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On the exponential and polynomial stability for a linear Bresse system

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In this paper, we consider a linear one-dimensional Bresse system consisting of three hyperbolic equations coupled in a certain manner under mixed homogeneous Dirichlet-Neumann boundary conditions. Here, we consider that only the longitudinal displacement is damped, and the vertical displacement and shear angle displacement are free. We prove the well-posedness of the system and some exponential, lack of exponential and polynomial stability results depending on the coefficients of the equations and the smoothness of initial data. At the end, we use some numerical approximations based on finite difference techniques to validate the theoretical results. The proof is based on the semigroup theory and a combination of the energy method and the frequency domain approach.

KEYWORDS

asymptotic behavior, Bresse system, energy method, frequency domain approach, well-posedness, semigroup theory, numerical approximation

MSC CLASSIFICATION

35B40; 35L45; 74H40; 93D20; 93D15

1 | INTRODUCTION

In this paper, we consider the one-dimensional Bresse system model consisting of three coupled hyperbolic equations as follows:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\,w)_x - lk_0\,(w_x - l\varphi) = F_1 & \text{in} \quad (0,L)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\,(\varphi_x + \psi + l\,w) = F_2 & \text{in} \quad (0,L)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk\,(\varphi_x + \psi + l\,w) = F_3 & \text{in} \quad (0,L)\times(0,\infty)\,, \end{cases} \tag{1.1}$$

along with the initial data

$$\begin{cases} \varphi\,(x,0) = \varphi_0\,(x)\,, \ \ \varphi_t\,(x,0) = \varphi_1\,(x) & \text{in} \quad (0,L)\,, \\ \psi\,(x,0) = \psi_0\,(x)\,, \ \ \psi_t\,(x,0) = \psi_1\,(x) & \text{in} \quad (0,L)\,, \\ w\,(x,0) = w_0\,(x)\,, \ \ w_t\,(x,0) = w_1\,(x) & \text{in} \quad (0,L)\,. \end{cases} \tag{1.2}$$

where $L > 0$, $F_i \ : \ (0,L)\times(0,\infty) \ \to \ \mathbb{R}$ are the external forces (controllers) and w, φ and ψ represent, respectively, the longitudinal, vertical, and shear angle displacements. For more details, we refer to Lagnese et al.[1,2] The coefficients $\rho_1,\ \rho_2,\ b,\ k,\ k_0$ and l are positive constants, the initial data $\varphi_0, \varphi_1, \psi_0, \psi_1, w_0$, and w_1 belong to a suitable Hilbert space. For the last few years, many researchers studied the well-posedness and the stability of Bresse systems (1.1). Under different types of controls F_i, various stability results have been yet obtained depending on the nature and the number of controls, the regularity of initial data, and the following parameters:

$$s_1 = \frac{k}{\rho_1}, \quad s_2 = \frac{b}{\rho_2} \quad \text{and} \quad s_3 = \frac{k_0}{\rho_1}; \tag{1.3}$$

for this purpose, we refer the reader to previous studies[3,4,6-10] in case of (local or global, linear or nonlinear) frictional dampings and De Lima Santos et al[11], Guesmia and Kafini,[12,13] and Guesmia[14] in case of memories. In some papers, it was proved that, when each equation of (1.1) is directly damped, that is, $F_1F_2F_3 \neq 0$, the stability of (1.1) holds regardless to s_1, s_2 and s_3. However, when at least one equation in (1.1) is free, that is, $F_1F_2F_3 = 0$ and $(F_1, F_2, F_3) \neq (0, 0, 0)$, system (1.1) is still stable depending on the relation between the coefficients s_1, s_2, and s_3 like

$$s_1 = s_2 = s_3. \tag{1.4}$$

When $(F_1, F_2, F_3) = (0, 0, 0)$, system (1.1) is conservative, which means that the energy is conserved and equal to the energy of initial data along the trajectory of solutions.

We need to mention here that similar conditions on the coefficients were established in the case of Cauchy problem for the Bresse system (see Ghoul et al[15]). Also recent new decay and new conditions on the coefficients were introduced and established in the case of thermoelastic Bresse system with second sound (see Afilall et al[16]). Notice that, when the three hyperbolic equations in Bresse system are (all or some of them) directly damped; that is, $(F_1, F_2, F_3) \neq (0, 0, 0)$, system (1.1) is dissipative. In this paper, we consider (1.1) and (1.2) along with a mixed boundary conditions of the form:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = 0 \text{ in } (0,\infty), \\ \varphi_x(L,t) = \psi(L,t) = w(L,t) = 0 \text{ in } (0,\infty), \end{cases} \tag{1.5}$$

and here, only the third hyperbolic equation is damped, that is,

$$(F_1, F_2, F_3) = (0, 0, -\delta w_t),$$

where δ is a positive constant . We prove the well-posedness, and we establish some decay rates for the solutions (like exponential stability, nonexponential stability, and polynomial stability) depending on a new relationship between the coefficients of (1.1) and the smoothness of the initial data.

Without loss of generality, we consider the domain $(0, 1)$ instead of $(0, L)$. The proof of the well-posedness is based on the semigroup theory. However, the stability results are proved using the energy method combining with the frequency domain approach. The paper is organized as follows. In Section 2, we prove the well-posedness of (1.1), (1.2), and (1.5). In Sections 3 and 4, we show, respectively, the lack of exponential stability and the exponential stability results for (1.1), (1.2), and (1.5). The polynomial decay result for (1.1), (1.2), and (1.5) is proved in Section 5. Numerical approximations using finite difference techniques is introduced in Section 6. Concluding remarks and open questions are given in Section 7.

2 | WELL-POSEDNESS

In this section, we prove the existence, uniqueness and smoothness of solutions for (1.1), (1.2), and (1.5) using the semigroup theory. In order to transform (1.1), (1.2), and (1.5) into a first-order evolution system on a suitable Hilbert space, we introduce the vector functions

$$\Phi = (\varphi,\ \tilde{\varphi},\ \psi,\ \tilde{\psi},\ w,\ \tilde{w})^T \quad \text{and} \quad \Phi_0 = (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ w_0,\ w_1)^T,$$

where $\tilde{\varphi} = \varphi_t$, $\tilde{\psi} = \psi_t$ and $\tilde{w} = w_t$. System (1.1) with initial data (1.2) can be written as

$$\begin{cases} \Phi_t = \mathcal{A}\Phi & \text{in } (0,\infty), \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.1}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\Phi = \begin{pmatrix} \tilde{\varphi} \\ \frac{k}{\rho_1}(\varphi_x + \psi + l\,w)_x + \frac{lk_0}{\rho_1}(w_x - l\varphi) \\ \tilde{\psi} \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + l\,w) \\ \tilde{w} \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{lk}{\rho_1}(\varphi_x + \psi + l\,w) - \frac{\delta}{\rho_1}w_t \end{pmatrix}. \tag{2.2}$$

Now, we introduce the following spaces:

$$\begin{cases} H_*^1(0,1) = \left\{ f \in H^1(0,1) : f(0) = 0 \right\}, \\ \tilde{H}_*^1(0,1) = \left\{ f \in H^1(0,1) : f(1) = 0 \right\}, \\ H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1), \\ \tilde{H}_*^2(0,1) = H^2(0,1) \cap \tilde{H}_*^1(0,1) \end{cases}$$

and the energy space is given by

$$\mathcal{H} = H_*^1(0,1) \times L^2(0,1) \times (\tilde{H}_*^1(0,1) \times L^2(0,1))^2,$$

equipped with the inner product, for $\Phi_j = (\varphi_j,\ \tilde{\varphi}_j,\ \psi_j,\ \tilde{\psi}_j,\ w_j,\ \tilde{w}_j)^T \in \mathcal{H},\ j = 1,2,$

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} = {} & k\langle (\varphi_{1x} + \psi_1 + l\,w_1), (\varphi_{2x} + \psi_2 + l\,w_2)\rangle_{L^2(0,1)} + b\langle \psi_{1x}, \psi_{2x}\rangle_{L^2(0,1)} \\ & + k_0\langle (w_{1x} - l\varphi_1), (w_{2x} - l\varphi_2)\rangle_{L^2(0,1)} + \rho_1\langle \tilde{\varphi}_1, \tilde{\varphi}_2\rangle_{L^2(0,1)} \\ & + \rho_2\langle \tilde{\psi}_1, \tilde{\psi}_2\rangle_{L^2(0,1)} + \rho_1\langle \tilde{w}_1, \tilde{w}_2\rangle_{L^2(0,1)}, \end{aligned}$$

and the corresponding norm in the energy space will be given by

$$\begin{aligned} \|\varphi\|_{\mathcal{H}}^2 = {} & k\,\|\varphi_x + \psi + l\,w\|_{L^2(0,1)}^2 + b\,\|\psi_x\|_{L^2(0,1)}^2 + k_0\,\|w_x - l\varphi\|_{L^2(0,1)}^2 \\ & + \rho_1\,\|\tilde{\varphi}\|_{L^2(0,1)}^2 + \rho_2\|\tilde{\psi}\|_{L^2(0,1)}^2 + \rho_1\,\|\tilde{w}\|_{L^2(0,1)}^2. \end{aligned}$$

The domain of the operator $\mathcal{A}$ will be

$$D(\mathcal{A}) = \{\Phi \in \mathcal{H} \mid \mathcal{A}\Phi \in \mathcal{H},\ \varphi_x(1) = \psi_x(0) = w_x(0) = 0\}.$$

Based on the definition of $\mathcal{A}$ and $\mathcal{H}$, one can see that

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} |\ \varphi \in H_*^2(0,1);\ \psi,\ w \in \tilde{H}_*^2(0,1);\ \tilde{\varphi} \in H_*^1(0,1); \\ \tilde{\psi},\ \tilde{w} \in \tilde{H}_*^1(0,1);\ \varphi_x(1) = \psi_x(0) = w_x(0) = 0 \end{array} \right\}.$$

Since the homogeneous Dirichlet-Neumann boundary conditions in (1.5) are included in the definition of $H_*^1(0,1)$, $\tilde{H}_*^1(0,1)$, and $D(\mathcal{A})$, it follows that, if $\varphi \in D(\mathcal{A})$ and satisfies (2.1), then (1.1), (1.2), and (1.5) holds.

