Obstruction to the bilinear control of the Gross-Pitaevskii equation: an example with an unbounded potential ⋆

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Abstract: In 1982, Ball, Marsden, and Slemrod proved an obstruction to the controllability of linear dynamics with a bounded bilinear control term. This note presents an example of nonlinear dynamics with respect to the state for which this obstruction still holds while the bilinear control term is not continuous.

Keywords: Nonlinear control system, controllability, bilinear control, Gross-Pitaevskii equation

1. INTRODUCTION AND RESULTS

1.1 Introduction

On the Euclidean space $\mathbf{R}^3$ endowed with its natural norm $|\cdot|$, we study the control problem:

$$\begin{cases} i\partial_t\psi + H\psi = u(t)K(x)\psi - \sigma|\psi|^2\psi, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^3, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1}$$

where $H = -\Delta + |x|^2$ is the Hamiltonian of the quantum harmonic oscillator on $\mathbf{R}^3$, $u : \mathbf{R} \longrightarrow \mathbf{R}$ is the control, $K : \mathbf{R}^3 \longrightarrow \mathbf{R}$ is a given potential and $\sigma \in \{0,1\}$.

The Sobolev spaces based on the domain of the harmonic oscillator are instrumental in the study of dynamics (1). They are defined, for $s \geq 0$ and $p \geq 1$ by

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbf{R}^3) = \big\{ f \in L^p(\mathbf{R}^3),\ H^{s/2} f \in L^p(\mathbf{R}^3) \big\},$$
$$\mathcal{H}^s = \mathcal{H}^s(\mathbf{R}^3) = \mathcal{W}^{s,2}.$$

The natural norms are denoted by $\|f\|_{\mathcal{W}^{s,p}}$ and up to equivalence of norms (see *e.g.* (Yajima and Zhang, 2004, Lemma 2.4)), for $1 < p < +\infty$, we have

$$\|f\|_{\mathcal{W}^{s,p}} = \|H^{s/2} f\|_{L^p} \equiv \|(-\Delta)^{s/2} f\|_{L^p} + \|\langle x\rangle^s f\|_{L^p}, \tag{2}$$

with the notation $\langle x\rangle = (1+|x|^2)^{1/2}$.

1.2 Ball-Marsden-Slemrod obstructions

The dynamical system (1) is called the bilinear Gross-Pitaevskii equation. It is a nonlinear version of dynamics of the type $\dot\psi = A\psi + u(t)B\psi$, where A and B are linear operators in a Banach space X and $u : \mathbf{R} \to \mathbf{R}$ is a real-valued control, which involves a control term $uB\psi$ that is bilinear in (u,ψ). Such dynamics play a major role in physics and are the subject of a vast literature Khapalov (2010). In Ball et al. (1982), Ball, Marsden, and Slemrod have proven that if A generates a C^0 semi-group in X and B is bounded on X, then the attainable set from any source ψ_0 in X with L^r controls, $r > 1$, is contained in a countable union of compact sets of X. This represents a deep obstruction to the controllability of bilinear control systems in infinite dimensional Banach spaces, since this result implies that the attainable set is meager in Baire sense and has empty interior.

The original result of Ball et al. (1982) (and its adaptation to the Schrödinger equation Turinici (2000)) has been extended to the case of L^1 controls in Boussaïd et al. (2017). More recently, the case where A is non-linear has been investigated in Chambrion and Thomann (2018b) (for the Klein-Gordon equation) and Chambrion and Thomann (2018a) (for the Gross-Pitaevskii equation (1)).

In (Chambrion and Thomann, 2018a, Theorem 1.6) we showed in particular that if $K \in \mathcal{W}^{1,\infty}(\mathbf{R}^3)$, the dynamics (1) is non controllable. Under this assumption, the map

$$\begin{aligned} \mathcal{H}^1(\mathbf{R}^3) &\longrightarrow \mathcal{H}^1(\mathbf{R}^3) \\ \psi &\longmapsto K\psi, \end{aligned}$$

is continuous and this was used in the heart of the proof.

The main result of this note, Theorem 1 below, provides an example of potential $K \notin L^\infty(\mathbf{R}^3)$ where this condition is violated, but where the obstruction to controllability result still holds true.

1.3 Main result

Our main result reads as follows

Theorem 1. Let $K(x) = \ln(|x|)\mathbf{1}_{\{|x|\leq 1\}}$ and $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$. Assume that $u \in \bigcup_{r>1} L^r_{loc}(\mathbf{R})$, then the equation (1) admits a global flow $\psi(t) = \Phi^u(t)(\psi_0)$.

Moreover, for every $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$, the attainable set

$$\bigcup_{t\in\mathbf{R}} \ \bigcup_{\substack{u\in L^r_{loc}(\mathbf{R}),\\ r>1}} \big\{\Phi^u(t)(\psi_0)\big\} \tag{3}$$

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is a countable union of compact subsets of $\mathcal{H}^1(\mathbf{R}^3)$.

