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Non-exponential and polynomial stability results of a Bresse system with one infinite memory in the vertical displacement

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Abstract: The asymptotic stability of one-dimensional linear Bresse systems under infinite memories was obtained by Guesmia and Kafini [10] (three infinite memories), Guesmia and Kirane [11] (two infinite memories), Guesmia [9] (one infinite memory acting on the longitudinal displacement) and De Lima Santos et al. [6] (one infinite memory acting on the shear angle displacement). When the kernel functions have an exponential decay at infinity, the obtained stability estimates in these papers lead to the exponential stability of the system if the speeds of wave propagations are the same, and to the polynomial one with decay rate $t^{\frac{-1}{2}}$ otherwise. The subject of this paper is to study the case where only one infinite memory is considered and it is acting on the vertical displacement. As far as we know, this case has never studied before in the literature. We show that this case is deeply different from the previous ones cited above by proving that the exponential stability does not hold even if the speeds of wave propagations are the same and the kernel function has an exponential decay at infinity. Moreover, we prove that the system is still stable at least polynomially where the decay rate depends on the smoothness of the initial data. For classical solutions, this decay rate is arbitrarily close to $t^{\frac{-1}{4}}$. The proof is based on a combination of the energy method and the frequency domain approach to overcome the new mathematical difficulties generated by our system.

Keywords: Bresse system, Infinite memory, Asymptotic behavior, Energy method, Frequency domain approach

MSC: 35B40, 35L45, 74H40, 93D20, 93D15

1 Introduction

The Bresse system [4], known as the circular arch problem, is the following coupled three hyperbolic equations:

$$\begin{cases} \rho_1\varphi_{tt} - k\,(\varphi_x + \psi + l\,w)_x - lk_0\,(w_x - l\varphi) = F_1 & \text{in } (0,L)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\,(\varphi_x + \psi + l\,w) = F_2 & \text{in } (0,L)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0\,(w_x - l\varphi)_x + lk\,(\varphi_x + \psi + l\,w) = F_3 & \text{in } (0,L)\times(0,\infty)\,, \end{cases} \tag{1.1}$$

where $\rho_1,\ \rho_2,\ b,\ k,\ k_0,\ l$ and L are positive constants,

$$F_j:\ (0,L)\times(0,\infty)\to\mathbb{R},\quad j=1,2,3,$$

are given external forces, which play the role of controls, and $\varphi,\ \psi$ and w represent, respectively, the vertical, shear angle and longitudinal displacements. For more details, see for example [14] and [15].

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The stability of Bresse systems with (local or global) frictional dampings

$$F_1 = -\gamma_1(x)\varphi_t, \quad F_2 = -\gamma_2(x)\psi_t \quad \text{and} \quad F_3 = -\gamma_3(x)w_t,$$

where $\gamma_j : (0,1) \to \mathbb{R}$, $j = 1,2,3$, are given functions, was obtained by several researchers in the last few years; see [1] for the case of one frictional damping acting on the longitudinal displacement (that is $\gamma_1 = \gamma_2 = 0$), [2], [7], [19] and [22] for the case of one frictional damping acting on the shear angle displacement (that is $\gamma_1 = \gamma_3 = 0$), [3] and [24] for the case of two frictional dampings, and [5], [21] and [23] for the case of three frictional dampings. When each equation is controlled by a frictional damping, the exponential stability of Bresse systems was proved regardless to the speeds of wave propagations given by

$$s_1 = \sqrt{\frac{k}{\rho_1}}, \quad s_2 = \sqrt{\frac{b}{\rho_2}} \quad \text{and} \quad s_3 = \sqrt{\frac{k_0}{\rho_1}}. \tag{1.2}$$

When at least one equation is free, the obtained stability estimate is of exponential or polynomial type depending on some relations between s_i. When only one frictional damping is considered on the longitudinal or shear angle displacement (that is $\gamma_1 = \gamma_2 = 0$ or $\gamma_1 = \gamma_3 = 0$), it was proved that the exponential stability is equivalent to

$$s_1 = s_2 = s_3. \tag{1.3}$$

Similar stability results were proved in [1], [8], [13], [17] and [18] in case where the Bresse system is coupled with one or two heat equations in a certain manner so that at least the longitudinal or shear angle displacement is indirectely damped via the heat equations.

The stability of Bresse systems with memories was also recently studied. When the three equations are controlled via infinite memories of the form

$$F_1 = -\int_0^\infty g_1(s)\varphi_{xx}(x,t-s)\,ds, \quad F_2 = -\int_0^\infty g_2(s)\psi_{xx}(x,t-s)\,ds$$

and

$$F_3 = -\int_0^\infty g_3(s)w_{xx}(x,t-s)\,ds,$$

where $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1,2,3$, are differentiable, non-increasing and integrable functions on $\mathbb{R}_+$, the stability was proved in [10] regardless to s_i, where the obtained decay rate depends only on the arbitrary growth at infinity of the kernels g_j. When only two memories are considered, the stability of Bresse systems was proved in [11], where the decay rate depends also on s_i and on the smoothness of initial data.

Similar stability results to the ones of [11] were also proved in [9] under one infinite memory acting on the longitudinal displacement (that is $g_1 = g_2 = 0$) with kernels having a general decay at infinity, and in [6] under one infinite memory acting on the shear angle displacement (that is $g_1 = g_3 = 0$) with kernels having an esponential decay at infinity.

Our objective in this paper is studying the last case which, as far as we know, has never been considered before conserning Bresse system under only one infinite memory acting on the vertical displacement (that is $g_2 = g_3 = 0$), more precisely, we consider the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + \int_0^\infty g(s)\varphi_{xx}(x,t-s)\,ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + lw) = 0, \end{cases} \tag{1.4}$$

where $(x,t) \in (0,1) \times (0,\infty)$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, along with the initial data

$$\begin{cases} \varphi(x,-t) = \varphi_0(x,t),\ \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,1)\times(0,\infty), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x) & \text{in } (0,1), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{in } (0,1) \end{cases} \tag{1.5}$$

and the homogeneous Dirichlet-Neumann-Neumann boundary conditions

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = 0 & \text{in } (0,\infty), \\ \varphi(1,t) = \psi_x(1,t) = w_x(1,t) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.6}$$

Without lose of generality, we consider the domain $(0,1)$ instead of $(0,L)$ to simplify the computations.

In (1.4), only the vertical displacement is damped via the dissipation from the infinite memory, and the shear angle and longitudinal displacements are free. Our first main result in this paper is proving that the dissipation generated by the infinite memory in (1.4) can not stabilize exponentially the overall system even if (1.3) holds and g converges exponentially to zero at infinity. Our second main result is showing a polynomial stability estimte where the decay rate of solutions depends on the smoothness of initial data. For classical solutions, this decay rate is arbitrarily close to $t^{\frac{-1}{4}}$.

The paper is organized as follows: in Section 2, we present our hypotheses and state our non-exponential and polynomial stability results. The proof of these results will be given in Sections 3 and 4. Concluding comments and open questions are given in Section 5.

2 Hypotheses and main results

2.1 Well-posedness.

We give here a brief idea about the well-posedness of (1.4) – (1.6). As in [11], (1.4) – (1.6) can be formulated as a first order system of the form

$$\begin{cases} U_t = \mathcal{A}U & \text{in } (0,\infty), \\ U(t=0) = U_0, \end{cases} \tag{2.1}$$

where

$$\begin{cases} U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w}, \eta)^T, \quad U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \eta_0)^T, \\ \tilde{\varphi} = \varphi_t, \quad \tilde{\psi} = \psi_t, \quad \tilde{w} = w_t, \\ \eta(x,t,s) = \varphi(x,t) - \varphi(x,t-s), \quad \eta_0(x,s) = \eta(x,0,s), \end{cases}$$

$$\mathcal{A}U = \begin{pmatrix} \tilde{\varphi} \\ \frac{k}{\rho_1}(\varphi_x + \psi + l\,w)_x + \frac{lk_0}{\rho_1}(w_x - l\varphi) - \frac{g_0}{\rho_1}\varphi_{xx} + \frac{1}{\rho_1}\int_0^\infty g\eta_{xx}\,ds \\ \tilde{\psi} \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + l\,w) \\ \tilde{w} \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{lk}{\rho_1}(\varphi_x + \psi + l\,w) \\ \tilde{\varphi} - \eta_s \end{pmatrix} \tag{2.2}$$

and

$$g_0 = \int_0^\infty g(s)\,ds. \tag{2.3}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ V = (v_1, \cdots, v_7)^T \in \mathcal{H},\ \mathcal{A}V \in \mathcal{H},\ v_7(0) = \partial_x v_3(0) = \partial_x v_5(0) = \partial_x v_3(1) = \partial_x v_5(1) = 0 \right\},$$

where

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_\star^1(0,1) \times L_\star^2(0,1) \times H_\star^1(0,1) \times L_\star^2(0,1) \times L_g,$$

$$L_\star^2(0,1) = \left\{ v \in L^2(0,1), \int_0^1 v\,dx = 0 \right\}, \quad H_\star^1(0,1) = H^1(0,1) \cap L_\star^2(0,1)$$

and

$$L_g = \left\{ v : \mathbb{R}_+ \to H_0^1(0,1), \int_0^1 \int_0^\infty g|v_x|^2\,ds\,dx < \infty \right\}.$$

