Well-posedness and asymptotic behavior of a wave equation with time delay and Neumann boundary conditions

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Abstract

This paper is concerned with the asymptotic behavior analysis of solutions to a multi-dimensional wave equation. Assuming that there is no displacement term in the system and taking into consideration the presence of distributed or discrete time delay, we show that the solutions exponentially converge to their stationary state. The proof mainly consists in utilizing the resolvent method. The approach adopted in this work is also used to other physical systems.

Keywords: Wave equation; time delay; long time behavior; exponential convergence; resolvent method.

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1 Introduction

Let $N \in \mathbb{N}$ be a positive integer and Ω be a bounded open connected set of $\mathbb{R}^N$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. We consider in this paper the following wave equation with distributed time delay:

$$u_{tt}(x,t) + Au(x,t) + a(x)u_t(x,t) + \int_0^{+\infty} f(s)u_t(x,t-s)ds = 0, \tag{1.1}$$

for any $x \in \Omega$ and $t > 0$, as well as the homogeneous Neumann boundary condition on the whole boundary Γ

$$\partial_A u(x,t) := \sum_{i,j=1}^{N} a_{ij}(x)\nu_j \partial_i u(x,t) = 0, \quad \forall x \in \Gamma,\ \forall t > 0, \tag{1.2}$$

and initial conditions

$$\begin{cases} u(x,-t) = u_0(x,t), & \forall x \in \Omega,\ \forall t \in \mathbb{R}_+, \\ u_t(x,-t) = u_1(x,t), & \forall x \in \Omega,\ \forall t \in \mathbb{R}_+, \end{cases} \tag{1.3}$$

where u is the displacement and the differential operator with domain $D(A) = H^2(\Omega)$ is given by

$$A = -\sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j), \tag{1.4}$$

in which $\partial_k = \frac{\partial}{\partial x_k}$. In turn, $a_{ij} \in C^1(\bar{\Omega})$ and $a \in C(\bar{\Omega})$ such that there exist α_0, $a_0 > 0$ satisfying

$$a_{ij}(x) = a_{ji}(x), \quad \sum_{k,l=1}^{N} a_{kl}(x)\epsilon_k\epsilon_l \geq \alpha_0 \sum_{k=1}^{N} \epsilon_k^2, \quad \forall x \in \Omega,\ \forall\, \epsilon_k,\ \epsilon_l \in \mathbb{R},\ \forall\, i,j = 1,\cdots,N \tag{1.5}$$

and

$$a(x) \geq a_0, \quad \forall x \in \Omega. \tag{1.6}$$

Moreover, $\nu = (\nu_1, \cdots, \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω and (u_0, u_1) are given initial data belonging to a suitable space. Lastly, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function.

Let $H = L^2(\Omega)$ equipped with its standard norm $\|\cdot\|$ generated by the inner product

$$\langle p, q\rangle = \int_\Omega p(x)q(x)\,dx.$$

Let $V = H^1(\Omega)$ equipped with its classical norm $\|\cdot\|_V$ generated by the inner product

$$\langle p, q\rangle_V = \int_\Omega \left(p(x)q(x) + \sum_{i=1}^{N} \partial_i p(x)\partial_i q(x) \right) dx.$$

The main concern of our work is to compensate the effect of the delay term in (1.1) via the action of the damping $a(x)u_t$.

The study of long time behavior of the wave equation has been the subject of an active research and mathematical endeavor. Thereby, a huge number of research articles have been appeared. In order to highlight the main contribution and the novelty of the present paper, we will merely point out the articles whose content is closely related to our problem. Indeed, since the pioneer work of [10, 11, 12], where it has been shown that the presence of a delay in a one-dimensional wave equation could create an instability phenomenon, delayed wave equation has come under the spotlight (see for instance [3, 4, 5, 13, 14, 16, 21, 23, 24, 25] and the references therein). Recently, the authors in [1] studied the asymptotic behavior of solutions of a delayed wave equation without neither the well-known (BLR) geometric condition [6, 19] on the domain nor the presence of any displacement term in the system. In such an event, the solutions of the wave equation with a boundary delay term converge with a logarithmic rate to an equilibrium state. Naturally, the convergence can be improved to the exponential one when the (BLR) geometric condition holds [2]. In turn, the internal delayed wave equation in a three-dimensional domain with a trapped ray and hence the geometric control condition is violated can have a polynomial convergence of solutions [2].

In the present paper, we shall continue in the direction of [1] by considering a distributed delay (1.1) appearing as an infinite memory. Unlike the discrete time delay case which does not take in account the inherent memory effects, we shall deal with the distributed time delay as an infinite past history of the solution. Indeed, the discrete case will be treated as a special case which corresponds to the Dirac delta distribution kernel (at some time τ).

The rest of this work is organized as follows. In Section 2, the system (1.1)-(1.3) is shown to be well-posed in the sense of semigroups theory of linear operators. Section 3 is devoted to the analysis of the asymptotic behavior of (1.1)-(1.3) via LaSalle's principle. The proof of the exponential convergence of solutions to (1.1)-(1.3) is provided in Section 4, by using the resolvent method. We treat in Section 5 the discrete time delay case. The results are illustrated via a set of applications to other physical system in Section 6. Finally, this note ends with a conclusion.

2 Well-posedness of (1.1)-(1.3)

In this section, we state our assumptions on a and f, and prove the well-posedness of (1.1)-(1.3). We assume that

(A1) The function f is of class $C^1(\mathbb{R}_+, \mathbb{R}_+)$, non-increasing and satisfies, for some positive constants α_1 and α_2,

$$-\alpha_1 f(s) \leq f'(s) \leq -\alpha_2 f(s), \quad \forall s \in \mathbb{R}_+. \tag{2.1}$$

(A2) The positive function a appearing in (1.1) satisfies

$$\int_0^{+\infty} f(s)ds := f_0 < \frac{1}{\alpha_1}(a_0\alpha_2 + f(0)). \tag{2.2}$$

Remarks 2.1. *The class of functions f satisfying* (**A1**) *and* (**A2**) *is very wide and contains the ones which converge to zero exponentially as*

$$f(s) = de^{-qs},$$

where d, $q > 0$. Condition (2.1) *is satisfied with $\alpha_1 = \alpha_2 = q$, and condition* (2.2) *is automatically satisfied because it is equivalent to $qa_0 > 0$.*

In order to express system (1.1)-(1.3) in a linear first-order system, we consider the new state variable η and its initial data η_0 introduced in [9] and given by

$$\begin{cases} \eta(x,t,s) = u(x,t) - u(x,t-s), & \forall x \in \Omega,\ \forall t,\ s \in \mathbb{R}_+, \\ \eta_0(x,s) = \eta(x,0,s) = u_0(x,0) - u_0(x,s), & \forall x \in \Omega,\ \forall s \in \mathbb{R}_+. \end{cases} \tag{2.3}$$

This variable η satisfies

$$\begin{cases} \eta_t(x,t,s) + \eta_s(x,t,s) = u_t(x,t), & \forall x \in \Omega,\ \forall t,\ s \in \mathbb{R}_+, \\ \eta_s(x,t,s) = u_t(x,t-s), & \forall x \in \Omega,\ \forall t,\ s \in \mathbb{R}_+, \\ \eta(x,t,0) = 0, & \forall x \in \Omega,\ \forall t \in \mathbb{R}_+. \end{cases} \tag{2.4}$$

To simplify the formulas, the variables x, t and s are omitted whenever there is no ambiguity. Let

$$\begin{cases} y = u_t, \\ \Psi = (u, y, \eta), \\ \Psi_0 = (u_0(\cdot,0), u_1(\cdot,0), \eta_0) \end{cases}$$

and

$$\mathcal{H} = V \times H \times L_f,$$

where L_f is the weighted space with respect to the measure $f(s)ds$ defined by

$$L_f = \left\{ \eta : \mathbb{R}_+ \to H; \quad \int_0^{+\infty} f(s) \|\eta(s)\|^2 \, ds < +\infty \right\}$$

and endowed with the inner product

$$\langle \eta, \tilde{\eta} \rangle_{L_f} = \int_0^{+\infty} f(s) \langle \eta(s), \tilde{\eta}(s) \rangle \, ds.$$

Then (1.1)-(1.3) can be formulated as

$$\begin{cases} \Psi_t(x,t) = \mathcal{A}\Psi(x,t), & \forall x \in \Omega,\ \forall t > 0, \\ \Psi(x,0) = \Psi_0(x), & \forall x \in \Omega, \end{cases} \tag{2.5}$$

where $\mathcal{A}$ is a linear operator given by

$$\mathcal{A} \begin{pmatrix} u \\ y \\ \eta \end{pmatrix} = \begin{pmatrix} y \\ -Au - ay - \displaystyle\int_0^{+\infty} f(s)\eta_s(s)\, ds \\ -\eta_s + y \end{pmatrix}. \tag{2.6}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \Psi = (u, y, \eta) \in \mathcal{H},\ Au \in H,\ y \in V,\ \eta_s \in L_f,\ \eta(0) = 0,\ \partial_A u = 0 \,\text{on}\, \Gamma \right\}. \tag{2.7}$$

Therefore, we conclude from (2.4) and (2.6) that the systems (1.1)-(1.3) and (2.5) are equivalent.

Because the Neumann boundary condition is considered on the whole boundary Γ and no displacement term is considered neither on Ω nor on Γ, the classical energy norm $\|\cdot\|_E$ given by

$$\|(u,y,\eta)\|_E^2 = \int_\Omega \sum_{i=1,j}^{N} a_{ij}\partial_i u \partial_j u \, dx + \|y\|^2 + \|\eta\|_{L_f}^2$$

is only a semi-norm in our case, since $\|(u,y,\eta)\|_E^2 = 0$ does not imply that $u = 0$. Hence we need to define an inner product on $\mathcal{H}$ which makes it a Hilbert space. By integrating (1.1) on Ω, we get

$$\partial_t \left(\int_\Omega \left(u_t + au + \int_0^{+\infty} f(s)u(t-s)\, ds \right) dx \right) = 0.$$

Using the definition of y, η and f_0, we get

$$\partial_t \left(\int_\Omega \left((f_0+a)u + y - \int_0^{+\infty} f(s)\eta(s)\, ds \right) dx \right) = 0. \tag{2.8}$$

This will help to introduce the appropriate inner product on $\mathcal{H}$. In fact, we pick up an arbitrary positive constant ϵ_0 and consider on $\mathcal{H}$ the inner product

$$\langle (u,y,\eta),(\tilde{u},\tilde{y},\tilde{\eta})\rangle_{\mathcal{H}} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j \tilde{u} \right) dx + \langle y,\tilde{y}\rangle + \alpha_1 \langle \eta,\tilde{\eta}\rangle_{L_f}$$

$$+\epsilon_0 \left(\int_\Omega \left((f_0+a)u + y - \int_0^{+\infty} f(s)\eta(s)\, ds \right) dx \right) \int_\Omega \left((f_0+a)\tilde{u} + \tilde{y} - \int_0^{+\infty} f(s)\tilde{\eta}(s)\, ds \right) dx,$$

where α_1 is the positive constant appearing in (2.1). To prove that $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space, it is enough to prove that there exists $\epsilon_0 > 0$ such that the norm $\|\cdot\|_{V\times H\times L_f}$ and the one $\|\cdot\|_{\mathcal{H}}$ generated by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ are equivalent. This will be done in the following lemma.

