2 directly applies with $\mathcal{X} = H^1(\mathbf{R}) \times L^2(\mathbf{R})$, $A = \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}$, $D(A) = H^2(\mathbf{R}) \times H^1(\mathbf{R})$, $B \in L^\infty(\mathbf{R})$ and $K(\psi,\varphi) = (0; -\sin(\psi))$.

3.2. The wave equation in dimension 3. The result of Theorem 1.2 also applies to nonlinear equations, with local Lipschitz nonlinear terms. We develop here the examples of the wave and Klein-Gordon equations. Denote by $\mathcal{M}$ a compact manifold of dimension 3 without boundary, or $\mathcal{M} = \mathbf{R}^3$. We consider the defocusing cubic wave equation

$$\begin{cases} \partial_t^2 \psi - \Delta\psi + m\psi = u(t)B(x)\partial_t\psi - \psi^3, & (t,x) \in \mathbf{R}\times\mathcal{M}, \\ \psi(0,.) = \psi_0 \in H^1(\mathcal{M}), \\ \partial_t\psi(0,.) = \psi_1 \in L^2(\mathcal{M}), \end{cases} \tag{3.2}$$

with $m \ge 0$ and $B \in L^\infty(\mathcal{M})$. Positive exact controllability results for such non-linear dynamics in the case $\mathcal{M} = (0,1)$ were obtained by Beauchard [4, Theorem 1].

Let the control function u be in $L^1_{loc}(\mathbf{R})$; the mild solution reads

$$\psi(t) = S_0(t)\psi_0 + S_1(t)\psi_1 + \int_0^t S_1(t-s)\big(u(s)B(x)\partial_s\psi(s) - \psi^3(s)\big)ds$$

where

$$S_0(t) = \cos(t\sqrt{-\Delta+m}) \quad \text{and} \quad S_1(t) = \frac{\sin(t\sqrt{-\Delta+m})}{\sqrt{-\Delta+m}}. \tag{3.3}$$

3.2.1. *The obstruction result for controllability of the wave equation.* We state the main result of this section, which is the analogue of Theorem 1.2 for equation (3.2).

Theorem 3.1. *For all $(\psi_0,\psi_1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$ and $u \in L^1(\mathbf{R})$, there exists a unique solution to* (3.2)
$$\psi \in \mathcal{C}^0\big(\mathbf{R}; H^1(\mathcal{M})\big) \cap \mathcal{C}^1\big(\mathbf{R}; L^2(\mathcal{M})\big).$$

This enables us to define a global flow

$$\Phi=(\Phi_1,\Phi_2): \begin{array}{ccc} H^1(\mathcal{M})\times L^2(\mathcal{M})\times L^1(\mathbf{R}) & \longrightarrow & \mathcal{C}^0\big(\mathbf{R};H^1(\mathcal{M})\big)\times\mathcal{C}^0(\mathbf{R};L^2(\mathcal{M})) \\ (\psi_0,\psi_1,u) & \longmapsto & (\psi,\psi_t) \end{array}.$$

Moreover, for every $(\psi_0,\psi_1)\in H^1(\mathcal{M})\times L^2(\mathcal{M})$, the attainable set

$$\bigcup_{t\in\mathbf{R}}\bigcup_{u\in L^1}\big\{\Phi^u(t)(\psi_0,\psi_1)\big\}$$

is contained in a countable union of compact subsets of $H^1(\mathcal{M})\times L^2(\mathcal{M})$.

While we decided to illustrate our method for the equation (3.2), our approach can be applied to other wave-type equations, such as

$$\partial_t^2\psi-\Delta\psi+m\psi=u(t)B(x)\psi-\psi^3,$$

with a given potential $B\in L^3(\mathcal{M})$. We omit the details.

3.2.2. *Local and global existence results.* Since equation (3.2) is reversible, in the sequel, we ony consider non-negative times. Let $T>0$, $u\in L^1([0,T])$ and $t_0\geq 0$ be given. We define by induction on $p\geq 0$,

$$\begin{cases} \widetilde{Y}^u_{0,t,t_0}=0 \\ \widetilde{Y}^u_{p+1,t,t_0}(\psi_0,\psi_1)=S_0(t)\psi(t_0)+S_1(t)\partial_t\psi(t_0)+\displaystyle\int_0^t S_1(t-s)\Big[u(s+t_0)B(x)\partial_s\widetilde{Y}^u_{p,s,t_0}-(\widetilde{Y}^u_{p,s,t_0})^3\Big]\,\mathrm{d}s \end{cases}$$

with $\widetilde{Y}^u_{p,s,t_0}=\widetilde{Y}^u_{p,s,t_0}(\psi_0,\psi_1)$, and where S_0 and S_1 are defined in (3.3).