Now let us consider the following hypothesis:
(**H1**) Assume that the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, non-increasing and integrable on $\mathbb{R}_+$, and there exists a postive constant $\tilde{k}$ such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(0,1) \times H_\star^1(0,1) \times H_\star^1(0,1),$$

we have

$$\int_0^1 \left(|\varphi_x|^2 + |\psi_x|^2 + |w_x|^2 \right) dx \leq \tilde{k} \int_0^1 \left(b|\psi_x|^2 + k|\varphi_x + \psi + lw|^2 + k_0|w_x - l\varphi|^2 - g_0|\varphi_x|^2 \right) dx. \tag{2.4}$$

Moreover, assume that there exists a positive constant β_1 such that

$$-\beta_1 g(s) \leq g'(s), \quad \forall s \in \mathbb{R}_+. \tag{2.5}$$

Remark 1. *Condition* (2.4) *holds if the constants l and g_0 are small enough. Under condition* (2.4), *the sets L_g and $\mathcal{H}$ are Hilbert spaces equipped with the inner products, for*

$$\Phi_1 = (\varphi_1, \tilde{\varphi}_1, \psi_1, \tilde{\psi}_1, w_1, \tilde{w}_1, \eta)^T, \quad \Phi_2 = (\varphi_2, \tilde{\varphi}_2, \psi_2, \tilde{\psi}_2, w_2, \tilde{w}_2, \tilde{\eta})^T \in \mathcal{H},$$

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} &= k \langle (\varphi_{1x} + \psi_1 + l w_1), (\varphi_{2x} + \psi_2 + l w_2) \rangle_{L^2(0,1)} + b \langle \psi_{1x}, \psi_{2x} \rangle_{L^2(0,1)} \\ &\quad + k_0 \langle (w_{1x} - l\varphi_1), (w_{2x} - l\varphi_2) \rangle_{L^2(0,1)} - g_0 \langle \varphi_{1x}, \varphi_{2x} \rangle_{L^2(0,1)} \\ &\quad + \rho_1 \langle \tilde{\varphi}_1, \tilde{\varphi}_2 \rangle_{L^2(0,1)} + \rho_2 \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle_{L^2(0,1)} + \rho_1 \langle \tilde{w}_1, \tilde{w}_2 \rangle_{L^2(0,1)} + \langle \eta, \tilde{\eta} \rangle_{L_g} \end{aligned}$$

and

$$\langle \eta, \tilde{\eta} \rangle_{L_g} = \int_0^\infty g \langle \eta_x, \tilde{\eta}_x \rangle_{L^2(0,1)}\,ds.$$

The corresponding energy will be defined as follows, for $\Phi = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w}, \eta)^T$ in $\mathcal{H}$:

$$\begin{aligned} \|\Phi\|_{\mathcal{H}}^2 &= k \|\varphi_x + \psi + l w\|_{L^2(0,1)}^2 + b \|\psi_x\|_{L^2(0,1)}^2 + k_0 \|w_x - l\varphi\|_{L^2(0,1)}^2 \\ &\quad - g_0 \|\varphi_x\|_{L^2(0,1)}^2 + \rho_1 \|\tilde{\varphi}\|_{L^2(0,1)}^2 + \rho_2 \|\tilde{\psi}\|_{L^2(0,1)}^2 + \rho_1 \|\tilde{w}\|_{L^2(0,1)}^2 + \|\eta\|_{L_g}^2 . \end{aligned}$$

Theorem 2. *We assume that* (**H1**) *holds. Let $n \in \mathbb{N}$ and $U_0 \in D(\mathcal{A}^n)$. Then (2.1) has a unique solution*

$$U \in \cap_{m=0}^{n} C^{n-m} \left(\mathbb{R}_+; D\left(\mathcal{A}^m\right) \right). \tag{2.6}$$

Proof. Ecxactely as in [11] one can prove that the linear operator $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$ by proving that $-\mathcal{A}$ is maximal monotone (it is enough to neglect the second memory in the second system considered in [11]). So, we deduce (2.6) (see Theorem 2.3 [11] and its proof). □

2.2 Lack of exponential stability.

Our first main result is that the semigroup associated with Bresse system (2.1) is not exponentially stable.

Theorem 3. *We assume that* (**H1**) *holds. Then the semigroup associated with* (2.1) *is not exponentially stable.*

2.3 Polynomial stability.

Our second result is that the semigroup associated with Bresse system (2.1) is polynomially stable under the following additional two hypotheses:
(**H2**) Assume that $g_0 > 0$ and there exists a positive constant β_2 such that

$$g^{'}(s) \leq -\beta_2 g(s), \quad \forall s \in \mathbb{R}_+. \tag{2.7}$$

(**H3**) Assume that l satisfies

$$l^2 \neq \frac{k_0\rho_2 - b\rho_1}{k_0\rho_2}(m\pi)^2 - \frac{k\rho_1}{\rho_2(k+k_0)}, \quad \forall m \in \mathbb{Z}. \tag{2.8}$$

Our second main result is stated as follow:

Theorem 4. *We assume that* (**H1**) – (**H3**) *hold. Then, for any* $m \in \mathbb{N}^*$, *there exists a constant* $c_m > 0$ *such that*

$$\forall \Phi_0 \in D\left(\mathcal{A}^m\right), \ \forall t > 0, \ \left\| e^{t\mathcal{A}}\Phi_0 \right\|_{\mathcal{H}} \leq c_m \|\Phi_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t}\right)^{\frac{m}{4}} \ln t. \tag{2.9}$$

Remark 5. *1. Typical simple examples of g satisfying* (**H1**) *and* (**H2**) *are*

$$g(s) = b_1 e^{-b_2 s},$$

where b_1 and b_2 are psotive constants.
2. The estimate (2.9) *gives*

$$\forall m \in \mathbb{N}^*, \ \forall \Phi_0 \in D\left(\mathcal{A}^m\right), \ \forall \epsilon > 0, \ \exists C_{m,\epsilon,\Phi_0} > 0: \ \left\| e^{t\mathcal{A}}\Phi_0 \right\|_{\mathcal{H}} \leq C_{m,\epsilon,\Phi_0} t^{\frac{-m}{4}+\epsilon}. \quad \forall t > 0.$$

So, for classical solutions (that is $m = 1$), the decay rate of $t \to \left\| e^{t\mathcal{A}}\Phi_0 \right\|_{\mathcal{H}}$ is arbitrarily close to $t^{\frac{-1}{4}}$.

The proof of our non-exponential and polynomial stability for (2.1) is based on the following frequency domain theorems:

Theorem 6. *([12] and [20]) A C_0 semigroup of contractions on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.10}$$

Theorem 7. *([16]) If a bounded C_0 semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ satisfies, for some $j \in \mathbb{N}^*$,*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\lambda| \geq 1} \frac{1}{\lambda^j} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.11}$$

Then, for any $m \in \mathbb{N}^*$, there exists a positive constant c_m such that

$$\forall \Phi_0 \in D\left(\mathcal{A}^m\right), \ \forall t > 0, \quad \left\| e^{t\mathcal{A}}\Phi_0 \right\|_{\mathcal{H}} \leq c_m \|\Phi_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t}\right)^{\frac{m}{j}} \ln t. \tag{2.12}$$

3 Lack of exponential stability of (2.1)

We use Theorem 6 by proving that the second condition in (2.10) is not satisfied. We have to prove that there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \left\| (i\lambda_n I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} = \infty,$$

which is equivalent to find a sequence $(F_n)_n \subset \mathcal{H}$ satisfying

$$\|F_n\|_{\mathcal{H}} \leq 1, \quad \forall n \in \mathbb{N} \tag{3.1}$$

and

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})^{-1} F_n\|_{\mathcal{H}} = \infty. \tag{3.2}$$