Lemma 2.2. *There exists $\epsilon_1 > 0$ such that for any $\epsilon_0 \in (0,\epsilon_1)$, and for any $(u,y,\eta) \in \mathcal{H}$, we have:*

$$c_1\|(u,y,\eta)\|_{V\times H\times L_f}^2 \le \|(u,y,\eta)\|_{\mathcal{H}}^2 \le c_2\|(u,y,\eta)\|_{V\times H\times L_f}^2, \tag{2.9}$$

where c_1, c_2 are positive constants.

Proof. Firstly, note that there exists $k_1 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}\epsilon_i\epsilon_j \le k_1 \sum_{i=1}^{N} \epsilon_i^2, \quad \forall\ (\epsilon_1,\cdots,\epsilon_N) \in \mathbb{R}^N. \tag{2.10}$$

It is clear that, using Hölder's and Young's inequalities, the direct inequality in (2.9) holds, for any $\epsilon_0 > 0$ and where c_2 depends on ϵ_0, k_1, a, f and Ω.

To prove the reverse inequality in (2.9), we see that

$$\left(\int_\Omega \left((f_0+a)u + y - \int_0^{+\infty} f(s)\eta(s)\, ds \right) dx \right)^2 = \left(\int_\Omega (f_0+a)u\, dx \right)^2 \tag{2.11}$$

$$+\left(\int_\Omega \left(y-\int_0^{+\infty} f(s)\eta(s)\,ds\right)\,dx\right)^2+2\left(\int_\Omega (f_0+a)u\,dx\right)\int_\Omega \left(y-\int_0^{+\infty} f(s)\eta(s)\,ds\right)\,dx.$$

Using Young's and Hölder's inequalities, we have

$$2\left(\int_\Omega (f_0+a)u\,dx\right)\left(\int_\Omega \left(y-\int_0^{+\infty} f(s)\eta(s)\,ds\right)\,dx\right)$$

$$\geq -\frac{1}{2}\left(\int_\Omega (f_0+a)u\,dx\right)^2-2\left(\int_\Omega \left(y-\int_0^{+\infty} f(s)\eta(s)\,ds\right)\,dx\right)^2$$

and there exists a positive constant c such that

$$\left(\int_\Omega \left(y-\int_0^{+\infty} f(s)\eta(s)\,ds\right)\,dx\right)^2 \leq c\left(\|y\|^2+\|\eta\|_{L_f}^2\right), \quad \forall (y,\eta)\in H\times L_f.$$

On the other hand, using a classical compactness argument, one can show the following generalized Poincaré's inequality (thanks to (1.6)): there exists a positive constant ϖ_0 depending on Ω, f_0 and a such that

$$\|u\|^2 \leq \varpi_0\left(\|\nabla u\|^2+\left(\int_\Omega (f_0+a)u\,dx\right)^2\right), \quad \forall u\in V. \tag{2.12}$$

Combining the above three inequalities and (2.11), we get

$$\epsilon_0\left(\int_\Omega \left((f_0+a)u+y-\int_0^{+\infty} f(s)\eta(s)\,ds\right)\,dx\right)^2 \geq \frac{\epsilon_0}{2\varpi_0}\|u\|^2-\frac{\epsilon_0}{2}\|\nabla u\|^2-\epsilon_0 c\|y\|^2-\epsilon_0 c\|\eta\|_{L_f}^2.$$

Therefore, we have after recalling (1.5)

$$\|(u,y,\eta)\|_{\mathcal{H}}^2 \geq \left(\alpha_0-\frac{\epsilon_0}{2}\right)\|\nabla u\|^2+\frac{\epsilon_0}{2\varpi_0}\|u\|^2+(1-\epsilon_0 c)\|y\|^2+(\alpha_1-\epsilon_0 c)\|\eta\|_{L_f}^2.$$

Thereafter, for ϵ_0 satisfying

$$0<\epsilon_0<\epsilon_1:=\min\left\{2\alpha_0,\frac{1}{c},\frac{\alpha_1}{c}\right\},$$

we obtain the reverse inequality in (2.9) with

$$c_1=\min\left\{\alpha_0-\frac{\epsilon_0}{2},\frac{\epsilon_0}{2\varpi_0},1-\epsilon_0 c,\alpha_1-\epsilon_0 c\right\}>0.$$

□

The well-posedness of problem (2.5) is ensured by the following theorem:

Theorem 2.3. *Assume that* **(A1)-(A2)** *hold. Then, the operator $\mathcal{A}$ is the infinitesimal generator of a linear C_0 semigroup $S(t)$ of contractions on $\mathcal{H}$ and its domain $D(\mathcal{A})$ is dense in $\mathcal{H}$. Moreover, for any $\Psi_0 \in \mathcal{H}$, the system (2.5) has a unique weak solution*

$$\Psi \in C(\mathbb{R}_+, \mathcal{H}). \tag{2.13}$$

Moreover, if $\Psi_0 \in D(\mathcal{A})$, then the solution of (2.5) satisfies (classical solution)

$$\Psi \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, D(\mathcal{A})). \tag{2.14}$$

Proof. We will prove that $\mathcal{A}$ is m-dissipative, that is $\mathcal{A}$ is dissipative and $I - \mathcal{A}$ is onto. Let $\Psi \in D(\mathcal{A})$. Using the definition of Ψ, $\mathcal{A}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and integrating by parts, we find

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} = -\|\sqrt{a}y\|^2 + \alpha_1 \int_\Omega y \int_0^{+\infty} f(s)\eta(s)\, ds\, dx$$

$$- \int_\Omega y \int_0^{+\infty} f(s)\eta_s(s)\, ds\, dx - \alpha_1 \int_\Omega \int_0^{+\infty} f(s)\eta(s)\eta_s(s)\, ds\, dx.$$

Then , integrating with respect to s the last two integrals and noting that $\eta(0) = 0$, we obtain

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} = -\|\sqrt{a}y\|^2 + \frac{\alpha_1}{2} \int_0^{+\infty} f'(s)\|\eta(s)\|^2\, ds + \int_\Omega y \int_0^{+\infty} (f'(s) + \alpha_1 f(s))\eta(s)\, ds\, dx.$$

Using Young's and Hölder's inequalities for the last integral of the above identity, we deduce that, for any $\delta > 0$,

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} \leq \left(\frac{\delta}{2} - a_0\right) \|y\|^2 + \frac{1}{2} \int_0^{+\infty} \left(\int_0^{+\infty} \frac{(f'(\tau) + \alpha_1 f(\tau))^2}{\delta f(\tau)}\, d\tau + \alpha_1 \frac{f'(s)}{f(s)} \right) f(s)\|\eta\|^2\, ds,$$

where a_0 is given in (1.6). By exploiting the condition (2.1) and the definition of f_0 in (2.2), we remark that

$$\int_0^{+\infty} \frac{(f'(\tau) + \alpha_1 f(\tau))^2}{\delta f(\tau)}\, d\tau + \alpha_1 \frac{f'(s)}{f(s)} = \frac{1}{\delta} \int_0^{+\infty} \left(\frac{(f'(\tau))^2}{f(\tau)} + 2\alpha_1 f'(\tau) + \alpha_1^2 f(\tau) \right) d\tau + \alpha_1 \frac{f'(s)}{f(s)}$$

$$\leq \frac{2}{\delta} \left(\alpha_1^2 f_0 - \alpha_1 f(0) \right) - \alpha_1 \alpha_2.$$

Therefore

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} \leq \left(\frac{\delta}{2} - a_0\right) \|y\|^2 + \frac{\alpha_1}{2} \left(\frac{2}{\delta} (\alpha_1 f_0 - f(0)) - \alpha_2 \right) \|\eta\|^2_{L_f}.$$

Thus, choosing

$$\frac{2(\alpha_1 f_0 - f(0))}{\alpha_2} < \delta < 2a_0,$$

which is possible thanks to (2.2), we get

$$\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}} \leq -d_0 \left(\|y\|^2 + \|\eta\|^2_{L_f} \right), \tag{2.15}$$

where

$$d_0 = \frac{1}{2}\min\left\{2a_0 - \delta, \alpha_1\left(\alpha_2 - \frac{2}{\delta}\left(\alpha_1 f_0 - f(0)\right)\right)\right\} > 0. \tag{2.16}$$

Thereby $\mathcal{A}$ is dissipative.