We now state a global existence result, which is an application of the Picard fixed point theorem.

Proposition 3.2. (*i*) *For all $(\psi_0,\psi_1)\in H^1(\mathcal{M})\times L^2(\mathcal{M})$ there exists a unique solution to* (3.2)

$$\psi\in\mathcal{C}^0\big(\mathbf{R};H^1(\mathcal{M})\big)\cap\mathcal{C}^1\big(\mathbf{R};L^2(\mathcal{M})\big).$$

(*ii*) *Moreover, for all $T>0$, for all $L>0$ and u such that $\int_0^T|u(s)|ds\leq L$,*

$$\sup_{0\leq t\leq T}\|(\psi,\partial_t\psi)(t)\|_{H^1(\mathcal{M})\times L^2(\mathcal{M})}\leq C\left(\|\psi_0\|_{H^1},\|\psi_1\|_{L^2},L,T\right),$$

where C is a continuous function.

(*iii*) *Furthermore, for all $T>0$, and $L>0$, there exist $k\geq 1$, $0<c_0<1$ and a continuous function $\tau=\tau(\|\psi_0\|_{H^1},\|\psi_1\|_{L^2},L,T)>0$ such that, for all $0\leq t_0\leq T$, $p\geq 0$ and u with $\int_0^T|u(s)|ds\leq L$,*

$$\sup_{t\in[0,\tau]}\|\big(\psi(t+t_0)-\widetilde{Y}^u_{kp,t,t_0},\partial_t\psi(t+t_0)-\partial_t\widetilde{Y}^u_{kp,t,t_0}\big)\|_{H^1(\mathcal{M})\times L^2(\mathcal{M})}\leq Cc_0^p. \tag{3.4}$$

In the previous result, it is crucial that we obtain a time $\tau=\tau(\|\psi_0\|_{H^1},\|\psi_1\|_{L^2},L,T)$ which only depends on the norms of ψ_0, ψ_1 and u (and not ψ_0, ψ_1 or u themselves). This fact will be used in the compactness argument (see Section 3.2.3).

Proof. A first local existence result: Let $t_0\geq 0$. We prove a local in time existence result for the problem

$$\begin{cases} \partial_t^2\widetilde{\psi}-\Delta\widetilde{\psi}+m\widetilde{\psi}=u(t)B(x)\partial_t\widetilde{\psi}-\widetilde{\psi}^3, & (t,x)\in\mathbf{R}\times\mathcal{M}, \\ \widetilde{\psi}(t_0,.)=\widetilde{\psi}_0\in H^1(\mathcal{M}), \\ \partial_t\widetilde{\psi}(t_0,.)=\widetilde{\psi}_1\in L^2(\mathcal{M}). \end{cases} \tag{3.5}$$

We consider the map

$$F(\psi)(t) = S_0(t)\widetilde{\psi}_0 + S_1(t)\widetilde{\psi}_1 + \int_0^t S_1(t-s)\left[u(s+t_0)B(x)\partial_s\psi(s) - (\psi(s))^3\right] \mathrm{d}s,$$

and we will show that, for $t > 0$ small enough, it is a contraction in some Banach space. Then, by the Picard theorem, there will exist a unique fixed point ψ, and $\widetilde{\psi}(t) = \psi(t-t_0)$ will be the unique solution to (3.5).

We define the norm $\|\psi\|_T = \|\psi\|_{L^\infty_T H^1} + \|\partial_t\psi\|_{L^\infty_T L^2}$ and the space

$$X_{T,R} = \{\|\psi\|_T \le R\},$$

with $R > 0$ and $T > 0$ to be determined.

