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Supplement to "Integration with respect to the non-commutative fractional Brownian motion"

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This supplemental article is devoted to the presentation of the proof of [3, Proposition 2.8]. We assume in the sequel that the reader is familiar with the notations of [3].

For the sake of clarity, let us briefly recall the statement of the result under consideration:

Proposition 0.1. *Assume that $H \in (\frac{1}{4}, \frac{1}{2})$. Then, for all $0 \leq s \leq t \leq 1$ and $U \in \mathcal{A}_s$, the sequence $\mathbb{X}^{2,(N)}_{st}[U]$ converges in $\mathcal{A}$ as $N \to \infty$, and the limit, that we denote by $\mathbb{X}^2_{st}[U]$, satisfies the following properties:*

(i) For all $0 \leq s \leq t \leq 1$, $\mathbb{X}^2_{st} \in \mathcal{L}(\mathcal{A}_s, \mathcal{A})$.

(ii) For all $0 \leq s \leq u \leq t \leq 1$ and $U \in \mathcal{A}_s$,

$$\mathbb{X}^2_{st}[U] - \mathbb{X}^2_{su}[U] - \mathbb{X}^2_{ut}[U] = \delta X_{su} U \delta X_{ut} \ . \tag{1}$$

(iii) For all $\varepsilon \in (0, 2H - \frac{1}{2})$, $\varepsilon' \in [0, H)$, there exist constants $c_{H,\varepsilon}, c_{H,\varepsilon,\varepsilon'} > 0$ such that for all $0 \leq s \leq t \leq 1$, $N \geq 0$, $m \geq 0$, $N \leq N_1, \ldots, N_m \leq \infty$, $1 \leq \iota \leq m$ and $0 \leq u_j \leq v_j \leq s$ $(j = 1, \ldots, m)$,

$$\big\| \big\{ \mathbb{X}^2_{st} - \mathbb{X}^{2,(N)}_{st} \big\} \big[\delta X_{u_1 v_1} \cdots \delta X_{u_m v_m} \big] \big\| \leq (c_{H,\varepsilon})^{m+1} \frac{|t-s|^{2H-\varepsilon}}{2^{N\varepsilon}} \prod_{j=1,\ldots,m} |u_j - v_j|^H \ , \tag{2}$$

and

$$\begin{aligned} &\big\| \mathbb{X}^{2,(N)}_{st} \big[\delta X^{(N_1)}_{u_1 v_1} \cdots \delta (X^{(N_\iota)} - X)_{u_\iota v_\iota} \cdots \delta X^{(N_m)}_{u_m v_m} \big] \big\| \\ &\qquad \leq (c_{H,\varepsilon,\varepsilon'})^{m+1} |t-s|^{2H-\varepsilon} \frac{|v_\iota - u_\iota|^{H-\varepsilon'}}{2^{N_\iota \varepsilon'}} \prod_{\substack{j=1,\ldots,m \\ j \neq \iota}} |u_j - v_j|^H \ , \end{aligned} \tag{3}$$

where we have used the convention $X^{(\infty)} := X$.

Proof. Throughout the proof, we will denote by c_H, resp. $c_{H,\varepsilon}, c_{H,\varepsilon,\varepsilon'}, c_{H,\varepsilon,\varepsilon',m}$, any positive quantity that depends only on H, resp. $(H,\varepsilon), (H,\varepsilon,\varepsilon'), (H,\varepsilon,\varepsilon',m)$.

Our main task will be to find a suitable estimate of the difference

$$\mathbb{X}^{2,(n+1)}_{st}[U] - \mathbb{X}^{2,(n)}_{st}[U] \ ,$$

for $0 \leq n \leq N-1$, $0 \leq s \leq t \leq 1$ and $U \in \mathcal{A}_s$.

If $|t-s| \leq 2^{-n+1}$, then we can check explicitly (see the beginning of the proof of [2, Theorem 3.1] for details) that

$$\big\| \mathbb{X}^{2,(n+1)}_{st}[U] - \mathbb{X}^{2,(n)}_{st}[U] \big\| \leq c_{H,\varepsilon} \frac{|t-s|^{(2H-\varepsilon)}}{2^{n\varepsilon}} \|U\| \ . \tag{4}$$

Let us assume from now on that

$$t^n_{k-1} \leq s < t^n_k < t^n_\ell \leq t < t^n_{\ell+1} \quad \text{with} \quad \ell - k \geq 1 \ . \tag{5}$$

Using first-order controls only, and more precisely the two bounds (valid for all $\varepsilon \in (0,H)$, $0 \leq u \leq v \leq 1$)

$$\|X^{(n)}_v - X^{(n)}_u\| \leq c_H |v-u|^H \text{ and } \|(X^{(n)}-X)_v - (X^{(n)}-X)_u\| \leq c_{H,\varepsilon} |v-u|^{H-\varepsilon} 2^{-n\varepsilon} \ , \tag{6}$$

it can be shown that (see again the beginning of the proof of [2, Theorem 3.1] for details)

$$\big\| \big\{ \mathbb{X}^{2,(n+1)}_{st}[U] - \mathbb{X}^{2,(n)}_{st}[U] \big\} - \big\{ \mathbb{X}^{2,(n+1)}_{t^n_k t^n_\ell}[U] - \mathbb{X}^{2,(n)}_{t^n_k t^n_\ell}[U] \big\} \big\| \leq c_{H,\varepsilon} \frac{|t-s|^{(2H-\varepsilon)}}{2^{n\varepsilon}} \|U\| \ , \tag{7}$$

and we are thus left with the estimation of $\mathbb{X}^{2,(n+1)}_{t^n_k t^n_\ell}[U] - \mathbb{X}^{2,(n)}_{t^n_k t^n_\ell}[U]$. Setting $Y_i = Y^{(n)}_i := \delta X_{t^{n+1}_i t^{n+1}_{i+1}}$, this difference actually reduces to

$$\begin{aligned} \mathbb{X}^{2,(n+1)}_{t^n_k t^n_\ell}[U] - \mathbb{X}^{2,(n)}_{t^n_k t^n_\ell}[U] &= \int_{t^n_k}^{t^n_\ell} \delta X^{(n+1)}_{t^n_k u} U \mathrm{d}X^{(n+1)}_u - \int_{t^n_k}^{t^n_\ell} \delta X^{(n)}_{t^n_k u} U \mathrm{d}X^{(n)}_u \\ &= \frac{1}{2} \sum_{i=k}^{\ell-1} \big[Y_{2i} U Y_{2i+1} - Y_{2i+1} U Y_{2i} \big] \ . \end{aligned} \tag{8}$$

