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A general decay result for a viscoelastic equation in the presence of past and finite history memories

Aissa Guesmia[a], Salim A. Messaoudi[b,*]

[a] LMAM, Bat. A, Université Paul Verlaine - Metz, Ile du Saulcy, 57045 Metz Cedex 01, France
[b] King Fahd University of Petroleum and Minerals, Department of Mathematical Sciences, Dhahran 31261, Saudi Arabia

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ABSTRACT

In this work, we are concerned with the following equation

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)div\,(a_1(x)\,\nabla u(s))\,ds + \int_0^{+\infty} g_2(s)div\,(a_2(x)\nabla u(t-s))\,ds = 0$$

in a bounded domain Ω. Under suitable conditions on a_1 and a_2 and for a wide class of relaxation functions g_1 and g_2, we establish a general decay result, from which the usual exponential and polynomial decay rates are only special cases.

1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\mathrm{div}\,(a_1(x)\nabla u(s))\,ds + \int_0^{+\infty} g_2(s)\mathrm{div}\,(a_2(x)\nabla u(t-s))\,ds = 0, & \forall x \in \Omega,\ \forall t > 0 \\ u(x,t) = 0, & \forall x \in \partial\Omega,\ \forall t \geq 0 \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & \forall x \in \Omega,\ \forall t \geq 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, g_1 and g_2 are two positive non-increasing functions defined on $\mathbb{R}^+$, a_1 and a_2 are essentially bounded non-negative functions defined on Ω, and u_0 and u_1 are given initial data. This type of problems arise in viscoelasticity. For the thermodynamics of materials with fading memory, we refer the reader to the early work of Coleman and Mizel [1] and the references therein.

We start our literature review with the pioneer work of Dafermos [2], in 1970, where the author discussed a certain one-dimensional viscoelastic problem, established some existence results, and then proved that, for smooth monotone decreasing relaxation functions, the solutions go to zero as t goes to infinity. However, no rate of decay has been specified. In Dafermos [3], a similar result, under a convexity condition on the kernel, has been established. After that a great deal of attention has been devoted to the study of viscoelastic problems and many existence and long-time behavior results have been established. Hrusa [4] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)\,(\psi(u_x(x,s)))_x\,ds = f(x,t)$$

* Corresponding author.
E-mail addresses: guesmia@univ-metz.fr (A. Guesmia), messaoud@kfupm.edu.sa (S.A. Messaoudi).

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and proved several global existence results for large data. He also proved an exponential decay result for strong solutions when $m(s) = e^{-s}$ and ψ satisfies certain conditions. Dassios and Zafiropoulos [5] studied a viscoelastic problem in $\mathbb{R}^3$ and proved a polynomial decay result for exponentially decaying kernels. After that, a very important contribution by Rivera was introduced. In 1994, Rivera [6] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$. In the bounded-domain case, and for exponentially decaying memory kernels and regular solutions, he showed that the sum of the first and the second energy decays exponentially. For the whole-space case and for exponentially decaying memory kernels, he showed that the rate of decay of energy is of algebraic type and depends on the regularity of the solution. This result was later generalized to a situation, where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [7]. In their paper, the authors considered the case of bounded domains as well as the case when the material is occupying the entire space and showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. This latter result was later improved by Baretto et al. [8], where equations related to linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera and Salvatierra [9] showed that the energy decays exponentially, provided the relaxation function decays in a similar fashion and the dissipation is acting on a part of the domain near to the boundary. See also, in this direction, the work of Rivera and Oquendo [10].

For an equation with a localized frictional damping cooperating with the dissipation induced by the viscoelastic term, we mention the work of Cavalcanti et al. [11], where an exponential rate of decay has been proved for a relaxation function satisfying

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t), \quad t \ge 0$$

and under some geometry restriction on the domain. Berrimi and Messaoudi [12] improved Cavalcanti's result [11] by showing, similarly to [6], that the viscoelastic dissipation alone is enough to stabilize the system. To achieve their goal, Berrimi and Messaoudi [12] introduced a different functional, which allowed them to weaken the conditions on g, imposed in both [6,11]. This result has been later extended to a situation, where a source is competing with the viscoelastic dissipation, by Berrimi and Messaoudi [13]. Also, Cavalcanti and Oquendo [14] considered

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}[a(x) g(t-\tau) \nabla u(\tau)] \mathrm{d}\tau + b(x) h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \ge \delta > 0$, and improved the result in [11]. They established an exponential stability when g is decaying exponentially and h is linear, and a polynomial stability when g is decaying polynomially and h is non-linear. For quasilinear problems, Cavalcanti et al. [15] studied, in a bounded domain, the following equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau) \Delta u(\tau) \mathrm{d}\tau - \gamma \Delta u_t = 0,$$

for $\rho > 0$. A global existence result for $\gamma \ge 0$, as well as an exponential decay result for $\gamma > 0$, have been established. This latter result was then extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [16,17], and exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term.