For this purpose, let

$$\Phi_n = (i\lambda_n I - \mathcal{A})^{-1} F_n, \quad \forall n \in \mathbb{N}.$$

Then we have to find sequences $(\lambda_n)_n \subset \mathbb{R}$, $(F_n)_n \subset \mathcal{H}$ and $(\Phi_n)_n \subset D(\mathcal{A})$ satisfying (3.1),

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = \infty \quad \text{and} \quad i\lambda_n \Phi_n - \mathcal{A}\Phi_n = F_n, \ \forall n \in \mathbb{N}. \tag{3.3}$$

Taking

$$\Phi_n = \left(\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n, \eta_n\right)^T \quad \text{and} \quad F_n = (f_{1n}, \cdots, f_{7n})^T.$$

then we have the following system:

$$\begin{cases} i\lambda_n \varphi_n - \tilde{\varphi}_n = f_{1n}, \\ i\rho_1 \lambda_n \tilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l\, w_n)_x - lk_0(w_{nx} - l\varphi_n) + g_0 \varphi_{nxx} - \int_0^\infty g \eta_{nxx}\, ds = \rho_1 f_{2n}, \\ i\lambda_n \psi_n - \tilde{\psi}_n = f_{3n}, \\ i\rho_2 \lambda_n \tilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l\, w_n) = \rho_2 f_{4n}, \\ i\lambda_n w_n - \tilde{w}_n = f_{5n}, \\ i\rho_1 \lambda_n \tilde{w}_n - k_0 (w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + l\, w_n) = \rho_1 f_{6n}, \\ i\lambda_n \eta_n + \eta_{ns} - \tilde{\varphi}_n = f_{7n}. \end{cases} \tag{3.4}$$

Choosing

$$f_{1n} = f_{3n} = f_{5n} = f_{7n} = 0. \tag{3.5}$$

Then system (3.4) becomes

$$\begin{cases} \tilde{\varphi}_n = i\lambda_n \varphi_n, \quad \tilde{\psi}_n = i\lambda_n \psi_n, \quad \tilde{w}_n = i\lambda_n w_n, \\ -\rho_1 \lambda_n^2 \varphi_n - k(\varphi_{nx} + \psi_n + l\, w_n)_x - lk_0(w_{nx} - l\varphi_n) + g_0 \varphi_{nxx} - \int_0^\infty g \eta_{nxx}\, ds = \rho_1 f_{2n}, \\ -\rho_2 \lambda_n^2 \psi_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l\, w_n) = \rho_2 f_{4n}, \\ -\rho_1 \lambda_n^2 w_n - k_0 (w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + l\, w_n) = \rho_1 f_{6n}, \\ i\lambda_n \eta_n + \eta_{ns} - i\lambda_n \varphi_n = 0. \end{cases} \tag{3.6}$$

To simplify the calculations, we put $N = n\pi$. We use here some ideas of [1], where some of the next computations are addapted to our problem. Now we consider three cases.

Case 1: $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$. We choose

$$\begin{cases} \varphi_n = \tilde{\varphi}_n = \eta_n = 0, \\ \psi_n(x) = \alpha_1 \cos(Nx), \quad \tilde{\psi}_n(x) = i\alpha_1 \lambda_n \cos(Nx), \\ w_n(x) = \alpha_2 \cos(Nx), \quad \tilde{w}_n(x) = i\alpha_2 \lambda_n \cos(Nx), \end{cases} \tag{3.7}$$

$$f_{2n} = 0, \quad f_{4n}(x) = -\frac{lk_0}{\rho_2}\alpha_2 \cos(Nx), \quad f_{6n}(x) = -\frac{l^2 k_0}{\rho_1}\alpha_2 \cos(Nx) \tag{3.8}$$

and

$$\lambda_n = N\sqrt{\frac{k_0}{\rho_1}}, \tag{3.9}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. We have $\Phi_n \in D(\mathcal{A})$ and $F_n \in \mathcal{H}$. On the other hand, (3.6) is satisfied if and only if

$$\begin{cases} k\alpha_1 + l(k+k_0)\alpha_2 = 0, \\ \left[-\lambda_n^2 + \frac{b}{\rho_2}N^2 + \frac{k}{\rho_2}\right]\alpha_1 + \frac{lk}{\rho_2}\alpha_2 = -\frac{lk_0}{\rho_2}\alpha_2, \\ \frac{lk}{\rho_1}\alpha_1 + \left[-\lambda_n^2 + \frac{k_0}{\rho_1}N^2 + \frac{l^2 k}{\rho_1}\right]\alpha_2 = -\frac{l^2 k_0}{\rho_1}\alpha_2. \end{cases} \tag{3.10}$$

According to (3.9) and because $\frac{b}{\rho_2} = \frac{k_0}{\rho_1}$, we have

$$-\lambda_n^2 + \frac{k_0}{\rho_1}N^2 = -\lambda_n^2 + \frac{b}{\rho_2}N^2 = 0,$$

and therefore, the system (3.10) is equivalent to

$$\alpha_1 = -l\left(1 + \frac{k_0}{k}\right)\alpha_2. \tag{3.11}$$

Choosing

$$\alpha_2 = \frac{\rho_1\rho_2}{lk_0\sqrt{\rho_1^2 + l^2\rho_2^2}}$$

and using (3.5) and (3.8), we obtain

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{4n}\|_{L^2(0,1)}^2 + \|f_{6n}\|_{L^2(0,1)}^2 = \left(\frac{lk_0}{\rho_2}\right)^2\left[1 + \left(\frac{l\rho_2}{\rho_1}\right)^2\right]\alpha_2^2 \int_0^1 \cos^2(Nx)\,dx$$

$$\leq \left(\frac{lk_0}{\rho_2}\right)^2\left[1 + \left(\frac{l\rho_2}{\rho_1}\right)^2\right]\alpha_2^2 = 1.$$

On the other hand, from (2.4), we have

$$\|\Phi_n\|_{\mathcal{H}}^2 \geq \frac{1}{\tilde{k}}\|w_{nx} - l\varphi_n\|_{L^2(0,1)}^2 = \frac{1}{\tilde{k}}\|w_{nx}\|_{L^2(0,1)}^2 = \frac{\alpha_2^2}{2\tilde{k}}N^2 \int_0^1 [1 - \cos(2Nx)]\,dx = \frac{\alpha_2^2}{2\tilde{k}}N^2,$$

hence 5

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = \infty. \tag{3.12}$$

Case 2: $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$ and $k \neq k_0$. We choose

$$f_{2n} = f_{4n} = 0, \quad f_{6n}(x) = \cos(Nx), \tag{3.13}$$

$$\begin{cases} \varphi_n(x) = \alpha_1 \sin(Nx), \quad \tilde{\varphi}_n(x) = i\alpha_1\lambda_n \sin(Nx), \\ \psi_n(x) = \alpha_2 \cos(Nx), \quad \tilde{\psi}_n(x) = i\alpha_2\lambda_n \cos(Nx), \\ w_n(x) = \alpha_3 \cos(Nx), \quad \tilde{w}_n(x) = i\alpha_3\lambda_n \cos(Nx), \\ \eta_n(x,s) = \alpha_1\left(1 - e^{-i\lambda_n s}\right)\sin(Nx) \end{cases} \tag{3.14}$$

and

$$\lambda_n = \sqrt{\frac{k_0}{\rho_1}N^2 + \frac{l^2 k}{\rho_1}}, \tag{3.15}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Notice that, according to these choices, $\Phi_n \in D(\mathcal{A})$, $F_n \in \mathcal{H}$ and

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{6n}\|_{L^2(0,1)}^2 = \int_0^1 \cos^2(Nx)\,dx \le 1. \tag{3.16}$$