Keeping in mind the desired estimates (2)-(3), we henceforth consider U of the two following possible forms:

Situation A: $U := U_1 \cdots U_m \ , \quad m \geq 1 \ , \quad U_j := \delta X_{u_j v_j} \ , \ 0 \leq u_j \leq v_j \leq s \ ;$
Situation B: $U := U_1 \cdots U_\iota \cdots U_m \ , \quad 1 \leq \iota \leq m \ ,$

$$U_j := \delta X^{(N_j)}_{u_j v_j} \ \text{ for } j \neq \iota \ , \ U_\iota := \delta (X^{(N_\iota)} - X)_{u_\iota v_\iota} \ , \quad 0 \leq u_j \leq v_j \leq s \ .$$

For each of these two situations, our aim is thus to estimate

$$\Big\|\sum_{i=k}^{\ell-1}\big[Y_{2i}UY_{2i+1}-Y_{2i+1}UY_{2i}\big]\Big\|$$
$$=\lim_{r\to\infty}\varphi\Big(\Big(\Big(\sum_{i=k}^{\ell-1}\big[Y_{2i}UY_{2i+1}-Y_{2i+1}UY_{2i}\big]\Big)\Big(\sum_{i=k}^{\ell-1}\big[Y_{2i}UY_{2i+1}-Y_{2i+1}UY_{2i}\big]\Big)^*\Big)^r\Big)^{1/(2r)}.$$

For the sake of clarity, let us introduce the additional notations: for all $V,V_1,\ldots,V_m\in\mathcal{A}$,

$$\mathbf{Y}_{1,i}[V]:=Y_{2i}VY_{2i+1}\quad,\quad\mathbf{Y}_{-1,i}[V]:=Y_{2i+1}VY_{2i}$$

and

$$\mathbb{Y}_{1,i}[V_1,\ldots,V_m]:=(Y_{2i},V_1,\ldots,V_m,Y_{2i+1})\quad,\quad\mathbb{Y}_{-1,i}[V_1,\ldots,V_m]:=(Y_{2i+1},V_1,\ldots,V_m,Y_{2i}).$$

The above r-th moment can then be expanded as

$$(-1)^r\sum_{i_1,\ldots,i_{2r}}\varphi\big(\big\{\big[\mathbf{Y}_{1,i_1}(U)-\mathbf{Y}_{-1,i_1}(U)\big]\big[\mathbf{Y}_{1,i_2}(U^*)-\mathbf{Y}_{-1,i_2}(U^*)\big]\big\}\cdots$$
$$\big\{\big[\mathbf{Y}_{1,i_{2r-1}}(U)-\mathbf{Y}_{-1,i_{2r-1}}(U)\big]\big[\mathbf{Y}_{1,i_{2r}}(U^*)-\mathbf{Y}_{-1,i_{2r}}(U^*)\big]\big\}\big)$$
$$=(-1)^r\sum_{\sigma\in\{-1,1\}^{2r}}(-1)^{N(\sigma)}\sum_{i_1,\ldots,i_{2r}}\varphi\big(\big(\mathbf{Y}_{\sigma_1,i_1}[U]\mathbf{Y}_{\sigma_2,i_2}[U^*]\big)\cdots\big(\mathbf{Y}_{\sigma_{2r-1},i_{2r-1}}[U]\mathbf{Y}_{\sigma_{2r},i_{2r}}[U^*]\big)\big),$$

where $N(\sigma)$ denotes the number of (-1) in σ. At this point, recall that (Y_i,U_j) is a semicircular family, and so, by the "non-commutative Wick formula" (see [3, Definition 1.3]), we can go ahead with our expansion and write the previous quantity as

$$(-1)^r\sum_{\sigma\in\{-1,1\}^{2r}}(-1)^{N(\sigma)}$$
$$\sum_{i_1,\ldots,i_{2r}}\sum_{\pi\in NC_2(2r(m+2))}\kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}],\mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big),\ldots,\big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}],\mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big), \tag{9}$$

where we have set $\mathbb{U}:=(U_1,\ldots,U_m)$ and $\mathbb{U}^*:=(U_m,\ldots,U_1)$.

Let us now consider the subset of $NC_2(2r(m+2))$ given by

$$\mathcal{E}_1:=\{\pi\in NC_2(2r(m+2)):\ (1,m+2)\in\pi\}.$$

With expression (9) in mind, $\mathcal{E}_1$ thus corresponds to the set of pairings π for which the variables Y_{2i_1} and Y_{2i_1+1} are "connected" within $\kappa_\pi(...)$. Observe in particular that, for fixed σ and $i_1, \ldots, i_{2r}$,

$$\sum_{\pi \in \mathcal{E}_1} \kappa_\pi\big(\big(\mathbb{Y}_{1,i_1}[\mathbb{U}], \mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$= \varphi\big(Y_{2i_1} Y_{2i_1+1}\big)\varphi\big(U\big)$$
$$\sum_{\pi \in NC_2((2r-1)(m+2))} \kappa_\pi\big(\mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*], \big(\mathbb{Y}_{\sigma_3,i_3}[\mathbb{U}], \mathbb{Y}_{\sigma_4,i_4}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$= \sum_{\pi \in \mathcal{E}_1} \kappa_\pi\big(\big(\mathbb{Y}_{-1,i_1}[\mathbb{U}], \mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big) \ ,$$

and as result

$$\sum_{\sigma \in \{-1,1\}^{2r}} (-1)^{N(\sigma)} \sum_{i_1,\ldots,i_{2r}} \sum_{\pi \in \mathcal{E}_1} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}], \mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$= \sum_{\substack{\sigma \in \{-1,1\}^{2r} \\ \sigma_1 = 1}} \ldots + \sum_{\substack{\sigma \in \{-1,1\}^{2r} \\ \sigma_1 = -1}} \ldots \ = \ 0 \ .$$

