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On the stability of a laminated Timoshenko problem with interfacial slip in the whole space under frictional dampings or infinite memories

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Abstract: The author of the present paper considered in [16] a model describing a vibrating strucure of an interfacial slip and consists of three coupled hyperbolic equations in one-dimensional bounded interval, where the dissipation is generated by either a frictional damping or an infinite memory, and it is acting only on one component. Some strong, polynomial, exponential and non exponential stability results were proved in [16] depending on the values of the parameters and the regularity of the initial data. The objective of the present paper is to compelete the study of [16] by considering this model in the whole line $\mathbb{R}$ and under only one control given by a frictional damping or an infinite memory. When the system is controled via its second or third component (rotation angle displacement or dynamic of the slip), we show that this control alone is sufficient to stabilize our system and get different polynomial stability estimates in the L^2-norm of the solution and its higher order derivatives with respect to the space variable. The decay rate depends on the regularity of the initial data, the nature of the control and the parameters in the system. However, when the system is controled via its first component (transversal displacement), we found a new stability condition depending on the parameters in the system. This condition defines a limit between the stability and instability of the system in the sense that, when this condition is staisfied, the system is polynomially stable. Otherwise, when this condition is not satisfied, we prove that the solution does not converge to zero at all. The proofs are based on the energy method and Fourier analysis combined with judicious choices of weight functions.

Keywords: Timoshenko beam with interfacial slip, Frictional damping, Infinite memory, Asymptotic behavior, Energy method, Fourier analysis

MSC: 34B05, 34D05, 34H05

1 Introduction

The structures known as the laminated Timoshenko beams in one-dimensional domains are composed of two layered identical beams of uniform thikness and attached together on top of each other subject to transversal and rotational vibrations, and taking account the longitudinal dislacement; see, for example, [28] for more details.

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A model of laminated Timoshenko beams of length L and with interfacial slip based on the Timoshenko theory is mathematically formulated by the system (see [22, 23])

$$\begin{cases} \rho\, W_{tt} + G\,(\psi - W_x)_x = 0, \\ I_\rho\,(3S - \psi)_{tt} - G\,(\psi - W_x) - D\,(3S - \psi)_{xx} = 0, \\ 3I_\rho\, S_{tt} + 3G\,(\psi - W_x) + 4\gamma S + 4\beta S_t - 3DS_{xx} = 0, \end{cases} \tag{1.1}$$

where the subscripts x and t denote the derivative with respect to space and time variables x and t, respectively, $x \in]0, L[$ and $t > 0$, combininig with some initial data and boundary conditions at $x = 0$ and $x = L$. The parameters $L, \rho,\ G,\ I_\rho,\ D,\ \gamma$ and β are positive constants and denote, respectively, the length, density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping parameter. The functions $W = W(x, t)$ and $\psi = \psi(x, t)$ represent, respectively, the transverse dispalcement and rotation angle, and the function $S = S(x, t)$ is proportional to the amount of slip along the interface, so the third equation in (1.1) describes the dynamics of the slip and contains already the internal frictional damping $4\beta S_t$. Without loss of generality, the length L of the beam can be assumed to be equal to 1.

The model (1.1) can be derived from the following more general one of Bresse-type [5] (known as the circular arch problem):

$$\begin{cases} \rho_1\, W_{tt} - k_1\,(W_x + \psi + lS)_x - lk_3(S_x - lW) = 0, \\ \rho_2\, \psi_{tt} - k_2\, \psi_{xx} + k_1\,(W_x + \psi + lS) = 0, \\ \rho_3\, S_{tt} - k_3\,(S_x - lW)_x + l\,k_1\,(W_x + \psi + lS) = 0, \end{cases} \tag{1.2}$$

where l, ρ_j and k_j, $j = 1, 2, 3$, are positive constants. When l and the third equation in (1.2) are not taken in consideration (i.e. $S = l = 0$); (1.2) reduces to the following Timoshenko-type system [38]:

$$\begin{cases} \rho_1\, W_{tt} - k_1\,(W_x + \psi)_x = 0, \\ \rho_2\, \psi_{tt} - k_2\, \psi_{xx} + k_1\,(W_x + \psi) = 0. \end{cases} \tag{1.3}$$

During the last thirty years, the models (1.1), (1.2) and (1.3) were the subject of various studies in the literature tackling well-posedness and stability questions by adding some controls (dampings, memories, heat conduction, ...) and/or boundary conditions (Dirichlet, Neaumann, mixed, ...). Let us mention here some of these studies related to our objectives in this paper.

Bounded domains. In the case of bounded domains, the well-posedness and stability of (1.1), (1.2) and (1.3) were widely treated in a huge number of works; see, for example, [1–4, 6–8, 11, 13–15, 17–20, 26–33, 36, 39] and the refereces therein. The obtained stability results in these papers depend on the nature, number and position of the controls, and some relations between the constants of the model, where the decay rate depends, in some situations, on the regularity of the initial data.

Observe that (1.1) is already damped via the control $4\beta S_t$ and, moreover, the speeds of the wave propagations of the last two equations in (1.1) are both asummed to be equal to $\sqrt{\frac{D}{I_\rho}}$. The author of the present paper considered in [16] a more general form of (1.1) by assuming that the three speeds of the wave propagations are not necessarily equal, and investigated the well-posedness and stability under a unique control represented by an infinite memory or a frictional damping. More precisely, he considered

$$\begin{cases} \rho_1\, \varphi_{tt} + k\,(u - \varphi_x)_x + \tau_1 F = 0, \\ \rho_2\,(3v - u)_{tt} - b\,(3v - u)_{xx} - k\,(u - \varphi_x) + \tau_2 F = 0, \\ \tilde{\rho}_3\, v_{tt} - \tilde{k}_0\, v_{xx} + 3\,k\,(u - \varphi_x) + 4\,\tilde{\delta}\, v + \tau_3 F = 0, \end{cases} \tag{1.4}$$

where $x \in]0, 1[$, $t > 0$, all the coefficients are positive constants except $\tilde{\delta}$, which is nonnegative, and

$$(\tau_1, \tau_2, \tau_3) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \tag{1.5}$$

and the external force F, which plays the role of control, is given by (frictional damping)

$$F = \gamma\left[\tau_1\varphi_t + \tau_2(3v-u)_t + \tau_3 v_t\right] \tag{1.6}$$

or (infinite memory)

$$F = \int_0^{+\infty} g(s)\left[\tau_1\varphi_{xx} + \tau_2(3v-u)_{xx} + \tau_3 v_{xx}\right](x, t-s)\,ds, \tag{1.7}$$

where $g:\mathbb{R}_+ \to \mathbb{R}_+$ is a given relaxation function satisfying some hypotheses and γ is a positive constant. The results in [16] show that, first, (1.4) in case

$$(\tau_1, \tau_2, \tau_3) \in \{(0,1,0), (0,0,1)\}$$

is strongly and polynomially stable without any resrictions on the parameters, and its exponential stability holds if and only if the three speeds of wave propagations are equal; that is

$$\frac{k}{\rho_1} = \frac{b}{\rho_2} = \frac{\tilde{k}_0}{\tilde{\rho}_3}.$$

Second, in case $(\tau_1, \tau_2, \tau_3) = (1,0,0)$, the strong and polynomial stability of (1.4) hold provided that the speeds of wave propagations of the last two equations in (1.4) are equal; that is

$$\frac{b}{\rho_2} = \frac{\tilde{k}_0}{\tilde{\rho}_3},$$

or $\tilde{\delta}$ does not belong to a given sequence of real numbers. Third, (1.4) in case $(\tau_1, \tau_2, \tau_3) = (1,0,0)$ is not exponentially stable whichever the values of the parameters. These stability results extend some similar ones known in the literature; see the references cited above.

Unbounded domains. The stability of (1.2) and (1.3) in unbounded domains has been extensively treated in the last few years. In this direction, we mention the papers [10, 12, 24, 25, 34, 35] (see also the references therien), where some polynomial stability estimates for L^2-norm of solutions have been proved using frictinal dampings or heat conduction effects or memory controls. In some particular cases, the optimality of the decay rate was also proved.

To the best of our knowledge, in the literature there is no stability results for laminated Timeshenko beams (1.1) in unbounded domains. Our main goal in the present paper is to consider the general model (1.4) in the whole line $\mathbb{R}$ and under one control of frictional damping or infinite memory type acting only on one equation; that is (1.4) with (1.6) or with (1.7). When the contol is active on the second or third equation, we show the asymptotic stability of the system in both cases by proving polynomial stability estimates for the L^2-norm of solutions and their higher derivatives with respect to the space varaible x, where the decay rate depends on the nature of the control (frictional damping (1.6) or infinite memory (1.7)), the regularity of the initial data and some relations between the coefficients k_1, k_2 and k_3. However, when the frictional damping $\gamma\varphi_t$ is active on the first equation, we present a new condition, which defines a border between the stability and instability of the system. More precisely, if this condition is satisfied, we show that the system is stable and has a similar polynomial stability estimate but with a weaker decay rate. Otherwise, when this condition is not satisfied, we prove, despite the presence of the dissipation $\gamma\varphi_t$, that the solution does not converge to zero at all. This last result can be explained by the weakness of the role played by the first equation in comparaison with the one played by the other two equations.

Our results give extensions from the bounded to the unbounded domain case of some of the works cited above, and in particular, they complete the results in [16]. The proof is based on the energy method combined with the Fourier analysis (by using the transformation in the Fourier space) and well chosen weight functions.

The paper is organized as follows. In Section 2, we formulate (1.4) in a first order Cauchy system, consider some assumptions and give some preliminaries. In Sections 3 and 4, we prove our polynomial stability estimates in cases (1.6) and (1.7), respectively. We end our paper by some general comments and related issues in Section 5.

2 Preliminaries and formulation of the problems

First, in order to simplify the form of (1.4), let us introduce the following change of variables:

$$\begin{cases} \rho_3 = \frac{1}{9}\tilde{\rho}_3, \quad k_1 = k, \quad k_2 = b, \quad k_3 = \frac{1}{9}\tilde{k}_0, \quad \delta = \frac{4}{9}\tilde{\delta}, \\ w = -3v, \quad \psi = 3v - u. \end{cases}$$

Then, (1.4) with (1.6) can be rewritten as

$$\begin{cases} \rho_1\varphi_{tt} - k_1\,(\varphi_x + \psi + w)_x + \tau_1\gamma\varphi_t = 0, \\ \rho_2\psi_{tt} - k_2\,\psi_{xx} + k_1\,(\varphi_x + \psi + w) + \tau_2\gamma\psi_t = 0, \\ \rho_3 w_{tt} - k_3\,w_{xx} + k_1(\varphi_x + \psi + w) + \delta w + \tau_3\gamma w_t = 0, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}$, $t > 0$, $\rho_1, \rho_2, \rho_3, k_1, k_2, k_3, \gamma > 0$, $\delta \geq 0$ and (τ_1, τ_2, τ_3) is defined by (1.5). For (1.4) with (1.7), we consider the two cases where the memory is acting on the second or third equation, that is

$$\begin{cases} \rho_1\varphi_{tt} - k_1\,(\varphi_x + \psi + w)_x = 0, \\ \rho_2\psi_{tt} - k_2\,\psi_{xx} + k_1\,(\varphi_x + \psi + w) + \tilde{\tau}_1 \int\limits_0^{+\infty} g(s)\psi_{xx}(x, t-s)\,ds = 0, \\ \rho_3 w_{tt} - k_3\,w_{xx} + k_1(\varphi_x + \psi + w) + \delta w + \tilde{\tau}_2 \int\limits_0^{+\infty} g(s)w_{xx}(x, t-s)\,ds = 0, \end{cases} \tag{2.2}$$

where

$$(\tilde{\tau}_1, \tilde{\tau}_2) \in \{(1, 0), (0, 1)\} \tag{2.3}$$

and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable and integrable on $\mathbb{R}_+$ such that

$$0 < g_0 := \int\limits_0^{+\infty} g(s)\,ds < \begin{cases} k_2 & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (1, 0), \\ k_3 & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (0, 1), \end{cases} \tag{2.4}$$

and there exist positive constants β_1 and β_2 satisfying

$$-\beta_1 g \leq g' \leq -\beta_2 g. \tag{2.5}$$

The systems (2.1) and (2.2) are supplemented by the initial conditions

$$\begin{cases} (\varphi(x, 0), \psi(x, 0), w(x, 0)) = (\varphi_0(x), \psi_0(x), w_0(x)), \\ (\varphi_t(x, 0), \psi_t(x, 0), w_t(x, 0)) = (\varphi_1(x), \psi_1(x), w_1(x)) \end{cases} \tag{2.6}$$

and

$$\begin{cases} (\varphi(x, 0), \psi(x, -\tilde{\tau}_1 t), w(x, -\tilde{\tau}_2 t)) = (\varphi_0(x), \psi_0(x, \tilde{\tau}_1 t), w_0(x, \tilde{\tau}_2 t)), \\ (\varphi_t(x, 0), \psi_t(x, 0), w_t(x, 0)) = (\varphi_1(x), \psi_1(x), w_1(x)), \end{cases} \tag{2.7}$$

respectively. To formulate (2.1) and (2.2) in an abstract first-order system, we consider the new variables

$$u = \varphi_t, \quad y = \psi_t, \quad \theta = w_t, \quad v = \varphi_x + \psi + w, \quad z = \psi_x \quad \text{and} \quad \phi = w_x. \tag{2.8}$$

For simplicity and without loss of generality, we take $\rho_1 = \rho_2 = \rho_3 = 1$ and $\delta = 0$. Using (2.8), (2.1) leads to the following system:

$$\begin{cases} v_t - u_x - y - \theta = 0, \\ u_t - k_1\,v_x + \tau_1\gamma\,u = 0, \\ z_t - y_x = 0, \\ y_t - k_2\,z_x + k_1 v + \tau_2\gamma\,y = 0, \\ \phi_t - \theta_x = 0, \\ \theta_t - k_3\,\phi_x + k_1\,v + \tau_3\gamma\theta = 0. \end{cases} \tag{2.9}$$