In this paper, the solutions to (1) are understood in the mild sense

$$\psi(t) = e^{itH}\psi_0 - i\int_0^t u(\tau)e^{i(t-\tau)H}(K\psi(\tau))d\tau + i\sigma\int_0^t e^{i(t-\tau)H}(|\psi|^2\psi)d\tau.$$

1.4 Content of the paper

The rest of this note provides a proof of Theorem 1. The proof crucially relies on classical Strichartz estimates, which we recall in Section 2. The proof itself is split in two parts. The global well-posedness of the problem (1) is established in Section 3.1, using among other some energy estimates. The proof of the obstruction result follows the strategy used in the paper Ball et al. (1982) and is given in Section 3.2.

2. STRICHARTZ ESTIMATES

As in Chambrion and Thomann (2018a), the Strichartz estimates play a major role in the argument, let us recall them in the three-dimensional case. A couple $(q,r) \in [2,+\infty]^2$ is called admissible if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2},$$

and if one defines

$$X_T^1 := \bigcap_{(q,r)\ admissible} L^q\big([-T,T]\,;\mathcal{W}^{1,r}(\mathbf{R}^3)\big),$$

then for all $T > 0$ there exists $C_T > 0$ so that for all $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$ we have

$$\|e^{itH}\psi_0\|_{X_T^1} \leq C_T\|\psi_0\|_{\mathcal{H}^1(\mathbf{R}^3)}. \tag{4}$$

Using interpolation theory one can prove that

$$X_T^1 = L^\infty\big([-T,T]\,;\mathcal{H}^1(\mathbf{R}^3)\big) \cap L^2\big([-T,T]\,;\mathcal{W}^{1,6}(\mathbf{R}^3)\big),$$

so that one can define

$$\|\psi\|_{X_T^1} = \|\psi\|_{L^\infty([-T,T];\mathcal{H}^1(\mathbf{R}^3))} + \|\psi\|_{L^2([-T,T];\mathcal{W}^{1,6}(\mathbf{R}^3))}.$$

We will also need the inhomogeneous version of the Strichartz estimates: for all $T > 0$, there exists $C_T > 0$ so that for all admissible couple (q,r) and function $F \in L^{q'}([-T,T];\mathcal{W}^{1,r'}(\mathbf{R}^3))$,

$$\Big\|\int_0^t e^{i(t-\tau)H}F(\tau)d\tau\Big\|_{X_T^1} \leq C_T\|F\|_{L^{q'}([-T,T],\mathcal{W}^{1,r'}(\mathbf{R}^3))}, \tag{5}$$

where q' and r' are the Hölder conjugate of q and r. In particular, they are related by

$$\frac{2}{q'} + \frac{3}{r'} = \frac{7}{2}. \tag{6}$$

We refer to Poiret (2012) for a proof. Let us point out that (5) implies that

$$\Big\|\int_0^t e^{i(t-\tau)H}F(\tau)d\tau\Big\|_{X_T^1} \leq C_T\big(\|F_1\|_{L^1([-T,T],\mathcal{H}^1(\mathbf{R}^3))} + \|F_2\|_{L^2([-T,T],\mathcal{W}^{1,6/5}(\mathbf{R}^3))}\big), \tag{7}$$

for any F_1, F_2 such that $F_1 + F_2 = F$, which will prove useful.

In the sequel $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded. For $x \in \mathbf{R}^3$, we write $\langle x\rangle = (1+|x|^2)^{1/2}$. We will sometimes use the notations $L_T^p = L^p([0,T])$ and $L_T^pX = L^p([0,T];X)$ for $T > 0$.

3. PROOF OF THEOREM 1

3.1 Global existence theory for dynamics (1)

Notice that using the reversibility of the equation (1), it is enough to consider non-negative times in the proofs.

We first state two technical lemmas which will be useful to control the bilinear term in (1).

Lemma 2. Let $K(x) = \ln(|x|)\mathbf{1}_{\{|x|\leq 1\}}$. Then for all $1 \leq p < 2$ there exists $C > 0$ such that

$$\|K\psi\|_{\mathcal{W}^{1,p}(\mathbf{R}^3))} \leq C\|\psi\|_{\mathcal{H}^1(\mathbf{R}^3)}.$$

Proof. Let $1 \leq p < 2$, then by (Chambrion and Thomann, 2018a, Lemma A.1),

$$\big\|K\psi\big\|_{\mathcal{W}^{1,p}} \leq C\|K\|_{L^{q_1}}\|\psi\|_{\mathcal{H}^1} + C\|K\|_{\mathcal{W}^{1,q_2}}\|\psi\|_{L^6}, \tag{8}$$

with $1/q_1 = 1/p - 1/2$ and $1/q_2 = 1/p - 1/6$. Now we use the expression of K to get $\|K\|_{L^{q_1}} < \infty$ (since $q_1 < \infty$). Then we write, by (2)

$$\|K\|_{\mathcal{W}^{1,q_2}} \leq C\|\langle x\rangle K\|_{L^{q_2}} + C\|\nabla K\|_{L^{q_2}},$$

and since $|\nabla K| \leq C|x|^{-1}$, we check that $\|K\|_{\mathcal{W}^{1,q_2}} < \infty$ because $q_2 < 3$. □

The case $p = 2$ does not hold true in the previous statement, but we have the following bound after averaging in time.