It is clear from the homogeneous Dirichlet boundary conditions in $H_*^1(0,1)$ and $\widetilde{H}_*^1(0,1)$ that, if $(\varphi,\psi,w) \in H_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1)$ satisfying

$$k\,\|(\varphi_x+\psi+l\,w)\|_{L^2(0,1)}^2 + b\,\|\psi_x\|_{L^2(0,1)}^2 + k_0\,\|(w_x-l\varphi)\|_{L^2(0,1)}^2 = 0,$$

then $\psi = 0$, $\varphi = -c\sin(lx)$ and $w = c\cos(lx)$, where c is a constant such that $c = 0$ or $l = \frac{\pi}{2} + m\pi$, for some $m \in \mathbb{N}$. Furthermore, we get $\varphi = \psi = w = 0$ if

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}. \tag{2.3}$$

Here and after we assume that (2.3) is satisfied. Thus, $\mathcal{H}$ is a Hilbert space and $D(\mathcal{A})$ is dense in $\mathcal{H}$. If the domain $(0,1)$ is replaced by $(0,L)$, then (2.3) becomes

$$lL \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}. \tag{2.4}$$

Now, we prove that the operator $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$. For this purpose, it is sufficient to prove that $\mathcal{A}$ is maximal monotone. A direct calculation gives

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = -\delta\,\|\tilde{w}\|_{L^2(0,1)}^2 \leq 0, \tag{2.5}$$

which implies that $\mathcal{A}$ is dissipative in $\mathcal{H}$. On the other hand, based on the theory elliptic equations, it is easy (see Pazy[17]) to show that $0 \in \rho(\mathcal{A})$; that is, for any $F = (f_1, \ldots, f_6)^T \in \mathcal{H}$, there exists $Z = (z_1, \ldots, z_6)^T \in D(\mathcal{A})$ satisfying

$$\mathcal{A}Z = F. \tag{2.6}$$

Thus, the well-posedness result for (2.1) is stated in the following:

Theorem 1. *Assume that (2.3) holds. For any $p \in \mathbb{N}$ and $\varphi_0 \in D(\mathcal{A}^p)$, system (2.1) admits a unique solution*

$$\Phi \in \cap_{j=0}^{p} C^{p-j}\left(\mathbb{R}_+; D\left(\mathcal{A}^j\right)\right), \tag{2.7}$$

where $D\left(\mathcal{A}^j\right)$ is endowed by the graph norm $\|\cdot\|_{D(\mathcal{A}^j)} = \sum_{r=0}^{j} \|\mathcal{A}^r\cdot\|_{\mathcal{H}}$.

In the coming sections, the following stated theorems will play an essential role in our proof:

Theorem 2. *([18] and[19]) A C_0 semigroup of contractions on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{\lambda\in\mathbb{R}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.8}$$

Theorem 3 (Liu and Rao[18])**.** *If a bounded C_0 semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ satisfies, for some $j \in \mathbb{N}^*$,*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\lambda|\geq 1} \frac{1}{\lambda^j} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.9}$$

Then, for any $p \in \mathbb{N}^*$, there exists a positive constant c_p such that

$$\left\|e^{t\mathcal{A}} z_0\right\|_{\mathcal{H}} \leq c_p \|z_0\|_{D(\mathcal{A}^p)} \left(\frac{\ln\, t}{t}\right)^{\frac{p}{j}} \ln\, t, \quad \forall z_0 \in D\left(\mathcal{A}^p\right), \quad \forall t > 0. \tag{2.10}$$

3 | LACK OF EXPONENTIAL STABILITY

In this section, we prove that the semigroup associated with the Bresse system with frictional damping is not exponentially stable if (1.4) or

$$l^2 \neq \frac{\rho_2 k_0 + \rho_1 b}{\rho_2 k_0}\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k}{\rho_2 (k + k_0)}, \quad \forall m \in \mathbb{Z}, \tag{3.1}$$

does not hold. Our main result is given as follows:

Theorem 4. *Let us assume that*

$$\text{does not hold} \quad \text{or} \quad \frac{b}{\rho_2} \neq \frac{k}{\rho_1} \quad \text{or} \quad \frac{b}{\rho_2} \neq \frac{k_0}{\rho_1} \quad \text{or} \quad k \neq k_0.$$

Then, the semigroup associated with (1.1) and (1.2) is not exponentially stable.

Proof. We prove that the first condition in (2.8) is equivalent to (3.1) by proving that the unique

$$\Phi = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T \in D(\mathcal{A})$$

satisfying

$$\mathcal{A}\ \Phi = i\ \lambda\ \Phi \tag{3.2}$$

is $\Phi = 0$ if and only if (3.1) holds. Equation (3.2) is equivalent to

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \\ \frac{k}{\rho_1}(\varphi_x + \psi + l\ w)_x + \frac{lk_0}{\rho_1}(w_x - l\varphi) = i\lambda\tilde{\varphi}, \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + l\ w) = i\lambda\tilde{\psi}, \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{lk}{\rho_1}(\varphi_x + \psi + l\ w) - \frac{\delta}{\rho_1}\tilde{w} = i\lambda\tilde{w}. \end{cases} \tag{3.3}$$

Using (2.5), we find

$$-\delta\ \|\tilde{w}\|^2_{L^2(0,1)} = Re\ \langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = Re\ \langle i\lambda\ \Phi, \Phi\rangle_{\mathcal{H}} = Re\ i\lambda\ \|\Phi\|^2_{\mathcal{H}} = 0.$$

So $\tilde{w} = 0$. From the third equation in (3.3), we get $w = 0$. Therefore, (3.3) is reduced into the following:

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \\ k(\varphi_x + \psi)_x - l^2k_0\varphi = -\rho_1\lambda^2\varphi, \\ b\psi_{xx} - k(\varphi_x + \psi) = -\rho_2\lambda^2\psi, \\ -k_0\varphi_x - k(\varphi_x + \psi) = 0, \end{cases} \tag{3.4}$$

which is equivalent to $\tilde{\varphi} = i\lambda\varphi$, $\tilde{\psi} = i\lambda\psi$ and

$$\begin{cases} \left(l^2k_0 - \rho_1\lambda^2\right)\varphi - k(\varphi_x + \psi)_x = 0, \\ -\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x + \psi) = 0, \\ \varphi_x + \psi = -\frac{k_0}{k}\varphi_x. \end{cases} \tag{3.5}$$

By deriving $(3.5)_3$ and combining with $(3.5)_1$, we see that φ satisfy the following equation:

$$\varphi_{xx} + \alpha\varphi = 0, \tag{3.6}$$

where $\alpha = \frac{l^2k_0 - \rho_1\lambda^2}{k_0}$. At this stage, we distinguish three cases.

Case 1:

$\lambda^2 = \frac{l^2k_0}{\rho_1}$. Then,

$$\varphi(x) = c_1x + c_2,$$

for $c_1,\ c_2 \in \mathbb{C}$. Using the boundary conditions

$$\varphi(0) = \varphi_x(1) = 0, \tag{3.7}$$

we find

$$\varphi = 0, \tag{3.8}$$

which implies that, using the first two equations in (3.4) and the last one in (3.5),

$$\tilde{\varphi} = 0 \tag{3.9}$$

and

$$\psi = \tilde{\psi} = 0. \tag{3.10}$$

Consequently, we get

$$\Phi = 0. \tag{3.11}$$

Case 2: $\lambda^2 > \frac{l^2 k_0}{\rho_1}$. Then,

$$\varphi(x) = c_1 e^{\sqrt{-\alpha}x} + c_2 e^{-\sqrt{-\alpha}x}.$$

Using again the boundary conditions (3.7), we find (3.8), and similarly as before, we arrive at (3.11).

Case 3: $\lambda^2 < \frac{l^2 k_0}{\rho_1}$. Then,

$$\varphi(x) = c_1 \cos\left(\sqrt{\alpha}x\right) + c_2 \sin\left(\sqrt{\alpha}x\right).$$

Using the boundary conditions (3.7), we deduce that $c_1 = 0$, and

$$c_2 = 0 \quad \text{or} \quad \exists\, m \in \mathbb{Z}: \quad \alpha = \left(\frac{\pi}{2} + m\pi\right)^2. \tag{3.12}$$

If $c_2 = 0$, then (3.8) holds, and as before, we find (3.11). If $c_2 \neq 0$, then, by (3.12),

$$\exists\, m \in \mathbb{Z}: \quad \frac{l^2 k_0 - \rho_1 \lambda^2}{k_0} = \left(\frac{\pi}{2} + m\pi\right)^2. \tag{3.13}$$

Therefore, $(3.5)_3$ is equivalent to

$$\psi(x) = -c_2 \left(1 + \frac{k_0}{k}\right) \sqrt{\alpha} \cos\left(\sqrt{\alpha}x\right), \tag{3.14}$$

and then the first two equations in (3.5) are reduced to

$$\lambda^2 = \frac{k_0 \left[k k_0 + b l^2 (k + k_0)\right]}{(k + k_0)(k_0 \rho_2 + b \rho_1)}. \tag{3.15}$$

We see that (3.13) and (3.15) lead to

$$\exists\, m \in \mathbb{Z}: \quad l^2 = \frac{\rho_2 k_0 + \rho_1 b}{\rho_2 k_0} \left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k}{\rho_2 (k + k_0)};$$

that is (3.1) does not hold. So, if (3.1) holds, we get a contradiction, and hence, $c_2 = 0$ and, as before, we find (3.11). If (3.1) does not hold, then, for $\lambda \in \mathbb{R}$ satisfying (3.15), the function

$$\Phi(x) = c_2 \left(\sin\left(\sqrt{\alpha}x\right), i\lambda \sin\left(\sqrt{\alpha}x\right), -\left(1 + \frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right),\right.$$
$$\left. -i\lambda \left(1 + \frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right), 0, 0, 0, 0\right)^T$$

is a solution of (3.2), for any $c_2 \in \mathbb{C}$, and then $i\lambda \notin \rho(\mathcal{A})$. Thus, we proved that $i\,\mathbb{R} \subset \rho(\mathcal{A})$ is equivalent to (3.1).