Now, we show that $I - \mathcal{A}$ is surjective. Let $F = (f_1, f_2, f_3) \in \mathcal{H}$. We prove that there exists $\Psi = (u, y, \eta) \in D(\mathcal{A})$ satisfying

$$\Psi - \mathcal{A}\Psi = F. \tag{2.17}$$

The first equation in (2.17) is equivalent to

$$y = u - f_1. \tag{2.18}$$

From (2.18), we see that the last equation in (2.17) is reduced to

$$\eta_s + \eta = u - f_1 + f_3. \tag{2.19}$$

Integrating with respect to s and noting that η should satisfy $\eta(0) = 0$, we get

$$\eta(s) = (1 - e^{-s})(u - f_1) + \int_0^s e^{\tau - s} f_3(\tau)\, d\tau. \tag{2.20}$$

Using (2.18) and (2.20), we find that the second equation in (2.17) is reduced to

$$Au + (a + 1 + \tilde{f}_0)u = f_2 + (a + 1 + \tilde{f}_0)f_1 + \int_0^{+\infty} f(s)\left(\int_0^s e^{\tau - s} f_3(\tau)\, d\tau - f_3(s)\right) ds, \tag{2.21}$$

where

$$\tilde{f}_0 = \int_0^{+\infty} e^{-s} f(s)\, ds.$$

We see that, if (2.21) admits a solution $u \in V$, then (2.18) implies that $y \in V$. Moreover, from (2.19), we remark that, if $\eta \in L_f$, then $\eta_s \in L_f$. So it is enough to prove that $\eta \in L_f$ to get the required regularity on η in $D(\mathcal{A})$. Indeed,

$$s \mapsto (1 - e^{-s})(u - f_1) \in L_f \tag{2.22}$$

because $f_1 \in H$ and we supposed that $u \in V$. On the other hand, using Fubini theorem and Hölder inequality, we get

$$\begin{aligned}
\int_0^{+\infty} f(s)\left\|\int_0^s e^{\tau - s} f_3(\tau)\, d\tau\right\|^2 ds &\le \int_0^{+\infty} e^{-2s} f(s)\left(\int_0^s e^{\tau}\, d\tau\right)\int_0^s e^{\tau}\|f_3(\tau)\|^2\, d\tau\, ds \\
&\le \int_0^{+\infty} e^{-s}(1 - e^{-s})\, f(s)\int_0^s e^{\tau}\|f_3(\tau)\|^2\, d\tau\, ds \\
&\le \int_0^{+\infty} e^{-s} f(s)\int_0^s e^{\tau}\|f_3(\tau)\|^2\, d\tau\, ds \\
&\le \int_0^{+\infty} e^{\tau}\|f_3(\tau)\|^2\int_{\tau}^{+\infty} e^{-s} f(s)\, ds\, d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{+\infty} e^{\tau} f(\tau) \|f_3(\tau)\|^2 \int_{\tau}^{+\infty} e^{-s} \, ds \, d\tau \\
&\leq \int_0^{+\infty} f(\tau) \|f_3(\tau)\|^2 \, d\tau \\
&\leq \|f_3\|_{L_f}^2 < +\infty,
\end{aligned}$$

since $f_3 \in L_f$. Thereafter

$$s \mapsto \int_0^s e^{\tau - s} f_3(\tau) \, d\tau \in L_f. \tag{2.23}$$

Hence (2.20), (2.22) and (2.23) imply that $\eta \in L_f$. On the other hand, because $f_2 \in H$, $f_1 \in V$, $f_3 \in L_f$ and thanks to (2.23), we have

$$\tilde{f} := f_2 + (a + 1 + \tilde{f}_0) f_1 + \int_0^{+\infty} f(s) \left(\int_0^s e^{\tau - s} f_3(\tau) \, d\tau - f_3(s) \right) ds \in H. \tag{2.24}$$

Finally, we just have to prove that (2.21) admits a solution $u \in V$ satisfying $\Delta u \in H$ and $u_\nu = 0$ on Γ. To do so, we consider the variational formulation of (2.21)

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij} \partial_i u \partial_j v + (a + 1 + \tilde{f}_0) uv \right) dx = \int_\Omega \tilde{f} v \, dx, \quad \forall v \in C_c^\infty(\Omega),$$

and using (1.5), (1.6), (2.24), the Lax-Milgram theorem and classical elliptic regularity arguments, we deduce that (2.21) admits a unique solution $u \in V$ satisfying

$$Au \in H \quad \text{and} \quad \partial_A u = 0 \text{ on } \Gamma.$$

This proves that (2.17) has a unique solution $\Psi \in D(\mathcal{A})$. By the resolvent identity, we have $\lambda I - \mathcal{A}$ is surjective, for any $\lambda > 0$ (see [20]). Consequently, the Lumer-Phillips theorem implies that $\mathcal{A}$ is the infinitesimal generator of a linear C_0 semigroup $S(t)$ of contractions on $\mathcal{H}$ and its domain $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see [26]). Our Theorem 2.3 is a direct consequence of semigroup theory (see [20] and [26]). $\square$

3 Asymptotic stability of (2.5)

We show in this section an asymptotic behavior result for the unique solution Ψ of (2.5) in $\mathcal{H}$. Namely, we prove that, as time goes to infinity, Ψ converges to an equilibrium point.

Theorem 3.1. *Assume that the assumptions* **(A1)-(A2)** *are fulfilled. Then, for any initial data $\Psi_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, the solution $\Psi = (u, u_t, \eta)$ of (2.5) converges in $\mathcal{H}$ to the equilibrium state $(\mathcal{C}, 0, 0)$ as $t \longrightarrow +\infty$, where*

$$\mathcal{C} = \left(\int_\Omega (f_0 + a) \, dx \right)^{-1} \int_\Omega \left[(f_0 + a) u_0 + u_1 - \int_0^{+\infty} f(s) \eta_0(s) \, ds \right] dx. \tag{3.1}$$

Proof. By a standard argument of density of $D(\mathcal{A})$ in $\mathcal{H}$ and the contraction of the semigroup $S(t)$ generated by $\mathcal{A}$, it suffices to prove Theorem 3.1 for smooth initial data

$$\Psi_0 = (u_0, u_1, \eta_0) \in D(\mathcal{A}).$$

Let $\Psi = (u, u_t, \eta) = S(\cdot)\Psi_0$ be the solution of (2.5). It follows from the dissipativity of $\mathcal{A}$ that the trajectory of solution $\{\Psi(t)\}_{t\in\mathbb{R}_+}$ is a bounded and precompact set. Applying LaSalle's principle, we deduce that the $\omega-$limit set $\omega\left(\Psi_0\right)$ is non empty, compact and invariant under the semigroup $S(t)$. In addition (see [17])

$$S(t)\Psi_0 \longrightarrow \omega\left(\Psi_0\right) \quad \text{as } t \to +\infty. \tag{3.2}$$

In order to prove our result, it is sufficient to show that $\omega\left(\Psi_0\right)$ is reduced to $(\mathcal{C}, 0, 0)$. To this end, let

$$\tilde{\Psi}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0) \in \omega\left(\Psi_0\right) \subset D(\mathcal{A})$$

and

$$\tilde{\Psi}(t) = (\tilde{u}(t), \tilde{u}_t(t), \tilde{\eta}(t)) = S(t)\tilde{\Psi}_0 \in D(\mathcal{A})$$

the unique classical solution of (2.5). Recall that it is well-known that $t \mapsto \|\tilde{\Psi}(t)\|_{\mathcal{H}}$ is a constant function [17] and thus

$$\frac{d}{dt}\left(\frac{1}{2}\|\tilde{\Psi}(t)\|_{\mathcal{H}}^2\right) = 0.$$

Then, using (2.5) and (2.15),

$$0 = \left\langle \tilde{\Psi}_t, \tilde{\Psi} \right\rangle_{\mathcal{H}} = \left\langle \mathcal{A}\tilde{\Psi}, \tilde{\Psi} \right\rangle_{\mathcal{H}} \leq -d_0 \left(\|\tilde{u}_t\|^2 + \|\tilde{\eta}\|_{L_f}^2\right) \leq 0. \tag{3.3}$$

This implies that $\tilde{u}_t = \tilde{\eta} = 0$, and therefore $\tilde{u}$ is a solution of the following system:

$$\begin{cases} A\tilde{u} = 0, & \forall x \in \Omega, \\ \tilde{u}_\nu = 0, & \forall x \in \Gamma, \end{cases} \tag{3.4}$$

which clearly yields that $\tilde{u} \equiv$ constant. Thus, we have proved that for any $\tilde{\Psi}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0) \in \omega\left(\tilde{\Psi}_0\right) \subset D(\mathcal{A})$, the solution $\tilde{\Psi}(t) = (\tilde{u}, \tilde{u}_t, \tilde{\eta}) = S(t)\tilde{\Psi}_0 \in D(\mathcal{A})$ satisfies

$$\tilde{\Psi}(t) = (\mathcal{C}, 0, 0), \quad \forall t \in \mathbb{R}_+,$$

where $\mathcal{C}$ is a constant. In particular, $\tilde{\Psi}_0 = (\mathcal{C}, 0, 0)$, and hence the ω-limit set $\omega\left(\Psi_0\right)$ consists of constants elements $(\mathcal{C}, 0, 0)$. Now, we shall prove that $\mathcal{C}$ is unique and it is given by (3.1). Let $(\mathcal{C}, 0, 0) \in \omega\left(\Psi_0\right)$. This implies that there exists $\{t_n\} \to +\infty$, as $n \to +\infty$ such that

$$\Psi(t_n) = (u(t_n), u_t(t_n), \eta(t_n)) = S(t_n)\Psi_0 \longrightarrow (\mathcal{C}, 0, 0) \tag{3.5}$$

in the state space $\mathcal{H}$. Furthermore, using (2.8), we see that the function

$$t \mapsto \int_\Omega \left((f_0 + a)u + u_t - \int_0^{+\infty} f(s)\eta(s)\, ds\right) dx$$

is a constant function and so

$$\int_\Omega \left((f_0 + a)u + u_t - \int_0^{+\infty} f(s)\eta(s)\,ds \right) dx = \int_\Omega \left((f_0 + a)u_0 + u_1 - \int_0^{+\infty} f(s)\eta_0(s)\,ds \right) dx. \tag{3.6}$$

Finally, let $t = t_n$ in (3.6) with $n \to +\infty$ and use (3.5) to obtain

$$\int_\Omega (f_0 + a)\mathcal{C}\,dx = \int_\Omega \left((f_0 + a)u_0 + u_1 - \int_0^{+\infty} f(s)\eta_0(s)\,ds \right) dx,$$

which gives (3.1). This achieves the proof of Theorem 3.1. □

4 Exponential stability of (2.5)

This section is concerned with the convergence rate of solutions of our system. Indeed, we will prove that the solutions exponentially converge to their equilibrium state given in Theorem 3.1. To do so, we shall use the following frequency domain theorem (see Huang [18] and Prüss [27]):

Theorem 4.1. *There exist two positive constants C and ω such that a C_0-semigroup $e^{t\mathcal{A}}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies the estimate*

$$||e^{t\mathcal{A}}||_{\mathcal{L}(\mathcal{H})} \le Ce^{-\omega t}, \quad \forall t > 0,$$

if and only if

$$\rho(\mathcal{A}) \supset \{i\gamma;\ \gamma \in \mathbb{R}\} \equiv i\mathbb{R} \tag{4.1}$$

and

$$\limsup_{|\gamma| \to +\infty} \|(i\gamma I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty, \tag{4.2}$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