On the other hand, thanks to (3.5), (3.13) and (3.14), the first three equations and the last one in (3.6) are satisfied, and the other three equations are equivalent to

$$\begin{cases} \left[(k-\mu_n)N^2 - \rho_1\lambda_n^2 + l^2 k_0\right]\alpha_1 + kN\alpha_2 + l(k+k_0)N\alpha_3 = 0, \\ kN\alpha_1 + \left(bN^2 - \rho_2\lambda_n^2 + k\right)\alpha_2 + kl\alpha_3 = 0, \\ l(k+k_0)N\alpha_1 + lk\alpha_2 + \left(k_0N^2 - \rho_1\lambda_n^2 + l^2 k\right)\alpha_3 = \rho_1, \end{cases} \tag{3.17}$$

where we note

$$\mu_n = \int_0^\infty g(s)e^{-i\lambda_n s}\,ds$$

(μ_n exists because g is integrable on $\mathbb{R}_+$ and $|e^{-i\lambda_n s}| = 1$). From the choice (3.15), we see that the last equation in (3.17) is equivalent to

$$\alpha_2 = -\frac{k+k_0}{k}N\alpha_1 + \frac{\rho_1}{lk}, \tag{3.18}$$

so, substituting in the first two equations in (3.17), we get

$$\alpha_3 = a_1 N\alpha_1 + a_2 \tag{3.19}$$

and

$$\alpha_1 = \frac{\left[l(k+k_0)a_2 + \frac{\rho_1}{l}\right]N}{\left[2k_0 + \mu_n - l(k+k_0)a_1\right]N^2 + l^2(k-k_0)}, \tag{3.20}$$

where

$$\begin{cases} a_1 = \frac{k+k_0}{lk^2}\left(b - \frac{\rho_2 k_0}{\rho_1}\right)N^2 + \frac{k_0}{lk} - \frac{l\rho_2(k+k_0)}{\rho_1 k}, \\ a_2 = \frac{\rho_1}{(lk)^2}\left[\left(\frac{\rho_2 k_0}{\rho_1} - b\right)N^2 + \frac{l^2\rho_2 k}{\rho_1} - k\right]. \end{cases}$$

To simplify the computations, we put

$$\begin{cases} a_3 = \frac{\rho_1(k+k_0)}{lk^2}\left(\frac{\rho_2 k_0}{\rho_1} - b\right), \quad a_4 = \frac{(k+k_0)^2}{k^2}\left(\frac{\rho_2 k_0}{\rho_1} - b\right), \\ a_5 = \frac{l\rho_2(k+k_0)}{k} - \frac{k_0\rho_1}{lk}, \quad a_6 = \frac{l^2\rho_2(k+k_0)^2}{\rho_1 k} + \frac{k_0(k-k_0)}{k} \end{cases}$$

and

$$\begin{cases} d_0 = \frac{k+k_0}{lk^2}\left(b - \frac{\rho_2 k_0}{\rho_1}\right), \quad d_1 = \frac{\rho_1}{(lk)^2}\left(\frac{\rho_2 k_0}{\rho_1} - b\right), \\ d_2 = \frac{k_0}{lk} - \frac{l\rho_2(k+k_0)}{\rho_1 k}, \quad d_3 = \frac{\rho_1}{l^2 k}\left(\frac{l^2\rho_2}{\rho_1} - 1\right). \end{cases}$$

Then

$$N\alpha_1 = \frac{a_3N^4 + a_5N^2}{a_4N^4 + (\mu_n + a_6)N^2 + l^2(k-k_0)}$$

and (notice that $d_0a_3 + d_1a_4 = 0$)

$$\alpha_3 = \frac{\left(d_0N^2 + d_2\right)\left(a_3N^4 + a_5N^2\right)}{a_4N^4 + (\mu_n + a_6)N^2 + l^2(k-k_0)} + d_1N^2 + d_3 \tag{3.21}$$

$$= \frac{(d_0 a_5 + d_2 a_3 + d_3 a_4 + d_1 a_6 + d_1 \mu_n) N^4 + \left(d_2 a_5 + d_3 a_6 + l^2 (k - k_0) d_1 + d_3 \mu_n\right) N^2 + l^2 (k - k_0) d_3}{a_4 N^4 + (\mu_n + a_6) N^2 + l^2 (k - k_0)},$$

Because $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$ and $k \neq k_0$, then $a_4 \neq 0$ and

$$d_0 a_5 + d_2 a_3 + d_3 a_4 + d_1 a_6 = \frac{\rho_1}{(lk)^2} \left(\frac{\rho_2 k_0}{\rho_1} - b\right) (k_0 - k) \neq 0. \tag{3.22}$$

On the other hand, integrating by parts and using the fact that g is non-increasing and $\lim_{s \to \infty} g(s) = 0$, we get

$$\begin{aligned} |\mu_n| &= \left| \frac{1}{i\lambda_n} \left(g(0) + \int_0^\infty g^{'}(s) e^{-i\lambda_n s}\, ds \right) \right| \\ &\leq \frac{1}{\lambda_n} \left(g(0) + \int_0^\infty |g^{'}(s)|\, ds \right) \\ &\leq \frac{1}{\lambda_n} \left(g(0) - \int_0^\infty g^{'}(s)\, ds \right) \\ &\leq \frac{2g(0)}{\lambda_n}, \end{aligned}$$

therefore

$$\lim_{n \to \infty} \mu_n = 0. \tag{3.23}$$

Then we deduce from (3.21), (3.22) and (3.23) that

$$\lim_{n \to \infty} \alpha_3 = \frac{d_0 a_5 + d_2 a_3 + d_3 a_4 + d_1 a_6}{a_4} \neq 0, \tag{3.24}$$

hence

$$\lim_{n \to \infty} |\alpha_3| N = \infty. \tag{3.25}$$

Now, in virtue of (2.4), we have

$$\begin{aligned} \|\Phi_n\|^2_{\mathcal{H}} &\geq \frac{1}{\tilde{k}} \|w_{nx}\|^2_{L^2(0,1)} = \frac{(|\alpha_3| N)^2}{\tilde{k}} \int_0^1 \sin^2 (Nx)\, dx \\ &\geq \frac{(|\alpha_3| N)^2}{2\tilde{k}} \int_0^1 [1 - \cos (2Nx)]\, dx = \frac{(|\alpha_3| N)^2}{2\tilde{k}}, \end{aligned}$$

then by (3.25) we get (3.12). 5

Case 3: $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$ and $k = k_0$. We consider the choices (3.5),

$$\lambda_n = \sqrt{\frac{b}{\rho_2} N^2 + \frac{k}{2\rho_2}}, \tag{3.26}$$

$$f_{2n} = 0, \quad f_{4n}(x) = \alpha_2 C_n \cos (Nx), \quad f_{6n}(x) = \alpha_2 D_n \cos (Nx) \tag{3.27}$$

and (3.14) with

$$\alpha_1 = \left(\frac{\rho_1 D_n}{2lk} - \frac{1}{2} \right) \frac{\alpha_2}{N} \quad \text{and} \quad \alpha_3 = 0, \tag{3.28}$$

where

$$C_n = \frac{\rho_1}{2l\rho_2} D_n \quad \text{and} \quad D_n = \frac{2lk}{\rho_1} \left(\frac{1}{2} - \frac{k}{k + \frac{l^2 k}{N^2} - \mu_n - \frac{\rho_1 \lambda_n^2}{N^2}} \right).$$

According to (3.23) and (3.26), we remark that

$$\lim_{n\to\infty} D_n = \frac{2lk}{\rho_1}\left(\frac{1}{2} - \frac{k}{k-\frac{\rho_1 b}{\rho_2}}\right) \quad \text{and} \quad \lim_{n\to\infty} C_n = \frac{k}{\rho_2}\left(\frac{1}{2} - \frac{k}{k-\frac{\rho_1 b}{\rho_2}}\right)$$

(these limits exist since $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$ and $k = k_0$), so the sequence $\left(|C_n|^2 + |D_n|^2\right)_n$ is bounded. Then we choose

$$\alpha_2 = \frac{1}{\sqrt{sup_{n\in\mathbb{N}}\left(|C_n|^2 + |D_n|^2\right)}}. \tag{3.29}$$

According to these choices, we see that $\Phi_n \in D(\mathcal{A})$, $F_n \in \mathcal{H}$ and, using (3.5), (3.27) and (3.29), we find

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{4n}\|_{L^2(0,1)}^2 + \|f_{6n}\|_{L^2(0,1)}^2 = \left(|C_n|^2 + |D_n|^2\right)\alpha_2^2 \int_0^1 \cos^2(Nx)\,dx$$

$$\leq \left(|C_n|^2 + |D_n|^2\right)\alpha_2^2 \leq 1.$$

On the other hand, thanks to (3.5), (3.14) and (3.27), the first three equations and the last one in (3.6) are satisfied, and because $\alpha_3 = 0$ and $k = k_0$, the other three equations are equivalent to

$$\begin{cases} \left[(k-\mu_n)N^2 - \rho_1\lambda_n^2 + l^2 k\right]\alpha_1 + kN\alpha_2 = 0, \\ kN\alpha_1 + \left(bN^2 - \rho_2\lambda_n^2 + k\right)\alpha_2 = \rho_2\alpha_2 C_n, \\ 2lkN\alpha_1 + lk\alpha_2 = \rho_1\alpha_2 D_n. \end{cases} \tag{3.30}$$

The first equation in (3.30) is satisfied thanks to the definition of α_1 and D_n, the second equation in (3.30) holds according to the definition of λ_n, α_1 and C_n, and the last equation in (3.30) is satisfied from the definition of α_1.