In case (2.2), we consider the additional new variable introduced in [9]

$$\eta(x,t,s) = \begin{cases} \psi(x,t) - \psi(x,t-s) & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (1,0), \\ w(x,t) - w(x,t-s) & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (0,1) \end{cases} \tag{2.10}$$

with its initial data $\eta_0(x,s) = \eta(x,0,s)$. The variable η satisfies

$$\eta_t(x,t,s) + \eta_s(x,t,s) = \begin{cases} \psi_t(x,t) & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (1,0), \\ w_t(x,t) & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (0,1) \end{cases} \tag{2.11}$$

and

$$\int_0^{+\infty} g(s)\eta_{xx}(x,t,s)\,ds = \begin{cases} g_0\psi_{xx}(x,t) - \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x,t-s)\,ds & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (1,0), \\ g_0 w_{xx}(x,t) - \displaystyle\int_0^{+\infty} g(s) w_{xx}(x,t-s)\,ds & \text{if } (\tilde{\tau}_1, \tilde{\tau}_2) = (0,1). \end{cases} \tag{2.12}$$

Then, we get from (2.2) (with $\rho_1 = \rho_2 = \rho_3 = 1$ and $\delta = 0$), (2.8), (2.11) and (2.12) the following system:

$$\begin{cases} v_t - u_x - y - \theta = 0, \\ u_t - k_1 v_x = 0, \\ z_t - y_x = 0, \\ y_t - (k_2 - \tilde{\tau}_1 g_0) z_x + k_1 v - \tilde{\tau}_1 \displaystyle\int_0^{+\infty} g(s)\eta_{xx}\,ds = 0, \\ \phi_t - \theta_x = 0, \\ \theta_t - (k_3 - \tilde{\tau}_2 g_0)\phi_x + k_1 v - \tilde{\tau}_2 \displaystyle\int_0^{+\infty} g(s)\eta_{xx}\,ds = 0, \\ \eta_t + \eta_s - \tilde{\tau}_1 y - \tilde{\tau}_2 \theta = 0. \end{cases} \tag{2.13}$$

Let define the variable U and its initial data U_0 by

$$U = \begin{cases} (v, u, z, y, \phi, \theta)^T & \text{in case (2.9)}, \\ (v, u, z, y, \phi, \theta, \eta)^T & \text{in case (2.13)} \end{cases} \quad \text{and} \quad U_0 = \begin{cases} (v, u, z, y, \phi, \theta)^T(\cdot, 0) & \text{in case (2.9)}, \\ (v, u, z, y, \phi, \theta, \eta)^T(\cdot, 0) & \text{in case (2.13)}. \end{cases}$$

The systems (2.9) and (2.13) with the initial conditions (2.6) and (2.7), respectively, are equivalent to

$$\begin{cases} U_t(x,t) + A_1 U_{xx}(x,t) + A_2 U_x(x,t) + A_3 U(x,t) = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{2.14}$$

where, for (2.9),

$$A_1 = 0, \quad A_2 U_x = \begin{pmatrix} -u_x \\ -k_1 v_x \\ -y_x \\ -k_2 z_x \\ -\theta_x \\ -k_3 \phi_x \end{pmatrix} \quad \text{and} \quad A_3 U = \begin{pmatrix} -y - \theta \\ \tau_1 \gamma u \\ 0 \\ k_1 v + \tau_2 \gamma y \\ 0 \\ k_1 v + \tau_3 \gamma \theta \end{pmatrix}, \tag{2.15}$$

and for (2.13)

$$
A_1 U_{xx} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\tilde{\tau}_1 \displaystyle\int_0^{+\infty} g(s)\eta_{xx}\, ds \\ 0 \\ -\tilde{\tau}_2 \displaystyle\int_0^{+\infty} g(s)\eta_{xx}\, ds \\ 0 \end{pmatrix}, \quad A_2 U_x = \begin{pmatrix} -u_x \\ -k_1 v_x \\ -y_x \\ -(k_2 - \tilde{\tau}_1 g_0)\, z_x \\ -\theta_x \\ -(k_3 - \tilde{\tau}_2 g_0)\,\phi_x \\ 0 \end{pmatrix} \quad \text{and } A_3 U = \begin{pmatrix} -y-\theta \\ 0 \\ 0 \\ k_1 v \\ 0 \\ k_1 v \\ \eta_s - \tilde{\tau}_1 y - \tilde{\tau}_2 \theta \end{pmatrix}.
$$

For a given function $h : \mathbb{R} \to \mathbb{C}$, we use the classical notations $Re\, h$, $Im\, h$, $\bar{h}$ and $\widehat{h}$ to denote, respectively, the real part of h, the imaginary part of h, the conjugate of h and the Fourier transformation of h given by

$$
\widehat{h}(\xi) = \int_{-\infty}^{+\infty} e^{-i\xi x} h(x)\, dx, \quad \xi \in \mathbb{R}.
$$

Applying the Fourier transformation (with respect to the space variable x) to (2.14), we obtain the following first-order Cauchy system in the Fourier space, for any $(\xi, t) \in \mathbb{R} \times \mathbb{R}_+$:

$$
\begin{cases} \widehat{U}_t(\xi,\, t) - \xi^2 A_1 \widehat{U}(\xi,\, t) + i\,\xi A_2 \widehat{U}(\xi,\, t) + A_3 \widehat{U}(\xi,\, t) = 0, \\ \widehat{U}(\xi,\, 0) = \widehat{U}_0(\xi). \end{cases} \tag{2.16}
$$

The solution of (2.16) is given by

$$
\widehat{U}(\xi,\, t) = e^{-(-\xi^2 A_1 + i\,\xi A_2 + A_3)\, t}\, \widehat{U}_0(\xi). \tag{2.17}
$$

Let $\widehat{E}$ be the total energy associated with system (2.16) given by

$$
\widehat{E}(\xi,\, t) = \frac{1}{2}\left[k_1 |\widehat{v}|^2 + |\widehat{u}|^2 + k_2 |\widehat{z}|^2 + |\widehat{y}|^2 + k_3 |\widehat{\phi}|^2 + |\widehat{\theta}|^2\right] \tag{2.18}
$$

in case (2.9), and

$$
\begin{aligned} \widehat{E}(\xi,\, t) \;=\; & \frac{1}{2}\left[k_1 |\widehat{v}|^2 + |\widehat{u}|^2 + (k_2 - \tilde{\tau}_1 g_0)\, |\widehat{z}|^2 + |\widehat{y}|^2 + (k_3 - \tilde{\tau}_2 g_0) |\widehat{\phi}|^2 + |\widehat{\theta}|^2\right] \\ & + \frac{\xi^2}{2} \int_0^{+\infty} g(s) |\widehat{\eta}|^2\, ds \end{aligned} \tag{2.19}
$$

in case (2.13).

We finish this section by establishing four results to be used in the next two sections.

Lemma 2.1. *Let $h,\ d : \mathbb{R} \to \mathbb{C}$ be two differentiable functions. Then*

$$
\frac{d}{dt} Re\,(h\bar{d}) = Re\,(h_t \bar{d} + d_t \bar{h}) \tag{2.20}
$$

and

$$
\frac{d}{dt} Re\,(ih\bar{d}) = Re\,\left[i(h_t \bar{d} - d_t \bar{h})\right]. \tag{2.21}
$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}Re\,(h\bar{d}) &= \frac{d}{dt}\left((Re\,h)\,(Re\,d) + (Im\,h)\,(Im\,d)\right)\\ &= (Re\,h_t)\,(Re\,d) + (Re\,h)\,(Re\,d_t) + (Im\,h_t)\,(Im\,d) + (Im\,h)\,(Im\,d_t)\\ &= Re\,(h_t\bar{d} + d_t\bar{h}),\end{aligned}$$

so we get (2.20). Using (2.20) with ih instead of h, we find (2.21). □

Lemma 2.2. *The energy functional* $\widehat{E}$ *satisfies*

$$\frac{d}{dt}\widehat{E}(\xi,\, t) = -\gamma\,\left[\tau_1\,|\widehat{u}|^2 + \tau_2\,|\widehat{y}|^2 + \tau_3\,|\widehat{\theta}|^2\right] \tag{2.22}$$

in case (2.9), *and*

$$\frac{d}{dt}\widehat{E}(\xi,\, t) = \frac{\xi^2}{2}\int_0^{+\infty} g'(s)|\widehat{\eta}|^2\, ds \tag{2.23}$$

in case (2.13).

Proof. The first equation of (2.16) in case (2.9) is equivalent to

$$\begin{cases}\widehat{v}_t - i\xi\widehat{u} - \widehat{y} - \widehat{\theta} = 0,\\ \widehat{u}_t - ik_1\xi\,\widehat{v} + \tau_1\gamma\,\widehat{u} = 0,\\ \widehat{z}_t - i\xi\widehat{y} = 0,\\ \widehat{y}_t - ik_2\xi\,\widehat{z} + k_1\widehat{v} + \tau_2\gamma\,\widehat{y} = 0,\\ \widehat{\phi}_t - i\xi\widehat{\theta} = 0,\\ \widehat{\theta}_t - ik_3\,\xi\widehat{\phi} + k_1\,\widehat{v} + \tau_3\gamma\widehat{\theta} = 0.\end{cases} \tag{2.24}$$

Multiplying the equations in (2.24) by $k_1\bar{\widehat{v}}$, $\bar{\widehat{u}}$, $k_2\bar{\widehat{z}}$, $\bar{\widehat{y}}$, $k_3\bar{\widehat{\phi}}$ and $\bar{\widehat{\theta}}$, respectively, adding the obtained equations, taking the real part of the resulting expression and using (2.20), we obtain (2.22).

Similarily, the first equation of (2.16) in case (2.13) is reduced to

$$\begin{cases}\widehat{v}_t - i\xi\widehat{u} - \widehat{y} - \widehat{\theta} = 0,\\ \widehat{u}_t - ik_1\xi\,\widehat{v} = 0,\\ \widehat{z}_t - i\xi\widehat{y} = 0,\\ \widehat{y}_t - i(k_2 - \tilde{\tau}_1 g_0)\xi\,\widehat{z} + k_1\widehat{v} + \tilde{\tau}_1\xi^2\displaystyle\int_0^{+\infty} g(s)\widehat{\eta}\, ds = 0,\\ \widehat{\phi}_t - i\xi\widehat{\theta} = 0,\\ \widehat{\theta}_t - i(k_3 - \tilde{\tau}_2 g_0)\,\xi\widehat{\phi} + k_1\,\widehat{v} + \tilde{\tau}_2\xi^2\displaystyle\int_0^{+\infty} g(s)\widehat{\eta}\, ds = 0,\\ \widehat{\eta}_t + \widehat{\eta}_s - \tilde{\tau}_1\,\widehat{y} - \tilde{\tau}_2\,\widehat{\theta} = 0.\end{cases} \tag{2.25}$$

Multiplying the first six equations in (2.25) by $k_1\bar{\widehat{v}}$, $\bar{\widehat{u}}$, $(k_2 - \tilde{\tau}_1 g_0)\bar{\widehat{z}}$, $\bar{\widehat{y}}$, $(k_3 - \tilde{\tau}_2 g_0)\bar{\widehat{\phi}}$ and $\bar{\widehat{\theta}}$, respectively, multiplying the last equation in (2.25) by $\xi^2 g(s)\bar{\widehat{\eta}}$ and integrating on $\mathbb{R}_+$ with respect to s, adding all the obtained equations, taking the real part of the resulting expression and using (2.20), we get

$$\frac{d}{dt}\widehat{E}(\xi,\, t) = -\frac{\xi^2}{2}\int_0^{+\infty} g(s)\frac{d}{ds}|\widehat{\eta}|^2\, ds.$$

Then, by integrating with respect to s and exploiting the properties of g, we find (2.23). □

Lemma 2.3. *Let σ, p and q be real numbers such that $\sigma > -1$ and $p,\ q > 0$. Then there exists $C_{\sigma,p,q} > 0$ such that*

$$\int_0^1 \xi^{\sigma}\, e^{-q\,t\,\xi^{p}}\, d\xi \le C_{\sigma,p,q}\,(1+t)^{-(\sigma+1)/p}, \quad \forall\, t \in \mathbb{R}_+. \tag{2.26}$$

Proof. For $0 \le t \le 1$, (2.26) is true, for any $C_{\sigma,p,q} \ge \frac{2^{(\sigma+1)/p}}{\sigma+1}$. For $t > 1$, we have

$$\int_0^1 \xi^{\sigma}\, e^{-q\,t\,\xi^{p}}\, d\xi = \int_0^1 \xi^{\sigma+1-p}\, e^{-q\,t\,\xi^{p}}\, \xi^{p-1}\, d\xi = \int_0^1 (\xi^{p})^{(\sigma+1-p)/p}\, e^{-q\,t\,\xi^{p}}\, \xi^{p-1}\, d\xi.$$

Taking $\tau = q\,t\,\xi^{p}$. Then

$$\xi^{p} = \frac{\tau}{q\,t} \quad \text{and} \quad \xi^{p-1}\, d\xi = \frac{1}{p\,q\,t}\, d\tau.$$

Replacing in the above integral, we find

$$\begin{aligned}\int_0^1 (\xi^{p})^{(\sigma+1-p)/p}\, e^{-q\,t\,\xi^{p}}\, \xi^{p-1}\, d\xi &= \int_0^{q\,t} \left(\frac{\tau}{q\,t}\right)^{(\sigma+1-p)/p} e^{-\tau}\, \frac{1}{p\,q\,t}\, d\tau \\ &\le \frac{1}{p\,(q\,t)^{(\sigma+1)/p}} \int_0^{+\infty} \tau^{(\sigma+1-p)/p}\, e^{-\tau}\, d\tau \le \frac{2^{(\sigma+1)/p}}{p\,q^{(\sigma+1)/p}} C_{\sigma,p}\,(t+1)^{-(\sigma+1)/p},\end{aligned}$$

where

$$C_{\sigma,p} = \int_0^{+\infty} \tau^{(\sigma+1-p)/p}\, e^{-\tau}\, d\tau,$$

which is a convergent integral, for any $\sigma > -1$ and $p > 0$. This completes the proof of (2.26), where

$$C_{\sigma,p,q} = \max\left\{\frac{2^{(\sigma+1)/p}}{\sigma+1}, \frac{2^{(\sigma+1)/p}}{p\,q^{(\sigma+1)/p}} C_{\sigma,p}\right\}.$$