In order to proceed, let us consider the closed subspace $\hat{\mathcal{H}}$ of $\mathcal{H}$ given by

$$\hat{\mathcal{H}} = \left\{ (u, y, \eta) \in \mathcal{H};\ \int_\Omega \left((f_0 + a)u + y - \int_0^{+\infty} f(s)\eta(s)\,ds \right) dx = 0 \right\}$$

and equipped with the inner product

$$\langle (u, y, \eta), (\tilde{u}, \tilde{y}, \tilde{\eta}) \rangle_{\hat{\mathcal{H}}} = \int_\Omega \left(\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j \tilde{u} \right) dx + \langle y, \tilde{y} \rangle + \alpha_1 \langle \eta, \tilde{\eta} \rangle_{L_f},$$

where α_1 is the positive constant appearing in (2.1). Let us denote by $\hat{\mathcal{A}}$ a new operator defined as follows

$$\begin{aligned} &\hat{\mathcal{A}} : D(\hat{\mathcal{A}}) := D(\mathcal{A}) \cap \hat{\mathcal{H}} \subset \hat{\mathcal{H}} \to \hat{\mathcal{H}}, \\ &\hat{\mathcal{A}}(u, y, \eta) = \mathcal{A}(u, y, \eta),\ \forall\, (u, y, \eta) \in D(\hat{\mathcal{A}}). \end{aligned} \tag{4.3}$$

Clearly, going back to the previous sections, one can claim the operator $\hat{\mathcal{A}}$ defined by (4.3) generates on $\hat{\mathcal{H}}$ a C_0-semigroup of contractions $e^{t\hat{\mathcal{A}}}$. Moreover, $\sigma(\hat{\mathcal{A}})$, the spectrum of $\hat{\mathcal{A}}$, consists of isolated eigenvalues of finite algebraic multiplicity only.

Now we are in a position to state and prove that the semigroup operator $e^{t\hat{\mathcal{A}}}$ is exponentially stable on $\hat{\mathcal{H}}$ and hence the solutions of the system exponentially converge to the equilibrium state.

Theorem 4.2. *Suppose that the assumptions* **(A1)**-**(A2)** *hold. Then, there exist two positive constants k and ω such that we have*

$$\left\| e^{t\hat{\mathcal{A}}} \right\|_{\mathcal{L}(\hat{\mathcal{H}})} \leq \kappa e^{-\omega t}, \quad \forall t > 0. \tag{4.4}$$

Proof. For sake of clarity, we will proceed by steps.

Step 1: The immediate objective is to show (4.1) holds for $\hat{\mathcal{A}}$, which is equivalent to prove that, given a real number γ, the equation

$$\hat{\mathcal{A}}\Psi = i\gamma\Psi \tag{4.5}$$

with $\Psi = (u, y, \eta) \in D(\hat{\mathcal{A}})$ has only the trivial solution. It follows from (4.5) that

$$y = i\gamma u, \tag{4.6}$$

$$-Au - ay - \int_0^{+\infty} f(s)\eta_s(s)\, ds = i\gamma y, \tag{4.7}$$

$$-\eta_s + y = i\gamma\eta, \tag{4.8}$$

$$\int_\Omega \left((f_0 + a)u + y - \int_0^{+\infty} f(s)\eta(s)\, ds \right) dx = 0, \tag{4.9}$$

$$\partial_A u = 0 \ \text{ on}\, \Gamma, \tag{4.10}$$

$$\eta(0) = 0. \tag{4.11}$$

In the event that $\gamma = 0$, then (4.6) implies that $y = 0$ in $L^2(\Omega)$, which together with (4.8) and (4.11), lead to $\eta = 0$ in L_f. Going back to (4.7) and using (4.9)-(4.10), we deduce that $u = 0$ in $H^1(\Omega)$. Thereby, (4.5) has only the trivial solution.

Suppose now that $\gamma \neq 0$. Then, taking the inner product of (4.5) with Ψ and using (2.15), we obtain

$$0 = \Re\left(i\gamma\|\Psi\|^2_{\hat{\mathcal{H}}}\right) = \Re\left(\langle i\gamma\Psi, \Psi\rangle_{\hat{\mathcal{H}}}\right) = \Re\left(\langle \hat{\mathcal{A}}\Psi, \Psi\rangle_{\hat{\mathcal{H}}}\right) \leq -d_0\left(\|y\|^2 + \|\eta\|^2_{L_f}\right), \tag{4.12}$$

where d_0 is a positive constant (see (2.16)). Whence $y = 0$ in $L^2(\Omega)$ and $\eta = 0$ in L_f. Invoking (4.6)-(4.11), we also conclude that the only solution of (4.5) is the trivial one.

Step 2: Our goal is to prove that the resolvent operator of $\hat{\mathcal{A}}$ satisfies condition (4.2). If that was not the case, then Banach-Steinhaus Theorem (see [7]) leads us to claim that there exist a sequence of real numbers

$$\gamma_n \to +\infty \tag{4.13}$$

and a sequence of vectors $\Psi_n = (u_n, y_n, \eta_n) \in D(\hat{\mathcal{A}})$ with

$$\|\Psi_n\|_{\hat{\mathcal{H}}} = 1, \quad \forall n \in \mathbb{N} \tag{4.14}$$

such that

$$\left\|(i\gamma_n I - \hat{\mathcal{A}})\Psi_n\right\|_{\hat{\mathcal{H}}} \to 0 \quad \text{as} \quad n \to +\infty, \tag{4.15}$$

that is,

$$i\gamma_n u_n - y_n \equiv P_n \to 0 \quad \text{in} \quad H^1(\Omega), \tag{4.16}$$

$$i\gamma_n y_n + Au_n + ay_n + \int_0^{+\infty} f(s)(\eta_n)_s(s)\, ds \equiv Q_n \to 0 \quad \text{in} \quad L^2(\Omega), \tag{4.17}$$

$$i\gamma_n \eta_n + (\eta_n)_s - y_n \equiv R_n \to 0 \quad \text{in} \quad L_f \tag{4.18}$$

and

$$\int_\Omega \left((f_0 + a)u_n + y_n - \int_0^{+\infty} f(s)\eta_n(s)\, ds \right) dx = 0, \tag{4.19}$$

$$\partial_A u_n = 0 \;\text{ on}\, \Gamma, \tag{4.20}$$

$$\eta_n(0) = 0. \tag{4.21}$$

Exploring (4.14) and the fact that

$$\|\Psi_n\|_{\hat{\mathcal{H}}} \left\|(i\gamma_n I - \hat{\mathcal{A}})\Psi_n\right\|_{\hat{\mathcal{H}}} \geq \Re\left(\langle (i\gamma_n I - \hat{\mathcal{A}})\Psi_n, \Psi_n\rangle_{\hat{\mathcal{H}}}\right) = -\Re\left(\langle \hat{\mathcal{A}}\Psi_n, \Psi_n\rangle_{\hat{\mathcal{H}}}\right), \tag{4.22}$$

and recalling (4.12) as well as (4.15), we get

$$y_n \to 0 \text{ in } L^2(\Omega) \quad \text{and} \quad \eta_n \to 0 \text{ in } L_f. \tag{4.23}$$

Thenceforth

$$\gamma_n u_n \to 0 \quad \text{and} \quad u_n \to 0 \text{ in } L^2(\Omega), \tag{4.24}$$

by means of (4.13), (4.16) and (4.23).

Next, taking the inner product of (4.17) with u_n in $L^2(\Omega)$ and applying Green formula, it follows that

$$\int_\Omega (i\gamma_n + a)y_n u_n\, dx + \int_\Omega \sum_{i,j=1}^N a_{ij}\partial_i u_n \partial_j u_n\, dx + \int_\Omega u_n \int_0^{+\infty} f(s)(\eta_n)_s(s)\, ds\, dx \to 0.$$

This, together with (4.23) and (4.24), implies that

$$\int_\Omega \sum_{i,j=1}^N a_{ij}\partial_i u_n \partial_j u_n\, dx + \int_\Omega u_n \int_0^{+\infty} f(s)(\eta_n)_s(s)\, ds\, dx \to 0. \tag{4.25}$$

On the other hand, taking the inner product of (4.18) with u_n in L_f, we get

$$i\gamma_n \int_\Omega u_n \int_0^{+\infty} f_n(s)\eta_n(s)\,ds\,dx + \int_\Omega u_n \int_0^{+\infty} f_n(s)(\eta_n)_s(s)\,ds\,dx - f_0 \int_\Omega u_n y_n\,dx \to 0.$$

Thereby, from (4.23) and (4.24), we obtain that

$$\int_\Omega u_n \int_0^{+\infty} f_n(s)(\eta_n)_s(s)\,ds\,dx \to 0. \tag{4.26}$$

The limits (4.25) and (4.26) lead to

$$\sum_{i,j=1}^{N} a_{ij}\partial_i u_n \partial_j u_n \to 0 \text{ in } L^2(\Omega).$$

Then (1.5) and (2.10) imply that

$$\nabla u_n \to 0 \text{ in } L^2(\Omega). \tag{4.27}$$

Combining (4.23), (4.24) and (4.27), we deduce that $\|\Psi_n\|_{\hat{\mathcal{H}}}$ converges to zero in $\hat{\mathcal{H}}$, which contradicts (4.14).