Along the same idea, consider the subset

$$\mathcal{E}_2 := \{\pi \in NC_2(2r(m+2)) : \ (1, m+2) \notin \pi \ , \ (m+3, 2(m+2)) \in \pi\} \ ,$$

so that, just as above,

$$\sum_{\pi \in \mathcal{E}_2} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}], \mathbb{Y}_{1,i_2}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$= \varphi\big(Y_{2i_2} Y_{2i_2+1}\big)\varphi\big(U^*\big)$$
$$\sum_{\pi \in \tilde{\mathcal{E}}_2} \kappa_\pi\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}], \big(\mathbb{Y}_{\sigma_3,i_3}[\mathbb{U}], \mathbb{Y}_{\sigma_4,i_4}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$= \sum_{\pi \in \mathcal{E}_2} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}], \mathbb{Y}_{-1,i_2}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big) \ ,$$

where $\tilde{\mathcal{E}}_2 := \{\pi \in NC_2((2r-1)(m+2)) : \ (1, m+2) \notin \pi\}$, and accordingly

$$\sum_{\sigma \in \{-1,1\}^{2r}} (-1)^{N(\sigma)} \sum_{i_1,\ldots,i_{2r}} \sum_{\pi \in \mathcal{E}_2} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}], \mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big), \ldots, \big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$= \sum_{\substack{\sigma \in \{-1,1\}^{2r} \\ \sigma_2 = 1}} \ldots + \sum_{\substack{\sigma \in \{-1,1\}^{2r} \\ \sigma_2 = -1}} \ldots \ = \ 0 \ .$$

Iterating the procedure, we see that the quantity (9) reduces in fact to

$$(-1)^r \sum_{\pi\in\mathcal{E}} \sum_{\sigma\in\{-1,1\}^{2r}} (-1)^{N(\sigma)} \sum_{i_1,\ldots,i_{2r}} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}],\mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big),\ldots,\big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}],\mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)\ , \quad (10)$$

where

$$\mathcal{E} := \{\pi \in NC_2(2r(m+2)) : \text{ for all } j=1,\ldots,2r\,,\ ((j-1)(m+2)+1, j(m+2)) \notin \pi\}\ .$$

As a next step, observe that for each fixed $\pi\in\mathcal{E}$ and $\sigma\in\{-1,1\}^{2r}$, the sum

$$\sum_{i_1,\ldots,i_{2r}} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}],\mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big),\ldots,\big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}],\mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$

can always be written as a product of three terms P_i of the form

$$P_1 = \prod_{j=1}^{q_1}\Bigg(\sum_{i_1,\ldots,i_{p_j}} \varphi\big(Z_{(j,1),i_1}Z'_{(j,2),i_2}\big)\varphi\big(Z_{(j,2),i_2}Z'_{(j,3),i_3}\big)\cdots\varphi\big(Z_{(j,p_j),i_{p_j}}Z'_{(j,1),i_1}\big)\Bigg)\ ,$$

$$P_2 = \prod_{j=1}^{q_2}\Bigg(\sum_{i_1,\ldots,i_{r_j}} \varphi\big(U_{\alpha_j}W'_{(j,1),i_1}\big)\varphi\big(W_{(j,1),i_1}W'_{(j,2),i_2}\big)\varphi\big(W_{(j,2),i_2}W'_{(j,3),i_3}\big)\cdots\varphi\big(W_{(j,r_j),i_{r_j}}U_{\beta_j}\big)\Bigg)\ ,$$

$$P_3 = \prod_{j=1}^{q_3}\varphi\big(U_{\eta_j}U_{\lambda_j}\big)\ ,$$

where:

(a) the integers q_1,q_2,q_3,p_j,r_j are such that $(p_1+\ldots+p_{q_1})+(r_1+\ldots+r_{q_2})=2r$ and $2q_2+2q_3=2mr$;

(b) the variables (Z,Z') (resp. (W,W')) are such that for all $j=1,\ldots,q_1$, $p=1,\ldots,p_j$ (resp. $j=1,\ldots,q_2$, $p=1,\ldots,r_j$) and $i=k,\ldots,\ell-1$, $\{Z_{(j,p),i},Z'_{(j,p),i}\}=\{Y_{2i},Y_{2i+1}\}$ (resp. $\{W_{(j,p),i},W'_{(j,p),i}\}=\{Y_{2i},Y_{2i+1}\}$).

(c) the coefficients $1\le\alpha_j,\beta_j,\eta_j,\lambda_j\le m$ are such that each variable U_α $(1\le\alpha\le m)$ appears exactly $2r$-times in the product P_2P_3.

Let us now bound the product $P_1P_2P_3$ in each of the two above-described situations for U.

Situation A.

