THE WKB METHOD AND GEOMETRIC INSTABILITY FOR NON LINEAR SCHRÖDINGER EQUATIONS ON SURFACES

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ABSTRACT. — In this paper we are interested in constructing WKB approximations for the non linear cubic Schrödinger equation on a Riemannian surface which has a stable geodesic. These approximate solutions will lead to some instability properties of the equation.

RÉSUMÉ (*Méthode WKB et instabilité géométrique pour Schrödinger non linéaire sur des surfaces*)

À l'aide de la méthode WKB nous construisons des solutions approchées à l'équation de Schrödinger cubique sur une variété qui possède une géodésique stable. Cette construction permet d'obtenir des résultats d'instabilités dans des espaces de Sobolev.

1. Introduction

Let (M, g) be a Riemannian surface (i.e. a Riemannian manifold of dimension 2), orientable or not. We assume that M is either compact or a compact perturbation of the euclidian space, so that the Sobolev embeddings are true. Consider $\Delta = \Delta_g$ the Laplace-Beltrami operator. In this paper we are interested in constructing WKB approximations for the non linear cubic Schrödinger equation

$$(1) \qquad \begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = \varepsilon |u|^2 u(t,x), & \varepsilon = \pm 1, \\ u(0,x) = u_0(x) \in H^\sigma(M), \end{cases}$$

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that is, given a small parameter $0 < h < 1$ and an integer N, functions $u_N(h)$ satisfying

$$(2) \qquad i\partial_t u_N(h) + \Delta u_N(h) = \varepsilon |u_N(h)|^2 u_N(h) + R_N(h),$$

with $\|u_N(h)\|_{H^\sigma} \sim 1$ and $\|R_N(h)\|_{H^\sigma} \leq C_N h^N$.

Here h is introduced so that $u_N(h)$ oscillates with frequency $\sim \frac{1}{h}$.

These approximate solutions to (1) will lead to some instability properties in the following sense (where h^{-1} will play the role of n):

DEFINITION 1.1. — We say that the Cauchy problem (1) is unstable near 0 in $H^\sigma(M)$, if for all $C > 0$ there exist times $t_n \longrightarrow 0$ and $u_{1,n}, u_{2,n} \in H^\sigma(M)$ solutions of (1) so that

$$\|u_{1,n}(0)\|_{H^\sigma(M)}, \ \|u_{2,n}(0)\|_{H^\sigma(M)} \ \leq \ C,$$

$$\|u_{1,n}(0) - u_{2,n}(0)\|_{H^\sigma(M)} \longrightarrow 0,$$

$$\limsup \|u_{1,n}(t_n) - u_{2,n}(t_n)\|_{H^\sigma(M)} \ \geq \ \frac{1}{2}C,$$

when $n \longrightarrow +\infty$.

This means that the problem is not uniformly well-posed, if we refer to the following definition:

DEFINITION 1.2. — Let $\sigma \in \mathbb{R}$. Denote by $B_{R,\sigma}$ the ball of radius R in H^σ. We say that the Cauchy problem (1) is uniformly well-posed in H^σ if the flow map

$$u_0 \in B_{R,\sigma} \cap H^1(M) \longmapsto \Phi_t(u_0) \in H^\sigma(M),$$

is uniformly continuous for any t.

We now state our instability result:

PROPOSITION 1.3. — *Let* $0 < \sigma < \frac{1}{4}$, *and assume that* M *has a stable and non degenerated periodic geodesic (see Assumptions 1 and 2), then the Cauchy problem* (1) *is not uniformly well-posed.*

This problem is motivated by the following results: Let (M, g) be a riemannian compact surface, then in [5], N. Burq, P. Gérard and N. Tzvetkov prove that (1) is uniformly well-posed in $H^\sigma(M)$ for $\sigma > \frac{1}{2}$. Whereas, in [4], they show that (1) is unstable on the sphere $\mathbb{S}^2$ for $0 < \sigma < \frac{1}{4}$. In fact they construct solutions of (1) of the form

$$(3) \qquad u_n^\kappa(t, x) = \kappa e^{i\lambda_n^\kappa t}(n^{\frac{1}{4}-\sigma}\psi_n(x) + r_n(t, x)),$$

where $0 < \kappa < 1$, $\psi_n = (x_1 + ix_2)^n$ is a spherical harmonic which concentrates on the equator of the sphere when $n \longrightarrow +\infty$ and where r_n is an error term which is small. To obtain instability, they consider $\kappa_n \longrightarrow \kappa$, then

$$\|u_n^\kappa(0) - u_n^{\kappa_n}(0)\|_{H^\sigma(\mathbb{S}^2)} \lesssim |\kappa - \kappa_n| \longrightarrow 0,$$

but

$$\|u_n^\kappa(t_n) - u_n^{\kappa_n}(t_n)\|_{H^\sigma(\mathbb{S}^2)} \gtrsim \kappa |e^{i\lambda_n^\kappa t_n} - e^{i\lambda_n^{\kappa_n} t_n}| \longrightarrow 2\kappa,$$

with a suitable choice of $t_n \longrightarrow 0$.

We follow this strategy but as the surface is not rotation invariant, the ansatz will be more complicated than (3).

This result is sharp, because in [6] they show that (1) is uniformly well-posed on $\mathbb{S}^2$ when $\sigma > \frac{1}{4}$.

On the other hand, in [3] J. Bourgain shows that (1) is uniformly well-posed on the rational torus $\mathbb{T}^2$ when $\sigma > 0$.

These results show how the geometry of M can lead to instability for the equation (1). Therefore it seems reasonable to obtain a result like Proposition 1.3 with purely geometric assumptions.

We first make the following assumption on M:

ASSUMPTION 1. — *The manifold M has a periodic geodesic.*

Denote by γ such a geodesic, then there exists a system of coordinates (s, r) near γ, say for $(s, r) \in \mathbb{S}^1 \times] - r_0, r_0[$, called Fermi coordinates such that (see [12], p. 80)

1. The curve $r = 0$ is the geodesic γ parametrized by arclength and
2. The curves $s = $ constant are geodesics parametrized by arclength. The curves $r = $ constant meet these curves perpendicularly.
3. In this system the metric writes

$$g = \begin{pmatrix} 1 & 0 \\ 0 & a^2(s, r) \end{pmatrix}.$$

We set the length of γ equal to 2π. Denote by $R(s, r)$ the Gauss curvature at (s, r), then a is the unique solution of

(4)
$$\begin{cases} \dfrac{\partial^2 a}{\partial r^2} + R(s, r)a = 0, \\ a(s, 0) = 1, \quad \dfrac{\partial a}{\partial r}(s, 0) = 0. \end{cases}$$

The initial conditions traduce the fact that the curve $r = 0$ is a unit-speed geodesic. In these coordinates the Laplace-Beltrami operator is

$$\Delta := \frac{1}{\sqrt{\det g}} \operatorname{div}(\sqrt{\det g}\, g^{-1} \nabla) = \frac{1}{a} \partial_s(\frac{1}{a} \partial_s) + \frac{1}{a} \partial_r(a \partial_r).$$

A function on M, defined locally near γ, can be identified with a function of $[0, 2\pi] \times] - r_0, r_0[$ such that

$$\forall (s, r) \in [0, 2\pi] \times] - r_0, r_0[\quad f(s + 2\pi, r) = f(s, \omega r)$$

where $\omega = 1$ if M is orientable and $\omega = -1$ if M is not. Define

$$(5) \qquad \omega_1 = \frac{1}{2}(\omega - 1) \in \{-1, 0\}.$$

From (4) we deduce that a admits the Taylor expansion

$$(6) \qquad a = 1 - \frac{1}{2}R(s)r^2 + R_3(s)r^3 + \cdots + R_p(s)r^p + o(r^p),$$

with $R(s) = R(s, 0)$ and

$$(7) \qquad R_k(s) = \frac{1}{k!}\frac{\partial^k a}{\partial r^k}(s, 0),$$

for $k \geq 3$.

As $a(s + 2\pi, r) = a(s, \omega r)$, we deduce $R(s + 2\pi) = R(s)$ and for all $j \geq 3$, $R_j(s + 2\pi) = \omega^j R_j(s)$.

