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General decay of solutions of a nonlinear system of viscoelastic wave equations

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Abstract. This work is concerned with a system of two viscoelastic wave equations with nonlinear damping and source terms acting in both equations. Under some restrictions on the nonlinearity of the damping and the source terms, we prove that, for certain class of relaxation functions and for some restrictions on the initial data, the rate of decay of the total energy depends on those of the relaxation functions. This result improves many results in the literature, such as the ones in Messaoudi and Tatar (Appl. Anal. 87(3):247–263, 2008) and Liu (Nonlinear Anal. 71:2257–2267, 2009) in which only the exponential and polynomial decay rates are considered.

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1. Introduction

In this paper, we consider the following system:

$$u_{tt}-\Delta u+\int_0^t g\,(t-s)\,\Delta u\,(x,s)\;ds+|u_t|^{m-1}u_t=f_1(u,v),\quad x\in\Omega,\;t>0 \tag{1.1a}$$

$$v_{tt}-\Delta v+\int_0^t h\,(t-s)\,\Delta v\,(x,s)\;ds+|v_t|^{r-1}u_t=f_2(u,v),\quad x\in\Omega,\;t>0 \tag{1.1b}$$

$$u(x,t)=v(x,t)=0,\quad x\in\partial\Omega,\;t\geq 0 \tag{1.1c}$$

$$(u(0),v(0))=(u_0,v_0),\quad (u_t(0),v_t(0))=(u_1,v_1),\quad x\in\Omega, \tag{1.1d}$$

where

$$\begin{aligned} f_1(u,v)&=a|u+v|^{2(\rho+1)}(u+v)+b|u|^{\rho}u|v|^{(\rho+2)}\\ f_2(u,v)&=a|u+v|^{2(\rho+1)}(u+v)+b|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{1.2}$$

$u=u(t,x), v=v(t,x), t\in\mathbb{R}^{+}, x\in\Omega$ a bounded domain of $\mathbb{R}^{N}$ $(N\geq 1)$ with a smooth boundary $\partial\Omega, \rho, m$ and r are constants satisfying (2.5) and (2.6), and the kernel functions g and h are satisfying (2.3) and (2.4).

To motivate our work, we present some results related to viscoelastic wave equations.

We start with the pioneer works of Dafermos [9,10], in 1970, where the author discussed certain one-dimensional viscoelastic problems, established some existence results, and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as t goes to infinity. However, no rate of decay has been specified.

After that, the single equation of the form

$$u_{tt}-\Delta u+\int_{0}^{t}g\left(t-s\right)\Delta u\left(x,s\right)\,ds+h(u_{t})=f(u) \tag{1.3}$$

in $\Omega\times(0,+\infty)$, with initial and boundary conditions, has been extensively studied and many results concerning existence, nonexistence and asymptotic behavior have been proved. See in this regard [2,4,7,14,16,17,20,21] and the references therein.

Cavalcanti et al. [7] considered

$$u_{tt}-\Delta u+\int_{0}^{t}g\left(t-s\right)\Delta u\left(x,s\right)ds+a(x)u_{t}+|u|^{p-1}u=0,\quad \text{in } \Omega\times(0,\infty), \tag{1.4}$$

where $a:\Omega\rightarrow\mathbb{R}^{+}$ is a function, which may vanish on a part of the domain Ω.

By assuming $a(x)\geq a_{0}$ on $\omega\subset\Omega$ and for two positive constants ξ_{1} and ξ_{2} such that

$$-\zeta_{1}g(t)\leq g^{\prime}(t)\leq-\zeta_{2}g(t),\quad t\geq 0,$$

the authors established an exponential decay result under some geometry restrictions on ω. Berrimi and Messaoudi [4] established the result of [7], under weaker conditions on both a and g, to a system where a source term is competing with the damping term. In [2], an abstract version of the Eq. (1.3) has been considered and a uniform stability result has been obtained. By using the piecewise multipliers method, Cavalcanti and Oquendo [8] investigated the following problem:

$$u_{tt}-k_{0}\Delta u+\int_{0}^{t}div[a(x)g\left(t-s\right)\Delta u\left(x,s\right)]\,ds+b(x)h(u_{t})+f(u)=0 \tag{1.5}$$

and established, under the same conditions on the function g and for $a(x)+b(x)\geq\rho>0$, an exponential stability result for g decaying exponentially and h linear, and a polynomial stability result for g decaying polynomially and h nonlinear. Cabanillas and Rivera [5] considered an anisotropic and inhomogeneous viscoelastic equation, in a bounded domain, and proved that the sum of the first and the second energies decays polynomially when the relaxation function is of polynomial decay type. A similar result has also been obtained for an isotropic and homogeneous equation in the case of the whole $\mathbb{R}^{N}$. Their result depends on both the dissipation, the L^{p} regularity of the kernel and on an extra assumption on g''. This result was later improved by Baretto et al. [3], where equations of linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function, which is either exponential or polynomial. For partially viscoelastic materials, Rivera et al. [26] showed that solutions decay exponentially to zero, provided the relaxation function decays in a similar fashion, regardless to the size of the viscoelastic part of the material. In [25], a class of abstract viscoelastic systems of the form

$$\begin{aligned} &u_{tt}(t) + \mathscr{A}u(t) + \beta u(t) - (g * \mathscr{A}^{\alpha} u)(t) = 0 \\ &u(0) = u_0, \quad u_t(0) = u_1 \end{aligned} \tag{1.6}$$

for $0 \le \alpha \le 1$, $\beta \ge 0$, were investigated. The main focus was on the case when $0 < \alpha < 1$ and the main result was that solutions of (1.6) decay polynomially even if the kernel g decay exponentially. This result is sharp (see Theorem 12 [25]). This result has been improved by Rivera et al. [24], where the authors studied a more general abstract problem than (1.6) and established a necessary and sufficient condition to obtain an exponential decay. In the case of lack of exponential decay, a polynomial decay has been proved. In the latter case they showed that the rate of decay can be improved by taking more regular initial data. Also applications to concrete examples have been presented.

For viscoelastic systems, Messaoudi and Tatar [22] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\,\Delta u(x,s)\,ds + f(u,v) = 0, & x \in \Omega,\ t > 0, \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\,\Delta v(x,s)\,ds + k(u,v) = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.7}$$

where the functions f and k satisfying for all $(u, v) \in \mathbb{R}^2$ the following assumptions:

$$\begin{cases} |f(u,v)| \le d(|u|^{\beta_1} + |v|^{\beta_2}) \\ |k(u,v)| \le d(|u|^{\beta_3} + |v|^{\beta_4}) \end{cases}$$

for some constant $d > 0$ and

$$\beta_i \ge 1, \quad (N-2)\beta_i \le N, \quad i = 1, 2, 3, 4.$$

They proved an exponential decay result if both g and h are decaying exponentially and a polynomial decay result otherwise. Their result improves the one in [29], where (1.7) was considered with

$$f(u,v) = \alpha(u - v), \quad k(u,v) = -\alpha(u - v),$$

for α a positive constant, and only exponentially decaying relaxation functions g and h. In addition, some extra conditions on g'' and h'' were imposed. Also, a system similar to (1.1a)–(1.1d) was studied by Han and Wang [13] and some global existence and blow-up results have been established. However, the decay issue was not discussed.

In the absence of the viscoelasticity ($g = h = 0$), Agre and Rammaha [1] studied a system of wave equations of the form

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v) \\ v_{tt} - \Delta v + |v_t|^{r-1}u_t = f_2(u,v) \end{cases} \tag{1.8}$$

in $\Omega \times (0,T)$, with initial and boundary conditions of Dirichlet type and the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ as given in (1.2), and proved, under some appropriate conditions on $f_1(u,v)$, $f_2(u,v)$ and the initial data, several results on local and global existence. They also showed that any weak solution with negative initial energy blows up in finite time, using the same techniques as in [11]. Recently, the result of [1] has been improved by Said-Houari [28] by considering a certain class of initial data with positive initial energy.

The work in [28] has been followed by the one in [27] in which the author showed that, under some restrictions on the nonlinearities of the damping and the source terms, problem (1.8) has a unique global solution provided that the initial data are small enough. In addition, he proved that the rate of decay of the total energy is exponential or polynomial depending on the exponents of the damping terms in both equations.

In this work, we study problem (1.1a)–(1.1d). We state first the local existence result. Then, we show that this local solution is global, provided that the initial data are small enough. After that, we show that, for a certain class of relaxation functions and for some restrictions on the initial data, the rate of decay of the total energy is similar to those of the relaxation functions. This result improves many results in the literature, such as the results in [15,22] in which only the exponential and polynomial decay rates are considered. To achieve our goal we use a Lyapunov type technique for some perturbation energy following the method introduced in [18]. In fact this method allows us to weaken some of the technical assumptions for the convolution kernels.

