RESONANT DYNAMICS FOR THE QUINTIC NON LINEAR SCHRÖDINGER EQUATION

by

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Abstract. — We consider the quintic nonlinear Schrödinger equation (NLS) on the circle

$$i\partial_t u + \partial_x^2 u = \pm\nu \ |u|^4 u, \quad \nu \ll 1, \ x \in \mathbb{S}^1, \ t \in \mathbb{R}.$$

We prove that there exist solutions corresponding to an initial datum built on four Fourier modes which form a resonant set (see definition 1.1), which have a non trivial dynamic that involves periodic energy exchanges between the modes initially excited. It is noticeable that this nonlinear phenomena does not depend on the choice of the resonant set.
The dynamical result is obtained by calculating a resonant normal form up to order 10 of the Hamiltonian of the quintic NLS and then by isolating an effective term of order 6. Notice that this phenomena can not occur in the cubic NLS case for which the amplitudes of the Fourier modes are almost actions, i.e. they are almost constant.

Résumé. — Nous considérons l'équation de Schrödinger non linéaire (NLS) quintique sur le cercle

$$i\partial_t u + \partial_x^2 u = \pm\nu\,|u|^4 u, \quad \nu \ll 1,\ x \in \mathbb{S}^1,\ t \in \mathbb{R}.$$

Nous montrons qu'il existe des solutions issues d'une condition initiale construite sur quatre modes de Fourier formant un ensemble résonant (voir définition 1.1) ont une dynamique non triviale mettant en jeu des échanges périodiques d'énergie entre ces quatre modes initialement excités. Il est remarquable que ce phénomène non linéaire soit indépendant du choix de l'ensemble résonant.

Le résultat dynamique est obtenu en mettant d'abord sous forme normale résonante jusqu'à l'ordre 10 l'Hamiltonien de NLS quintique puis en isolant un terme effectif d'ordre 6. Il est à noter que ce phénomène ne peut pas se produire pour NLS cubique pour lequel les amplitudes des modes de Fourier sont des presque-actions et donc ne varient quasiment pas au cours du temps.

1. Introduction

1.1. General introduction. — Denote by $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ the circle, and let $\nu > 0$ be a small parameter. In this paper we are concerned with the following quintic non linear Schrödinger equation

$$(1.1) \qquad \begin{cases} i\partial_t u + \partial_x^2 u = \pm\nu|u|^4 u, & (t,x) \in \mathbb{R} \times \mathbb{S}^1, \\ u(0,x) = u_0(x). \end{cases}$$

If $u_0 \in H^1(\mathbb{S}^1)$, thanks to the conservation of the energy, we show that the equation admits a unique global solution $u \in H^1(S^1)$. In this work we want to describe some particular examples of nonlinear dynamics which can be generated by (1.1).

For the linear Schrödinger equation ($\nu = 0$ in (1.1)) we can compute the solution explicitly in the Fourier basis: Assume that $u_0(x) = \sum_{j \in \mathbb{Z}} \xi_j^0 e^{ijx}$, then $u(t,x) = \sum_{j \in \mathbb{Z}} \xi_j(t) e^{ijx}$ with $\xi_j(t) = \xi_j^0 e^{-ij^2 t}$. In particular, for all $j \in \mathbb{Z}$, the quantity $|\xi_j|$ remains constant. Now, let $\nu > 0$, then a natural question is: do there exist solutions so that the $|\xi_j|$ have a nontrivial dynamic. First we review some known results.

Consider a general Hamiltonian perturbation where we add a linear term and a nonlinear term:

$$(1.2) \qquad i\partial_t u + \partial_x^2 u + V \star u = \nu\partial_{\bar{u}} g(x, u, \bar{u}), \quad x \in \mathbb{S}^1,\ t \in \mathbb{R}$$

where V is a smooth periodic potential and g is analytic and at least of order three. In that case the frequencies are $\omega_j = j^2 + \hat{V}(j)$, $j \in \mathbb{Z}$ where $\hat{V}(j)$ denote the Fourier coefficients of V. Under a non resonant condition on these frequencies, it has been established by D. Bambusi and the first author [2] (see also [6]) that the linear actions $|\xi_j|^2$, $j \in \mathbb{Z}$ are almost invariant during very long time, or more precisely, that for all $N \geq 1$

$$|\xi_j(t)|^2 = |\xi_j(0)|^2 + \mathcal{O}(\nu), \quad \text{for } |t| \leq \nu^{-N}.$$

Therefore in this non resonant case, the dynamics of NLS are very close to the linear dynamics. On the other hand, assuming more restrictive non resonant conditions on the frequencies, it is possible to develop the KAM machinery to equation (1.2). Actually, for the quintic nonlinearity J. Bourgain [3] has shown the existence of invariant tori of full dimension which are close to the invariant tori of the linear part of the equation.

Another very interesting case is the classical cubic NLS

$$(1.3) \qquad i\partial_t u + \partial_x^2 u = \pm \nu |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{S}^1$$

and for this equation again nothing moves:

$$|\xi_j(t)|^2 = |\xi_j(0)|^2 + \mathcal{O}(\nu), \quad \text{for all } t \in \mathbb{R}.$$

This last result is a consequence of the existence of action angle variables (I, θ) for the cubic NLS equation (there are globally defined in the defocusing case and locally defined around the origin in the focusing case, see respectively [7, 8] and [10]) and that the actions are close to the Fourier mode amplitudes to the square: $I_j = |\xi_j|^2(1 + \mathcal{O}(\nu))$.

Thus, in these two examples, the linear actions $|\xi_j|^2$ are almost constant in time, but for different reasons.

Notice that in both previous cases, the Sobolev norms of the solutions, $\left(\sum_{j \in \mathbb{Z}} j^{2s} |\xi_j(t)|^2 \right)^{1/2}$ are almost constant for all $s \geq 0$.

On the other hand, recently C. Villegas-Blas and the first author consider the following cubic NLS equation

$$(1.4) \qquad i\partial_t u + \partial_x^2 u = \pm \nu \cos 2x \, |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{S}^1$$

and prove that this special nonlinearity generates a nonlinear effect: if $u_0(x) = Ae^{ix} + \bar{A}e^{-ix}$ then the modes 1 and -1 exchange energy periodically (see [9]). For instance if $u_0(x) = \cos x + \sin x$, a total beating is proved for $|t| \leq \nu^{-5/4}$:

$$|\xi_1(t)|^2 = \frac{1 \pm \sin 2\nu t}{2} + \mathcal{O}(\nu^{3/4}), \quad |\xi_{-1}(t)|^2 = \frac{1 \mp \sin 2\nu t}{2} + \mathcal{O}(\nu^{3/4}).$$

Of course in (1.4) the interaction between the mode 1 and the mode -1 is induced by the $\cos 2x$ in front of the nonlinearity.

In the present work we consider the quintic NLS equation (1.1). Notice that Liang and You have proved in [**12**] that, in the neighborhood of the origin, there exist many quasi periodic solutions of (1.1). The basic approach is to apply the KAM method and vary the amplitude of the solutions in order to avoid resonances in the spirit of the pioneer work of Kuksin-Pöschel ([**11**]). Here we want to take advantage of the resonances in the linear part of the equation to construct solutions that exchange energy between different Fourier modes.

Formally, by the Duhamel formula

$$u(t) = e^{it\partial_x^2} u_0 - i\nu \int_0^t e^{i(t-s)\partial_x^2} \big(|u|^4 u\big)(s)\mathrm{d}s,$$

and we deduce that $|\xi_j|^2$ cannot move as long as $t \ll \nu^{-1}$. In this paper we prove that for a large class of convenient initial data, certain of the $|\xi_j|^2$ effectively move after a time of order $t \sim \nu^{-1}$.

Definition 1.1. — A set $\mathcal{A}$ of the form

$$\mathcal{A} = \big\{n, n + 3k, n + 4k, n + k\big\}, \quad k \in \mathbb{Z}\backslash\{0\} \text{ and } n \in \mathbb{Z},$$

is called a resonant set. In the sequel we will use the notation

$$a_2 = n, \ \ a_1 = n + 3k, \ \ b_2 = n + 4k, \ \ b_1 = n + k.$$

We are interested in these resonant sets, since they correspond to resonant monomials of order 6 in the normal form of the Hamiltonian (1.1), namely $\xi_{a_1}^2 \xi_{a_2} \bar{\xi}_{b_1}^2 \bar{\xi}_{b_2}$. See Sections 2 and 3 for more details.

Example 1.2. — For $(n, k) = (-2, 1)$, we obtain $(a_2, a_1, b_2, b_1) = (-2, 1, 2, -1)$; for $(n, k) = (-1, 2)$, we obtain $(a_2, a_1, b_2, b_1) = (-1, 5, 7, 1)$.

1.2. The main result. — Our first result is the following:

Theorem 1.3. — *There exist $T > 0$, $\nu_0 > 0$ and a $2T-$periodic function $K_\star : \mathbb{R} \longmapsto]0, 1[$ which satisfies $K_\star(0) \leq 1/4$ and $K_\star(T) \geq 3/4$ so that if $\mathcal{A}$ is a resonant set and if $0 < \nu < \nu_0$, there exists a solution to (1.1) satisfying for all $0 \leq t \leq \nu^{-3/2}$*

$$u(t, x) = \sum_{j \in \mathcal{A}} u_j(t) e^{ijx} + \nu^{1/4} q_1(t, x) + \nu^{3/2} t q_2(t, x),$$

with

$$|u_{a_1}(t)|^2 \ = \ 2|u_{a_2}(t)|^2 \ = \ \quad K_\star(\nu t)$$

$$|u_{b_1}(t)|^2 \ = \ 2|u_{b_2}(t)|^2 \ = \ 1 - K_\star(\nu t),$$

and where for all $s \in \mathbb{R}$, $\|q_1(t, \cdot)\|_{H^s(\mathbb{S}^1)} \leq C_s$ for all $t \in \mathbb{R}_+$, and $\|q_2(t, \cdot)\|_{H^s(\mathbb{S}^1)} \leq C_s$ for all $0 \leq t \leq \nu^{-3/2}$.

Theorem 1.3 shows that there is an exchange between the two modes a_1 and a_2 and the two modes b_1 and b_2. It is remarkable that this nonlinear effect is universal in the sense that this dynamic does not depend on the choice of the resonant set $\mathcal{A}$.

In Section 2, we will see that such a result does not hold for any set $\mathcal{A}$ with $\#\mathcal{A} \leq 3$. However, three modes of a resonant set $\mathcal{A}$ can excite the fourth mode of $\mathcal{A}$ if this one was initially arbitrary small but not zero. More precisely :

Theorem 1.4. — *For all $0 < \gamma < 1/10$, there exist $T_\gamma > 0$, a $2T_\gamma-$periodic function $K_\gamma : \mathbb{R} \longmapsto]0, 1[$ which satisfies $K_\gamma(0) = \gamma$ and $K_\gamma(T_\gamma) \geq 1/10$, and there exists $\nu_0 > 0$ so that if $\mathcal{A}$ is a resonant set and if $0 < \nu < \nu_0$, there exists a solution to (1.1) satisfying for all $0 \leq t \leq \nu^{-3/2}$*

$$u(t, x) = \sum_{j \in \mathcal{A}} u_j(t)e^{ijx} + \nu^{1/4}q_1(t, x) + \nu^{3/2}tq_2(t, x),$$

with
$$|u_{a_1}(t)|^2 = K_\gamma(\nu t); \quad 2|u_{a_2}(t)|^2 = 7 + K_\gamma(\nu t)$$
$$|u_{b_1}(t)|^2 = 1 - K_\gamma(\nu t); \quad 2|u_{b_2}(t)|^2 = 1 - K_\gamma(\nu t),$$

and where for all $s \in \mathbb{R}$, $\|q_1(t, \cdot)\|_{H^s(\mathbb{S}^1)} \leq C_s$ for all $t \in \mathbb{R}_+$, and $\|q_2(t, \cdot)\|_{H^s(\mathbb{S}^1)} \leq C_s$ for all $0 \leq t \leq \nu^{-3/2}$.