As far as P_1 is concerned, we can first use the subsequent elementary Lemma 0.2 to assert that for each $j=1,\ldots,q_1$,

$$\sum_{i_1,\ldots,i_{p_j}} \varphi\big(Z_{(j,1),i_1} Z'_{(j,2),i_2}\big)\varphi\big(Z_{(j,2),i_2} Z'_{(j,3),i_3}\big)\cdots\varphi\big(Z_{(j,p_j),i_{p_j}} Z'_{(j,1),i_1}\big)$$
$$\leq \bigg[\prod_{q=1}^{p_j-1}\bigg(\sum_{i_1,i_2}\varphi\big(Z_{(j,q),i_1} Z'_{(j,q+1),i_2}\big)^2\bigg)\bigg]^{1/2}\bigg(\sum_{i_1,i_2}\varphi\big(Z_{(j,p_j),i_1} Z'_{(j,1),i_2}\big)^2\bigg)^{1/2} .$$

We are here in a position to apply Lemma 0.3 below and deduce that for each $j=1,\ldots,q_1$,

$$\sum_{i_1,\ldots,i_{p_j}} \varphi\big(Z_{(j,1),i_1} Z'_{(j,2),i_2}\big)\varphi\big(Z_{(j,2),i_2} Z'_{(j,3),i_3}\big)\cdots\varphi\big(Z_{(j,p_j),i_{p_j}} Z'_{(j,1),i_1}\big)\leq (c_{H,\varepsilon})^{p_j}\frac{|t-s|^{(2H-\varepsilon)p_j}}{2^{n\varepsilon p_j}} ,$$

and therefore

$$P_1\leq (c_{H,\varepsilon})^{p_1+\ldots+p_{q_1}}\frac{|t-s|^{(2H-\varepsilon)(p_1+\ldots+p_{q_1})}}{2^{n\varepsilon(p_1+\ldots+p_{q_1})}} . \tag{11}$$

In order to estimate P_2, let us first write, with the help of Lemma 0.2, and for each $j=1,\ldots,q_2$,

$$\sum_{i_1,\ldots,i_{r_j}} \varphi\big(U_{\alpha_j} W'_{(j,1),i_1}\big)\varphi\big(W_{(j,1),i_1} W'_{(j,2),i_2}\big)\varphi\big(W_{(j,2),i_2} W'_{(j,3),i_3}\big)\cdots\varphi\big(W_{(j,r_j),i_{r_j}} U_{\beta_j}\big)$$
$$\leq \bigg(\sum_{i'_1,i'_{r_j}}\varphi\big(U_{\alpha_j} W'_{(j,1),i'_1}\big)^2\varphi\big(W_{(j,r_j),i'_{r_j}} U_{\beta_j}\big)^2\bigg)^{1/2}$$
$$\bigg(\sum_{i_1,i_{r_j}}\bigg[\sum_{i_2,\ldots,i_{r_j-1}}\varphi\big(W_{(j,1),i_1} W'_{(j,2),i_2}\big)\varphi\big(W_{(j,2),i_2} W'_{(j,3),i_3}\big)\cdots\varphi\big(W_{(j,r_j-1),i_{r_j-1}} W'_{(j,r_j),i_{r_j}}\big)\bigg]^2\bigg)^{1/2}$$
$$\leq \bigg(\sum_{i'_1}\varphi\big(U_{\alpha_j} W'_{(j,1),i'_1}\big)^2\bigg)^{1/2}\bigg(\sum_{i'_{r_j}}\varphi\big(W_{(j,r_j),i'_{r_j}} U_{\beta_j}\big)^2\bigg)^{1/2}\bigg[\prod_{q=1}^{r_j-1}\bigg(\sum_{i_1,i_2}\varphi\big(W_{(j,q),i_1} W'_{(j,q+1),i_2}\big)^2\bigg)\bigg]^{1/2} . \tag{12}$$

We can now combine the results of the subsequent Lemmas 0.3 and 0.4 to deduce that for each $j=1,\ldots,q_2$,

$$\sum_{i_1,\ldots,i_{r_j}} \varphi\big(U_{\alpha_j} W'_{(j,1),i_1}\big)\varphi\big(W_{(j,1),i_1} W'_{(j,2),i_2}\big)\varphi\big(W_{(j,2),i_2} W'_{(j,3),i_3}\big)\cdots\varphi\big(W_{(j,r_j),i_{r_j}} U_{\beta_j}\big)$$
$$\leq (c_{H,\varepsilon})^{r_j}|v_{\alpha_j}-u_{\alpha_j}|^H|v_{\beta_j}-u_{\beta_j}|^H\frac{|t-s|^{(2H-\varepsilon)r_j}}{2^{n\varepsilon r_j}} ,$$

and as a result

$$P_2 \leq (c_{H,\varepsilon})^{r_1+\ldots+r_{q_2}} \bigg(\prod_{j=1}^{q_2} |v_{\alpha_j} - u_{\alpha_j}|^H |v_{\beta_j} - u_{\beta_j}|^H \bigg) \frac{|t-s|^{(2H-\varepsilon)(r_1+\ldots+r_{q_2})}}{2^{n\varepsilon(r_1+\ldots+r_{q_2})}} \ . \tag{13}$$

Finally, the estimation of P_3 is an immediate consequence of the H-Hölder regularity of X:

$$P_3 \leq \prod_{j=1}^{q_3} \|U_{\eta_j}\| \|U_{\lambda_j}\| \leq (c_H)^{q_3} \prod_{j=1}^{q_3} |v_{\eta_j} - u_{\eta_j}|^H |v_{\lambda_j} - u_{\lambda_j}|^H \ . \tag{14}$$

Combining (11)-(13)-(14) with the above constraints (a)-(b)-(c), we obtain that for each fixed $\pi \in \mathcal{E}$ and $\sigma \in \{-1,1\}^{2r}$,

$$\sum_{i_1,\ldots,i_{2r}} \kappa_\pi\big((\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}], \mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]), \ldots, (\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}], \mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*])\big)$$
$$\leq (c_{H,\varepsilon})^{r(m+1)} \frac{|t-s|^{2r(2H-\varepsilon)}}{2^{2rn\varepsilon}} \prod_{j=1}^m |v_j - u_j|^{2rH} \ .$$