□

Lemma 2.4. *For any positive real numbers $\sigma_1,\ \sigma_2$ and σ_3, we have*

$$\sup_{|\xi|\ge 1} |\xi|^{-\sigma_1}\, e^{-\sigma_2\,t\,|\xi|^{-\sigma_3}} \le \left(1+\sigma_1/(\sigma_2\sigma_3)\right)^{\sigma_1/\sigma_3} (1+t)^{-\sigma_1/\sigma_3}, \quad \forall t \in \mathbb{R}_+. \tag{2.27}$$

Proof. It is clear that (2.27) is satisfied for $t = 0$. Let $t > 0$ and $h(x) = x^{-\sigma_1}\, e^{-\sigma_2\,t\,x^{-\sigma_3}}$, for $x \ge 1$. Direct and simple computations show that

$$h'(x) = (\sigma_2\sigma_3 t x^{-\sigma_3} - \sigma_1) x^{-\sigma_1-1}\, e^{-\sigma_2\,t\,x^{-\sigma_3}}.$$

If $t \ge \sigma_1/(\sigma_2\sigma_3)$, then

$$\begin{aligned} h(x) &\le h(((\sigma_2\sigma_3 t)/\sigma_1)^{1/\sigma_3}) = ((\sigma_2\sigma_3)/\sigma_1)^{-\sigma_1/\sigma_3} e^{-\sigma_1/\sigma_3} \left(1+1/t\right)^{\sigma_1/\sigma_3} (1+t)^{-\sigma_1/\sigma_3} \\ &\le ((\sigma_2\sigma_3)/\sigma_1)^{-\sigma_1/\sigma_3} \left(1+(\sigma_2\sigma_3)/\sigma_1\right)^{\sigma_1/\sigma_3} (1+t)^{-\sigma_1/\sigma_3} = \left(1+\sigma_1/(\sigma_2\sigma_3)\right)^{\sigma_1/\sigma_3} (1+t)^{-\sigma_1/\sigma_3}, \end{aligned}$$

which gives (2.27) by taking $x = |\xi|$. If $0 < t < \sigma_1/(\sigma_2\sigma_3)$, then

$$h(x) \le h(1) = e^{-\sigma_2 t}\,(1+t)^{\sigma_1/\sigma_3}\,(1+t)^{-\sigma_1/\sigma_3} \le (1+\sigma_1/(\sigma_2\sigma_3))^{\sigma_1/\sigma_3}\,(1+t)^{-\sigma_1/\sigma_3},$$

so, (2.27) holds true, for $x = |\xi|$. □

Remark 1. *1. The expressions* (2.22) *and* (2.23) *show the dissipativeness of* (2.16)*:*

$$\widehat{E}(\xi,\ t) \le \widehat{E}(\xi,\ 0), \quad \forall\, t \in \mathbb{R}_+,$$

since $\gamma > 0$ *and* $g' \le 0$. *If* $\gamma = 0$ *and* $g = 0$, *then* (2.16) *is conservative:*

$$\widehat{E}(\xi,\ t) = \widehat{E}(\xi,\ 0), \quad \forall\, t \in \mathbb{R}_+.$$

2. The left inequality in (2.5) *guarantees the implication*

$$\left|\int_0^{+\infty} g(s)|\widehat{\eta}|^2\, ds\right| < +\infty \quad \to \quad \left|\int_0^{+\infty} g'(s)|\widehat{\eta}|^2\, ds\right| < +\infty.$$

The right inequality in (2.5) *will be used in Section 4 to prove our stability estimates in case of infinite memory.*

3. We have in case (2.24)

$$|\widehat{U}(\xi,\ t)|^2 = |\widehat{v}|^2 + |\widehat{u}|^2 + |\widehat{z}|^2 + |\widehat{y}|^2 + |\widehat{\phi}|^2 + |\widehat{\theta}|^2,$$

and in case (2.25)

$$|\widehat{U}(\xi,\ t)|^2 = |\widehat{v}|^2 + |\widehat{u}|^2 + |\widehat{z}|^2 + |\widehat{y}|^2 + |\widehat{\phi}|^2 + |\widehat{\theta}|^2 + \xi^2 \int_0^{+\infty} g(s)|\widehat{\eta}|^2\, ds.$$

So we deduce that

$$\alpha_1|\widehat{U}(\xi,\ t)|^2 \le \widehat{E}(\xi,\ t) \le \alpha_2|\widehat{U}(\xi,\ t)|^2, \quad \forall \xi \in \mathbb{R},\ \forall t \in \mathbb{R}_+, \tag{2.28}$$

where $\alpha_2 = \frac{1}{2}\max\{k_1, k_2, k_3, 1\}$ *and*

$$\alpha_1 = \begin{cases} \frac{1}{2}\min\{k_1, k_2, k_3, 1\} & \textit{in case } (2.24), \\ \frac{1}{2}\min\{k_1, k_2 - \tilde{\tau}_1 g_0, k_3 - \tilde{\tau}_2 g_0, 1\} & \textit{in case } (2.25). \end{cases}$$

3 Stability under frictional damping

This section is dedicated to the investigation of the asymptotic behavior, when time t goes to infinity, of the solution U of (2.14) in case of frictional damping (2.9). We will prove some polynomial decay estimates on $\|\partial_x^k U\|_{L^2(\mathbb{R})}$, where $k \in \mathbb{N}$ and the decay rate depends on the smoothness of initial data U_0. To get such polynomial decay estimates, we prove, first, that $|\widehat{U}|^2$ converges exponentially to zero with respect to time t.

In this section and in the next one, C denotes a generic positive constant, and C_ε denotes a generic positive constant depending on some positive constant ε, which can be different from line to line. Before distinguishing the three cases (1.5), we prove several equalities which will play a crucial role.

Multiplying $(2.24)_4$ and $(2.24)_3$ by $i\,\xi\,\overline{\widehat{z}}$ and $-i\,\xi\,\overline{\widehat{y}}$, respectively, adding the resulting equations, taking the real part and using (2.21), we obtain

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{z}}\right) = \xi^2\left(|\widehat{y}|^2 - k_2|\widehat{z}|^2\right) - k_1\, Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right) - \tau_2\gamma\, Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{z}}\right). \tag{3.1}$$

Multiplying $(2.24)_2$ and $(2.24)_1$ by $i\,\xi\,\overline{\widehat{v}}$ and $-i\,\xi\,\overline{\widehat{u}}$, respectively, adding the resulting equations, taking the real part and using (2.21), we find

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{u}\,\overline{\widehat{v}}\right) = \xi^2\left(|\widehat{u}|^2 - k_1|\widehat{v}|^2\right) - Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{u}}\right) - Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{u}}\right) - \tau_1\gamma\, Re\left(i\,\xi\,\widehat{u}\,\overline{\widehat{v}}\right). \tag{3.2}$$

Multiplying $(2.24)_6$ and $(2.24)_5$ by $i\,\xi\,\overline{\widehat{\phi}}$ and $-i\,\xi\,\overline{\widehat{\theta}}$, respectively, adding the resulting equations, taking the real part and using (2.21), we get

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{\phi}}\right) = \xi^2\left(|\widehat{\theta}|^2 - k_3|\widehat{\phi}|^2\right) - k_1\, Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{\phi}}\right) - \tau_3\gamma\, Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{\phi}}\right). \tag{3.3}$$

Multiplying $(2.24)_6$ and $(2.24)_1$ by $-\xi^2\,\overline{\widehat{v}}$ and $-\xi^2\,\overline{\widehat{\theta}}$, respectively, adding the resulting equations, taking the real part and using (2.20), we arrive at

$$\begin{aligned}\frac{d}{dt}Re\left(-\xi^2\,\widehat{\theta}\,\overline{\widehat{v}}\right) &= \xi^2\left(k_1\,|\widehat{v}|^2-|\widehat{\theta}|^2\right)-\xi^2\,Re\left(i\,\xi\,\widehat{u}\,\overline{\widehat{\theta}}\right)-k_3\,\xi^2\,Re\left(i\,\xi\,\widehat{\phi}\,\overline{\widehat{v}}\right)\\ &\quad -\xi^2\,Re\left(\widehat{y}\,\overline{\widehat{\theta}}\right)+\tau_3\gamma\,\xi^2\,Re\left(\widehat{\theta}\,\overline{\widehat{v}}\right). \end{aligned}\tag{3.4}$$

Multiplying $(2.24)_4$ and $(2.24)_1$ by $-\xi^2\,\overline{\widehat{v}}$ and $-\xi^2\,\overline{\widehat{y}}$, respectively, adding the resulting equations, taking the real part and using (2.20), we infer that

$$\begin{aligned}\frac{d}{dt}Re\left(-\xi^2\,\widehat{y}\,\overline{\widehat{v}}\right) &= \xi^2\left(k_1\,|\widehat{v}|^2-|\widehat{y}|^2\right)-\xi^2\,Re\left(i\,\xi\,\widehat{u}\,\overline{\widehat{y}}\right)-k_2\,\xi^2\,Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{v}}\right)\\ &\quad -\xi^2\,Re\left(\widehat{\theta}\,\overline{\widehat{y}}\right)+\tau_2\gamma\,\xi^2\,Re\left(\widehat{y}\,\overline{\widehat{v}}\right). \end{aligned}\tag{3.5}$$

Multiplying $(2.24)_3$ and $(2.24)_6$ by $i\,\xi\,\overline{\widehat{\theta}}$ and $-i\,\xi\,\overline{\widehat{z}}$, respectively, adding the resulting equations, taking the real part and using (2.21), we entail

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{\theta}}\right)=-\xi^2\,Re\left(\widehat{y}\,\overline{\widehat{\theta}}\right)+k_3\,\xi^2\,Re\left(\widehat{\phi}\,\overline{\widehat{z}}\right)+k_1\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right)+\tau_3\gamma\,Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{z}}\right).\tag{3.6}$$

Mltiplying $(2.24)_5$ and $(2.24)_4$ by $i\,\xi\,\overline{\widehat{y}}$ and $-i\,\xi\,\overline{\widehat{\phi}}$, respectively, adding the resulting equations, taking the real part and using (2.21), it follows that

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{\phi}\,\overline{\widehat{y}}\right)=-\xi^2\,Re\left(\widehat{\theta}\,\overline{\widehat{y}}\right)+k_2\,\xi^2\,Re\left(\widehat{z}\,\overline{\widehat{\phi}}\right)+k_1\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{\phi}}\right)+\tau_2\gamma\,Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{\phi}}\right).\tag{3.7}$$

Multiplying $(2.24)_2$ and $(2.24)_3$ by $-\,\overline{\widehat{z}}$ and $-\,\overline{\widehat{u}}$, respectively, adding the resulting equations, taking the real part and using (2.20), it appears that

$$\frac{d}{dt}Re\left(-\,\widehat{u}\,\overline{\widehat{z}}\right)=-k_1\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right)-\,Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{u}}\right)+\tau_1\gamma\,Re\left(\widehat{u}\,\overline{\widehat{z}}\right).\tag{3.8}$$

Finally, multiplying $(2.24)_2$ and $(2.24)_5$ by $-\,\overline{\widehat{\phi}}$ and $-\,\overline{\widehat{u}}$, respectively, adding the resulting equations, taking the real part and using (2.20), we see that

$$\frac{d}{dt}Re\left(-\,\widehat{u}\,\overline{\widehat{\phi}}\right)=-k_1\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{\phi}}\right)-\,Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{u}}\right)+\tau_1\gamma\,Re\left(\widehat{u}\,\overline{\widehat{\phi}}\right).\tag{3.9}$$

3.1 Case 1: $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$

We start by presenting the exponential stability result of (2.16) in the next lemma.

Lemma 3.1. *Assume that* $k_2 \neq k_3$. *Let* $\widehat{U}$ *be the solution of* (2.16). *Then there exist* $c,\ \widetilde{c} > 0$ *such that*

$$|\widehat{U}(\xi,\ t)|^2 \leq \widetilde{c}\,e^{-c\,f(\xi)\,t}\,|\widehat{U}_0(\xi)|^2, \quad \forall\,\xi\in\mathbb{R}, \quad \forall\,t\in\mathbb{R}_+,\tag{3.10}$$

where

$$f(\xi)=\frac{\xi^4}{1+\xi^2+\xi^4+\xi^6}.\tag{3.11}$$

Proof. Let us define the functional F_0 as follows:

$$\begin{aligned}F_0(\xi,\ t) &= Re\left[i\,\xi\,\left(\lambda_1\,\widehat{y}\,\overline{\widehat{z}}+\lambda_3\,\widehat{\theta}\,\overline{\widehat{\phi}}+i\lambda_4\xi\,\widehat{\theta}\,\overline{\widehat{v}}-\frac{(\lambda_4+1)k_2}{k_2-k_3}\,\widehat{z}\,\overline{\widehat{\theta}}+\frac{(\lambda_4+1)k_3}{k_2-k_3}\,\widehat{\phi}\,\overline{\widehat{y}}\right)\right]\\ &\quad +Re\left(-\xi^2\,\widehat{y}\,\overline{\widehat{v}}+i\,\lambda_2\,\xi\widehat{u}\,\overline{\widehat{v}}\right), \end{aligned}\tag{3.12}$$

where λ_1, λ_2, λ_3 and λ_4 are positive constants to be specified later (F_0 is well defined since $k_2 \neq k_3$). By multiplying (3.1)-(3.4), (3.6) and (3.7) by $\lambda_1, \lambda_2, \lambda_3, \lambda_4, -\frac{(\lambda_4+1)k_2}{k_2-k_3}$ and $\frac{(\lambda_4+1)k_3}{k_2-k_3}$, respectively, adding the obtained equations and adding (3.5), we deduce that

$$\begin{aligned}\frac{d}{dt}F_0(\xi,\, t) \;&=\; -\xi^2\left(k_2\lambda_1\,|\widehat{z}|^2 + k_3\lambda_3\,|\widehat{\phi}|^2 + (1-\lambda_1)\,|\widehat{y}|^2 + (\lambda_4-\lambda_3)\,|\widehat{\theta}|^2 + (k_1\lambda_2 - k_1\lambda_4 - k_1)\,|\widehat{v}|^2\right)\\ &+I_1 Re(i\xi\widehat{v}\,\overline{\widehat{z}}) + I_2 Re(i\xi\widehat{v}\,\overline{\widehat{\phi}}) + \xi^2\left[\lambda_2|\widehat{u}|^2 - Re\left(i\xi\,\left(\lambda_4\widehat{u}\,\overline{\widehat{\theta}} + \widehat{u}\,\overline{\widehat{y}}\right)\right)\right]\\ &-\lambda_2 Re\left[i\xi\,\left(\widehat{y}\,\overline{\widehat{u}} + \widehat{\theta}\,\overline{\widehat{u}} + \gamma\widehat{u}\,\overline{\widehat{v}}\right)\right],\end{aligned} \tag{3.13}$$

where

$$I_1 = k_2\xi^2 - k_1\lambda_1 - \frac{(\lambda_4+1)k_1k_2}{k_2-k_3} \quad \text{and} \quad I_2 = k_3\lambda_4\xi^2 - k_1\lambda_3 + \frac{(\lambda_4+1)k_1k_3}{k_2-k_3}.$$