By the Sobolev embedding $H^1(\mathcal{M}) \subset L^6(\mathcal{M})$ (see Proposition (A.1) with $p = 2$ and $n = 3$), there exists $c = c(m,T) > 0$ such that

$$\begin{aligned} \|F(\psi)\|_T &\le 2(\|\widetilde{\psi}_0\|_{H^1} + \|\widetilde{\psi}_1\|_{L^2}) + c\int_0^T \left(\|u(s+t_0)B\partial_s\psi\|_{L^2} + \|\psi(s)\|^3_{L^6}\right)ds \\ &\le 2(\|\widetilde{\psi}_0\|_{H^1} + \|\widetilde{\psi}_1\|_{L^2}) + c\Big(\int_0^T |u(s+t_0)|ds\Big)\big\|B\big\|_{L^\infty}\big\|\partial_t\psi\big\|_{L^\infty_T L^2} + cT\|\psi\|^3_{L^\infty_T H^1}. \end{aligned} \tag{3.6}$$

Let us set $R = 4(\|\widetilde{\psi}_0\|_{H^1} + \|\widetilde{\psi}_1\|_{L^2})$. Then we choose $T_1 = c_1 R^{-2}$ with $c_1 > 0$ small enough such that $cT_1R^2 \le 1/4$ and we choose $T_2 > 0$ such that $c\int_0^{T_2} |u(s+t_0)|ds \le \big\|B\big\|^{-1}_{L^\infty}/4$. Therefore, for $T = \min(T_1, T_2)$, F maps $X_{T,R}$ into itself. With similar estimates we can show that F is a contraction in $X_{T,R}$, namely

$$\|F(\psi_1) - F(\psi_2)\|_T \le \left[cTR^2 + c\Big(\int_0^T |u(s+t_0)|ds\Big)\big\|B\big\|_{L^\infty}\right]\|\psi_1 - \psi_2\|_T.$$

As a consequence, there exists a unique, local in time solution to (3.5), with the time of existence, τ, depending on the norms of $\widetilde{\psi}_0$, $\widetilde{\psi}_1$ and u.

Energy bound: We define

$$E(\psi)(t) = \frac{1}{2}\int_{\mathcal{M}} \left((\partial_t\psi)^2 + |\nabla\psi|^2 + m\psi^2\right) + \frac{1}{4}\int_{\mathcal{M}} \psi^4.$$

By differentiation with respect to time, we get

$$\begin{aligned} \frac{d}{dt}E(\psi)(t) &= \int_{\mathcal{M}} \partial_t\psi\left(\partial_t^2\psi - \Delta\psi + m\psi + \psi^3\right)dx \\ &= u(t)\int_{\mathcal{M}} \partial_t\psi . B\partial_t\psi dx. \end{aligned}$$

Next, since $B \in L^\infty(\mathcal{M})$, we get

$$\begin{aligned} \frac{d}{dt}E(\psi)(t) &\le |u(t)|\|B\|_{L^\infty}\|\partial_t\psi\|^2_{L^2} \\ &\le C|u(t)|E(\psi)(t) \end{aligned}$$

which implies

$$E(\psi)(t) \le E(\psi)(0)e^{C\int_0^t |u(s)|ds}. \tag{3.7}$$

In the particular case $m = 0$, the energy E does not control the term $\int_{\mathcal{M}} \psi^2$, and we bound this latter term as follows. We set $M(\psi)(t) = \left(\int_{\mathcal{M}} \psi^2\right)^{1/2}$. Then

$$\frac{d}{dt}M(\psi)(t) \le \|\partial_t\psi\|_{L^2} \le 2E^{1/2}(\psi)(t),$$

and by integration in time together with (3.7) we obtain

$$E(\psi)(t)+\int_{\mathcal{M}}\psi^2\le C_0(t,\int_0^t|u(s)|ds,\|\psi_0\|_{H^1},\|\psi_1\|_{L^2}). \tag{3.8}$$

Proof of (i) *and* (ii)*:* Assume that one can solve (3.5) on $[0,T^\star)$, starting from $t_0=0$. By (3.8), there is a time $T_1^\star>0$ such that $cT_1^\star(R^\star)^2\le 1/4$ with $R^\star=4c(\|\psi\|_{L^\infty_{T^\star}H^1}+\|\partial_t\psi\|_{L^\infty_{T^\star}L^2})$. Then we choose $T_2^\star>0$ such that

$$c\left(\int_{T^\star-\frac{T_2^\star}{2}}^{T^\star+\frac{T_2^\star}{2}}|u(s)|ds\right)\big\|B\big\|_{L^\infty}\le 1/4.$$

As a consequence, using the arguments of the previous (local) step, we are able to solve the equation (3.5), with an initial condition at $t_0=T^\star-\min(T_1^\star,T_2^\star)/2$, on the time interval $[T^\star-\min(T_1^\star,T_2^\star)/2,T^\star+\min(T_1^\star,T_2^\star)/2]$. This shows that the maximal solution is global in time.