We prove now that the second condition in (2.8) is not satisfied. We have to prove that there exist sequences $(\lambda_n)_n \subset \mathbb{R}$ and $(F_n)_n \subset \mathcal{H}$, with $\|F_n\|_{\mathcal{H}} \leq 1$, for which we have

$$||\underbrace{(\lambda_n I - \mathcal{A})^{-1} F_n}_{\Phi_n}||_{\mathcal{H}} \to \infty, \tag{3.16}$$

therefore, we have

$$\lambda_n \Phi_n - \mathcal{A}\Phi_n = F_n. \tag{3.17}$$

Rewriting the spectral equation in terms of its components, we have

$$\begin{cases} i\lambda_n \varphi_n - \tilde{\varphi}_n = f_{1n}, \\ i\lambda_n \rho_1 \tilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l\, w_n)_x - lk_0\,(w_{nx} - l\varphi_n) = \rho_1 f_{2n}, \\ i\lambda_n \psi_n - \tilde{\psi}_n = f_{3n}, \\ i\lambda_n \rho_2 \tilde{\psi}_n - b\psi_{nxx} + k\,(\varphi_{nx} + \psi_n + l\, w_n) = \rho_2 f_{4n}, \\ i\lambda_n w_n - \tilde{w}_n = f_{5n}, \\ i\lambda_n \rho_1 \tilde{w}_n - k_0(w_{nx} - l\varphi_n)_x + lk\,(\varphi_{nx} + \psi_n + l\, w_n) + \delta\tilde{w}_n = \rho_1 f_{6n}. \end{cases} \tag{3.18}$$

Let us take

$$f_{4n}(x) = c\cos(Nx), \quad \text{and} \quad f_{1n} = f_{2n} = f_{3n} = f_{5n} = f_{6n} = 0, \tag{3.19}$$

where $N = \frac{(2n+1)\pi}{2}$ and c is a constant satisfying $0 < |c| \leq \frac{1}{\sqrt{\rho_2}}$, so

$$\|F_n\|^2_{\mathcal{H}} = \rho_2\, \|f_{4n}\|^2_{L^2(0,1)} = \rho_2 |c|^2 \int_0^1 \cos^2\,(Nx)\, dx \leq 1.$$

Choosing

$$\begin{cases} \varphi_n(x) = \alpha_1 \sin\,(Nx), \quad \psi_n(x) = \alpha_2 \cos\,(Nx), \quad w_n(x) = \alpha_3 \cos\,(Nx), \\ \tilde{\varphi}_n(x) = i\lambda_n \alpha_1 \sin\,(Nx), \quad \tilde{\psi}_n(x) = i\lambda_n \alpha_2 \cos\,(Nx), \quad \tilde{w}_n(x) = i\lambda_n \alpha_3 \cos\,(Nx), \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants depending on N (will be fixed later). Then, we get

$$\begin{cases} \left[kN^2 + l^2 k_0 - \lambda_n^2 \rho_1\right] \alpha_1 + kN\alpha_2 + l\,(k + k_0)\, N\alpha_3 = 0, \\ \left[bN^2 + k - \lambda_n^2 \rho_2\right] \alpha_2 + kN\alpha_1 + lk\alpha_3 = \rho_2 c, \\ \left[k_0 N^2 + i\delta\lambda_n + l^2 k - \lambda_n^2 \rho_1\right] \alpha_3 + l\,(k + k_0)\, N\alpha_1 + lk\alpha_2 = 0. \end{cases} \tag{3.20}$$

We have to discuss the two cases

$$\rho_1 b - \rho_2 k \neq 0 \quad \text{and} \quad [\rho_1 b - \rho_2 k = 0 \;\; \text{and} \;\; k - k_0 \neq 0],$$

Case 1: $\rho_1 b - \rho_2 k \neq 0$. Let us choose $\lambda_n = \sqrt{\frac{b}{\rho_2} N^2 + \frac{kk_0}{\rho_2(k+k_0)}}$. Then, (3.20) becomes

$$\begin{cases} \left[\left(k - \frac{\rho_1 b}{\rho_2}\right) N^2 + l^2 k_0 - \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right] \alpha_1 + kN\alpha_2 + l\,(k + k_0)\, N\alpha_3 = 0, \\ \frac{k^2}{k+k_0} \alpha_2 + kN\alpha_1 + lk\alpha_3 = \rho_2 c, \\ \left[\left(k_0 - \frac{\rho_1 b}{\rho_2}\right) N^2 + i\delta\lambda_n + l^2 k - \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right] \alpha_3 + l\,(k + k_0)\, N\alpha_1 + lk\alpha_2 = 0. \end{cases} \tag{3.21}$$

From $(3.21)_2$, we get

$$\alpha_1 = \frac{\rho_2 c - lk\alpha_3 - \frac{k^2}{k+k_0}\alpha_2}{kN}. \tag{3.22}$$

By substituting (3.22) into $(3.21)_3$ and into $(3.21)_1$, we obtain, respectively,

$$\alpha_3 = \frac{\rho_2 lc(k + k_0)}{k\left[\left(\frac{\rho_1 b}{\rho_2} - k_0\right) N^2 - i\delta\lambda_n + l^2 k_0 + \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right]} \tag{3.23}$$

and

$$\alpha_2 = \frac{\left[(\rho_2 c - lk\alpha_3)\left(k - \frac{\rho_1 b}{\rho_2}\right) + lk(k+k_0)\alpha_3\right] N^2 + (\rho_2 c - lk\alpha_3)\left[l^2 k_0 - \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right]}{\frac{k^2}{k+k_0}\left[-\left(\frac{\rho_1 b}{\rho_2} + k_0\right) N^2 + l^2 k_0 - \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right]}. \tag{3.24}$$

We see that (3.23) and the fact that $\lim_{n\to\infty} \lambda_n = \infty$ imply that

$$\lim_{n\to\infty} \alpha_3 = 0, \tag{3.25}$$

therefore,

$$\lim_{n\to\infty} \alpha_2 = \frac{c(k+k_0)(\rho_1 b - \rho_2 k)}{k^2\left(\frac{\rho_1 b}{\rho_2} + k_0\right)} \neq 0 \tag{3.26}$$

since $\rho_1 b - \rho_2 k \neq 0$. Then,

$$\lim_{n\to\infty} |\alpha_2| N = \infty. \tag{3.27}$$

Finally, using the norm of ψ_{nx} in $L^2(0,1)$, we obtain

$$\begin{aligned} \|\Phi_n\|^2_{\mathcal{H}} &\geq b\, \|\psi_{nx}\|^2_{L^2(0,1)} = b|\alpha_2|^2 N^2 \int_0^1 \sin^2 (Nx)\, dx \\ &\geq \frac{b}{2}|\alpha_2|^2 N^2 \int_0^1 (1-\cos(2Nx))\; dx = \frac{b}{2}|\alpha_2|^2 N^2 \to \infty. \end{aligned} \tag{3.28}$$

Case 2: $\rho_1 b - \rho_2 k = 0$ and $k - k_0 \neq 0$. Let us choose $\lambda_n = \sqrt{\frac{k}{\rho_1} N^2 + \frac{k}{\sqrt{\rho_1\rho_2}} N}$. Then, (3.20) becomes

$$\begin{cases} \left(-\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N + l^2 k_0\right)\alpha_1 + kN\alpha_2 + l(k+k_0)N\alpha_3 = 0, \\ \left(-\frac{\rho_2 k}{\sqrt{\rho_1\rho_2}} N + k\right)\alpha_2 + kN\alpha_1 + lk\alpha_3 = \rho_2 c, \\ \left[(k_0 - k)N^2 + i\delta\lambda_n - \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N + l^2 k\right]\alpha_3 + l(k+k_0)N\alpha_1 + lk\alpha_2 = 0. \end{cases} \tag{3.29}$$

From $(3.29)_1$ we get, for $N > \frac{l^2 k_0 \sqrt{\rho_1\rho_2}}{\rho_1 k}$,

$$\alpha_1 = \frac{kN\alpha_2 + l(k+k_0)N\alpha_3}{\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N - l^2 k_0}. \tag{3.30}$$

By substituting (3.30) into $(3.29)_3$, we find, for $N > \frac{l^2 k_0 \sqrt{\rho_1\rho_2}}{\rho_1 k}$,

$$\alpha_3 = \frac{lk\left[(k+k_0)N^2 + \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N - l^2 k_0\right]\alpha_2}{\left(-\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N + l^2 k_0\right)\left[(k_0 - k)N^2 + i\delta\lambda_n - \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N + l^2 k\right] - l^2(k+k_0)^2 N^2}. \tag{3.31}$$

By substituting (3.30) and (3.31) into $(3.29)_2$, we obtain, for $N > \frac{l^2 k_0 \sqrt{\rho_1\rho_2}}{\rho_1 k}$,

$$\alpha_2 = \frac{a_1}{a_2}, \tag{3.32}$$

where

$$a_1 = -\rho_2 c\left(\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N - l^2 k_0\right)^2 \left[(k_0 - k)N^2 + i\delta\lambda_n - \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}} N + l^2 k\right]$$

$$+\rho_2 c l^2 (k+k_0)^2 \left(-\frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N + l^2 k_0 \right) N^2$$

and

$$a_2 = l^2 k^2 \left[(k+k_0) N^2 + \frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N - l^2 k_0 \right]^2 + l^2 (k+k_0)^2 \left(l^2 k k_0 - \frac{\rho_1 k^2 + l^2 k k_0 \rho_2}{\sqrt{\rho_1 \rho_2}} N \right) N^2$$
$$+ \left(l^2 k k_0 - \frac{\rho_1 k^2 + l^2 k k_0 \rho_2}{\sqrt{\rho_1 \rho_2}} N \right) \left(\frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N - l^2 k_0 \right) \left[(k_0 - k) N^2 + i \delta \lambda_n - \frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N + l^2 k \right].$$

We see that (3.32) implies that

$$\lim_{n\to\infty} |\alpha_2| = \begin{cases} \left| \frac{c \rho_1 \rho_2 (k-k_0)}{k[\rho_2 l^2 (k+3k_0) + \rho_1 (k-k_0)]} \right|, & if \quad \rho_2 l^2 (k+3k_0) + \rho_1 (k-k_0) \neq 0, \\ \infty, & if \quad \rho_2 l^2 (k+3k_0) + \rho_1 (k-k_0) = 0. \end{cases} \tag{3.33}$$

Because $k - k_0 \neq 0$, then (3.27) holds. Consequently, (3.28) remains valid. □

4 | EXPONENTIAL STABILITY

In this section, we use Theorem 2 to prove that the semigroup associated to system (1.1) to (1.5) is exponentially stable provided that (1.4), (2.3), and (3.1) hold.