In all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. For more general decaying relaxation functions, Messaoudi [18,19] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau) \mathrm{d}\tau = b|u|^{p-2} u,$$

for $p \ge 2$ and $b \in \{0, 1\}$, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases. After that, a considerable literature in this direction has appeared (see in this regards the papers [20–28]).

For past (infinite) history problems, all the relaxation functions are either of polynomial or exponential decay (see for example [29,30]). In fact, the argument introduced by Messaoudi [18,19] cannot be extended to this case. Recently, Guesmia [31] introduced a new approach which allows a larger class of past-history kernels and consequently a more general decay result for a class of hyperbolic problems with past history is obtained.

In the present work, we consider (1.1), with relaxation functions g_1, g_2 that are not necessarily decaying in a polynomial or exponential fashion and establish a general decay result. In fact, our result allows a larger class of relaxation functions and improves the decay rates in some special cases (see examples in Section 4). The paper is organized as follows. In Section 2, we present some material needed for our work. Section 3 contains the statement and the proof of our main result. We end our paper by giving some illustrating examples in Section 4.

2. Preliminaries

In this section, we present some material needed in the proof of our main result. We start with the following assumptions:

(G1) $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable non-increasing functions such that

$$g_i(0) > 0, \quad i = 1, 2, \qquad 1 - \|a_1\|_\infty \int_0^{+\infty} g_1(s)\mathrm{d}s - \|a_2\|_\infty \int_0^{+\infty} g_2(s)\mathrm{d}s = l > 0.$$

(G2) There exists a positive differentiable non-increasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g_1'(t) \le -\xi(t)g_1(t), \quad \forall t \ge 0.$$

(G3) There exists a positive constant σ and an increasing strictly convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty$$

such that

$$g_2'(t) \le -\sigma g_2(t), \quad \forall t \ge 0 \tag{2.1}$$

or

$$\int_0^{+\infty} \frac{g_2(t)}{G^{-1}(-g_2'(t))}\mathrm{d}t + \sup_{t\in\mathbb{R}^+} \frac{g_2(t)}{G^{-1}(-g_2'(t))} < +\infty. \tag{2.2}$$

(G4) $a_i : \overline{\Omega} \to \mathbb{R}^+$ are in $C^1(\overline{\Omega})$ such that, for positive constants δ and a_0 and for $\Gamma_1, \Gamma_2 \subset \partial\Omega$ with $\mathrm{meas}(\Gamma_i) > 0,\ i = 1, 2,$

$$\mathrm{Inf}_{x\in\overline{\Omega}}\,(a_1(x) + a_2(x)) \ge \delta$$

and

$$a_i = 0 \quad \text{or} \quad \mathrm{Inf}_{\Gamma_i} a_i(x) \ge 2a_0, \quad i = 1, 2.$$

Remark 2.1. If $a_i \ne 0,\ i = 1, 2$, there exist neighborhoods w_i of $\Gamma_i,\ i = 1, 2$, such that

$$\mathrm{Inf}_{\overline{\Omega\cap w_i}} a_i(x) \ge a_0 > 0, \quad i = 1, 2.$$

As in [14], let $d = \min\{a_0, \delta\}$ and let $\alpha_i \in C^1(\bar{\Omega}),\ i = 1, 2$, be such that

$$\begin{cases} 0 \le \alpha_i(x) \le a_i(x) \\ \alpha_i(x) = 0, & \text{if } a_i(x) \le \dfrac{d}{4} \\ \alpha_i(x) = a_i(x), & \text{if } a_i(x) \ge \dfrac{d}{2}. \end{cases} \tag{2.3}$$

Lemma 2.1. *The functions $\alpha_i,\ i = 1, 2$, are not identically zero and satisfy*

$$\alpha_1(x) + \alpha_2(x) \ge \frac{d}{2}.$$

Proof. (1) For $x \in \Omega \cap w_i$, we have $a_i(x) \ge a_0 \ge d$, which implies, by (2.3), that $\alpha_i(x) = a_i(x) \ge d$. Thus α_i is not identically zero.

(2) If $a_1(x) \ge \frac{d}{2}$, then $\alpha_1(x) = a_1(x)$. Consequently $\alpha_1(x) + \alpha_2(x) \ge a_1(x) \ge \frac{d}{2}$. If $a_1(x) < \frac{d}{2}$, then $a_2(x) > \frac{d}{2}$ which implies, by (2.3), $\alpha_2(x) = a_2(x) > \frac{d}{2}$. Consequently $\alpha_1(x) + \alpha_2(x) > \frac{d}{2}$. This completes the proof. □