We put

$$F(\xi,\, t) = \xi^2 F_0(\xi,\, t) - \frac{1}{k_1}\xi^2 Re\left(I_1\,\widehat{u}\,\overline{\widehat{z}} + I_2\,\widehat{u}\,\overline{\widehat{\phi}}\right). \tag{3.14}$$

Multiplying (3.8), (3.9) and (3.13) by $\frac{I_1}{k_1}\xi^2$, $\frac{I_2}{k_1}\xi^2$ and ξ^2, respectively, and adding the obtained equations, we find

$$\begin{aligned}\frac{d}{dt}F(\xi,\, t) \;&=\; -\xi^4\left(k_2\lambda_1\,|\widehat{z}|^2 + k_3\lambda_3\,|\widehat{\phi}|^2 + (1-\lambda_1)\,|\widehat{y}|^2 + (\lambda_4-\lambda_3)\,|\widehat{\theta}|^2 + (k_1\lambda_2 - k_1\lambda_4 - k_1)\,|\widehat{v}|^2\right)\\ &+\xi^4\left[\lambda_2|\widehat{u}|^2 - Re\left(i\xi\,\left(\lambda_4\widehat{u}\,\overline{\widehat{\theta}} + \widehat{u}\,\overline{\widehat{y}}\right)\right)\right] - \lambda_2\xi^2 Re\left[i\xi\,\left(\widehat{y}\,\overline{\widehat{u}} + \widehat{\theta}\,\overline{\widehat{u}} + \gamma\widehat{u}\,\overline{\widehat{v}}\right)\right]\\ &+\frac{I_1}{k_1}\xi^2 Re\left(\gamma\widehat{u}\,\overline{\widehat{z}} - i\xi\widehat{y}\,\overline{\widehat{u}}\right) + \frac{I_2}{k_1}\xi^2 Re\left(\gamma\widehat{u}\,\overline{\widehat{\phi}} - i\xi\widehat{\theta}\,\overline{\widehat{u}}\right).\end{aligned} \tag{3.15}$$

Applying Young's inequality for the terms depending on $\widehat{u}$ in (3.15), we get, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt}F(\xi,\, t) \le -(k_2\lambda_1 - \varepsilon_0)\xi^4\,|\widehat{z}|^2 - (k_3\lambda_3 - \varepsilon_0)\xi^4\,|\widehat{\phi}|^2 - (1-\lambda_1-\varepsilon_0)\,\xi^4\,|\widehat{y}|^2 \tag{3.16}$$

$$-(\lambda_4-\lambda_3-\varepsilon_0)\,\xi^4\,|\widehat{\theta}|^2 - (k_1\lambda_2 - k_1\lambda_4 - k_1 - \varepsilon_0)\xi^4\,|\widehat{v}|^2 + C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4}(1+\xi^2+\xi^4+\xi^6)|\widehat{u}|^2.$$

We choose $0 < \lambda_1 < 1$, $\lambda_2 > 1$, $0 < \lambda_3 < \lambda_4 < \lambda_2 - 1$ and

$$0 < \varepsilon_0 < \min\{k_2\lambda_1, k_3\lambda_3, 1-\lambda_1, \lambda_4-\lambda_3, k_1\lambda_2 - k_1\lambda_4 - k_1\}.$$

Hence, using the definition (2.18) of $\widehat{E}$, (3.16) leads, for some positive constant c_1, to

$$\frac{d}{dt}F(\xi,\, t) \le -c_1\xi^4\widehat{E}(\xi,\, t) + C\,\left(1+\xi^2+\xi^4+\xi^6\right)|\widehat{u}|^2. \tag{3.17}$$

Now, we ntroduce the *Perturbed Energy L* as follows:

$$L(\xi,\, t) = \lambda\,\widehat{E}(\xi,\, t) + \frac{1}{1+\xi^2+\xi^4+\xi^6}\,F(\xi,\, t), \tag{3.18}$$

where λ is a positive constant to be fixed later. Then, from (2.22), (3.17) and (3.18), we have

$$\frac{d}{dt}L(\xi,\, t) \le -c_1 f(\xi)\widehat{E}(\xi,\, t) - (\gamma\,\lambda - C)\,|\widehat{u}|^2, \tag{3.19}$$

where f is defined in (3.11). Moreover, using the definitions (2.18), (3.14) and (3.18) of $\widehat{E}$, F and L, respectively, we arrive at, for some $c_2 > 0$ (independent of λ),

$$|L(\xi,\, t) - \lambda\,\widehat{E}(\xi,\, t)| \le \frac{c_2\,(\xi^2+|\xi|^3+\xi^4)}{1+\xi^2+\xi^4+\xi^6}\,\widehat{E}(\xi,\, t) \le 3c_2\,\widehat{E}(\xi,\, t). \tag{3.20}$$

Therefore, for λ large enough such that $\lambda > \max\left\{\frac{C}{\gamma},\, 3c_2\right\}$, it follows from (3.19) and (3.20) that

$$\frac{d}{dt}L(\xi,\, t) + c_1\,f(\xi)\,\widehat{E}(\xi,\, t) \le 0 \tag{3.21}$$

and

$$c_3\,\widehat{E}(\xi,\ t) \le L(\xi,\ t) \le c_4\,\widehat{E}(\xi,\ t), \tag{3.22}$$

where $c_3 = \lambda - 3c_2 > 0$ and $c_4 = \lambda + 3c_2 > 0$. Consequently, a combination of (3.21) and the second inequality in (3.22) leads, for $c = \frac{c_1}{c_4}$, to

$$\frac{d}{dt}L(\xi,\ t) + c\,f(\xi)\,L(\xi,\ t) \le 0. \tag{3.23}$$

Finally, by integration of (3.23) with respect to t and using (2.28) and (3.22), estimate (3.10) follows with $\widetilde{c} = \frac{c_4\alpha_2}{c_3\alpha_1}$. □

Theorem 3.2. *Assume that $k_2 \neq k_3$. Let $N,\ \ell \in \mathbb{N}^\star$ such that $\ell \le N$, $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$ and U be the solution of* (2.14)*. Then, for any $j = 0,\ \dots,\ N-\ell$, there exists $c_0 > 0$ such that*

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \le c_0\,(1+t)^{-1/8-j/4}\,\|U_0\|_{L^1(\mathbb{R})} + c_0\,(1+t)^{-\ell/2}\,\|\partial_x^{j+\ell}U_0\|_{L^2(\mathbb{R})}, \quad \forall t \in \mathbb{R}_+. \tag{3.24}$$

Proof. From (3.11), we have (low and high frequences)

$$f(\xi) \ge \begin{cases} \frac{1}{4}\,\xi^4 & \text{if} \quad |\xi| \le 1, \\ \frac{1}{4}\,\xi^{-2} & \text{if} \quad |\xi| > 1. \end{cases} \tag{3.25}$$

Applying Plancherel's theorem and (3.10), it follows that

$$\|\partial_x^j U\|^2_{L^2(\mathbb{R})} = \left\|\widehat{\partial_x^j U}(x,\ t)\right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \xi^{2j}\,|\widehat{U}(\xi,\ t)|^2 d\xi \tag{3.26}$$

$$\begin{aligned} &\le \widetilde{c}\int_{\mathbb{R}} \xi^{2j}\,e^{-c f(\xi)\,t}\,|\widehat{U}_0(\xi)|^2 d\xi \\ &\le \widetilde{c}\int_{|\xi|\le 1} \xi^{2j}\,e^{-c f(\xi)\,t}\,|\widehat{U}_0(\xi)|^2 d\xi + \widetilde{c}\int_{|\xi|>1} \xi^{2j}\,e^{-c f(\xi)\,t}\,|\widehat{U}_0(\xi)|^2\,d\xi := J_1 + J_2. \end{aligned}$$

Using (2.26) (with $\sigma = 2j$ and $p = 4$) and (3.25), it appears, for the low frequency region, that

$$J_1 \le C\,\|\widehat{U}_0\|^2_{L^\infty(\mathbb{R})} \int_{|\xi|\le 1} \xi^{2j}\,e^{-\frac{c}{4}\,t\,\xi^4}\,d\xi \le C\,(1+t)^{-\frac{1}{4}(1+2j)}\,\|U_0\|^2_{L^1(\mathbb{R})}. \tag{3.27}$$

In the high frequency region, using (3.25), we entail

$$\begin{aligned} J_2 &\le C\int_{|\xi|>1} |\xi|^{2j}\,e^{-\frac{c}{4}\,t\,\xi^{-2}}\,|\widehat{U}_0(\xi)|^2\,d\xi \\ &\le C\sup_{|\xi|>1}\left\{|\xi|^{-2\ell}\,e^{-\frac{c}{4}\,t\,|\xi|^{-2}}\right\}\int_{\mathbb{R}} |\xi|^{2(j+\ell)}\,|\widehat{U}_0(\xi)|^2\,d\xi, \end{aligned}$$

then, using (2.27) (with $\sigma_1 = 2l$, $\sigma_2 = \frac{c}{4}$ and $\sigma_3 = 2$),

$$J_2 \le C\,(1+t)^{-\ell}\,\|\,\partial_x^{j+\ell}U_0\,\|^2_{L^2(\mathbb{R})}, \tag{3.28}$$

and so, by combining (3.26)-(3.28), we get (3.24). □

We finish this subsection by proving that the condition $k_2 \neq k_3$ is necessary for the stability of (2.16).

Theorem 3.3. *Assume that $k_2 = k_3$. Then $|\widehat{U}(\xi,\ t)|$ doesn't converge to zero when time t goes to infinity.*

Proof. We show that, for any $\xi \in \mathbb{R}$, the matrix

$$A := -(-\xi^2 A_1 + i\xi A_2 + A_3)$$

has at least a pure imaginary eigenvalue; that is

$$\forall \xi \in \mathbb{R},\ \exists \lambda \in \mathbb{C}:\ Re\,(\lambda) = 0, \quad Im\,(\lambda) \neq 0 \quad \text{and} \quad det(\lambda I - A) = 0,$$

where I denotes the identity matrix. From (2.15) with $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ and $k_2 = k_3$, we have

$$\lambda I - A = \begin{pmatrix} \lambda & -i\xi & 0 & -1 & 0 & -1 \\ -ik_1\xi & \lambda + \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & -i\xi & 0 & 0 \\ k_1 & 0 & -ik_2\xi & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -i\xi \\ k_1 & 0 & 0 & 0 & -ik_2\xi & \lambda \end{pmatrix}.$$

A direct computation shows that

$$det(\lambda I - A) = (\lambda^2 + k_2\xi^2)\left[k_1\xi^2(\lambda^2 + k_2\xi^2) + \lambda(\lambda + \gamma)(\lambda^2 + k_2\xi^2 + 2k_1)\right].$$

It is clear that, if $\xi \neq 0$, then $\lambda = i\sqrt{k_2}\xi$ is a pure imaginary eigenvalue of A. If $\xi = 0$, then $\lambda = i\sqrt{2k_1}$ is also a pure imaginary eigenvalue of A. Consequently, according to (2.17) (see [37]), the solution of (2.16) doesn't converge to zero when time t goes to infinity. □

3.2 Case 2: $(\tau_1, \tau_2, \tau_3) = (0, 1, 0)$

As in the previous case 1, we present, first, our exponential stability result for (2.16).

Lemma 3.4. *Let $\widehat{U}$ be the solution of* (2.16). *Then there exist* $c,\ \widetilde{c} > 0$ *such that* (3.10) *is staisfied with*

$$f(\xi) = \begin{cases} \frac{\xi^2}{1+\xi^2} & \text{if } k_1 = k_2 = k_3, \\ \frac{\xi^2}{1+\xi^2+\xi^4+\xi^6} & \text{if not.} \end{cases} \tag{3.29}$$

Proof. Let us introduce the functionals

$$F_0(\xi,\ t) = Re\left[i\,\xi\,\left(\lambda_1\widehat{y}\,\overline{\widehat{z}} - \lambda_2\,\widehat{u}\,\overline{\widehat{v}} + \lambda_3\,\widehat{\theta}\,\overline{\widehat{\phi}}\right) - \lambda_4\,\xi^2\ \widehat{\theta}\,\overline{\widehat{v}} + \xi^2\,\widehat{y}\,\overline{\widehat{v}}\right], \tag{3.30}$$

$$F_1(\xi,\ t) = \left(\frac{k_2}{k_1}\xi^2 + \lambda_1\right) Re\left(\widehat{u}\,\overline{\widehat{z}}\right), \tag{3.31}$$

$$F_2(\xi,\ t) = \frac{k_2}{k_1k_3}\left(k_3\lambda_4\xi^2 - k_1\lambda_3\right) Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{\theta}} - \widehat{u}\,\overline{\widehat{z}} - i\,\frac{k_3}{k_2}\,\xi\,\widehat{\phi}\,\overline{\widehat{y}}\right) \tag{3.32}$$

and

$$F_3(\xi,\ t) = -\frac{k_2}{k_3}\left(\lambda_4\xi^2 + \lambda_2\right) Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{\theta}} - \widehat{u}\,\overline{\widehat{z}} - i\,\frac{k_3}{k_2}\,\xi\,\widehat{\phi}\,\overline{\widehat{y}} + \frac{k_3}{k_2}\,\widehat{u}\,\overline{\widehat{\phi}}\right), \tag{3.33}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ and λ_4 are positive constants to be fixed later. Multiplying (3.1)-(3.5) by $\lambda_1, -\lambda_2, \lambda_3, \lambda_4$ and -1, respectively, and adding the obtained equations, we get

$$\frac{d}{dt}F_0(\xi,\ t) = (\lambda_1 + 1)\xi^2|\widehat{y}|^2 + Re\left((1 - \lambda_4)\xi^2\widehat{\theta}\,\overline{\widehat{y}} + (\xi^2 - \lambda_2)i\xi\widehat{u}\,\overline{\widehat{y}} - \gamma\xi^2\widehat{y}\overline{\widehat{v}} - i\gamma\lambda_1\xi\widehat{y}\overline{\widehat{z}}\right) \tag{3.34}$$

$$-\xi^2\ \left((k_1 - k_1\lambda_2 - k_1\lambda_4)|\widehat{v}|^2 + k_2\lambda_1|\widehat{z}|^2 + (\lambda_4 - \lambda_3)|\widehat{\theta}|^2 + \lambda_2|\widehat{u}|^2 + k_3\lambda_3|\widehat{\phi}|^2\right)$$

$$+\left(k_2\xi^2 + k_1\lambda_1\right)\ Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{v}}\right) + \left(k_3\lambda_4\xi^2 - k_1\lambda_3\right)\ Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{\phi}}\right) + \left(\lambda_4\xi^2 + \lambda_2\right)\ Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{u}}\right).$$