Now, in virtue of (2.4), we have

$$\|\Phi_n\|_{\mathcal{H}}^2 \geq \frac{1}{\tilde{k}}\|\psi_{nx}\|_{L^2(0,1)}^2 = \frac{(\alpha_2 N)^2}{\tilde{k}}\int_0^1 \sin^2(Nx)\,dx$$

$$\geq \frac{(\alpha_2 N)^2}{2\tilde{k}}\int_0^1 [1-\cos(2Nx)]\,dx = \frac{(\alpha_2 N)^2}{2\tilde{k}},$$

consequently, (3.12) holds.

Finally, there exist sequences $(F_n)_n \subset \mathcal{H}$, $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying (3.1) and (3.3). Hence, Theorem 6 implies that system (2.1) is not exponentially stable.

4 Polynomial stability of (2.1)

Using Theorem 7, we need to show that

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{4.1}$$

and

$$\sup_{|\lambda|\geq 1}\frac{1}{\lambda^4}\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{H}} < \infty. \tag{4.2}$$

We start by proving (4.1). Notice that, according to the fact that $0 \in \rho(\mathcal{A})$ (see [11] for the second system with a neglected second memory), $\mathcal{A}^{-1}$ is bounded and it is a bijection between $\mathcal{H}$ and $D(\mathcal{A})$. Since $D(\mathcal{A})$ has a compact embedding into $\mathcal{H}$, so it follows that $\mathcal{A}^{-1}$ is a compact operator, which implies that the spectrum of $\mathcal{A}$ is discrete.

From subsection 2.1, we have $0 \in \rho(\mathcal{A})$. Let $\lambda \in \mathbb{R}^*$ and

$$\Phi = \left(\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w}, \eta\right)^T \in D(\mathcal{A}).$$

We prove that $i\lambda$ is not an eigenvalue of $\mathcal{A}$ by proving that the equation

$$\mathcal{A}\,\Phi = i\,\lambda\,\Phi \tag{4.3}$$

has a unique solution $\Phi = 0$. Assume that (4.3) is true, then we have

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \\ \dfrac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \dfrac{lk_0}{\rho_1}\left(w_x - l\varphi\right) - \dfrac{g_0}{\rho_1}\varphi_{xx} + \dfrac{1}{\rho_1}\displaystyle\int_0^\infty g(s)\eta_{xx}\,ds = i\lambda\tilde{\varphi}, \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) = i\lambda\tilde{\psi}, \\ \dfrac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \dfrac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) = i\lambda\tilde{w}, \\ \tilde{\varphi} - \eta_s = i\lambda\eta. \end{cases} \tag{4.4}$$

A simple computations implies that (see (44) [11])

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^\infty g'(s)\|\eta_x\|^2_{L^2(0,1)}\,ds, \tag{4.5}$$

then

$$0 = Re\, i\lambda\,\|\Phi\|^2_{\mathcal{H}} = Re\,\langle i\lambda\Phi, \Phi\rangle_{\mathcal{H}} = Re\,\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^\infty g'(s)\|\eta_x\|^2_{L^2(0,1)}\,ds.$$

Therefore, using (2.7),

$$0 \le \|\eta\|^2_{L_g} = \int_0^\infty g(s)\|\eta_x\|^2_{L^2(0,1)}\,ds \le \frac{-1}{\beta_2}\int_0^\infty g'(s)\|\eta_x\|^2_{L^2(0,1)}\,ds = 0,$$

so

$$\eta = 0. \tag{4.6}$$

By the first and last equations in (4.4), we find

$$\varphi = \tilde{\varphi} = 0. \tag{4.7}$$

Using (4.6) and (4.7), we see that (4.4) leads to

$$\begin{cases} \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \\ k\psi_x + l\left(k + k_0\right)w_x = 0, \\ b\psi_{xx} - k\left(\psi + l\,w\right) = -\rho_2\lambda^2\psi, \\ k_0 w_{xx} - lk\left(\psi + l\,w\right) = -\rho_1\lambda^2 w. \end{cases} \tag{4.8}$$

The third equation in (4.8) implies that $k\psi + l\left(k + k_0\right)w$ is a constant, then, thanks to the definition of $L^2_*(0,1)$, we get

$$\psi = -l\left(1 + \frac{k_0}{k}\right)w. \tag{4.9}$$

Using the last two equations in (4.8), we obtain

$$lb\psi_{xx} - k_0 w_{xx} = -\rho_2 l\lambda^2\psi + \rho_1\lambda^2 w. \tag{4.10}$$

Then, combining with (4.9), we find

$$w_{xx} + \alpha^2\lambda^2 w = 0,$$

where

$$\alpha = \sqrt{\frac{\rho_2 l^2 (k + k_0) + k\rho_1}{bl^2 (k + k_0) + kk_0}}. \tag{4.11}$$

This implies that, for $c_1,\ c_2 \in \mathbb{C}$,

$$w(x) = c_1 \cos(\alpha\lambda x) + c_2 \sin(\alpha\lambda x).$$

The boundary condition $w_x(0) = 0$ leads to $c_2 = 0$, and then, using (4.9),

$$\psi(x) = -l\left(1 + \frac{k_0}{k}\right) c_1 \cos(\alpha\lambda x) \quad \text{and} \quad w(x) = c_1 \cos(\alpha\lambda x). \tag{4.12}$$

Because $\psi_x(1) = w_x(1) = 0$, we have

$$c_1 = 0 \quad \text{or} \quad \exists m \in \mathbb{Z}:\ \alpha\lambda = m\pi.$$

Assume by contradiction that

$$\exists m \in \mathbb{Z}:\ \alpha\lambda = m\pi. \tag{4.13}$$

Therefore, using (4.11) and (4.12), we get that the last two equations in (4.8) are equivalent to

$$(k_0\rho_2 - b\rho_1)\lambda^2 = \frac{k_0}{k + k_0}\left[bl^2 (k + k_0) + kk_0\right]. \tag{4.14}$$

So, combining (4.11), (4.13) and (4.14), we get

$$\exists m \in \mathbb{Z}:\ l^2 = \frac{k_0\rho_2 - b\rho_1}{k_0\rho_2}(m\pi)^2 - \frac{k\rho_1}{\rho_2 (k + k_0)},$$

which is a contradiction with (2.8). Consequentely, $c_1 = 0$ and hence

$$\psi = w = 0. \tag{4.15}$$

Using (4.15) and the first two equations in (4.8), we obtain

$$\tilde{\psi} = \tilde{w} = 0.$$

Finally, $\Phi = 0$ and thus

$$i\lambda \in \rho(\mathcal{A}). \tag{4.16}$$

This ends the proof of (4.1).

Now we establish (4.2) by contradiction. Assume that (4.2) is false, then there exist sequences $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}} = 1, \quad \forall n \in \mathbb{N}, \tag{4.17}$$

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{4.18}$$

and

$$\lim_{n\to\infty} \lambda_n^4 \|(i\lambda_n I - \mathcal{A})\Phi_n\|_{\mathcal{H}} = 0. \tag{4.19}$$

Let $\Phi_n = \left(\varphi_n, \widetilde{\varphi}_n, \psi_n, \widetilde{\psi}_n, w_n, \widetilde{w}_n, \eta_n\right)^T$. The limit (4.19) implies that

$$\begin{cases} \lambda_n^4 \left[i\lambda_n\varphi_n - \widetilde{\varphi}_n\right] \to 0 \text{ in } H_0^1(0,1), \\ \lambda_n^4 \left[i\lambda_n\rho_1\widetilde{\varphi}_n - k(\varphi_{nx} + \psi_n + lw_n)_x - lk_0(w_{nx} - l\varphi_n) + g_0\varphi_{nxx} - \int\limits_0^\infty g(s)\eta_{nxx}\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n\psi_n - \widetilde{\psi}_n\right] \to 0 \text{ in } H_\star^1(0,1), \\ \lambda_n^4 \left[i\lambda_n\rho_2\widetilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + lw_n)\right] \to 0 \text{ in } L_\star^2(0,1), \\ \lambda_n^4 \left[i\lambda_n w_n - \widetilde{w}_n\right] \to 0 \text{ in } H_\star^1(0,1), \\ \lambda_n^4 \left[i\lambda_n\rho_1\widetilde{w}_n - k_0(w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + lw_n)\right] \to 0 \text{ in } L_\star^2(0,1), \\ \lambda_n^4 \left[i\lambda_n\eta_n + \eta_{ns} - \widetilde{\varphi}\right] \to 0 \text{ in } L_g. \end{cases} \tag{4.20}$$

We will prove that $\|\Phi_n\|_{\mathcal{H}} \to 0$, which gives a contradiction with (4.17). To do so, we will use several multipliers, where some of them are used in [1] .