Proof of (iii)*:* To prove this last statement, we will find a time of existence which does not depend on $t_0\in[0,T]$ and which only depends on u through the quantity $\int_0^T|u(s)|ds$. Assume that $\int_0^T|u(s)|ds\le L$.

For $k\ge 0$, we denote by $F^k=F\circ F\circ\cdots\circ F$ the kth iterate of F. From (3.9) and (3.10) (see Lemma 3.3 below) we obtain the bounds (with $L=L(T)$)

$$\|F^k(\psi)\|_{T_1}\le C_k(L,\|\psi\|_0)+\frac{(CL)^k}{k!}\|\psi\|_{T_1}+T_1P_k(T_1,L,\|\psi\|_{T_1})$$

and

$$\|F^k(\psi)-F^k(\varphi)\|_{T_1}\le\Big[\frac{(CL)^k}{k!}+T_1Q_k(T_1,L,\|\psi\|_{T_1},\|\varphi\|_{T_1})\Big]\|\psi-\varphi\|_{T_1}.$$

Set $k\ge 0$ such that $\frac{(CL)^k}{k!}\le 1/2$. Let $R_1=\max\big(2C_k,C_0\big)$, where $C_k=C_k(L,\|\psi\|_0)$ is given in (3.9) and $C_0=C_0(T,L,\|\psi\|_0)$ is given in (3.8). Set

$$X_{T_1,R_1}=\big\{\|\varphi\|_{T_1}\le R_1\big\}.$$

Then from the two previous estimates we infer that $F^k:X_{T_1,R_1}\longrightarrow X_{T_1,R_1}$ is a contraction, provided that $T_1=T_1(L,R_1)$ is small enough. As a consequence, there exists a unique solution in X_{T_1,R_1} to the equation $\varphi=F^k(\varphi)$. However, it is not clear whether F does map X_{T_1,R_1} into X_{T_1,R_1}, and we cannot conclude directly that $\varphi=F(\varphi)$, in other words that φ satisfies (3.2). By the global well-posedness result, there exists a unique $\psi=F(\psi)$ for $t\in[0,T_1]$. Let us prove that $\varphi\equiv\psi$ on $[0,T_1]$. Observe that we have $\psi=F^k(\psi)$. To conclude the proof, by uniqueness of the fixed point of F^k in X_{T_1,R_1}, it is enough to check that $\psi\in X_{T_1,R_1}$. By (3.8), $\|\psi\|_{T_1}\le C_0(T,L,\|\psi\|_0)\le R_1$, hence the result.

Finally the bound (3.4) directly follows from the Picard iteration procedure, since

$$\widetilde{Y}^u_{k(p+1),t,t_0}(\psi_0,\psi_1)=F^k\big(\widetilde{Y}^u_{kp,t,t_0}(\psi_0,\psi_1)\big).$$

□

Recall that $\|\psi\|_T=\|\psi\|_{L^\infty_TH^1}+\|\partial_t\psi\|_{L^\infty_TL^2}$.

Lemma 3.3. *Let $0<T_1\le T$. For $0\le t\le T$, set $L(t)=\int_0^t|u(s)|ds$ and $L=L(T)$. Then there exists a constant $C>0$ such that for all $k\ge 0$ and $0\le t+t_0\le T$, there exist polynomials C_k, P_k and Q_k such that*

$$\|F^k(\psi)\|_t\le C_k(L,\|\psi\|_0)+\frac{\big(CL(t+t_0)\big)^k}{k!}\|\psi\|_{T_1}+T_1P_k(T_1,L,\|\psi\|_{T_1}) \tag{3.9}$$

and

$$\|F^k(\psi)-F^k(\varphi)\|_t \le \Big[\frac{\big(CL(t+t_0)\big)^k}{k!}+T_1Q_k(T_1,L,\|\psi\|_{T_1},\|\varphi\|_{T_1})\Big]\|\psi-\varphi\|_{T_1}. \tag{3.10}$$