Theorem 5. *We assume that (1.4), (2.3), and (3.1) hold. Then, the semigroup associated with (1.1) and (1.2) is exponentially stable.*

Proof. We have proved that the first condition in (2.8) is equivalent to (3.1). Now, by contradiction, we will prove the second condition in (2.8). So we assume that the second condition in (2.8) is false, then there exist sequences $(\Phi_n)_n \subset \mathcal{D}(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}} = 1, \quad \forall\, n \geq 0, \tag{4.1}$$

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{4.2}$$

and

$$\lim_{n\to\infty} \|(i\,\lambda_n\, I - \mathcal{A})\ \Phi_n\|_{\mathcal{H}} = 0, \tag{4.3}$$

which implies that

$$\begin{cases} i\lambda_n \varphi_n - \tilde{\varphi}_n \ \to 0 \ \text{ in } \ H^1_*(0,1), \\ i\lambda_n \rho_1 \tilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l w_n)_x - l k_0 (w_{nx} - l\varphi_n) \ \to 0 \ \text{ in } \ L^2(0,1), \\ i\lambda_n \psi_n - \tilde{\psi}_n \ \to 0 \ \text{ in } \ \tilde{H}^1_*(0,1), \\ i\lambda_n \rho_2 \tilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l w_n) \ \to 0 \ \text{ in } \ L^2(0,1), \\ i\lambda_n w_n - \tilde{w}_n \ \to 0 \ \text{ in } \ \tilde{H}^1_*(0,1), \\ i\lambda_n \rho_1 \tilde{w}_n - k_0 (w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + l w_n) + \delta \tilde{w}_n \ \to 0 \ \text{ in } \ L^2(0,1). \end{cases} \tag{4.4}$$

We will check the second condition in (2.8) by finding a contradiction with (4.1). Our proof is divided into several steps.

Step 1. Taking the inner product of $(i\ \lambda_n\ I - \mathcal{A})\ \Phi_n$ with Φ_n in $\mathcal{H}$ and using (2.5), we get

$$\mathcal{R}]\langle (i\,\lambda_n\, I - \mathcal{A})\,\Phi_n, \Phi_n\rangle_{\mathcal{H}} = \delta \left\| \tilde{w}_n \right\|^2_{L^2(0,1)}.$$

Using (4.1) and (4.3), we deduce that

$$\tilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{4.5}$$

Step 2. Using $(4.4)_1$, $(4.4)_3$, $(4.4)_5$,(4.1), (4.2), and (4.5), we deduce that

$$\begin{cases} \varphi_n \to 0 \text{ in } L^2(0,1), \\ \psi_n \to 0 \text{ in } L^2(0,1), \\ \lambda_n w_n \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.6}$$

Step 3. Taking the inner product of $(5.4)_6$ with w_n in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$k_0 \|w_{nx}\|^2_{L^2(0,1)} + \rho_1 \left\langle i\tilde{w}_n \lambda_n w_n \right\rangle_{L^2(0,1)} + \left\langle \left[lk(\varphi_{nx} + \psi_n + lw_n) + k_0 l\varphi_{nx} + \delta \tilde{w}_n \right], w_n \right\rangle_{L^2(0,1)} \to 0.$$

Using (4.1), (4.2), and $(4.6)_3$, we deduce that

$$w_{nx} \to 0 \text{ in } L^2(0,1), \tag{4.7}$$

and by $(4.4)_5$, we deduce that

$$\frac{\tilde{w}_{nx}}{\lambda_n} \to 0 \text{ in } L^2(0,1). \tag{4.8}$$

Step 4. Taking the inner product of $(\varphi_{nx} + \psi_n + lw_n)$ with $i\lambda_n \tilde{w}_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}
\left\langle (\varphi_{nx} + \psi_n + lw_n), i\lambda_n \tilde{w}_n \right\rangle_{L^2(0,1)} &= -\left\langle i\lambda_n \varphi_{nx}, \tilde{w}_n \right\rangle_{L^2(0,1)} - \left\langle i\lambda_n \psi_n, \tilde{w}_n \right\rangle_{L^2(0,1)} - l\left\langle i\lambda_n w_n, \tilde{w}_n \right\rangle_{L^2(0,1)} \\
&= \left\langle \left(i\lambda_n \varphi_n - \tilde{\varphi}_n \right), \tilde{w}_{nx} \right\rangle_{L^2(0,1)} + \left\langle \tilde{\varphi}_n, \tilde{w}_{nx} \right\rangle_{L^2(0,1)} - \left\langle \left(i\lambda_n \psi_n - \tilde{\psi}_n \right), \tilde{w}_n \right\rangle_{L^2(0,1)} \\
&\quad - \left\langle \tilde{\psi}_n, \tilde{w}_n \right\rangle_{L^2(0,1)} - l\left\langle \left(i\lambda_n w_n - \tilde{w}_n \right), \tilde{w}_n \right\rangle_{L^2(0,1)} - l \left\| \tilde{w}_n \right\|^2_{L^2(0,1)} \\
&= -\left\langle \left(i\lambda_n \varphi_{nx} - \tilde{\varphi}_{nx} \right), \tilde{w}_n \right\rangle_{L^2(0,1)} + \left\langle \tilde{\varphi}_n, \tilde{w}_{nx} \right\rangle_{L^2(0,1)} - \left\langle \left(i\lambda_n \psi_n - \tilde{\psi}_n \right), \tilde{w}_n \right\rangle_{L^2(0,1)} \\
&\quad - \left\langle \tilde{\psi}_n, \tilde{w}_n \right\rangle_{L^2(0,1)} - l\left\langle \left(i\lambda_n w_n - \tilde{w}_n \right), \tilde{w}_n \right\rangle_{L^2(0,1)} - l \left\| \tilde{w}_n \right\|^2_{L^2(0,1)}.
\end{aligned}$$

Then, by using (4.1), $(4.4)_1$, $(4.4)_3$, $(4.4)_5$, and (4.5), we deduce that

$$\left\langle (\varphi_{nx} + \psi_n + lw_n), i\lambda_n \tilde{w}_n \right\rangle_{L^2(0,1)} - \left\langle \tilde{\varphi}_n, \tilde{w}_{nx} \right\rangle_{L^2(0,1)} \to 0. \tag{4.9}$$

Taking the inner product of $\tilde{\varphi}_n$ with $\tilde{w}_{nx}$ in $L^2(0,1)$, we get

$$\begin{aligned}
\left\langle \tilde{\varphi}_n, \tilde{w}_{nx} \right\rangle_{L^2(0,1)} &= \left\langle \tilde{\varphi}_n, \left(\tilde{w}_{nx} - \tilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} + \left\| \tilde{\varphi}_n \right\|^2_{L^2(0,1)} \\
&= -\left\langle \tilde{\varphi}_n, \left(i\lambda_n w_{nx} - \tilde{w}_{nx} \right) \right\rangle_{L^2(0,1)} + \left\langle \tilde{\varphi}_n, \left(i\lambda_n \varphi_n - \tilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} \\
&\quad + \left\langle \tilde{\varphi}_n, i\lambda_n (w_{nx} - \varphi_n) \right\rangle_{L^2(0,1)} + \left\| \tilde{\varphi}_n \right\|^2_{L^2(0,1)},
\end{aligned}$$

then, by (4.1), $(4.4)_1$, and $(4.4)_5$, we have

$$\lambda_n \left\langle \tilde{\varphi}_n, i(w_{nx} - \varphi_n) \right\rangle_{L^2(0,1)} - \left\langle \tilde{\varphi}_n, \tilde{w}_{nx} \right\rangle_{L^2(0,1)} + \left\| \tilde{\varphi}_n \right\|^2_{L^2(0,1)} \to 0. \tag{4.10}$$

On the other hand, taking the inner product of $(4.4)_2$ with $(w_{nx} - l\varphi_n)$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\left\langle i\lambda_n \rho_1 \tilde{\varphi}_n, (w_{nx} - l\varphi_n) \right\rangle_{L^2(0,1)} + k\langle (\varphi_{nx} + \psi_n + lw_n), (w_{nx} - l\varphi_n)_x \rangle_{L^2(0,1)}$$

$$-lk_0 \left\|(w_{nx} - l\varphi_n)\right\|^2_{L^2(0,1)} \to 0,$$

then,

$$\begin{aligned}&\lambda_n\rho_1\left\langle i\tilde{\varphi}_n, (w_{nx} - l\varphi_n)\right\rangle_{L^2(0,1)}\\ &-\frac{k}{k_0}\left\langle(\varphi_{nx} + \psi_n + lw_n), \left[\lambda_n\rho_1\tilde{w}_n - k_0(w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + lw_n) + \delta\tilde{w}_n\right]\right\rangle_{L^2(0,1)}\\ &+\frac{k\rho_1}{k_0}\left\langle(\varphi_{nx} + \psi_n + lw_n),\ i\lambda_n\tilde{w}_n\right\rangle_{L^2(0,1)} + \frac{lk^2}{k_0}\left\|(\varphi_{nx} + \psi_n + lw_n)\right\|^2_{L^2(0,1)}\\ &+\frac{\delta k}{k_0}\left\langle(\varphi_{nx} + \psi_n + lw_n), \tilde{w}_n\right\rangle_{L^2(0,1)} - lk_0\left\|(w_{nx} - l\varphi_n)\right\|^2_{L^2(0,1)} \to 0.\end{aligned}$$

Using (4.1), (4.2), $(4.4)_6$, (4.5), (4.6), and (4.7), we get

$$\begin{aligned}&-\lambda_n\rho_1\left\langle\tilde{\varphi}_n, i\ (w_{nx} - l\varphi_n)\right\rangle_{L^2(0,1)} + \frac{k\rho_1}{k_0}\left\langle(\varphi_{nx} + \psi_n + lw_n), i\lambda_n\tilde{w}_n\right\rangle_{L^2(0,1)}\\ &+\frac{lk^2}{k_0}\left\|(\varphi_{nx} + \psi_n + lw_n)\right\|^2_{L^2(0,1)} \to 0,\end{aligned} \tag{4.11}$$

then, by (4.9), (4.10), and (4.11), we obtain

$$\left(\frac{k}{k_0} - 1\right)\rho_1\left\langle\tilde{\varphi}_n, \tilde{w}_{nx}\right\rangle_{L^2(0,1)} + \frac{lk^2}{k_0}\left\|(\varphi_{nx} + \psi_n + lw_n)\right\|^2_{L^2(0,1)} + \rho_1\left\|\tilde{\varphi}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.12}$$