Going back to (10), we have thus shown that for every $r \geq 1$,

$$\varphi\bigg(\bigg(\bigg(\sum_{i=k}^{\ell-1} \big[Y_{2i} U Y_{2i+1} - Y_{2i+1} U Y_{2i} \big] \bigg) \bigg(\sum_{i=k}^{\ell-1} \big[Y_{2i} U Y_{2i+1} - Y_{2i+1} U Y_{2i} \big] \bigg)^* \bigg)^r \bigg)^{1/(2r)}$$
$$\leq (c_{H,\varepsilon})^{m+1} \big(|NC_2(2r(m+2))|^{1/(2r(m+2))} \big)^{m+2} \frac{|t-s|^{(2H-\varepsilon)}}{2^{n\varepsilon}} \prod_{j=1}^m |v_j - u_j|^H \ .$$

By letting r tend to infinity, we get the desired estimate, namely

$$\big\| \big\{ \mathbb{X}^{2,(n+1)}_{t^n_k t^n_\ell} - \mathbb{X}^{2,(n)}_{t^n_k t^n_\ell} \big\} \big[\delta X_{u_1 v_1} \cdots \delta X_{u_m v_m} \big] \big\| \leq (c_{H,\varepsilon})^{m+1} \frac{|t-s|^{(2H-\varepsilon)}}{2^{n\varepsilon}} \prod_{j=1}^m |v_j - u_j|^H \ . \tag{15}$$

It is readily checked that the above procedure can also be applied in the case $m=0$, yielding

$$\big\| \big\{ \mathbb{X}^{2,(n+1)}_{t^n_k t^n_\ell} - \mathbb{X}^{2,(n)}_{t^n_k t^n_\ell} \big\} [1] \big\| \leq c_{H,\varepsilon} \frac{|t-s|^{(2H-\varepsilon)}}{2^{n\varepsilon}} \ . \tag{16}$$

Situation B.

The expression of P_1 is of course the same as in Situation A, and thus, just as above, we have

$$P_1 \leq (c_{H,\varepsilon})^{p_1+\ldots+p_{q_1}} \frac{|t-s|^{(2H-\varepsilon)(p_1+\ldots+p_{q_1})}}{2^{n\varepsilon(p_1+\ldots+p_{q_1})}} \ . \tag{17}$$

In order to estimate P_2, let us first write, just as in (12), and for each $j = 1, \dots, q_2$,

$$\sum_{i_1,\dots,i_{r_j}} \varphi\big(U_{\alpha_j} W'_{(j,1),i_1}\big)\varphi\big(W_{(j,1),i_1} W'_{(j,2),i_2}\big)\varphi\big(W_{(j,2),i_2} W'_{(j,3),i_3}\big)\cdots\varphi\big(W_{(j,r_j),i_{r_j}} U_{\beta_j}\big)$$

$$\leq \Big(\sum_{i'_1} \varphi\big(U_{\alpha_j} W'_{(j,1),i'_1}\big)^2\Big)^{1/2}\Big(\sum_{i'_{r_j}} \varphi\big(W_{(j,r_j),i'_{r_j}} U_{\beta_j}\big)^2\Big)^{1/2}\Big[\prod_{q=1}^{r_j-1}\Big(\sum_{i_1,i_2} \varphi\big(W_{(j,q),i_1} W'_{(j,q+1),i_2}\big)^2\Big)\Big]^{1/2} .$$

We can now combine Lemma 0.3 and Corollary 0.5 below to obtain that for each $j = 1, \dots, q_2$,

$$\sum_{i_1,\dots,i_{r_j}} \varphi\big(U_{\alpha_j} W'_{(j,1),i_1}\big)\varphi\big(W_{(j,1),i_1} W'_{(j,2),i_2}\big)\varphi\big(W_{(j,2),i_2} W'_{(j,3),i_3}\big)\cdots\varphi\big(W_{(j,r_j),i_{r_j}} U_{\beta_j}\big)$$
$$\leq (c_{H,\varepsilon,\varepsilon'})^{r_j} \frac{|t-s|^{(2H-\varepsilon)r_j}}{2^{n\varepsilon r_j}} Q_{\alpha_j,\beta_j} ,$$

with

$$Q_{\alpha_j,\beta_j} := \mathbf{1}_{\{\alpha_j\neq\iota,\beta_j\neq\iota\}}|v_{\alpha_j}-u_{\alpha_j}|^H|v_{\beta_j}-u_{\beta_j}|^H + \mathbf{1}_{\{\alpha_j=\iota,\beta_j=\iota\}}\frac{|v_\iota-u_\iota|^{2H-2\varepsilon'}}{2^{2n_\iota\varepsilon'}}$$
$$+ \mathbf{1}_{\{\alpha_j=\iota,\beta_j\neq\iota\}}\frac{|v_\iota-u_\iota|^{H-\varepsilon'}}{2^{n_\iota\varepsilon'}}|v_{\beta_j}-u_{\beta_j}|^H + \mathbf{1}_{\{\alpha_j\neq\iota,\beta_j=\iota\}}|v_{\alpha_j}-u_{\alpha_j}|^H\frac{|v_\iota-u_\iota|^{H-\varepsilon'}}{2^{n_\iota\varepsilon'}} ,$$

and accordingly

$$P_2 \leq (c_{H,\varepsilon,\varepsilon'})^{r_1+\ldots+r_{q_2}}\Big(\prod_{j=1}^{q_2} Q_{\alpha_j,\beta_j}\Big)\frac{|t-s|^{(2H-\varepsilon)(r_1+\ldots+r_{q_2})}}{2^{n\varepsilon(r_1+\ldots+r_{q_2})}} . \tag{18}$$