Lemma 3. Let $K(x) = \ln(|x|)\mathbf{1}_{\{|x|\leq 1\}}$ and $T > 0$. Then for all $2 \leq q < \infty$, there exists $C_T > 0$ such that for all $\psi \in X_T^1$

$$\|K\psi\|_{L^q([-T,T];\mathcal{H}^1(\mathbf{R}^3))} \leq C_T\|\psi\|_{X_T^1}.$$

Proof. Firstly by (2) we have

$$\big\|K\psi\big\|_{\mathcal{H}^1} \leq c\big\|\nabla K\psi\big\|_{L^2} + c\big\|K\nabla\psi\big\|_{L^2} + c\big\|K\langle x\rangle\psi\big\|_{L^2}. \tag{9}$$

• Let us study the first term in (9). Since $|\nabla K| \leq C|x|^{-1}$, we can use the Hardy inequality

$$\big\||x|^{-1}\psi\big\|_{L^2(\mathbf{R}^3)} \leq C\big\|\psi\big\|_{H^1(\mathbf{R}^3)} \tag{10}$$

(we refer to (Tao, 2006, Lemma A.2) for the general statement and proof of this inequality), and therefore the contribution of the first term reads $\big\|\nabla K\psi\big\|_{L_T^qL^2} \leq CT^{1/q}\|\psi\|_{L_T^\infty\mathcal{H}^1} \leq C_T\|\psi\|_{X_T^1}$.

• To bound the contribution of the two last terms in (9), we will use that $K \in L^p(\mathbf{R}^3)$ for any $1 \leq p < \infty$. Given $2 \leq q < \infty$, we choose $r > 2$ such that the couple (q,r) is (Strichartz) admissible and write, using Hölder

$$\big\|K\nabla\psi\big\|_{L^2} \leq \|K\|_{L^p}\|\nabla\psi\|_{L^r} \leq c\|K\|_{L^p}\|\psi\|_{\mathcal{W}^{1,r}},$$

with $1/p + 1/r = 1/2$. Thus

$$\big\|K\nabla\psi\big\|_{L_T^qL^2} \leq c\|K\|_{L^p}\|\psi\|_{L_T^q\mathcal{W}^{1,r}} \leq c\|K\|_{L^p}\|\psi\|_{X_T^1}.$$

Similarly,

$$\big\|K\langle x\rangle\psi\big\|_{L_T^qL^2} \leq c\|K\|_{L^p}\|\psi\|_{L_T^q\mathcal{W}^{1,r}} \leq c\|K\|_{L^p}\|\psi\|_{X_T^1},$$

which achieves the proof. □

We now state a global existence result for (1) adapted to our control problem.

Proposition 4. Let $r > 1$ and $u \in L^r_{loc}(\mathbf{R})$. Set $K(x) = \ln(|x|)\mathbf{1}_{\{|x|\le 1\}}$. Let $\psi_0 \in \mathcal{H}^1(\mathbf{R}^3)$, then the equation (1) admits a unique global solution $\psi \in \mathcal{C}(\mathbf{R};\mathcal{H}^1(\mathbf{R}^3)) \cap L^2_{loc}(\mathbf{R};\mathcal{W}^{1,6}(\mathbf{R}^3))$ which moreover satisfies the bound

$$\|\psi\|_{L^\infty([-T,T];\mathcal{H}^1(\mathbf{R}^3))} + \|\psi\|_{L^2([-T,T];\mathcal{W}^{1,6}(\mathbf{R}^3))} \le C\big(T, \|\psi_0\|_{\mathcal{H}^1(\mathbf{R}^3)}, \|u\|_{L^r(-T,T)}\big). \quad (11)$$

Proof. The proof is in the spirit of the proof of (Chambrion and Thomann, 2018a, Proposition 1.5), but here we use moreover the Hardy inequality (10) to control the bilinear term in (1).

Energy bound: Assume for a moment that the solution exists on a time interval $[0, T]$. For $0 \le t \le T$, we define

$$\begin{aligned} E(t) &= \int_{\mathbf{R}^3} \big(\overline{\psi} H\psi + |\psi|^2 + \frac{\sigma}{2}|\psi|^4\big)dx \\ &= \int_{\mathbf{R}^3} \big(|\nabla\psi|^2 + |x|^2|\psi|^2 + |\psi|^2 + \frac{\sigma}{2}|\psi|^4\big)dx. \end{aligned}$$

Then, using that $\partial_t\overline{\psi} = -i(H\overline{\psi} + \sigma|\psi|^2\overline{\psi}) + iu(t)K(x)\overline{\psi}$, we get

$$\begin{aligned} E'(t) &= 2\Re\int_{\mathbb{R}^3} \partial_t\overline{\psi}\big(\psi + H\psi + \sigma|\psi|^2\psi\big)dx \\ &= -2u(t)\Im\int_{\mathbf{R}^3} K\overline{\psi}H\psi dx \\ &= 2u(t)\Im\int_{\mathbf{R}^3} \overline{\psi}\nabla K\cdot\nabla\psi dx. \end{aligned}$$