Let $p_2 = \frac{1}{a^2}\sigma^2 + \rho^2$ be the principal symbol of Δ, and

$$(8) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}s(t) = \dfrac{\partial p_2}{\partial \sigma} = \dfrac{2\sigma}{a^2}, \; \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = -\dfrac{\partial p_2}{\partial s} = -\partial_s(\dfrac{1}{a^2})\sigma^2, \\[2mm] \dfrac{\mathrm{d}}{\mathrm{d}t}r(t) = \dfrac{\partial p_2}{\partial \rho} = 2\rho, \; \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\dfrac{\partial p_2}{\partial r} = -\partial_r(\dfrac{1}{a^2})\sigma^2, \\[2mm] s(0) = s_0, \; \sigma(0) = \sigma_0, r(0) = r_0, \; \rho(0) = \rho_0, \end{cases}$$

its associated hamiltonian system, where $p_2 = p_2(s(t), r(t), \sigma(t), \rho(t))$. The system (8) admits a unique solution and defines the hamiltonian flow

$$\Phi_t : (s_0, \sigma_0, r_0, \rho_0) \longmapsto (s(t), \sigma(t), r(t), \rho(t)).$$

The curve $\Gamma = \{(s(t) = t, \sigma(t) = 1/2, r(t) = 0, \rho(t) = 0), t \in [0, 2\pi]\}$ is solution of (8) and its projection in the (s, r) space is the curve γ. Now denote by ϕ the Poincaré map associated to the trajectory Γ and to the hyperplane $\Sigma = \{s = 0\}$. There exists a neighborhood $\mathcal{N}$ of $(\sigma = 1/2, r = 0, \rho = 0)$ such that the following makes sense: solve the system (8) with the initial conditions $(0, \sigma_0, r_0, \rho_0) \in \{0\} \times \mathcal{N}$ and let T be such that $s(T) = 2\pi$, then ϕ is the application

$$\phi : (r_0, \rho_0) \longmapsto (r(T), \rho(T)).$$

Moreover, the Poincaré map is continuously differentiable (see [13] p. 193). To obtain its differential $\mathrm{d}\phi(0, 0)$ at $(0, 0)$, we linearize the system (8) about the orbit Γ, i.e.

$$(9) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}s(t) = 2\sigma, \; \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = 0, \\[2mm] \dfrac{\mathrm{d}}{\mathrm{d}t}r(t) = 2\rho, \; \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\dfrac{1}{2}R(s(t))r, \end{cases}$$

then $\sigma = \frac{1}{2}$, $s(t) = t$ and

$$(10) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -R/2 & 0 \end{pmatrix}\begin{pmatrix} r \\ \rho \end{pmatrix}.$$

Hence the application $\mathrm{d}\phi(0,0)$ is

$$(11) \qquad \mathrm{d}\phi(0,0) : (r_0, \rho_0) \longmapsto \big(r(2\pi), \rho(2\pi)\big),$$

where (r, ρ) solves (10). As $\mathrm{d}\phi(0,0)$ is symplectic, it admits two eigenvalues Λ and Λ^{-1} that are called the characteristic multipliers of the system (10). We add the following assumption on γ, which can be formulated in terms of the eigenvalues of $\mathrm{d}\phi(0,0)$:

ASSUMPTION 2. — *The geodesic γ is stable, i.e. $\mathrm{d}\phi(0,0)$ is a rotation. Then the multipliers take the form $\Lambda = e^{i\lambda}$ and $\Lambda^{-1} = e^{-i\lambda}$ with $\lambda \in \mathbb{R}$. We assume moreover that there exist $\tau, \mu > 0$ such that*

$$(12) \qquad \forall (p,q) \in \mathbb{Z} \times \mathbb{N} \quad |p - q\frac{\lambda}{\pi}| \geq \frac{\mu}{|(p,q)|^\tau},$$

where $|(p,q)| = |p| + |q|$. When this condition is fulfilled, we say that γ is non degenerated.

REMARK 1.4. — Almost every $\lambda \in \mathbb{R}$ satisfies (12) with $\tau > 1$. This is an easy consequence of [1] p. 159, e.g.

EXAMPLES 1. — *Let M be a surface which has a periodic geodesic γ. In the general case, the eigenvalues of $\mathrm{d}\phi(0,0)$ defined by (11) are $\Lambda = \rho e^{i\lambda}$ and $\Lambda^{-1} = \rho^{-1}e^{-i\lambda}$, with $\Lambda + \Lambda^{-1} \in \mathbb{R}_+$, i.e.*

$$(13) \qquad (\rho - \rho^{-1})\sin\lambda = 0.$$

Assume that M is a surface of revolution and that $R > 0$ on γ. Then the characteristic multipliers are

$$\Lambda = \rho e^{2\pi i\sqrt{R}} \quad and \quad \Lambda^{-1} = \rho^{-1}e^{-2\pi i\sqrt{R}}.$$

i) If $\lambda = 2\pi\sqrt{R}$ satisfies (12) then $\rho = 1$ and M satisfies the assumptions.
ii) Let $2\sqrt{R} \notin \mathbb{N}$. Let $\tilde{M}$ be a perturbation of M, and denote by

$$\tilde{\Lambda} = \tilde{\rho}e^{i\tilde{\lambda}} \quad and \quad \tilde{\Lambda}^{-1} = \tilde{\rho}^{-1}e^{-i\tilde{\lambda}},$$

the new characteristic multipliers.
By (13), $\tilde{\rho} = 1$, and Assumption 2 is satisfied almost surely.
iii) Let $a > 0$, then the torus $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/a\mathbb{Z}$ is not under the hypotheses : in this case $\mathrm{d}\phi(0,0)$ is not diagonalizable.

Notice that the function r which satisfies (10) is solution of

$$(14) \qquad \qquad \ddot{y}(s) + R(s)y(s) = 0.$$

Consider a_0 the solution of (14) with initial conditions $a_0(0) = 1$ and $\dot{a}_0(0) = i$. Then, from the Floquet theory, there exists a 2π-periodic function P so that

$$a_0(s) = \mathrm{e}^{i \frac{\lambda}{2\pi} s} P(s)$$

(or $a_0(s) = \exp{(-i \frac{\lambda}{2\pi} s)} P(s)$, but λ can be replaced with $-\lambda$).

Here, and in all the paper we denote by $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}s} f$ if f is differentiable. This notation is motived by the fact that s will play the role of a time variable (see section 2).

In order to prove Proposition 1.3, we construct stationnary approximate solutions of (1), as stated in the following theorem

THEOREM 1.5. — *Assume 1 and 2. Let* $h \in]0, 1]$ *such that* $\frac{1}{h} \in \mathbb{N}$, *let* $\kappa, \sigma > 0$ *and* $k \in \mathbb{N}$. *Let* λ *be given by Assumption 2 and* ω_1 *by* (5).
Define $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$.
Then for all $N \in \mathbb{N}$, *there exist* $\lambda_N(k) \in \mathbb{R}$ *and a family* $u_N(h)$ *such that* $C_1 h^\sigma \leq \|u_N(h)\|_{L^2(M)} \leq C_2 h^\sigma$ *with* $C_1, C_2 > 0$ *independent of* N *and* h, *and*

$$(15) \qquad -\Delta u_N(h) = \lambda_N(k)u_N(h) - \varepsilon|u_N(h)|^2 u_N(h) + h^N g_N(h)$$

with for all $N \in \mathbb{N}$

$$\|h^N g_N(h)\|_{H^n(M)} \lesssim h^{N-n}.$$

Moreover

$$\lambda_N(k) = \frac{1}{h^2} - \frac{2}{h} E_0(k) + \frac{1}{\sqrt{h}} \varepsilon \kappa^2 h^{2\sigma} C_0 + \mathcal{O}(1),$$

where $C_0 > 0$ *is independent of* ε, κ *and* σ.

REMARK 1.6. — The analog of Theorem 1.5 was proved by J. Ralston in [14] for the linear case ($\varepsilon = 0$), with the same type of assumptions.

REMARK 1.7. — Consider the more general equations

$$(16) \qquad \qquad i\partial_t u + \Delta u = F(u),$$

where $F : \mathbb{C} \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function. The result of Theorem 1.5 is likely to hold with other nonlinearities $F(u)$, for example for $F(z) = z^3$, $F(z) = z^4$ or $F(z) = (1 + |z|^2)^\alpha z$ with $\alpha < 1$. However, the instability phenomenon is strongly related to the gauge invariance of the equation (16).

The scheme of the paper is the following: Thanks to a scaling, we reduce the problem (15) to the resolution of linear Schrödinger equations with a harmonic time dependent potential, and we will see, using Assumption 2, that these equations have periodic solutions. To prove Proposition 1.3 we show that the

family $u_N(h)$ provides good approximations of (1) in times where instability occurs.

NOTATIONS 1.8. — *In this paper c, C denote constants the value of which may change from line to line. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \le a \le Cb$, $a \le Cb$ respectively. By $\delta_{i,j}$ we mean the Kronecker symbol, i.e. $\delta_{i,j} = 0$ for $i \ne j$ and $\delta_{i,i} = 1$.*

REMARK 1.9. — In the sequel we do not always mention the dependence on h of the functions: we will write u, f, r_i, ... instead of u_h, f_h, $r_{i,h}$, ...

2. The WKB construction

Consider the equation

$$(17) \qquad -\Delta u = \lambda u - \varepsilon |u|^2 u.$$

Given $h > 0$, we are looking for a solution of the form

$$(18) \qquad u = \delta h^{-\frac{1}{4}} \mathrm{e}^{i\frac{s}{h}} f(s, r, h),$$

where $\delta = \kappa h^\sigma$, with $\kappa > 0$ and $0 \le \sigma \le \frac{1}{4}$. In all this section, δ will play the role of a parameter.

We try to find a solution (u, λ) of (17) of the form

$$u \sim \sum_{j \ge 0} h^{j/2} u_j, \quad \lambda \sim h^{-2} \sum_{j \ge 0} h^{j/2} \lambda_j.$$

As we will see, identifying each power of h will lead to a linear equation which can be solved with a suitable choice of λ_j.