Of course the solutions satisfy the three conservation laws : the mass, the momentum and the energy are constant quantities. Denote by $L_j = |u_j|^2$, then we have

• Conservation of the mass: $\int |u|^2$

(1.5) $$L_{a_1} + L_{a_2} + L_{b_1} + I_{b_2} = cst.$$

• Conservation of the momentum: $\text{Im} \int \overline{u}\partial_x u$

(1.6) $$a_1 L_{a_1} + a_2 L_{a_2} + b_1 L_{b_1} + b_2 L_{b_2} = cst.$$

• Conservation of the energy : $\int |\partial_x u|^2 + \frac{\nu}{3}\int |u|^6$

(1.7) $$a_1^2 L_{a_1} + a_2^2 L_{a_2} + b_1^2 L_{b_1} + b_2^2 L_{b_2} = cst.$$

On the other hand, the solutions given by Theorem 1.3 satisfy for $0 \leq t \leq \nu^{-5/4}$ and $s \geq 0$
(1.8)
$$\|u(t, \cdot)\|_{\dot{H}^s}^2 = \frac{K_\star(\nu t)}{2}\left(2|a_1|^{2s} + |a_2|^{2s} - 2|b_1|^{2s} - |b_2|^{2s}\right) + |b_1|^{2s} + \frac{1}{2}|b_2|^{2s} + \mathcal{O}(\nu^{1/4}).$$

Remark that (1.8) for $s = 0, 1$ is compatible with respectively (1.5) and (1.7), since, for these values of s, the coefficient $\left(2|a_1|^{2s} + |a_2|^{2s} - 2|b_1|^{2s} - |b_2|^{2s}\right)$ vanishes for $(a_1, a_2, b_1, b_2) \in \mathcal{A}$.

But for $s \geq 2$, this coefficient is no more zero, except for some symmetric choices of $\mathcal{A}$ like $(-2, 1, 2, -1)$. Thus in the other cases $\|u(t, \cdot)\|_{\dot{H}^s}^2$ is not constant. Actually, a computation shows that, choosing $n = -k$ in the definition of $\mathcal{A}$, the ratio between $\|u(T, \cdot)\|_{H^s}^2$ and $\|u(0, \cdot)\|_{H^s}^2$ is larger than 2 for $s \geq 4$. Very recently, Colliander, Keel, Staffilani, Takaoka and Tao [**4**] have proved a very nice result on the transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation on the 2 dimensional torus. Of course their result is more powerful ; in particular they allow a ratio between the initial H^s-norm and the H^s norm for long time arbitrarily large. On the contrary our result only allows transfers of energy from modes $\{n, n + 3k\}$ to modes $\{n + 4k, n + k\}$ and thus the possibility of growing of the H^s-norm is bounded by c^s for some constant c. Nevertheless our approach is much more simple, it applies in 1-d and it is somehow universal (the dynamics we describe are not at all exceptional).

Remark 1.5. — Consider a resonant set $\mathcal{A}$, and let u be given by Theorem 1.3. Then by the scaling properties of the equation, for all $N \in \mathbb{N}^*$, u_N defined by $u_N(t, x) = N^{\frac{1}{2}} u(N^2 t, Nx)$ is also a solution of (1.1) and we have

$$u_N(t, x) = N^{1/2} \sum_{j \in \mathcal{A}} u_j(N^2 t) \mathrm{e}^{ijNx} + \nu^{1/4} q_1(N^2 t, Nx) + \nu^{3/2} t q_2(N^2 t, Nx).$$

Next, for any $N \in \mathbb{N}^*$, the set $N\mathcal{A}$ is also a resonant set, and thus we can apply Theorem 1.3, which gives the existence of a solution to (1.1) which reads

$$\widetilde{u}_N(t, x) = \sum_{j \in \mathcal{A}} \widetilde{u}_j(t) \mathrm{e}^{ijNx} + \nu^{1/4} \widetilde{q}_1(t, x) + \nu^{3/2} t \widetilde{q}_2(t, x).$$

Observe however that they are not the same.

Theorem 1.3 is obtained by calculating a resonant normal form up to order 10 of the Hamiltonian of the quintic NLS and then by isolating an effective term of order 6. Roughly speaking we obtain in the new variables $H = N + Z^i + Z_6^e + R$ where $N + Z^i$ depends only on the actions, Z_6^e, the effective part, is a polynomial homogeneous of order 6 which depends on one angle and R is a remainder term.
We first prove that, reduced to the resonant set, $N + Z^i + Z_6^e$ generates the nonlinear dynamic that we expect. Then we have to prove that adding the remainder term R and considering all the modes, this nonlinear dynamic persists beyond the local time (here $t \gtrsim \nu^{-1}$). In general this is a hard problem. Nevertheless in our case, the nonlinear dynamic corresponds to a stable orbit around a elliptic equilibrium point. So we explicitly calculate the action-angle variables $(K, \varphi) \in \mathbb{R}^4 \times \mathbb{T}^4$ for the finite dimensional system in such way that our nonlinear dynamics reads $\dot{K} = 0$. Then for the complete system, we obtain

$\dot{K} = O(\nu^{5/2})$ and we are essentially done.

In [**9**], this construction was much more simpler since the finite dimensional nonlinear dynamics was in fact linear (after a change of variable) and linear dynamics are more stable by perturbation than nonlinear ones.

1.3. Plan of the paper. — We begin in Section 2 with some arithmetical preliminaries. In Section 3 we reinterpret equation (1.1) as a Hamiltonian equation and we compute a completely resonant normal form at order 6. In Section 4 we study the equation (the model equation) obtained by the previous normal form after truncation of the error terms. In Section 5 we show that the model equation gives a good approximation of some particular solutions of (1.1).

2. Preliminaries: Arithmetic

We are interested in sets $\mathcal{A}$ of small cardinality so that there exist $(j_1, j_2, j_3, \ell_1, \ell_2, \ell_3) \in \mathcal{A}^6$ satisfying the following resonance condition

$$(2.1) \qquad \begin{cases} j_1^2 + j_2^2 + j_3^2 = \ell_1^2 + \ell_2^2 + \ell_3^2, \\ j_1 + j_2 + j_3 = \ell_1 + \ell_2 + \ell_3, \end{cases} \quad \text{and} \quad \{j_1, j_2, j_3\} \neq \{\ell_1, \ell_2, \ell_3\}.$$

To begin with, let us recall a classical result

Lemma 2.1. — *Assume that* $(j_1, j_2, j_3, \ell_1, \ell_2, \ell_3) \in \mathbb{Z}^6$ *satisfy* (2.1). *Then* $\{j_1, j_2, j_3\} \cap \{\ell_1, \ell_2, \ell_3\} = \emptyset$.

Proof. — If, say $j_1 = \ell_1$, then we have the relation

$$j_2 + j_3 = \ell_2 + \ell_3 \text{ and } j_2^2 + j_3^2 = \ell_2^2 + \ell_3^2,$$

and this implies that $(j_2, j_3) = (l_2, l_3)$ or $(j_2, j_3) = (l_3, l_2)$. Squaring the first equality yields $(j_2 + j_3)^2 = (l_2 + l_3)^2$. To this equality we subtract $j_2^2 + j_3^2 = \ell_2^2 + \ell_3^2$, which implies $j_2 j_3 = \ell_2 \ell_3$. Now compute

$$(\ell_2 - j_2)(\ell_2 - j_3) = \ell_2^2 + j_2 j_3 - j_2 \ell_2 - j_3 \ell_2 = \ell_2(\ell_2 + \ell_3 - j_2 - j_3) = 0,$$

hence the result. $\qquad\square$

Lemma 2.2. — *Assume that there exist integers* $(j_1, j_2, j_3, \ell_1, \ell_2, \ell_3) \in \mathcal{A}^6$ *which satisfy* (2.1). *Then the cardinal of* $\mathcal{A}$ *is greater or equal than 4.*

Proof. — Assume that $\#\mathcal{A} \leq 3$. Then by Lemma 2.1 we can assume that $\mathcal{A} = \{j_1, j_2, \ell_1\}$ and that

$$2j_1 + j_2 = 3\ell_1 ; \ \ 2j_1^2 + j_2^2 = 3\ell_1^2.$$

Let $k \in \mathbb{Z}$ so that $j_1 = \ell_1 + k$, then from the first equation we deduce that $j_2 = \ell_1 - 2k$. Finally, inserting the last relation in the second equation, we deduce that $k = 0$ which implies that $j_1 = j_2 = \ell_1$. □

The next result describes the sets $\mathcal{A}$ of cardinal 4 and which contain non trivial solutions to (2.1). According to definition 1.1, these sets are called resonant sets.

Lemma 2.3 (Description of the resonant sets). — *The resonance sets are the*

$$\mathcal{A} = \big\{n, n+3k, n+4k, n+k\big\}, \quad k \in \mathbb{Z}\backslash\{0\} \ \ and \ n \in \mathbb{Z}.$$

Proof. — By Lemma 2.1, we know that either $\big\{j_1, j_2, j_3\big\} = \big\{\ell_1, \ell_2, \ell_3\big\}$ or $\big\{j_1, j_2, j_3\big\} \cap \big\{\ell_1, \ell_2, \ell_3\big\} = \emptyset$. We consider the second case.
• First we exclude the case $j_1 = j_2 = j_3 = j$. In that case we have to solve

$$(2.2) \qquad \begin{cases} 3j^2 = \ell_1^2 + \ell_2^2 + \ell_3^2, \\ 3j = \ell_1 + \ell_2 + \ell_3. \end{cases}$$

We will show that (2.2) implies $\ell_1 = \ell_2 = \ell_3 = j$. Set $\ell_1 = j + p$ and $\ell_2 = j + q$. Then by the second line $\ell_3 = j - p - q$. Now, we plug in the first line and get $p^2 + q^2 + pq = 0$. This in turn implies that $p = q = 0$ thanks to the inequality $p^2 + q^2 \geq 2|pq|$.
• Then we can assume that $j_2 = j_3$ and $\ell_2 = \ell_3$, and $\sharp\{j_1, j_2, \ell_1, \ell_2\} = 4$. Thus we have to solve

$$\begin{cases} j_1^2 + 2j_2^2 = \ell_1^2 + 2\ell_2^2, \\ j_1 + 2j_2 = \ell_1 + 2\ell_2. \end{cases}$$

From the first line, we infer that $(j_1 - \ell_1)(j_1 + \ell_1) = 2(\ell_2 - j_2)(\ell_2 + j_2)$. The second gives $j_1 - \ell_1 = 2(\ell_2 - j_2)$, thus $j_1 + \ell_1 = j_2 + \ell_2$. Hence we are led to solve the system

$$\begin{cases} \ell_1 - \ell_2 = -j_1 + j_2, \\ \ell_1 + 2\ell_2 = j_1 + 2j_2 \end{cases}$$

where the integers j_1 and j_2 are considered as parameters. The solutions are

$$\ell_1 = \frac{1}{3}(-j_1 + 4j_2), \qquad \ell_2 = \frac{1}{3}(2j_1 + j_2)$$

with the restriction, $j_1 \equiv j_2 \mod 3$, in order to obtain integer solutions. Let $n \in \mathbb{Z}$, $k \in \mathbb{Z}^*$ so that $j_1 = n$ and $j_2 = n + 3k$, the solutions then reads $\ell_1 = n + 4k$ and $\ell_2 = n + k$, as claimed. □

Define the set

$$\mathcal{R} = \{(j_1, j_2, j_3, \ell_1, \ell_2, \ell_3) \in \mathbb{Z}^6 \text{ s.t.}$$
$$j_1 + j_2 + j_3 = \ell_1 + \ell_2 + \ell_3 \text{ and } j_1^2 + j_2^2 + j_3^2 = \ell_1^2 + \ell_2^2 + \ell_3^2\}.$$

The following result will be useful in the sequel

Lemma 2.4. — *Let $(j_1, j_2, j_3, \ell_1, p_1, p_2) \in \mathcal{R}$. Assume that $j_1, j_2, j_3, \ell_1 \in \mathcal{A}$. Then $p_1, p_2 \in \mathcal{A}$.*

Proof. — Let $j_1, j_2, j_3, \ell_1 \in \mathcal{A}$ and $p_1, p_2 \in \mathbb{N}$ so that

$$(2.3) \qquad \begin{cases} p_1 + p_2 = j_1 + j_2 + j_3 - \ell_1, \\ p_1^2 + p_2^2 = j_1^2 + j_2^2 + j_3^2 - \ell_1^2. \end{cases}$$