The outline of this paper is the following: in Sect. 2, we fix notations and we prove some technical lemmas. In Sect. 3, we state a local existence result. While Sects. 4 and 5 are devoted to the global existence and general decay of solutions, respectively.

2. Preliminaries

In this section, we introduce some notations and establish some technical lemmas to be used throughout this paper. By $\|.\|_q$, we denote the usual $L^q(\Omega)$-norm and for $\varphi \in H_0^1(\Omega)$, we denote by $\|\nabla\varphi\|_2$ the equivalent norm of φ in $H_0^1(\Omega)$, and we mean by H the following energy space $H = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ endowed with the norm:

$$\|(u_0,u_1,u_2,u_3)\|_H^2 = \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 + \big\|u_3\|_2^2 + \|u_4\big\|_2^2.$$

The following Sobolev embedding will be used frequently in this paper:

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega), \; 2 \le q \le \overline{q} = \begin{cases} \dfrac{2N}{N-2}, & N \ge 3, \\ +\infty, & N = 1,2. \end{cases} \tag{2.1}$$

Furthermore, we use the following notations:

$$(\phi,\psi) = \int_\Omega \phi(x)\psi(x)\,dx,$$

$$(\phi*\psi)(t) := \int_0^t \phi(t-\tau)\,\psi(\tau)\,d\tau,$$

$$(\phi\diamond\psi)(t) := \int_0^t \phi(t-\tau)\,(\psi(t)-\psi(\tau))\,d\tau$$

and

$$(\phi\circ\psi)(t) := \int_0^t \phi(t-\tau)\int_\Omega |\psi(t)-\psi(\tau)|^2\,dx\,d\tau.$$

The constant C used throughout this paper is a positive generic constant, which may be different in various occurrences. Concerning the functions f_1 and f_2, we note that

$$uf_1(u,v)+vf_2(u,v) = 2(\rho+2)F(u,v), \quad \forall (u,v)\in\mathbb{R}^2$$

for

$$F(u,v) = \frac{1}{2(\rho+2)}\Big[a|u+v|^{2(\rho+2)} + 2b|uv|^{\rho+2}\Big].$$

Lemma 2.1. *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\Big(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\Big) \le F(u,v) \le \frac{c_1}{2(\rho+2)}\Big(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\Big). \tag{2.2}$$

Proof. The right-hand side of inequality (2.2) is trivial. For the left-hand side, the result is also trivial if $u=v=0$.
If, without loss of generality, $v\neq 0$, then either $|u|\le|v|$ or $|u|>|v|$.
For $|u|\le|v|$, we get

$$F(u,v) = \frac{1}{2(\rho+2)}\,|v|^{2(\rho+2)}\Big[a|1+\frac{u}{v}|^{2(\rho+2)} + 2b|\frac{u}{v}|^{\rho+2}\Big].$$

Consider the continuous function

$$j(s) = a|1+s|^{2(\rho+2)} + 2b|s|^{\rho+2} \quad \text{over } [-1,1].$$

So $\min j(s)\ge 0$. If $\min j(s)=0$ then, for some $s_0\in[-1,1]$, we have

$$j(s_0) = a|1+s_0|^{2(\rho+2)} + 2b|s_0|^{\rho+2} = 0.$$

This implies that $|1+s_0| = |s_0| = 0$, which is impossible. Thus $\min j(s) = 2c_0>0$. Therefore

$$F(u,v) \ge \frac{c_0}{\rho+2}\,|v|^{2(\rho+2)} \ge \frac{c_0}{\rho+2}\,|u|^{2(\rho+2)}.$$

Consequently,

$$2F(u,v) \ge \frac{c_0}{\rho+2}\Big\{|v|^{2(\rho+2)} + |u|^{2(\rho+2)}\Big\},$$

and then

$$\frac{c_0}{2(\rho+2)}\left\{|v|^{2(\rho+2)}+|u|^{2(\rho+2)}\right\}\leq F(u,v).$$

If $|u|\geq|v|$, then

$$\begin{aligned}F(u,v)&=\frac{1}{2\left(\rho+2\right)}\left|u\right|^{2(\rho+2)}\left[a|1+\frac{v}{u}|^{2(\rho+2)}+2b|\frac{v}{u}|^{\rho+2}\right]\\&\geq\frac{c_0}{\rho+2}\left|u\right|^{\rho+2}\geq\frac{c_0}{\rho+2}\left|v\right|^{\rho+2}.\end{aligned}$$

This leads to the desired result and completes the proof of Lemma 2.1.

We assume that the relaxation functions g and h are of class C^1 and satisfying

$$\begin{cases}g\left(s\right)\geq0, & 1-\displaystyle\int_0^{+\infty}g\left(s\right)ds=l>0\\ h\left(s\right)\geq0, & 1-\displaystyle\int_0^{+\infty}h\left(s\right)ds=k>0\end{cases}\tag{2.3}$$

and

$$g'\left(s\right),\quad h'\left(s\right)\leq0,\quad\forall s\geq0.\tag{2.4}$$

For the nonlinearity, we suppose that

$$-1<\rho\quad\text{if } N=1,2\quad\text{and}\quad-1<\rho\leq\frac{3-N}{N-2}\quad\text{if } N\geq3,\tag{2.5}$$

and

$$1\leq r,m\quad\text{if } N=1,2\quad\text{and}\quad1\leq r,m\leq\frac{N+2}{N-2}\quad\text{if } N\geq3.\tag{2.6}$$

The following Lemma will be used in the proof of Theorem 5.1.

Lemma 2.2. *There exist two positive constants* Λ_1 *and* Λ_2 *such that*

$$\int_\Omega|f_i(u,v)|^2\,dx\leq\Lambda_i\left(l\left\|\nabla u\right\|_2^2+k\left\|\nabla v\right\|_2^2\right)^{2\rho+3},\quad i=1,2.\tag{2.7}$$

Proof. We prove inequality (2.7) for f_1 and the same result also holds for f_2.

It's clear that

$$\begin{aligned}\left|f_1\left(u,v\right)\right|&\leq C\left(\left|u+v\right|^{2\rho+3}+|u|^{\rho+1}|v|^{(\rho+2)}\right)\\&\leq C[|u|^{2\rho+3}+|v|^{2\rho+3}+|u|^{\rho+1}|v|^{\rho+2}].\end{aligned}\tag{2.8}$$

From (2.8) and Young's inequality, with

$$q=\frac{2\rho+3}{\rho+1},\quad q'=\frac{2\rho+3}{\rho+2},$$

we get

$$|u|^{\rho+1}|v|^{\rho+2}\leq c_1|u|^{2\rho+3}+c_2|v|^{2\rho+3},$$

hence

$$|f_1(u,v)|\leq C[|u|^{2\rho+3}+|v|^{2\rho+3}].$$

Consequently, by using Poincaré's inequality and (2.5), we obtain

$$\int_\Omega |f_1(u,v)|^2\,dx \le C\left(\|\nabla u\|_2^{2(2\rho+3)} + \|\nabla v\|_2^{2(2\rho+3)}\right)$$
$$\le \Lambda_1\left(l\,\|\nabla u\|_2^2 + k\,\|\nabla v\|_2^2\right)^{2\rho+3}.$$

This completes the proof of Lemma 2.2.

The Lemma below, introduced in [23], plays a crucial role in the construction of the "modified" functional energy E associated to system (1.1a)–(1.1d).

Lemma 2.3. *For any function $\phi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,T)$, we have*

$$(\phi * \psi)(t)\,\psi_t(t) = -\frac{1}{2}\phi(t)\,|\psi(t)|^2 + \frac{1}{2}(\phi' \diamond \psi)(t)$$
$$-\frac{1}{2}\frac{d}{dt}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)\,d\tau\right)|\psi(t)|^2\right\}.$$

3. Local existence

In this section, we state the local existence and the uniqueness of the solution of problem (1.1a)–(1.1d). The proof of this result was given in [13], in which the authors adopted the technique of [1] which consists of constructing approximations by the Faedo–Galerkin procedure without imposing the usual smallness conditions on the initial data in order to handle the source terms. Unfortunately, due to the strong nonlinearities on f_1 and f_2, the techniques used [1,13] allowed them to prove the local existence result only for $N \le 3$. We note that the local existence result in the case of $N > 3$ is still an open problem.