Using that $|\nabla K| \le C|x|^{-1}$, by the Hardy inequality (10) we get

$$\begin{aligned} E'(t) &\le 2|u(t)|\big\|\psi\nabla K\big\|_{L^2}\big\|\nabla\psi\big\|_{L^2} \\ &\le C|u(t)|\big\||x|^{-1}\psi\big\|_{L^2}\big\|\nabla\psi\big\|_{L^2} \\ &\le C|u(t)|\big\|\psi\big\|^2_{\mathcal{H}^1} \\ &\le C|u(t)|E(t). \end{aligned}$$

Thus, using that $\sigma \ge 0$, we deduce the first part of the bound (11).

Local existence and global existence: We consider the map

$$\begin{aligned} \Phi(\psi)(t) = e^{itH}\psi_0 &+ i\sigma\int_0^t e^{i(t-\tau)H}(|\psi|^2\psi)d\tau \\ &- i\int_0^t u(\tau)e^{i(t-\tau)H}(K\psi)d\tau, \end{aligned}$$

and we will show that it is a contraction in the space

$$B_{T,R} := \big\{\|\psi\|_{X^1_T} \le R\big\},$$

with $R > 0$ and $T > 0$ to be fixed.

By the Strichartz inequalities (4) and (5)

$$\|\Phi(\psi)\|_{X^1_T} \le c\|\psi_0\|_{\mathcal{H}^1} + c\big\||\psi|^2\psi\big\|_{L^1_T\mathcal{H}^1} + c\|uK\psi\|_{L^1_T\mathcal{H}^1}.$$

Then, for all $r > 1$, there exists $q < \infty$ sur that $1/r + 1/q = 1$, and we have by Lemma 3

$$\|uK\psi\|_{L^1_T\mathcal{H}^1} \le \|u\|_{L^r_T}\|K\psi\|_{L^q_T\mathcal{H}^1} \le C_T\|u\|_{L^r_T}\|\psi\|_{X^1_T}.$$

As a consequence,

$$\begin{aligned} \|\Phi(\psi)\|_{X^1_T} \le c\|\psi_0\|_{\mathcal{H}^1} &+ c\|\psi\|_{L^\infty_T\mathcal{H}^1}\|\psi\|^2_{L^2_T L^\infty} \\ &+ C_T\|u\|_{L^r_T}\|\psi\|_{L^\infty_T\mathcal{H}^1}. \end{aligned}$$

By the Gagliardo-Nirenberg and Sobolev inequalities on $\mathbf{R}^3$,

$$\|\psi\|_{L^\infty} \le C\|\psi\|^{1/2}_{L^6}\|\psi\|^{1/2}_{\mathcal{W}^{1,6}} \le C\|\psi\|^{1/2}_{\mathcal{H}^1}\|\psi\|^{1/2}_{\mathcal{W}^{1,6}},$$

thus $\|\psi\|_{L^2_T L^\infty} \le cT^{1/4}\|\psi\|^{1/2}_{L^\infty_T\mathcal{H}^1}\|\psi\|^{1/2}_{L^2_T\mathcal{W}^{1,6}}$, and for $\psi \in B_{T,R}$ we get

$$\|\Phi(\psi)\|_{X^1_T} \le c\|\psi_0\|_{\mathcal{H}^1} + cT^{1/2}R^3 + C_T R\|u\|_{L^r_T}.$$

We now choose $R = 2c\|\psi_0\|_{\mathcal{H}^1}$. Then for $T > 0$ small enough, Φ maps $B_{T,R}$ into itself. With similar estimates we can show that Φ is a contraction in $B_{T,R}$, namely

$$\|\Phi(\psi_1) - \Phi(\psi_2)\|_{X^1_T} \le \big[cT^{1/2}R^2 + C_T\|u\|_{L^r_T}\big]\|\psi_1 - \psi_2\|_{X^1_T}.$$

As a conclusion there exists a unique fixed point to Φ, which is a local solution to (1).

The local time of existence only depends on u and on the $\mathcal{H}^1$-norm. Therefore one can use the energy bound to show the global existence.