By Lemma 2.3, there exist $n, k \in \mathbb{Z}$ and $(m_s)_{1 \leq s \leq 4}$ with $m_s \in \{0, 1, 3, 4\}$ so that $j_s = n + m_s k$ and $\ell_1 = n + m_4 k$. We also write $p_1 = n + q_1$ and $p_2 = n + q_2$. We plug these expressions in (2.3) which gives

$$\begin{cases} q_1 + q_2 = (m_1 + m_2 + m_3 - m_4)k, \\ q_1^2 + q_2^2 + 2n(q_1 + q_2) = 2n(m_1 + m_2 + m_3 - m_4)k + (m_1^2 + m_2^2 + m_3^2 - m_4^2)k^2, \end{cases}$$

and is equivalent to

$$\begin{cases} q_1 + q_2 = (m_1 + m_2 + m_3 - m_4)k, \\ q_1^2 + q_2^2 = (m_1^2 + m_2^2 + m_3^2 - m_4^2)k^2. \end{cases}$$

We write $q_1 = r_1 k$ and $q_2 = r_2 k$, then $r_1, r_2 \in \mathbb{Q}$ satisfy

$$(2.4) \qquad \begin{cases} r_1 + r_2 = m_1 + m_2 + m_3 - m_4 := S, \\ r_1^2 + r_2^2 = m_1^2 + m_2^2 + m_3^2 - m_4^2 := T. \end{cases}$$

Next, we observe that indeed $r_1, r_2 \in \mathbb{Z}$: In fact (2.4) is equivalent to

$$(2.5) \qquad r_1 + r_2 = S, \qquad r_1 r_2 = \frac{1}{2}(S^2 - T) := U,$$

($U \in \mathbb{Z}$ since S and T have same parity) and r_1, r_2 are the roots of the polynomial $X^2 - SX + U$. Thus if $r = \alpha/\beta$ with $\alpha \wedge \beta = 1$, we have that $\beta | 1$ and then $r \in \mathbb{Z}$.

We are finally reduced to solve (2.4) where $m_s \in \{0, 1, 3, 4\}$. We list all possible cases in the following array : By symmetry we only need to consider the cases $m_1 \geq m_2 \geq m_3$. We denote by $m_1 m_2 m_3 m_4$ a possible choice and by $T = m_1^2 + m_2^2 + m_3^2 - m_4^2$.

Values of m_s	Value of T	Values of m_s	Value of T	m_s	Value of T
4440	48	4441	47	4443	39
4430	41	4431	40=36+4	4410	33
4413	24	4401	31	4403	23
4330	34=25+9	4331	33	4310	26=25+1
4301	24	4110	18=9+9	4113	9=9+0
4103	8=4+4	4001	15	4003	7
3330	27	3331	26=25+1	3334	11
3310	19	3314	3	3301	17=16+1
3304	2	3110	11	3114	-5
3104	-6	3001	8=4+4	3004	-7
1110	3	1113	-6	1114	-13

In this array, we read all the possible solutions to (2.4) which are (assuming that $m_1 \geq m_2 \geq m_3$ and $r_1 \geq r_2$)
(2.6)
$$(r_1, r_2, m_1, m_2, m_3, m_4) = (3, 3, 4, 1, 1, 0), \quad (3, 0, 4, 1, 1, 3), \quad (4, 1, 3, 3, 0, 1).$$

Now we observe that we always have $r_1, r_2 \in \{0, 1, 3, 4\}$, so that if we come back to (2.3), $p_1 = n + r_1 k$, $p_2 = n + r_2 k$ and $p_1, p_2 \in \mathcal{A}$. □

3. The normal form

3.1. Hamiltonian formulation. — From now, and until the end of the paper, we set $\varepsilon = \nu^{1/4}$. In the sequel, it will be more convenient to deal with small initial conditions to (1.1), thus we make the change of unknown $v = \varepsilon u$ and we obtain

(3.1)
$$\begin{cases} i\partial_t v + \partial_x^2 v = |v|^4 v, & (t, x) \in \mathbb{R} \times \mathbb{S}^1, \\ v(0, x) = v_0(x) = \varepsilon u_0(x). \end{cases}$$

Let us expand v and $\bar{v}$ in Fourier modes:
$$v(x) = \sum_{j \in \mathbb{Z}} \xi_j e^{ijx}, \quad \bar{v}(x) = \sum_{j \in \mathbb{Z}} \eta_j e^{-ijx}.$$

We define
$$P(\xi, \eta) = \frac{1}{3} \int_{\mathbb{S}^1} |v(x)|^6 \mathrm{d}x = \frac{1}{3} \sum_{\substack{j, \ell \in \mathbb{Z}^3 \\ \mathcal{M}(j, \ell) = 0}} \xi_{j_1} \xi_{j_2} \xi_{j_3} \eta_{\ell_1} \eta_{\ell_2} \eta_{\ell_3},$$

where $\mathcal{M}(j, \ell) = j_1 + j_2 + \cdots + j_p - \ell_1 - \ell_2 - \cdots - \ell_p$ denotes the momentum of the multi-index $(j, l) \in \mathbb{Z}^{2p}$ or equivalently the momentum of the monomial $\xi_{j_1} \xi_{j_2} \cdots \xi_{j_p} \eta_{\ell_1} \eta_{\ell_2} \cdots \eta_{\ell_p}$.

In this Fourier setting the equation (3.1) reads as an infinite Hamiltonian system

$$(3.2) \qquad \begin{cases} i\dot{\xi}_j &= j^2\xi_j + \frac{\partial P}{\partial \eta_j} \qquad j \in \mathbb{Z}, \\ -i\dot{\eta}_j &= j^2\eta_j + \frac{\partial P}{\partial \xi_j} \qquad j \in \mathbb{Z}. \end{cases}$$

Since the regularity is not an issue in this work, we will work in the following analytic phase space ($\rho \geq 0$)

$$\mathcal{A}_\rho = \{(\xi, \eta) \in \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \mid \|(\xi, \eta)\|_\rho := \sum_{j \in \mathbb{Z}} e^{\rho|j|}(|\xi_j| + |\eta_j|) < \infty\}$$

which we endow with the canonical symplectic structure $-i\sum_j d\xi_j \wedge \eta_j$. Notice that this Fourier space corresponds to functions $u(z)$ analytic on a strip $|\Im z| < \rho$ around the real axis.

According to this symplectic structure, the Poisson bracket between two functions f and g of (ξ, η) is defined by

$$\{f, g\} = -i \sum_{j \in \mathbb{Z}} \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} - \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j}.$$

In particular, if $(\xi(t), \eta(t))$ is a solution of (3.2) and F is some regular Hamiltonian function, we have

$$\frac{d}{dt} F(\xi(t), \eta(t)) = \{F, H\}(\xi(t), \eta(t))$$

where

$$H = N + P = \sum_{j \in \mathbb{Z}} j^2 \xi_j \eta_j + \frac{1}{3} \sum_{\substack{j,\ell \in \mathbb{Z}^3 \\ \mathcal{M}(j,\ell)=0}} \xi_{j_1} \xi_{j_2} \xi_{j_3} \eta_{\ell_1} \eta_{\ell_2} \eta_{\ell_3},$$

is the total Hamiltonian of the system. It is convenient to work in the symplectic polar coordinates $\left(\xi_j = \sqrt{I_j}e^{i\theta_j}, \eta_j = \sqrt{I_j}e^{-i\theta_j}\right)_{j \in \mathbb{Z}}$. Since we have $d\xi \wedge d\eta = id\theta \wedge dI$, the system (3.1) is equivalent to

$$\begin{cases} \dot{\theta}_j = & -\frac{\partial H}{\partial I_j} \qquad j \in \mathbb{Z}, \\ \dot{I}_j = & \frac{\partial H}{\partial \theta_j} \qquad j \in \mathbb{Z}. \end{cases}$$

Finally, we define

$$(3.3) \qquad J = \sum_{j \in \mathbb{Z}} I_j = \sum_{j \in \mathbb{Z}} \xi_j \eta_j = \|v\|^2_{L^2(\mathbb{S}^1)},$$

which is a constant of motion for (3.1) and (3.2).

3.2. The Birkhoff normal form procedure. — We denote by $B_\rho(r)$ the ball of radius r centred at the origin in $\mathcal{A}_\rho$. Recall the definition

$$\mathcal{R} = \{(j_1, j_2, j_3, \ell_1, \ell_2, \ell_3) \in \mathbb{Z}^6 \ s.t.$$
$$j_1 + j_2 + j_3 = \ell_1 + \ell_2 + \ell_3 \ \text{and} \ j_1^2 + j_2^2 + j_3^2 = \ell_1^2 + \ell_2^2 + \ell_3^2\}$$

and its subset

$$\mathcal{R}_0 = \mathcal{R} \cap \Big\{ \{j_1, j_2, j_3\} = \{\ell_1, \ell_2, \ell_3\} \Big\}.$$

We are now able to state the main result of this section, which is a normal form result at order 10 for the Hamiltonian H.

Proposition 3.1. — *There exists a canonical change of variable τ from $B_\rho(\varepsilon)$ into $B_\rho(2\varepsilon)$ with ε small enough such that*

(3.4) $$\overline{H} := H \circ \tau = N + Z_6 + R_{10},$$

where

(i) N is the term $N(I) = \sum_{j \in \mathbb{Z}} j^2 I_j$;

(ii) Z_6 is the homogeneous polynomial of degree 6

$$Z_6 = \sum_{\mathcal{R}} \xi_{j_1} \xi_{j_2} \xi_{j_3} \eta_{\ell_1} \eta_{\ell_2} \eta_{\ell_3}.$$

(iii) R_{10} is the remainder of order 10, i.e. a Hamiltonian satisfying $\|X_{R_{10}}(z)\|_\rho \le C\|z\|_\rho^9$ for $z = (\xi, \eta) \in B_\rho(\varepsilon)$;

(iv) τ is close to the identity: there exist a constant C_ρ such that $\|\tau(z) - z\|_\rho \le C_\rho \|z\|_\rho^2$ for all $z \in B_\rho(\varepsilon)$.

By abuse of notation, in the proposition and in the sequel, the new variables $(\xi', \eta') = \tau^{-1}(\xi, \eta)$ are still denoted by (ξ, η).