Remark 2.2. Following the idea of Dafermos [2], we introduce

$$\eta^t(x, s) = u(x, t) - u(x, t - s), \quad \forall x \in \Omega,\ \forall s,\ t \ge 0; \tag{2.4}$$

consequently we obtain the following initial and boundary conditions

$$\begin{cases} \eta^t(x, 0) = 0, \quad \forall x \in \Omega,\ \forall t \ge 0 \\ \eta^t(x, s) = 0, \quad \forall x \in \partial\Omega,\ \forall s,\ t \ge 0 \\ \eta^0(x, s) = \eta_0(x, s) = u_0(x, 0) - u_0(x, s), \quad \forall x \in \Omega,\ \forall s \ge 0. \end{cases} \tag{2.5}$$

Clearly, (2.4) gives

$$\eta_t^t(x, s) + \eta_s^t(x, s) = u_t(x, t). \tag{2.6}$$

By combining (1.1), (2.5), (2.6), we obtain the following system

$$\begin{cases} u_{tt} - \operatorname{div}\left[\left(1 - a_2(x)\int_0^{+\infty} g_2(s)\mathrm{d}s\right)\nabla u\right] + \int_0^t g_1(t-s)\operatorname{div}\left(a_1(x)\nabla u(s)\right)\mathrm{d}s \\ \quad -\int_0^{+\infty} g_2(s)\operatorname{div}\left(a_2(x)\nabla\eta^t(s)\right)\mathrm{d}s = 0, & \forall x\in\Omega,\ \forall t>0 \\ \eta_t^t(x,s) + \eta_s^t(x,s) - u_t(x,t) = 0, & \forall x\in\Omega,\ \forall s,\ t\geq 0 \end{cases} \tag{2.7}$$

together with the following initial and boundary conditions

$$\begin{cases} u(x,t) = \eta^t(x,s) = 0, \quad \forall x\in\partial\Omega,\ \forall s,\ t\geq 0 \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), \quad \forall x\in\Omega,\ \forall t\geq 0 \\ \eta^t(x,0) = 0, \qquad \eta^0(x,s) = \eta_0(x,s) = u_0(x,0) - u_0(x,s), \quad \forall x\in\Omega,\ \forall s,\ t\geq 0. \end{cases} \tag{2.8}$$

The existence and uniqueness of the solution of problem (2.7), (2.8) can be established by using the Galerkin method.

We define the "modified" energy functional of the weak solution by

$$E(t) = \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x + \frac{1}{2}\int_\Omega\left[1 - a_1(x)\int_0^t g_1(s)\mathrm{d}s - a_2(x)\int_0^{+\infty} g_2(s)\mathrm{d}s\right]|\nabla u|^2\mathrm{d}x + \frac{1}{2}g_1\circ\nabla u + \frac{1}{2}g_2\circ\nabla\eta^t, \tag{2.9}$$

where

$$g_1\circ\nabla u = \int_\Omega a_1(x)\int_0^t g_1(t-s)|\nabla u(t)-\nabla u(s)|^2\mathrm{d}s\mathrm{d}x,$$
$$g_2\circ\nabla\eta^t = \int_\Omega a_2(x)\int_0^{+\infty} g_2(s)|\nabla\eta^t|^2\mathrm{d}s\mathrm{d}x.$$

Lemma 2.2. *The "modified" energy functional satisfies, along the solution of (2.7), (2.8),*

$$E'(t) = -\frac{1}{2}g_1(t)\int_\Omega|\nabla u(t)|^2\mathrm{d}x + \frac{1}{2}g_1'\circ\nabla u + \frac{1}{2}g_2'\circ\nabla\eta^t \leq 0, \tag{2.10}$$

where

$$g_1'\circ\nabla u = \int_\Omega a_1(x)\int_0^t g_1'(t-s)|\nabla u(t)-\nabla u(s)|^2\mathrm{d}s\mathrm{d}x,$$
$$g_2'\circ\nabla\eta^t = \int_\Omega a_2(x)\int_0^{+\infty} g_2'(s)|\nabla\eta^t|^2\mathrm{d}s\mathrm{d}x.$$

Proof. By multiplying Eq. $(2.7)_1$ by u_t and integrating over Ω, using integration by parts, hypotheses (G1)–(G4) and some manipulations as in [7,11] and others, we obtain (2.10) for regular solutions. This inequality remains valid for weak solutions by a simple density argument. □

We define

$$g_1\odot\nabla u = \int_\Omega a_1(x)\int_0^t g_1(t-s)\left(\nabla u(s)-\nabla u(t)\right)\mathrm{d}s\mathrm{d}x,$$
$$g_2\odot\nabla\eta^t = \int_\Omega a_2(x)\int_0^{+\infty} g_2(s)\nabla\eta^t\mathrm{d}s\mathrm{d}x.$$

Lemma 2.3. *There exists a positive constant c such that*

$$\begin{cases} |g_1\odot\nabla u|^2 \leq c g_1\circ\nabla u \\ \left|g_2\odot\nabla\eta^t\right|^2 \leq c g_2\circ\nabla\eta^t, \end{cases} \tag{2.11}$$

and

$$\begin{cases} \left|g_1'\odot\nabla u\right|^2 \leq -c g_1'\circ\nabla u \\ \left|g_2'\odot\nabla\eta^t\right|^2 \leq -c g_2'\circ\nabla\eta^t, \end{cases} \tag{2.12}$$

for all u, $\eta^t(.,s)\in H^1(\Omega)$.