Choose h such that $h^{-1} \in \mathbb{N}$, this ensures that $\exp i\frac{s}{h}$ is $2\pi-$periodic. Such a condition on h is natural and is known as a Bohr-Sommerfeld quantification condition.

With the ansatz (18), equation (17) becomes

$$-\frac{1}{a^2}(\frac{2i}{h}\partial_s f + \partial_s^2 f - \frac{1}{h^2}f) - \frac{1}{a}\partial_s(\frac{1}{a})(\frac{i}{h}f + \partial_s f)$$

$$(19) \qquad -\partial_r^2 f - \frac{\partial_r a}{a}\partial_r f = \lambda f - \varepsilon \delta^2 h^{-\frac{1}{2}}|f|^2 f.$$

We make the change of variables $x = \frac{r}{\sqrt{h}}$ and set $v(s, x, h) = f(s, \sqrt{h}x, h)$. Thus $\partial_r f = \frac{1}{\sqrt{h}}\partial_x v$ and $\partial_r^2 f = \frac{1}{h}\partial_x^2 v$.

Therefore we now have to find $v \sim \sum_{j \geq 0} h^{j/2} v_j$.

Using (6) we obtain the following Taylor expansions in h

$$\frac{1}{a^2} = 1 + hRx^2 - 2h^{\frac{3}{2}} R_3 x^3 + \mathcal{O}(h^2),$$

$$a^{-1} \partial_s (a^{-1}) = \mathcal{O}(h) \quad \text{and} \quad a^{-1} \partial_r a = \mathcal{O}(h^{\frac{1}{2}}).$$

Equation (19) can therefore be written, after multiplication by $\frac{1}{2} h$

$$
\begin{aligned}
(20) \qquad & i\partial_s v + \frac{1}{2} \partial_x^2 v - \frac{1}{2} Rx^2 v \\
& = \frac{1 - \lambda h^2}{2h} v + h^{\frac{1}{2}} R_3 x^3 v + \frac{1}{2} \varepsilon \delta^2 h^{\frac{1}{2}} |v|^2 v + hPv,
\end{aligned}
$$

where

$$(21) \qquad P = A_1 \partial_s^2 + A_2 \partial_s + A_3 \partial_x + A_4$$

is a second order differential operator with coefficients $A_j = A_j(s, x, h)$ satisfying $A_j(s + 2\pi, x, h) = A_j(s, \omega x, h)$ for $0 \leq j \leq 4$.

Denote by $E = \frac{1 - \lambda h^2}{2h} = E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p + o(h^{\frac{p}{2}})$ and write $v = v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p + o(h^{\frac{p}{2}})$ and by identifying the powers of h we obtain the system of equations:

$$(22) \qquad \left(i\partial_s + \frac{1}{2} \partial_x^2 - \frac{1}{2} Rx^2 - E_0 \right) v_0 = 0,$$

$$(23) \qquad \left(i\partial_s + \frac{1}{2} \partial_x^2 - \frac{1}{2} Rx^2 - E_0 \right) v_1 = E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2} \varepsilon \delta^2 |v_0|^2 v_0,$$

$$\cdots = \cdots$$

$$(24) \qquad \left(i\partial_s + \frac{1}{2} \partial_x^2 - \frac{1}{2} Rx^2 - E_0 \right) v_p = E_p v_0 + Q_p.$$

$$\cdots = \cdots$$

so that the $(j + 1)$th equation of unknown (v_j, E_j) corresponds to the annihilation of the coefficient of $h^{\frac{j}{2}}$ in (20).

Here Q_p is a function which only depends on x, s, $(v_j)_{j \leq p-1}$ and $(E_j)_{j \leq p-1}$.

REMARK 2.1. — Notice that thanks to the scaling, we have reduced the problem (17) to the resolution of linear equations. However we have to solve them exactly; no smallness assumption on x is possible, as x can be of size $\sim \frac{1}{\sqrt{h}}$.

In this section we will show

PROPOSITION 2.2. — *For all $p \in \mathbb{N}$, there exist $(E_0, \cdots, E_p) \in \mathbb{R}^{p+1}$ and $(v_0, \cdots, v_p) \in \left(\mathcal{C}^\infty \left([0, 2\pi], \mathcal{S}(\mathbb{R}) \right) \right)^{p+1}$ with $v_0 \neq 0$, which solve the system (22)-(24).*

This permits us to construct approximate solutions of (17); more precisely, we will obtain the following proposition, which is the main result of this section.

PROPOSITION 2.3. — *Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be such that $0 \le \chi \le 1$, $\chi = 1$ on $[-r_0/2, r_0/2]$ and suppose moreover that χ is an even function. Let $\delta > 0$. Denote by*

$$
(25) \qquad u_p(s, r) = \delta h^{-\frac{1}{4}} \chi(r) e^{i\frac{s}{h}} (v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p)(s, \frac{r}{\sqrt{h}})
$$

and by

$$
(26) \qquad \lambda_p = \frac{1}{h^2} - \frac{2}{h}(E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p).
$$

Then u_p satisfies $\|u_p\|_{L^2(M)} \sim \delta$ and

$$
(27) \qquad -\Delta u_p = \lambda_p u_p - \varepsilon |u_p|^2 u_p + h^{\frac{p-1}{2}} g_p(h)
$$

with

$$
\forall h \in]0, 1], \ \forall n \in \mathbb{N}, \quad \|h^{\frac{p-1}{2}} g_p(h)\|_{H^n([0, 2\pi] \times \mathbb{R})} \lesssim \delta h^{\frac{p-1}{2} - n}.
$$

2.1. Preliminaries: the analysis of the linear equations. — We will solve the system (22)-(24) for $x \in \mathbb{R}$. Notice that the Fermi coordinates are only defined for $|r| \le r_0$ i.e. for $x \le \frac{r_0}{\sqrt{h}}$. That's the reason why we need the cutoff which appears in the Proposition 2.3.

We first give an expansion of the operator P defined by (21).

LEMMA 2.4. — *Let*

$$
P(s, x, h) = A_1(s, x, h)\partial_s^2 + A_2(s, x, h)\partial_s + A_3(s, x, h)\partial_x + A_4(s, x, h),
$$

be the differential operator defined by (21). Then for all $p \ge 2$, P can be written

$$
(28) \qquad P(s, x, h) = \sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k(s, x) + h^{\frac{p}{2}} \tilde{P}_p(s, x, h),
$$

so that
i) For all $0 \le k \le p - 1$,

$$
P_k(s, x) = A_1^k(s, x)\partial_s^2 + A_2^k(s, x)\partial_s + A_3^k(s, x)\partial_x + A_4^k(s, x),
$$

where $A_j^k \in \mathcal{C}^\infty([0, 2\pi] \times \mathbb{R})$, for all $s \in [0, 2\pi]$ the function $x \mapsto A_j^k(s, x)$ is a polynomial and $A_j^k(s + 2\pi, x) = A_j^k(s, \omega x)$.
ii) Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ and $v \in \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R}))$, then for all $n \in \mathbb{N}$, there exists $C = C(p, n)$ independent of $h \in]0, 1]$ so that

$$
(29) \qquad \|\chi(h^{\frac{1}{2}} x) \tilde{P}_p v(s, x)\|_{H^n([0, 2\pi] \times \mathbb{R})} \le C.
$$

Proof. — We first compute the coefficients of P.
By the Taylor formula near $r = 0$ we have

$$\frac{1}{a^2}(s,r) = 1 + R(s)r^2 - 2R_3(s)r^3 + \sum_{k=4}^{p+3} r^k R_k(s)$$
$$+ \frac{r^{p+4}}{(p+3)!} \int_0^1 (1-t)^{p+3} \frac{\partial^{p+4}}{\partial r^{p+4}}\Big(\frac{1}{a^2}\Big)(s,tr)\mathrm{d}t,$$

where R_k is given by (7).
Now write $r = \sqrt{h}x$ and obtain

(30) $$\frac{1}{a^2}(s,\sqrt{h}x) = 1 + hR(s)x^2 - 2h^{\frac{3}{2}}R_3(s)x^3 + h^2 I_1(s,x,h),$$

where
(31)
$$I_1(s,x,h) = \sum_{k=4}^{p+3} h^{\frac{k-4}{2}} x^k R_k(s) + h^{\frac{p}{2}} \frac{x^{p+4}}{(p+3)!} \int_0^1 (1-t)^{p+3} \frac{\partial^{p+4}}{\partial r^{p+4}}\Big(\frac{1}{a^2}\Big)(s,\sqrt{h}xt)\mathrm{d}t.$$

Similarly

(32) $$\frac{1}{a}\partial_s\Big(\frac{1}{a}\Big)(s,\sqrt{h}x) = hI_2(s,x,h),$$

with

$$I_2(s,x,h) = \sum_{k=2}^{p+1} h^{\frac{k-2}{2}} \frac{x^k}{k!} \frac{1}{a}\partial_s\Big(\frac{1}{a}\Big)(s,0)$$