Step 1. Using (4.5), we get

$$\begin{aligned} Re\left\langle\lambda_n^4(i\lambda_n I - \mathcal{A})\ \Phi_n, \Phi_n\right\rangle_{\mathcal{H}} &= Re\left(i\lambda_n^5\|\Phi_n\|_{L^2(0,1)}^2 - \lambda_n^4\langle\mathcal{A}\,\Phi_n, \Phi_n\rangle_{\mathcal{H}}\right) \\ &= \frac{-\lambda_n^4}{2}\int\limits_0^\infty g'(s)\|\eta_{nx}\|_{L^2(0,1)}^2\,ds. \end{aligned}$$

So (4.17) and (4.19) imply that

$$\lambda_n^4\int\limits_0^\infty g'(s)\|\eta_{nx}\|_{L^2(0,1)}^2\,ds \longrightarrow 0.$$

But, in virtue of (2.7), we have

$$0 \le \lambda_n^4\int\limits_0^\infty g(s)\|\eta_{nx}\|_{L^2(0,1)}^2\,ds \le \frac{-\lambda_n^4}{\beta_2}\int\limits_0^\infty g'(s)\|\eta_{nx}\|_{L^2(0,1)}^2\,ds,$$

then

$$\lambda_n^4\int\limits_0^\infty g(s)\|\eta_{nx}\|_{L^2(0,1)}^2\,ds \longrightarrow 0,$$

hence

$$\lambda_n^2\eta_n \longrightarrow 0 \text{ in } L_g. \tag{4.21}$$

Step 2. Using (4.17) and (4.18), we get from the last limit in (4.20) that

$$\lambda_n\left\langle\left(i\lambda_n\eta_n + \eta_{ns} - i\lambda_n\varphi_n + i\lambda_n\varphi_n - \widetilde{\varphi}_n\right), i\varphi_n\right\rangle_{L_g} \longrightarrow 0,$$

so, using the first limit in (4.20), 5

$$\left\langle\lambda_n^2\eta_n, \varphi_n\right\rangle_{L_g} - i\lambda_n\langle\eta_{ns}, \varphi_n\rangle_{L_g} - \lambda_n^2\langle\varphi_n, \varphi_n\rangle_{L_g} \longrightarrow 0. \tag{4.22}$$

We see that

$$-\lambda_n^2\langle\varphi_n, \varphi_n\rangle_{L_g} = -g_0\lambda_n^2\|\varphi_{nx}\|_{L^2(0,1)}^2, \tag{4.23}$$

and (4.17) and (4.21) imply that

$$\left\langle \lambda_n^2 \eta_n, \varphi_n \right\rangle_{L_g} \longrightarrow 0. \tag{4.24}$$

On the other hand, integrating by part with respect to s, applying Cauchy-Schwartz inequality and using (2.5) and the fact that

$$\eta_{nx}(x,0) = 0 \quad \text{and} \quad \lim_{s\to\infty} g(s) = 0,$$

we find

$$\begin{aligned} \left| -i\lambda_n \left\langle \eta_{ns}, \varphi_n \right\rangle_{L_g} \right| &= |\lambda_n| \left| \left\langle \varphi_{nx}, \int_0^\infty (-g'(s)) \eta_{nx}\, ds \right\rangle_{L^2(0,1)} \right| \\ &\le |\lambda_n| \|\varphi_{nx}\|_{L^2(0,1)} \int_0^\infty (-g'(s)) \|\eta_{nx}\|_{L^2(0,1)}\, ds \\ &\le \sqrt{g(0)} |\lambda_n| \|\varphi_{nx}\|_{L^2(0,1)} \left(\int_0^\infty (-g'(s)) \|\eta_{nx}\|^2_{L^2(0,1)}\, ds \right)^{\frac{1}{2}} \\ &\le \sqrt{\beta_1 g(0)} |\lambda_n| \|\varphi_{nx}\|_{L^2(0,1)} \|\eta_n\|_{L_g}, \end{aligned}$$

and then, according to (4.17), (4.18) and (4.21),

$$\lambda_n \left\langle \eta_{ns}, \varphi_n \right\rangle_{L_g} \longrightarrow 0. \tag{4.25}$$

Consequentely, (4.22) – (4.25) and since $g_0 > 0$ (hypothesis (**H2**)) lead to

$$\lambda_n \varphi_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.26}$$

Moreover, because $\varphi_n \in H_0^1(0,1)$, then

$$\lambda_n \varphi_n \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.27}$$

and by $(4.20)_1$, we find

$$\widetilde{\varphi}_{nx} \to 0 \text{ in } L^2(0,1). \tag{4.28}$$

Therefore, since $\widetilde{\varphi}_n \in H_0^1(0,1)$,

$$\widetilde{\varphi}_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.29}$$

Step 3. Multiplying $(4.20)_3$ and $(4.20)_5$ by $\frac{1}{\lambda_n^5}$, and using (4.17) and (4.18), we obtain

$$\begin{cases} \psi_n \longrightarrow 0 \text{ in } L^2(0,1), \\ w_n \longrightarrow 0 \text{ in } L^2(0,1). \end{cases} \tag{4.30}$$

Step 4. Taking the inner product of $(4.20)_2$ with $\frac{i\widetilde{\varphi}_n}{\lambda_n^3}$ in $L^2(0,1)$, using (4.18), integrating by parts and using the boundary conditions, we get

$$\begin{aligned} &\rho_1 \left\| \lambda_n \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} + \left\langle k\lambda_n (\varphi_{nx} + \psi_n + l w_n) - g_0 \lambda_n \varphi_{nx}, i\widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} \\ &+ l k_0 \left\langle \lambda_n w_n, i\widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} + l^2 k_0 \left\langle \lambda_n \varphi_n, i\widetilde{\varphi}_n \right\rangle_{L^2(0,1)} + \left\langle \lambda_n \int_0^\infty g(s) \eta_{nx}\, ds, i\widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} \to 0. \end{aligned} \tag{4.31}$$

Multiplying $(4.20)_3$ and $(4.20)_5$ by $\frac{1}{\lambda_n^4}$, and using (4.17) and (4.18), we have

$$(\lambda_n \psi_n)_n \text{ and } (\lambda_n w_n)_n \text{ are bounded in } L^2(0,1). \tag{4.32}$$

So, using (4.17), (4.21), (4.26), (4.27), (4.28), (4.31) and (4.32), we deduce that

$$\lambda_n \widetilde{\varphi}_n \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.33}$$

and by (4.18) and $(4.20)_1$, we find

$$\lambda_n^2 \varphi_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.34}$$

Step 5. Taking the inner product of $(4.20)_2$ with $\frac{1}{\lambda_n^4}\left[k\psi_{nx} + l(k+k_0)w_{nx}\right]$ in $L^2(0,1)$ and using (4.18), we get

$$\begin{aligned}\rho_1 \left\langle i\lambda_n \widetilde{\varphi}_n, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} &+ (g_0 - k)\left\langle \varphi_{nxx}, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} \\ - \left\|k\psi_{nx} + l(k+k_0)w_{nx}\right\|^2_{L^2(0,1)} &+ l^2 k_0 \left\langle \varphi_n, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} \\ - \left\langle \int_0^\infty g(s)\eta_{nxx}\,ds, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} &\to 0.\end{aligned} \tag{4.35}$$

Integrating by parts and using the boundary conditions, we get

$$\left\langle \varphi_{nxx}, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} = -\left\langle \lambda_n \varphi_{nx}, \left[k\frac{\psi_{nxx}}{\lambda_n} + l(k+k_0)\frac{w_{nxx}}{\lambda_n}\right]\right\rangle_{L^2(0,1)} \tag{4.36}$$

and

$$\begin{aligned}&\left\langle \int_0^\infty g(s)\eta_{nxx}\,ds, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} \\ &= -\left\langle \lambda_n \int_0^\infty g(s)\eta_{nx}\,ds, \left[k\frac{\psi_{nxx}}{\lambda_n} + l(k+k_0)\frac{w_{nxx}}{\lambda_n}\right]\right\rangle_{L^2(0,1)}.\end{aligned} \tag{4.37}$$