Proof. By using (G1)–(G4), and Hölder's inequality, we get

$$\begin{aligned}|g_1 \odot \nabla u|^2 &= \left|\int_\Omega \int_0^t a_1^{\frac{1}{2}}(x) g_1^{\frac{1}{2}}(t-s) a_1^{\frac{1}{2}}(x) g_1^{\frac{1}{2}}(t-s)\left(\nabla u(s) - \nabla u(t)\right) \mathrm{d}s\mathrm{d}x\right|^2 \\ &\le \|a_1\|_\infty \left(\int_0^{+\infty} g_1(s)\mathrm{d}s\right) \int_\Omega a_1(x) \int_0^t g_1(t-s)\left|\nabla u(s) - \nabla u(t)\right|^2 \mathrm{d}s\mathrm{d}x \\ &\le c g_1 \circ \nabla u.\end{aligned}$$

Similarly,

$$\left|g_2 \odot \nabla \eta^t\right|^2 \le c g_2 \circ \nabla \eta^t$$

and (2.12) can be established. □

3. General decay

In this section, we state and prove our main result. For this purpose, we introduce three "auxiliary" functionals and establish three related lemmas. We will use c to denote a positive generic constant and assume that $E(t) > 0,\ \forall t \ge 0$ (if $E(t_0) = 0$, for some $t_0 \ge 0$, then $E(t) = 0,\ \forall t \ge t_0$; consequently, by (2.10), estimate (3.8) below holds).

Lemma 3.1. *Under the assumptions* (G1)–(G4), *the functional*

$$\phi(t) = \int_\Omega u u_t \mathrm{d}x \tag{3.1}$$

satisfies, along the solution of (2.7), (2.8) *and for any* $\varepsilon_1 > 0$,

$$\begin{aligned}\phi'(t) &\le \int_\Omega u_t^2 \mathrm{d}x - \left[1 - \varepsilon_1 - \|a_1\|_\infty \int_0^{+\infty} g_1(s)\mathrm{d}s - \|a_2\|_\infty \int_0^{+\infty} g_2(s)\mathrm{d}s\right] \int_\Omega |\nabla u|^2 \mathrm{d}x \\ &\quad + \frac{c}{\varepsilon_1}\left(g_1 \circ \nabla u + g_2 \circ \nabla \eta^t\right), \quad \forall t \ge 0.\end{aligned} \tag{3.2}$$

Proof. By differentiating (3.1) and using (2.7), (2.8), we easily see that

$$\begin{aligned}\phi'(t) &= \int_\Omega u_t^2 \mathrm{d}x - \int_\Omega \left(1 - a_2(x) \int_0^{+\infty} g_2(s)\mathrm{d}s\right) |\nabla u|^2 \mathrm{d}x + \int_\Omega \nabla u \cdot \int_0^t a_1(x) g_1(t-s) \nabla u(s) \mathrm{d}s\mathrm{d}x \\ &\quad - \int_\Omega \nabla u \cdot \int_0^{+\infty} g_2(s) a_2(x) \nabla \eta^t(s) \mathrm{d}s\mathrm{d}x \\ &= \int_\Omega u_t^2 \mathrm{d}x - \int_\Omega \left(1 - a_1(x) \int_0^t g_1(s)\mathrm{d}s - a_2(x) \int_0^{+\infty} g_2(s)\mathrm{d}s\right) |\nabla u|^2 \mathrm{d}x \\ &\quad + \int_\Omega \nabla u. \int_0^t a_1(x) g_1(t-s)\left(\nabla u(s) - \nabla u(t)\right) \mathrm{d}s\mathrm{d}x - \int_\Omega \nabla u \cdot \int_0^{+\infty} g_2(s) a_2(x) \nabla \eta^t(s) \mathrm{d}s\mathrm{d}x \\ &\le \int_\Omega u_t^2 \mathrm{d}x - \int_\Omega \left(1 - a_1(x) \int_0^t g_1(s)\mathrm{d}s - a_2(x) \int_0^{+\infty} g_2(s)\mathrm{d}s\right) |\nabla u|^2 \mathrm{d}x \\ &\quad + \varepsilon_1 \int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2\varepsilon_1} |g_1 \odot \nabla u|^2 + \frac{1}{2\varepsilon_1} \left|g_2 \odot \nabla \eta^t\right|^2.\end{aligned}$$