The two conditions of Theorem 4.1 are proved. Thereby, the proof of Theorem 4.2 is achieved. ☐

Remarks 4.3. *(i) Let* $\Psi_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$ *and* $\Psi = (u, u_t, \eta)$ *be the solution of* (2.5). *We put*

$$\hat{\Psi} = \Psi - (\mathcal{C}, 0, 0) = (u - \mathcal{C}, u_t, \eta) \quad \textit{and} \quad \hat{\Psi}_0 = \Psi_0 - (\mathcal{C}, 0, 0) = (u_0 - \mathcal{C}, u_1, \eta_0),$$

where $\mathcal{C}$ *is defined in* (3.1). *Then* $\hat{\Psi}$ *is a solution of*

$$\begin{cases} \hat{\Psi}_t(x,t) = \mathcal{A}\hat{\Psi}(x,t), & \forall x \in \Omega,\ \forall t > 0, \\ \hat{\Psi}(x,0) = \hat{\Psi}_0(x), & \forall x \in \Omega. \end{cases}$$

On the other hand, we see that

$$\int_\Omega \left((f_0 + a)(u_0 - \mathcal{C}) + u_1 - \int_0^{+\infty} f(s)\eta_0(s)\,ds \right) dx$$

$$= \int_\Omega \left((f_0 + a)u_0 + u_1 - \int_0^{+\infty} f(s)\eta_0(s)\,ds \right) dx - \mathcal{C} \int_0^{+\infty} (f_0 + a)\,dx = 0,$$

thus $\hat{\Psi}_0 \in \hat{\mathcal{H}}$. *Consequently, Theorem 4.2 implies that there exist two positive constants* k *and* ω *such that*

$$\|\Psi - (\mathcal{C}, 0, 0)\|_{\hat{\mathcal{H}}} \le \kappa e^{-\omega t}, \quad \forall t > 0. \tag{4.28}$$

(ii) Note that the condition (2.2) has a similar form even for ordinary differential equations (ODEs) with distributed delay (see for instance [22, equation (1.13), p. 107] for scalar ODEs and [28] for linear systems of ODEs where Linear Matrix Inequalities are needed).

5 Case of discrete time delay

In this section, we consider the case of discrete time delay

$$u_{tt}(x,t) + Au(x,t) + a(x)u_t(x,t) + b(x)u_t(x,t-\tau) = 0, \quad \forall x \in \Omega, \ \forall t > 0, \tag{5.1}$$

with, as in the previous sections, the homogeneous Neumann boundary condition on the whole boundary Γ

$$\partial_A u(x,t) := \sum_{i,j=1}^{N} a_{ij}(x)\nu_j \partial_i u(x,t) = 0, \quad \forall x \in \Gamma, \ \forall t > 0 \tag{5.2}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x), & \forall x \in \Omega, \\ u_t(x,0) = u_1(x), & \forall x \in \Omega, \\ u_t(x,t) = u_2(x,t), & \forall x \in \Omega, \ \forall t \in (-\tau, 0), \end{cases} \tag{5.3}$$

where A and a are defined in (1.4), (1.5) and (1.6). Moreover, $\tau \in (0,+\infty)$, and (u_0, u_1, u_2) are given initial data. Finally, b is a real space variable function of $C(\bar{\Omega})$ so that

$$\|b\|_\infty < a_0. \tag{5.4}$$

5.1 Well-posedness of (5.1)-(5.3)

Arguing as in [23] when dealing with a discrete delay term, we consider the variable z and its initial data z_0 given by

$$\begin{cases} z(x,t,p) = u_t(x,t-\tau p), & \forall x \in \Omega, \ \forall t \in \mathbb{R}_+, \ \forall p \in (0,1), \\ z_0(x,p) = z(x,0,p) = u_2(x,-\tau p), & \forall x \in \Omega, \ \forall p \in (0,1). \end{cases} \tag{5.5}$$

Then z satisfies

$$\begin{cases} \tau z_t(x,t,p) + z_p(x,t,p) = 0, & \forall x \in \Omega, \ \forall t \in \mathbb{R}_+, \ \forall p \in (0,1), \\ z(x,t,0) = u_t(x,t), & \forall x \in \Omega, \ \forall t \in \mathbb{R}_+. \end{cases} \tag{5.6}$$

Let $y = u_t$, $\Phi = (u, y, z)$ and $\Phi_0 = (u_0, u_1, z_0) \in \mathcal{X}$, where

$$\mathcal{X} = V \times H \times L^2((0,1), H)$$

and

$$L^2((0,1), H) = \left\{ w : (0,1) \to H; \quad \int_0^1 \|w(p)\|^2 \, dp < +\infty \right\}$$

endowed with the inner product

$$\langle w_1, w_2 \rangle_{L^2((0,1),H)} = \int_0^1 \langle w_1(p), w_2(p) \rangle \, dp.$$

We can write the system (5.1)-(5.3) in the abstract form

$$\begin{cases} \Phi_t(x,t) = \mathcal{B}\Phi(x,t), & \forall x \in \Omega, \ \forall t > 0, \\ \Phi(x,0) = \Phi_0(x), & \forall x \in \Omega, \end{cases} \tag{5.7}$$

the operator $\mathcal{B}$ given by

$$\mathcal{B}\begin{pmatrix} u \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -Au - ay - bz(1) \\ -\dfrac{z_p}{\tau} \end{pmatrix}, \tag{5.8}$$

and the domain

$$D(\mathcal{B}) = \left\{(u,y,z) \in \mathcal{X};\ Au \in H,\ y \in V,\ z_p \in L^2((0,1),H),\ z(0) = y,\ \partial_A u = 0 \text{ on } \Gamma\right\}. \tag{5.9}$$

Whereupon, the systems (5.1)-(5.3) and (5.7) are equivalent. As in the case of distributed delay (see previous sections), one should introduce an inner product on $\mathcal{X}$ which makes $\mathcal{X}$ a Hilbert space. Because of the Neumann boundary condition is considered on the whole boundary Γ and no displacement term is considered neither on Ω nor on Γ, the classical energy norm $\|\cdot\|_E$ given by

$$\|(u,y,z)\|_E^2 = \int_\Omega \sum_{i,j=1}^{N} a_{ij}\partial_i u \partial_j u \, dx + \|y\|^2 + \|z\|^2_{L^2((0,1),H)}$$

is only a semi-norm in our case, since $\|(u,y,z)\|_E^2 = 0$ does not imply that $u = 0$. By integrating (5.1) on Ω, we get

$$\partial_t \left(\int_\Omega (u_t + au + bu(x,t-\tau))\, dx \right) = 0.$$

Using the definition of y and z, we get

$$\partial_t \left(\int_\Omega \left((a+b)u + y - \tau b \int_0^1 z(x,p)\, dp \right) dx \right) = 0. \tag{5.10}$$

As for the memory case, we fix a positive constant ϵ_2 and consider on $\mathcal{X}$ the inner product

$$\langle (u,y,z), (\tilde{u},\tilde{y},\tilde{z}) \rangle_{\mathcal{X}} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j \tilde{u} \right) dx + \langle y, \tilde{y} \rangle + \xi \langle z, \tilde{z} \rangle_{L^2((0,1),H)}$$

$$+\epsilon_2 \left(\int_\Omega \left((a+b)u + y - \tau b \int_0^1 z(x,p)\, dp \right) dx \right) \int_\Omega \left((a+b)\tilde{u} + \tilde{y} - \tau b \int_0^1 \tilde{z}(x,p)\, dp \right) dx,$$

where ξ is a positive constant satisfying

$$\tau \|b\|_\infty < \xi < \tau \left(2a_0 - \|b\|_\infty\right) \tag{5.11}$$

and a_0 is given in (1.6). Note that ξ exists in light of to (5.4). The space $(\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}})$ is a Hilbert space if $\|\cdot\|_{V \times H \times L^2((0,1),H)}$ and $\|\cdot\|_{\mathcal{X}}$ are equivalent. So it is sufficient to prove the next lemma.

Lemma 5.1. *There exists $\epsilon_3 > 0$ such that for any $\epsilon_2 \in (0, \epsilon_3)$ and for each $(u, y, z) \in \mathcal{X}$, we have:*

$$\zeta_1 \|(u, y, z)\|^2_{V\times H\times L^2((0,1),H)} \le \|(u, y, z)\|^2_{\mathcal{X}} \le \zeta_2 \|(u, y, z)\|^2_{V\times H\times L^2((0,1),H)}, \tag{5.12}$$

in which ζ_1 and ζ_2 are positive constants.

Proof. Invoking Hölder's and Young's inequalities, and using (2.10), one can show that the right inequality in (5.12) holds, for any $\epsilon_2 > 0$, and for some ζ_2 depending on ϵ_2, k_1, ξ, Ω, a and b. Concerning the proof of the left inequality in (5.12), one can proceed in a similar way as for (2.9). Indeed, we see that

$$\left(\int_\Omega \left((a+b)u + y - \tau b \int_0^1 z(p)\,dp\right) dx\right)^2 = \left(\int_\Omega (a+b)u\,dx\right)^2 \tag{5.13}$$

$$+\left(\int_\Omega \left(y - \tau b \int_0^1 z(p)\,dp\right) dx\right)^2 + 2\left(\int_\Omega (a+b)u\,dx\right)\int_\Omega \left(y - \tau b\int_0^1 z(p)\,dp\right) dx.$$

Using Young's and Hölder's inequalities, we have

$$2\left(\int_\Omega (a+b)u\,dx\right)\left(\int_\Omega \left(y - \tau b\int_0^1 z(p)\,dp\right) dx\right)$$

$$\ge -\frac{1}{2}\left(\int_\Omega (a+b)u\,dx\right)^2 - 2\left(\int_\Omega \left(y - \tau b\int_0^1 z(p)\,dp\right) dx\right)^2$$

and there exists a positive constant c such that

$$\left(\int_\Omega \left(y - \tau b\int_0^1 z(p)\,dp\right) dx\right)^2 \le c\left(\|y\|^2 + \|z\|^2_{L^2((0,1),H)}\right), \quad \forall (y,z) \in H \times L^2((0,1), H).$$

By combining the above two inequalities, (5.13) and the generalized Poincaré's inequality (thanks to (5.4) and similar to (2.12))

$$\|u\|^2 \le \varpi_1 \left(\|\nabla u\|^2 + \left(\int_\Omega (a+b)u\,dx\right)^2\right), \quad \forall u \in V,$$

we get

$$\epsilon_2\left(\int_\Omega \left((a+b)u + y - \tau b\int_0^1 z(p)\,dp\right) dx\right)^2 \ge \frac{\epsilon_2}{2\varpi_1}\|u\|^2 - \frac{\epsilon_2}{2}\|\nabla u\|^2 - \epsilon_2 c\|y\|^2 - \epsilon_2 c\|z\|^2_{L^2_{((0,1),H)}}.$$

Here ϖ_1 depends on Ω, a and b. Therefore

$$\|(u,y,z)\|^2_{\mathcal{X}} \ge \left(\alpha_0 - \frac{\epsilon_2}{2}\right)\|\nabla u\|^2 + \frac{\epsilon_2}{2\varpi_1}\|u\|^2 + (1-\epsilon_2 c)\|y\|^2 + (\xi - \epsilon_2 c)\|z\|^2_{L^2((0,1),H)}.$$