Multiplying $(4.20)_4$ and $(4.20)_6$ by $\frac{1}{\lambda_n^5}$ and using (4.18), we obtain

$$\begin{cases} i\rho_2 \widetilde{\psi}_n - b\dfrac{\psi_{nxx}}{\lambda_n} + \dfrac{k}{\lambda_n}(\varphi_{nx} + \psi_n + lw_n) \to 0 \text{ in } L^2(0,1), \\ i\rho_1 \widetilde{w}_n - k_0\dfrac{w_{nxx}}{\lambda_n} + lk_0\dfrac{\varphi_{nx}}{\lambda_n} + \dfrac{lk}{\lambda_n}(\varphi_{nx} + \psi_n + lw_n) \to 0 \text{ in } L^2(0,1). \end{cases}$$

Exploiting (4.17), we get

$$\left(\frac{1}{\lambda_n}\psi_{nxx}\right)_n \text{ and } \left(\frac{1}{\lambda_n}w_{nxx}\right)_n \text{ are bounded in } L^2(0,1), \tag{4.38}$$

then, using (4.21), (4.26), (4.36), (4.37) and (4.38), we deduce that

$$\left\langle \varphi_{nxx}, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} \to 0 \tag{4.39}$$

and

$$\left\langle \int_0^\infty g(s)\eta_{nxx}\,ds, \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} \to 0,$$

so, exploiting (4.17), (4.27), (4.33) and (4.35), we have

$$k\psi_{nx} + l(k+k_0)w_{nx} \to 0 \text{ in } L^2(0,1). \tag{4.40}$$

Step 6. Taking the inner product of $(4.20)_4$ with $\frac{\psi_n}{\lambda_n^4}$ in $L^2(0,1)$, using (4.18), integrating by parts and using the boundary conditions, we obtain

$$-\rho_2 \left\langle \widetilde{\psi}_n, \left(i\lambda_n\psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2 \left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)}$$

$$+b\left\|\psi_{nx}\right\|^2_{L^2(0,1)} + k\left\langle(\varphi_{nx}+\psi_n+lw_n),\psi_n\right\rangle_{L^2(0,1)} \to 0,$$

then, using (4.17), (4.18), $(4.20)_3$ and (4.30), we find

$$b\left\|\psi_{nx}\right\|^2_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.41}$$

On the other hand, taking the inner product of $(4.20)_6$ with $\frac{w_n}{\lambda_n^4}$ in $L^2(0,1)$, using (4.18), integrating by parts and using the boundary conditions, we observe that

$$-\rho_1\left\langle\widetilde{w}_n,\left(i\lambda_n w_n-\widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\left\|\widetilde{w}_n\right\|^2_{L^2(0,1)} + k_0\left\|w_{nx}\right\|^2_{L^2(0,1)}$$
$$+lk_0\left\langle\varphi_{nx},w_n\right\rangle_{L^2(0,1)} + lk\left\langle(\varphi_{nx}+\psi_n+lw_n),w_n\right\rangle_{L^2(0,1)} \to 0.$$

By (4.17), (4.18), $(4.20)_5$ and (4.30), we deduce that

$$k_0\left\|w_{nx}\right\|^2_{L^2(0,1)} - \rho_1\left\|\widetilde{w}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.42}$$

Step 7. Taking the inner product of $(4.20)_4$ with $\frac{w_n}{\lambda_n^4}$ and of $(4.20)_6$ with $\frac{\psi_n}{\lambda_n^4}$ in $L^2(0,1)$, and using (4.18), we get

$$\begin{cases} \left\langle\left[i\lambda_n\rho_2\widetilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx}+\psi_n+lw_n)\right], w_n\right\rangle_{L^2(0,1)} \to 0, \\ \left\langle\left[i\lambda_n\rho_1\widetilde{w}_n - k_0(w_{nx}-l\varphi_n)_x + lk(\varphi_{nx}+\psi_n+lw_n)\right], \psi_n\right\rangle_{L^2(0,1)} \to 0. \end{cases}$$

Integrating by parts and using the boundary conditions, we obtain

$$-\rho_2\left\langle\widetilde{\psi}_n,\left(i\lambda_n w_n-\widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - \rho_2\left\langle\widetilde{\psi}_n,\widetilde{w}_n\right\rangle_{L^2(0,1)}$$
$$+b\left\langle\psi_{nx},w_{nx}\right\rangle_{L^2(0,1)} + k\left\langle(\varphi_{nx}+\psi_n+lw_n),w_n\right\rangle_{L^2(0,1)} \to 0$$

and

$$-\rho_1\left\langle\widetilde{w}_n,\left(i\lambda_n\psi_n-\widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\left\langle\widetilde{w}_n,\widetilde{\psi}_n\right\rangle_{L^2(0,1)}$$
$$+k_0\left\langle(w_{nx}-l\varphi_n),\psi_{nx}\right\rangle_{L^2(0,1)} + lk\left\langle(\varphi_{nx}+\psi_n+lw_n),\psi_n\right\rangle_{L^2(0,1)} \to 0,$$

then, using (4.17), (4.18), $(4.20)_3$, $(4.20)_5$ and (4.30), we obtain

$$\begin{cases} -\rho_2\left\langle\widetilde{\psi}_n,\widetilde{w}_n\right\rangle_{L^2(0,1)} + b\left\langle\psi_{nx},w_{nx}\right\rangle_{L^2(0,1)} \to 0, \\ -\rho_1\left\langle\widetilde{\psi}_n,\widetilde{w}_n\right\rangle_{L^2(0,1)} + k_0\left\langle\psi_{nx},w_{nx}\right\rangle_{L^2(0,1)} \to 0, \end{cases}$$

which implies that

$$\left(\frac{\rho_2}{b}-\frac{\rho_1}{k_0}\right)\left\langle\widetilde{\psi}_n,\widetilde{w}_n\right\rangle_{L^2(0,1)} \to 0 \tag{4.43}$$

and

$$\left(\frac{b}{\rho_2}-\frac{k_0}{\rho_1}\right)\left\langle\psi_{nx},w_{nx}\right\rangle_{L^2(0,1)} \to 0. \tag{4.44}$$

Step 8. We distinguish in this step two cases.

Case 1: $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$. From (4.43) and (4.44), we see that

$$\left\langle\widetilde{\psi}_n,\widetilde{w}_n\right\rangle_{L^2(0,1)} \to 0 \quad \text{and} \quad \left\langle\psi_{nx},w_{nx}\right\rangle_{L^2(0,1)} \to 0. \tag{4.45}$$

Therefore, taking the inner product in $L^2(0,1)$ of (4.40), first, with ψ_{nx}, and second, with w_{nx}, we obtain

$$\psi_{nx} \to 0 \quad \text{and} \quad w_{nx} \to 0 \text{ in } L^2(0,1), \tag{4.46}$$

and then, by (4.41), (4.42) and (4.46),

$$\widetilde{\psi}_n \to 0 \quad \text{and} \quad \widetilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{4.47}$$

Finally, combining (4.21), (4.26), (4.27), (4.29), (4.30), (4.46) and (4.47), we get

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow 0, \tag{4.48}$$

which is a contradiction with (4.17), so (4.2) holds. Consequentely, (2.9) is satisfied.

Case 2: $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$. Using (4.18), $(4.20)_4$ and $(4.20)_6$, we obtain

$$\begin{cases} \lambda_n^2 \left[-\dfrac{i\rho_2}{b}\lambda_n \left(i\lambda_n\psi_n - \tilde{\psi}_n \right) - \dfrac{\rho_2}{b}\lambda_n^2\psi_n - \psi_{nxx} + \dfrac{k}{b}\left(\varphi_{nx} + \psi_n + lw_n\right) \right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^2 \left[-\dfrac{i\rho_2}{b}\lambda_n \left(i\lambda_n w_n - \tilde{w}_n \right) - \dfrac{\rho_2}{b}\lambda_n^2 w_n - \left(w_{nx} - l\varphi_n\right)_x + \dfrac{lk}{k_0}\left(\varphi_{nx} + \psi_n + lw_n\right) \right] \to 0 \text{ in } L^2(0,1), \end{cases}$$

so, using $(4.20)_3$ and $(4.20)_5$, we find 5

$$\begin{cases} \lambda_n^2 \left[-\dfrac{\rho_2}{b}\lambda_n^2\psi_n - \psi_{nxx} + \dfrac{k}{b}\left(\varphi_{nx} + \psi_n + lw_n\right) \right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^2 \left[-\dfrac{\rho_2}{b}\lambda_n^2 w_n - \left(w_{nx} - l\varphi_n\right)_x + \dfrac{lk}{k_0}\left(\varphi_{nx} + \psi_n + lw_n\right) \right] \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.49}$$