By using (2.11), the assertion of the lemma is proved. □

Lemma 3.2. *Under the assumptions* (G1)–(G4), *the functional*

$$\psi_1(t) = -\int_\Omega \alpha_1(x) u_t \int_0^t g_1(t-s)(u(t) - u(s)) \mathrm{d}s\mathrm{d}x \tag{3.3}$$

satisfies, along the solution of (2.7), (2.8) *and for any* $\varepsilon_2, \varepsilon_3 > 0$,

$$\begin{aligned}\psi_1'(t) &\le -\left[\int_0^t g_1(s)\mathrm{d}s - \varepsilon_2\right] \int_\Omega \alpha_1(x) u_t^2 \mathrm{d}x + \frac{\varepsilon_3}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{c}{\varepsilon_2} g_1' \circ \nabla u \\ &\quad + \frac{c}{\varepsilon_3}\left(g_1 \circ \nabla u + g_2 \circ \nabla \eta^t\right), \quad \forall t \ge 0.\end{aligned} \tag{3.4}$$

Proof. By differentiating (3.3) and using (2.7), (2.8), we easily see that

$$\begin{aligned}
\psi_1'(t) &= -\left(\int_\Omega \alpha_1(x)u_t^2\mathrm{d}x\right)\left(\int_0^t g_1(s)\mathrm{d}s\right) - \int_\Omega \alpha_1(x)u_t \int_0^t g_1'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
&\quad - \int_\Omega \alpha_1(x)\left[\int_0^t g_1(t-s)(u(t)-u(s))\mathrm{d}s\right]\mathrm{div}\left[\left(1-a_2(x)\int_0^{+\infty} g_2(s)\mathrm{d}s\right)\nabla u\right]\mathrm{d}x \\
&\quad + \int_\Omega \alpha_1(x)\left[\int_0^t g_1(t-s)(u(t)-u(s))\mathrm{d}s\right]\left[\int_0^t g_1(t-s)\mathrm{div}\left(a_1(x)\nabla u(s)\right)\mathrm{d}s\right]\mathrm{d}x \\
&\quad - \int_\Omega \alpha_1(x)\left[\int_0^t g_1(t-s)(u(t)-u(s))\mathrm{d}s\right]\int_0^{+\infty} g_2(s)\mathrm{div}\left(a_2(x)\nabla\eta^t(s)\right)\mathrm{d}s\mathrm{d}x.
\end{aligned}$$

Since $\mathrm{supp}\,\alpha_1 \supset \overline{\Omega\cap w_1} \supset \Gamma_1$ and $u=0$ on Γ_1, then

$$\begin{aligned}
\int_\Omega\left(\alpha_1(x)\int_0^t g_1'(t-s)(u(t)-u(s))\mathrm{d}s\right)^2\mathrm{d}x &= \int_{\mathrm{supp}\,\alpha_1}\left(\alpha_1(x)\int_0^t g_1'(t-s)(u(t)-u(s))\mathrm{d}s\right)^2\mathrm{d}x \\
&\le -c\int_{\mathrm{supp}\,\alpha_1}\int_0^t g_1'(t-s)\,(u(t)-u(s))^2\,\mathrm{d}s.
\end{aligned}$$

Hence, using a version of Poincaré's inequality [14] and (2.3), we obtain

$$\begin{aligned}
\int_\Omega\left(\alpha_1(x)\int_0^t g_1'(t-s)(u(t)-u(s))\mathrm{d}s\right)^2\mathrm{d}x &\le -c\int_{\mathrm{supp}\,\alpha_1}\int_0^t g_1'(t-s)|\nabla u(t)-\nabla u(s)|^2\mathrm{d}s\mathrm{d}x \\
&\le -c\int_{\mathrm{supp}\,\alpha_1} a_1(x)\int_0^t g_1'(t-s)|\nabla u(t)-\nabla u(s)|^2\mathrm{d}s\mathrm{d}x \\
&\le -cg_1'\circ\nabla u.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\psi_1'(t) &\le -\left(\int_0^t g_1(s)\mathrm{d}s-\varepsilon_2\right)\int_\Omega\alpha_1(x)u_t^2\mathrm{d}x - \frac{c}{\varepsilon_2}g_1'\circ\nabla u \\
&\quad + \int_\Omega\alpha_1(x)\left(1-a_2(x)\int_0^{+\infty}g_2(s)\right)\nabla u(t)\cdot\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\
&\quad + \int_\Omega\left(1-a_2(x)\int_0^{+\infty}g_2(s)\right)\nabla\alpha_1\cdot\nabla u(t)\int_0^t g_1(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
&\quad - \int_\Omega a_1\nabla\alpha_1\cdot\left(\int_0^t g_1(t-s)\nabla u(s)\mathrm{d}s\right)\left(\int_0^t g_1(t-s)\,(u(t)-u(s))\,\mathrm{d}s\right)\mathrm{d}x \\
&\quad - \int_\Omega a_1\alpha_1\left(\int_0^t g_1(t-s)\nabla u(s)\mathrm{d}s\right)\cdot\left(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\right)\mathrm{d}x \\
&\quad + \int_\Omega a_2\nabla\alpha_1\cdot\left(\int_0^{+\infty}g_2(s)\nabla\eta^t(s)\mathrm{d}s\right)\left(\int_0^t g_1(t-s)(u(t)-u(s))\mathrm{d}s\right)\mathrm{d}x \\
&\quad + \int_\Omega a_2\alpha_1\left(\int_0^{+\infty}g_2(s)\nabla\eta^t(s)\mathrm{d}s\right)\cdot\left(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\right)\mathrm{d}x.
\end{aligned}$$