Here, we use the fact that $k = k_0$ (condition 1.4), we deduce that

$$\frac{lk^2}{k_0}\left\|(\varphi_{nx} + \psi_n + lw_n)\right\|^2_{L^2(0,1)} + \rho_1\left\|\tilde{\varphi}_n\right\|^2_{L^2(0,1)} \to 0,$$

and then using (4.6), we have

$$\varphi_{nx} \to 0 \ \text{ in } \ L^2(0,1) \tag{4.13}$$

and

$$\tilde{\varphi}_n \to 0 \ \text{ in } \ L^2(0,1), \tag{4.14}$$

and by (4.1) , (4.2), $(4.4)_1$, (4.13), and (4.14), we have

$$\lambda_n\varphi_n \to 0 \ \text{ in } \ L^2(0,1) \tag{4.15}$$

and

$$\frac{\tilde{\varphi}_{nx}}{\lambda_n} \to 0 \ \text{ in } \ L^2(0,1). \tag{4.16}$$

Step 5. Taking the inner product of $(4.4)_4$ with $(\varphi_{nx} + \psi_n + lw_n)$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}&\left\langle i\lambda_n\rho_2\tilde{\psi}_n, \varphi_{nx}\right\rangle_{L^2(0,1)} + \left\langle i\lambda_n\rho_2\tilde{\psi}_n, \psi_n\right\rangle_{L^2(0,1)} + l\left\langle i\lambda_n\rho_2\tilde{\psi}_n, w_n\right\rangle_{L^2(0,1)}\\ &+b\langle\psi_{nx}, (\varphi_{nx} + \psi_n + lw_n)_x\rangle_{L^2(0,1)} + k\left\|(\varphi_{nx} + \psi_n + lw_n)\right\|^2_{L^2(0,1)} \to 0,\end{aligned}$$

then,

$$\begin{aligned}&-\lambda_n\rho_2\left\langle\tilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \rho_2\left\langle\tilde{\psi}_n, \left(i\lambda_n\psi_n - \tilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2\left\|\tilde{\psi}_n\right\|^2_{L^2(0,1)}\\ &-l\rho_2\left\langle\tilde{\psi}_n, \left(i\lambda_n w_n - \tilde{w}_n\right)\right\rangle_{L^2(0,1)} - l\rho_2\left\langle\tilde{\psi}_n, \tilde{w}_n\right\rangle_{L^2(0,1)}\\ &-\frac{b}{k}\left\langle\psi_{nx}, \left[i\lambda_n\rho_1\tilde{\varphi}_n - k(\varphi_{nx} + \psi_n + lw_n)_x - lk_0(w_{nx} - l\varphi_n)\right]\right\rangle_{L^2(0,1)}\\ &+\frac{b}{k}\left\langle\psi_{nx}, i\lambda_n\rho_1\tilde{\varphi}_n\right\rangle_{L^2(0,1)} - \frac{lk_0 b}{k}\langle\psi_{nx}, (w_{nx} - l\varphi_n)\rangle_{L^2(0,1)} + k\left\|\varphi_{nx} + \psi_n + lw_n\right\|^2_{L^2(0,1)} \to 0,\end{aligned}$$

using (4.1), $(4.4)_2$, $(4.4)_3$, $(4.4)_5$, (4.5) , (4.6), (4.7), and (4.13), we get

$$-\lambda_n\rho_2\left\langle\tilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \rho_2\left\|\tilde{\psi}_n\right\|^2_{L^2(0,1)} + \frac{b\rho_1}{k}\lambda_n\left\langle\psi_{nx},\ i\tilde{\varphi}_n\right\rangle_{L^2(0,1)} \to 0. \tag{4.17}$$

Now, we use that

$$\lambda_n\left\langle \psi_{nx}, i\tilde{\varphi}_n\right\rangle_{L^2(0,1)} = -\left\langle \left(i\lambda_n\psi_{nx} - \tilde{\psi}_{nx}\right), \tilde{\varphi}_n\right\rangle_{L^2(0,1)} - \left\langle \tilde{\psi}_{nx}, \tilde{\varphi}_n\right\rangle_{L^2(0,1)},$$

and by integrating by parts and using the boundary conditions, we have

$$\begin{aligned}\lambda_n\left\langle \psi_{nx}, i\tilde{\varphi}_n\right\rangle_{L^2(0,1)} &= -\left\langle i\lambda_n\psi_{nx} - \tilde{\psi}_{nx}, \tilde{\varphi}_n\right\rangle_{L^2(0,1)} + \left\langle \tilde{\psi}_n, \tilde{\varphi}_{nx}\right\rangle_{L^2(0,1)} \\ &= -\left\langle \left(i\lambda_n\psi_{nx} - \tilde{\psi}_{nx}\right), \tilde{\varphi}_n\right\rangle_{L^2(0,1)} - \left\langle \tilde{\psi}_n, \left(i\lambda_n\varphi_{nx} - \tilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)} + \left\langle \tilde{\psi}_n, i\lambda_n\varphi_{nx}\right\rangle_{L^2(0,1)},\end{aligned}$$

therefore, from (4.1), $(4.4)_1$, and $(4.4)_3$, we see that

$$\lambda_n\left\langle \psi_{nx}, i\tilde{\varphi}_n\right\rangle_{L^2(0,1)} - \lambda_n\left\langle \tilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} \to 0, \tag{4.18}$$

so, inserting (4.18) into (4.17), we obtain

$$\frac{\lambda_n}{k}(b\rho_1 - k\rho_2)\left\langle \psi_{nx}, i\tilde{\varphi}_n\right\rangle_{L^2(0,1)} - \rho_2\left\|\tilde{\psi}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.19}$$

At this stage, we use the fact that $b\rho_1 - k\rho_2 = 0$, then we have from (4.22)

$$\tilde{\psi}_n \to 0 \text{ in } L^2(0,1), \tag{4.20}$$

and by $(4.4)_3$, we deduce that

$$\lambda_n\psi_n \to 0 \text{ in } L^2(0,1). \tag{4.21}$$

so, inserting (4.18) into (4.17), we obtain

$$\frac{\lambda_n}{k}(b\rho_1 - k\rho_2)\left\langle \psi_{nx}, i\tilde{\varphi}_n\right\rangle_{L^2(0,1)} - \rho_2\left\|\tilde{\psi}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.22}$$

Here, again, using $b\rho_1 - k\rho_2 = 0$, then we have

$$\tilde{\psi}_n \to 0 \text{ in } L^2(0,1), \tag{4.23}$$

and by $(4.4)_3$, we deduce that

$$\lambda_n\psi_n \to 0 \text{ in } L^2(0,1). \tag{4.24}$$

Step 6. Taking the inner product of $(4.4)_4$ with ψ_n in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$-\rho_2\left\langle \tilde{\psi}_n, i\lambda_n\psi_n\right\rangle_{L^2(0,1)} + b\left\|\psi_{nx}\right\|^2_{L^2(0,1)} + k\langle(\varphi_{nx} + \psi_n + lw_n), \psi_n\rangle_{L^2(0,1)} \to 0,$$

and by using (4.6), (4.23), and (4.24), we obtain

$$\psi_{nx} \to 0 \text{ in } L^2(0,1). \tag{4.25}$$

A combination of (4.5) ,(4.6), (4.7) , (4.13), (4.14), (4.23), and (4.24) leads to

$$\|\Phi_n\|_{\mathcal{H}} \to 0,$$

which is a contradiction with (4.1). Hence, the proof of Theorem 5 is completed. □

5 | POLYNOMIAL STABILITY

In this section, we prove the polynomial decay of the solutions of (1.1) to (1.5) using Theorem 3. Our main result is stated as follow:

Theorem 6. *We assume that (2.3) and (3.1) hold. Then, for each $p \in \mathbb{N}^*$, there exists a constant $c_p > 0$ such that*

$$\forall \Phi_0 \in D\left(\mathcal{A}^p\right), \ \forall t > 0, \ \left\| e^{t\mathcal{A}}\Phi_0 \right\|_{\mathcal{H}} \leq c_p \|\Phi_0\|_{D(\mathcal{A}^p)} \left(\frac{\ln t}{t}\right)^{\frac{p}{8}} \ln t. \tag{5.1}$$

Proof. In section 4, we have proved that the first condition in (2.9) is satisfied if (3.1) holds. Now, we need to show that

$$\sup_{|\lambda| \geq 1} \frac{1}{\lambda^8} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{H}} < \infty. \tag{5.2}$$

We establish (5.2) by contradiction. So, if (5.2) is false, then there exist sequences $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying (4.1) , (4.2) and

$$\lim_{n\to\infty} \lambda_n^8 \|(i\lambda_n I - \mathcal{A}) \ \Phi_n\|_{\mathcal{H}} = 0, \tag{5.3}$$

which implies that

$$\begin{cases} \lambda_n^8 \left(i\lambda_n \varphi_n - \tilde{\varphi}_n\right) \to 0 \ \text{ in } \ H_*^1(0,1), \\ \lambda_n^8 \left[i\lambda_n \rho_1 \tilde{\varphi}_n - k(\varphi_{nx} + \psi_n + lw_n)_x - lk_0 \left(w_{nx} - l\varphi_n\right)\right] \to 0 \ \text{ in } \ L^2(0,1), \\ \lambda_n^8 \left(i\lambda_n \psi_n - \tilde{\psi}_n\right) \to 0 \ \text{ in } \ \tilde{H}_*^1(0,1), \\ \lambda_n^8 \left[i\lambda_n \rho_2 \tilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + lw_n\right)\right] \to 0 \ \text{ in } \ L^2(0,1), \\ \lambda_n^8 \left(i\lambda_n w_n - \tilde{w}_n\right) \to 0 \ \text{ in } \ \tilde{H}_*^1(0,1), \\ \lambda_n^8 \left[i\lambda_n \rho_1 \tilde{w}_n - k_0(w_{nx} - l\varphi_n)_x + lk\left(\varphi_{nx} + \psi_n + lw_n\right) + \delta \tilde{w}_n\right] \to 0 \ \text{ in } \ L^2(0,1). \end{cases} \tag{5.4}$$