Definition 3.1. A pair of functions (u,v) is said to be a week solution of (1.1a) –(1.1d) on $[0,T]$ if $u,v \in C_w([0,T],H_0^1(\Omega)), u_t,v_t \in C_w([0,T],L^2(\Omega)), u_t \in L^{m+1}(\Omega\times(0,T))$, $v_t \in L^{r+1}(\Omega\times(0,T)), (u(x,0),v(x,0)) = (u_0,v_0) \in H_0^1(\Omega) \times H_0^1(\Omega), (u_t(x,0),v_t(x,0) = (u_1,v_1) \in L^2(\Omega)\times L^2(\Omega)$ and (u,v) satisfies

$$\int_\Omega u'(t)\phi\,dx - \int_\Omega u_1\phi\,dx + \int_0^t\int_\Omega |u'|^{m-1}u'\phi\,dx\,d\tau - \int_0^t\int_\Omega \nabla\phi\,(g*\nabla u)\,dx\,d\tau$$
$$+\int_0^t\int_\Omega \nabla\phi\nabla u\,dx$$
$$= \int_0^t\int_\Omega f_1(u(\tau),v(\tau))\phi\,dx\,d\tau$$

and

$$\int_\Omega v'(t)\psi\,dx - \int_\Omega v_1\psi\,dx + \int_0^t\int_\Omega |v'|^{r-1}v'\psi\,dx\,d\tau - \int_0^t\int_\Omega \nabla\psi\,(h*\nabla v)\,dx\,d\tau$$
$$+\int_0^t\int_\Omega \nabla\psi\nabla v\,dx = \int_0^t\int_\Omega f_2(u(\tau),v(\tau))\psi\,dx\,d\tau$$

for all test functions $\phi \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$, $\psi \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$ and for almost all $t \in [0, T]$.

Now, we state the local existence theorem.

Theorem 3.2. *Let $N = 1, 2, 3$. Assume that (2.3)–(2.6) hold. Then for any initial data $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$, there exists a unique local weak solution (u, v) of (1.1a)–(1.1d) (in the sense of Definition 3.1) defined in $[0, T]$ for some $T > 0$, and satisfies the energy inequality*

$$E(t) + \int_s^t \left(\|u_t(\tau)\|_{m+1}^{m+1} \, d\tau + \|v_t(\tau)\|_{r+1}^{r+1} \right) d\tau - \frac{1}{2} \int_s^t ((g' \circ \nabla u)(\tau) + (h' \circ \nabla v)(\tau)) \, d\tau \leq E(s)$$

for $0 \leq s \leq t \leq T$, where E is defined in (4.3) below.

4. Global existence

In this section, we state and prove the global existence of the solution of problem (1.1a)–(1.1d). In order to do so, a suitable choice of a Lyapunov functional will be made.

First, we introduce the following functionals:

$$J(t) = \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\, ds\right) \|\nabla v\|_2^2 + \frac{1}{2}\left[(g \circ \nabla u)(t) + (h \circ \nabla v)(t)\right] - \int_\Omega F(u, v)\, dx \tag{4.1}$$

and

$$I(t) = \left(1 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 - 2(\rho + 2) \int_\Omega F(u, v)\, dx + \left(1 - \int_0^t h(s)\, ds\right) \|\nabla v\|_2^2 + (g \circ \nabla u)(t) + (h \circ \nabla v)(t). \tag{4.2}$$

The "modified" energy functional E associated to our system (1.1a)–(1.1d) is

$$E(t) = \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 - \int_\Omega F(u, v)\, dx + \frac{1}{2}\left(1 - \int_0^t h(s)\, ds\right) \|\nabla v\|_2^2 + \frac{1}{2}\left[(g \circ \nabla u)(t) + (h \circ \nabla v)(t)\right]. \tag{4.3}$$

In the next Lemma, we show that the energy functional (4.3) is a non-increasing function along solutions of (1.1a)–(1.1d).

Lemma 4.1. *Suppose that (2.3)–(2.6) hold. Let (u, v) be the solution of the system (1.1a)–(1.1d), then the energy functional is a non-increasing function,* *that is*

$$\frac{dE\left(t\right)}{dt}\leq-\left[\|u_t\|_{m+1}^{m+1}+\|v_t\|_{r+1}^{r+1}-\frac{1}{2}\left(g'\circ\nabla u\right)\left(t\right)-\frac{1}{2}\left(h'\circ\nabla v\right)\left(t\right)\right]\leq 0,\quad\forall t\geq 0. \tag{4.4}$$

Proof. By multiplying Eq. (1.1a) by u_t and Eq. (1.1b) by v_t, integrating over Ω, using integration by parts and summing up the results, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2-\int_\Omega F\left(u,v\right)\,dx\right)\\&\quad=-\|u_t\|_{m+1}^{m+1}-\|v_t\|_{r+1}^{r+1}+\int_0^t g\left(t-s\right)\int_\Omega\nabla u_t\left(t\right)\cdot\nabla u\left(\tau\right)\,dx\,d\tau\\&\qquad+\int_0^t h\left(t-s\right)\int_\Omega\nabla v_t\left(t\right)\cdot\nabla v\left(\tau\right)\,dx\,d\tau.\end{aligned} \tag{4.5}$$

Now, applying Lemma 2.3, the last two terms in the right hand side of (4.5) can be rewritten as follows

$$\begin{aligned}\int_0^t g\left(t-\tau\right)\int_\Omega\nabla u_t\left(t\right)\cdot\nabla u\left(\tau\right)\,dx\,d\tau&=-\frac{1}{2}g\left(t\right)\|\nabla u\|_2^2+\frac{1}{2}\left(g'\circ\nabla u\right)\left(t\right)\\&\quad-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g\left(\tau\right)\|\nabla u\|_2^2\,d\tau-\left(g\circ\nabla u\right)\left(t\right)\right]\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\int_0^t h\left(t-\tau\right)\int_\Omega\nabla v_t\left(t\right)\cdot\nabla v\left(\tau\right)\,dx\,d\tau&=-\frac{1}{2}h\left(t\right)\|\nabla v\|_2^2+\frac{1}{2}\left(h'\circ\nabla v\right)\left(t\right)\\&\quad-\frac{1}{2}\frac{d}{dt}\left[\int_0^t h\left(\tau\right)\|\nabla v\|_2^2\,d\tau-\left(h\circ\nabla v\right)\left(t\right)\right].\end{aligned} \tag{4.7}$$

Consequently, inserting (4.6) and (4.7) into (4.5), estimate (4.4) follows.

The inequality below is a key element in proving the global existence of solution. (cf. [28]).

Lemma 4.2. *Suppose that* (2.5) *holds. Then there exists* $\eta>0$ *such that for any* $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\leq\eta(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2)^{\rho+2} \tag{4.8}$$

holds.

Proof. It is clear that by using the Minkowski inequality we get

$$\|u+v\|_{2(\rho+2)}^2\leq 2(\|u\|_{2(\rho+2)}^2+\|v\|_{2(\rho+2)}^2).$$

Also, Hölder's and Young's inequalities give us

$$\|uv\|_{(\rho+2)}\leq\|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)}\leq c(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2).$$

A combination of the two last inequalities and the embedding $H_0^1(\Omega)\hookrightarrow L^{2(\rho+2)}(\Omega)$ yields (4.8).

Lemma 4.3. *Suppose that* (2.3)–(2.6) *hold. Then for any* $(u_0, v_0, u_1, v_1) \in H$ *satisfying*

$$\begin{cases} \beta = \eta \left[\frac{2(\rho+2)}{\rho+1} E(0)\right]^{\rho+1} < 1, \\ I(0) = I(u_0, v_0) > 0, \end{cases} \tag{4.9}$$

we have

$$I(t) = I(u(t),\ v(t)) > 0, \quad \forall t > 0. \tag{4.10}$$

Proof. Since $I(0) > 0$, then by continuity,

$$I(t) \geq 0, \quad \text{on } (0, \delta),\ \delta > 0.$$

Let T_m be such that

$$\{I(T_m) = 0 \quad \text{and} \quad I(t) > 0, \quad \forall\, 0 \leq t < T_m\} \tag{4.11}$$

which implies that, for all $t \in [0, T_m]$,

$$\begin{aligned} J(t) &= \frac{1}{2(\rho+2)} I(t) + \frac{\rho+1}{2(\rho+2)} \left\{ \left(1 - \int_0^t g(s)\ ds\right) \|\nabla u\|_2^2 \right. \\ &\quad \left. + \left(1 - \int_0^t h(s)\ ds\right) \|\nabla v\|_2^2 + (g \circ \nabla u)(t) + (h \circ \nabla v)(t) \right\} \\ &\geq \frac{\rho+1}{2(\rho+2)} \left\{ \left(1 - \int_0^t g(s)\ ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right. \\ &\quad \left. + \left(1 - \int_0^t h(s)\ ds\right) \|\nabla v\|_2^2 + (h \circ \nabla v)(t) \right\}. \end{aligned} \tag{4.12}$$