By using Young' s inequality, Poincaré's inequality, (2.11) and the fact that $|\nabla\alpha_1(x)|\le ca_1(x)$ (thanks to (2.3)), estimate (3.4) follows. □

Similar computations yield the following:

Lemma 3.3. *Under the assumptions* (G1)–(G4), *the functional*

$$\psi_2(t) = -\int_\Omega\alpha_2(x)u_t\int_0^{+\infty}g_2(s)\eta^t(s)\mathrm{d}s\mathrm{d}x \tag{3.5}$$

satisfies, along the solution of (2.7), (2.8) *and for any* $\varepsilon_2,\varepsilon_3>0$,

$$\psi_2'(t) \le -\left[\int_0^{+\infty} g_2(s)\mathrm{d}s - \varepsilon_2\right]\int_\Omega \alpha_2(x)u_t^2\mathrm{d}x + \frac{\varepsilon_3}{2}\int_\Omega |\nabla u|^2\mathrm{d}x - \frac{c}{\varepsilon_2}g_2' \circ \nabla\eta^t + \frac{c}{\varepsilon_3}\left(g_1 \circ \nabla u + g_2 \circ \nabla\eta^t\right), \quad \forall t \ge 0. \tag{3.6}$$

Now, we state and prove our main result.

Theorem 3.4. *Assume that* (G1)–(G4) *hold. Assume further that, in case of* (2.2)*, there exists* $M_0 > 0$*, for which*

$$\int_\Omega |\nabla u_0(x,s)|^2\mathrm{d}x \le M_0, \quad \forall s > 0. \tag{3.7}$$

Then, there exist positive constants ε_0, c', c'' *such that the solution of* (2.7), (2.8) *satisfies*

$$E(t) \le c'' G_1^{-1}\left(c'\int_0^t \xi(s)\mathrm{d}s\right), \quad \forall t \ge 0, \tag{3.8}$$

where $G_1(t) = \int_t^1 \frac{1}{G_0(s)}\mathrm{d}s$ *and*

$$G_0(t) = \begin{cases} t & \text{if (2.1) holds} \\ tG'(\varepsilon_0 t) & \text{if (2.2) holds.} \end{cases}$$

(We can take $\xi = 1$ *if* $a_1 = 0$*, and* $G_0 = Id$ *if* $a_2 = 0$*).*

Proof. Let $L = NE + M\phi + \psi_1 + \psi_2$, for $M, N > 0$, and let, for $t_0 > 0$ fixed,

$$g_0 = \min\left\{\int_0^{t_0} g_1(s)\mathrm{d}s, \int_0^{+\infty} g_2(s)\mathrm{d}s\right\}.$$

A differentiation of L, using (2.10), (3.2), (3.4), (3.6), leads to

$$L'(t) \le \left(\frac{N}{2} - \frac{c}{\varepsilon_2}\right)(g_1' \circ \nabla u + g_2' \circ \nabla\eta^t) - \int_\Omega [(g_0 - \varepsilon_2)(\alpha_1 + \alpha_2) - M]\,u_t^2\mathrm{d}x \times \left(\frac{c}{\varepsilon_3} + \frac{Mc}{\varepsilon_1}\right)(g_1 \circ \nabla u + g_2 \circ \nabla\eta^t) - [(l - \varepsilon_1)M - \varepsilon_3]\int_\Omega |\nabla u|^2\mathrm{d}x, \quad \forall t \ge t_0. \tag{3.9}$$

By using the fact that $(\alpha_1 + \alpha_2)(x) \ge \frac{d}{2}$ and choosing

$$\varepsilon_1 = \frac{l}{2}, \qquad \varepsilon_2 = \frac{1}{2}g_0, \qquad M = \frac{dg_0}{8}, \qquad \varepsilon_3 = \frac{ldg_0}{32},$$

we obtain, for some $\beta > 0$,

$$L'(t) \le -\beta E(t) + \left(\frac{N}{2} - c\right)(g_1' \circ \nabla u + g_2' \circ \nabla\eta^t) + c(g_1 \circ \nabla u + g_2 \circ \nabla\eta^t), \quad \forall t \ge t_0. \tag{3.10}$$

Then, we choose N large enough so that $\frac{N}{2} - c \ge 0$ and $L \sim E$ since

$$|M\phi + \psi_1 + \psi_2| \le cE.$$

Consequently, we get

$$L'(t) \le -\beta E(t) + c(g_1 \circ \nabla u + g_2 \circ \nabla\eta^t), \quad \forall t \ge t_0. \tag{3.11}$$

To this end, we distinguish two cases to estimate $g_2 \circ \nabla\eta^t$.