Multiplying (3.8) by $-\left(\frac{k_2}{k_1}\xi^2+\lambda_1\right)$, we obtain

$$\frac{d}{dt}F_1(\xi,\,t)=-\left(k_2\xi^2+k_1\lambda_1\right)\,Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{v}}\right)+\left(\frac{k_2}{k_1}\xi^2+\lambda_1\right)Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{u}}\right). \tag{3.35}$$

Multiplying (3.7) by $-\frac{k_3}{k_2}$, adding (3.6) and (3.8), and multiplying the obtained equation by

$$\frac{k_2}{k_1k_3}\left(k_3\lambda_4\xi^2-k_1\lambda_3\right),$$

we find

$$\begin{aligned}\frac{d}{dt}F_2(\xi,\,t) &= \frac{k_2}{k_1k_3}\left(k_3\lambda_4\xi^2-k_1\lambda_3\right)Re\left[\left(\frac{k_3}{k_2}-1\right)\xi^2\widehat{y}\,\overline{\widehat{\theta}}-i\xi\widehat{y}\,\overline{\widehat{u}}-i\frac{\gamma k_3}{k_2}\xi\widehat{y}\,\overline{\widehat{\phi}}\right]\\ &\quad-\left(k_3\lambda_4\xi^2-k_1\lambda_3\right)\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{\phi}}\right).\end{aligned} \tag{3.36}$$

Similarily, adding (3.7) and (3.9), multiplying by $-\frac{k_3}{k_2}$, adding (3.6) and (3.8), and multiplying the obtained formula by $-\frac{k_2}{k_3}\left(\lambda_4\xi^2+\lambda_2\right)$, we deduce that

$$\begin{aligned}\frac{d}{dt}F_3(\xi,\,t) &= \frac{k_2}{k_3}\left(\lambda_4\xi^2+\lambda_2\right)Re\left[\left(1-\frac{k_3}{k_2}\right)\xi^2\widehat{y}\,\overline{\widehat{\theta}}+i\xi\widehat{y}\,\overline{\widehat{u}}+i\frac{\gamma k_3}{k_2}\xi\widehat{y}\,\overline{\widehat{\phi}}\right]\\ &\quad-\left(\lambda_4\xi^2+\lambda_2\right)\,Re\left(i\,\xi\,\widehat{\theta}\,\overline{\widehat{u}}\right).\end{aligned} \tag{3.37}$$

Now, let us define the functionals F and L by

$$F(\xi,\,t)=F_0(\xi,\,t)+F_1(\xi,\,t)+F_2(\xi,\,t)+F_3(\xi,\,t) \tag{3.38}$$

and

$$L(\xi,\,t)=\lambda\,\widehat{E}(\xi,\,t)+\frac{1}{\tilde{f}(\xi)}F(\xi,\,t), \tag{3.39}$$

where λ is a positive constant to be specified after and

$$\tilde{f}(\xi)=\begin{cases}1+\xi^2 & \text{if } k_1=k_2=k_3,\\ 1+\xi^2+\xi^4+\xi^6 & \text{if not.}\end{cases} \tag{3.40}$$

By combining (3.34)-(3.37), we infer that

$$\begin{aligned}\frac{d}{dt}F(\xi,\,t) &= -\xi^2\,\left((k_1-k_1\lambda_2-k_1\lambda_4)|\widehat{v}|^2+k_2\lambda_1|\widehat{z}|^2+(\lambda_4-\lambda_3)|\widehat{\theta}|^2+\lambda_2|\widehat{u}|^2+k_3\lambda_3|\widehat{\phi}|^2\right)\\ &\quad+F_4(\xi,\,t),\end{aligned} \tag{3.41}$$

where

$$F_4(\xi,\,t)=Re\left(I_1\widehat{\theta}\,\overline{\widehat{y}}+iI_2\xi\widehat{y}\,\overline{\widehat{u}}+iI_3\xi\widehat{y}\,\overline{\widehat{\phi}}-\gamma\xi^2\widehat{y}\,\overline{\widehat{v}}-i\gamma\lambda_1\xi\widehat{y}\,\overline{\widehat{z}}\right)+(\lambda_1+1)\xi^2|\widehat{y}|^2, \tag{3.42}$$

$$I_1=\left[1-\lambda_4+\frac{k_2}{k_1k_3}\left(\frac{k_3}{k_2}-1\right)(k_3\lambda_4\xi^2-k_1\lambda_3)+\left(\frac{k_2}{k_3}-1\right)\lambda_4\xi^2+\left(\frac{k_2}{k_3}-1\right)\lambda_2\right]\xi^2,$$

$$I_2=\left[\left(\frac{k_2}{k_3}-\frac{k_2}{k_1}\right)\lambda_4+\frac{k_2}{k_1}-1\right]\xi^2+\lambda_1+\left(\frac{k_2}{k_3}+1\right)\lambda_2+\frac{k_2}{k_3}\lambda_3$$

and

$$I_3=\gamma\left(1-\frac{k_3}{k_1}\right)\lambda_4\xi^2+\gamma(\lambda_2+\lambda_3).$$

Noticing that, if $k_1=k_2=k_3$, then I_2 and I_3 are constants, and $I_1=(1-\lambda_4)\xi^2$. Otherwise, I_2 and I_3 are of the form $const\,\xi^2+const$, and I_1 is of the form $(const\,\xi^2+const)\xi^2$. Then, by applying Young's inequality, we see that, for any $\varepsilon_0>0$,

$$F_4(\xi,\,t)\le\varepsilon_0\xi^2\left(|\widehat{\theta}|^2+|\widehat{u}|^2+|\widehat{\phi}|^2+|\widehat{v}|^2+|\widehat{z}|^2\right)+C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4}\tilde{f}(\xi)|\widehat{y}|^2. \tag{3.43}$$

Thus, we conclude from (3.41) and (3.43) that

$$\frac{d}{dt}F(\xi,\ t) \le C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4}\tilde{f}(\xi)|\widehat{y}|^2 \tag{3.44}$$
$$-\xi^2\left((k_1-k_1\lambda_2-k_1\lambda_4-\varepsilon_0)|\widehat{v}|^2+(k_2\lambda_1-\varepsilon_0)|\widehat{z}|^2+(\lambda_4-\lambda_3-\varepsilon_0)|\widehat{\theta}|^2+(\lambda_2-\varepsilon_0)|\widehat{u}|^2+(k_3\lambda_3-\varepsilon_0)|\widehat{\phi}|^2\right).$$

We choose $0 < \lambda_1$, $0 < \lambda_2 < 1$, $0 < \lambda_3 < \lambda_4 < 1-\lambda_2$ and

$$0 < \varepsilon_0 < \min\{k_1-k_1\lambda_2-k_1\lambda_4,\, k_2\lambda_1,\, \lambda_4-\lambda_3,\, \lambda_2,\, k_3\lambda_3\}.$$

Therefore, using the definition of $\widehat{E}$, (3.44) leads, for some positive constant c_1, to

$$\frac{d}{dt}F(\xi,\ t) \le -c_1\xi^2\widehat{E}(\xi,\ t)+C\tilde{f}(\xi)|\widehat{y}|^2. \tag{3.45}$$

Then from (2.22), (3.39) and (3.45) we have

$$\frac{d}{dt}L(\xi,\ t) \le -c_1 f(\xi)\widehat{E}(\xi,\ t)-(\gamma\lambda-C)\,|\widehat{y}|^2, \tag{3.46}$$

where f is given in (3.29). On the other hand, the definitions (2.18), (3.38) and (3.39) of $\widehat{E}$, F and L, respectively, imply that there exists $c_2 > 0$ (independent of λ) such that

$$\left|L(\xi,\ t)-\lambda\widehat{E}(\xi,\ t)\right| \le \begin{cases} c_2\frac{1+|\xi|+\xi^2}{1+\xi^2}\widehat{E}(\xi,\ t) & \text{if } k_1=k_2=k_3,\\ c_2\frac{1+|\xi|+\xi^2+|\xi|^3}{1+\xi^2+\xi^4+\xi^6}\widehat{E}(\xi,\ t) & \text{if not} \end{cases} \le 4c_2\widehat{E}(\xi,\ t).$$

So, we choose $\lambda > \max\left\{\frac{C}{\gamma}, 4c_2\right\}$ to get (3.21) and (3.22) with $c_3 = \lambda-4c_2 > 0$ and $c_4 = \lambda+4c_2 > 0$. The proof can be completed as for Lemma 3.1. □

Theorem 3.5. *Let $N,\ \ell \in \mathbb{N}^\star$ such that $\ell \le N$, $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$ and U be the solution of (2.14). Then, for any $j = 0, \ldots, N-\ell$, there exist $c_0,\ \tilde{c}_0 > 0$ such that, for any $t \in \mathbb{R}_+$,*

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \le c_0\,(1+t)^{-1/4-j/2}\,\|U_0\|_{L^1(\mathbb{R})}+c_0e^{-\tilde{c}_0 t}\,\|\partial_x^{j+\ell}U_0\|_{L^2(\mathbb{R})} \quad \text{if } k_1=k_2=k_3, \tag{3.47}$$

and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \le c_0\,(1+t)^{-1/4-j/2}\,\|U_0\|_{L^1(\mathbb{R})}+c_0\,(1+t)^{-\ell/4}\,\|\partial_x^{j+\ell}U_0\|_{L^2(\mathbb{R})} \quad \textit{if not}. \tag{3.48}$$

Proof. From (3.29), we have (low and high frequences)

$$f(\xi) \ge \begin{cases} \frac{1}{2}\xi^2 & \text{if } |\xi| \le 1,\\ \frac{1}{2} & \text{if } |\xi| > 1 \end{cases} \quad \text{if } k_1=k_2=k_3, \tag{3.49}$$

and

$$f(\xi) \ge \begin{cases} \frac{1}{4}\xi^2 & \text{if } |\xi| \le 1,\\ \frac{1}{4}\xi^{-4} & \text{if } |\xi| > 1 \end{cases} \quad \text{if not}. \tag{3.50}$$

The proof of (3.48) is identical to the one of Theorem 3.2 by using (3.50) and applying (2.26) (with $\sigma = 2j$ and $p = 2$) and (2.27) (with $\sigma_1 = 2l$, $\sigma_2 = \frac{c}{4}$ and $\sigma_3 = 4$). To get (3.47), noticing that the low frequency region can be teated as in the proof of Theorem 3.2. For the high frequencies, we have just to note that (3.49) implies that

$$\begin{aligned}\int\limits_{|\xi|>1} |\xi|^{2j}\,e^{-cf(\xi)t}\,|\widehat{U}(\xi,\ 0)|^2\,d\xi &\le \int\limits_{|\xi|>1} |\xi|^{2j}\,e^{-\frac{c}{2}t}\,|\widehat{U}(\xi,\ 0)|^2\,d\xi\\ &\le \sup_{|\xi|>1}\left\{|\xi|^{-2\ell}\,e^{-\frac{c}{2}t}\right\}\int\limits_{\mathbb{R}} |\xi|^{2(j+\ell)}\,|\widehat{U}(\xi,\ 0)|^2\,d\xi\\ &\le e^{-\frac{c}{2}t}\,\|\,\partial_x^{j+\ell}U_0\,\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

□

3.3 Case 3: $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$

In this case, we prove the same stability results for (2.16) and (2.14) given in Subsection 3.2. Moreover, the proofs are very similar.

Lemma 3.6. *The result of Lemma 3.4 holds true also when* $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$.