Finally, the estimation of P_3 in this situation follows from the two controls in (6):

$$P_3 \leq \prod_{j=1}^{q_3}\|U_{\eta_j}\|\|U_{\lambda_j}\| \leq (c_H)^{q_3}\prod_{j=1}^{q_3} Q_{\eta_j,\lambda_j} . \tag{19}$$

Combining (17)-(18)-(19) with the above constraints (a)-(b)-(c), we deduce, for each fixed $\pi \in \mathcal{E}$ and $\sigma \in \{-1,1\}^{2r}$,

$$\sum_{i_1,\dots,i_{2r}} \kappa_\pi\big(\big(\mathbb{Y}_{\sigma_1,i_1}[\mathbb{U}],\mathbb{Y}_{\sigma_2,i_2}[\mathbb{U}^*]\big),\dots,\big(\mathbb{Y}_{\sigma_{2r-1},i_{2r-1}}[\mathbb{U}],\mathbb{Y}_{\sigma_{2r},i_{2r}}[\mathbb{U}^*]\big)\big)$$
$$\leq (c_{H,\varepsilon,\varepsilon'})^{r(m+1)}\frac{|t-s|^{2r(2H-\varepsilon)}}{2^{2rn\varepsilon}}\frac{|v_\iota-u_\iota|^{2r(H-\varepsilon')}}{2^{2rN_\iota\varepsilon'}}\prod_{\substack{j=1\\j\neq\iota}}^{m}|v_j-u_j|^{2rH} ,$$

and we can then use the same arguments as in Situation A to derive that

$$\big\|\big\{\mathbb{X}^{2,(n+1)}_{t^n_k t^n_\ell} - \mathbb{X}^{2,(n)}_{t^n_k t^n_\ell}\big\}\big[\delta X^{(N_1)}_{u_1v_1}\cdots\delta(X^{(N_\iota)}-X)_{u_\iota v_\iota}\cdots\delta X^{(N_m)}_{u_m v_m}\big]\big\|$$
$$\leq (c_{H,\varepsilon,\varepsilon'})^{m+1}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}}\frac{|v_\iota-u_\iota|^{H-\varepsilon'}}{2^{N_\iota\varepsilon'}}\prod_{\substack{j=1,\dots,m\\j\neq\iota}}|u_j-v_j|^H . \tag{20}$$

Conclusion.

First, based on (4), (7), (15) and (16), we can assert that for all fixed $0 \leq s \leq t \leq 1$ and $U \in \mathcal{A}_s$, $(\mathbb{X}^{2,(n)}_{st}[U])_{n\geq 1}$ is a Cauchy sequence in $(\mathcal{A}, \|.\|)$, and therefore it converges to an element $\mathbb{X}^2_{st}[U]$, as desired.

The fact that $\mathbb{X}^2_{st}$ is linear (as a function of U) follows immediately from the linearity of $\mathbb{X}^{2,(n)}_{st}$, and in the same way, identity (1) is a straightforward consequence of the (readily-checked) relation

$$\mathbb{X}^{2,(n)}_{st}[U] - \mathbb{X}^{2,(n)}_{su}[U] - \mathbb{X}^{2,(n)}_{ut}[U] = \delta X^{(n)}_{su} U \delta X^{(n)}_{ut} .$$

Finally, estimate (2), resp. estimate (3), follows at once from (4), (7) and (15), resp. (4), (7) and (20).

□

We are now left with the proof of the few technical results related to the control of the covariances.

Lemma 0.2. *Given a finite set I, an integer $p \geq 1$ and real quantities $A^{(q)}_{i_1 i_2}$ $(1 \leq q \leq p$, $i_1, i_2 \in I)$, it holds that*

$$\sum_{i_1, i_p \in I} \Big[\sum_{i_2,\ldots,i_{p-1} \in I} A^{(1)}_{i_1 i_2} A^{(2)}_{i_2 i_3} \cdots A^{(p-1)}_{i_{p-1} i_p} \Big]^2 \leq \prod_{q=1}^{p-1} \Big(\sum_{i_1, i_2 \in I} \big(A^{(q)}_{i_1 i_2}\big)^2 \Big) ,$$

and as a particular consequence

$$\sum_{i_1,\ldots,i_p \in I} A^{(1)}_{i_1 i_2} A^{(2)}_{i_2 i_3} \cdots A^{(p-1)}_{i_{p-1} i_p} A^{(p)}_{i_p i_1} \leq \prod_{q=1}^{p} \Big(\sum_{i_1, i_2 \in I} \big(A^{(q)}_{i_1 i_2}\big)^2 \Big)^{1/2} .$$

Proof. These results can actually be shown through an easy iteration of Cauchy-Schwarz inequality (we have labeled them for the sake of clarity only). □

Lemma 0.3. *With the notations of the above proof, one has, for every $\varepsilon \in (0, 2H - \frac{1}{2})$,*

$$\max\Big(\sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i} Y_{2j}\big)^2, \sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i} Y_{2j+1}\big)^2, \sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i+1} Y_{2j+1}\big)^2 \Big) \leq c_{H,\varepsilon} \frac{|t-s|^{4H-2\varepsilon}}{2^{2n\varepsilon}} .$$

Proof. Let us naturally write

$$\sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i} Y_{2j+1}\big)^2 = \sum_{i=k}^{\ell-1} \varphi\big(Y_{2i} Y_{2i+1}\big)^2 + \sum_{\substack{i,j=k \\ i<j}}^{\ell-1} \varphi\big(Y_{2i} Y_{2j+1}\big)^2 + \sum_{\substack{i,j=k \\ j<i}}^{\ell-1} \varphi\big(Y_{2i} Y_{2j+1}\big)^2 .$$

First, one has obviously

$$\sum_{i=k}^{\ell-1} \varphi\big(Y_{2i}Y_{2i+1}\big)^2 = \sum_{i=k}^{\ell-1} \frac{c_H}{2^{4Hn}} = c_H \frac{\ell-k}{2^{4Hn}} \leq c_H \frac{|t-s|}{2^{n(4H-1)}} \leq c_H \frac{|t-s|^{4H-2\varepsilon}}{2^{2n\varepsilon}} ,$$

where we have used the fact that $2\varepsilon < 4H-1$ and $2^{-n} \leq |t-s|$ (by (5)) to get the last inequality.