Proof of the bound (11)*:* It remains to prove that

$$\|\psi\|_{L^2([-T,T];\mathcal{W}^{1,6}(\mathbf{R}^3))} \le C\big(T, \|\psi_0\|_{\mathcal{H}^1(\mathbf{R}^3)}, \|u\|_{L^r(-T,T)}\big). \quad (12)$$

The argument follows the main lines of (Chambrion and Thomann, 2018a, Bound (1.18)), so we only give the key steps. Similarly to Chambrion and Thomann (2018a), for any $\delta > 0$ small enough we can prove the bound

$$\begin{aligned} \|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})} \le c\|\psi\|_{L^\infty_T\mathcal{H}^1} &+ c\delta^{1/2}\big\|\psi\big\|^2_{L^\infty_T\mathcal{H}^1}\|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})} \\ &+ C_T\|\psi\|_{L^\infty_T\mathcal{H}^1}\|u\|_{L^r(-T,T)}. \end{aligned}$$

Then we choose $\delta = \delta(T) > 0$ such that

$$c\delta^{1/2}\big\|\psi\big\|^2_{L^\infty_T\mathcal{H}^1} = \frac{1}{2},$$

thus the previous estimate gives

$$\|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})} \le 2\|\psi\|_{L^\infty_T\mathcal{H}^1}\big(c + C_T\|u\|_{L^r(-T,T)}\big),$$

and by summing up in δ, we get (12). □

3.2 Meagerness of the attainable set

Let $\epsilon > 0$ and let $u, u_n \in L^{1+\epsilon}([0,T];\mathbf{R})$ such that $u_n \rightharpoonup u$ weakly in $L^{1+\epsilon}([0,T];\mathbf{R})$. This implies a bound $\|u_n\|_{L^{1+\epsilon}_T} \le C(T)$ for some $C(T) > 0$, uniformly in $n \ge 1$. We have

$$\begin{aligned} \psi(t) = e^{itH}\psi_0 &- i\int_0^t u(\tau)e^{i(t-\tau)H}(K\psi(\tau))d\tau \\ &+ i\sigma\int_0^t e^{i(t-\tau)H}(|\psi|^2\psi)d\tau, \end{aligned}$$

and

$$\begin{aligned} \psi_n(t) = e^{itH}\psi_0 &- i\int_0^t u_n(\tau)e^{i(t-\tau)H}(K\psi_n(\tau))d\tau \\ &+ i\sigma\int_0^t e^{i(t-\tau)H}(|\psi_n|^2\psi_n)d\tau. \end{aligned}$$

We set $z_n = \psi - \psi_n$, then z_n satisfies

$$z_n = \mathcal{L}(\psi, \psi_n) + \mathcal{N}(\psi, \psi_n), \tag{13}$$

with

$$\mathcal{L}(\psi, \psi_n) = -i\int_0^t \big(u(\tau) - u_n(\tau)\big)e^{i(t-\tau)H}(K\psi)d\tau \\ -i\int_0^t u_n(\tau)e^{i(t-\tau)H}\big(K(\psi-\psi_n)\big)d\tau$$

and

$$\mathcal{N}(\psi, \psi_n) = i\sigma\int_0^t e^{i(t-\tau)H}\big((\psi-\psi_n)(\psi+\psi_n)\overline{\psi}\big)d\tau \\ +i\sigma\int_0^t e^{i(t-\tau)H}\big((\overline{\psi}-\overline{\psi_n})\psi_n^2\big)d\tau.$$

Let us prove that z_n tends to 0 in $L^\infty([0,T];\mathcal{H}^1(\mathbf{R}^3))$.

Lemma 5. Denote by

$$\epsilon_n := \Big\|\int_0^t \big(u_n(\tau) - u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau\Big\|_{L_T^\infty\mathcal{H}^1(\mathbf{R}^3)}.$$

Then ϵ_n tends to 0, when n tends to $+\infty$.

Proof. We proceed by contradiction. Assume that there exists $\epsilon > 0$, a subsequence of u_n (still denoted by u_n) and a sequence $t_n \longrightarrow t \in [0,T]$ such that

$$\Big\|\int_0^{t_n} \big(u_n(\tau) - u(\tau)\big)e^{i(t_n-\tau)H}(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^1(\mathbf{R}^3)} \geq \epsilon. \tag{14}$$

Up to a subsequence, we can assume that for all $n \geq 1$, $t_n \leq t$ or $t_n \geq t$. We only consider the first case, since the second is similar. By the Minkowski inequality and the unitarity of $e^{i\tau H}$

$$\Big\|\int_0^{t_n}\big(u_n(\tau) - u(\tau)\big)\big(e^{i(t_n-\tau)H} - e^{i(t-\tau)H}\big)(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ \leq \int_0^{t_n}\big|u_n(\tau) - u(\tau)\big|\Big\|\big(e^{i(t_n-\tau)H} - e^{i(t-\tau)H}\big)(K\psi(\tau))\Big\|_{\mathcal{H}^1(\mathbf{R}^3)}d\tau \\ = \int_0^{t_n}\big|u_n(\tau) - u(\tau)\big|\Big\|\big(e^{it_nH} - e^{itH}\big)(K\psi(\tau))\Big\|_{\mathcal{H}^1(\mathbf{R}^3)}d\tau.$$