(33) $$+ h^{\frac{p}{2}} \frac{x^{p+2}}{(p+1)!} \int_0^1 (1-t)^{p+1} \frac{\partial^{p+2}}{\partial r^{p+2}}\Big(\frac{1}{a}\partial_s\Big(\frac{1}{a}\Big)\Big)(s,\sqrt{h}xt)\mathrm{d}t,$$

and

(34) $$\frac{\partial_r a}{a}(s,\sqrt{h}x) = h^{\frac{1}{2}} I_3(s,x,h),$$

where

$$I_3(s,x,h) = \sum_{k=1}^{p} h^{\frac{k-1}{2}} \frac{x^k}{k!} \frac{\partial^k}{\partial r^k}\Big(\frac{\partial_r a}{a}\Big)(s,0)$$

(35) $$+ h^{\frac{p}{2}} \frac{x^{p+1}}{p!} \int_0^1 (1-t)^p \frac{\partial^{p+1}}{\partial r^{p+1}}\Big(\frac{\partial_r a}{a}\Big)(s,\sqrt{h}xt)\mathrm{d}t.$$

Plug the expressions (30), (32) and (34) in equation (20), and deduce that
coefficients A_j are

$$A_1 = \frac{1}{2}(-1 - hRx^2 + 2h^{\frac{3}{2}}R_3x^3 - h^2I_1),$$

$$A_2 = -iRx^2 + 2ih^{\frac{1}{2}}R_3x^3 - ihI_1 - \frac{1}{2}I_2,$$

$$A_3 = -\frac{1}{2}I_3,$$

$$A_4 = \frac{1}{2}(I_1 - iI_2).$$

Then with the developments (31), (33) and (35), we see that for all $1 \leq j \leq 4$ and $0 \leq k \leq p - 1$, $x \mapsto A_j^k(s, x)$ is a polynomial. Moreover as $a(s + 2\pi, x) = a(s, \omega x)$, we also have $A_j^k(s + 2\pi, x) = A_j^k(s, \omega x)$.

To obtain the bound (29), we now have to control the integral rests which appear in (31),(33) and (35).

Let $q \in \mathbb{N}^*$ and let $(s, r) \mapsto f(s, r)$ be one of the functions a^{-2}, $a^{-1}\partial_s(a^{-1})$ or $a^{-1}\partial_r$. Let $\chi \in \mathcal{C}^\infty(]-r_0, r_0[)$ and define F_q by

$$F_q(s, x) = \chi(\sqrt{h}x) \int_0^1 (1 - t)^{q-1} \frac{\partial^q}{\partial r^q} f(s, \sqrt{h}xt)\mathrm{d}t.$$

As $f \in \mathcal{C}^\infty([0, 2\pi] \times]-r_0, r_0[)$, we deduce that for all $n_1, n_2 \in \mathbb{N}$ there exists $C = C(q, n_1, n_2)$, independent of $h \in]0, 1]$ so that

(36) $\forall (s, x) \in [0, 2\pi] \times \mathbb{R}, \qquad |\partial_s^{n_1}\partial_x^{n_2}F_q(s, x)| \leq C.$

Now let $v \in \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R}))$ and $n \in \mathbb{N}$. We can assume that $n \geq 2$, so that H^n is an algebra. Then by (36)

(37) $\|x^q F_q v\|_{H^n([0, 2\pi] \times \mathbb{R})} \leq C\|F_q\|_{H^n([0, 2\pi] \times \mathbb{R})}\|x^q v\|_{H^n([0, 2\pi] \times \mathbb{R})} \leq C,$

and this yields ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Consider the Hilbertian basis of $L^2(\mathbb{R})$ composed of the Hermite functions $(\varphi_k)_{k \geq 0}$ which are the eigenfunctions of the harmonic oscillator $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$, i.e. $H\varphi_k = (k + \frac{1}{2})\varphi_k$. Moreover $\varphi_k(x) = P_k(x)\mathrm{e}^{-x^2/2}$ where P_k is a polynomial of degree k with $P_k(-x) = (-1)^k P_k(x)$. The link between the s-dependent operator $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$ and H is given by the following result proved by M. Combescure in [10].

THEOREM 2.5. — *Let $a_0 : \mathbb{R} \longrightarrow \mathbb{C}$ be the solution of (14) with $a_0(0) = 1$, $\dot{a}_0(0) = i$. Define*

$$\alpha = \log|a_0|, \; \beta = \frac{1}{2i}\log\frac{a_0}{\bar{a}_0},$$

let the unitary transform $T(s)$ be defined by

$$T(s) = e^{i\dot{\alpha}(s)x^2/2}e^{-i\alpha(s)D}, \; where \; D = -\frac{i}{2}(x \cdot \nabla + \nabla \cdot x),$$

and let $U(s, \tau)$ be the unitary evolution operator for $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$, i.e.
$U(s, \tau)\varphi$ is the unique solution of the problem

$$\begin{cases} \left(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}R(s)x^2\right)u = 0, \\ u(\tau, x) = \varphi(x) \in L^2(\mathbb{R}). \end{cases}$$

Then we have for any $s, \tau \in \mathbb{R}$

$$U(s, \tau) = T(s)e^{-i(\beta(s)-(\beta(\tau))H}T(\tau)^{-1}.$$

REMARK 2.6. — The functions α and β are well defined: suppose that there exists s_0 such that $a_0(s_0) = 0$, then $\operatorname{Re} a_0$ and $\operatorname{Im} a_0$ are linearly dependent, which is impossible with this choice of the initial conditions.

REMARK 2.7. — Define $\theta(s) = \beta(s) - \frac{\lambda}{2\pi}s$ where λ is given by Assumption 2. Then α and θ are 2π-periodic real functions. Moreover $\alpha(0) = \dot{\alpha}(0) = \beta(0) = \theta(0) = 0$.

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space, i.e. the space of smooth functions which are fast decreasing and their derivatives too.

PROPOSITION 2.8. — *Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ and $E \in \mathbb{C}$. Let $f \in \mathcal{C}^\infty([0, 2\pi] \times \mathbb{R}, \mathbb{R})$ be such that*

$$\forall n \in \mathbb{N}, \ \forall s \in [0, 2\pi], \quad \partial_s^n f(s, \cdot) \in \mathcal{S}(\mathbb{R}),$$

in other words $f \in \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R}))$.
Let $\psi \in \mathcal{C}^1([0, 2\pi], L^2(\mathbb{R})) \cap \mathcal{C}^0([0, 2\pi], H^2(\mathbb{R}))$ be the solution of

$$(38) \qquad \begin{cases} i\partial_s\psi + \frac{1}{2}\partial_x^2\psi - \frac{1}{2}R(s)x^2\psi - E\psi = f, \\ \psi(0, x) = \psi_0(x). \end{cases}$$

Then $\psi \in \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R}))$.

Proof. — By replacing ψ with $e^{iEt}\psi$, we can assume that $E = 0$. The solution of equation (38) is given by

$$\psi(s, \cdot) = U(s, 0)\psi_0 - i\int_0^s U(s, \tau)f(\tau, \cdot)\mathrm{d}\tau$$

$$(39) \qquad = T(s)e^{-i\beta(s)H}\left(\psi_0 - i\int_0^s e^{i\beta(\tau)H}T(\tau)^{-1}f(\tau, \cdot)\mathrm{d}\tau\right).$$

As D is a transport operator, we have

$$T, T^{-1} : \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R})),$$

we only have to show that

$$e^{i\beta H} : \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R})).$$

This follows from the fact that β is regular and $e^{iH} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$. $\qquad\square$

The description of U given in Theorem 2.5 yields the following representation of $U(s,0)\varphi_k$:

PROPOSITION 2.9. — *For all $k \in \mathbb{N}$ and $s, x \in \mathbb{R}$ we have*

$$(40) \qquad U(s,0)\varphi_k(x) = e^{i\dot{\alpha}(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k\big(xe^{-\alpha(s)}\big).$$

Proof. — According to Theorem 2.5, and as $H\varphi_k = (k + \frac{1}{2})\varphi_k$,

$$U(s,0)\varphi_k = e^{i\dot{\alpha}(s)x^2/2} e^{-i(k+\frac{1}{2})\beta(s)} e^{-i\alpha(s)D} \varphi_k.$$

Denote by $f(s) = e^{-i\alpha(s)D}\varphi_k$. Then f is solution of the transport equation

$$\partial_s f = -\frac{1}{2}\dot{\alpha}(s)\big(x\partial_x f + \partial_x(xf)\big) = -\frac{1}{2}\dot{\alpha}(s)\big(f + 2x\partial_x f\big)$$

with Cauchy data $f(0,x) = \varphi_k(x)$. Make the change of variables $\sigma = \alpha(s)$ and set $g(\sigma) = f(s)$. Therefore g satisfies $\partial_\sigma g = -\frac{1}{2}(g + 2x\partial_x g)$. The equation $x = \dot{x}$, $x(0) = x_0$ admits the solution $x(\tau) = x_0 e^{\tau}$ and the characteristics method gives $g(\tau, x(\tau)) = e^{-\frac{1}{2}\tau}\varphi_k(x_0) = e^{-\frac{1}{2}\tau}\varphi_k(x(\tau)e^{-\tau})$, hence

$$f(s) = e^{-\frac{1}{2}\alpha(s)}\varphi_k(xe^{-\alpha(s)}).$$

$\square$

COROLLARY 2.10. — *Let $k \in \mathbb{N}$, define $\omega_1 = \frac{1}{2}(\omega - 1)$ and $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$. Then*

$$w_k = e^{-isE_0(k)}U(s,0)\varphi_k$$

$$(41) \qquad = e^{-isE_0(k)} e^{i\dot{\alpha}(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k\big(xe^{-\alpha(s)}\big)$$

is solution of the equation

$$\big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}R(s)x^2 - E_0(k)\big)w_k(s,x) = 0.$$

Proof. — On the one hand, from Proposition 2.9 we deduce

$$w_k(s + 2\pi, x) = e^{-2i\pi E_0(k)} e^{-i\lambda(\frac{1}{2}+k)} w_k(s,x) = e^{-ik\omega_1\pi} w_k(s,x)$$

$$= (-1)^{k\omega_1} w_k(s,x) = w_k(s, \omega x).$$

On the other hand, w_k satisfies (22) because of the definition of $U(s,0)$. $\square$

Fix $k_0 \in \mathbb{N}$ and take $v_0 = w_{k_0}$ with the previous choice of $E_0(k_0)$. This choice corresponds to the k_0th level of energy for the harmonic oscillator.