We will prove that $\| \Phi_n\|_{\mathcal{H}} \to 0$ as a contradiction with (4.1). This will be established through several steps. Step 1. Taking the inner product of $\lambda_n^8 (i \ \lambda_n I - \mathcal{A}) \ \ \Phi_n$ with Φ_n in $\mathcal{H}$, we get

$$\mathcal{R}] \left\langle \lambda_n^8 (i \ \lambda_n I - \mathcal{A}) \ \ \Phi_n, \Phi_n \right\rangle_{L^2(0,1)} = \delta \left\| \lambda_n^4 \tilde{w} \right\|_{L^2(0,1)}^2,$$

so we have

$$\lambda_n^4 \tilde{w}_n \to 0 \ \text{ in } \ L^2(0,1). \tag{5.5}$$

Step 2. Using (4.1), (4.2), $(5.4)_1$, $(5.4)_3$, $(5.4)_5$, and (5.5), we obtain

$$\begin{cases} \varphi_n, \ \psi_n, \ \lambda_n^5 w_n \to 0 \ \text{ in } \ L^2(0,1), \\ (\lambda_n \varphi_n)_n \ \text{ and } \ (\lambda_n \psi_n)_n \ \text{ are uniformly bounded in } \ L^2(0,1). \end{cases} \tag{5.6}$$

Step 3. Taking the inner product of $(5.4)_6$ with $\frac{w_n}{\lambda_n^3}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\rho_1 \left\langle i\lambda_n \tilde{w}_n, \lambda_n^5 w_n \right\rangle_{L^2(0,1)} + k_0 \lambda_n^5 \|w_{nx}\|_{L^2(0,1)}^2 + l(k + k_0) \left\langle \varphi_{nx}, \lambda_n^5 w_n \right\rangle_{L^2(0,1)}$$
$$+ lk\langle (\psi_n + lw_n), w_n \rangle_{L^2(0,1)} + \delta \left\langle \tilde{w}_n, \lambda_n^5 w_n \right\rangle_{L^2(0,1)} \to 0,$$

then, using (4.1), (4.2), (5.5) and (5.6) , we obtain

$$|\lambda_n|^{\frac{5}{2}} w_{nx} \to 0 \quad \text{in} \quad L^2(0,1). \tag{5.7}$$

So, from $(5.4)_5$, we find

$$|\lambda_n|^{\frac{3}{2}} \tilde{w}_{nx} \to 0 \quad \text{in} \quad L^2(0,1). \tag{5.8}$$

Step 4. Applying triangle inequality, we have

$$\begin{aligned} \left\| \frac{\varphi_{nxx}}{\lambda_n} \right\|_{L^2(0,1)} \leq \frac{1}{k} \left\| \frac{1}{\lambda_n} \left[i\lambda_n \rho_1 \tilde{\varphi}_n - k(\varphi_{nx} + \psi_n + lw_n)_x - lk_0 (w_{nx} - l\varphi_n) \right] \right\|_{L^2(0,1)} \\ + \frac{1}{k} \left\| i\rho_1 \tilde{\varphi}_n - \frac{k}{\lambda_n} (\psi_{nx} + lw_{nx}) - \frac{lk_0}{\lambda_n} (w_{nx} - l\varphi_n) \right\|_{L^2(0,1)}, \end{aligned}$$

and using (4.1), (4.2) and $(5.4)_2$, we deduce that

$$\left(\frac{\varphi_{nxx}}{\lambda_n} \right)_n \text{ is uniformly bounded in } L^2(0,1). \tag{5.9}$$

Taking the inner product of $(5.4)_6$ with $\frac{\varphi_{nx}}{\lambda_n^8}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned} \rho_1 \left\langle i\lambda_n \tilde{w}_n, \varphi_{nx} \right\rangle_{L^2(0,1)} + k_0 \left\langle \lambda_n w_{nx}, \frac{\varphi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} + l(k + k_0) \|\varphi_{nx}\|^2_{L^2(0,1)} \\ + lk \langle (\psi_n + lw_n), \varphi_{nx} \rangle_{L^2(0,1)} + \delta \left\langle \tilde{w}_n, \varphi_{nx} \right\rangle_{L^2(0,1)} \to 0, \end{aligned}$$

then, from (4.1), (4.2), (5.6), (5.7) ,(5.8), and (5.9), we have

$$\varphi_{nx} \to 0 \quad \text{in} \quad L^2(0,1). \tag{5.10}$$

Step 5. Taking the inner product of $(5.10)_6$ with $\frac{\varphi_{nx}}{\lambda_n^7}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned} -\rho_1 \left\langle \tilde{w}_n, \lambda_n \left(i\lambda_n \varphi_{nx} - \tilde{\varphi}_{nx} \right) \right\rangle_{L^2(0,1)} + \rho_1 \left\langle \lambda_n \tilde{w}_{nx}, \tilde{\varphi}_n \right\rangle_{L^2(0,1)} \\ + k_0 \left\langle \lambda_n^2 w_{nx}, \frac{\varphi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} + l(k + k_0) \lambda_n \|\varphi_{nx}\|^2_{L^2(0,1)} \\ + lk \langle \lambda_n (\psi_n + lw_n), \varphi_{nx} \rangle_{L^2(0,1)} + \delta \left\langle \lambda_n \tilde{w}_n, \varphi_{nx} \right\rangle_{L^2(0,1)} \to 0, \end{aligned}$$

hence, using (4.1) , (4.2), $(5.4)_1$, (5.6), (5.7), (5.9), and (5.10), we obtain

$$\lambda_n \|\varphi_{nx}\|^2_{L^2(0,1)} \to 0. \tag{5.11}$$

Taking the inner product of $(5.4)_2$ with $\frac{\varphi_n}{\lambda_n^7}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$-\rho_1 \lambda_n \left\langle \tilde{\varphi}_n, \left(i\lambda_n \varphi_n - \tilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} - \rho_1 \lambda_n \left\| \tilde{\varphi}_n \right\|^2_{L^2(0,1)}$$

$$+ k\lambda_n \langle (\varphi_{nx} + \psi_n + lw_n), \varphi_{nx} \rangle_{L^2(0,1)} - lk_0 \lambda_n \langle (w_{nx} - l\varphi_n), \varphi_n \rangle_{L^2(0,1)} \to 0,$$

which implies

$$\begin{aligned} -\rho_1 \left\langle \tilde{\varphi}_n, \lambda_n \left(i\lambda_n \varphi_n - \tilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} - \rho_1 \lambda_n \left\| \tilde{\varphi}_n \right\|^2_{L^2(0,1)} \\ + k\lambda_n \|\varphi_{nx}\|^2_{L^2(0,1)} + k \langle (\lambda_n \psi_n + l\lambda_n w_n), \varphi_{nx} \rangle_{L^2(0,1)} \\ - lk_0 \langle (\lambda_n w_{nx} - l\lambda_n \varphi_n), \varphi_n \rangle_{L^2(0,1)} \to 0, \end{aligned}$$

so, using (4.1), (4.2), $(5.4)_1$, (5.6) , (5.7), and (5.11), we deduce that

$$\lambda_n \left\|\tilde{\varphi}_n\right\|^2_{L^2(0,1)} \to 0, \tag{5.12}$$

and from $(5.4)_1$, we obtain that

$$\lambda_n^3 \|\varphi_n\|^2 \to 0. \tag{5.13}$$

Step 6. Multiplying $(5.4)_2$ by $\frac{1}{|\lambda_n|^{\frac{1}{2}}\lambda_n^8}$, we get

$$i\frac{\lambda_n}{|\lambda_n|}\rho_1|\lambda_n|^{\frac{1}{2}}\tilde{\varphi}_n - k\frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} - k\frac{\psi_{nx}}{|\lambda_n|^{\frac{1}{2}}} - l(k+k_0)\frac{w_{nx}}{|\lambda_n|^{\frac{1}{2}}} + l^2k_0\frac{\varphi_n}{|\lambda_n|^{\frac{1}{2}}} \to 0 \ \text{ in } \ L^2(0,1),$$

then, using (4.1), (4.2) and (5.12), we deduce that

$$\frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} \to 0 \ \text{ in } \ L^2(0,1). \tag{5.14}$$

On the other hand, by integrating by parts and using the boundary conditions, we see that

$$\begin{aligned}\lambda_n\langle w_{nxx}, i\lambda_n\varphi_{nx}\rangle_{L^2(0,1)} &= \lambda_n^2\langle iw_{nx}, \varphi_{nxx}\rangle_{L^2(0,1)}\\ &= \left\langle \lambda_n\left(i\lambda_n w_{nx} - \tilde{w}_{nx}\right), \varphi_{nxx}\right\rangle_{L^2(0,1)} + \lambda_n\left\langle \tilde{w}_{nx}, \varphi_{nxx}\right\rangle_{L^2(0,1)}\\ &= \left\langle \lambda_n^2\left(i\lambda_n w_{nx} - \tilde{w}_{nx}\right), \frac{\varphi_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)} + \left\langle \lambda_n|\lambda_n|^{\frac{1}{2}}\tilde{w}_{nx}, \frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}}\right\rangle_{L^2(0,1)},\end{aligned}$$

then, using (4.2), $(5.4)_5$, (5.8) , and (5.14), we obtain

$$\lambda_n\langle w_{nxx}, i\lambda_n\varphi_{nx}\rangle_{L^2(0,1)} \to 0. \tag{5.15}$$

Furthermore, integrating by parts and using the boundary conditions,

$$\begin{aligned}\lambda_n\left\langle(\varphi_{nx}+\psi_n+lw_n)_x, \tilde{\varphi}_n\right\rangle_{L^2(0,1)} &= -\lambda_n\left\langle(\varphi_{nx}+\psi_n+lw_n), \tilde{\varphi}_{nx}\right\rangle_{L^2(0,1)}\\ &= -\frac{1}{lk}\left\langle \lambda_n^2\left[i\lambda_n\rho_1\tilde{w}_n - k_0(w_{nx}-l\varphi_n)_x + lk(\varphi_{nx}+\psi_n+lw_n) + \delta\theta_{nx}\right], \frac{\tilde{\varphi}_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)}\\ &\quad -\frac{1}{lk}\left\langle\left(i\lambda_n\rho_1\tilde{w}_n + \delta\theta_{nx}\right), \lambda_n\left(i\lambda_n\varphi_{nx} - \tilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)}\\ &\quad +\frac{k_0}{lk}\left\langle(w_{nx}-l\varphi_n)_x, \lambda_n\left(i\lambda_n\varphi_{nx} - \tilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)} - \frac{\lambda_n^3}{lk}\left\langle i\rho_1\tilde{w}_{nx}, i\varphi_n\right\rangle_{L^2(0,1)}\\ &\quad +\frac{\delta}{lk}\left\langle\lambda_n^2\theta_{nx}, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \frac{k_0\lambda_n}{lk}\langle w_{nxx}, i\lambda_n\varphi_{nx}\rangle_{L^2(0,1)} - \frac{k_0\lambda_n^2}{k}i\|\varphi_{nx}\|^2_{L^2(0,1)},\end{aligned}$$

then, using $(5.4)_1$, $(5.4)_6$, (5.7) , (5.8), (5.13), and (5.15), we find

$$\lambda_n\left\langle(\varphi_{nx}+\psi_n+lw_n)_x, \tilde{\varphi}_n\right\rangle_{L^2(0,1)} + \frac{k_0}{k}i\|\lambda_n\varphi_{nx}\|^2_{L^2(0,1)} \to 0. \tag{5.16}$$