Proof. We introduce the functionals

$$F_0(\xi,\ t) = Re\left[i\,\xi\,\left(\lambda_1 \hat{y}\,\overline{\hat{z}} - \lambda_2\,\hat{u}\,\overline{\hat{v}} + \lambda_3\,\hat{\theta}\,\overline{\hat{\phi}}\right) + \lambda_4\,\xi^2\,\hat{\theta}\,\overline{\hat{v}} - \xi^2\,\hat{y}\,\overline{\hat{v}}\right], \tag{3.51}$$

$$F_1(\xi,\ t) = \left(\frac{k_3}{k_1}\lambda_4 \xi^2 + \lambda_3\right) Re\left(\hat{u}\,\overline{\hat{\phi}}\right), \tag{3.52}$$

$$F_2(\xi,\ t) = -\frac{1}{k_1}\left(k_2\xi^2 - k_1\lambda_1\right) Re\left(i\,\xi\,\hat{z}\,\overline{\hat{\theta}} - i\frac{k_3}{k_2}\xi\hat{\phi}\,\overline{\hat{y}} + \frac{k_3}{k_2}\hat{u}\,\overline{\hat{\phi}}\right) \tag{3.53}$$

and

$$F_3(\xi,\ t) = \left(\xi^2 + \lambda_2\right) Re\left(i\,\xi\,\hat{z}\,\overline{\hat{\theta}} - i\frac{k_3}{k_2}\xi\hat{\phi}\,\overline{\hat{y}} + \frac{k_3}{k_2}\hat{u}\,\overline{\hat{\phi}} - \hat{u}\,\overline{\hat{z}}\right), \tag{3.54}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ and λ_4 are positive constants to be fixed later. Multiplying (3.1), (3.2), (3.3) and (3.4) by λ_1, $-\lambda_2$, λ_3 and $-\lambda_4$, respectively, adding the obtained equations in addition to (3.5), we get

$$\begin{aligned}\frac{d}{dt}F_0(\xi,\ t) &= Re\left[i\xi\left(\left(\lambda_2 - \lambda_4\xi^2\right)\hat{\theta}\,\overline{\hat{u}} - \gamma\lambda_3\hat{\theta}\,\overline{\hat{\phi}}\right) + (\lambda_4 - 1)\xi^2\hat{\theta}\,\overline{\hat{y}} - \gamma\lambda_4\xi^2\hat{\theta}\,\overline{\hat{v}}\right] \\ &+(\lambda_3 + \lambda_4)\xi^2|\hat{\theta}|^2 - \xi^2\left((k_1\lambda_4 - k_1\lambda_2 - k_1)|\hat{v}|^2 + k_2\lambda_1|\hat{z}|^2 + (1-\lambda_1)|\hat{y}|^2 + \lambda_2|\hat{u}|^2 + k_3\lambda_3|\hat{\phi}|^2\right) \\ &+(k_2\xi^2 - k_1\lambda_1)\,Re\left(i\,\xi\,\hat{v}\,\overline{\hat{z}}\right) + (k_3\lambda_4\xi^2 + k_1\lambda_3)\,Re\left(i\,\xi\,\hat{\phi}\,\overline{\hat{v}}\right) + (\xi^2 + \lambda_2)\,Re\left(i\,\xi\,\hat{y}\,\overline{\hat{u}}\right).\end{aligned} \tag{3.55}$$

Multiplying (3.9) by $-\left(\frac{k_3}{k_1}\lambda_4\xi^2 + \lambda_3\right)$, we find

$$\frac{d}{dt}F_1(\xi,\ t) = \left(\frac{k_3}{k_2}\lambda_4\xi^2 + \lambda_3\right) Re\left(i\,\xi\,\hat{\theta}\,\overline{\hat{u}}\right) - \left(k_3\lambda_4\xi^2 + k_1\lambda_3\right) Re\left(i\,\xi\,\hat{\phi}\,\overline{\hat{v}}\right). \tag{3.56}$$

Adding (3.7) and (3.9), multiplying the obtained equation by $-\frac{k_3}{k_2}$, adding (3.6) and multiplying the reuslting equation by $-\left(\frac{k_2}{k_1}\xi^2 - \lambda_1\right)$, we infer that

$$\begin{aligned}\frac{d}{dt}F_2(\xi,\ t) &= \left(\frac{k_2}{k_1}\xi^2 - \lambda_1\right) Re\left[i\xi\left(-\frac{k_3}{k_2}\hat{\theta}\,\overline{\hat{u}} - \gamma\hat{\theta}\,\overline{\hat{z}}\right) + \left(1 - \frac{k_3}{k_2}\right)\xi^2\hat{\theta}\,\overline{\hat{y}}\right] \\ &\quad - \left(k_2\xi^2 - k_1\lambda_1\right) Re\left(i\,\xi\,\hat{v}\,\overline{\hat{z}}\right).\end{aligned} \tag{3.57}$$

Similarily, adding (3.7) and (3.9), multiplying the obtained equation by $-\frac{k_3}{k_2}$, adding (3.6) and (3.8), and multiplying the reuslting equation by $\xi^2 + \lambda_2$, we entail

$$\begin{aligned}\frac{d}{dt}F_3(\xi,\ t) &= \left(\xi^2 + \lambda_2\right) Re\left[i\xi\left(\frac{k_3}{k_2}\hat{\theta}\,\overline{\hat{u}} + \gamma\hat{\theta}\,\overline{\hat{z}}\right) + \left(\frac{k_3}{k_2} - 1\right)\xi^2\hat{\theta}\,\overline{\hat{y}}\right] \\ &\quad - \left(\xi^2 + \lambda_2\right) Re\left(i\,\xi\,\hat{y}\,\overline{\hat{u}}\right).\end{aligned} \tag{3.58}$$

Now, as in Subsection 3.2, we define the functionals F and L by (3.38) and (3.39) with the same function $\tilde{f}$ defined by (3.40). By combining (3.55)-(3.58), we deduce that

$$\frac{d}{dt}F(\xi,\ t) = F_4(\xi,\ t) - \xi^2\left((k_1\lambda_4 - k_1\lambda_2 - k_1)|\hat{v}|^2 + k_2\lambda_1|\hat{z}|^2 + (1-\lambda_1)|\hat{y}|^2 + \lambda_2|\hat{u}|^2 + k_3\lambda_3|\hat{\phi}|^2\right), \tag{3.59}$$

where

$$F_4(\xi,\ t) = Re\left(iI_1\xi\widehat{\theta}\,\overline{\widehat{u}} + iI_2\xi\widehat{\theta}\,\overline{\widehat{z}} + I_3\widehat{\theta}\,\overline{\widehat{y}} - \gamma\lambda_4\xi^2\widehat{\theta}\,\overline{\widehat{v}} - i\gamma\lambda_3\xi\widehat{\theta}\,\overline{\widehat{\phi}}\right) + (\lambda_3+\lambda_4)\xi^2|\widehat{\theta}|^2, \tag{3.60}$$

$$I_1 = \left[\left(\frac{k_3}{k_2}-1\right)\lambda_4 + \frac{k_3}{k_2} - \frac{k_3}{k_1}\right]\xi^2 + \frac{k_3}{k_2}\lambda_1 + \left(\frac{k_3}{k_2}+1\right)\lambda_2 + \lambda_3,$$

$$I_2 = \gamma\left[\left(1-\frac{k_2}{k_1}\right)\xi^2 + \lambda_1 + \lambda_2\right]$$

and

$$I_3 = \left[\left(\frac{k_3}{k_2}-1\right)(\xi^2+\lambda_2) + \left(1-\frac{k_3}{k_2}\right)\left(\frac{k_2}{k_1}\xi^2-\lambda_1\right) + \lambda_4 - 1\right]\xi^2.$$

We see that, if $k_1 = k_2 = k_3$, then I_1 and I_2 are constants, and $I_3 = (\lambda_4-1)\xi^2$. Otherwise, I_1 and I_2 are of the form $const\,\xi^2 + const$, and I_3 is of the form $(const\,\xi^2 + const)\xi^2$. Then, by applying Young's inequality, we arrive at, for any $\varepsilon_0 > 0$,

$$F_4(\xi,\ t) \le \varepsilon_0\xi^2\left(|\widehat{y}|^2 + |\widehat{u}|^2 + |\widehat{\phi}|^2 + |\widehat{v}|^2 + |\widehat{z}|^2\right) + C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4}\tilde{f}(\xi)|\widehat{\theta}|^2. \tag{3.61}$$

Therefore, we conclude from (3.60) and (3.61) that

$$\begin{aligned}\frac{d}{dt}F(\xi,\ t) \quad\le\quad & C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4}\tilde{f}(\xi)|\widehat{\theta}|^2 - \xi^2\left((k_1\lambda_4 - k_1\lambda_2 - k_1 - \varepsilon_0)|\widehat{v}|^2 + (k_2\lambda_1 - \varepsilon_0)|\widehat{z}|^2\right)\\ & -\xi^2\left((1-\lambda_1-\varepsilon_0)|\widehat{y}|^2 + (\lambda_2-\varepsilon_0)|\widehat{u}|^2 + (k_3\lambda_3-\varepsilon_0)|\widehat{\phi}|^2\right).\end{aligned} \tag{3.62}$$

We choose $0 < \lambda_3$, $0 < \lambda_1 < 1$, $\lambda_4 > 1$, $0 < \lambda_2 < \lambda_4 - 1$ and

$$0 < \varepsilon_0 < \min\{k_3\lambda_3, \lambda_2, 1-\lambda_1, k_2\lambda_1, k_1\lambda_4 - k_1\lambda_2 - k_1\}.$$

Then, using the definition of $\widehat{E}$, (3.62) leads, for some $c_1 > 0$, to

$$\frac{d}{dt}F(\xi,\ t) \le -c_1\xi^2\widehat{E}(\xi,\ t) + C\tilde{f}(\xi)|\widehat{\theta}|^2,$$

which is similar to (3.45). Consequentely, the proof can be finalized as that of Lemma 3.4. □

Theorem 3.7. *The stability result given in Theorem 3.5 is satisfied when* $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$.

Proof. The proof is identical to the one of Theorem 3.5. □

4 Stability under infinite memory

This section is devoted to the study of the asymptotic behavior, when time t goes to infinity, of the solution U of (2.14) in the case of infinite memory (2.13). We will prove two polynomial decay estimates on $\|\partial_x^k U\|_{L^2(\mathbb{R})}$ similar to the ones proved in the Subsections 3.2 and 3.3. We start by this lemma.

Lemma 4.1. *Let* $\varepsilon_0 > 0$, $\tilde{\eta} \in \{\widehat{\eta}, \overline{\widehat{\eta}}\}$ *and* $h,\ d : \mathbb{R} \to \mathbb{C}$ *be two functions. Then*

$$Re\left(h(\xi)d(\xi)\int_0^{+\infty} g(s)\tilde{\eta}(\xi,s)\,ds\right) \le \varepsilon_0|h(\xi)|^2 - C_{\varepsilon_0}|d(\xi)|^2\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds, \quad \forall\xi\in\mathbb{R} \tag{4.1}$$

and

$$Re\left(h(\xi)d(\xi)\int_0^{+\infty} g'(s)\tilde{\eta}(\xi,s)\,ds\right) \le \varepsilon_0|h(\xi)|^2 - C_{\varepsilon_0}|d(\xi)|^2\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds, \quad \forall\xi\in\mathbb{R}. \tag{4.2}$$

Proof. Using Hölder's inequality and the right inequality in (2.5), we see that

$$\begin{aligned}\left|\int_0^{+\infty} g(s)\tilde{\eta}(\xi,s)\,ds\right|^2 &= \left|\int_0^{+\infty} \sqrt{g(s)}\sqrt{g(s)}\tilde{\eta}(\xi,s)\,ds\right|^2\\ &\le \left(\int_0^{+\infty} g(s)\,ds\right)\int_0^{+\infty} g(s)|\tilde{\eta}(\xi,s)|^2\,ds\\ &\le -\frac{g_0}{\beta_2}\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds.\end{aligned}$$

Similarily, we have

$$\begin{aligned}\left|\int_0^{+\infty} g'(s)\tilde{\eta}(\xi,s)\,ds\right|^2 &= \left|\int_0^{+\infty} \sqrt{-g'(s)}\sqrt{-g'(s)}\tilde{\eta}(\xi,s)\,ds\right|^2\\ &\le \left(-\int_0^{+\infty} g'(s)\,ds\right)\int_0^{+\infty} (-g'(s))|\tilde{\eta}(\xi,s)|^2\,ds\\ &\le -g(0)\int_0^{+\infty} g'(s)|\tilde{\eta}(\xi,s)|^2\,ds.\end{aligned}$$

Then, using these two inequalities and Young's inequality, we get (4.1) (with $C_{\varepsilon_0} = \frac{g_0}{4\varepsilon_0\beta_2}$) and (4.2) (with $C_{\varepsilon_0} = \frac{g(0)}{4\varepsilon_0}$). □

We distinguishing the two cases (2.3).

4.1 Case 1: $(\tilde{\tau}_1, \tilde{\tau}_2) = (1, 0)$

Multiplying $(2.25)_4$ by $-\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds$, multiplying $(2.25)_7$ by $-g(s)\,\overline{\hat{y}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.20), we find

$$\begin{aligned}\frac{d}{dt}Re\left(-\hat{y}\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right) &= -g_0|\hat{y}|^2-(k_2-g_0)Re\left(i\xi\hat{z}\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right)+k_1Re\left(\hat{v}\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right)\\ &\quad+\xi^2\left|\int_0^{+\infty} g(s)\hat{\eta}\,ds\right|^2-Re\left(\overline{\hat{y}}\int_0^{+\infty} g'(s)\hat{\eta}\,ds\right). \qquad (4.3)\end{aligned}$$

Multiplying $(2.25)_6$ by $\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds$, multiplying $(2.25)_7$ by $g(s)\overline{\hat{\theta}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.20), we get

$$\begin{aligned}\frac{d}{dt}Re\left(\overline{\hat{\theta}}\int_0^{+\infty} g(s)\hat{\eta}\,ds\right) &= k_3Re\left(i\xi\hat{\phi}\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right)+g_0Re\left(\hat{y}\overline{\hat{\theta}}\right)\\ &\quad-k_1Re\left(\hat{v}\int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right)+Re\left(\overline{\hat{\theta}}\int_0^{+\infty} g'(s)\hat{\eta}\,ds\right). \qquad (4.4)\end{aligned}$$

Similarly, multiplying $(2.25)_2$ by $-i\xi \int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds$, multiplying $(2.25)_7$ by $i\xi g(s)\overline{\widehat{u}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.21), we obtain

$$\begin{aligned} \frac{d}{dt} Re\left(i\xi\overline{\widehat{u}} \int_0^{+\infty} g(s)\widehat{\eta}\, ds \right) &= k_1\xi^2 Re\left(\widehat{v} \int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds \right) + g_0 Re\left(i\xi\widehat{y}\overline{\widehat{u}} \right) \\ &\quad + Re\left(i\xi\overline{\widehat{u}} \int_0^{+\infty} g^{'}(s)\widehat{\eta}\, ds \right). \end{aligned} \tag{4.5}$$

Also, multiplying $(2.25)_5$ by $-i\xi \int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds$, multiplying $(2.25)_7$ by $i\xi g(s)\overline{\widehat{\phi}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.21), we arrive at

$$\begin{aligned} \frac{d}{dt} Re\left(i\xi\overline{\widehat{\phi}} \int_0^{+\infty} g(s)\widehat{\eta}\, ds \right) &= \xi^2 Re\left(\widehat{\theta} \int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds \right) + g_0 Re\left(i\xi\widehat{y}\overline{\widehat{\phi}} \right) \\ &\quad + Re\left(i\xi\overline{\widehat{\phi}} \int_0^{+\infty} g^{'}(s)\widehat{\eta}\, ds \right). \end{aligned} \tag{4.6}$$

Now, we present our stability estimate for (2.16).

Lemma 4.2. *Let $\widehat{U}$ be the solution of* (2.16). *Then there exist $c,\ \widetilde{c} > 0$ such that* (3.10) *holds true with the same function f defined by* (3.11).