Case 1: Condition (2.1) holds. At this point we use (2.10) to get

$$g_2 \circ \nabla\eta^t \le -\frac{1}{\sigma}g_2' \circ \nabla\eta^t \le -\frac{2}{\sigma}E'(t). \tag{3.12}$$

Case 2: Condition (2.2) holds. In this case, following the approach of [31], let G^* be the dual function of the convex function G defined by $G^*(t) = \sup_{s\ge0}\{ts - G(s)\}$, and let $\tau_1, \tau_2 > 0$. By using the fact that $s \mapsto \frac{s}{G^{-1}(s)}$ is non-decreasing then (2.9), (2.10) and (3.7) yield

$$\int_\Omega |\nabla\eta^t(s)|^2\mathrm{d}x \le c,$$

hence, using Young's inequality $ts \le G(t) + G^*(s)$ and the facts that $s \mapsto \frac{s}{G^{-1}(s)}$ and G^* are non-decreasing we obtain

$$\begin{aligned} g_2 \circ \nabla\eta^t &= \frac{1}{\tau_1 G'(\varepsilon_0 E(t))} \int_0^{+\infty} G^{-1}\left(-\tau_2 g_2'(s) \int_\Omega |\nabla\eta^t(s)|^2 \mathrm{d}x\right) \frac{\tau_1 G'(\varepsilon_0 E(t)) g_2(s) \int_\Omega |\nabla\eta^t(s)|^2 \mathrm{d}x}{G^{-1}\left(-\tau_2 g_2'(s) \int_\Omega |\nabla\eta^t(s)|^2 \mathrm{d}x\right)} \mathrm{d}s \\ &\le -\frac{\tau_2}{\tau_1 G'(\varepsilon_0 E(t))} g_2' \circ \nabla\eta^t + \frac{1}{\tau_1 G'(\varepsilon_0 E(t))} \int_0^{+\infty} G^*\left(\frac{c\tau_1 G'(\varepsilon_0 E(t)) g_2(s)}{G^{-1}(-c\tau_2 g_2'(s))}\right) \mathrm{d}s. \end{aligned}$$

By exploiting (2.10) and

$$G^*(s) = s(G')^{-1}(s) - G\left((G')^{-1}(s)\right) \le s(G')^{-1}(s),$$

we get

$$g_2 \circ \nabla\eta^t \le -\frac{2\tau_2}{\tau_1 G'(\varepsilon_0 E(t))} E'(t) + c \int_0^{+\infty} \frac{g_2(s)}{G^{-1}(-c\tau_2 g_2'(s))} (G')^{-1}\left(\frac{c\tau_1 G'(\varepsilon_0 E(t)) g_2(s)}{G^{-1}(-c\tau_2 g_2'(s))}\right) \mathrm{d}s.$$

Choosing $\tau_2 = \frac{1}{c}$ and recalling (2.2), we arrive at

$$g_2 \circ \nabla\eta^t \le -\frac{c}{\tau_1 G'(\varepsilon_0 E(t))} E'(t) + c(G')^{-1}\left(c\tau_1 G'(\varepsilon_0 E(t))\right) \int_0^{+\infty} \frac{g_2(s)}{G^{-1}(-g_2'(s))} \mathrm{d}s.$$

Now, choosing $\tau_1 = \frac{1}{c}$ and using again (2.2), we obtain

$$G'(\varepsilon_0 E(t)) g_2 \circ \nabla\eta^t \le -cE'(t) + c\varepsilon_0 E(t) G'(\varepsilon_0 E(t)). \tag{3.13}$$

Then, we deduce, from (3.12) and (3.13), that

$$\frac{G_0(E(t))}{E(t)} g_2 \circ \nabla\eta^t \le -cE'(t) + c\varepsilon_0 G_0(E(t)), \tag{3.14}$$

where G_0 is defined in Theorem 3.4. Therefore, multiplying (3.11) by $\frac{G_0(E(t))}{E(t)}$, using (3.14) and choosing ε_0 small enough, we arrive at

$$\frac{G_0(E(t))}{E(t)} L'(t) + cE'(t) \le -cG_0(E(t)) + c\frac{G_0(E(t))}{E(t)} g_1 \circ \nabla u, \quad \forall t \ge t_0. \tag{3.15}$$

Let

$$I(t) = \frac{G_0(E(t))}{E(t)} L(t) + cE(t), \quad \forall t \ge 0.$$

By recalling the fact that $t \longmapsto \frac{G_0(E(t))}{E(t)}$ is non-increasing, we deduce that $I \sim E$ and by exploiting (3.15), we conclude that

$$I'(t) \le -cG_0(E(t)) + c\frac{G_0(E(t))}{E(t)} g_1 \circ \nabla u, \quad \forall t \ge t_0. \tag{3.16}$$