Proof. Let us prove (3.9) by induction. For $k=0$ the result holds true. Let $k\ge 0$ so that we have (3.9). As in (3.6), we get

$$\|F^{k+1}(\psi)\|_t \le 2(\|\widetilde{\psi}_0\|_{H^1}+\|\widetilde{\psi}_1\|_{L^2})+c\big\|B\big\|_{L^\infty}\Big(\int_0^t |u(s+t_0)|\big\|F^k(\psi)\big\|_s ds\Big)+cT_1\|F^k(\psi)\|_{T_1}^3, \tag{3.11}$$

where $c>0$ is a universal constant. Moreover, by (3.8),

$$\|\widetilde{\psi}_0\|_{H^1}+\|\widetilde{\psi}_1\|_{L^2}\le D(L,\|\psi\|_0).$$

Next, by (3.9)

$$\begin{aligned}&\int_0^t |u(s+t_0)|\big\|F^k(\psi)\big\|_s ds \le\\ &\qquad\le C_k(L,\|\psi\|_0)L+\|\psi\|_{T_1}\int_0^t |u(s+t_0)|\frac{\big(CL(s+t_0)\big)^k}{k!}ds+T_1LP_k(T_1,L,\|\psi\|_{T_1})\\ &\qquad\le C_k(L,\|\psi\|_0)L+C^k\frac{\big(L(t+t_0)\big)^{k+1}}{(k+1)!}\|\psi\|_{T_1}+T_1LP_k(T_1,L,\|\psi\|_{T_1}).\end{aligned}$$

The term $\|F^k(\psi)\|_{T_1}^3$ is directly controlled by (3.9). Now we make the choice $C=c\big\|B\big\|_{L^\infty}$, and, thanks to (3.11) we get (3.9) for $k+1$.

The proof of (3.10) is similar and omitted. □

As in the abstract result, a major ingredient of the proof is a Dyson expansion of the form (2.3). However, since the nonlinearity is stronger than in our abstract result, the expansion only holds for finite times. Set

$$\widetilde{Z}^u_{p,t,t_0}(\psi_0,\psi_1):=\widetilde{Y}^u_{k(p+1),t,t_0}(\psi_0,\psi_1)-\widetilde{Y}^u_{kp,t,t_0}(\psi_0,\psi_1),$$

where $k\ge 0$ is given by the proof of Proposition 3.2.

Proposition 3.4. *Let $T>0$ and $u\in L^1([0,T],\mathbf{R})$ such that $\int_0^T|u(s)|ds\le L$. Consider $\tau=\tau(\|\psi_0\|_{H^1},\|\psi_1\|_{L^2},L,T)>0$ given by Proposition 3.2 (iii). Then for all $t\in[0,\tau]$*

$$\Phi^u(t+t_0)\big(\psi_0,\psi_1\big)=\Big(\sum_{j=0}^\infty \widetilde{Z}^u_{j,t,t_0}(\psi_0,\psi_1),\sum_{j=0}^\infty \partial_t\widetilde{Z}^u_{j,t,t_0}(\psi_0,\psi_1)\Big).$$

Proof. This result is a direct consequence of (3.4). □

3.2.3. *Proof of the compactness result.* We now proceed to the end of the proof of Theorem 3.1. For every $(\widetilde{\psi}_0,\widetilde{\psi}_1)$ in $H^1(\mathcal{M})\times L^2(\mathcal{M})$, we define the attainable set from $(\widetilde{\psi}_0,\widetilde{\psi}_1)$ in time less than T with controls whose L^1 norm is less than L:

$$\mathcal{V}^{T,L}(\widetilde{\psi}_0,\widetilde{\psi}_1)=\big\{\Phi^u(t)(\widetilde{\psi}_0,\widetilde{\psi}_1)\mid u\in L^1([0,T],\mathbf{R}),\|u\|_{L^1([0,T],\mathbf{R})}\le L,0\le t\le T\big\}.$$

Proposition 3.5. *For every $(\widetilde{\psi}_0,\widetilde{\psi}_1)$ in $H^1(\mathcal{M})\times L^2(\mathcal{M})$, for every $L>0$, for every $T\le\tau$ (defined in Proposition 3.2 (iii)), $\mathcal{V}^{T,L}(\widetilde{\psi}_0,\widetilde{\psi}_1)$ is contained in a compact set of $H^1(\mathcal{M})\times L^2(\mathcal{M})$.*