Proof. Noticing that, if we replace in (2.24) k_2, k_3, $\tau_1\gamma\widehat{u}$, $\tau_2\gamma\widehat{y}$ and $\tau_3\gamma\widehat{\theta}$ by $k_2 - \tilde{\tau}_1 g_0$, $k_3 - \tilde{\tau}_2 g_0$, 0, $\tilde{\tau}_1\xi^2 \int_0^{+\infty} g(s)\widehat{\eta}\, ds$ and $\tilde{\tau}_2\xi^2 \int_0^{+\infty} g(s)\widehat{\eta}\, ds$, respectively, we get the first six equations in (2.25). Then we define F_0 by (3.30), and as in (3.31)-(3.33), we define F_1, F_2 and F_3 by ($g_0 \in]0, k_2[$ because of (2.4))

$$F_1(\xi,\ t) = \left(\frac{k_2 - g_0}{k_1}\xi^2 + \lambda_1 \right) Re\left(\widehat{u}\,\overline{\widehat{z}} \right), \tag{4.7}$$

$$F_2(\xi,\ t) = \frac{k_2 - g_0}{k_1 k_3}\left(k_3\lambda_4\xi^2 - k_1\lambda_3 \right) Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{\theta}} - \widehat{u}\,\overline{\widehat{z}} - i\,\frac{k_3}{k_2 - g_0}\,\xi\,\widehat{\phi}\,\overline{\widehat{y}} \right) \tag{4.8}$$

and

$$F_3(\xi,\ t) = -\frac{k_2 - g_0}{k_3}\left(\lambda_4\xi^2 + \lambda_2 \right) Re\left(i\,\xi\,\widehat{z}\,\overline{\widehat{\theta}} - \widehat{u}\,\overline{\widehat{z}} - i\,\frac{k_3}{k_2 - g_0}\,\xi\,\widehat{\phi}\,\overline{\widehat{y}} + \frac{k_3}{k_2 - g_0}\,\widehat{u}\,\overline{\widehat{\phi}} \right). \tag{4.9}$$

Then, we get (3.34)-(3.37) with $k_2 - g_0$ and $\xi^2 \int_0^{+\infty} g(s)\widehat{\eta}\, ds$ instead of k_2 and $\gamma\widehat{y}$, respectively. Let us define the functional F by

$$\begin{aligned} F(\xi,\ t) &= \xi^2(F_0(\xi,\ t) + F_1(\xi,\ t) + F_2(\xi,\ t) + F_3(\xi,\ t)) - \lambda_5\xi^4 Re\left(\widehat{y} \int_0^{+\infty} g(s)\overline{\widehat{\eta}} ds \right) \\ &\quad -\frac{1}{g_0}\xi^2 Re\left[\left(I_1\overline{\widehat{\theta}} + iI_2\xi\overline{\widehat{u}} + iI_3\xi\,\overline{\widehat{\phi}} \right) \int_0^{+\infty} g(s)\widehat{\eta} ds \right] \end{aligned} \tag{4.10}$$

and let L be the functional given by (3.39) with

$$\tilde{f}(\xi) = 1 + \xi^2 + \xi^4 + \xi^6 \tag{4.11}$$

instead of (3.40), where $\lambda_5 > 0$, and I_1, I_2 and I_3 are defined as in Subsection 3.2 with $k_2 - g_0$ instead of k_2. By combining (3.34)-(3.37) and (4.3)-(4.6), we deduce that

$$\begin{aligned} \frac{d}{dt}F(\xi,\ t) &= \xi^2(F_4(\xi,\ t) + F_5(\xi,\ t)) - g_0\lambda_5\xi^4|\widehat{y}|^2 \\ &\quad -\xi^4\left((k_1 - k_1\lambda_2 - k_1\lambda_4)|\widehat{v}|^2 + (k_2 - g_0)\lambda_1|\widehat{z}|^2 + (\lambda_4 - \lambda_3)|\widehat{\theta}|^2 + \lambda_2|\widehat{u}|^2 + k_3\lambda_3|\widehat{\phi}|^2\right), \end{aligned} \tag{4.12}$$

where F_4 is defined by (3.42) with $k_2 - g_0$ and $\xi^2 \int_0^{+\infty} g(s)\widehat{\eta}\,ds$ instead of k_2 and $\gamma\widehat{y}$, respectively, and

$$\begin{aligned} F_5(\xi,\ t) &= Re\left[\left(\frac{1}{g_0}I_1(k_1\widehat{v} - ik_3\xi\widehat{\phi}) - \frac{k_1}{g_0}I_2\xi^2\widehat{v} - \frac{1}{g_0}I_3\xi^2\widehat{\theta}\right)\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right] \\ &\quad + Re\left[\lambda_5\xi^2\left(-i(k_2 - g_0)\xi\widehat{z} + k_1\widehat{v} + \xi^2\int_0^{+\infty} g(s)\widehat{\eta}\,ds\right)\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right] \\ &\quad - Re\left[\left(\frac{1}{g_0}\left(I_1\overline{\widehat{\theta}} + iI_2\xi\overline{\widehat{u}} + iI_3\xi\overline{\widehat{\phi}}\right) + \lambda_5\xi^2\overline{\widehat{y}}\right)\int_0^{+\infty} g'(s)\widehat{\eta}\,ds\right] - Re\left(I_1\widehat{\theta}\overline{\widehat{y}} + iI_2\xi\widehat{y}\overline{\widehat{u}} + iI_3\xi\widehat{y}\overline{\widehat{\phi}}\right). \end{aligned}$$

Therefore, from (3.42), we have

$$F_4(\xi,\ t) - Re\left(I_1\widehat{\theta}\overline{\widehat{y}} + iI_2\xi\widehat{y}\overline{\widehat{u}} + iI_3\xi\widehat{y}\overline{\widehat{\phi}}\right) = (\lambda_1 + 1)\xi^2|\widehat{y}|^2 - Re\left[\left(\xi^2\widehat{v} + i\lambda_1\xi\widehat{z}\right)\xi^2\int_0^{+\infty} g(s)\widehat{\eta}\,ds\right],$$

which implies that all the terms appearing in the real part in $F_4 + F_5$ depend on η. Because I_1, I_2 and I_3 are defined as in Subsection 3.2 with $k_2 - g_0$ instead of k_2, then

$$|I_1|^2 \le C\xi^4(1 + \xi^2)^2, \quad |I_2|^2 \le C(1 + \xi^2)^2 \quad \text{and} \quad |I_3|^2 \le C(1 + \xi^2)^2.$$

Thus, by applying Young's inequality and using (4.1) and (4.2), we see that, for any $\varepsilon_0 > 0$,

$$\begin{aligned} \xi^2\left(F_4(\xi,\ t) + F_5(\xi,\ t)\right) &\le \varepsilon_0\xi^4\left(|\widehat{\theta}|^2 + |\widehat{u}|^2 + |\widehat{\phi}|^2 + |\widehat{v}|^2 + |\widehat{z}|^2\right) + (\lambda_1 + 1 + \varepsilon_0)\xi^4|\widehat{y}|^2 \\ &\quad - C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}\tilde{f}(\xi)\xi^2\int_0^{+\infty} g'(s)|\widehat{\eta}|^2\,ds. \end{aligned} \tag{4.13}$$

So, we conclude, from (4.12) and (4.13), that

$$\begin{aligned} \frac{d}{dt}F(\xi,\ t) &\le -C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}\tilde{f}(\xi)\xi^2\int_0^{+\infty} g'(s)|\eta|^2\,ds - [g_0\lambda_5 - (\lambda_1 + 1 + \varepsilon_0)]\xi^4|\widehat{y}|^2 \\ &\quad -\xi^4\left((k_1 - k_1\lambda_2 - k_1\lambda_4 - \varepsilon_0)|\widehat{v}|^2 + ((k_2 - g_0)\lambda_1 - \varepsilon_0)|\widehat{z}|^2 + (\lambda_4 - \lambda_3 - \varepsilon_0)|\widehat{\theta}|^2\right) \\ &\quad -\xi^4\left((\lambda_2 - \varepsilon_0)|\widehat{u}|^2 + (k_3\lambda_3 - \varepsilon_0)|\widehat{\phi}|^2\right). \end{aligned} \tag{4.14}$$

We choose $0 < \lambda_1$, $0 < \lambda_2 < 1$, $0 < \lambda_3 < \lambda_4 < 1 - \lambda_2$ and

$$0 < \varepsilon_0 < \min\left\{k_1 - k_1\lambda_2 - k_1\lambda_4,\ (k_2 - g_0)\lambda_1,\ \lambda_4 - \lambda_3,\ \lambda_2,\ k_3\lambda_3\right\}$$

(recall that $g_0 \in]0, k_2[$ since (2.4)). Then, we choose $\lambda_5 > \frac{1}{g_0}(\lambda_1 + 1 + \varepsilon_0)$. Therefore, using the definition (2.19) of $\widehat{E}$ and the second inequality in (2.5), estimate (4.14) implies that, for some positive constant c_1,

$$\frac{d}{dt}F(\xi,\, t) \le -c_1\xi^4\widehat{E}(\xi,\, t) - C\tilde{f}(\xi)\xi^2 \int_0^{+\infty} g'(s)|\eta|^2\, ds. \tag{4.15}$$

Hence, from (2.23), (3.39) and (4.15), it follows that

$$\frac{d}{dt}L(\xi,\, t) \le -c_1 f(\xi)\widehat{E}(\xi,\, t) - \left(\frac{1}{2}\lambda - C\right)\xi^2 \int_0^{+\infty} g'(s)|\eta|^2\, ds. \tag{4.16}$$

where f is defined in (3.11). As in Subsections 3.1 and 3.2, we conclude that there exists $c_2 > 0$ (independent of λ) such that

$$|L(\xi,\, t) - \lambda\,\widehat{E}(\xi,\, t)| \le c_2\,\widehat{E}(\xi,\, t). \tag{4.17}$$

Therefore, by choosing $\lambda > \max\{2C,\, c_2\}$, we get (3.21) and (3.22). Consequentely, the proof can be ended as in the proof of Lemma 3.1. □

Theorem 4.3. *Let $N,\, \ell \in \mathbb{N}^*$ such that $\ell \le N$, $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$ and U be the solution of* (2.14). *Then, for any $j = 0,\, \dots,\, N - \ell$, there exist $c_0 > 0$ such that* (3.24) *holds true.*

Proof. The proof is identical to that of Theorem 3.2. □

4.2 Case 2: $(\tilde{\tau}_1,\, \tilde{\tau}_2) = (0,\, 1)$

Multiplying $(2.25)_6$ by $-\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds$, multiplying $(2.25)_7$ by $-g(s)\,\overline{\widehat{\theta}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.20), we have

$$\begin{aligned}\frac{d}{dt}Re\left(-\widehat{\theta}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds\right) \quad = \quad & -g_0|\theta|^2 - (k_3 - g_0)Re\left(i\xi\widehat{\phi}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds\right)\\ & + k_1 Re\left(\widehat{v}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds\right) + \xi^2\left|\int_0^{+\infty} g(s)\widehat{\eta}\, ds\right|^2\\ & - Re\left(\overline{\widehat{\theta}}\int_0^{+\infty} g'(s)\widehat{\eta}\, ds\right).\end{aligned} \tag{4.18}$$

Multiplying $(2.25)_4$ by $\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds$, multiplying $(2.25)_7$ by $g(s)\overline{\widehat{y}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.20), we obtain

$$\begin{aligned}\frac{d}{dt}Re\left(\overline{\widehat{y}}\int_0^{+\infty} g(s)\widehat{\eta}\, ds\right) \quad = \quad & k_2 Re\left(i\xi\widehat{z}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds\right) + g_0 Re\left(\widehat{y}\overline{\widehat{\theta}}\right)\\ & - k_1 Re\left(\widehat{v}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\, ds\right) + Re\left(\overline{\widehat{y}}\int_0^{+\infty} g'(s)\widehat{\eta}\, ds\right).\end{aligned} \tag{4.19}$$

Similarily, multiplying $(2.25)_3$ by $-i\xi\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds$, multiplying $(2.25)_7$ by $i\xi g(s)\overline{\widehat{z}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.21), we entail

$$\begin{aligned}\frac{d}{dt}Re\left(i\xi\overline{\widehat{z}}\int_0^{+\infty} g(s)\widehat{\eta}\,ds\right) &= \xi^2 Re\left(\widehat{y}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right) + g_0 Re\left(i\xi\widehat{\theta}\overline{\widehat{z}}\right)\\ &\quad + Re\left(i\xi\overline{\widehat{z}}\int_0^{+\infty} g'(s)\widehat{\eta}\,ds\right).\end{aligned} \tag{4.20}$$

Also, multiplying $(2.25)_2$ by $-i\xi\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds$, multiplying $(2.25)_7$ by $i\xi g(s)\overline{\widehat{u}}$ and integrating over $\mathbb{R}_+$ with respect to s, adding the resulting equations, taking the real part and using (2.21), it appears that

$$\begin{aligned}\frac{d}{dt}Re\left(i\xi\overline{\widehat{u}}\int_0^{+\infty} g(s)\widehat{\eta}\,ds\right) &= k_1\xi^2 Re\left(\widehat{v}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right) + g_0 Re\left(i\xi\widehat{\theta}\overline{\widehat{u}}\right)\\ &\quad + Re\left(i\xi\overline{\widehat{u}}\int_0^{+\infty} g'(s)\widehat{\eta}\,ds\right).\end{aligned} \tag{4.21}$$

Lemma 4.4. *The result of Lemma 4.2 holds also in case* $(\tilde{\tau}_1, \tilde{\tau}_2) = (0, 1)$.