Then by Hölder

$$\Big\|\int_0^{t_n}\big(u_n(\tau) - u(\tau)\big)\big(e^{i(t_n-\tau)H} - e^{i(t-\tau)H}\big)(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ \leq \|u_n - u\|_{L_T^{1+\epsilon}}\Big\|\big(e^{it_nH} - e^{itH}\big)(K\psi(\tau))\Big\|_{L^{q_\epsilon}_{\tau\in[0,T]}\mathcal{H}^1(\mathbf{R}^3)},$$

where $1 < q_\epsilon < \infty$ is such that $1/(1+\epsilon) + 1/q_\epsilon = 1$. Now, by Lemma 3, we have

$$\|K\psi\|_{L_T^{q_\epsilon}\mathcal{H}^1(\mathbf{R}^3)} \leq C_{T,\epsilon}\|\psi\|_{X_T^1} < \infty. \tag{15}$$

Now we apply (Chambrion and Thomann, 2018a, Lemma 3.2) (with $d = 3$ and $s = 1$) together with the previous lines, and we get that

$$\Big\|\int_0^{t_n}\big(u_n(\tau) - u(\tau)\big)\big(e^{i(t_n-\tau)H} - e^{i(t-\tau)H}\big)(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^1(\mathbf{R}^3)} \tag{16}$$

tends to zero when n tends to $+\infty$.

By the Minkowski inequality, the unitarity of $e^{i\tau H}$ and the Hölder inequality

$$\Big\|\int_{t_n}^t \big(u_n(\tau) - u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^1(\mathbf{R}^3)} \\ \leq \int_{t_n}^t \big|u_n(\tau) - u(\tau)\big|\big\|K\psi(\tau)\big\|_{\mathcal{H}^1(\mathbf{R}^3)}d\tau \\ \leq \|u_n - u\|_{L_T^{1+\epsilon}}\|K\psi\|_{L^{q_\epsilon}_{\tau\in[t_n,t]}\mathcal{H}^1(\mathbf{R}^3)} \\ \leq |t - t_n|^{1/q_\epsilon}\|u_n - u\|_{L_T^{1+\epsilon}}\|K\psi\|_{L_T^{2q_\epsilon}\mathcal{H}^1(\mathbf{R}^3)}. \tag{17}$$

Using Lemma 3 and the fact that $\|u_n - u\|_{L_T^{1+\epsilon}} \leq C$, we deduce that the term (17) tends to 0. We combine this with (16) to deduce

$$\Big\|\int_0^{t_n}\big(u_n(\tau) - u(\tau)\big)e^{i(t_n-\tau)H}(K\psi(\tau))d\tau - \\ \int_0^t \big(u_n(\tau) - u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^1(\mathbf{R}^3)} \longrightarrow 0 \tag{18}$$

Let us now prove that $\int_0^t \big(u_n(\tau) - u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau$ tends to 0 in $\mathcal{H}^1(\mathbf{R}^3)$, to reach a contradiction with (14). We set $v(\tau) = e^{i(t-\tau)H}(K\psi(\tau))$. Then by the unitarity of H, we have $\|v(\tau)\|_{\mathcal{H}^1} = \|K\psi(\tau)\|_{\mathcal{H}^1}$, thus by (15), $v \in L^{q_\epsilon}([0,T];\mathcal{H}^1(\mathbf{R}^3))$. We expand v on the Hermite functions $(h_k)_{k\geq 0}$ (which are the eigenfunctions of H) which form a Hilbertian basis of $L^2(\mathbf{R}^3)$

$$v(\tau, x) = \sum_{k=0}^{+\infty}\alpha_k(\tau)h_k(x),$$

so that we have $\|v(\tau,\cdot)\|^2_{\mathcal{H}^1} = \sum_{k=0}^{+\infty}(2k+1)|\alpha_k(\tau)|^2$ and

$$\|v\|_{L_T^{q_\epsilon}\mathcal{H}^1} = \Big[\int_0^T \Big(\sum_{k=0}^{+\infty}(2k+1)|\alpha_k(\tau)|^2\Big)^{q_\epsilon/2}d\tau\Big]^{1/q_\epsilon}.$$

This implies in particular that

$$\Big(\sum_{k=0}^{+\infty}(2k+1)|\alpha_k(\tau)|^2\Big)^{q_\epsilon/2} \in L_T^{q_\epsilon}. \tag{19}$$

Denote by $\rho = \sup_{n\geq 0}\|u_n - u\|_{L_T^{1+\epsilon}}$. We claim that there exists $M > 0$ large enough such that the function $g(\tau, x) = \sum_{k=0}^M \alpha_k(\tau)h_k(x)$ satisfies $\|v - g\|_{L^{q_\epsilon}([0,T];\mathcal{H}^1(\mathbf{R}^3))} \leq \epsilon/(4\rho)$. Actually,

$$\|v - g\|_{L_T^{q_\epsilon}\mathcal{H}^1} = \Big[\int_0^T \Big(\sum_{k=M+1}^{+\infty}(2k+1)|\alpha_k(\tau)|^2\Big)^{q_\epsilon/2}d\tau\Big]^{1/q_\epsilon}$$

tends to zero when $M \longrightarrow +\infty$, by the Lebesgue theorem and (19), hence the claim.