Proof. — For convenience of the reader, we briefly recall the Birkhoff normal form method. Let us search τ as time one flow of χ a polynomial Hamiltonian of order 6,

$$\chi = \sum_{\substack{j, \ell \in \mathbb{Z}^3 \\ \mathcal{M}(j, \ell) = 0}} a_{j, \ell} \ \xi_{j_1} \xi_{j_2} \xi_{j_3} \eta_{\ell_1} \eta_{\ell_2} \eta_{\ell_3}.$$

For any smooth function F, the Taylor expansion of $F \circ \Phi_\chi^t$ between $t = 0$ and $t = 1$ gives

$$F \circ \tau = F + \{F, \chi\} + \frac{1}{2} \int_0^1 (1 - t)\{\{F, \chi\}, \chi\} \circ \Phi_\chi^t \mathrm{d}t.$$

Applying this formula to $H = N + P$ we get

$$H \circ \tau = N + P + \{N, \chi\} + \{P, \chi\} + \frac{1}{2} \int_0^1 (1-t)\{\{H, \chi\}, \chi\} \circ \Phi_\chi^t \mathrm{d}t.$$

Therefore in order to obtain $H \circ \tau = N + Z_6 + R_{10}$ we define

(3.5) $Z_6 = P + \{N, \chi\}$

and

(3.6) $R_{10} = \{P, \chi\} + \frac{1}{2} \int_0^1 (1-t)\{\{H, \chi\}, \chi\} \circ \Phi_\chi^t \mathrm{d}t.$

For $j, \ell \in \mathbb{Z}^3$ we define the associated divisor by

$$\Omega(j, \ell) = j_1^2 + j_2^2 + j_3^2 - \ell_1^2 - \ell_2^2 - \ell_3^2.$$

The homological equation 3.5 is solved by defining

$$\chi := \sum_{\substack{j,\ell \in \mathbb{Z}^3 \\ \mathcal{M}(j,\ell)=0, \Omega(j,\ell)\neq 0}} \frac{1}{i\Omega(j,\ell)} \xi_{j_1} \xi_{j_2} \xi_{j_3} \eta_{\ell_1} \eta_{\ell_2} \eta_{\ell_3}$$

and thus $Z_6 = \displaystyle\sum_{\substack{j,\ell \in \mathbb{Z}^3 \\ \mathcal{M}(j,\ell)=0, \Omega(j,\ell)=0}} \xi_{j_1} \xi_{j_2} \xi_{j_3} \eta_{\ell_1} \eta_{\ell_2} \eta_{\ell_3}$. At this stage we define the

class $\mathcal{P}_p$ of formal polynomial

$$Q = \sum_{\substack{j,l \in \mathbb{Z}^p \\ \mathcal{M}(j,\ell)=0}} a_{j,\ell} \xi_{j_1} \xi_{j_2} \cdots \xi_{j_p} \eta_{\ell_1} \eta_{\ell_2} \cdots \eta_{\ell_p}$$

where the $a_{j\ell}$ form a bounded family and we define $[Q] = \sup_{j,\ell} |a_{j\ell}|$. We recall the following result from [**5**]

Lemma 3.2. — *Let $P \in \mathcal{P}_p$. Then*

(i) *P is well defined and continuous (and thus analytic) on $\mathcal{A}_\rho$ and*

$$|P(\xi, \eta)| \leq [P] \|(\xi, \eta)\|_0^{2p} \leq [P] \|(\xi, \eta)\|_\rho^{2p}.$$

(ii) *The associated vector field X_P is bounded (and thus smooth) from $\mathcal{A}_\rho$ to $\mathcal{A}_\rho$ and*

$$\|X_P(\xi, \eta)\|_\rho \leq 2p[P] \|(\xi, \eta)\|_\rho^{2p-1}.$$

(iii) *Let $Q \in \mathcal{P}_q$ then $\{P, Q\} \in \mathcal{P}_{p+q-2}$ and*

$$[\{P, Q\}] \leq 2qp[P][Q].$$

For convenience of the reader the proof of this lemma is recalled in the appendix A.

By using this Lemma and since there are no small divisors in this resonant case, Z_6 and χ have analytic vector fields on $\mathcal{A}_\rho$. On the other hand, since χ is homogeneous of order 6, for ε sufficiently small, the time one flow generated by χ maps the ball $B_\rho(\varepsilon)$ into the ball $B_\rho(2\varepsilon)$ and is close to the identity in the sense of assertion (iv).

Concerning R_{10}, by construction it is a Hamiltonian function which is of order at least 10. To obtain assertion (iii) it remains to prove that the vector field $X_{R_{10}}$ is smooth from $B_\rho(\varepsilon)$ into $\mathcal{A}_\rho$ in such a way we can Taylor expand $X_{R_{10}}$ at the origin. This is clear for the first term of (3.6): $\{P, \chi\}$ have a smooth vector field as a consequence of Lemma 3.2 assertions (ii) and (iii). For the second term, notice that $\{H, \chi\} = Z_6 - P + \{P, \chi\}$ which is a polynomial on $\mathcal{A}_\rho$ having bounded coefficients and the same is true for $Q = \big\{\{H, \chi\}, \chi\big\}$. Therefore, in view of Lemma 3.2, X_Q is smooth. Now, since for ε small enough Φ_χ^t maps smoothly the ball $B_\rho(\varepsilon)$ into the ball $B_\rho(2\varepsilon)$ for all $0 \le t \le 1$, we conclude that $\int_0^1 (1-t)\{\{H, \chi\}, \chi\} \circ \Phi_\chi^t \mathrm{d}t$ has a smooth vector field. $\qquad\square$

3.3. Description of the resonant normal form. — In this subsection we study the resonant part of the normal form given by Proposition 3.1

$$Z_6 = \sum_{\mathcal{R}} \xi_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_1}\eta_{\ell_2}\eta_{\ell_3}.$$

We have

Proposition 3.3. — *The polynomial Z_6 reads*

(3.7) $$Z_6 = Z_6^i + Z_6^e + Z_{6,2} + Z_{6,3},$$

where

   (i) Z_6^i *is a homogeneous polynomial of degree 6 which only depends on the actions (recall the definition (3.3) of J):*

$$Z_6^i(I) = \sum_{\mathcal{R}_0} \xi_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_1}\eta_{\ell_2}\eta_{\ell_3} = 6J^3 - 9J\sum_{k\in\mathbb{Z}} I_k^2 + 4\sum_{k\in\mathbb{Z}} I_k^3;$$

   (ii) Z_6^e *is the effective Hamiltonian, it is a homogeneous polynomial of degree 6 which involves only modes in the resonant set $\mathcal{A}$:*

$$Z_6^e(\xi, \eta) = 9(\xi_{a_2}\xi_{a_1}^2\eta_{b_2}\eta_{b_1}^2 + \xi_{b_2}\xi_{b_1}^2\eta_{a_2}\eta_{a_1}^2);$$

   (iii) $Z_{6,2}$ *is an homogeneous polynomial of degree 6 which contains all the terms involving exactly two modes which are not in $\mathcal{A}$;*

   (iv) $Z_{6,3}$ *is an homogeneous polynomial of degree 6 which contains all the terms involving at least three modes which are not in $\mathcal{A}$.*

Example 3.4. — Assume that $\mathcal{A} = \{-2, 1, 2, -1\}$. Then we have $Z_6^e(\xi, \eta) = 9(\xi_{-2}\xi_1^2\eta_2\eta_{-1}^2 + \xi_2\xi_{-1}^2\eta_{-2}\eta_1^2)$, and we can compute (see Example (B.1))

$$\begin{aligned} Z_{6,2}(\xi, \eta) &= 36(\xi_3\xi_{-2}\xi_{-1}\eta_{-3}\eta_2\eta_1 + \xi_{-3}\xi_2\xi_1\eta_3\eta_{-2}\eta_{-1}) \\ &\quad + 9(\xi_4\xi_{-2}^2\eta_{-4}\eta_2^2 + \xi_{-4}\xi_2^2\eta_4\eta_{-2}^2). \end{aligned}$$

If $\mathcal{A} = \{-1, 5, 7, 1\}$, the term $Z_{6,2}$ is much more complicated (see Example (B.1)).

Proof. — (*of Proposition 3.3*) A priori, in (3.7) there should also be a polynomial $Z_{6,1}$ composed of the terms involving exactly one mode which is not in $\mathcal{A}$. An important fact of Proposition 3.3 is that $Z_{6,1} = 0$, and this is a consequence of Lemma 2.4.

The specific form of the effective Hamiltonian announced in (ii) follows from the proof of Lemma 2.3.

It remains to compute Z_6^i. This is done in the two following lemmas.

Denote by

$$\mathcal{Q} = \{(j_1, j_2, \ell_1, \ell_2) \in \mathbb{Z}^4 \text{ s.t. } j_1 + j_2 = \ell_1 + \ell_2 \text{ and } j_1^2 + j_2^2 = \ell_1^2 + \ell_2^2\}.$$

Observe that if $(j_1, j_2, \ell_1, \ell_2) \in \mathcal{Q}$, then $\{j_1, j_2\} = \{\ell_1, \ell_2\}$ (see the proof of Lemma 2.1). Next, we can state

Lemma 3.5. — *The two following identities hold true*

$$(3.8) \qquad Z_4(I) := \sum_{(j_1, j_2, \ell_1, \ell_2) \in \mathcal{Q}} \xi_{j_1}\xi_{j_2}\eta_{\ell_1}\eta_{\ell_2} = 2J^2 - \sum_{j \in \mathbb{Z}} I_j^2,$$

$$(3.9) \qquad W_4^{(k)}(I) := \sum_{(j_1, \ell_2, \ell_3) \in \Omega^{(k)}} \xi_k\xi_{j_1}\eta_{\ell_2}\eta_{\ell_3} = 2I_k(J - I_k),$$

where $\Omega^{(k)} = \{(j_1, \ell_2, \ell_3) \in \mathbb{Z}^3 \text{ s.t. } (k, j_1, \ell_2, \ell_3) \in \mathcal{Q} \text{ and } j_1 \neq k\}$.

Proof. — First we prove (3.8). Thanks to the previous remark and the fact that $\xi_j\eta_j = I_j$, we have

$$\begin{aligned} Z_4(I) &= \sum_{\mathcal{Q}, j_1 = \ell_1} \xi_{j_1}\xi_{j_2}\eta_{\ell_1}\eta_{\ell_2} + \sum_{\mathcal{Q}, j_1 \neq \ell_1} \xi_{j_1}\xi_{j_2}\eta_{\ell_1}\eta_{\ell_2} \\ &= \sum_{(j_1, j_2) \in \mathbb{Z}^2} I_{j_1}I_{j_2} + \sum_{\substack{(j_1, j_2) \in \mathbb{Z}^2 \\ j_1 \neq j_2}} I_{j_1}I_{j_2} \\ &= 2\Big(\sum_{j \in \mathbb{Z}} I_j\Big)^2 - \sum_{j \in \mathbb{Z}} I_j^2 = 2J^2 - \sum_{j \in \mathbb{Z}} I_j^2, \end{aligned}$$

which was the claim.

We now turn to (3.9). Again we split the sum in two

$$
\begin{aligned}
W_4^{(k)}(I) &= \sum_{\substack{(j_1,\ell_2,\ell_3)\in\Omega^{(k)} \\ j_1=\ell_2}} \xi_k\xi_{j_1}\eta_{\ell_2}\eta_{\ell_3} + \sum_{\substack{(j_1,\ell_2,\ell_3)\in\Omega^{(k)} \\ j_1\neq\ell_2}} \xi_k\xi_{j_1}\eta_{\ell_2}\eta_{\ell_3} \\
&= I_k \sum_{j_1\in\mathbb{Z}\setminus\{k\}} I_{j_1} + I_k \sum_{j_1\in\mathbb{Z}\setminus\{k\}} I_{j_1} \\
&= 2I_k(J-I_k),
\end{aligned}
$$

hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.6. — *The following identity holds true*

$$
Z_6^i(I) := \sum_{\mathcal{R}_0} \xi_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_1}\eta_{\ell_2}\eta_{\ell_3} = 6J^3 - 9J\sum_{k\in\mathbb{Z}} I_k^2 + 4\sum_{k\in\mathbb{Z}} I_k^3.
$$

Proof. — First we split the sum into three parts

$$
\begin{aligned}
Z_6^i(I) &= \sum_{\mathcal{R}_0, j_1=\ell_1} \xi_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_1}\eta_{\ell_2}\eta_{\ell_3} + \sum_{\mathcal{R}_0, j_1\neq\ell_1, j_2=\ell_1} \xi_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_1}\eta_{\ell_2}\eta_{\ell_3} \\
&\quad + \sum_{\substack{\mathcal{R}_0, j_1\neq\ell_1, \\ j_2\neq\ell_1, j_3=\ell_1}} \xi_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_1}\eta_{\ell_2}\eta_{\ell_3} := \Sigma_1 + \Sigma_2 + \Sigma_3.
\end{aligned}
$$

For the first sum, we use (3.8) to write

$$
(3.10) \qquad \Sigma_1 = \sum_{\substack{(j_2,j_3,\ell_2,\ell_3)\in\mathcal{Q} \\ j_1\in\mathbb{Z}}} I_{j_1}\xi_{j_2}\xi_{j_3}\eta_{\ell_2}\eta_{\ell_3} = JZ_4(I) = 2J^3 - J\sum_{k\in\mathbb{Z}} I_k^2.
$$

Now we deal with the sum Σ_3. Denote by

$$
\mathcal{Q}^{(k)} = \big\{(j_1,j_2,\ell_1,\ell_2)\in\big(\mathbb{Z}\setminus\{k\}\big)^4 \text{ s.t. } j_1+j_2=\ell_1+\ell_2 \ \text{ and } \ j_1^2+j_2^2=\ell_1^2+\ell_2^2\big\},
$$

then from (3.8) we deduce that

$$
Z_4^{(k)}(I) := \sum_{(j_1,j_2,\ell_1,\ell_2)\in\mathcal{Q}^{(k)}} \xi_{j_1}\xi_{j_2}\eta_{\ell_1}\eta_{\ell_2} = 2(J-I_k)^2 - \sum_{j\in\mathbb{Z}} I_j^2 + I_k^2.
$$