REMARK 2.11. — Until now we didn't use the restriction (12), but it will be crucial in the following.

PROPOSITION 2.12. — *For all $p \geq 0$, there exist $E_p \in \mathbb{C}$ and $v_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$ which solve (24).*

REMARK 2.13. — As stated in Theorem 1.5, the E_j's are in fact real numbers. This will be proved in Lemma 2.17.

Proof. — We proceed by induction on $p \in \mathbb{N}$.
For $p = 0$ the result was proved in Corollary 2.10.
Let $p \geq 1$, and suppose that for all $j \leq p - 1$ there exist $E_j \in \mathbb{C}$ and $v_j \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$ which solve the $(j+1)$th equation of (22). When $p \geq 2$, set

$$\tilde{v}_{p-1} = h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p-1}{2}} v_{p-1},$$
$$\tilde{E}_{p-1} = h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p-1}{2}} E_{p-1}$$

and $\tilde{v}_0 = \tilde{E}_0 = 0$. By (28), the function Q_p given by (24) is the coefficient of $h^{\frac{p}{2}}$ in the expansion in h of

$$\tilde{E}_{p-1} \tilde{v}_{p-1} + \frac{1}{2} \varepsilon \delta^2 |v_0 + \tilde{v}_{p-1}|^2 (v_0 + \tilde{v}_{p-1}) + h \Big(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k \Big)(v_0 + \tilde{v}_{p-1}).$$

Now using the regularity of the v_j's and the fact that for all $0 \leq k \leq p - 1$, P_k is an operator

$$P_k : \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big) \longrightarrow \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big),$$

we obtain $Q_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$.
Moreover Q_p satisfies, $\forall (s, x) \in [0, 2\pi] \times \mathbb{R}$

$$Q_p(s + 2\pi, x) = Q_p(s, \omega x)$$

because this property holds for the v_j's, and a.
Define $F_p(s, x) = e^{-i\dot{\alpha}(s) e^{2\alpha(s)} x^2/2} Q_p(s, x e^{\alpha(s)})$, then $F_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$ and satisfies $Q_p(s, x) = e^{i\dot{\alpha}(s) x^2/2} F_p(s, x e^{-\alpha(s)})$ and $F_p(s + 2\pi, x) = F_p(s, \omega x)$. Let us decompose F_p on the basis $(\varphi_j)_{j \geq 0}$: there exists a unique family of smooth functions $(g_j^p(s))_{j \geq 0} \in l^2(\mathbb{N})$ so that

$$(42) \qquad\qquad F_p(s, y) = \sum_{j \geq 0} g_j^p(s) \varphi_j(y).$$

Then

$$(43) \qquad Q_p(s, x) = \sum_{j \geq 0} g_j^p(s) e^{i\dot{\alpha}(s) x^2/2} \varphi_j(x e^{-\alpha(s)}) = \sum_{j \geq 0} h_j^p(s) w_j(s, x),$$

where according to (41)

$$(44) \qquad\qquad h_j^p(s) = e^{is E_0(j)} e^{i(\frac{1}{2}+j)\beta(s)} e^{\frac{1}{2}\alpha(s)} g_j^p(s).$$

We have

$$Q_p(s, \omega x) = \sum_{j \geq 0} h_j^p(s) w_j(s, \omega x),$$

but also

$$Q_p(s, \omega x) = Q_p(s + 2\pi, x) = \sum_{j \geq 0} h_j^p(s + 2\pi) w_j(s + 2\pi, x)$$

$$= \sum_{j \geq 0} h_j^p(s + 2\pi) w_j(s, \omega x),$$

and from the uniqueness of the h_j^p's we deduce $h_j^p(s + 2\pi) = h_j^p(s)$.
We are now looking for a solution of (24) of the form

$$(45) \qquad v_p(s, x) = \sum_{j \geq 0} e_j^p(s) w_j(s, x)$$

where the e_j^p's are 2π-periodic functions. For all $j \geq 0$, by Corollary 2.10 we have

$$\left(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2\right)\left(e_j^p w_j\right) = i\dot{e}_j^p w_j + \left(E_0(k_0) - E_0(j)\right)e_j^p w_j,$$

hence we have to solve the equations

$$(46) \qquad i\dot{e}_j^p + \left(E_0(k_0) - E_0(j)\right)e_j^p = h_j^p + \delta_{j,k_0} E_p.$$

As $E_0(k_0) - E_0(j) = \frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})$, the solutions of (46) take the form

$$(47) \qquad e_j^p(s) = e^{\frac{1}{2}i(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})s}\left(C_j^p - i\int_0^s h_j^p(\tau)e^{-\frac{1}{2}i(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})\tau}d\tau\right)$$

for $j \neq k_0$, and

$$e_{k_0}^p(s) = C_{k_0}^p - i\int_0^s h_{k_0}^p(\tau)d\tau - iE_p s.$$

The constants $C_j^p \in \mathbb{C}$ and $E_p \in \mathbb{C}$ have to be determined such that $e_j^p(s+2\pi) = e_j^p(s)$.

• Case $j = k_0$:

$$e_{k_0}^p(s + 2\pi) = -i\int_0^{2\pi} h_{k_0}^p(\tau)d\tau - 2\pi i E_p + e_{k_0}^p(s),$$

thus $e_{k_0}^p$ is 2π-periodic iff

$$(48) \qquad E_p = -\frac{1}{2\pi}\int_0^{2\pi} h_{k_0}^p(\tau)d\tau.$$

• Case $j \neq k_0$:
Denote by $\tilde{h}_j^p : \tau \longmapsto h_j^p(\tau)e^{-i\frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})\tau}$ and by $K = e^{i(k_0 - j)(\pi\omega_1 - \lambda)}$. Then

$$\int_0^{s+2\pi} \tilde{h}_j^p(\tau)d\tau = \int_0^{2\pi} \tilde{h}_j^p(\tau)d\tau + \int_{2\pi}^{s+2\pi} \tilde{h}_j^p(\tau)d\tau$$

$$= \int_0^{2\pi} \tilde{h}_j^p(\tau)d\tau + K^{-1}\int_0^s \tilde{h}_j^p(\tau)d\tau,$$

and by (47)

$$e_j^p(s + 2\pi) = K e^{i\frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})s} \Big(C_j^p - i \int_0^{s+2\pi} \tilde{h}_j^p(\tau) \mathrm{d}\tau \Big)$$

$$(49) \qquad = e^{i\frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})s} \Big(K C_j^p - iK \int_0^{2\pi} \tilde{h}_j^p(\tau) \mathrm{d}\tau - i \int_0^s \tilde{h}_j^p(\tau) \mathrm{d}\tau \Big).$$

Notice that $K \neq 1$, as $\lambda \notin \pi\mathbb{Q}$ and choose

$$C_j^p = \frac{iK}{K - 1} \int_0^{2\pi} \tilde{h}_j^p(\tau) \mathrm{d}\tau,$$

then according to (47) and (49), the function e_j^p is 2π−periodic.

Now, we show that the constants C_j^p are uniformly bounded in $j \geq 0$, so that the function v_p given by (45) is well defined. We first need the

LEMMA 2.14. — *Let* $(h_j^p)_{j\geq 0} \in l^2(\mathbb{N})$ *be the family of* 2π−*periodic functions defined by* (44) *and* $h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$ *its Fourier decomposition. Then for all* $n_1, n_2 \in \mathbb{N}$ *there exists* $C^p > 0$ *such that for all* $j \in \mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c_{l,j}^p|^2 \leq C^p.$$

Proof. — Consider the function $F_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$ which defines the family $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ with (42). Denote by $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$. Let $n_1, n_2 \in \mathbb{N}$ and decompose the function $\partial_s^{n_2} H^{n_1} F_p$ on the basis $(\varphi_j)_{j\geq 0}$

$$\partial_s^{n_2} H^{n_1} F_p(s, y) = \sum_{j\geq 0} \tilde{g}_j^p(s) \varphi_j(y)$$

where $(\tilde{g}_j^p)_{j\geq 0}$ is a smooth family of functions in $l^2(\mathbb{N})$.