Whereupon, for ϵ_2 satisfying

$$0 < \epsilon_2 < \epsilon_3 := \min\left\{2\alpha_0, \frac{1}{c}, \frac{\xi}{c}\right\},$$

we obtain the reverse inequality in (5.12) with

$$\zeta_1 = \min\left\{\alpha_0 - \frac{\epsilon_2}{2}, \frac{\epsilon_2}{2\varpi_1}, 1 - \epsilon_2 c, \xi - \epsilon_2 c\right\} > 0.$$

$\square$

Our well-posedness result is announced in the following theorem:

Theorem 5.2. *Assume that (5.4) holds. Then, for any $\Phi_0 \in \mathcal{X}$, the system (5.7) has a unique weak solution*

$$\Phi \in C(\mathbb{R}_+, \mathcal{X}). \tag{5.14}$$

Moreover, if $\Phi_0 \in D(\mathcal{B})$, then the solution of (5.7) satisfies (classical solution)

$$\Phi \in C^1(\mathbb{R}_+, \mathcal{X}) \cap C(\mathbb{R}_+, D(\mathcal{B})). \tag{5.15}$$

Proof. As in section 2, we will prove that $\mathcal{B}$ is a maximal monotone operator. Direct computations lead to

$$\langle \mathcal{B}\Phi, \Phi\rangle_{\mathcal{X}} = -\|\sqrt{a}y\|^2 - \int_\Omega byz(1)\, dx - \frac{\xi}{\tau}\int_\Omega\int_0^1 zz_p\, dp\, dx. \tag{5.16}$$

By integrating with respect to p the last integral of the above equality, noting that $z(0) = y$ and using Young's inequality for $-byz(1)$, we obtain

$$\langle \mathcal{B}\Phi, \Phi\rangle_{\mathcal{X}} \leq -\frac{1}{2}\left(2a_0 - \|b\|_\infty - \frac{\xi}{\tau}\right)\|y\|^2 - \frac{1}{2}\left(\frac{\xi}{\tau} - \|b\|_\infty\right)\|z(1)\|^2,$$

and then

$$\langle \mathcal{B}\Phi, \Phi\rangle_{\mathcal{X}} \leq -r_0\left(\|y\|^2 + \|z(1)\|^2\right), \tag{5.17}$$

where

$$r_0 = \frac{1}{2}\min\left\{2a_0 - \|b\|_\infty - \frac{\xi}{\tau}, \frac{\xi}{\tau} - \|b\|_\infty\right\} > 0. \tag{5.18}$$

Note that $r_0 > 0$ thanks to (5.11). This proves that $\mathcal{B}$ is dissipative. To prove that $I - \mathcal{B}$ is surjective, let $F = (f_1,\, f_2,\, f_3) \in \mathcal{X}$ and we prove that there exists $\Phi = (u, y, z) \in D(\mathcal{B})$ satisfying

$$\Phi - \mathcal{B}\Phi = F. \tag{5.19}$$

The first equation in (5.19) is equivalent to

$$y = u - f_1. \tag{5.20}$$

The last equation in (5.19) is reduced to

$$z_p + \tau z = \tau f_3. \tag{5.21}$$

Integrating with respect to p and noting that z should satisfy $z(0) = y = u - f_1$, we get

$$z(p) = e^{-\tau p}(u - f_1) + \tau\int_0^p e^{\tau(\theta - p)}\, f_3(\theta)\; d\theta. \tag{5.22}$$

Using (5.20) and (5.22), we find that the second equation in (5.19) is reduced to

$$Au + (a + 1 + be^{-\tau})u = \tilde{f}, \tag{5.23}$$

where

$$\tilde{f} = f_2 + (a + 1 + be^{-\tau})f_1 - \tau e^{-\tau} \int_0^1 e^{\tau\theta} f_3(\theta)\, d\theta \in H$$

($\tilde{f} \in H$ since $f_1 \in V$, $f_2 \in H$, $f_3 \in L^2((0,1), H)$ and $e^{\tau\theta} \le e^{\tau}$, for $\theta \in (0,1)$). It is worth mentioning that, according to (1.6) and (5.4), we have

$$a + 1 + be^{-\tau} \ge a_0 + 1 - \|b\|_\infty > 1.$$

Then, considering the variational formulation of (5.23) and using the Lax-Milgram theorem and classical elliptic regularity arguments, we deduce that (5.23) has a unique solution $u \in V$ satisfying

$$Au \in H \quad \text{and} \quad \partial_A u = 0 \text{ on } \Gamma.$$

Therefore (5.20) and (5.22) imply that

$$y \in V, \quad z \in L^2((0,1), H) \quad \text{and} \quad z(0) = y.$$

From (5.21), we have $z_p \in L^2((0,1), H)$. This proves that (5.19) has a unique solution $\Phi \in D(\mathcal{B})$; that is $I - \mathcal{B}$ is onto. Consequently, one can conclude that $\mathcal{B}$ is the infinitesimal generator of a linear C_0 semigroup $S(t)$ of contractions on $\mathcal{X}$ and its domain $D(\mathcal{B})$ is dense in $\mathcal{X}$. Whereupon, Theorem 5.2 is shown. □

5.2 Asymptotic stability of (5.7)

As in section 3, we prove the following convergence result:

Theorem 5.3. *Suppose that (5.4) is satisfied. Then for any initial data $\Phi_0 = (u_0, u_1, z_0) \in \mathcal{X}$, the solution $\Phi = (u, y, z)$ of (5.7) tends in $\mathcal{X}$ to $(\mathcal{K}, 0, 0)$ as $t \longrightarrow +\infty$, where*

$$\mathcal{K} = \left(\int_\Omega (a + b)\, dx \right)^{-1} \int_\Omega \left[(a + b)u_0 + u_1 - \tau b \int_0^1 z_0(p)\, dp \right] dx. \tag{5.24}$$

Proof. The proof of (5.24) is identical to the one of (3.1) given in section 3 (using (5.10) and (5.17)); we omit the details here. □

5.3 Exponential stability of (5.7)

As for the previous case, we will use Theorem 4.1 to prove that the solutions of (5.7) converge to $(\mathcal{K}, 0, 0)$ (see Theorem 5.3) with an exponential rate. So it suffices to proceed as in section 4. Thus, let the subspace $\hat{\mathcal{X}}$ of $\mathcal{X}$ given by

$$\hat{\mathcal{X}} = \left\{ (u, y, z) \in \mathcal{X};\ \int_\Omega \left((a + b)u + y - \tau b \int_0^1 z(x, p)\, dp \right) dx = 0 \right\} \tag{5.25}$$

with inner product

$$\langle (u,y,z),(\tilde{u},\tilde{y},\tilde{z})\rangle_{\hat{\mathcal{X}}} = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i u \partial_j \tilde{u}\right)\, dx + \langle y,\tilde{y}\rangle + \xi \langle z,\tilde{z}\rangle_{L^2((0,1),H)}, \tag{5.26}$$

where ξ is a positive constant satisfying (5.11). Then, define the operator $\hat{\mathcal{B}}$ by

$$\begin{aligned} &\hat{\mathcal{B}} : D(\hat{\mathcal{B}}) := D(\mathcal{B}) \cap \hat{\mathcal{X}} \subset \hat{\mathcal{X}} \to \hat{\mathcal{X}}, \\ &\hat{\mathcal{B}}(u,y,z) = \mathcal{B}(u,y,z),\ \forall\,(u,y,z) \in D(\hat{\mathcal{B}}), \end{aligned} \tag{5.27}$$

which generates a C_0-semigroup of contractions $e^{t\hat{\mathcal{B}}}$ on $\hat{\mathcal{X}}$.

Theorem 5.4. *Suppose that the condition (5.4) holds. Then, the solutions of (5.7) exponentially converge to their equilibrium state $(\mathcal{K},0,0)$ in $\mathcal{X}$.*

Proof. Using Remark 4.3 with $\mathcal{K}$ instead of $\mathcal{C}$, it is clear that, in order to prove the desired result, it amounts to showing the exponential stability of the semigroup operator $e^{t\hat{\mathcal{B}}}$ on $\hat{\mathcal{X}}$. To do so, let us first check that $\rho(\hat{\mathcal{B}}) \supset i\mathbb{R}$, which means that $\mathcal{B}$ satisfies (4.1). In fact, given $\gamma \in \mathbb{R}$, let us seek $\Phi = (u,y,z) \in D(\mathcal{B})$ such that $\mathcal{B}\Phi = i\gamma\Phi$, that is

$$y = i\gamma u, \tag{5.28}$$
$$-Au - ay - bz(1) = i\gamma y, \tag{5.29}$$
$$z_p = -i\tau\gamma z, \tag{5.30}$$
$$\int_\Omega \left((a+b)u + y - \tau b \int_0^1 z(x,p)\, dp\right) dx = 0, \tag{5.31}$$
$$\partial_A u = 0 \ \text{ on}\,\Gamma, \tag{5.32}$$
$$z(0) = y. \tag{5.33}$$

If $\gamma = 0$, then $y = 0 = z(0)$ in $L^2(\Omega)$ by (5.28) and (5.33), which together with (5.30), implies that $z = 0$ in $L^2((0,1),H)$. Using (5.29) and (5.31)-(5.32), we can conclude that $u = 0$ in $H^1(\Omega)$. Hence (4.1) is verified when $\gamma = 0$.

If $\gamma \neq 0$, then taking the inner product of $\mathcal{B}\Phi = i\gamma\Phi$ with Φ and using (5.17), we obtain

$$0 = \Re\left(\langle \hat{\mathcal{B}}\Phi, \Phi\rangle_{\hat{\mathcal{X}}}\right) \leq -r_0\left(\|y\|^2 + \|z(1)\|^2\right), \tag{5.34}$$

where r_0 is a positive constant (see (5.18)). Thus

$$y = z(1) = 0 \quad \text{in } L^2(\Omega),$$

and hence $u = 0$ in $L^2(\Omega)$ thanks to (5.28). Next, solving (5.30), we get

$$z_n(p) = e^{-i\tau\gamma_n p}\, z_n(0).$$

Because $z(1) = 0$, then $z_n(0) = 0$, and hence $z_n = 0$ in $L^2((0,1),H)$. Lastly, amalgamating these findings with (5.29) and (5.31)-(5.32), we obtain $u = 0$ in $H^1(\Omega)$. Thus (4.1) is also satisfied if $\gamma \neq 0$.