By using (2.3), (4.1), (4.4) and (4.12), we easily get

$$\begin{aligned} l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2 &\leq \left(1 - \int_0^t g(s)\ ds\right) \|\nabla u\|_2^2 + \left(1 - \int_0^t h(s)\ ds\right) \|\nabla v\|_2^2 \\ &\leq \frac{2(\rho+2)}{\rho+1} J(t) \\ &\leq \frac{2(\rho+2)}{\rho+1} E(t) \leq \frac{2(\rho+2)}{\rho+1} E(0), \quad \forall t \in [0, T_m]. \end{aligned} \tag{4.13}$$

By exploiting (4.8) and (4.9), we obtain

$$\begin{aligned} 2(\rho+2) \int_\Omega F(u(T_m), v(T_m))\ dx &\leq \eta (l\|\nabla u(T_m)\|_2^2 + k\|\nabla v(T_m)\|_2^2)^{\rho+2} \\ &= \eta (l\|\nabla u(T_m)\|_2^2 + k\|\nabla v(T_m)\|_2^2)^{\rho+1} \\ &\quad \times \left(l\|\nabla u(T_m)\|_2^2 + k\|\nabla v(T_m)\|_2^2\right) \\ &\leq \eta \left[\frac{2(\rho+2)}{\rho+1} E(0)\right]^{\rho+1} \\ &\quad \times \left(l\|\nabla u(T_m)\|_2^2 + k\|\nabla v(T_m)\|_2^2\right). \end{aligned}$$

Consequently,

$$\begin{aligned} 2(\rho+2)\int_\Omega F\left(u(T_m),v(T_m)\right)\,dx &\le \beta\left(l\|\nabla u(T_m)\|_2^2+k\|\nabla v(T_m)\|_2^2\right)\\ &\le \beta\left(1-\int_0^t g\left(s\right)\,ds\right)\|\nabla u(T_m)\|_2^2\\ &\quad+\beta\left(1-\int_0^t h\left(s\right)\,ds\right)\|\nabla v(T_m)\|_2^2\\ &< \left(1-\int_0^t g\left(s\right)\,ds\right)\|\nabla u(T_m)\|_2^2\\ &\quad+\left(1-\int_0^t h\left(s\right)\,ds\right)\|\nabla v(T_m)\|_2^2. \end{aligned}$$

Therefore, by using (4.2), we conclude that

$$I\left(T_m\right)>0,$$

which contradicts our hypothesis (4.11). So $I(t)>0$ for all $t\ge 0$.

Now, we state and prove our global existence result.

Theorem 4.4. *Suppose that* (2.3)–(2.6) *hold. If* $(u_0,v_0,u_1,v_1)\in H$ *and satisfies* (4.9)*. Then the solution of* (1.1a)–(1.1d) *is global and bounded.*

Proof. To prove Theorem 4.4, it suffices to show that the energy norm of the solution (u,v) is bounded, that is the norm

$$\|(u,v)\|_H^2=\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|u_t\|_2^2+\|v_t\|_2^2$$

is bounded independently of t. To achieve this we use (4.1), (4.3), (4.4) and (4.13) to get

$$\begin{aligned} E\left(0\right)\ge E\left(t\right)&=J\left(t\right)+\frac{1}{2}(\|u_t\|_2^2+\|v_t\|_2^2)\\ &\ge\frac{\rho+1}{2(\rho+2)}\left(l\left\|\nabla u\right\|_2^2+k\left\|\nabla v\right\|_2^2\right)+\frac{1}{2}(\|u_t\|_2^2+\|v_t\|_2^2). \end{aligned}\tag{4.14}$$

Therefore,

$$\|(u,v)\|_H^2\le CE\left(0\right),$$

where C is a positive constant, which depends only on ρ,k and l.

Remark 4.5. When $f_1(u,v)\le 0$ and $f_2(u,v)\le 0$, then any solution of (1.1a)–(1.1d) with arbitrary initial data in H is global in time and Theorems 4.4 and 5.1 hold without condition (4.9).

5. Asymptotic stability

In this section, we are interested in the study of the asymptotic behavior (when $t \to +\infty$) of the solutions to (1.1a)–(1.1d). In addition to (2.3) and (2.4), we also assume the existence of two positive non-increasing differentiable functions ξ_1 and ξ_2 such that

$$g'(t) \leq -\xi_1(t)\, g(t), \quad h'(t) \leq -\xi_2(t)\, h(t), \quad \forall t \geq 0 \tag{5.1}$$

$g(0), h(0) > 0$ and

$$\int_0^{+\infty} \xi_i(t)\, dt = +\infty, \quad i = 1, 2. \tag{5.2}$$

We then state our main stability result.

Theorem 5.1. *Suppose that (2.3)–(2.6), (5.1) and (5.2) hold. Let $(u_0, v_0, u_1, v_1) \in H$ be given and satisfying (4.9). Then for any $t_0 > 0$, there exist positive constants K and λ such that the solution (u, v) of (1.1a)–(1.1d) satisfies*

$$E(t) \leq K e^{-\lambda \int_{t_0}^{t} \xi(s)\, ds}, \quad \forall t \geq t_0, \tag{5.3}$$

where $\xi(t) = \min(\xi_1(t), \xi_2(t))$, $\forall t \geq 0$.

Example 5.2. Let

$$g(t) = a_1 e^{-b_1(1+t)^{\nu_1}}, \quad h(t) = a_2 e^{-b_2(1+t)^{\nu_2}}, \quad \text{with } a_i, b_i, \nu_i > 0 \ (i = 1, 2).$$

Then it's clear that (5.1) holds for $\xi_i(t) = b_i \nu_i (1+t)^{\min(0, \nu_i - 1)} (i = 1, 2)$. Consequently, applying (5.3), we obtain the following exponential decay:

$$E(t) \leq K e^{-\lambda b_0 (1+t)^{\min(1, \nu_1, \nu_2)}},$$

where

$$b_0 = \begin{cases} b_1 & \text{if } \nu_2 > \nu_1 \\ b_2 & \text{if } \nu_1 > \nu_2 \\ \min(b_1, b_2) & \text{if } \nu_1 = \nu_2 \end{cases}. \tag{5.4}$$

Example 5.3. If

$$g(t) = a_1 e^{-b_1 [\ln(1+t)]^{\nu_1}} \quad \textit{and} \quad h(t) = a_2 e^{-b_2 [\ln(1+t)]^{\nu_2}},$$
$$\textit{with } a_i, b_i > 0, \ \nu_i > 1 \quad (i = 1, 2)$$

Then for

$$\xi_i(t) = \frac{b_i \nu_i (\ln(1+t))^{\nu_i - 1}}{1+t} \quad (i = 1, 2)$$

the inequality (5.3) gives

$$E(t) \leq K e^{-\lambda b_0 (\ln(1+t))^{\min(\nu_1, \nu_2)}},$$

where b_0 is as in (5.4).

Example 5.4. If

$$g(t) = \frac{a_1}{(2+t)^{\nu_1} (\ln(2+t))^{b_1}}, \quad h(t) = \frac{a_2}{(2+t)^{\nu_2} (\ln(2+t))^{b_2}},$$

where

$$a_i > 0 \text{ and } \begin{cases} \nu_i > 1 \text{ and } b_i \in \mathbb{R} \\ \text{or} \\ \nu_i = 1 \text{ and } b_i > 1 \end{cases} \quad (i = 1, 2).$$

Then for

$$\xi_i(t) = \frac{\nu_i \left(\ln\left(2+t\right)\right) + b_i}{\left(2+t\right)\left(\ln\left(2+t\right)\right)} \quad (i = 1, 2)$$

we obtain from (5.3)

$$E(t) \leq \frac{K}{\left[\left(2+t\right)^{\min(\nu_1,\nu_2)}\left(\ln\left(2+t\right)\right)^{b_0}\right]^{\lambda}},$$

where b_0 is as in (5.4).

Remark 5.5. If the functions g and h decay faster than exponentially, then the energy decays exponentially. If one of the functions g or h does not decay faster than exponentially, then the energy has the same decay rate of the slower one of the relaxation functions g and h.