Therefore by the previous equality

$$
\begin{aligned}
\Sigma_3 &= \sum_{\substack{(j_1,j_2,\ell_2,\ell_3)\in\mathcal{Q}^{(\ell_1)} \\ \ell_1\in\mathbb{Z}}} I_{\ell_1}\xi_{j_1}\xi_{j_2}\eta_{\ell_2}\eta_{\ell_3} = \sum_{k\in\mathbb{Z}} I_k Z_4^{(k)}(I) \\
&= \sum_{k\in\mathbb{Z}} I_k\Big(2J^2 - 4JI_k + 2I_k^2 - \sum_{j\in\mathbb{Z}} I_j^2 + I_k^2\Big) \\
(3.11) \qquad\qquad &= 2J^3 - 5J\sum_{k\in\mathbb{Z}} I_k^2 + 3\sum_{k\in\mathbb{Z}} I_k^3.
\end{aligned}
$$

Now we consider Σ_2. By (3.9) and (3.11)

$$
\begin{aligned}
\Sigma_2 &= \sum_{\mathcal{R}_0, j_1 \neq j_2} I_{j_2} \xi_{j_1} \xi_{j_3} \eta_{\ell_2} \eta_{\ell_3} \\
&= \sum_{\mathcal{R}_0, j_1 \neq j_2, j_3 \neq j_2} I_{j_2} \xi_{j_1} \xi_{j_3} \eta_{\ell_2} \eta_{\ell_3} + \sum_{\mathcal{R}_0, j_1 \neq j_2, j_3 = j_2} I_{j_2} \xi_{j_1} \xi_{j_2} \eta_{\ell_2} \eta_{\ell_3} \\
&= \Sigma_3 + \sum_{j_2 \in \mathbb{Z}} I_{j_2} W^{j_2}(I) \\
&= 2J^3 - 5J \sum_{k \in \mathbb{Z}} I_k^2 + 3 \sum_{k \in \mathbb{Z}} I_k^3 + 2J \sum_{k \in \mathbb{Z}} I_k^2 - 2 \sum_{k \in \mathbb{Z}} I_k^3 \\
(3.12) \qquad &= 2J^3 - 3J \sum_{k \in \mathbb{Z}} I_k^2 + \sum_{k \in \mathbb{Z}} I_k^3.
\end{aligned}
$$

Finally, (3.10), (3.11) and (3.12) yield the result. $\qquad \square$

4. The model equation

We want to describe the dynamic of a solution to (3.2) so that $\xi_j^0 = \eta_j^0 = 0$ when $j \notin \mathcal{A}$. In view of the result of Propositions 3.1 and 3.3 we hope that such a solution will be close to the solution (with same initial condition) of the Hamiltonian flow of $N + Z_6^i + Z_6^e$ reduced to the four modes of the resonant set, i.e.

$$(4.1) \quad \widehat{H} = \sum_{j \in \mathcal{A}} j^2 I_j + 6J^3 - 9J \sum_{k \in \mathcal{A}} I_k^2 + 4 \sum_{k \in \mathcal{A}} I_k^3 + 18 I_{a_2}^{1/2} I_{b_2}^{1/2} I_{a_1} I_{b_1} \cos(2\varphi_0),$$

with $\varphi_0 = \theta_{a_1} - \theta_{b_1} + \frac{1}{2}\theta_{a_2} - \frac{1}{2}\theta_{b_2}$.

The Hamiltonian system associated to $\widehat{H}$ is defined on the phase space $\mathbb{T}^4 \times \mathbb{R}^4 \ni (\theta_{a_1}, \theta_{a_2}, \theta_{a_3}, \theta_{a_4}; I_{a_1}, I_{a_2}, I_{a_3}, I_{a_4})$ by

$$(4.2) \qquad
\begin{cases}
\dot{\theta}_{a_j} = -\frac{\partial \widehat{H}}{\partial I_{a_j}} & j = 1, 2, \\[2mm]
\dot{I}_{a_j} = \frac{\partial \widehat{H}}{\partial \theta_{a_j}} & j = 1, 2, \\[2mm]
\dot{\theta}_{b_j} = -\frac{\partial \widehat{H}}{\partial I_{b_j}} & j = 1, 2, \\[2mm]
\dot{I}_{b_j} = \frac{\partial \widehat{H}}{\partial \theta_{b_j}} & j = 1, 2,
\end{cases}
$$

This finite dimensional system turns out to be completely integrable.

Lemma 4.1. — *The system (4.2) is completely integrable.*

Proof. — It is straightforward to check that

$$K_1 = I_{a_1} + I_{b_1}, \quad K_2 = I_{a_2} + I_{b_2} \text{ and } K_{1/2} = I_{b_2} + \frac{1}{2} I_{a_1},$$

are constants of motion. Furthermore we verify

$$\{ K_1, \widehat{H} \} = \{ K_2, \widehat{H} \} = \{ K_{1/2}, \widehat{H} \} = 0,$$

as well as

$$\{ K_1, K_2 \} = \{ K_2, K_{1/2} \} = \{ K_{1/2}, K_1 \} = 0.$$

Moreover the previous quantities are independent. So $\widehat{H}$ admits four integrals of motions that are independent and in involution and thus $\widehat{H}$ is completely integrable. □

4.1. Action angle variables for $\widehat{H}$. — In this section we construct action angle variables for $\widehat{H}$ in two particular regimes corresponding to two particular set of initial data.

We begin with a partial construction common to both cases. The previous considerations suggest that we make the following symplectic change of variables: Denote by

$$\theta = {}^t(\theta_{a_1}, \theta_{b_1}, \theta_{b_2}, \theta_{a_2}), \quad I = {}^t(I_{a_1}, I_{b_1}, I_{b_2}, I_{a_2}).$$

Then we define the new variables

$$\varphi = {}^t(\varphi_0, \varphi_1, \varphi_2, \varphi_{1/2}), \quad K = {}^t(K_0, K_1, K_2, K_{1/2}),$$

by the linear transform

(4.3)
$$\begin{pmatrix} \varphi \\ K \end{pmatrix} = \begin{pmatrix} {}^tB^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \theta \\ I \end{pmatrix},$$

where the matrix B is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{1}{2} & 0 & 1 & 0 \end{pmatrix} \quad \text{and thus} \quad {}^tB^{-1} = \begin{pmatrix} 1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

In the new variables (4.2) reads

(4.4)
$$\begin{cases} \dot{\varphi}_0 = -\frac{\partial \widehat{H}}{\partial K_0} \\ \dot{K}_0 = \frac{\partial \widehat{H}}{\partial \varphi_0} \end{cases}, \quad \begin{cases} \dot{\varphi}_j = -\frac{\partial \widehat{H}}{\partial K_j} \\ \dot{K}_j = 0 \end{cases}, \quad \text{for} \quad j = 1, 2, 3.$$

In the sequel, we will need the explicit expression of $\widehat{H}$ in these new coordinates. Observe that for $j = 1, 2$ we have

$$I_{a_j}^2 + I_{b_j}^2 = K_j^2 - 2I_{a_j}I_{b_j} \text{ and } I_{a_j}^3 + I_{b_j}^3 = K_j(K_j^2 - 3I_{a_j}I_{b_j}),$$

then if we introduce the notation

$$F(K_1, K_2) = K_1 + 4K_2 + (K_1 + K_2)(K_1^2 + K_2^2 + 8K_1K_2),$$

the Hamiltonian $\widehat{H}$ reads

(4.5) $\widehat{H} = \widehat{H}(\varphi_0, K_0, K_1, K_2, K_{1/2})$

$= F(K_1, K_2) + 6\big[(K_1 + 3K_2)I_{a_1}I_{b_1} + (K_2 + 3K_1)I_{a_2}I_{b_2} + 3I_{a_2}^{\frac{1}{2}}I_{b_2}^{\frac{1}{2}}I_{a_1}I_{b_1}\cos(2\varphi_0)\big],$

where

$$I_{a_1} = K_0, \;\; I_{b_1} = K_1 - K_0, \;\; I_{b_2} = K_{1/2} - \frac{1}{2}K_0, \;\; I_{a_2} = K_2 - K_{1/2} + \frac{1}{2}K_0.$$

We now want to exhibit some particular trajectories (φ_0, K_0), actually periodic orbits around stable equilibrium. For that we particularise the coefficients K_j for $j \neq 0$.

Let $A \geq 1/2$. We set $K_1 = \varepsilon^2$, $K_2 = A\varepsilon^2$ and $K_{1/2} = \frac{1}{2}\varepsilon^2$, and we denote by

$$\widehat{H}_0(\varphi_0, K_0) := \widehat{H}(\varphi_0, K_0, \varepsilon^2, A\varepsilon^2, \frac{1}{2}\varepsilon^2).$$

The evolution of (φ_0, K_0) is given by

$$\begin{cases} \dot{\varphi}_0 = & -\frac{\partial \widehat{H}_0}{\partial K_0} \\ \dot{K}_0 = & \frac{\partial \widehat{H}_0}{\partial \varphi_0}. \end{cases}$$

Then, we make the change of unknown

$$\varphi_0(t) = \varphi(\varepsilon^4 t) \quad \text{and} \quad K_0(t) = \varepsilon^2 K(\varepsilon^4 t).$$

An elementary computation shows that, the evolution of (φ, K) is given by

$$\begin{cases} \dot{\varphi} = & -\frac{\partial H_\star}{\partial K} \\ \dot{K} = & \frac{\partial H_\star}{\partial \varphi}. \end{cases}$$

where

(4.6) $H_\star = H_\star(\varphi, K) =$

$$\frac{3}{2}(1 - K)\Big[(A + 3)(2A - 1) + (7 + 13A)K + 6(1 - K)^{\frac{1}{2}}(2A - 1 + K)^{\frac{1}{2}}K\cos(2\varphi)\Big].$$

4.1.1. First regime: $A = 1/2$. — In that case we have

$$H_\star = H_\star(\varphi, K) = \frac{9}{4}K(1 - K)\Big[9 + 4K^{\frac{1}{2}}(1 - K)^{\frac{1}{2}}\cos(2\varphi)\Big],$$

and the evolution of (φ, K) is given by

(4.7) $$\begin{cases} \dot{\varphi} = -\frac{27}{4}(1 - 2K)\Big[3 + 2K^{\frac{1}{2}}(1 - K)^{\frac{1}{2}}\cos(2\varphi)\Big] \\ \dot{K} = -18K^{\frac{3}{2}}(1 - K)^{\frac{3}{2}}\sin(2\varphi). \end{cases}$$

The dynamical system (4.7) is of pendulum type. Let us define

(4.8) $$\kappa_\star = \frac{1}{2} - \frac{1}{8}\Big[2(7\sqrt{105} - 69)\Big]^{1/2} \approx 0.208...,$$

we have

Proposition 4.2. — *Let $\kappa_\star$ be given by (4.8). If $\kappa_\star < K(0) < 1 - \kappa_\star$ and $\varphi(0) = 0$, then there is $T > 0$ so that (φ, K) is a $2T-$periodic solution of (4.7) and*

$$K(0) + K(T) = 1.$$

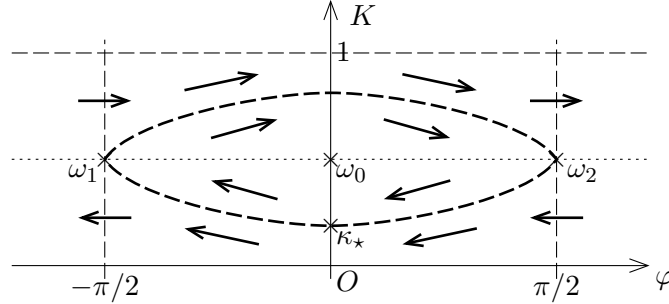We denote by $(\varphi_\star, K_\star)$ such a trajectory.