Proof. The proof of Proposition 3.5 proceeds in exactly the same way as the proof of Theorem 1.2, using the Dyson expansion (Proposition 3.4) and the fact that the mappings

$$\begin{array}{llll} G_1: & [0,T]\times L^2(\mathcal{M}) & \longrightarrow & H^1(\mathcal{M}) \\ & (s,\varphi) & \longmapsto & \dfrac{\sin((T-s)\sqrt{-\Delta+m})}{\sqrt{-\Delta+m}}B\varphi \end{array}$$

and

$$\begin{array}{llll} G_2: & [0,T]\times H^1(\mathcal{M}) & \longrightarrow & H^1(\mathcal{M}) \\ & (s,\psi) & \longmapsto & \dfrac{\sin((T-s)\sqrt{-\Delta+m})}{\sqrt{-\Delta+m}}\psi^3 \end{array}$$

are continuous. ☐

Proposition 3.6. *For every (ψ_0,ψ_1) in $H^1(\mathcal{M})\times L^2(\mathcal{M})$, and for every $L,T>0$, there exists $\tau^*>0$ such that, for every $(\widetilde{\psi_0},\widetilde{\psi_1})$ in the topological closure of $\mathcal{V}^{T,L}(\psi_0,\psi_1)$, the time τ given in Proposition 3.2 (iii) satisfies $\tau>\tau^*$.*

Proof. The time τ appearing in Proposition 3.2 (*iii*) is the time τ for which the Dyson expansion (Proposition 3.4) is valid. As proved in Proposition 3.2, this time depends on the norm of ψ_0 and ψ_1 (not on ψ_0 and ψ_1 themselves). The conclusion follows from the energy bound (3.8). ☐

Proposition 3.7. *For every $T,L>0$ and (ψ_0,ψ_1) in $H^1(\mathcal{M})\times L^2(\mathcal{M})$, the set $\mathcal{V}^{T,L}(\psi_0,\psi_1)$ is relatively compact in $H^1(\mathcal{M})\times L^2(\mathcal{M})$.*

Proof. In the following, for every real function $u:\mathbf{R}\to\mathbf{R}$ and every interval $I=[a,b]$ of $\mathbf{R}$, we define the function R_Iu by $R_Iu(x)=u(a+x)$ for x in $[0,b-a]$ and $R_Iu(x)=0$ otherwise.

Let τ^* be as defined in Proposition 3.6. We proceed by induction on p in $\mathbf{N}$ to prove Proposition 3.7 for $T\le p\tau^*$.

For $p=1$, this is just Proposition 3.5.

Assume the result holds for $p\ge 1$. Let T be in $(p\tau^*,(p+1)\tau^*]$ and $(A_n)_{n\in\mathbf{N}}=\big(\Phi^{u_n}(t_n)(\psi_0,\psi_1)\big)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{V}^{T,L}(\psi_0,\psi_1)$. We aim to find a convergent subsequence of $(A_n)_{n\in\mathbf{N}}$, which will prove the relative compactness of $\mathcal{V}^{T,L}(\psi_0,\psi_1)$.

By the induction hypothesis, the set $\mathcal{V}^{p\tau^*,L}(\psi_0,\psi_1)$ is relatively compact, hence up to extraction of a subsequence, one may assume that the sequence $\big(\Phi^{u_n}(p\tau^*)(\psi_0,\psi_1)\big)_{n\in\mathbf{N}}$ converges to some limit $A^\infty_{p\tau^*}$. By Proposition 3.6, $\tau(A^\infty_{p\tau^*},L)>\tau^*$. Hence, by Proposition 3.5, the set $\mathcal{V}^{\tau^*,L}(A^\infty_{p\tau^*})$ is relatively compact and, up to extraction of a subsequence, one may assume that the sequence $\big(\Phi^{R_{[p\tau^*,t_n]}u_n}(t_n-\tau^*)(A^\infty_{\tau^*})\big)_{n\in\mathbf{N}}$ converges to some limit $A^\infty_{T_\infty}$. By continuity of $\Phi^u(t)(\cdot,\cdot)$, the sequence $\big(\Phi^{u_n}(t_n)(\psi_0,\psi_1)\big)_{n\in\mathbf{N}}$ also converges to $A^\infty_{T_\infty}$, and that concludes the proof of Proposition 3.7. ☐

Proof of Theorem 3.1: It remains to prove the last statement of Theorem 3.1. This follows from Proposition 3.7 by noticing that $\bigcup_{t\in\mathbf{R}}\bigcup_{u\in L^1}\big\{\Phi^u(t)(\psi_0,\psi_1)\big\}\subset\bigcup_{\ell\in\mathbf{N}}\bigcup_{n\in\mathbf{N}}\mathcal{V}^{n,\ell}(\psi_0,\psi_1)$. ☐

Appendix A. Sobolev spaces

The aim of this appendix is to recall the classical Sobolev embedding theorem, which is instrumental in the proof of Proposition 3.2. For more details, the reader may refer to the classical reference [1, Theorem 5.4, statements (3) and (4)].