We turn now to the proof of (4.2). Suppose that this property does not hold for $\hat{\mathcal{B}}$. Thereby, there exist a sequence of real numbers

$$\gamma_n \to +\infty \tag{5.35}$$

and a sequence of vectors $\Phi_n = (u_n, y_n, z_n,) \in D(\hat{\mathcal{B}})$ with

$$\|\Phi_n\|_{\hat{\mathcal{X}}} = 1, \quad \forall n \in \mathbb{N}, \tag{5.36}$$

such that

$$\left\|(i\gamma_n I - \hat{\mathcal{B}})\Phi_n\right\|_{\hat{\mathcal{X}}} \to 0 \quad \text{as} \quad n \to +\infty; \tag{5.37}$$

that is

$$i\gamma_n u_n - y_n \equiv V_n \to 0 \quad \text{in} \quad H^1(\Omega), \tag{5.38}$$

$$i\gamma_n y_n + A u_n + a y_n + b z_n(1) \equiv W_n \to 0 \quad \text{in} \quad L^2(\Omega), \tag{5.39}$$

$$i\gamma_n z_n + \frac{1}{\tau}(z_n)_p \equiv X_n \to 0 \quad \text{in} \quad L^2((0,1), H), \tag{5.40}$$

together with

$$\int_\Omega \left((a+b)u_n + y - \tau b \int_0^1 z_n(x,p)\, dp \right) dx = 0, \tag{5.41}$$

$$\partial_A u_n = 0 \quad \text{on}\, \Gamma, \tag{5.42}$$

$$z_n(0) = y_n. \tag{5.43}$$

Combining

$$\|\Phi_n\|_{\hat{\mathcal{X}}} \left\|(i\gamma_n I - \hat{\mathcal{B}})\Phi_n\right\|_{\hat{\mathcal{X}}} \geq \Re\left(\langle (i\gamma_n I - \hat{\mathcal{B}})\Phi_n, \Phi_n\rangle_{\hat{\mathcal{X}}}\right) = -\Re\left(\langle \hat{\mathcal{B}}\Phi_n, \Phi_n\rangle_{\hat{\mathcal{X}}}\right), \tag{5.44}$$

with (5.34), (5.36), (5.37) and (5.43), we obtain

$$y_n = z_n(0) \to 0 \quad \text{and} \quad z_n(1) \to 0 \quad \text{in}\, L^2(\Omega). \tag{5.45}$$

This immediately yields, using (5.35) and (5.38),

$$u_n \to 0 \quad \text{and} \quad \gamma_n u_n \to 0 \quad \text{in}\, L^2(\Omega). \tag{5.46}$$

In the light of (5.40), we have

$$z_n(p) = z_n(0)\, e^{-i\tau\gamma_n p} + \tau \int_0^p e^{-i\tau\gamma_n(p-\theta)}\, X_n(\theta)\, d\theta,$$

and hence $z_n \to 0$ in $L^2((0,1), H)$. In turn, arguing as before by taking the inner product of (5.39) with u_n in $L^2(\Omega)$, we get (4.27), and then we conclude that $u_n \to 0$ in $H^1(\Omega)$. Thus $\|\Phi_n\|_{\hat{\mathcal{X}}} \to 0$ in $\hat{\mathcal{X}}$, which contradicts (5.36). This completes the proof of Theorem 5.4. □

Remarks 5.5. *Theorem 4.2, Remark 4.3-i and 5.4 ensure the exponential convergence of the solutions to their equilibrium state but no estimate of the exponential rate is provided. In order to obtain such an estimate, one should adopt another method (instead of the resolvent method) such that energy method (or Lyapunov method). Notwithstanding, this method requires an appropriate choice of an energy norm. Let us consider the system (5.1)-(5.3). The case of system (1.1)-(1.3) can be treated in a similar way.*

For simplicity and without loss of generality, let assume that $A = -\Delta$ (that is, $a_{ij} = \delta_{ij}$) and a and b are positive constants satisfying the condition (5.4), that is, $b < a$. In this case, the condition (5.11) becomes

$$\tau b < \xi < \tau (2a - b), \tag{5.47}$$

and the norm induced by the inner product (5.26) in $\hat{\mathcal{X}}$ is

$$\|(u, y, z)\|_{\hat{\mathcal{X}}}^2 = \|\nabla u\|^2 + \|y\|^2 + \xi \|z\|_{L^2((0,1),H)}^2 . \tag{5.48}$$

Without loss of generality, we can assume that

$$\int_\Omega u(x, t)dx = 0, \tag{5.49}$$

which implies that there exists a postive constant χ such that

$$\int_\Omega u^2(x, t)dx \leq \chi \int_\Omega |\nabla u(x, t)|^2 dx. \tag{5.50}$$

If (5.49) is not satisfied, we can consider, as in [[15], Remark 2.1-3],

$$\hat{u}(x, t) = u(x, t) - \frac{1}{|\Omega|} g(t)$$

and

$$(\hat{u}_0(x), \hat{u}_1(x), \hat{u}_2(x, t)) = (u_0(x), u_1(x), u_2(x, t)) - \frac{1}{|\Omega|}(g(0), g'(0), g'(t)),$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ is defined as in [[15], Remark 2.1-3 with $\psi = u$, $\rho_2 = 1$, $\lambda_2 = a$, $k_1 = 0$, $\mu_2 = b$, $\tau_2 = \tau$, $f_2 = u_2$, $\psi_0 = u_0$, $\psi_1 = u_1$ and Ω instead of $]0, L[$] and we see that $\hat{u}$ is also a solution of (5.1)-(5.3) (instead of u) for the initial data $(\hat{u}_0, \hat{u}_1, \hat{u}_2)$ (instead of (u_0, u_1, u_2)). Moreover, (5.49) is satisfied for $\hat{u}$ (instead of u).

For instance, one can consider the following energy functional:

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \int_\Omega \left(u_t^2(x, t) + |\nabla u(x, t)|^2 \right) dx + \frac{\xi}{2} \int_\Omega \int_0^1 u_t^2(x, t - \tau p)\, dp\, dx \\ &+ K \int_\Omega u(x, t) u_t(x, t) dx, \end{aligned} \tag{5.51}$$

where ξ satisfies (5.47), whereas K is a positive constant to be determined. Then, consider the solutions in $\hat{\mathcal{X}}$ (see (5.25)) of (5.1)-(5.3). Thereby, one can easily check that $\mathcal{E}(t)$ defines a new norm of (u, y, z) as long as K is sufficiently small, where

$$y(x, t) = u_t(x, t) \quad \textit{and} \quad z(x, t, p) = u_t(x, t - \tau p).$$

Indeed, using Young's inequality and using (5.50), we note that there exists a postive constant c_* *such that*

$$\left|\mathcal{E}(t) - \frac{1}{2}\|(u,y,z)\|_{\hat{\mathcal{X}}}^2\right| \leq K\int_\Omega |u||y|dx \leq Kc_*\|(u,y,z)\|_{\hat{\mathcal{X}}}^2,$$

and hence, for $0 < K < \frac{1}{2c_*}$*, we get, for* $e_1 = \frac{1}{2} - Kc_*$ *and* $e_2 = \frac{1}{2} + Kc_*$*,*

$$e_1 \|(u,y,z)\|_{\hat{\mathcal{X}}}^2 \leq \mathcal{E}(t) \leq e_2 \|(u,y,z)\|_{\hat{\mathcal{X}}}^2. \tag{5.52}$$

Next, differentiating $\mathcal{E}(t)$ *and integrating by parts, we obtain in a similar way as in (5.16):*

$$\begin{aligned}
\mathcal{E}_t(t) &= -a\int_\Omega u_t^2 dx - b\int_\Omega u_t u_t(x,t-\tau)\ dx - \frac{\xi}{2\tau}\int_\Omega u_t^2(x,t-\tau)\,dx \\
&+ \frac{\xi}{2\tau}\int_\Omega u_t^2\,dx + K\int_\Omega u_t^2\ dx - K\int_\Omega |\nabla u|^2\,dx \\
&- Ka\int_\Omega uu_t\ dx - Kb\int_\Omega uu_t(x,t-\tau)dx.
\end{aligned} \tag{5.53}$$

Applying once again Young's inequality and using (5.50) and (5.53) yields

$$\begin{aligned}
\mathcal{E}_t(t) &\leq -\left(a - \frac{b}{2} - \frac{\xi}{2\tau}\right)\int_\Omega u_t^2\ dx + K\left(a\frac{\delta}{2} + 1\right)\int_\Omega u_t^2\,dx \\
&- \frac{1}{2}\left(\frac{\xi}{\tau} - b\right)\int_\Omega u_t^2(x,t-\tau)\,dx + Kb\frac{\delta}{2}\int_\Omega\int_0^1 u_t^2(x,t-\tau)\,dx \\
&- \frac{K}{2}\int_\Omega |\nabla u|^2\,dx - \frac{K}{2}\int_\Omega |\nabla u|^2\,dx + \frac{K(a+b)}{2\delta}\int_\Omega u^2\ dx \\
&\leq -\left(a - \frac{b}{2} - \frac{\xi}{2\tau}\right)\int_\Omega u_t^2\ dx + K\left(a\frac{\delta}{2} + 1\right)\int_\Omega u_t^2\,dx \\
&- \frac{1}{2}\left(\frac{\xi}{\tau} - b\right)\int_\Omega u_t^2(x,t-\tau)\,dx + Kb\frac{\delta}{2}\int_\Omega\int_0^1 u_t^2(x,t-\tau)\,dx \\
&- \frac{K}{2}\int_\Omega |\nabla u|^2\,dx + \frac{K}{2}\left(\frac{\chi(a+b)}{\delta} - 1\right)\int_\Omega |\nabla u|^2\,dx,
\end{aligned} \tag{5.54}$$

where δ *is a positive constant to be fixed.*

One can choose δ *big so that*

$$\frac{\chi(a+b)}{\delta} - 1 \leq 0.$$

Lastly, bearing in mind (5.47), K *is picked up so that*

$$0 < K < \frac{1}{2c_*}, \quad K\left(a\frac{\delta}{2} + 1\right) - \left(a - \frac{b}{2} - \frac{\xi}{2\tau}\right) < 0 \quad \text{and} \quad Kb\frac{\delta}{2} - \frac{1}{2}\left(\frac{\xi}{\tau} - b\right) < 0.$$

Amalgamating theses choices with (5.52), it follows the existence of a positive constant ω *such that*

$$\mathcal{E}_t(t) \leq -2\omega\mathcal{E}(t).$$

By integrating with respect to t, we get $\mathcal{E}(t) \leq \mathcal{E}(0)e^{-2\omega t}$ and hence, using again (5.52),

$$\|(u,y,z)\|_{\hat{\mathcal{X}}} \leq \sqrt{\frac{e_2}{e_1}}\,\|(u_0,u_1,z_0)\|_{\hat{\mathcal{X}}}\, e^{-\omega t}.$$

This gives the desired result with $k = \sqrt{\frac{e_2}{e_1}}\,\|(u_0,u_1,z_0)\|_{\hat{\mathcal{X}}}$.