Figure 1 : The phase portrait of system (4.7)

Proof. — The line $K = 0$ and $K = 1$ are barriers and the phase portrait is π-periodic in φ so we restrict our study to the region $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, $0 < K < 1$. In this domain, there are exactly three equilibrium points : $\omega_0 = (0, 1/2)$ which is a centre and $\omega_1 = (-\pi/2, 1/2)$ and $\omega_2 = (\pi/2, 1/2)$ which are saddle points. The level set $H_\star(\varphi, K) = H_\star(\omega_1) = H_\star(\omega_2) = 63/16$, which corresponds to the equation

$$K(1 - K)\big(9 + 4K^{\frac{1}{2}}(1 - K)^{\frac{1}{2}}\cos(2\varphi)\big) = \frac{7}{4},$$

defines two heteroclinic orbits which link the points ω_1 and ω_2 : $\mathcal{C}_1$ in the region $\{K < 1/2\}$ and $\mathcal{C}_2$ in the region $\{K > 1/2\}$ (see the dashed curves in Figures 1&2). Moreover, we can explicitly compute the intersection $(0, \kappa_\star)$ of the curve $\mathcal{C}_1$ with the $K-$axis, and we obtain (4.8).

Let $\widetilde{U} \subset]-\pi/2, \pi/2[\times]\kappa_\star, 1 - \kappa_\star[\backslash\{\omega_0\}$ be the open domain delimited by the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ minus the point ω_0. Any solution issued from a point inside $\widetilde{U}$ is periodic and turns around the centre ω_0. Furthermore, let $2T$ be the period, by symmetry we have $(\varphi(T), K(T)) = (0, 1 - K(0))$.
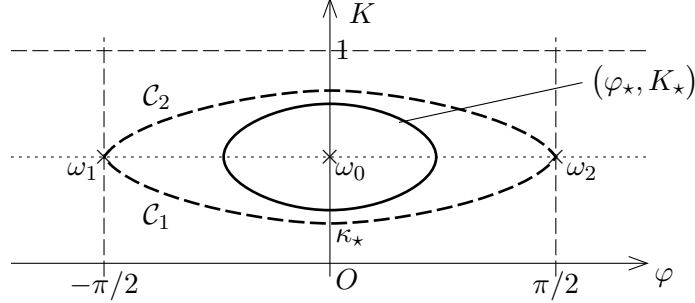
□

Figure 2 : An example of trajectory $(\varphi_\star, K_\star)$

By applying the Arnold-Liouville theorem (see e.g. [**1**]) inside $\widetilde{U}$ we obtain

Lemma 4.3. — *Let $U \subset\subset \widetilde{U}$, then there exists a symplectic change of variables $\Phi : U \ni (K, \varphi) \longmapsto (L, \alpha) \in \mathbb{R}_{>0} \times \mathbb{S}^1$ which defines action angle coordinates for* (4.7) *i.e.,* (4.7) *is equivalent to the system*

$$\dot{L} = -\frac{\partial H_\star}{\partial \alpha} = 0, \quad \dot{\alpha} = \frac{\partial H_\star}{\partial L}.$$

Moreover Φ is a $\mathcal{C}^1$-diffeomorphism, and there exists $C > 0$ depending on U so that

$$\|d\Phi\| \leq C, \quad \|d\Phi^{-1}\| \leq C.$$

4.1.2. Second regime: A=4. — In that case we obtain

$$H_\star = \frac{3}{2}(1 - K)\Big[49 + 59K + 6(1 - K)^{\frac{1}{2}}(7 + K)^{\frac{1}{2}}K\cos(2\varphi)\Big],$$

and the evolution of (φ, K) is given by
(4.9)
$$\begin{cases} \dot{\varphi} = & 3\Big[59K - 5 - 3(K + 7)^{-\frac{1}{2}}(1 - K)^{\frac{1}{2}}(-3K^2 - 16K + 7)\cos(2\varphi)\Big] \\ \dot{K} = & -18(1 - K)^{\frac{3}{2}}(7 + K)^{\frac{1}{2}}K\sin(2\varphi). \end{cases}$$

Proposition 4.4. —
Let $\gamma > 0$ arbitrary small, and set $(\varphi(0), K(0)) = (0, \gamma)$. Then there is $T_\gamma > 0$ so that (φ, K) is $2T_\gamma-$periodic and

$$K(T_\gamma) > \frac{1}{10}.$$

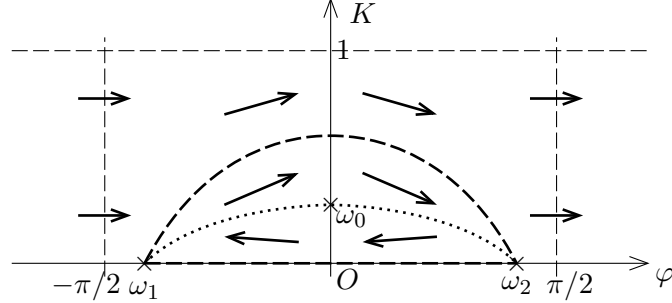We denote by $(\varphi_\star, K_\star)$ such a trajectory.

Figure 3 : The phase portrait of system (4.9)

Proof. — We restrict our study to the region $0 \leq \varphi \leq \frac{\pi}{2}$, $0 < K < 1$. First, we study the sign of $\dot{\varphi}$. To begin with, observe that $\dot{\varphi}$ has exactly the sign of $f(K) - \cos(2\varphi)$ where

$$f(K) = \frac{1}{3}(59K - 5)(K + 7)^{\frac{1}{2}}(1 - K)^{-\frac{1}{2}}(-3K^2 - 16K + 7)^{-1}.$$

We verify that there exists $1/10 < \kappa_0 < 1/5$ so that the function f is increasing and one to one $f : [0, \kappa_0] \longrightarrow [-5\sqrt{7}/21, 1]$. Thus, the curve $\mathcal{C}_0 := \{\dot{\varphi} = 0\}$ can be expressed as a decreasing function $K(\varphi) = f^{-1}\big(\cos(2\varphi)\big)$.

Thanks to this study, and the expression of $\dot{K}$, we deduce that the phase portrait has exactly three equilibrium points : $\omega_0 = (0, \kappa_0)$ which is a centre and $\omega_1 = (-\varphi_0, 0)$ and $\omega_2 = (\varphi_0, 0)$ which are saddle points (here $0 < \varphi_0 < \pi/2$ is defined by the equation $f(0) = -5\sqrt{7}/21 = \cos(2\varphi_0)$). The level set $H_\star(\varphi, K) = H_\star(\omega_1) = H_\star(\omega_2) = \frac{3}{2} \cdot 49$, which is defined by the equation

$$10 - 59K + 6(1 - K)^{\frac{3}{2}}(7 + K)^{\frac{1}{2}} \cos(2\varphi) = 0,$$

defines two heteroclinic orbits $\mathcal{C}_1 := \{K = 0\}$ and $\mathcal{C}_2$ that link the two saddle points (see the dashed curves in Figures 3&4).

Let $\widetilde{U}_2 \subset\,]-\pi/2, \pi/2[\times]0, 1[\backslash\{\omega_0\}$ be the open domain delimited by the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ minus the point ω_0. Any solution issued from a point inside $\widetilde{U}_2$ is periodic and turns around the centre ω_0. Furthermore, let $2T$ be the period, by symmetry we have $\varphi(T) = 0$ and $K(T) > \kappa_0$.
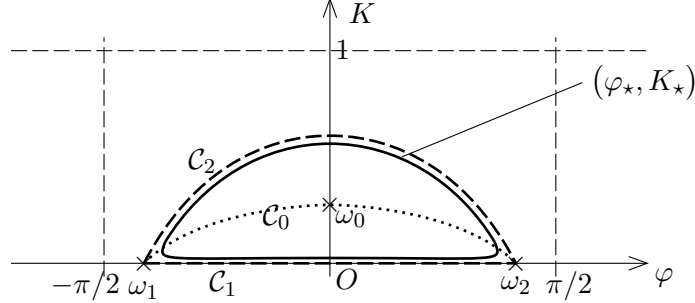
$\square$

Figure 4 : An example of trajectory $(\varphi_\star, K_\star)$

As in the first case, applying the Arnold-Liouville theorem inside $\widetilde{U}$ we obtain

Lemma 4.5. — *Let $U \subset\subset \widetilde{U}_2$, then there exists a symplectic change of variables $\Phi : U \ni (K, \varphi) \longmapsto (L, \alpha) \in \mathbb{R}_{>0} \times \mathbb{S}^1$ which defines action angle coordinates for* (4.9) *i.e.,* (4.9) *is equivalent to the system*

$$\dot{L} = -\frac{\partial H_\star}{\partial \alpha} = 0, \quad \dot{\alpha} = \frac{\partial H_\star}{\partial L}.$$

Moreover Φ is a $\mathcal{C}^1$-diffeomorphism, and there exists $C > 0$ depending on U so that

$$\|d\Phi\| \leq C, \quad \|d\Phi^{-1}\| \leq C.$$

4.1.3. On the other cases.— More generally, we can consider the case $K_1 = \varepsilon^2$, $K_2 = A\varepsilon^2$ and $K_{1/2} = B\varepsilon^2$, where the constants $A, B > 0$ satisfy the natural conditions $A \geq B$ and $B \geq 1/2$. Roughly speaking, the mechanism is the following. Consider the curve $\mathcal{C}_0 = \{\dot{\varphi} = 0\}$. If $\mathcal{C}_0$ has no intersection with $\{K = 0\}$ and $\{K = 1\}$, then the dynamic is essentially the one of $A = 1/2$, $B = 1/2$. On the contrary, if $\mathcal{C}_0$ has two intersections with $\{K = 0\}$ or $\{K = 1\}$, then the dynamic is essentially the one of $A = 4$, $B = 1/2$.

5. Proof of Theorems 1.3 and 1.4

Consider the Hamiltonian $\overline{H}$ given by (3.4), which is a function of $(\xi_j, \eta_j)_{j\in\mathbb{Z}}$. We make the linear change of variables given by (4.3) (the variables ξ_j, η_j remain unchanged for $j \notin \mathcal{A}$). In the sequel, the Hamiltonian in the new variables is still denoted by $\overline{H}$. Then $\overline{H}$ induces the system

$$(5.1) \qquad \begin{cases} \dot{\varphi}_j = -\frac{\partial \overline{H}}{\partial K_j} \\ \dot{K}_j = \frac{\partial \overline{H}}{\partial \varphi_j} \end{cases}, \qquad \begin{cases} i\dot{\xi}_p = \frac{\partial \overline{H}}{\partial \eta_p} \\ i\dot{\eta}_p = -\frac{\partial \overline{H}}{\partial \xi_p} \end{cases}, p \notin \mathcal{A}.$$

Next, we take some initial conditions to (5.1) which will be close to the initial conditions chosen for (4.4).

Observe that the K_j's aren't constants of motion of (5.1). However, they are almost preserved, and this is the result of the next lemma. Recall that $\mathcal{A} = \{a_2, a_1, b_2, b_1\}$,

$$K_1 = I_{a_1} + I_{b_1}, \quad K_2 = I_{a_2} + I_{b_2}, \quad \text{and} \quad K_{1/2} = I_{b_2} + \frac{1}{2} I_{a_1},$$

and recall the notations of Proposition (3.3).