Then, using (4.18), (4.26) and (4.30), we get

$$\begin{cases} \dfrac{\rho_2}{b}\lambda_n^2\psi_n + \psi_{nxx} \to 0 \text{ in } L^2(0,1), \\ \dfrac{\rho_2}{b}\lambda_n^2 w_n + w_{nxx} \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.50}$$

Multiplying $(4.50)_1$ by k and $(4.50)_2$ by $l(k+k_0)$ and adding the obtained limits, and multiplying $(4.50)_1$ by k and $(4.50)_2$ by $-l(k+k_0)$ and adding the limits, we obtain

$$\begin{cases} \dfrac{\rho_2}{b}\lambda_n^2 \left[k\psi_n + l(k+k_0)w_n\right] + \left[k\psi_{nxx} + l(k+k_0)w_{nxx}\right] \to 0 \text{ in } L^2(0,1), \\ \dfrac{\rho_2}{b}\lambda_n^2 \left[k\psi_n - l(k+k_0)w_n\right] + \left[k\psi_{nxx} - l(k+k_0)w_{nxx}\right] \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.51}$$

Taking the inner product in $L^2(0,1)$ of $(4.51)_1$ and $(4.51)_2$ with $\left[k\psi_n + l(k+k_0)w_n\right]$, integrating by parts and using the boundary conditions, we get

$$\frac{\rho_2}{b}\left\|k\lambda_n\psi_n + l(k+k_0)\lambda_n w_n\right\|_{L^2(0,1)}^2 - \|k\psi_{nx} + l(k+k_0)w_{nx}\|_{L^2(0,1)}^2 \to 0$$

and

$$\frac{\rho_2}{b}\left\langle \lambda_n^2\left[k\psi_n - l(k+k_0)w_n\right], \left[k\psi_n + l(k+k_0)w_n\right]\right\rangle_{L^2(0,1)}$$
$$-\left\langle \left[k\psi_{nx} - l(k+k_0)w_{nx}\right], \left[k\psi_{nx} + l(k+k_0)w_{nx}\right]\right\rangle_{L^2(0,1)} \to 0,$$

then, using (4.17) and (4.40), we obtain

$$\begin{cases} k\lambda_n\psi_n + l(k+k_0)\lambda_n w_n \to 0 \text{ in } L^2(0,1), \\ k^2\|\lambda_n\psi_n\|_{L^2(0,1)}^2 - l^2(k+k_0)^2\|\lambda_n w_n\|_{L^2(0,1)}^2 \to 0. \end{cases} \tag{4.52}$$

Taking the inner product in $L^2(0,1)$ of $(4.49)_1$ with w_n and $(4.49)_2$ with ψ_n, integrating by parts and using the boundary conditions, we get

$$-\frac{\rho_2}{b}\lambda_n^4\langle\psi_n,w_n\rangle_{L^2(0,1)}+\lambda_n^2\langle\psi_{nx},w_{nx}\rangle_{L^2(0,1)}-\frac{k}{b}\left\langle\lambda_n^2\varphi_n,w_{nx}\right\rangle_{L^2(0,1)} \tag{4.53}$$
$$+\frac{k}{b}\langle\lambda_n\psi_n,\lambda_nw_n\rangle_{L^2(0,1)}+\frac{lk}{b}\|\lambda_nw_n\|^2_{L^2(0,1)}\to 0$$

and

$$-\frac{\rho_2}{b}\lambda_n^4\langle\psi_n,w_n\rangle_{L^2(0,1)}+\lambda_n^2\langle\psi_{nx},w_{nx}\rangle_{L^2(0,1)}-l\left(1+\frac{k}{k_0}\right)\left\langle\psi_{nx},\lambda_n^2\varphi_n\right\rangle_{L^2(0,1)} \tag{4.54}$$
$$+\frac{lk}{k_0}\|\lambda_n\psi_n\|^2_{L^2(0,1)}+\frac{l^2k}{k_0}\langle\lambda_n\psi_n,\lambda_nw_n\rangle_{L^2(0,1)}\to 0,$$

then, multiplying (4.53) by $\frac{bk_0}{k}$, and (4.54) by $\frac{-bk_0}{k}$, adding the obtained limits and using (4.17) and (4.34), we find

$$lk_0\|\lambda_nw_n\|^2_{L^2(0,1)}-lb\|\lambda_n\psi_n\|^2_{L^2(0,1)}+\left(k_0-l^2b\right)\langle\lambda_n\psi_n,\lambda_nw_n\rangle_{L^2(0,1)}\to 0. \tag{4.55}$$

By taking the inner product in $L^2(0,1)$ of $(4.52)_1$ with $\lambda_n\psi_n$, and using (4.32), we have

$$k\|\lambda_n\psi_n\|^2_{L^2(0,1)}+l(k+k_0)\langle\lambda_nw_n,\lambda_n\psi_n\rangle_{L^2(0,1)}\to 0. \tag{4.56}$$

Combining $(4.52)_2$ and (4.55), we get

$$\frac{1}{l(k+k_0)^2}\left[k_0k^2-bl^2(k+k_0)^2\right]\|\lambda_n\psi_n\|^2_{L^2(0,1)}+\left(k_0-l^2b\right)\langle\lambda_nw_n,\lambda_n\psi_n\rangle_{L^2(0,1)}\to 0, \tag{4.57}$$

so, multiplying (4.56) by $\frac{(k+k_0)\left(k_0-l^2b\right)}{k_0}$, and (4.57) by $\frac{-l(k+k_0)^2}{k_0}$, adding the obtained limits and noting that $\frac{b}{\rho_2}=\frac{k_0}{\rho_1}$, we obtain

$$\left[kk_0+bl^2(k+k_0)\right]\|\lambda_n\psi_n\|^2_{L^2(0,1)}\to 0.$$

Then

$$\lambda_n\psi_n\to 0 \text{ in } L^2(0,1) \tag{4.58}$$

and, using $(4.52)_1$,

$$\lambda_nw_n\to 0 \text{ in } L^2(0,1). \tag{4.59}$$

Using (4.18), $(4.20)_3$, $(4.20)_5$, (4.58) and (4.59), we deduce that

$$\begin{cases}\widetilde{\psi}_n\to 0 \text{ in } L^2(0,1),\\ \widetilde{w}_n\to 0 \text{ in } L^2(0,1).\end{cases} \tag{4.60}$$

Taking the inner product in $L^2(0,1)$ of $(4.50)_1$ with ψ_n, and $(4.50)_2$ with w_n, integrating by parts and using the boundary conditions, we get

$$\begin{cases}\frac{\rho_2}{b}\|\lambda_n\psi_n\|^2_{L^2(0,1)}-\|\psi_{nx}\|^2_{L^2(0,1)}\to 0,\\ \frac{\rho_2}{b}\|\lambda_nw_n\|^2_{L^2(0,1)}-\|w_{nx}\|^2_{L^2(0,1)}\to 0,\end{cases}$$

then, from (4.58) and (4.59), we conclude that

$$\begin{cases}\psi_{nx}\to 0 \text{ in } L^2(0,1),\\ w_{nx}\to 0 \text{ in } L^2(0,1).\end{cases} \tag{4.61}$$

Finally, (4.21), (4.26), (4.27), (4.29), (4.30), (4.60) and (4.61) imply (4.48), which is a contradiction with (4.17). Consequentely, in both cases $\frac{b}{\rho_2}\neq\frac{k_0}{\rho_1}$ and $\frac{b}{\rho_2}=\frac{k_0}{\rho_1}$, (4.2) holds, and so (2.9) is satisfied. Hence, the proof of Theorem 4 is completed.

5 Conclusion and general remarks

41. Our first main result proved in this paper shows that the dissipation producing by the infinite memory in the first equation in (1.4) is not strong enough to stabilize (1.4) exponentially even if g converges exponentially to zero at infinity and regardless to the speeds of wave propagations. But this dissipation is strong enough to stabilize (1.4) at least polynomially. The natural question that we can ask is whether the obtained decay rate is optimal.
42. We have considered in this paper the homogeneous Dirichlet-Neumann-Neumann boundary conditions. The second interesting question we mention here is the extension of our results to the case of other boundary conditions, in particular, the homogeneous Dirichlet-Dirichlet-Dirichlet ones.
43. In [9], [10] and [11] where at least the second or the third equation of Bresse system is damped via an infinite memory, some stability estimates were proved with kernels having an arbitrary growth at infinity (not necessarily of exponential type). Showing the tability of (1.4) with such kernels is an important problem.
4. The last interesting question we note here is proving the tability of (1.4) in the whole space $\mathbb{R}$ (instead of $(0, 1)$).

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