We have

$$\int_0^t\big(u_n(\tau) - u(\tau)\big)g(\tau)d\tau = \sum_{k=0}^M h_k\int_0^t\big(u_n(\tau) - u(\tau)\big)\alpha_k(\tau)d\tau.$$

Then, by (19), for all $k \geq 0$, $\alpha_k \in L_T^{q_\epsilon}$, which implies

$$\|\int_0^t \big(u_n(\tau) - u(\tau)\big)g(\tau)d\tau\|^2_{\mathcal{H}^1(\mathbf{R}^3)} = \\ \sum_{k=0}^M (2k+1)^s\Big|\int_0^t \big(u_n(\tau) - u(\tau)\big)\alpha_k(\tau)d\tau\Big|^2 \longrightarrow 0,$$

by the weak convergence of (u_n). Finally, for n large enough,

$$\Big\| \int_0^t \big(u_n(\tau) - u(\tau)\big) v(\tau) d\tau \Big\|_{\mathcal{H}^1(\mathbf{R}^3)}$$
$$\leq \frac{\epsilon}{4\rho} \|u_n - u\|_{L_T^{1+\epsilon}} + \Big\| \int_0^t \big(u_n(\tau) - u(\tau)\big) g(\tau) d\tau \Big\|_{\mathcal{H}^1(\mathbf{R}^3)}$$
$$\leq \frac{\epsilon}{2},$$

which together with (14) and (18) gives the contradiction. □

Thanks to (Chambrion and Thomann, 2018a, Lemma A.3) we get

$$\begin{aligned}
&\|\mathcal{N}(\psi,\psi_n)(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} \\
&\leq \int_0^t \|(\psi - \psi_n)(\psi + \psi_n)\overline{\psi}\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \\
&\quad + \int_0^t \|(\overline{\psi} - \overline{\psi_n})\psi_n^2\|_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \\
&\leq c \int_0^t \|z_n\|_{\mathcal{H}^1(\mathbf{R}^3)} \big(\|\psi\|^2_{\mathcal{W}^{1,6}} + \|\psi_n\|^2_{\mathcal{W}^{1,6}}\big) d\tau. \qquad (20)
\end{aligned}$$

By (13) we have

$$H^{1/2} z_n(t) = -iH^{1/2} \int_0^t \big(u(\tau) - u_n(\tau)\big) e^{i(t-\tau)H} (K\psi) d\tau$$
$$- i \int_0^t u_n(\tau) e^{i(t-\tau)H} H^{1/2}\big(K z_n\big) d\tau + H^{1/2}\mathcal{N}(\psi,\psi_n)(t).$$

Thus from the Strichartz inequality (7) we deduce

$$\begin{aligned}
\|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} &\leq \epsilon_n + \|u_n H^{1/2}\big(K z_n\big)\|_{L_t^{1+\epsilon} L^{p_\epsilon}(\mathbf{R}^3)} \\
&\quad + \|\mathcal{N}(\psi,\psi_n)(t)\|_{\mathcal{H}^1(\mathbf{R}^3)}, \qquad (21)
\end{aligned}$$

where $p_\epsilon < 2$ is given by $\frac{1}{p_\epsilon} = \frac{1}{3}\big(\frac{7}{2} - \frac{2}{1+\epsilon}\big)$, see (6).

By Lemma 2
$\|H^{1/2}\big(K z_n\big)\|_{L^{p_\epsilon}(\mathbf{R}^3)} = \|K z_n\|_{\mathcal{W}^{1,p_\epsilon}(\mathbf{R}^3)} \leq C\|z_n\|_{\mathcal{H}^1(\mathbf{R}^3)}$,
which in turn implies

$$\begin{aligned}
&\|u_n H^{1/2}\big(K z_n\big)\|_{L_t^{1+\epsilon} L^{p_\epsilon}(\mathbf{R}^3)} \\
&\quad \leq C\Big(\int_0^t |u_n(\tau)|^{1+\epsilon} \|z_n(\tau)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \Big)^{1/(1+\epsilon)}. \qquad (22)
\end{aligned}$$

As a conclusion, from (20), (21) and (22) we infer

$$\begin{aligned}
&\|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} \leq \\
&\quad \epsilon_n + C\Big(\int_0^t |u_n(\tau)|^{1+\epsilon} \|z_n(\tau)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \Big)^{1/(1+\epsilon)} + \\
&+ c \int_0^t \|z_n(\tau)\|_{\mathcal{H}^1(\mathbf{R}^3)} \big(\|\psi(\tau)\|^2_{\mathcal{W}^{1,6}} + \|\psi_n(\tau)\|^2_{\mathcal{W}^{1,6}}\big) d\tau.
\end{aligned}$$