To handle the last term of (3.16), following the approach of [18,19], we multiply by $\xi(t)$. Hence, exploiting assumption (G2), (2.10) and the fact that $t \longmapsto \frac{G_0(E(t))}{E(t)}$ is non-increasing, we get

$$\begin{aligned} \xi(t) I'(t) &\le -c\xi(t) G_0(E(t)) + c\frac{G_0(E(t))}{E(t)} \xi(t) g_1 \circ \nabla u \\ &\le -c\xi(t) G_0(E(t)) + c(\xi g_1) \circ \nabla u \\ &\le -c\xi(t) G_0(E(t)) - cg_1' \circ \nabla u \\ &\le -c\xi(t) G_0(E(t)) - cE'(t), \quad \forall t \ge t_0. \end{aligned} \tag{3.17}$$

Finally, we introduce, for $\tau > 0$, the function $F = \tau\,(\xi I + cE)$, which is, clearly, equivalent to E and satisfies, thanks to (3.17) and the non-increasingness of ξ,

$$F'(t) \le -c\tau\xi(t) G_0(E(t)), \quad \forall t \ge t_0. \tag{3.18}$$

We choose $\tau > 0$ small enough so that

$$F \leq E \quad \text{and} \quad G_1\left(F(t_0)\right) \geq c\tau \int_0^{t_0} \xi(s)\mathrm{d}s, \tag{3.19}$$

where G_1 is given in Theorem 3.4. This choice is possible since

$$\lim_{\tau\to 0^+} F(\tau) = 0^+ \quad \text{and} \quad \lim_{s\to 0^+} G_1(s) = +\infty.$$

Therefore (3.18) becomes, for $c' = c\tau$,

$$F'(t) \leq -c'\xi(t)G_0(F(t)), \quad \forall t \geq t_0.$$

Consequently, we obtain

$$(G_1(F(t)))' \geq c'\xi(t), \quad \forall t \geq t_0. \tag{3.20}$$

By integrating (3.20) over $[t_0, t]$, we get

$$G_1(F(t)) \geq c' \int_0^t \xi(s)\mathrm{d}s + G_1(F(t_0)) - c' \int_0^{t_0} \xi(s)\mathrm{d}s, \quad \forall t \geq t_0.$$

Thanks to (3.19), we easily see that

$$G_1(F(t_0)) - c' \int_0^{t_0} \xi(s)\mathrm{d}s \geq 0$$

hence,

$$G_1(F(t)) \geq c' \int_0^t \xi(s)\mathrm{d}s, \quad \forall t \geq t_0.$$

By recalling that G_1 is non-increasing, we easily deduce

$$F(t) \leq G_1^{-1}\left(c' \int_0^t \xi(s)\mathrm{d}s\right), \quad \forall t \geq t_0$$

and by using $F \sim E$ and the boundedness of E, (3.8) is established. □

Remark 3.1. We note that our result cannot be deduced from the results of [9,11] since we require no condition on g''. Besides, our relaxation functions are of more general decay contrary to [9,11] where only the exponential decay has been considered.

4. Examples

In this section, we give two examples to illustrate our general decay estimate (3.8) and show how it generalizes and improves the results known in the literature related to the kernel g_2 (see [29,30]). For more examples concerning past history, see [31].

1. Let $g_2(t) = \frac{d}{(1+t)^q}$, for $q > 1$ and $d > 0$. The classical condition appeared in [29,30]

   $$g_2'(t) \leq -\sigma g_2^p(t), \quad \forall t \geq 0,$$

   where $\sigma > 0$ and $1 \leq p < 3/2$, is not satisfied if $1 < q \leq 2$, while (2.2) always holds with $G(t) = t^{\frac{1}{p}+1}$ and for any $p \in]0, \frac{q-1}{2}[$. In this case, (3.8) takes the form

   $$E(t) \leq \frac{c'}{\left(\int_0^t \xi(s)\mathrm{d}s + 1\right)^p}, \quad \forall t \geq 0,\ \forall p \in \left]0, \frac{q-1}{2}\right[\tag{4.1}$$

2. Let $g_2(t) = de^{-(1+t)^q}$, for $q > 1$ and $d > 0$. Condition (2.1) holds if $q \geq 1$ and condition (2.2) holds for $q \in]0, 1[$ with

   $$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{p}}\mathrm{d}s,$$

   for t near zero and for any $p \in]0, \frac{q}{2}[$. In this case (3.8) becomes, for all $t \geq 0$,

   $$E(t) \leq \begin{cases} c''e^{-c'\left(\int_0^t \xi(s)\mathrm{d}s\right)^p}, & \forall p \in \left]0, \frac{q}{2}\right[\text{ if } q \in]0, 1[\\ c''e^{-c'\int_0^t \xi(s)\mathrm{d}s}, & \text{if } q \geq 1. \end{cases} \tag{4.2}$$

Here, if $q \in]0, 1[$, the decay estimate (4.2) is stronger than the one obtained in [30], where only a polynomial rate was obtained.

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