Using that $H\varphi_j = (j + \frac{1}{2})\varphi_j$ and that $F_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$, we have for all $n_1, n_2 \in \mathbb{N}$

$$\partial_s^{n_2} H^{n_1} F_p(s, y) = \sum_{j\geq 0} (j + \frac{1}{2})^{n_1} (g_j^p)^{(n_2)}(s) \varphi_j(y).$$

By uniqueness of such a decomposition,

$$\Big((j + \frac{1}{2})^{n_1} (g_j^p)^{(n_2)} \Big)_{j\geq 0} = (\tilde{g}_j^p)_{j\geq 0} \in l^2(\mathbb{N}).$$

Then by the definition (44) of h_j^p, an easy induction on $n_1, n_2 \in \mathbb{N}$ shows that $\big(j^{n_1} (h_j^p)^{(n_2)} \big)_{j\geq 0} \in l^2(\mathbb{N})$. Write the Fourier decomposition of h_j^p

$$h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$$

and by Parseval

$$\sum_{j\geq 0}\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}(s)|c_{l,j}^p|^2 = \sum_{j\geq 0} j^{2n_1} \int_0^{2\pi} |(h_j^p)^{(n_2)}(s)|^2 \mathrm{d}s \leq C^p.$$

In particular, for all $j \in \mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}|c_{l,j}^p|^2 \leq C^p,$$

hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

End of the proof of Proposition 2.12: Using the Fourier decomposition of h_j we obtain

$$C_j^p = \frac{iK}{K-1} \int_0^{2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau$$

$$= \frac{iK}{K-1} \sum_{l\in\mathbb{Z}} c_{l,j}^p \int_0^{2\pi} \mathrm{e}^{i(l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi}))\tau}$$

(50)
$$= -i \sum_{l\in\mathbb{Z}} \frac{c_{l,j}^p}{l - \frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})}.$$

With Assumption 2 we have

$$\left|l - \frac{1}{2}(k_0-j)(\omega_1 - \frac{\lambda}{\pi})\right| = \frac{1}{2}\left|(2l - (k_0-j)\omega_1) + (k_0-j)\frac{\lambda}{\pi}\right|$$

$$\geq \frac{1}{2} \frac{\mu}{|(2l - (k_0-j)\omega_1, k_0-j)|^\tau},$$

and for $j \geq k_0$, $|2l - (k_0-j)\omega_1| + |k_0 - j| \leq 2(|l| + |j|)$, then

(51)
$$\left|l - \frac{1}{2}(k_0-j)(\omega_1 - \frac{\lambda}{\pi})\right| \geq \frac{c\mu}{(|l|+|j|)^\tau}.$$

Hence, from (50) and (51) we deduce

(52)
$$|C_j^p| \lesssim \sum_{l\in\mathbb{Z}} |c_{l,j}^p|(|j|+|l|)^\tau \lesssim \sum_{l\in\mathbb{Z}} |c_{l,j}^p|(|j|^\tau + |l|^\tau).$$

By Cauchy-Schwarz and Lemma 2.14, from (52) we obtain

$$|C_j^p| \lesssim \sum_{l\in\mathbb{Z}} \frac{1+|l|}{1+|l|} |c_{l,j}^p|(|j|^\tau + |l|^\tau)$$

$$\lesssim \Big(\sum_{l\in\mathbb{Z}} \frac{1}{(1+|l|)^2}\Big)^{\frac{1}{2}} \Big(\sum_{l\in\mathbb{Z}} |c_{l,j}^p|^2 (1+|l|)^2 (|j|^{2\tau} + |l|^{2\tau})\Big)^{\frac{1}{2}}$$

(53)
$$\leq C^p.$$

Set

$$v_p(s,x) = \sum_{j \geq 0} e_j^p(s) w_j(s,x).$$

For all $j \in \mathbb{N}$, $s \longmapsto e_j^p(s) w_j(s,x)$ is continuous and there exists $c > 0$ such that for all $j > k_0$, and for all $s \in [0, 2\pi]$

$$|e_j^p(s) w_j(s,x)| \lesssim |g_j^p(s)| |\varphi_j(cx)|$$

and this shows that $v_p \in C\big([0, 2\pi], L^2(\mathbb{R})\big)$. Now using Proposition 2.8 we conclude, by uniqueness of such a solution, that $v_p \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$. $\square$

2.2. The nonlinear analysis and proof of Proposition 2.3. —

LEMMA 2.15. — *The constant E_1 given by Proposition 2.12 writes $E_1 = -\varepsilon\delta^2 C_0$ where $C_0 > 0$ is independent of ε and δ.*

Proof. — Consider the equation

$$\Big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\Big)v_1 = E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0,$$

with

$$v_0(s,x) = \mathrm{e}^{-isE_0(k_0)} \mathrm{e}^{i\dot\alpha(s)x^2/2} \mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)} \mathrm{e}^{-\frac{1}{2}\alpha(s)} \varphi_{k_0}\big(x\mathrm{e}^{-\alpha(s)}\big).$$

By the definition of Q_p (see (24)),

$$Q_1(s,x) = R_3(s)x^3 v_0(s,x) + \frac{1}{2}\varepsilon\delta^2|v_0|^2 v_0(s,x),$$

and by (43), Q_1 can be written

$$Q_1(s,x) = \sum_{j \geq 0} h_j^1(s) w_j(s,x).$$

According to formula (48), we only have to compute the term $h_{k_0}^1$ in the previous expansion.
Write the expansion of $|\varphi_{k_0}|^2\varphi_{k_0}$ on the basis $(\varphi_j)_{j \geq 0}$:

$$(54) \qquad\qquad |\varphi_{k_0}|^2\varphi_{k_0} = \sum_{j \geq 0} p_j\varphi_j,$$

with $p_j \in \mathbb{R}$ and $p_j = 0$ for $j - k_0 = 1 \ mod \ 2$ as $\varphi_k(-x) = (-1)^k\varphi_k(x)$.
Then by (54) and the expression (41) of w_j

$$
\begin{aligned}
|v_0|^2 v_0(s,x) &= \mathrm{e}^{-isE_0(k_0)} \mathrm{e}^{i\dot\alpha(s)x^2/2} \mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)} \mathrm{e}^{-\frac{3}{2}\alpha(s)} |\varphi_{k_0}|^2\varphi_{k_0}\big(x\mathrm{e}^{-\alpha(s)}\big) \\
&= \sum_{j \geq 0} p_j \mathrm{e}^{-isE_0(k_0)} \mathrm{e}^{i\dot\alpha(s)x^2/2} \mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)} \mathrm{e}^{-\frac{3}{2}\alpha(s)} \varphi_j\big(x\mathrm{e}^{-\alpha(s)}\big) \\
&= \sum_{j \geq 0} f_j(s) w_j(s,x)
\end{aligned}
$$

where

$$f_j(s) = p_j \mathrm{e}^{-is(E_0(k_0)-E_0(j))} \mathrm{e}^{-i(k_0-j)\beta(s)} \mathrm{e}^{-\alpha(s)}$$
$$= p_j \mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)} \mathrm{e}^{-\alpha(s)}.$$

Therefore $f_{k_0}(s) = p_{k_0} \mathrm{e}^{-\alpha(s)}$ with, using (54), $p_{k_0} = \int_{\mathbb{R}} |\phi_{k_0}|^4 > 0$.
In the same manner we write

(55)
$$x^3 \varphi_{k_0}(x) = \sum_{j \geq 0} q_j \varphi_j(x),$$

with $q_j = 0$ when $j - k_0 = 0 \bmod 2$ and by (55) we have

$$R_3(s)x^3 v_0(s,x)$$
$$= R_3(s) \mathrm{e}^{-isE_0(k_0)} \mathrm{e}^{i\dot{\alpha}(s)x^2/2} \mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)} \mathrm{e}^{\frac{5}{2}\alpha(s)} \left(x\mathrm{e}^{-\alpha(s)} \right)^3 \varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)} \right)$$
$$= \sum_{j \geq 0} q_j R_3(s) \mathrm{e}^{-isE_0(k_0)} \mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)} \mathrm{e}^{\frac{5}{2}\alpha(s)} \mathrm{e}^{i\dot{\alpha}(s)x^2/2} \varphi_j\left(x\mathrm{e}^{-\alpha(s)} \right).$$

By (41) we have

$$\mathrm{e}^{i\dot{\alpha}(s)x^2/2} \varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)} \right) = \mathrm{e}^{isE_0(j)} \mathrm{e}^{i(\frac{1}{2}+j)\beta(s)} \mathrm{e}^{\frac{1}{2}\alpha(s)}.$$

Then

$$R_3(s)x^3 v_0(s,x) = \sum_{j \geq 0} \tilde{f}_j(s) w_j(s,x),$$

where

$$\tilde{f}_j(s) = q_j R_3(s) \mathrm{e}^{-is(E_0(k_0)-E_0(j))} \mathrm{e}^{-i(k_0-j)\beta(s)} \mathrm{e}^{3\alpha(s)}$$
$$= q_j R_3(s) \mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)} \mathrm{e}^{3\alpha(s)}.$$