A.1. **Definition.** Let $\mathcal{M}$ be an open subset of $\mathbf{R}^n$ or a compact Riemannian manifold of dimension n. For every k in $\mathbf{N}$ and every p in $[1,+\infty]$, the Sobolev space $W^{k,p}(\mathcal{M})$ is defined as the set of functions from $\mathcal{M}$ to $\mathbf{R}$ whose partial derivatives up to order k belongs to $L^p(\mathcal{M})$, that is:

$$W^{k,p}(\mathcal{M}) = \left\{ \psi \in L^p(\mathcal{M}) \mid D^\alpha \psi \in L^p(\mathcal{M}), \quad \forall |\alpha| \leq k \right\}.$$

When endowed with the norm $\|\psi\|_{W^{k,p}(\mathcal{M})} = \sum_{|\alpha|\leq p} \|D^\alpha \psi\|_{L^p}$, $W^{k,p}(\mathcal{M})$ turns into a Banach space.

In the case where $p=2$, $W^{k,2}(\mathcal{M})$ turns into a Hilbert space and is usually denoted by $H^2(\mathcal{M})$.

A.2. **Sobolev embedding theorem.** For every integers k,ℓ and every real numbers p,q such that $k>\ell$, $(k-\ell)p<n$, and $1\leq p<q\leq np/(n-(k-\ell)p)\leq +\infty$,

$$W^{k,p}(\mathcal{M}) \subset W^{\ell,q}(\mathcal{M})$$

and the embedding is continuous. In particular, there exists $C_{Sob}(p,q,k,n)>0$ such that

$$\|\psi\|_{W^{\ell,q}(\mathcal{M})} \leq C_{Sob}(p,q,k,n)\|\psi\|_{W^{k,p}(\mathcal{M})}.$$

In particular, if $k=1$ and $\ell=0$, one gets the following.

Proposition A.1 (Sobolev embedding). *If $1/p^* = 1/p - 1/n$ then $W^{1,p}(\mathcal{M}) \subset L^{p^*}(\mathcal{M})$.*

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References

[1] R. Adams. Sobolev Spaces. *Pure and Applied Mathematics* 65 (1975) Academic Press.
[2] J. Ball, J. Marsden and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.* 20 (1982), no. 4, 575–597.
[3] K. Beauchard. Local controllability of a 1D Schrödinger equation. *J. Math. Pures et Appl.*, 84 (2005), 851–956.
[4] K. Beauchard. Local controllability and non controllability of a 1D wave equation. *Journal of Differential Equations*, 250 (2011) 2064–2098.
[5] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl.* (9) 94 (2010), no. 5, 520–554.
[6] N. Boussaïd, M. Caponigro and T. Chambrion. Regular propagators of bilinear quantum systems. *Preprint : hal-01016299.*
[7] T. Chambrion and L. Thomann. On the bilinear control of the Gross-Pitaevskii equation. *arXiv:1810.09792.*
[8] J. Hansom, C. Schulte, Carsten, C. Le Gall, C. Matthiesen, C. Edmund, M. Hugues, Maxime, J. Taylor and A. Mete. Environment-assisted quantum control of a solid-state spin via coherent dark states. Nature Physics. 10. 10.1038/nphys3077, 2014.
[9] A. Khapalov. Controllability of partial differential equations governed by multiplicative controls. Lecture Notes in Mathematics, 1995. Springer-Verlag, Berlin, 2010. xvi+284 pp.
[10] M. Reed and B. Simon. Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. xv+361 pp.
[11] G. Turinici. On the controllability of bilinear quantum systems. Mathematical models and methods for ab initio quantum chemistry, 75–92, Lecture Notes in Chem., 74, Springer, Berlin, 2000.

Thomas Chambrion, Université de Lorraine, CNRS, INRIA, IECL, F-54000 Nancy, France
E-mail address: Thomas.Chambrion@univ-lorraine.fr

Laurent Thomann, Université de Lorraine, CNRS, IECL, F-54000 Nancy, France
E-mail address: Laurent.Thomann@univ-lorraine.fr