Then write

$$\sum_{\substack{i,j=k\\ i<j}}^{\ell-1} \varphi\big(Y_{2i}Y_{2j+1}\big)^2 = c_H \sum_{j=k}^{\ell-1}\sum_{i=k}^{j-1} \bigg(\int_{t^{n+1}_{2j+1}}^{t^{n+1}_{2j+2}} d\tau_1 \int_{t^{n+1}_{2i}}^{t^{n+1}_{2i+1}} \frac{d\tau_2}{(\tau_1-\tau_2)^{2-2H}} \bigg)^2$$

$$\leq c_H \sum_{j=k}^{\ell-1}\sum_{i=k}^{j-1} \bigg(\int_{t^n_j}^{t^n_{j+1}} d\tau_1 \int_{t^n_i}^{t^n_{i+1}} \frac{d\tau_2}{(\tau_1-\tau_2)^{2-2H}} \bigg)^2 .$$

Using elementary changes of variables, we can easily rewrite the latter quantity as

$$\sum_{j=k}^{\ell-1}\sum_{i=k}^{j-1} \bigg(\int_{t^n_j}^{t^n_{j+1}} d\tau_1 \int_{t^n_i}^{t^n_{i+1}} \frac{d\tau_2}{(\tau_1-\tau_2)^{2-2H}} \bigg)^2 = 2^{-4Hn} \sum_{j=1}^{\ell-k}\sum_{i=1}^{j-1} \bigg(\int_0^1 d\tau_1 \int_{i-1}^{i} \frac{d\tau_2}{(\tau_1+\tau_2)^{2-2H}} \bigg)^2 ,$$

which yields

$$\begin{aligned}
\sum_{\substack{i,j=k\\ i<j}}^{\ell-1} \varphi\big(Y_{2i}Y_{2j+1}\big)^2 &\leq \frac{c_H}{2^{4Hn}} \sum_{j=1}^{\ell-k}\sum_{i=1}^{j-1} \bigg(\int_0^1 \frac{d\tau_1}{\tau_1^{1-\varepsilon}} \int_{i-1}^{i} \frac{d\tau_2}{\tau_2^{1-2H+\varepsilon}} \bigg)^2 \\
&\leq \frac{c_{H,\varepsilon}}{2^{4Hn}} \sum_{j=1}^{\ell-k}\sum_{i=1}^{j-1} \int_{i-1}^{i} \frac{d\tau_2}{\tau_2^{2-4H+2\varepsilon}} \leq \frac{c_{H,\varepsilon}}{2^{4Hn}} \sum_{j=1}^{\ell-k} \int_0^j \frac{d\tau_2}{\tau_2^{2-4H+2\varepsilon}} \\
&\leq \frac{c_{H,\varepsilon}}{2^{4Hn}} \sum_{j=1}^{\ell-k} j^{4H-2\varepsilon-1} \leq c_{H,\varepsilon} \frac{|\ell-k|^{4H-2\varepsilon}}{2^{4Hn}} \leq c_{H,\varepsilon} \frac{|t-s|^{4H-2\varepsilon}}{2^{2n\varepsilon}} .
\end{aligned}$$

The same arguments can of course be used to bound $\sum_{i,j=k}^{\ell-1} 1_{\{j<i\}} \varphi\big(Y_{2i}Y_{2j+1}\big)^2$, so that

$$\sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i}Y_{2j+1}\big)^2 \leq c_{H,\varepsilon} \frac{|t-s|^{4H-2\varepsilon}}{2^{2n\varepsilon}} .$$

We can then estimate $\sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i}Y_{2j}\big)^2$ and $\sum_{i,j=k}^{\ell-1} \varphi\big(Y_{2i+1}Y_{2j+1}\big)^2$ along the same procedure. □

Lemma 0.4. *With the notations of the above proof, one has, for all $j = 1, \dots, m$ and $\varepsilon \in [0, H)$,*

$$\max\Big(\sum_{i=k}^{\ell-1}\varphi\big(\delta X_{u_jv_j}Y_{2i}\big)^2 \ , \ \sum_{i=k}^{\ell-1}\varphi\big(\delta X_{u_jv_j}Y_{2i+1}\big)^2\Big) \leq c_{H,\varepsilon}|v_j-u_j|^{2H}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}} \ . \tag{21}$$