Remark 5.6. Our results in Theorems 4.4 and 5.1 also hold for other nonlinearities in $f_1(u, v)$ and $f_2(u, v)$. For instance we can show the same results for $f_1(u, v) = |u|^{\rho} u|v|^{(\rho+2)} + |u|^{2(\rho+2)}$ and $f_2(u, v) = |v|^{\rho} v|u|^{(\rho+2)} + |v|^{2(\rho+2)}$.

Remark 5.7. It is clear that the coupling (1.2) is highly nonlinear compared to the one taken by Santos [29]. In addition, it is not a special case of the coupling of [22]. In fact, with this type of coupling, the well-posedness has been established only for domains in $\mathbb{R}^N$, $N = 1, 2, 3$. See [1,13,28].

Remark 5.8. In addition to the difference in the coupling types, the decay results of [13,22] are only special cases of our result. Precisely, the exponential decay is obtained when $\xi_i(t) = a_i > 0$ in (5.1) and the polynomial decay is obtained when $\xi_i(t) = \dfrac{a_i}{(1+t)^{b_i}}$, $a_i > 0$ and $b_i \geq 0$ such that $b_1 + b_2 > 0$.

Proof of Theorem 5.1. The proof of Theorem 5.1 will be done through several Lemmas. As usual, the key point in stability is the suitable choice of a Lyapunov functional. Let us first introduce the functional $\mathscr{F}(t)$ defined as

$$\mathscr{F}\left(t\right) := E\left(t\right) + \varepsilon_1 \Psi\left(t\right) + \varepsilon_2 \chi\left(t\right), \tag{5.5}$$

where ε_1 and ε_2 are positive constants,

$$\Psi\left(t\right) := \int_{\Omega} u_t u \, dx + \int_{\Omega} v_t v \, dx \tag{5.6}$$

and

$$\begin{aligned} \chi\left(t\right) := &-\int_{\Omega} u_t \int_0^t g\left(t-\tau\right)\left(u\left(t\right) - u\left(\tau\right)\right) d\tau \, dx \\ &-\int_{\Omega} v_t \int_0^t h\left(t-\tau\right)\left(v\left(t\right) - v\left(\tau\right)\right) d\tau \, dx. \end{aligned} \tag{5.7}$$

With the above choice of the functional $\mathscr{F}$, we shall prove an inequality of the form

$$\xi\left(t\right)\mathscr{F}'\left(t\right)\leq-\gamma_1\xi\left(t\right)E\left(t\right)-2\gamma_2E'\left(t\right),\quad\forall t\geq t_0,$$

for some positive constants t_0, γ_1 and γ_2.

Lemma 5.9. *For $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$, we have*

$$\int_\Omega\left(\int_0^t g\left(t-s\right)\left(u\left(t\right)-u\left(\tau\right)\right)\,d\tau\right)^2\leq\left(1-l\right)C_*^2\left(g\circ\nabla u\right)\left(t\right)\tag{5.8}$$

and

$$\int_\Omega\left(\int_0^t h\left(t-s\right)\left(v\left(t\right)-v\left(\tau\right)\right)\,d\tau\right)^2\leq\left(1-k\right)C_*^2\left(h\circ\nabla v\right)\left(t\right),\tag{5.9}$$

where C_* is the Poincaré constant.

For the proof of Lemma 5.9, we refer to [18].

In the following Lemma, we prove that the functional $\mathscr{F}$ is equivalent to the energy functional E, more precisely, we have:

Lemma 5.10. *Let (u,v) be the solution of (1.1a)–(1.1d) and assume that (4.9) holds. Then there exists a constant $\varepsilon_0>0$, such that for all $\varepsilon_1<\varepsilon_0$ and for all $\varepsilon_2<\varepsilon_0$, we have*

$$\frac{1}{2}E\left(t\right)\leq\mathscr{F}\left(t\right)\leq2E\left(t\right),\quad\forall t\geq0.\tag{5.10}$$

Proof. To prove Lemma 5.10, we follow the same techniques used in [18]. Therefore, using (4.3), (5.8) and (5.9), we get

$$\begin{aligned}\mathscr{F}\left(t\right)&\leq E\left(t\right)+\left(\varepsilon_1/2\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\left(\varepsilon_1/2\right)\left(\|u\|_2^2+\|v\|_2^2\right)\\&\quad+\left(\varepsilon_2/2\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\left(\varepsilon_2/2\right)\int_\Omega\left(\int_0^t g\left(t-s\right)\left(u\left(t\right)-u\left(\tau\right)\right)\,d\tau\right)^2\\&\quad+\left(\varepsilon_2/2\right)\int_\Omega\left(\int_0^t h\left(t-s\right)\left(v\left(t\right)-v\left(\tau\right)\right)\,d\tau\right)^2\\&\leq\frac{1}{2}\left[1+\varepsilon_1+\varepsilon_2\right]\left(\|u_t\|_2^2+\|v_t\|_2^2\right)-\int_\Omega F\left(u,v\right)\,dx\\&\quad+\frac{1}{2}\left[1-\int_0^t g\left(s\right)\,ds+\varepsilon_1C_*^2\right]\|\nabla u\|_2^2\\&\quad+\frac{1}{2}\left[1+\varepsilon_2C_*^2\left(1-l\right)\right]\left(g\circ\nabla u\right)\left(t\right)\\&\quad+\frac{1}{2}\left[1-\int_0^t h\left(s\right)ds+\varepsilon_1C_*^2\right]\|\nabla v\|_2^2\\&\quad+\frac{1}{2}\left[1+\varepsilon_2C_*^2\left(1-k\right)\right]\left(h\circ\nabla v\right)\left(t\right).\end{aligned}\tag{5.11}$$

Consequently, by using $1-\int_0^t g(s)\,ds \geq l$, $1-\int_0^t h(s)\,ds \geq k$ and (4.2), we conclude that

$$\begin{aligned}2E(t)-\mathscr{F}(t) \geq\ & \frac{1}{2}\left[1-(\varepsilon_1+\varepsilon_2)\right]\|u_t\|_2^2+\frac{1}{2}\left[1-(\varepsilon_1+\varepsilon_2)\right]\|v_t\|_2^2\\ &+\frac{1}{2(\rho+2)}I(t)\\ &+\left[\frac{\rho+1}{2(\rho+2)}-\frac{\varepsilon_2}{2}C_*^2(1-l)\right](g\circ\nabla u)(t)\\ &+\left[\frac{\rho+1}{2(\rho+2)}l-\frac{\varepsilon_1}{2}C_*^2\right]\|\nabla u\|_2^2\\ &+\left[\frac{\rho+1}{2(\rho+2)}-\frac{\varepsilon_2}{2}C_*^2(1-k)\right](h\circ\nabla v)(t)\\ &+\left[\frac{\rho+1}{2(\rho+2)}k-\frac{\varepsilon_1}{2}C_*^2\right]\|\nabla v\|_2^2.\end{aligned}$$

By fixing ε_1 and ε_2 small enough, we obtain $2E(t)-\mathscr{F}(t)\geq 0$. By the same method, we can show that

$$\mathscr{F}(t)-\frac{1}{2}E(t)\geq 0.$$

This completes the proof of Lemma 5.10.

Lemma 5.11. *Suppose that* (2.3), (2.4) *and* (5.1) *hold. Let* $(u_0,v_0,u_1,v_1)\in H$ *be given and satisfying (4.9). If* (u,v) *is the solution of* (1.1a)–(1.1d), *then we have*

$$\begin{aligned}\Psi'(t) \leq\ & \left(1+\frac{C_*}{l}\right)\|u_t\|_2^2+\left(1+\frac{C_*}{k}\right)\|v_t\|_2^2-\frac{l}{4}\|\nabla u\|_2^2-\frac{k}{4}\|\nabla v\|_2^2\\ &+\frac{1-l}{2l}(g\circ\nabla u)(t)+\frac{1-k}{2k}(h\circ\nabla v)(t)+\int_\Omega F(u,v)\,dx\\ &+\frac{m}{m+1}\beta_1^{-(m+1)/m}\|u_t\|_{m+1}^{m+1}+\frac{r}{r+1}\beta_2^{-(r+1)/r}\|v_t\|_{r+1}^{r+1}.\end{aligned}\tag{5.12}$$

Proof. By using (1.1a), direct computations lead to the following identity:

$$\begin{aligned}\Psi'(t) =\ & \|u_t\|_2^2+\|v_t\|_2^2-\|\nabla u\|_2^2-\|\nabla v\|_2^2+\int_\Omega uf_1(u,v)\,dx\\ &+\int_\Omega vf_2(u,v)\,dx-\int_\Omega u|u_t|^{m-1}u_t\,dx-\int_\Omega v|v_t|^{r-1}v_t\,dx\\ &+\int_\Omega \nabla u(t)\cdot\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\,dx\\ &+\int_\Omega \nabla v(t)\cdot\int_0^t h(t-\tau)\nabla v(\tau)\,d\tau\,dx.\end{aligned}\tag{5.13}$$