Then $\tilde{f}_{k_0} = 0$ as $q_j = 0$ when $j - k_0 = 0 \bmod 2$. Thus

$$h_{k_0}^1(s) = \frac{1}{2}\varepsilon\delta^2 f_{k_0}(s) = \frac{1}{2}\varepsilon\delta^2 p_{k_0} \mathrm{e}^{-\alpha(s)}.$$

Finally, from (48) we deduce

$$E_1 = -\frac{1}{4\pi}\varepsilon\delta^2 p_{k_0} \int_0^{2\pi} \mathrm{e}^{-\alpha(\tau)} \mathrm{d}\tau = -\varepsilon\delta^2 C_0,$$

where $C_0 > 0$ as $p_{k_0} > 0$. $\qquad\square$

LEMMA 2.16. — *Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\psi = 0$ near 0, and let $f \in \mathcal{S}(\mathbb{R})$.
Then for all $n, N \in \mathbb{N}$, there exists $C = C(n, N)$ so that*

(56)
$$\|\psi(h^{\frac{1}{2}}\cdot)f\|_{H^n(\mathbb{R})} \leq Ch^N.$$

Proof. — We only show (56) for $n = 0$, the general case follows from the Leibniz rule. We can assume that supp $\psi \subset [a, b]$ with $a > 0$. Then as $f \in \mathcal{S}(\mathbb{R})$, for all $N \in \mathbb{N}$, there exists $C_N > 0$ so that

$$|f(x)| \leq C_N \frac{1}{1 + |x|^N}.$$

Thus

$$\int_{\mathbb{R}} |\psi(h^{\frac{1}{2}} x)|^2 |f(x)|^2 \mathrm{d}x = h^{-\frac{1}{2}} \int_a^b |\psi(x)|^2 |f(h^{-\frac{1}{2}} x)|^2 \mathrm{d}x$$

$$\leq C_N h^{N-\frac{1}{2}} \int_a^b |\psi(x)|^2 \frac{1}{h^N + x^{2N}} \mathrm{d}x$$

$$\leq C_N h^{N-\frac{1}{2}},$$

hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 2.3. — Let $p \geq 1$, and consider

$$V_p(s, x) = \left(v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p \right)(s, x),$$

and

$$\tilde{E}_p = E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p,$$

where the v_j's and the E_j's are given by Proposition 2.12.
Let $\chi \in \mathcal{C}_0^\infty(] - r_0, r_0[)$ be an even function such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $[-r_0/2, r_0/2]$.
We claim that there exists $G_p(h) \in \mathcal{C}^\infty\big([0, 2\pi], \mathcal{S}(\mathbb{R})\big)$, so that

$$(57) \qquad\qquad \forall n \in \mathbb{N}, \quad \|G_p(h)\|_{H^n([0,2\pi] \times \mathbb{R})} \leq C_{n,p},$$

where $C_{n,p}$ is independent of $h \in]0, 1]$, and such that $G_p(h)$ satisfies

$$\chi(h^{\frac{1}{2}} x) \left((i\partial_s + \frac{1}{2} \partial_x^2 - \frac{1}{2} R x^2 - \tilde{E}_p) V_p \right.$$

$$(58) \qquad\qquad \left. - h^{\frac{1}{2}} R_3 x^3 V_p - \frac{1}{2} \varepsilon \delta^2 h^{\frac{1}{2}} |V_p|^2 V_p - hP V_p \right) = h^{\frac{p+1}{2}} G_p(h).$$

By construction of the v_j's and the E_j's, in the l.h.s. of (58), the coefficient of h^j cancels for $0 \leq j \leq p$.
Then write the expansion in powers of h

$$\frac{1}{2} \varepsilon \delta^2 |V_p|^2 V_p = \sum_{k=0}^{3p+1} h^{\frac{k}{2}} V_p^k,$$

and use (28) to obtain

$$hP V_p = h \Big(\sum_{k=0}^{p-1} h^{\frac{k}{2}} P_k + h^{\frac{p}{2}} \tilde{P}_p \Big) \Big(\sum_{k=0}^p h^{\frac{k}{2}} v_k \Big) := \sum_{k=0}^{2p+2} h^{\frac{k}{2}} W_p^k$$

We therefore obtain the explicit formula of $G_p(h)$

$$h^{\frac{p+1}{2}}G_p(h) := -\chi(h^{\frac{1}{2}}x)\sum_{k=p+1}^{2p+2}h^{\frac{k}{2}}W_p^k - \chi(h^{\frac{1}{2}}x)\sum_{k=p+1}^{3p+1}h^{\frac{k}{2}}V_p^k - \chi(h^{\frac{1}{2}}x)h^{\frac{p+1}{2}}R_3x^3v_p$$

$$= -h^{\frac{p+1}{2}}\chi(h^{\frac{1}{2}}x)\Big(\sum_{l=0}^{p+1}h^{\frac{l}{2}}W_p^{l+p+1}\sum_{l=0}^{2p}h^{\frac{l}{2}}V_p^{l+p+1} + R_3x^3v_p\Big).$$

The bound (57) then follows from an application of Lemma 2.4.

Denote by $\tilde{V}_p = \chi(h^{\frac{1}{2}}x)V_p$, and write

$$P\tilde{V}_p = (A_1\partial_s^2 + A_2\partial_s + A_3\partial_x + A_4)(\chi(h^{\frac{1}{2}}x)V_p)$$
$$= \chi(h^{\frac{1}{2}}x)PV_p + h^{\frac{1}{2}}\chi'(h^{\frac{1}{2}}x)A_3V_p.$$

By (58) we deduce that

$$(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - \tilde{E}_p)\tilde{V}_p - h^{\frac{1}{2}}R_3x^3\tilde{V}_p - \frac{1}{2}\varepsilon\delta^2h^{\frac{1}{2}}|\tilde{V}_p|^2\tilde{V}_p - hP\tilde{V}_p$$

$$= h^{\frac{p+1}{2}}G_h^p + h^{\frac{1}{2}}\chi'(h^{\frac{1}{2}}x)\partial_xV_p + \frac{1}{2}h\chi''(h^{\frac{1}{2}}x)V_p$$

$$+ \frac{1}{2}\varepsilon\delta^2h^{\frac{1}{2}}\chi(1-\chi^2)(h^{\frac{1}{2}}x)|V_p|^2V_p - h^{\frac{3}{2}}\chi'(h^{\frac{1}{2}}x)A_3V_p$$

$$:= h^{\frac{p+1}{2}}\tilde{G}_p(h).$$

Each of the functions χ', χ'' and $\chi(1-\chi^2)$ vanishes near 0, hence by Lemma 2.16 and (57)

(59) $$\forall n \in \mathbb{N}, \quad \|\tilde{G}_p(h)\|_{H^n([0,2\pi]\times\mathbb{R})} \leq C_{n,p}.$$

Finally, set

$$u_p = \delta h^{-\frac{1}{4}}e^{i\frac{s}{h}}V_p(s, \frac{r}{\sqrt{h}}),$$

then

$$-\Delta u_p - \lambda_p u_p + \varepsilon|u_p|^2u_p = \frac{2}{h}e^{i\frac{s}{h}}h^{\frac{p+1}{2}}\tilde{G}_p(h),$$

and $g_p(h) = 2e^{i\frac{s}{h}}\tilde{G}_p(h)$ satisfies the conclusion of Proposition 2.3 by (59).
$\square$

LEMMA 2.17. — *Let $p \geq 1$ and E_p given by Proposition (2.12). Then $E_p \in \mathbb{R}$.*

Proof. — We already know that $E_0, E_1 \in \mathbb{R}$. Let $p \geq 3$. Multiply (27) by $\overline{u}_p$, integrate on M and take the imaginary part

$$0 = \|u_p\|_{L^2}^2\text{Im}\,\lambda_p + h^{\frac{p-1}{2}}\text{Im}\int g_p(h)\overline{u}_p.$$

As $\|u_p\|_{L^2} \sim 1$ and $\|g_p\|_{L^2} \lesssim 1$, we obtain the estimate

$$|\text{Im}\,\lambda_p| \lesssim h^{\frac{p-1}{2}}\|g_p\|_{L^2}\|u_p\|_{L^2} \lesssim h^{\frac{p-1}{2}}$$

and as
$$\operatorname{Im}\lambda_p = -2(\operatorname{Im}E_2 + h^{\frac{1}{2}}\operatorname{Im}E_3 + \cdots + h^{\frac{p-1}{2}}\operatorname{Im}E_p)$$
it follows that for all $0 \le j \le p-1$, $\operatorname{Im}E_j = 0$, i.e. $E_j \in \mathbb{R}$. $\qquad\square$

3. The instability for the nonlinear Schrödinger equation

3.1. The error estimate. —

PROPOSITION 3.1. — *Let $\alpha > 0$, $\sigma \in]0, \frac{1}{4}]$ and let $v \in H^2(M)$ be such that*
$$\|v\|_{L^2} \lesssim 1, \quad \|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}, \quad \|\Delta v\|_{L^\infty} \lesssim h^{-\frac{9}{4}+\sigma},$$
and suppose that v satisfies
$$i\partial_t v + \Delta v = \varepsilon|v|^2 v + h^\alpha R(h),$$
with for all $\beta \in [0,2]$, $\|R(h)\|_{H^\beta} \lesssim h^{-\beta}$. Let u be solution of
$$\begin{cases} i\partial_t u + \Delta u = \varepsilon|u|^2 u, \\ u(0,x) = v(0,x). \end{cases}$$
Then, if $\alpha > \frac{1}{4} + 3\sigma$ we have
$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad when \quad h \longrightarrow 0,$$
where $t_h \sim h^{\frac{1}{2}-2\sigma}\log(\frac{1}{h})$.