Lemma 5.1. — *Assume that*

(5.2) $\xi_j(0), \eta_j(0) = \mathcal{O}(\varepsilon), \ \forall\, j \in \mathcal{A} \ \text{ and } \ \xi_p(0), \eta_p(0) = \mathcal{O}(\varepsilon^3), \ \forall\, p \notin \mathcal{A}.$

Then for all $0 \leq t \leq C\varepsilon^{-6}$,

(5.3) $I_p(t) = \mathcal{O}(\varepsilon^6) \ \text{ when } \ p \notin \mathcal{A},$

and

(5.4) $K_1(t) \quad = \quad K_1(0) + \mathcal{O}(\varepsilon^{10}) t$

(5.5) $K_2(t) \quad = \quad K_2(0) + \mathcal{O}(\varepsilon^{10}) t$

(5.6) $K_{1/2}(t) \quad = \quad K_{1/2}(0) + \mathcal{O}(\varepsilon^{10}) t.$

Proof. — We first remark that by the preservation of the L^2 norm in the NLS equation, we have

$$\sum_{p \in \mathbb{Z}} I_p(t) = \sum_{p \in \mathbb{Z}} I_p(0) \quad \text{ for all } t \in \mathbb{R},$$

and therefore by using (5.2)

$$I_p(t) = \mathcal{O}(\varepsilon^2) \quad \text{for all } p \in \mathbb{Z} \text{ and for all } t.$$

On the other hand by Propositions 3.1 and 3.3, we have for $p \in \mathbb{Z}$

(5.7) $\dot{I}_p = \{ I_p, \overline{H} \} = \{ I_p, Z_6^e \} + \{ I_p, Z_{6,2} \} + \{ I_p, Z_{6,3} \} + \{ I_p, R_{10} \}.$

• To prove (5.3), we first verify that for $p \notin \mathcal{A}$, $\{ I_p, Z_6^e \} = 0$. Then, we remark that, as a consequence of Lemma 2.4, all the monomials appearing in $Z_{6,2}$ have the form

$$\xi_{j_1} \xi_{j_2} \xi_{p_1} \eta_{\ell_1} \eta_{\ell_2} \eta_{p_2} \quad \text{or} \quad \xi_{\ell_1} \xi_{\ell_2} \xi_{p_2} \eta_{j_1} \eta_{j_2} \eta_{p_1}$$

where $(j_1, j_2, p_1, \ell_1, \ell_2, p_2) \in \mathcal{R}$, $j_1, j_2, \ell_1, \ell_2 \in \mathcal{A}$ and $p_1, p_2 \notin \mathcal{A}$. Furthermore, by straightforward computation,

(5.8) $\{ I_{p_1} + I_{p_2}, \xi_{j_1} \xi_{j_2} \xi_{p_1} \eta_{\ell_1} \eta_{\ell_2} \eta_{p_2} \} = \{ I_{p_1} + I_{p_2}, \xi_{\ell_1} \xi_{\ell_2} \xi_{p_2} \eta_{j_1} \eta_{j_2} \eta_{p_1} \} = 0.$

Then we define an equivalence relation : Let $p, \widetilde{p} \notin \mathcal{A}$. We say that p and $\widetilde{p}$ are linked and write $p \leftrightarrow \widetilde{p}$ if there exist $k \in \mathbb{N}^*$, a sequence $(q^{(i)})_{1 \leq i \leq k} \notin \mathcal{A}$ so that $q^{(1)} = p$, $q^{(k)} = \widetilde{p}$ and $j_1^{(i)}, j_2^{(i)}, \ell_1^{(i)}, \ell_2^{(i)} \in \mathcal{A}$ satisfying

$$(j_1^{(i)}, j_2^{(i)}, q^{(i)}, \ell_1^{(i)}, \ell_2^{(i)}, q^{(i+1)}) \in \mathcal{R}, \quad \text{for all} \quad 1 \leq i \leq k - 1.$$

For $p \notin \mathcal{A}$, we define $J_p = \sum\limits_{q \leftrightarrow p} I_q$ (the sum over all indexes q which are linked to p). We observe that J_p is a sum of positive quantities, one of them being I_p. So the control of J_p induces the control of I_p.

In view of (5.8) we have

$$\{ J_p, Z_{6,2} \} = 0$$

and thus

$$\dot{J}_p = \{ J_p, Z_{6,3} \} + \{ J_p, R_{10} \}, \text{ when } p \notin \mathcal{A}.$$

Furthermore all the monomials appearing in $\{ J_p, R_{10} \}$ are of order 10 and contains at least one mode out of $\mathcal{A}$. Therefore as soon as (5.3) remains valid, we have

$$\dot{J}_p(t) = \mathcal{O}(\varepsilon^{3+3\times 3}) + \mathcal{O}(\varepsilon^{9+3})$$

and thus

$$|J_p(t)| = \mathcal{O}(\varepsilon^6) + t \, \mathcal{O}(\varepsilon^{12}).$$

We then conclude by a classical bootstrap argument that (5.3) holds true for $t \le C\varepsilon^{-6}$.

• It remains to prove (5.4)-(5.6). Again this is proved by a bootstrap argument. To begin with, we verify by direct calculation that for all $p \in \{1/2, 1, 2\}$,

$$\{ K_p, Z_6^e \} = 0.$$

Therefore, by using (5.7) we deduce that for all $p \in \{1/2, 1, 2\}$

$$(5.9) \qquad \dot{K}_p = \{ K_p, Z_{6,2} \} + \{ K_p, Z_{6,3} \} + \{ K_p, R_{10} \}.$$

Then we use that each monomial of $Z_{6,2}$ contains at least two terms with indices $p' \notin \mathcal{A}$ (see Proposition 3.3). Therefore, as soon as (5.2) holds, we have $|\{ K_p, Z_{6,2} \}| \le C\varepsilon^{10}$. Furthermore $|\{ K_p, R_{10} \}| \le C\varepsilon^{10}$. Therefore, by (5.9),

$$K_p(t) = K_p(0) + t \, \mathcal{O}(\varepsilon^{10}).$$

Finally, to recover the bounds (5.2), we have to demand that t is so that $0 \le t \le \varepsilon^{-6}$, which was the claim. $\qquad\square$

From now, we fix the initial conditions

$$(5.10) \qquad \begin{aligned} &K_1(0) = \varepsilon^2, \ \ K_2(0) = A\varepsilon^2, \ \ K_{1/2}(0) = \varepsilon^2/2, \\ &\text{and } |\xi_j(0)|, |\eta_j(0)| \le C\varepsilon^3 \ \text{ for } \ j \notin \mathcal{A}. \end{aligned}$$

Let $\overline{H}$ be given by (3.4). Then according to the result of Lemma 5.1 which says that for a suitable long time we remain close to the regime of Section 4, we hope that we can write $\overline{H} = \widehat{H}_0 + R$, where R is an error term which remains small for times $0 \le t \le \varepsilon^{-6}$.

We focus on the motion of (φ_0, K_0) and as in the previous section, we make the change of unknown

(5.11) $$\varphi_0(t) = \varphi(\varepsilon^4 t) \quad \text{and} \quad K_0(t) = \varepsilon^2 K(\varepsilon^4 t),$$

and we work with the scaled time variable $\tau = \varepsilon^4 t$. Then we can state

Proposition 5.2. — *Consider the solution* (5.1) *with the initial conditions* (5.10). *Then* (φ, K) *defined by* (5.11) *satisfies for* $0 \leq \tau \leq \varepsilon^{-2}$

(5.12) $$\begin{cases} \dot{\varphi} = & -\frac{\partial H_\star}{\partial K} + \mathcal{O}(\varepsilon^2) \\ \dot{K} = & \frac{\partial H_\star}{\partial \varphi} + \mathcal{O}(\varepsilon^2), \end{cases}$$

where $H_\star$ *is the Hamiltonian* (4.6)

$$H_\star = \frac{3}{2}(1-K)\Big[(A+3)(2A-1)+(7+13A)K+6(1-K)^{\frac{1}{2}}(2A-1+K)^{\frac{1}{2}}K\cos(2\varphi)\Big].$$

Proof. — First recall that $\widehat{H} = \widehat{H}(\varphi_0, K_0, K_1, K_2, K_{1/2})$ is the reduced Hamiltonian given by (4.5). By Propositions 3.1 and 3.3 we have

(5.13) $$\overline{H} = \widehat{H} + R_I + Z_{6,2} + Z_{6,3} + R_{10},$$

where R_I is the polynomial function of the actions I_j defined by (recall that $J = \sum_{k \in \mathbb{N}} K_p$)

$$R_I = 6\big(J^3 - (K_1 + K_2)^3\big) - 9J \sum_{k \in \mathbb{Z}} I_k^2 + 9(K_1 + K_2)\sum_{k \in \mathcal{A}} I_k^2 +$$
$$+ \sum_{j \notin \mathcal{A}} j^2 I_j + 4\sum_{k \notin \mathcal{A}} I_k^3.$$

Notice that R_I vanishes when $I_k = 0$ for all $k \notin \mathcal{A}$ since R_I is in fact the part of $N + Z_6^i$ that does not depend only on the internal variables $(I_k)_{k \in \mathcal{A}}$. Thanks to the Taylor formula there is Q so that

$$\widehat{H}(\varphi_0, K_0, K_1, K_2, K_{1/2}) = \widehat{H}(\varphi_0, K_0, \varepsilon^2, A\varepsilon^2, \varepsilon^2/2) + Q$$
(5.14) $$= \widehat{H}_0 + Q.$$

Thus, by (5.13) and (5.14) we have $\overline{H} = \widehat{H}_0 + R$ with

$$R = Q + R_I + Z_{6,2} + Z_{6,3} + R_{10}.$$

By (5.1), (φ_0, K_0) satisfies the system

$$\begin{cases} \dot{\varphi}_0(t) = & -\frac{\partial \overline{H}}{\partial K_0}(\varphi_0(t), K_0(t), \dots) \\ \dot{K}_0(t) = & \frac{\partial \overline{H}}{\partial \varphi_0}(\varphi_0(t), K_0(t), \dots), \end{cases}$$

where the dots stand for the dependance of the Hamiltonian on the other coordinates. Then, after the change of variables (5.11) we obtain

$$
\begin{cases}
\dot{\varphi}(\tau) = & -\frac{1}{\varepsilon^6}\frac{\partial \overline{H}}{\partial K}(\varphi(\tau), \varepsilon^2 K(\tau), \dots) \\
\dot{K}(\tau) = & \frac{1}{\varepsilon^6}\frac{\partial \overline{H}}{\partial \varphi}(\varphi(\tau), \varepsilon^2 K(\tau), \dots).
\end{cases}
$$

Now write $\overline{H} = \widehat{H}_0 + R$ and observe that $\widehat{H}_0(\varphi, \varepsilon^2 K) = C_\varepsilon + \varepsilon^6 H_\star(\varphi, K)$. As a consequence, (φ, K) satisfies

$$
\begin{cases}
\dot{\varphi} = & -\frac{\partial H_\star}{\partial K} - \frac{1}{\varepsilon^6}\frac{\partial R(\varphi, \varepsilon^2 K, \dots)}{\partial K} \\
\dot{K} = & \frac{\partial H_\star}{\partial \varphi} + \frac{1}{\varepsilon^6}\frac{\partial R(\varphi, \varepsilon^2 K, \dots)}{\partial \varphi}.
\end{cases}
$$

Thus it remains to estimate $\partial_\varphi R(\varphi, \varepsilon^2 K, \dots)$ and $\partial_K R(\varphi, \varepsilon^2 K, \dots)$. Remark that φ and K are dimensionless variables. Thus, if P is a polynomial involving p internal modes, $(\xi_j, \eta_j)_{j \in \mathcal{A}}$, and q external modes, $(\xi_j, \eta_j)_{j \notin \mathcal{A}}$, we have by using Lemma 5.1

$$
\partial_\varphi P(\varphi, \varepsilon^2 K, \dots) = \mathcal{O}(\varepsilon^{p+3q}), \quad \partial_K P(\varphi, \varepsilon^2 K, \dots) = \mathcal{O}(\varepsilon^{p+3q}).
$$

Then notice that R_I contains only monomials involving at least one external actions $(I_k)_{k \notin \mathcal{A}}$. Therefore we get

$$
\begin{aligned}
\partial_\varphi R_I(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{10}), & \partial_K R_I(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{10}), \\
\partial_\varphi Z_{6,2}(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{10}), & \partial_K Z_{6,2}(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{10}), \\
\partial_\varphi Z_{6,3}(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{12}), & \partial_K Z_{6,3}(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{12}), \\
\partial_\varphi R_{10}(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{10}), & \partial_K R_{10}(\varphi, \varepsilon^2 K, \dots) &= \mathcal{O}(\varepsilon^{10}).
\end{aligned}
$$