Following the same approach as in Lemma 3.3 [18], the last two terms in the right hand side of (5.13) can be estimated as follows, for all $\mu_1, \mu_2 > 0$,

$$\int_\Omega \nabla u(t) \cdot \int_0^t g(t-\tau) \nabla u(\tau)\, d\tau\, dx \leq \frac{1}{2}\left\{ \|\nabla u\|_2^2 + \left(1+\frac{1}{\mu_1}\right)(1-l)(g \circ \nabla u)(t) + (1+\mu_1)(1-l)^2 \|u\|_2^2 \right\} \tag{5.14}$$

and

$$\int_\Omega \nabla v(t) \cdot \int_0^t h(t-\tau) \nabla v(\tau)\, d\tau\, dx \leq \frac{1}{2}\left\{ \|\nabla v\|_2^2 + \left(1+\frac{1}{\mu_2}\right)(1-k)(h \circ \nabla v)(t) + (1+\mu_2)(1-k)^2 \|\nabla v\|_2^2 \right\}.$$

Also, Young's inequality, Poincaré's inequality and (4.13) imply, for some β_1, $\beta_2 > 0$,

$$\begin{aligned} \left| \int_\Omega u|u_t|^{m-1}u_t\, dx \right| &\leq \frac{\beta_1^{m+1}}{m+1} \|u\|_{m+1}^{m+1} + \frac{m}{m+1}\beta_1^{-(m+1)/m} \|u_t\|_{m+1}^{m+1} \\ &\leq \frac{\beta_1^{m+1} C_*^{m+1}}{m+1} \left(\frac{2(\rho+2)}{l(\rho+1)} E(0) \right)^{(m-1)/2} \|\nabla u\|_2^2 \\ &\quad + \frac{m}{m+1}\beta_1^{-(m+1)/m} \|u_t\|_{m+1}^{m+1} \end{aligned} \tag{5.15}$$

and similarly,

$$\begin{aligned} \left| \int_\Omega v|v_t|^{r-1}v_t\, dx \right| &\leq \frac{\beta_2^{r+1} C_*^{r+1}}{r+1} \left(\frac{2(\rho+2)}{k(\rho+1)} E(0) \right)^{(r-1)/2} \|\nabla v\|_2^2 \\ &\quad + \frac{r}{r+1}\beta_2^{-(r+1)/r} \|v_t\|_{r+1}^{r+1}. \end{aligned} \tag{5.16}$$

Inserting estimates (5.14), (5.15) and (5.16) into (5.13), we arrive at

$$\begin{aligned} \Psi'(t) \leq\ & \|u_t\|_2^2 + \|v_t\|_2^2 - \kappa_1 \|\nabla u\|_2^2 - \kappa_2 \|\nabla v\|_2^2 \\ & + \frac{1}{2}\left(1+\frac{1}{\mu_1}\right)(1-l)(g \circ \nabla u)(t) \\ & + \frac{1}{2}\left(1+\frac{1}{\mu_2}\right)(1-k)(h \circ \nabla v)(t) + \int_\Omega F(u,v)\, dx \\ & + \left[\frac{m}{m+1}\beta_1^{-(m+1)/m} \right] \|u_t\|_{m+1}^{m+1} \\ & + \left[\frac{r}{r+1}\beta_2^{-(r+1)/r} \right] \|v_t\|_{r+1}^{r+1}, \end{aligned} \tag{5.17}$$

where

$$\kappa_1 = \frac{1}{2}\left(1 - (1+\mu_1)(1-l)^2\right) - \frac{\beta_1^{m+1} C_*^{m+1}}{m+1} \left(\frac{2(\rho+2)}{l(\rho+1)} E(0) \right)^{(m-1)/2}$$

and

$$\kappa_2 = \frac{1}{2}\left(1-(1+\mu_2)(1-k)^2\right) - \frac{\beta_2^{r+1}C_*^{r+1}}{r+1}\left(\frac{2(\rho+2)}{k(\rho+1)}E(0)\right)^{(r-1)/2}.$$

Choosing $\mu_1 = l/(1-l)$, $\mu_2 = k/(1-k)$ and piking β_1 and β_2 small enough such that

$$\frac{\beta_1^{m+1}C_*^{m+1}}{m+1}\left(\frac{2(\rho+2)}{l(\rho+1)}E(0)\right)^{(m-1)/2} \leq \frac{l}{4}$$

and

$$\frac{\beta_2^{r+1}C_*^{r+1}}{r+1}\left(\frac{2(\rho+2)}{k(\rho+1)}E(0)\right)^{(r-1)/2} \leq \frac{k}{4}$$

then, (5.12) is established.

Lemma 5.12. *Suppose that* (2.3), (2.4) *and* (5.1) *hold. Let* $(u_0, v_0, u_1, v_1) \in H$ *be given and satisfying (4.9). If* (u, v) *is the solution of* (1.1a)–(1.1d), *then the functional*

$$\chi_1(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx \tag{5.18}$$

satisfies, for all $\delta > 0$,

$$\begin{aligned}\chi_1'(t) \leq \Bigg[&\left(2\delta + \frac{1}{4\delta}\right)(1-l) + \frac{1-l}{4\delta}\\ &+ C_1\left(\frac{2(\rho+2)}{l(\rho+1)}E(0)\right)^{(m-1)/2} \frac{\delta^{m+1}}{m+1}(1-l)^m\Bigg](g\circ\nabla u)(t)\\ &+ \left(2\delta(1-l)^2+\delta\right)\|\nabla u\|_2^2 + \left(\delta - \int_0^t g(s)\, ds\right)\|u_t\|_2^2\\ &- \frac{g(0)}{4\delta}C_*^2(g'\circ\nabla u)(t) + \frac{m}{m+1}\delta^{-(m+1)/m}\|u_t\|_{m+1}^{m+1}\\ &+ \int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx.\end{aligned}$$

Proof. Differentiate (5.18) with respect to t to get by using Eq. (1.1a)

$$\begin{aligned}\chi_1'(t) = &\int_\Omega \nabla u(t).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\, d\tau\right) dx\\ &- \int_\Omega\left(\int_0^t g(t-\tau)\nabla u(\tau)\, d\tau\right).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\, d\tau\right) dx\\ &- \int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\, d\tau\, dx - \left(\int_0^t g(s)\, ds\right)\|u_t\|_2^2\\ &+ \int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx\\ &- \int_\Omega |u_t|^{m-1}u_t\int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx.\end{aligned} \tag{5.19}$$

Similarly as in (5.12), we estimate the right-hand side terms of (5.19) as follows:

First, by using Young's inequality and (5.8), we obtain for any $\delta > 0$

$$\left|\int_\Omega \nabla u(t) . \left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\, d\tau\right) dx\right| \\ \leq \delta \|\nabla u\|_2^2 + \frac{1-l}{4\delta}(g \circ \nabla u)(t). \tag{5.20}$$

Also, the second term can be estimated as follows (see [18]):

$$\int_\Omega \left(\int_0^t g(t-\tau)\nabla u(\tau)\, d\tau\right) . \left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\, d\tau\right) dx \\ \leq \left(2\delta + \frac{1}{4\delta}\right)(1-l)(g \circ \nabla u)(t) + 2\delta(1-l)^2 \|\nabla u\|_2^2. \tag{5.21}$$

Concerning the third term, we have

$$\int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\, d\tau\, dx \leq \delta \|u_t\|_2^2 - \frac{g(0)}{4\delta} C_*^2 (g' \circ \nabla u)(t). \tag{5.22}$$

To estimate the fifth term, we use Young's inequality and Poincaré's inequality to obtain

$$\int_\Omega |u_t|^{m-1} u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx \\ \leq \frac{m}{m+1}\delta^{-(m+1)/m}\|u_t\|_{m+1}^{m+1} + \frac{\delta^{m+1}}{m+1}\int_\Omega \left[\int_0^t g(t-\tau)|u(t)-u(\tau)|\, d\tau\right]^{m+1} dx \\ \leq \frac{m}{m+1}\delta^{-(m+1)/m}\|u_t\|_{m+1}^{m+1} \\ + \frac{\delta^{m+1}}{m+1}\left(\int_0^t g(\tau)\, d\tau\right)^m \int_\Omega \int_0^t g(t-\tau)|u(t)-u(\tau)|^{m+1}\, d\tau\, dx \\ \leq \frac{m}{m+1}\delta^{-(m+1)/m}\|u_t\|_{m+1}^{m+1} \\ + \frac{\delta^{m+1}}{m+1}(1-l)^m C_* \int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^{m+1}\, d\tau. \tag{5.23}$$

It's clear that by using (4.14) the last term in the right-hand side of (5.23) can be estimated as follows

$$\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^{m+1}\, d\tau \\ \leq \int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2 \|\nabla u(t)-\nabla u(\tau)\|_2^{m-1}\, d\tau \\ \leq C\left(\frac{2(\rho+2)}{l(\rho+1)}E(0)\right)^{(m-1)/2}(g \circ \nabla u)(t). \tag{5.24}$$

By combining (5.20)–(5.24), the assertion of Lemma 5.12 is established.