Proof. The proof is very similar to that of Lemma 4.2, using the arguments used in Subsection 3.3. We define F_0, F_1, F_2 and F_3 by (3.51)-(3.54), with k_3 replaced by $k_3 - g_0$, so we find (3.55)-(3.58) with $k_3 - g_0$ and $\xi^2\int_0^{+\infty} g(s)\widehat{\eta}\,ds$ instead of k_3 and $\gamma\widehat{\theta}$, respectively. As in Subsection 4.1, let us introduce the functional F given by

$$\begin{aligned}F(\xi, t) &= \xi^2(F_0(\xi, t) + F_1(\xi, t) + F_2(\xi, t) + F_3(\xi, t)) - \lambda_5\xi^4 Re\left(\widehat{\theta}\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right)\\ &\quad -\frac{1}{g_0}\xi^2 Re\left[\left(I_3\overline{\widehat{y}} + iI_1\xi\overline{\widehat{u}} + iI_2\xi\overline{\widehat{z}}\right)\int_0^{+\infty} g(s)\overline{\widehat{\eta}}\,ds\right]\end{aligned} \tag{4.22}$$

and let L be the functional defined by (3.39), where $\tilde{f}$ is given by (4.11) instead of (3.40), $\lambda_5 > 0$, and I_1, I_2 and I_3 are defined as in Subsection 3.3 with $k_3 - g_0$ instead of k_3. By combining (3.55)-(3.58) and (4.18)-(4.21), we deduce that

$$\begin{aligned}\frac{d}{dt}F(\xi, t) &= \xi^2(F_4(\xi, t) + F_5(\xi, t)) - g_0\lambda_5\xi^4|\widehat{\theta}|^2\\ &\quad -\xi^4\left((k_1\lambda_4 - k_1\lambda_2 - k_1)|\widehat{v}|^2 + k_2\lambda_1|\widehat{z}|^2 + (1-\lambda_1)|\widehat{y}|^2 + \lambda_2|\widehat{u}|^2 + (k_3 - g_0)\lambda_3|\widehat{\phi}|^2\right),\end{aligned} \tag{4.23}$$

where F_4 is defined by (3.60), with $k_3 - g_0$ and $\xi^2 \int_0^{+\infty} g(s)\hat{\eta}\,ds$ instead of k_3 and $\gamma\hat{\theta}$, respectively, and

$$
\begin{aligned}
F_5(\xi, t) = \; & Re\left[\left(\frac{1}{g_0} I_3(k_1\hat{v} - ik_2\xi\hat{z}) - \frac{k_1}{g_0} I_1\xi^2\hat{v} - \frac{1}{g_0} I_2\xi^2\hat{y}\right) \int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right] \\
& + Re\left[\lambda_5\xi^2\left(k_1\hat{v} - i(k_3 - g_0)\xi\hat{\phi} + \xi^2 \int_0^{+\infty} g(s)\hat{\eta}\,ds\right) \int_0^{+\infty} g(s)\overline{\hat{\eta}}\,ds\right] \\
& - Re\left[\left(\frac{1}{g_0}\left(I_3\overline{\hat{y}} + iI_1\xi\overline{\hat{u}} + iI_2\xi\overline{\hat{z}}\right) + \lambda_5\xi^2\overline{\hat{\theta}}\right) \int_0^{+\infty} g'(s)\hat{\eta}\,ds\right] - Re\left(I_3\hat{\theta}\overline{\hat{y}} + iI_2\xi\hat{\theta}\overline{\hat{z}} + iI_1\xi\hat{\theta}\overline{\hat{u}}\right).
\end{aligned}
$$

As in the previous Subsection 4.1, by applying Young's inequality and using (4.1) and (4.2), we see that, for any $\varepsilon_0 > 0$,

$$
\begin{aligned}
\xi^2\left(F_4(\xi, t) + F_5(\xi, t)\right) \leq \; & \varepsilon_0\xi^4\left(|\hat{y}|^2 + |\hat{u}|^2 + |\hat{\phi}|^2 + |\hat{v}|^2 + |\hat{z}|^2\right) + (\lambda_3 + \lambda_4 + \varepsilon_0)\xi^4|\hat{\theta}|^2 \\
& - C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}\tilde{f}(\xi)\xi^2 \int_0^{+\infty} g'(s)|\hat{\eta}|^2\,ds. \qquad (4.24)
\end{aligned}
$$

Hence, we conclude, from (4.23) and (4.24), that

$$
\begin{aligned}
\frac{d}{dt}F(\xi, t) \leq \; & -C_{\varepsilon_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}\tilde{f}(\xi)\xi^2 \int_0^{+\infty} g'(s)|\eta|^2\,ds - [g_0\lambda_5 - (\lambda_3 + \lambda_4 + \varepsilon_0)]\xi^4|\hat{\theta}|^2 \\
& -\xi^4\left((k_1\lambda_4 - k_1\lambda_2 - k_1 - \varepsilon_0)|\hat{v}|^2 + ((k_3 - g_0)\lambda_3 - \varepsilon_0)|\hat{\phi}|^2\right) \\
& -\xi^4\left((1 - \lambda_1 - \varepsilon_0)|\hat{y}|^2 + (\lambda_2 - \varepsilon_0)|\hat{u}|^2 + (k_2\lambda_1 - \varepsilon_0)|\hat{z}|^2\right). \qquad (4.25)
\end{aligned}
$$

We choose $0 < \lambda_3$, $0 < \lambda_1 < 1$, $\lambda_4 > 1$, $0 < \lambda_2 < \lambda_4 - 1$ and

$$0 < \varepsilon_0 < \min\left\{(k_3 - g_0)\lambda_3, \lambda_2, 1 - \lambda_1, k_2\lambda_1, k_1\lambda_4 - k_1\lambda_2 - k_1\right\}.$$

(notice that $g_0 \in]0, k_3[$ since (2.4)). Then we choose $\lambda_5 > \frac{1}{g_0}(\lambda_3 + \lambda_4 + \varepsilon_0)$, which implies (4.15), and therefore, (4.16) holds true. Finally, the proof can be ended as in the previous Subsection 4.1. □

Theorem 4.5. *The stability result of Theorem 4.3 is satisfied also in case* $(\tilde{\tau}_1, \tilde{\tau}_2) = (0, 1)$.

Proof. The proof is the same as the one of Theorem 4.3. □

5 Comments and issues

1. In case (3.29) with $k_1 = k_2 = k_3$, the function f tends to 1 when ξ goes to infinity, this means that, when the frictional damping is active on the second or third equation of (2.1), the resulting dissipation is very strong in the high frequency region, which avoid the regularity loss in the estimate on $\|\partial_x^j U\|_{L^2(\mathbb{R})}$; that is, we can take $j = \ell = 0$ and get the stability of (2.14), where the decay estimate on $\|U\|_{L^2(\mathbb{R})}$ depends only on $\|U_0\|_{L^1(\mathbb{R})}$ and $\|U_0\|_{L^2(\mathbb{R})}$. However, in the other cases, f tends to 0 when ξ goes to infinity, this means that the dissipation is very weak in the high frequency region, which leads to the regularity loss in the estimate on $\|\partial_x^j U\|_{L^2(\mathbb{R})}$. In all cases, the behavior of f in the low frequencies determines the decay rate of the solution.

2. Condition (2.5) means that g is between two exponentially decreasing functions. This class is the simplest standard one considered in the literature. Seeking the largest class possible of g was not amoung the

objectives of this paper. But we think that it will be possible to generalize our results to larger class of g thatn the one satisfying (2.5), and get the polynomial stability (perhaps with weaker decay rates than the ones given in this paper). In case of bounded domains, we refer the readers for this issue to [13, 19, 20] for (1.3), and to [14, 17, 18] for (1.2).

3. The optimality of the obtained decay rates on $\|\partial_x^j U\|_{L^2(\mathbb{R})}$ is an interesting open question. This question will be the focus of our attention in a future work.

4. The stability question in case of infinite memory acting on the first equation seems to be more delecate than the other ones treated in sections 3 and 4.

5. The decay rates of $\|\partial_x^j U\|_{L^2(\mathbb{R})}$ in cases $(\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\}$ are better than the one obtained in cases $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ and (2.3). On the other hand, the stability in case $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ holds true if and only if $k_2 \neq k_3$. Our stability results are, in some sense, compatible with the ones proved in [16], where $\mathbb{R}$ is replaced by a bounded domain $]0, L[$ and the obtained polynomial decay rate in case of frictional damping is better than the one obtained in case of infinite memory, and moreover, when $\delta = 0$ and the frictional damping is effective on the first equation, the polynomial stability was proved under the assumption $k_2 \neq k_3$.

References

[1] M. S. Alves, P. Gamboa, G. C. Gorain, A. Rambaud and O. Vera, Asymptotic behavior of a flexible structure with Cattaneo type of thermal effect, Indagationes Mathematicae, 27 (2016), 821-834.

[2] T. A. Apalara, Uniform stability of a laminated beam with structural damping and second sound, ZAMP, 68 (2017), 40-55.

[3] T. A. Apalara, On the stability of thermoelastic laminated beam, Acta. Math. Scie., 39 (2019), 1517-1524.

[4] C. F. Beards and I. M. A. Imam, The damping of plate vibration by interfacial slip between layers, Int. J. Mach. Tool. Des. Res., 18 (1978), 131-137.

[5] J. A. C. Bresse, Cours de Méchanique Appliquée, Mallet Bachelier, Paris, 1859.

[6] X. G. Cao, D. Y. Liu and G. Q. Xu, Easy test for stability of laminated beams with structural damping and boundary feedback controls, J. Dynamical Control Syst., 13 (2007), 313-336.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti, F. A. Falcao Nascimento, I. Lasiecka and J. H. Rodrigues, Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping, Z. Angew. Math. Phys., 65 (2014), 1189-1206.

[8] Z. Chen, W. Liu and D. Chen, General decay rates for a laminated beam with memory, Taiw. J. Math., 23 (2019), 1227-1252.

[9] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Rational Mech. Anal., 37 (1970), 297-308.

[10] L. Djouamai and B. Said-Houari, A new stability number of the Bresse-Cattaneo system, Math. Meth. Appl. Sci., 41 (2018), 2827-2847.

[11] B. Feng, T. E. Ma, R. N. Monteiro and C. A. Raposo, Dynamics of laminated Timoshenko beams, J. Dyn. Diff. Equa., 30 (2018), 1489-1507.

[12] T. E. Ghoul, M. Khenissi and B. Said-Houari, On the stability of the Bresse system with frictional damping, J. Math. Anal. Appl., 455 (2017), 1870-1898.

[13] A. Guesmia, Some well-posedness and general stability results in Timoshenko systems with infinite memory and distributed time delay, J. Math. Phys., 55 (2014), 1-40.

[14] A. Guesmia, Asymptotic stability of Bresse system with one infinite memory in the longitudinal displacements, Medi. J. Math., 14 (2017), 19 pages.

[15] A. Guesmia, Non-exponential and polynomial stability results of a Bresse system with one infinite memory in the vertical displacement, Nonauton. Dyn. Syst., 4 (2017), 78-97.

[16] A. Guesmia, Well-posedness and stability results for laminated Timoshenko beams with interfacial slip and infinite memory, IMA J. Math. Cont. Info., 37 (2020), 300-350.

[17] A. Guesmia and M. Kafini, Bresse system with infinite memories, Math. Meth. Appl. Sci., 38 (2015), 2389-2402.

[18] A. Guesmia and M. Kirane, Uniform and weak stability of Bresse system with two infinite memories, Z. Angew. Math. Phys., 67 (2016), 1-39.

[19] A. Guesmia and S. Messaoudi, A general stability result in a Timoshenko system with infinite memory: A new approach, Math. Meth. Appl. Scie., 37 (2014), 384-392.

[20] A. Guesmia, S. Messaoudi and A. Soufyane, On the stabilization for a linear Timoshenko system with infinite history and applications to the coupled Timoshenko-heat systems, Elec. J. Diff. Equa., 2012 (2012), 1-45.

[21] A. Guesmia, J. E. Muñoz Rivera, V. Poblete and O. Vera Villagrán, The effect of the wave speeds on the decay rate of the thermoelastic structure in the whole line with interfacial slip, submitted.
[22] S. W. Hansen, In control and estimation of distributed parameter systems: Non-linear phenomena, International Series of Numerical Analysis, 118 (1994), 143-170.
[23] S. W. Hansen and R. Spies, Structural damping in a laminated beams due to interfacial slip, J. Sound Vibration, 204 (1997), 183-202.
[24] M. Khader and B. Said-Houari, Decay rate of solutions to Timoshenko system with past history in unbounded domains, Appl. Math. Optim., 75 (2017), 403-428.
[25] M. Khader and B. Said-Houari, Optimal decay rate of solutions to Timoshenko system with past history in unbounded domains, Z. Anal. Anwend, 37 (2018), 435-459.
[26] G. Li, X. Kong and W. Liu, General decay for a laminated beam with structural damping and memory: the case of non-equal wave speeds, J. Inte. Equa., 30 (2018), 95-116.
[27] W. Liu and W. Zhao, Exponential and polynomial decay for a laminated beam with Fourier's type heat conduction, Preprints 2017, 2017020058, doi: 10.20944/preprints201702.0058.v1.
[28] A. Lo and N-E Tatar, Stabilization of laminated beams with interfacial slip, Elec. J. Diff. Equa., 2015 (2015), 1-14.
[29] A. Lo and N. E. Tatar, Uniform stability of a laminated beam with structural memory, Qual. Theory Dyn. Syst., 15 (2016), 517-540.
[30] A. Lo and N. E. Tatar, Exponential stabilization of a structure with interfacial slip, Discrete Contin. Dyn. Syst., 36 (2016), 6285-6306.
[31] M. I. Mustafa, Laminated Timoshenko beams with viscoelastic damping, J. Math. Anal. Appl., 466 (2018), 619-641.
[32] C. A. Raposo, Exponential stability for a structure with interfacial slip and frictional damping, Appl. Math. Lett., 53 (2016), 85-91.
[33] C. A. Raposo, O. V. Villagrán, J. E. Muñoz Rivera and M. S. Alves, Hybrid laminated Timoshenko beam, J. Math. Phys., 58 (2017), 11 pages.
[34] B. Said-Houari and R. Racke, Decay rates and global existence for semilinear dissipative Timoshenko systems, Quart. Appl. Math., 71 (2013), 229-266.
[35] B. Said-Houari and A. Soufyane, The effect of frictional damping terms on the decay rate of the Bresse system, Evol. Equa. Cont. Theory, 3 (2014), 713-738.
[36] N. E. Tatar, Stabilization of a laminated beam with interfacial slip by boundary controls, Boundary Value Problem, 2015, DOI: 10.1186/s13661-015-0432-3.
[37] G. Teschl, Ordinary differential equations and dynamical systems, American Mathematical Soc., 140 (2012), ISBN 978-0-8218-8328-0.
[38] S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismaticbars, Philisophical magazine, 41 (1921), 744-746.
[39] J. M. Wang, G. Q. Xu and S. P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, SIAM J. Control Optim., 44 (2005), 1575-1597.