Proof. Recall that for all $j = 1, \dots, m$ and $i = k, \dots, \ell-1$, one has

$$0 \leq u_j \leq v_j \leq s \leq t^{n+1}_{2i} \leq t^{n+1}_{2i+1} \ ,$$

and so

$$\sum_{i=k}^{\ell-1}\varphi\big(\delta X_{u_jv_j}Y_{2i}\big)^2 = c_H\sum_{i=k}^{\ell-1}\bigg(\int_{t^{n+1}_{2i}}^{t^{n+1}_{2i+1}}d\tau_1\int_{u_j}^{v_j}\frac{d\tau_2}{(\tau_1-\tau_2)^{2-2H}}\bigg)^2$$
$$\leq c_H\sum_{i=k}^{\ell-1}\bigg(\int_{t^{n+1}_{2i}}^{t^{n+1}_{2i+1}}\frac{d\tau_1}{(\tau_1-s)^{1-H}}\int_{u_j}^{v_j}\frac{d\tau_2}{(v_j-\tau_2)^{1-H}}\bigg)^2$$
$$\leq c_H|v_j-u_j|^{2H}\sum_{i=k}^{\ell-1}\bigg(\int_{t^{n+1}_{2i}}^{t^{n+1}_{2i+1}}\frac{d\tau_1}{(\tau_1-s)^{1-H}}\bigg)\bigg(\int_{t^{n+1}_{2i}}^{t^{n+1}_{2i+1}}\frac{d\tau_1}{(\tau_1-t^{n+1}_{2i})^{1-H}}\bigg)^{\frac{\varepsilon}{H}}\bigg(\int_s^t\frac{d\tau_1}{(\tau_1-s)^{1-H}}\bigg)^{1-\frac{\varepsilon}{H}}$$
$$\leq c_{H,\varepsilon}|v_j-u_j|^{2H}\frac{|t-s|^{H-\varepsilon}}{2^{n\varepsilon}}\sum_{i=k}^{\ell-1}\int_{t^{n+1}_{2i}}^{t^{n+1}_{2i+1}}\frac{d\tau_1}{(\tau_1-s)^{1-H}} \ \leq \ c_{H,\varepsilon}|v_j-u_j|^{2H}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}} \ .$$

The same arguments apply of course to $\sum_{i=k}^{\ell-1}\varphi\big(\delta X_{u_jv_j}Y_{2i+1}\big)^2$. □

Corollary 0.5. *With the notations of the above proof, one has, for all $j = 1, \dots, m$ and $\varepsilon, \varepsilon' \in [0, H)$,*

$$\max\Big(\sum_{i=k}^{\ell-1}\varphi\big(\delta X^{(N_j)}_{u_jv_j}Y_{2i}\big)^2 \ , \ \sum_{i=k}^{\ell-1}\varphi\big(\delta X^{(N_j)}_{u_jv_j}Y_{2i+1}\big)^2\Big) \leq c_{H,\varepsilon}|v_j-u_j|^{2H}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}} \tag{22}$$

and

$$\max\Big(\sum_{i=k}^{\ell-1}\varphi\big(\delta\{X^{(N_j)}-X\}_{u_jv_j}Y_{2i}\big)^2 \ , \ \sum_{i=k}^{\ell-1}\varphi\big(\delta\{X^{(N_j)}-X\}_{u_jv_j}Y_{2i+1}\big)^2\Big)$$
$$\leq c_{H,\varepsilon,\varepsilon'}\frac{|v_j-u_j|^{2H-\varepsilon'}}{2^{N_j\varepsilon'}}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}} \ . \tag{23}$$

Proof. For more clarity, set $t_r := t^{N_j}_r$ $(r = 0, \dots, 2^{N_j})$ for the whole proof. Assume that

$$t_p \leq u_j < t_{p+1} < \dots < t_q \leq v_j < t_{q+1} \quad \text{with } |v_j-u_j| \geq 2^{-N_j} \ , \tag{24}$$

and write

$$\delta X^{(N_j)}_{u_jv_j} = \delta X^{(N_j)}_{u_jt_{p+1}} + \delta X_{t_{p+1}t_q} + \delta X^{(N_j)}_{t_qv_j}$$
$$= \delta X_{t_pt_{p+1}}\big(1-2^{-N_j}(u_j-t_p)\big) + \delta X_{t_{p+1}t_q} + \delta X_{t_qt_{q+1}}2^{-N_j}(v_j-t_q)\ . \qquad (25)$$

If $t_{q+1} \leq s$, we can apply (21) to each of three above terms, which immediately gives (22). If $t_{q+1} > s$, we can still apply (21) to the first two summands in (25), but not to the third one. In this case, let us first write

$$\sum_{i=k}^{\ell-1} \varphi\big(\delta X^{(N_j)}_{t_qt_{q+1}} Y_{2i}\big)^2 = \varphi\big(\delta X^{(N_j)}_{t_qt_{q+1}} Y_{2k}\big)^2 + \sum_{i=k+1}^{\ell-1} \varphi\big(\delta X^{(N_j)}_{t_qt_{q+1}} Y_{2i}\big)^2$$
$$\leq \big\|\delta X^{(N_j)}_{t_qt_{q+1}}\big\|^2 \big\|Y_{2k}\big\|^2 + \sum_{i=k+1}^{\ell-1} \varphi\big(\delta X^{(N_j)}_{t_qt_{q+1}} Y_{2i}\big)^2$$
$$\leq c_{H,\varepsilon}|v_j-u_j|^{2H}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}} + \sum_{i=k+1}^{\ell-1} \varphi\big(\delta X^{(N_j)}_{t_qt_{q+1}} Y_{2i}\big)^2\ .$$

Due to $N_j \geq N \geq n$, we know that $t_{q+1} < t^{n+1}_{2k+2}$, and we can therefore apply the result of (21) to the remaining sum, which entails, as desired,

$$\sum_{i=k+1}^{\ell-1} \varphi\big(\delta X^{(N_j)}_{t_qt_{q+1}} Y_{2i}\big)^2 \leq c_{H,\varepsilon}\frac{|t-t^{n+1}_{2k+2}|^{2H-\varepsilon}}{2^{n\varepsilon}} 2^{-2HN_j} \leq c_{H,\varepsilon}\frac{|t-s|^{2H-\varepsilon}}{2^{n\varepsilon}}|v_j-u_j|^{2H}\ .$$

The case where $|v_j-u_j| \leq 2^{-N_j}$ can then be handled with elementary arguments, and this achieves the proof of (22).

The proof of (23) follows from similar considerations. Observe indeed that, in the situation (24),

$$\delta\{X^{(N_j)}-X\}_{u_jv_j} = \delta X_{t_pu_j} - \delta X_{t_qv_j} - 2^{-N_j}(u_j-t_p)\delta X_{t_pt_{p+1}} + 2^{-N_j}(v_j-t_q)\delta X_{t_qt_{q+1}}\ ,$$

and from here we can apply the above reasoning to each of the summands.

□

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