Proof. — Define $w = u - v$ and
$$E(t) = \|w\|_{L^2}^2 + \|h^2\Delta w\|_{L^2}^2.$$
We have $E(0) = 0$ and the following estimates:

(60) $\qquad \|w\|_{L^2} \le E^{\frac{1}{2}}, \quad \|\Delta w\|_{L^2} \le h^{-2}E^{\frac{1}{2}}, \quad \|\nabla w\|_{L^2} \le h^{-1}E^{\frac{1}{2}}.$

The function w satisfies the equation

(61) $\qquad i\partial_t w + \Delta w = \varepsilon(|w+v|^2(w+v) - |v|^2 v) - h^\alpha R(h).$

The energy method gives
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 = \operatorname{Im}\int \overline{w}\big(\varepsilon(|w+v|^2(w+v) - |v|^2 v) - h^\alpha R(h)\big)$$
$$\lesssim h^\alpha\|w\|_{L^2} + \|w\|_{L^4}^4 + \|w\|_{L^2}^2\|v\|_{L^\infty}^2.$$

The Gagliardo-Nirenberg inequality gives
$$\|w\|_{L^4}^4 \lesssim \|w\|_{L^2}^2\|\nabla w\|_{L^2}^2 \lesssim h^{-2}E^2,$$
and as $\|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}$, we obtain

(62) $\qquad \dfrac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma}E + h^{-2}E^2.$

Now, apply Δ to (61)

$$(63) \qquad i\partial_t \Delta w + \Delta^2 w = \varepsilon \Delta A - h^\alpha \Delta R(h),$$

with

$$\begin{aligned} A &= |w+v|^2(w+v) - |v|^2 v \\ &= 2w|v|^2 + \overline{w}v^2 + w^2\overline{v} + 2|w|^2 v + |w|^2 w, \end{aligned}$$

then

$$\begin{aligned} |\Delta A| &\lesssim |v|^2|\Delta w| + |v||\nabla v||\nabla w| + |\nabla v|^2|w| + |v||\Delta v||w| \\ &\quad + |\Delta v||w|^2 + |w|^2|\Delta w| + |w||\nabla w|^2, \end{aligned}$$

hence

$$\begin{aligned} \|\Delta A\|_{L^2} &\lesssim \|v\|_{L^\infty}^2 \|\Delta w\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^\infty}\|\nabla w\|_{L^2} + \|\nabla v\|_{L^\infty}^2\|w\|_{L^2} \\ (64) &\quad + \|v\|_{L^\infty}\|\Delta v\|_{L^\infty}\|w\|_{L^2} + \|\Delta v\|_{L^\infty}\|w\|_{L^4}^2 \\ &\quad + \|w\|_{L^\infty}^2\|\Delta w\|_{L^2} + \|w\|_{L^2}\|\nabla w\|_{L^4}^2. \end{aligned}$$

The following inequality holds in dimension 2

$$\|w\|_{L^\infty} \lesssim \|w\|_{L^2}^{\frac{1}{2}}\|\Delta w\|_{L^2}^{\frac{1}{2}} \lesssim h^{-1}E^{\frac{1}{2}},$$

and with (60) and (64) we deduce

$$\|\Delta A\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-\frac{13}{4}+\sigma}E + h^{-4}E^{\frac{3}{2}}.$$

But

$$h^{-\frac{13}{4}+\sigma}E = h^{-\frac{5}{4}+\sigma}E^{\frac{1}{4}}h^{-2}E^{\frac{3}{4}} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}},$$

and we obtain

$$(65) \qquad \|\Delta(A)\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}}.$$

Now, using (65) and $\|\Delta(R(h))\|_{L^2} \lesssim h^{-2}$, the energy method and the Cauchy-Schwarz inequality gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta w\|_{L^2}^2 &= \mathrm{Im}\int \Delta\overline{w}\big(\Delta A - h^\alpha \Delta R(h)\big) \\ (66) &\lesssim h^{-2}E^{\frac{1}{2}}\big(h^{\alpha-2} + h^{-\frac{5}{2}+2\sigma}E^{\frac{1}{2}} + h^{-4}E^{\frac{3}{2}}\big), \end{aligned}$$

therefore from (62) and (66) we have

$$\frac{d}{dt}E \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma}E + h^{-2}E^2.$$

Interpolation gives

$$\|w\|_{H^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{\dot{H}^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{L^2}^{1-\frac{\sigma}{2}}\|\Delta w\|_{L^2}^{\frac{\sigma}{2}} \lesssim h^{-\sigma}E^{\frac{1}{2}} := F.$$

The function F satisfies $F(0) = 0$ and

$$(67) \qquad \frac{d}{dt}F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma}F + h^{-2+2\sigma}F^3.$$

As long as $h^{-2+2\sigma}F^3 \lesssim h^{-\frac{1}{2}+2\sigma}F$, i.e. $F \lesssim h^{\frac{3}{4}}$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma}F,$$

and the Gronwall inequality yields

$$F \lesssim h^{\alpha+\frac{1}{2}-3\sigma}\mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma}t}.$$

The non linear term in (67) can be removed with the continuity argument for times such that

$$h^{\alpha+\frac{1}{2}-3\sigma}\mathrm{e}^{Ch^{-\frac{1}{2}+2\sigma}t} \lesssim h^{\frac{3}{4}+\eta},$$

with $\eta > 0$ i.e. for $t \lesssim (\alpha - \frac{1}{4} - 3\sigma - \eta)h^{\frac{1}{2}-2\sigma}\log\frac{1}{h}$, which is possible with η small enough as we assume $\alpha > \frac{1}{4} + 3\sigma$. $\qquad\square$

COROLLARY 3.2. — *Let* $\kappa > 0$, $0 < \sigma < \frac{1}{4}$ *and set* $\delta = \kappa h^\sigma$. *Denote by* $v = e^{-i\lambda_3 t}u_3$ *where* u_3 *and* λ_3 *are defined by* (25) *and* (26) *respectively.*
Let u *be solution of*

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon|u|^2 u, \\ u(0,x) = v(0,x). \end{cases}$$

Then $\|v\|_{H^\sigma} \sim 1$ *and*

$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad when \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma}\log(\frac{1}{h})$.

Proof. — The result directly follows from Propositions 2.3 and 3.1, as for all $0 < \sigma < \frac{1}{4}$, we have $\sigma + 1 > \frac{1}{4} + 3\sigma$. $\qquad\square$

3.2. The instability argument. — Let $\kappa, \kappa_h > 0$ and consider $v = v^1$ defined in Corollary 3.2 associated with κ and v^2 associated with κ_h. Let u be a solution of

$$\begin{cases} i\partial_t u^j + \Delta u^j = \varepsilon|u^j|^2 u^j, \\ u^j(0,x) = v^j(0,x), \end{cases}$$

and $t_h \sim h^{\frac{1}{2}-2\sigma}\log\frac{1}{h}$. Then

$$\|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \|(v^2 - v^1)(t_h)\|_{H^\sigma} - \|(u^2 - v^2)(t_h)\|_{H^\sigma}$$
(68)
$$-\|(u^1 - v^1)(t_h)\|_{H^\sigma}.$$

From Corollary 3.2 we deduce that for $j = 1, 2$

(69) $$\|(u^j - v^j)(t_h)\|_{H^\sigma} \longrightarrow 0.$$

Observe that

$$\|(v^2 - v^1)(t_h)\|_{H^\sigma} \sim \left|\mathrm{e}^{-i\lambda_3^2 t_h} - \mathrm{e}^{-i\lambda_3^1 t_h}\right| = \left|\mathrm{e}^{i(\lambda_3^2 - \lambda_3^1)t_h} - 1\right|,$$

from Lemma 2.15 we have

$$(\lambda_3^2 - \lambda_3^1)t_h \sim h^{2\sigma-1}(\kappa - \kappa_h)t_h \sim (\kappa - \kappa_h)\log\frac{1}{h}.$$

It is possible to choose κ_h such that $\kappa_h \longrightarrow \kappa$ and $(\kappa - \kappa_h)\log\frac{1}{h} \longrightarrow \infty$. Then using (68) and (69)

$$\limsup_{h \longrightarrow 0} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \limsup_{h \longrightarrow 0} \|(v^2 - v^1)(t_h)\|_{H^\sigma} \geq 2,$$

even though

$$\|(u^2 - u^1)(0)\|_{H^\sigma} = \|(v^2 - v^1)(0)\|_{H^\sigma} \sim |\kappa - \kappa_h|,$$

which tends to 0 with h. According to Definition 1.1, we have proved Proposition 1.3.

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