On the other hand, by construction Q reads $P_1 \Delta K_1 + P_2 \Delta K_2 + P_{1/2}\Delta_{1/2}$ where P_1, P_2 and $P_{1/2}$ are polynomials of order 2 in K_0, K_1, K_2, $K_{1/2}$ and ε^2 while ΔK_j denotes the variation of K_j: $\Delta K_j = K_j - K_j(0)$. Using again Lemma 5.1, we check that for $0 \le \tau \le \varepsilon^{-2}$

$$
\partial_\varphi Q = \mathcal{O}(\varepsilon^8), \quad \partial_K Q = \mathcal{O}(\varepsilon^8),
$$

hence the result. $\qquad\square$

Now we choose some precise initial conditions for (φ, K). We take $\varphi(0) = 0$ and $\kappa_\star < K(0) < 1 - \kappa_\star$ as in Theorem 1.3 or $K(0) = \gamma \ll 1$ as in Theorem 1.4. We also consider the solution $(\varphi_\star, K_\star)$ to (4.7) with initial condition $(\varphi_\star, K_\star)(0) = (\varphi, K)(0)$. Then

Lemma 5.3. — *For all $0 \le \tau \le \varepsilon^{-2}$ we have*

$$
(\varphi, K)(\tau) = (\varphi_\star, K_\star)(\tau) + \mathcal{O}(\varepsilon^2)\tau,
$$

Proof. — Consider the system (5.12), and apply the change of variable $(L, \alpha) = \Phi(K, \varphi)$ defined in Lemma 4.3. Using (5.12) and the fact that $d\Phi$ is bounded (cf. Lemma 4.3), we obtain that for $0 \leq \tau \leq \varepsilon^{-2}$

$$
\begin{aligned}
\frac{d}{d\tau}(L, \alpha) &= \frac{d}{d\tau}\Phi(K, \varphi) = d\Phi(K, \varphi).(\dot{K}, \dot{\varphi}) \\
&= d\Phi(K, \varphi).(\frac{\partial H_\star}{\partial \varphi}, -\frac{\partial H_\star}{\partial K}) + \mathcal{O}(\varepsilon^2) \\
&= (\frac{\partial H_\star}{\partial \alpha}, -\frac{\partial H_\star}{\partial L}) + \mathcal{O}(\varepsilon^2) \\
&= (0, -\frac{\partial H_\star}{\partial L}) + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

Therefore there exists $L_\star \in \mathbb{R}$ so that $L(\tau) = L_\star + \mathcal{O}(\varepsilon^2)\tau$ and if we define $\omega_\star = -\frac{\partial H_\star}{\partial L}(L_\star)$, we obtain $\alpha(\tau) = \omega_\star \tau + \mathcal{O}(\varepsilon^2)\tau$. Next, as $d\Phi^{-1}$ is bounded, we get

$$
\begin{aligned}
(\varphi, K)(\tau) = \Phi^{-1}\big(L(\tau), \alpha(\tau)\big) &= \Phi^{-1}\big(L_\star, \omega_\star \tau\big) + \mathcal{O}(\varepsilon^2)\tau \\
&= (\varphi_\star, K_\star)(\tau) + \mathcal{O}(\varepsilon^2)\tau,
\end{aligned}
$$

where $(\varphi_\star, K_\star)(\tau)$ is the solution of (4.7) so that $(\varphi_\star, K_\star)(0) = (\varphi, K)(0)$. $\square$

Proof of Theorems 1.3 and 1.4. — As a consequence of Lemma 5.3, the solution of (5.1) satisfies for $0 \leq t \leq \varepsilon^{-6}$

$$
\begin{aligned}
K_0(t) &= \varepsilon^2 K_\star(\varepsilon^4 t) + \mathcal{O}(\varepsilon^8)t \\
\varphi_0(t) &= \varphi_\star(\varepsilon^4 t) + \mathcal{O}(\varepsilon^6)t.
\end{aligned}
$$

This completes the proof of the main results : The error term q_1 comes from the normal form reduction (see Proposition 3.1), and the error term q_2 comes from the $\mathcal{O}(\varepsilon^6)$ above (recall that $\nu = \varepsilon^4$). $\square$

Appendix A

We prove Lemma 3.2:
The first assertion is trivial. Concerning the second one we have

$$||X_P(\xi,\eta)||_\rho = \sum_{k\in\mathbb{Z}} e^{\rho|k|}\left(\left|\frac{\partial Q}{\partial\xi_k}\right| + \left|\frac{\partial Q}{\partial\eta_k}\right|\right)$$

$$\le p[P]\sum_{k\in\mathbb{Z}} \mathrm{e}^{\rho|k|} \sum_{\substack{j_1,\cdots,j_{p-1},\ell_1,\cdots,\ell_p\in\mathbb{Z}\\ \mathcal{M}(j_1,\cdots,j_{p-1},k;\ell_1,\dots,\ell_p)=0}} \left|\xi_{j_1}\cdots\xi_{j_{p-1}}\eta_{\ell_1}\cdots\eta_{\ell_p}\right| + \left|\xi_{\ell_1}\cdots\xi_{\ell_p}\eta_{j_1}\cdots\eta_{j_{p-1}}\right|$$

$$\le p[P] \sum_{j_1,\cdots,j_{p-1},\ell_1,\cdots,\ell_p\in\mathbb{Z}} \left|\xi_{j_1}e^{\rho|j_1|}\cdots\xi_{j_{p-1}}\mathrm{e}^{\rho|j_{p-1}|}\eta_{\ell_1}\mathrm{e}^{\rho|\ell_1|}\cdots\eta_{\ell_p}\mathrm{e}^{\rho|\ell_p|}\right| +$$

$$+ p[P] \sum_{j_1,\cdots,j_{p-1},\ell_1,\cdots,\ell_p\in\mathbb{Z}} \left|\xi_{\ell_1}\mathrm{e}^{\rho|\ell_1|}\cdots\xi_{\ell_p}\mathrm{e}^{\rho|\ell_p|}\eta_{j_1}\mathrm{e}^{\rho|j_1|}\cdots\eta_{j_{p-1}}\mathrm{e}^{\rho|j_{p-1}|}\right|$$

$$\le 2p[P]||(\xi,\eta)||_\rho^{2p-1},$$

where we used,

$$\mathcal{M}(j_1,\cdots,j_{p-1},k;\ell_1,\dots,\ell_p)=0 \Rightarrow |k|\le |j_1|+\cdots+|j_{p-1}|+|\ell_1|+\cdots+|\ell_p|.$$

Assume now that $P\in\mathcal{P}_p$ and $Q\in\mathcal{P}_q$ with coefficients $a_{j\ell}$ and $b_{j\ell}$. It is clear that $\{P,Q\}$ is a monomial of degree $2p+2q-2$ satisfying the zero momentum condition. Furthermore writing

$$\{P,Q\}(\xi,\eta) = \sum_{(j,\ell)\in\mathbb{Z}^{2p+2q-2}} c_{j\ell}\xi_{j_1}\cdots\xi_{j_{p+q-1}}\eta_{\ell_1}\cdots\eta_{\ell_{p+q-1}},$$

where $c_{j\ell}$ is expressed as a sum of coefficients $a_{ik}b_{nm}$ for which there exists $s\in\mathbb{Z}$ such that

$$i\cup n \ \setminus\{s\} = j \text{ and } k\cup m \ \setminus\{s\} = \ell.$$

For instance if $s=i_1=m_1$ then necessarily $j=(i_2,\cdots,i_p,n_1,\cdots,n_q)$ and $\ell=k_1,\dots,k_p,m_2,\cdots,m_q$. Thus for fixed (j,ℓ), you just have to choose which of the indices i you excise and which of indices m you excise or, symmetrically, which of the indices n you excise and which of indices k you excise. Note that the value of s is automatically fixed by the zero momentum condition on (i,k) and on (n,m). So

$$|c_{j\ell}|\le 2pq[P][Q].$$

Appendix B

We give here a method to compute the terms which appear in $Z_{6,2}$ (see Proposition 3.3). Let $\mathcal{A}$ be a resonant set.
Let $(j_1,j_2,j_3,\ell_1,p_1,p_2)\in\mathcal{R}$. Assume that $j_1,j_2,j_3,\ell_1\in\mathcal{A}$. Then by Lemma

2.4, we deduce that $p_1, p_2 \in \mathcal{A}$. As a consequence, the only terms which will give a nontrivial contribution to $Z_{6,2}$ are of the form $(j_1, j_2, p_1, j_3, j_4, p_2) \in \mathcal{R}$, with $j_1, j_2, j_3, j_4 \in \mathcal{A}$ and $p_1, p_2 \notin \mathcal{A}$.

Let $j_1, j_2, \ell_1, \ell_2 \in \mathcal{A}$ and $p_1, p_2 \in \mathbb{N}$ so that

$$
(B.1) \qquad
\begin{cases}
p_2 - p_1 = j_1 + j_2 - \ell_1 - \ell_2, \\
p_2^2 - p_1^2 = j_1^2 + j_2^2 - \ell_1^2 - \ell_2^2.
\end{cases}
$$

By Lemma 2.3, there exist $k \in \mathbb{Z}^*$ and $n \in \mathbb{N}$ so that $\mathcal{A} = \{n, n+3k, n+4k, n+k\}$. Hence, there exist $n, k \in \mathbb{Z}$ and $(m_s)_{1 \le j \le 4}$ with $m_s \in \{0, 1, 3, 4\}$ so that $j_s = n + m_s k$ and $\ell_1 = n + m_3 k$, $\ell_2 = n + m_4 k$. We then define $q_1, q_2 \in \mathbb{Q}$ by $p_1 = n + q_1 k$ and $p_2 = n + q_2 k$. We plug these expressions in (B.1) which gives

$$
\begin{cases}
q_2 - q_1 = m_1 + m_2 - m_3 - m_4 := U, \\
q_2^2 - q_1^2 = m_1^2 + m_2^2 - m_3^2 - m_4^2 := V.
\end{cases}
$$

When $U \neq 0$, we can solve this latter equation and we obtain

$$
q_2 = \frac{1}{2}\left(\frac{V}{U} + U\right), \quad q_1 = \frac{1}{2}\left(\frac{V}{U} - U\right).
$$

By symmetry, we can assume that $m_1 \ge m_2$, $m_3 \ge m_4$. We also observe that $(m_1, m_2, p_1, m_3, m_4, p_2)$ is a solution iff $(m_3, m_4, p_2, m_1, m_2, p_1)$ is a solution.

Values of m_s	Value of V	Values of U	Value of q_2	Value of q_1
4400	32	8	6	-2
4401	31	7	40/7	-9/7
4411	30	6	11/2	-1/2
4431	22	4	19/4	3/4
4430	23	5	24/5	-1/5
4433	14	2	9/2	5/2
4300	25	7	37/7	-12/7
4301	24	6	5	-1
4311	23	5	24/5	-1/5
4100	17	5	21/5	-4/5
4103	8	2	3	1
4133	-1	-1	0	1
4011	14	2	9/2	5/2
4031	6	0	$\times$	$\times$
4033	-2	-2	-1/2	3/2
3300	18	6	9/2	-3/2
3301	17	5	21/5	-4/5
3311	16	4	4	0
3100	10	4	13/4	-3/4
3011	7	1	3	4
1100	2	2	3/2	-1/2

Example B.1. — Assume that $\mathcal{A} = \{-2, 1, 2, -1\}$. Then $n = -2$ and $k = 1$, so that $p_1 = -2 + q_1$ and $p_2 = -2 + q_2$. We only look at the integer values in the two last columns, and we find (up to permutation)

$$4400: (2, 2, -4, 2, 2, 4), \quad 4301: (2, 1, -3, -2, -1, 3).$$

Assume that $\mathcal{A} = \{-1, 5, 7, 1\}$. Then $n = -1$ and $k = 2$, so that $p_1 = -1 + 2q_1$ and $p_2 = -1 + 2q_2$. In this case, we look at the half-integer values in the two last columns, and we find (up to permutation)

$$4400: (7, 7, -5, -1, -1, 11), \quad 4411: (7, 7, -2, 1, 1, 10),$$
$$4433: (7, 7, 4, 5, 5, 8), \quad 4301: (7, 5, -3, -1, 1, 9),$$
$$4011: (7, -1, 4, 1, 1, 8), \quad 4033: (7, -1, 2, 5, 5, -2),$$
$$3300: (5, 5, -4, -1, -1, 8), \quad 1100: (1, 1, -2, -1, -1, 2).$$

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