We now apply Lemma 6 with

$$A(t) = \epsilon_n + C\Big(\int_0^t |u_n(\tau)|^{1+\epsilon} \|z_n(\tau)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \Big)^{1/(1+\epsilon)}$$

and

$$B(\tau) = c\big(\|\psi(\tau)\|^2_{\mathcal{W}^{1,6}} + \|\psi_n(\tau)\|^2_{\mathcal{W}^{1,6}}\big),$$

thus

$$\begin{aligned}
&\|z_n(t)\|_{\mathcal{H}^1(\mathbf{R}^3)} \\
&\leq \Big(\epsilon_n + C\Big(\int_0^t |u_n(\tau)|^{1+\epsilon} \|z_n(\tau)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \Big)^{1/(1+\epsilon)}\Big) \\
&\qquad e^{c\int_0^t \big(\|\psi(\tau)\|^2_{\mathcal{W}^{1,6}} + \|\psi_n(\tau)\|^2_{\mathcal{W}^{1,6}}\big) d\tau} \\
&\leq C_1(T)\Big(\epsilon_n + \Big(\int_0^t |u_n(\tau)|^{1+\epsilon} \|z_n(\tau)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} d\tau \Big)^{1/(1+\epsilon)}\Big),
\end{aligned}$$

where in the last line, we used (11). We raise the previous inequality to the power $(1+\epsilon)$, and get

$$\|z_n(t)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} \leq C_2(T)\Big(\epsilon_n + \int_0^t |u_n(\tau)|^{1+\epsilon} \|z_n(\tau)\|^{1+\epsilon}_{\mathcal{H}^1(\mathbf{R}^3)} d\tau\Big).$$

Finally, by the classical Grönwall inequality we deduce

$$\begin{aligned}
\|z_n\|^{1+\epsilon}_{L_T^\infty \mathcal{H}^1(\mathbf{R}^3)} &\leq C_2(T)\epsilon_n \exp\big(C_2(T) \int_0^T |u_n(\tau)|^{1+\epsilon} d\tau\big) \\
&\leq C_3(T)\epsilon_n,
\end{aligned}$$

and this latter quantity tends to 0 when n tends to $+\infty$.

Lemma 6. (Grönwall inequality). Let $f, A, B : [0,T] \longrightarrow \mathbf{R}_+$ be non-negative functions with $f(0) = 0$, and assume that A is an increasing function. Assume that

$$f(t) \leq A(t) + \int_0^t B(\tau) f(\tau) d\tau, \quad \forall\, 0 \leq t \leq T. \qquad (23)$$

Then for all $0 \leq t \leq T$,

$$f(t) \leq A(t) \exp\big(\int_0^t B(\tau) d\tau\big). \qquad (24)$$

Proof. In the case A constant, the result is classical (see e.g. (Tao, 2006, Theorem 1.10)). Now assume that A is an increasing function and let $0 \leq s \leq t \leq T$, then by (23)

$$f(s) \leq A(t) + \int_0^s B(\tau) f(\tau) d\tau, \quad \forall\, 0 \leq s \leq t.$$

Then by the classical Grönwall inequality (A is constant in s) we get

$$f(s) \leq A(t) \exp\big(\int_0^s B(\tau) d\tau\big),$$

for all $0 \leq s \leq t$, which implies (24) for $s = t$. □

4. CONCLUSION

This note provides an example of a Ball-Marsden-Slemrod like obstruction to the controllability of a nonlinear partial differential equation with a bilinear control term. The novelty of the result lies in the unboundedness of the biinear control term.

The possible relations of this obstruction result and the concepts introduced in Boussaïd et al. (2013) will be the subject of further investigations in future works.

REFERENCES

Ball, J.M., Marsden, J.E., and Slemrod, M. (1982). Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4), 575–597. doi:10.1137/0320042. URL `https://doi.org/10.1137/0320042`.

Boussaïd, N., Caponigro, M., and Chambrion, T. (2013). Weakly coupled systems in quantum control. *IEEE Trans. Automat. Control*, 58(9), 2205–2216. doi:10.1109/TAC.2013.2255948. URL `https://doi.org/10.1109/TAC.2013.2255948`.

Boussaïd, N., Caponigro, M., and Chambrion, T. (2017). On the ball–marsden–slemrod obstruction in bilinear control systems. Preprint hal-01537743.

Chambrion, T. and Thomann, L. (2018a). On the bilinear control of the gross-pitaevskii equation. Preprint arXiv:1810.09792.

Chambrion, T. and Thomann, L. (2018b). A topological obstruction to the controllability of nonlinear wave equations with bilinear control term. Preprint arXiv:1809.07107.

Khapalov, A.Y. (2010). *Controllability of partial differential equations governed by multiplicative controls*, volume 1995 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin. doi:10.1007/978-3-642-12413-6. URL `https://doi.org/10.1007/978-3-642-12413-6`.

Poiret, A. (2012). Solutions globales pour l'équation de schrödinger cubique en dimension 3. Preprint arXiv:1207.1578.

Tao, T. (2006). *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI. doi:10.1090/cbms/106. URL `https://doi.org/10.1090/cbms/106`. Local and global analysis.

Turinici, G. (2000). On the controllability of bilinear quantum systems. In *Mathematical models and methods for ab initio quantum chemistry*, volume 74 of *Lecture Notes in Chem.*, 75–92. Springer, Berlin.

Yajima, K. and Zhang, G. (2004). Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity. *J. Differential Equations*, 202(1), 81–110.