Taking the inner product of $(5.4)_2$ with $\frac{\tilde{\varphi}_n}{\lambda_n^7}$ in $L^2(0,1)$, we get

$$\rho_1 i\left\|\lambda_n\tilde{\varphi}_n\right\|^2_{L^2(0,1)} - k\lambda_n\left\langle(\varphi_{nx}+\psi_n+lw_n)_x, \tilde{\varphi}_n\right\rangle_{L^2(0,1)} - lk_0\left\langle(\lambda_n w_{nx} - l\lambda_n\varphi_n), \tilde{\varphi}_n\right\rangle_{L^2(0,1)} \to 0,$$

then, using (5.16), we obtain

$$\rho_1 i \left\| \lambda_n \tilde{\varphi}_n \right\|^2_{L^2(0,1)} + ik_0 \left\| \lambda_n \varphi_{nx} \right\|^2_{L^2(0,1)} - lk_0 \left\langle (\lambda_n w_{nx} - l\lambda_n \varphi_n), \tilde{\varphi}_n \right\rangle_{L^2(0,1)} \to 0,$$

and from (4.1), (4.2), $(5.4)_1$, (5.12), and (5.13), we deduce that

$$\lambda_n \tilde{\varphi}_n \to 0 \text{ in } L^2(0,1) \tag{5.17}$$

and

$$\lambda_n \varphi_{nx} \to 0 \text{ in } L^2(0,1). \tag{5.18}$$

Step 7. Multiplying $(5.4)_4$ by $\frac{1}{\lambda_n^9}$, we obtain

$$i\rho_2 \tilde{\psi}_n - b\frac{\psi_{nxx}}{\lambda_n} + \frac{k}{\lambda_n}(\varphi_{nx} + \psi_n + lw_n) \to 0 \text{ in } L^2(0,1).$$

By triangle inequality, we deduce from (4.1) and (4.2) that

$$\left(\frac{\psi_{nxx}}{\lambda_n} \right)_n \text{ is uniformly bounded in } L^2(0,1). \tag{5.19}$$

Taking the inner product of $(5.4)_2$ with $\frac{\psi_{nx}}{\lambda_n^8}$ in $L^2(0,1)$, we get

$$\rho_1 \left\langle i\lambda_n \tilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} - k\langle \varphi_{nxx}, \psi_{nx} \rangle_{L^2(0,1)} - k \left\| \psi_{nx} \right\|^2_{L^2(0,1)}$$

$$-l(k+k_0)\langle w_{nx}, \psi_{nx} \rangle_{L^2(0,1)} + l^2 k_0 \langle \varphi_n, \psi_{nx} \rangle_{L^2(0,1)} \to 0,$$

then, integrating by parts and using the boundary conditions, we obtain

$$\rho_1 \left\langle i\lambda_n \tilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} + k\left\langle \lambda_n \varphi_{nx}, \frac{\psi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} - k \left\| \psi_{nx} \right\|^2_{L^2(0,1)}$$

$$-l(k+k_0)\langle w_{nx}, \psi_{nx} \rangle_{L^2(0,1)} + l^2 k_0 \langle \varphi_n, \psi_{nx} \rangle_{L^2(0,1)} \to 0,$$

so, using (4.1), (4.2), (5.5), (5.6) , (5.7), (5.18), and (5.19), we deduce that

$$\psi_{nx} \to 0 \quad \text{in} \quad L^2(0,1). \tag{5.20}$$

Taking the inner product of $(5.4)_4$ with $\frac{\psi_n}{\lambda_n^8}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$-\rho_2 \left\langle \tilde{\psi}_n, \left(i\lambda_n \psi_n - \tilde{\psi}_n \right) \right\rangle_{L^2(0,1)} - \rho_2 \left\| \tilde{\psi}_n \right\|^2_{L^2(0,1)} + b \left\| \psi_{nx} \right\|^2_{L^2(0,1)}$$

$$+\langle k(\varphi_{nx} + \psi_n + lw_n), \psi_n \rangle_{L^2(0,1)} \to 0,$$

hence, using (4.1), (4.2), $(5.4)_3$, (5.6), and (5.20), we get

$$\tilde{\psi}_n \to 0 \text{ in } L^2(0,1). \tag{5.21}$$

A combination of (4.2) and all the above convergence leads to

$$\|\varphi_n\|_{\mathcal{H}} \to 0,$$

which is a contradiction with (4.1). Consequently, the proof of our Theorem 6 is completed. □

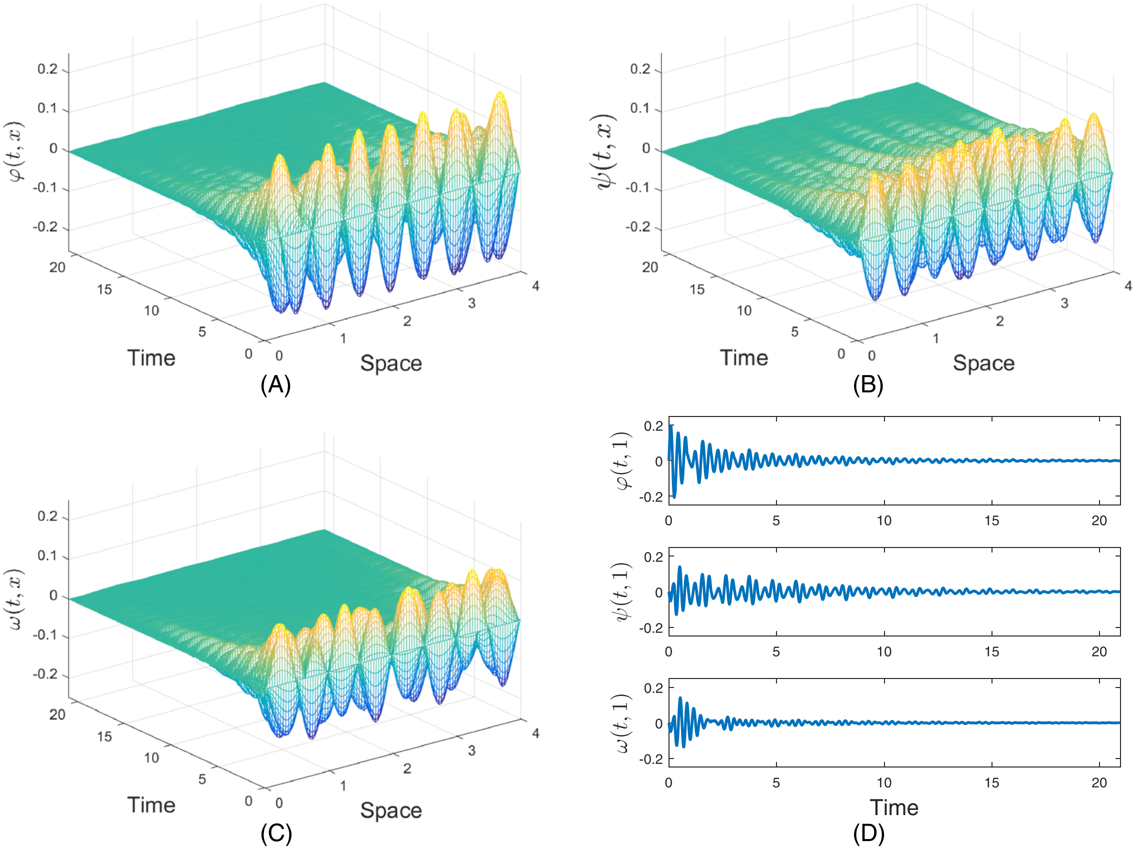

FIGURE 1 Test 1: Exponential decay

6 | NUMERICAL RESULTS

In the numerical part of this paper, we examine our theoretical results. We solve (1.1) using the finite difference method for discretizing the space-time domain $[0, L] \times [0, T_e]$. We implement a first-order approximation in time and a second order in space. For a similar construction, we refer to Santos and Junior.[19] We conduct three different tests as follow:

- Test 1: Here, we present the exponential decay case based on the result in Theorem 4.1 using the inputs parameters as follows: $\rho_1 = \rho_2 = 1$, $k = k_0 = 1$, $b = 1$, and $\ell = 1$.
- Test 2: In the second numerical test, we examine the polynomial decay case as in Theorem 6, for this case we choose the following parameters: $\rho_1 = \rho_2 = 1$, $k = 1$, $b = 1$, $\ell = 1$, and $k_0 = 0.5$.
- Test 3: In the third numerical test, we simulate the conservative case. We take out the damping source (ie, $\delta = 0$) from the third equation of the system (1.1) and we simulate the problem using the same value as in test 1.

In our numerical approach, we perform three tests using a fixed time steps Δt satisfying a stability condition according to the Courant-Friedrichs-Lewy (CFL) inequality.

For our numerical test, we uses a unform discretization of $[0, 4]$ into 500 subintervals and we fixed the time step by the condition $\Delta t = \Delta x/8$. The initial data are given by

$$\varphi(x,0) = \psi(x,0) = \omega(x,0) = 0 \quad \text{in} \quad [0,4], \tag{6.1}$$

and

$$\varphi_t(x,0) = \cos(2\pi x); \quad \psi_t(x,0) = \omega_t(x,0) = \sin(2\pi x) \quad \text{in} \quad [0,4]. \tag{6.2}$$

In Figure 1, we plot the result of the first test, namely, the exponential decay case. Figure 1A shows the two dimensional time-space evolution of φ, Figure 1B shows the evolution of ψ, and Figure 1C shows the evolution of ω. Figure 1D shows the corresponding cross-section cut at $x = 0.25$.