By repeating the same steps as in Lemma 5.12, we have the following result:

Lemma 5.13. *Suppose that* (2.3), (2.4) *and* (5.1) *hold. Let* $(u_0, v_0, u_1, v_1) \in H$ *be given satisfying* (4.9). *If* (u, v) *is the solution of* (1.1a)–(1.1d), *then the functional*

$$\chi_2(t) = -\int_\Omega v_t \int_0^t h(t-\tau)(v(t)-v(\tau))\, d\tau\, dx \tag{5.25}$$

satisfies, for any $\delta > 0$,

$$\begin{aligned}
\chi_2'(t) \le & \left[\left(2\delta + \frac{1}{4\delta}\right)(1-k) + \frac{1-k}{4\delta}\right.\\
& \left. + C_1\left(\frac{2(\rho+2)}{k(\rho+1)}E(0)\right)^{(r-1)/2} \frac{\delta^{r+1}}{r+1}(1-k)^r\right](h \circ \nabla v)(t)\\
& + \left(2\delta(1-k)^2 + \delta\right)\|\nabla v\|_2^2 + \left(\delta - \int_0^t h(s)\, ds\right)\|v_t\|_2^2\\
& - \frac{h(0)}{4\delta} C_*^2 (h' \circ \nabla v)(t)\\
& + \frac{r}{r+1}\delta^{-(r+1)/r}\|v_t\|_{r+1}^{r+1}\\
& + \int_\Omega f_2(u,v)\int_0^t h(t-\tau)(v(t)-v(\tau))\, d\tau\, dx.
\end{aligned}$$

Now, we are in a position to prove Theorem 5.1.

Proof of Theorem 5.1. Since the functions g and h are non-negative, continuous and $g(0), h(0) > 0$, then for any $t \ge t_0 > 0$, we have

$$\begin{aligned}
\int_0^t g(s)\, ds &\ge \int_0^{t_0} g(s)\, ds = g_0,\\
\int_0^t h(s)\, ds &\ge \int_0^{t_0} h(s)\, ds = h_0.
\end{aligned}$$

From (4.4), (5.5), (5.12), (5.19) and (5.26) we have, for all $t \ge t_0$,

$$\begin{aligned}
\mathscr{F}'(t) \le & -\left[\varepsilon_2\{g_0 - \delta\} - \varepsilon_1\left(1 + \frac{C_*}{l}\right)\right]\|u_t\|_2^2\\
& - \left[\varepsilon_2\{h_0 - \delta\} - \varepsilon_1\left(1 + \frac{C_*}{k}\right)\right]\|v_t\|_2^2\\
& + \left\{\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_*^2\right\}(g' \circ \nabla u)(t)\\
& + \left\{\frac{1}{2} - \varepsilon_2 \frac{h(0)}{4\delta} C_*^2\right\}(h' \circ \nabla v)(t)\\
& - \left[\varepsilon_1 \frac{l}{4} - \varepsilon_2 \delta\left\{2(1-l)^2 + 1\right\}\right]\|\nabla u\|_2^2\\
& - \left[\varepsilon_1 \frac{k}{4} - \varepsilon_2 \delta\left\{2(1-k)^2 + 1\right\}\right]\|\nabla v\|_2^2
\end{aligned}$$

$$+\left(\varepsilon_1 \frac{1-l}{2l}+\varepsilon_2\overline{\alpha}_1\right)(g\circ\nabla u)(t)+\left(\varepsilon_1 \frac{1-k}{2k}+\varepsilon_2\overline{\alpha}_2\right)(h\circ\nabla v)(t)$$
$$+\overline{\beta}_1(\varepsilon_1,\varepsilon_2)\|u_t\|_{m+1}^{m+1}+\overline{\beta}_2(\varepsilon_1,\varepsilon_2)\|v_t\|_{r+1}^{r+1}+\varepsilon_1\int_\Omega F(u,v)\,dx$$
$$+\varepsilon_2\int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx$$
$$+\varepsilon_2\int_\Omega f_2(u,v)\int_0^t h(t-\tau)(v(t)-v(\tau))\,d\tau\,dx, \tag{5.26}$$

where

$$\overline{\alpha}_1=\left(2\delta+\frac{1}{4\delta}\right)(1-l)+\frac{1-l}{4\delta}+C_1\left(\frac{2(\rho+2)}{l(\rho+1)}E(0)\right)^{(m-1)/2}\frac{\delta^{m+1}}{m+1}(1-l)^m,$$
$$\overline{\alpha}_2=\left(2\delta+\frac{1}{4\delta}\right)(1-k)+\frac{1-k}{4\delta}+C_1\left(\frac{2(\rho+2)}{k(\rho+1)}E(0)\right)^{(r-1)/2}\frac{\delta^{r+1}}{r+1}(1-k)^r$$

and

$$\overline{\beta}_1(\varepsilon_1,\varepsilon_2)=\left(\varepsilon_1\frac{m}{m+1}\beta_1^{-(m+1)/m}+\varepsilon_2\frac{m}{m+1}\delta^{-(m+1)/m}-1\right),$$
$$\overline{\beta}_2(\varepsilon_1,\varepsilon_2)=\left(\varepsilon_1\frac{r}{r+1}\beta_2^{-(r+1)/r}+\varepsilon_2\frac{r}{r+1}\delta^{-(r+1)/r}-1\right).$$

To estimate the last two terms in (5.26), we need the following lemma:

Lemma 5.14. *Suppose that (2.3), (2.4) and (2.5) hold. Let $(u_0, v_0, u_1, v_1)\in H$ be given and satisfying (4.9). Then, if (u,v) is the solution of (1.1a)–(1.1d), there exist two positive constant Λ_3 and Λ_4 such that for any $\delta>0$ and for all $t\geq 0$,*

$$\int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx\leq\Lambda_3\delta\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right)$$
$$+\frac{(1-l)C_*^2}{4\delta}(g\circ\nabla u)(t) \tag{5.27}$$

and

$$\int_\Omega f_2(u,v)\int_0^t h(t-\tau)(v(t)-v(\tau))\,d\tau\,dx\leq\Lambda_4\delta\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right)$$
$$+\frac{(1-k)C_*^2}{4\delta}(h\circ\nabla v)(t). \tag{5.28}$$

Proof. Using Young's inequality, we obtain

$$\int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx$$
$$\leq\delta\left(\int_\Omega|f_1(u,v)|^2\,dx\right)+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx\right)^2dx.$$

Application of (5.8) and (2.7) yields

$$\int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx$$
$$\leq \Lambda_1\delta\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right)^{2\rho+3}+\frac{(1-l)\,C_*^2}{4\delta}(g\circ\nabla u)(t). \quad (5.29)$$

By using (4.13), then inequality (5.29) takes the form

$$\int_\Omega f_1(u,v)\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx$$
$$\leq\Lambda_1\left(\frac{2(\rho+2)}{\rho+1}E(0)\right)^{2(\rho+1)}\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right)+\frac{(1-l)\,C_*^2}{4\delta}(g\circ\nabla u)(t)$$
$$=\Lambda_3\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right)+\frac{(1-l)\,C_*^2}{4\delta}(g\circ\nabla u)(t), \quad (5.30)$$

where

$$\Lambda_3=\Lambda_1\left(\frac{2(\rho+2)}{\rho+1}E(0)\right)^{2(\rho+1)}.$$

Analogously, we obtain (5.28) with

$$\Lambda_4=\Lambda_2\left(\frac{2(\rho+2)}{\rho+1}E(0)\right)^{2(\rho+1)}.$$

This completes the proof of Lemma 5.14.

Inserting estimates (5.27) and (5.28) into (5.26), we get

$$\begin{aligned}\mathscr{F}'(t)\leq&-\left[\varepsilon_2(g_0-\delta)-\varepsilon_1\left(1+\frac{C_*}{l}\right)\right]\|u_t\|_2^2\\&-\left[\varepsilon_2(h_0-\delta)-\varepsilon_1\left(1+\frac{C_*}{k}\right)\right]\|v_t\|_2^2\\&+\left\{\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_*^2\right\}(g'\circ\nabla u)(t)\\&+\left\{\frac{1}{2}-\varepsilon_2\frac{h(0)}{4\delta}C_*^2\right\}(h'\circ\nabla v)(t)\\&-\left[\varepsilon_1\frac{l}{4}-\varepsilon_2\delta\left\{2(1-l)^2+1+\Lambda_3 l+\Lambda_4 l\right\}\right]\|\nabla u\|_2^2\\&-\left[\varepsilon_1\frac{k}{4}-\varepsilon_2\delta\left\{2(1-k)^2+1+\Lambda_4 k+\Lambda_3 k\right\}\right]\|\nabla v\|_2^2\\&+\left(\varepsilon_1\frac{1-l}{2l}+\varepsilon_2\left(\overline{\alpha}_1+\frac{(1-l)\,C_*^2}{4\delta}\right)\right)(g\circ\nabla u)(t)\\&+\left(\varepsilon_1\frac{1-k}{2k}+\varepsilon_2\left(\overline{\alpha}_2+\frac{(1-k)\,C_*^2}{4\delta}\right)\right)(h\circ\nabla v)(t)\\&+\overline{\beta}_1(\varepsilon_1,\varepsilon_2)\|u_t\|_{m+1}^{m+1}+\overline{\beta}_2(\varepsilon_1,\varepsilon_2)\|v_t\|_{r+1}^{r+1}+\varepsilon_1\int_\Omega F(u,v)\,dx.\end{aligned} \quad (5.31)$$

At this point, we choose δ small enough so that

$$\delta \leq \frac{1}{2}\min\left(h_0, g_0\right)$$

and

$$\begin{cases} \frac{4}{l}\delta\left(2\left(1-l\right)^2+1+\Lambda_3 l+\Lambda_4 l\right) < \frac{1}{4\left(1+\frac{C_*}{l}\right)}g_0 \\ \frac{4}{k}\delta\left(2\left(1-k\right)^2+1+\Lambda_4 k+\Lambda_3 k\right) < \frac{1}{4\left(1+\frac{C_*}{k}\right)}h_0 \end{cases}.$$

Once δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\begin{cases} \frac{1}{4\left(1+\frac{C_*}{l}\right)}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2\left(1+\frac{C_*}{l}\right)}g_0\varepsilon_2 \\ \frac{1}{4\left(1+\frac{C_*}{k}\right)}h_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2\left(1+\frac{C_*}{k}\right)}h_0\varepsilon_2 \end{cases} \tag{5.32}$$

will make

$$\begin{aligned} k_1 &= \varepsilon_2\left(g_0-\delta\right)-\varepsilon_1\left(1+\frac{C_*}{l}\right) > 0, \\ k_2 &= \varepsilon_2\left(h_0-\delta\right)-\varepsilon_1\left(1+\frac{C_*}{k}\right) > 0, \\ k_3 &= \varepsilon_1\frac{l}{4}-\varepsilon_2\delta\left\{2\left(1-l\right)^2+1+\Lambda_3 l+\Lambda_4 l\right\} > 0, \\ k_4 &= \varepsilon_1\frac{k}{4}-\varepsilon_2\delta\left\{2\left(1-k\right)^2+1+\Lambda_4 k+\Lambda_3 k\right\} > 0. \end{aligned}$$

We then pick ε_1 and ε_2 so small that (5.10) and (5.32) remain valid and further

$$k_5 = \frac{1}{2}-\varepsilon_2\frac{g\left(0\right)}{4\delta}C_*^2 > 0, \quad k_6 = \frac{1}{2}-\varepsilon_2\frac{h\left(0\right)}{4\delta}C_*^2 > 0$$

and

$$\overline{\beta}_1\left(\varepsilon_1,\varepsilon_2\right) < 0, \qquad \overline{\beta}_2\left(\varepsilon_1,\varepsilon_2\right) < 0.$$

Therefore, there exist two positive constants γ_1 and γ_2 such that

$$\mathscr{F}'\left(t\right) \leq -\gamma_1 E\left(t\right)+\gamma_2\left[\left(g\circ\nabla u\right)\left(t\right)+\left(h\circ\nabla v\right)\left(t\right)\right], \quad \forall t\geq t_0. \tag{5.33}$$

Now, for all $t \geq 0$, let $\xi(t) = \min(\xi_1(t), \xi_2(t))$.

By multiplying (5.33) by $\xi(t)$ we arrive at

$$\xi\left(t\right)\mathscr{F}'\left(t\right) \leq -\gamma_1\xi\left(t\right)E\left(t\right)+\gamma_2\xi\left(t\right)\left[\left(g\circ\nabla u\right)\left(t\right)+\left(h\circ\nabla v\right)\left(t\right)\right], \qquad \forall t\geq t_0.$$

Recalling (5.1) and using (4.4), we get

$$\begin{aligned} \xi\left(t\right)\mathscr{F}'\left(t\right) &\leq -\gamma_1\xi\left(t\right)E\left(t\right)-\gamma_2\left[\left(g'\circ\nabla u\right)\left(t\right)+\left(h'\circ\nabla v\right)\left(t\right)\right], \\ &\leq -\gamma_1\xi\left(t\right)E\left(t\right)-2\gamma_2 E'\left(t\right), \quad \forall t\geq t_0. \end{aligned}$$

That is

$$\left(\xi\left(t\right)\mathscr{F}\left(t\right)+2\gamma_2 E\left(t\right)\right)'-\xi'\left(t\right)\mathscr{F}\left(t\right) \leq -\gamma_1\xi\left(t\right)E\left(t\right), \quad \forall \text{ a.e } t\ \geq t_0.$$

By using (5.10), the fact that $\xi'\left(t\right) \leq 0$ for a.e $t \geq t_0$ and $\xi > 0$ and letting

$$\mathscr{L}\left(t\right) = \xi\left(t\right)\mathscr{F}\left(t\right)+2\gamma_2 E\left(t\right) \sim E\left(t\right) \tag{5.34}$$

we obtain, for some $\lambda > 0$

$$\mathscr{L}'(t) \leq -\gamma_1 \xi(t) E(t) \leq -\lambda \xi(t) \mathscr{L}(t), \qquad \forall \text{ a.e } t \geq t_0 \tag{5.35}$$

A simple integration of (5.35) over (t_0, t) leads to

$$\mathscr{L}(t) \leq \mathscr{L}(t_0) e^{-\lambda \int_{t_0}^t \xi(s)\, ds}, \quad \forall t \geq t_0. \tag{5.36}$$

A combination of (5.34) and (5.36) leads to (5.3). The proof of Theorem 5.1 is thus completed.

Remark 5.15. Estimate (5.3) holds for all $t \geq 0$ by virtue of continuity and boundedness of E and ξ.

Remark 5.16. Our approach here allows us to weaken some of the technical assumptions of [6,7], for convolution kernels. We only need g and h to be differentiable satisfying (2.3) and (2.4).

Remark 5.17. One can easily generalize the results of this paper to the following system:

$$|u_t|^i u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(x,s)\, ds + |u_t|^{m-1} u_t = f_1(u,v), \tag{5.37a}$$

$$|v_t|^j v_{tt} - \Delta v - \Delta v_{tt} + \int_0^t h(t-s) \Delta v(x,s)\, ds + |v_t|^{r-1} u_t = f_2(u,v), \tag{5.37b}$$

with the initial conditions (1.1c) and the boundary conditions (1.1d), and in this case the result obtained in [15] will be only a particular case.

Remark 5.18. The results of this paper hold also if we consider two past history controls in (1.1a) and (1.1b); that is the integral $\int_0^t$ is replaced by $\int_{-\infty}^t$, where the functions g and h satisfy in this case the condition (5.1) with $\xi_1(t) = a_1 g^{p_1-1}(t)$ and $\xi_2(t) = a_2 h^{p_2-1}(t)$ with $a_i > 0$ and $p_i \in [1, 3/2)$, that is

$$g' \leq -a_1 g^{p_1} \quad \text{and} \quad h' \leq -a_2 h^{p_2}. \tag{5.38}$$

We can also consider the case of mixed memory-past history controls, that is $\int_0^t$ is replaced by $\int_{-\infty}^t$ only in (1.1a) and (5.37a). We also point out that kernels satisfying (5.38) has been considered by Messaoudi and Said-Houari [19] for Timoshenko systems. Recently, the assumption (5.38) has been further weakened by Guesmia [12], where he studied an abstract hyperbolic system with past history.

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