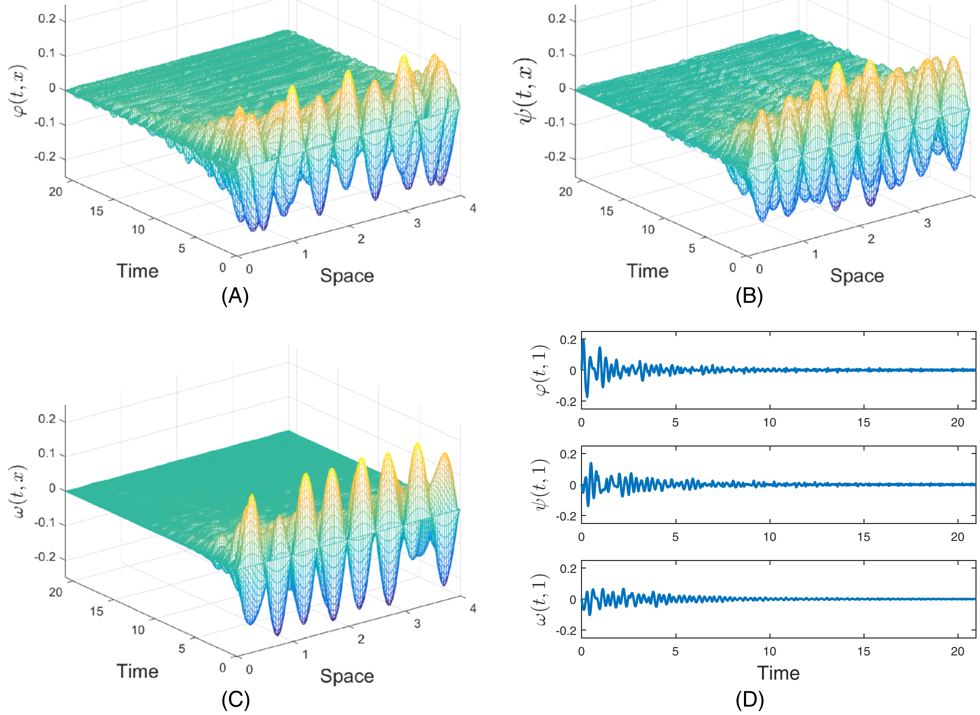

FIGURE 2 Test 2: Polynomial decay

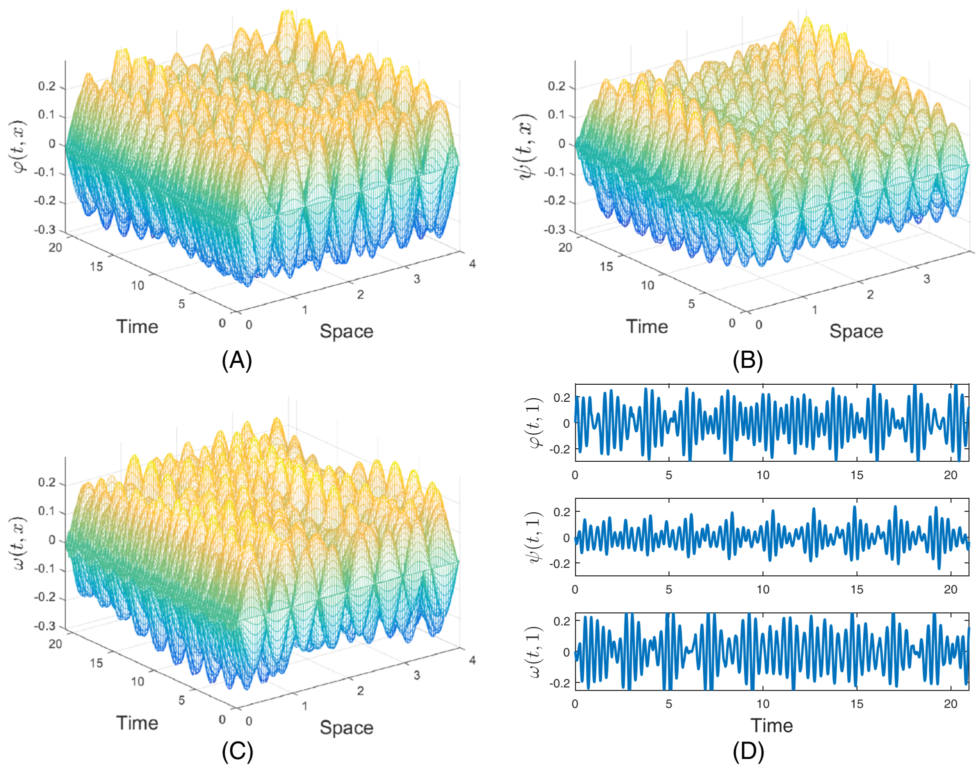

FIGURE 3 Test 3: Conservative system

In Figure 2, we show similar results to test 1. We need to mention here the difference between the decay behavior of this test in comparison with the first one.

In Figure 3, we plot the result of the conservative case, namely, after removing the damping term in the third equation. Figure 3A shows the nondecay of the two-dimensional time-space evolution of φ, Figure 3B shows a nonconverging periodic-like behavior of the evolution of ψ, and Figure 3C shows the nondamped behavior of ω. Figure 3D shows the corresponding cross-section cut at $x = 0.25$. where, the nondecay physical behavior is clearly demonstrated.

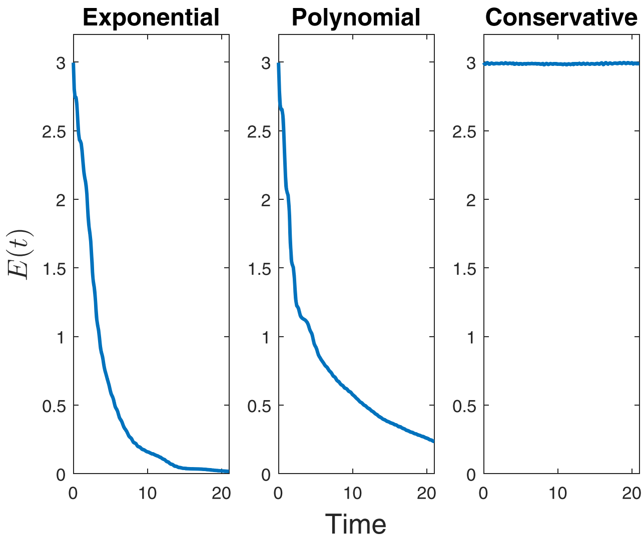

FIGURE 4 Energy plot of tests 1, 2, and 3

In Figure 4, we plot the energy for the three scenarios as mentioned in tests 1, 2, and 3. As it is seen in the figures, the asymptotic behavior through the curves clearly justify our new stability conditions proved in the previous sections.

7 | CONCLUDING REMARKS AND OPEN PROBLEMS

In this paper, Bresse system with one damping was considered, we proved the well-posedness, stability results were proved subject to a new relationship on the coefficients of the Bresse system, and some numerical approximations were presented to validate the theoretical results obtained in this paper. The proofs were based on a combination between frequency domain approach and multipliers techniques. We need to mention here that the optimality rates of decay in the polynomial stability is an open problem as well as the stability results in case of Dirichlet boundary conditions instead of a mixed boundary conditions considered in this paper is an open problem too.

This work does not have any conflict of interest.

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REFERENCES

1. Lagnese, JE, Leugering, G and Schmidt, JP, Modelling analysis and control of dynamic elastic multi-link structures, systems control found. Appl., 1994.
2. Alabau-Boussouira F, Muñoz Rivera JE, Almeida Júnior DS. Stability to weak dissipative Bresse system. *J Math Anal Appl*. 2011;374:481-498.
3. Alves MO, Fatori LH, Jorge Silva MA, Monteiro RN. Stability and optimality of decay rate for weakly dissipative Bresse system. *Math Meth Appl Sci*. 2015;38:898-908.
4. Noun N, Wehbe A. Weakly locally internal stabilization of elastic Bresse system. *C R Acad Scie Paris, Sér I*. 2012;350:493-498.
5. Soriano JA, Charles W, Schulz R. Asymptotic stability for Bresse systems. *J Math Anal Appl*. 2014;412:369-380.
6. Soriano JA, Rivera JM, Fatori LH. Bresse system with indefinite damping. *J Math Anal Appl*. 2012;387:284-290.
7. Soufyane A, Said-Houari B. The effect of the wave speeds and the frictional damping terms on the decay rate of the Bresse system. *Evol Equa Cont Theory*. 2014;3:713-738.
8. Wehbe A, Youssef W. Exponential and polynomial stability of an elastic Bresse system with two locally distributed feedbacks. *J Math Phys*. 2010;51:1-17.
9. De Lima Santos M. A. Soufyane and D. Da Silva Almeida Júnior, Asymptotic behavior to Bresse system with past history. *Quart Appl Math*. 2015;73:23-54.
10. Guesmia A, Kafini M. Bresse system with infinite memories. *Math Meth Appl Sci*. 2015;38:2389-2402.
11. Guesmia A, Kirane M. Uniform and weak stability of Bresse system with two infinite memories. *Z Angew Math Phys*. 2016;67:1-39.
12. Guesmia A. Asymptotic stability of Bresse system with one infinite memory in the longitudinal displacements. *Mediterr J Math*. 2017;14:49.
13. Ghoul TE, Khenissi M, Said-Houari B. On the stability of the Bresse system with frictional damping. *Math Anal Appl*. 2017;455:1870-1898.

14. Afilall M, Guesmia A, Soufyane A. New stability results for a linear thermoelastic Bresse system with second sound. *Appl Math Opt*. 2019;1-40. https://doi.org/10.1007/s00245-019-09560-7
15. Pazy A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York: Springer-Verlag; 1983.
16. Huang FL. Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces. *Ann Diff Equa*. 1985;1:43-56.
17. Pruss J. On the spectrum of C_0 semigroups. *Trans Amer Math Soc*. 1984;284:847-857.
18. Liu Z, Rao B. Characterization of polymomial decay rate for the solution of linear evolution equation. *Z Angew Math Phys*. 2005;56:630-644.
19. Santos, ML and Dilberto da Junior, SA, A numerical exponential decay to dissipative Bresse system, *J Appl Math*, vol. 2010, Article ID 848620, 17 pages, 2010. https://doi.org/10.1155/2010/848620

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