6 Extensions

Our well-posedness and stability results can be extended to an abstract form by considering in (1.1) and (5.1) a self-adjoint linear positive definite operator $A: D(A) \to H$ and a Hilbert space H with dense and compact embedding $D(A) \subset H$. This abstract form includes various hyperbolic systems. We present in this section some of them.

6.1 Petrovsky equation

It is possible to consider the case

$$A = \beta \Delta^2, \tag{6.1}$$

where $\beta > 0$ such that (1.2) is replaced by

$$u_\nu(x,t) = (\Delta u)_\nu(x,t) = 0, \quad \forall x \in \Gamma,\ \forall t > 0, \tag{6.2}$$

and Γ is assumed being of class C^4. Here, $V = H^2(\Omega)$ and the equilibrium points $(\mathcal{C},0,0)$ and $(\mathcal{K},0,0)$, as well as the exponential decay, are the same as in sections 3 and 5, respectively.

6.2 Coupled equations

The following two coupled wave and/or Petrovsky equations can also be considered:

$$\begin{cases} u_{tt}(x,t) + Au(x,t) + au_t(x,t) + d\tilde{u}_{tt}(x,t) + G(x,t) = 0, & \forall\, x \in \Omega,\ \forall\, t > 0, \\ \tilde{u}_{tt}(x,t) + \tilde{A}\tilde{u}(x,t) + \tilde{a}\tilde{u}_t(x,t) + du_{tt}(x,t) + \tilde{G}(x,t) = 0, & \forall\, x \in \Omega,\ \forall\, t > 0 \end{cases} \tag{6.3}$$

and

$$\begin{cases} u_{tt}(x,t) + Au(x,t) + au_t(x,t) + d\tilde{u}_t(x,t) + G(x,t) = 0, & \forall\, x \in \Omega,\ \forall\, t > 0, \\ \tilde{u}_{tt}(x,t) + \tilde{A}\tilde{u}(x,t) + \tilde{a}\tilde{u}_t(x,t) + du_t(x,t) + \tilde{G}(x,t) = 0, & \forall\, x \in \Omega,\ \forall\, t > 0, \end{cases} \tag{6.4}$$

where A and $\tilde{A}$ are of the form (1.4) and/or (6.1), a, $\tilde{a} \in C(\bar{\Omega})$ such that

$$\inf_\Omega a > 0 \quad \text{and} \quad \inf_\Omega \tilde{a} > 0,$$

$d \in C(\bar{\Omega})$ satisfying $\|d\|_\infty < 1$ in case (6.3), and

$$\inf_\Omega \{a - |d|\} > 0 \quad \text{and} \quad \inf_\Omega \{\tilde{a} - |d|\} > 0$$

in case (6.4), the boundary condition for u is of the form (1.2) if A is of the form (1.4), of the form (6.2) if A is of the form (6.1), and similarly for $\tilde{u}$ depending on the form of $\tilde{A}$, and G and $\tilde{G}$ are distributed and/or discrete time delays; that is

$$G(x,t) = \int_0^{+\infty} f(s)u_t(x,t-s)ds \quad \text{and} \quad \tilde{G}(x,t) = \int_0^{+\infty} \tilde{f}(s)\tilde{u}_t(x,t-s)ds \tag{6.5}$$

or

$$G(x,t) = bu_t(x,t-\tau) \quad \text{and} \quad \tilde{G}(x,t) = \tilde{b}\tilde{u}_t(x,t-\tilde{\tau}) \tag{6.6}$$

or

$$G(x,t) = \int_0^{+\infty} f(s)u_t(x,t-s)ds \quad \text{and} \quad \tilde{G}(x,t) = \tilde{b}\tilde{u}_t(x,t-\tilde{\tau}), \tag{6.7}$$

where τ, $\tilde{\tau} > 0$ and b, $\tilde{b} \in C(\bar{\Omega})$ such that

$$\|b\|_\infty < \inf_\Omega a \quad \text{and} \quad \|\tilde{b}\|_\infty < \inf_\Omega \tilde{a},$$

and $\tilde{f}$ is as f in section 2. For the convergence, we get that $(y,\, \tilde{y},\, \eta, \tilde{\eta})$, $(y,\, \tilde{y},\, z, \tilde{z})$ and $(y,\, \tilde{y},\, \eta,\, \tilde{z})$ converge exponentially to zero, respectively, in

$$H \times H \times L_f \times L_{\tilde{f}}, \quad H \times H \times L^2((0,1),H) \times L^2((0,1),H) \quad \text{and} \quad H \times H \times L_f \times L^2((0,1),H)$$

(corresponding to (6.5), (6.6) and (6.7), respectively), where $y = u_t$, $\tilde{y} = \tilde{u}_t$, $\tilde{\eta}$ and $\tilde{z}$ are defined in term of $\tilde{u}$ as η and z in sections 2 and 5, respectively. On the other hand, $(u,\tilde{u})$ converges exponentially to $(\mathcal{C},\tilde{\mathcal{C}})$ in $H^i(\Omega) \times H^j(\Omega)$, $i = 1$ if A is of the form (1.4), $i = 2$ if A is of the form (6.1), $j = 1$ if $\tilde{A}$ is of the form (1.4), and $j = 2$ if $\tilde{A}$ is of the form (6.1), where $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are constants satisfying

$$\mathcal{C}\int_\Omega (a+\lambda)\,dx + \tilde{\mathcal{C}}\int_\Omega (\tilde{a}+\tilde{\lambda})\,dx = \int_\Omega \Big[(a+\lambda)u_0 + (\tilde{a}+\tilde{\lambda})\tilde{u}_0 + (1+d)(u_1+\tilde{u}_1) - F_0(x)\Big]\,dx$$

in case (6.3), and

$$\mathcal{C}\int_\Omega (a+d+\lambda)\,dx + \tilde{\mathcal{C}}\int_\Omega (\tilde{a}+d+\tilde{\lambda})\,dx = \int_\Omega \Big[(a+d+\lambda)u_0 + (\tilde{a}+d+\tilde{\lambda})\tilde{u}_0 + u_1 + \tilde{u}_1 - F_0(x)\Big]\,dx$$

in case (6.4), where $\lambda = f_0$ and $\tilde{\lambda} = \tilde{f}_0$ in case (6.5), $\lambda = b$ and $\tilde{\lambda} = \tilde{b}$ in case (6.6), $\lambda = f_0$ and $\tilde{\lambda} = \tilde{b}$ in case (6.7), $\tilde{u}_0$ and $\tilde{u}_1$ are the initial data of $\tilde{u}$,

$$\tilde{f}_0 = \int_0^{+\infty} \tilde{f}(s)\,ds \quad \text{and} \quad F_0(x) = \begin{cases} \displaystyle\int_0^{+\infty} (f(s)\eta_0(s) + \tilde{f}(s)\tilde{\eta}_0(s))\,ds & \text{in case (6.5)}, \\ \displaystyle\int_0^1 (\tau b z_0(p) + \tilde{\tau}\tilde{b}\tilde{z}_0(p))\,dp & \text{in case (6.6)}, \\ \displaystyle\int_0^{+\infty} f(s)\eta_0(s)\,ds + \tilde{\tau}\tilde{b}\int_0^1 \tilde{z}_0(p)\,dp & \text{in case (6.7)}. \end{cases}$$

Moreover, the convergence of $(u,\, \tilde{u},\, y,\, \tilde{y},\, \eta, \tilde{\eta})$, $(u,\, \tilde{u},\, y,\, \tilde{y},\, z, \tilde{z})$ and $(u,\, \tilde{u},\, y,\, \tilde{y},\, \eta, \tilde{z})$ to zero is of exponential type in $\hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is the space containing the elements of

$$\mathcal{H} := H^i(\Omega) \times H^j(\Omega) \times H \times H \times \begin{cases} L_f \times L_{\tilde{f}} \quad \text{in case (6.5)}, \\ L^2((0,1), H) \times L^2((0,1), H) \quad \text{in case (6.6)}, \\ H \times H \times L_f \times L^2((0,1), H) \quad \text{in case (6.7)} \end{cases}$$

satisfying

$$\int_\Omega [(a+\lambda)u + y + d\tilde{y} - L(x)]\, dx = \int_\Omega \left[(\tilde{a}+\tilde{\lambda})\tilde{u} + \tilde{y} + dy - \tilde{L}(x)\right] dx = 0$$

in case (6.3), and

$$\int_\Omega [(a+\lambda)u + y + d\tilde{u} - L(x)]\, dx = \int_\Omega \left[(\tilde{a}+\tilde{\lambda})\tilde{u} + \tilde{y} + du - \tilde{L}(x)\right] dx = 0$$

in case (6.4), where L and $\tilde{L}$ are defined by

$$\begin{cases} L(x) = \displaystyle\int_0^{+\infty} f(s)\eta(s)\, ds, \quad \tilde{L}(x) = \int_0^{+\infty} \tilde{f}(s)\tilde{\eta}(s)\, ds \quad \text{in case (6.5)}, \\ L(x) = \tau b \displaystyle\int_0^1 z(p)\, dp, \quad \tilde{L}(x) = \tilde{\tau}\tilde{b} \int_0^1 \tilde{z}(p)\, dp \quad \text{in case (6.6)}, \\ L(x) = \displaystyle\int_0^{+\infty} f(s)\eta(s)\, ds, \quad \tilde{L}(x) = \tilde{\tau}\tilde{b} \int_0^1 \tilde{z}(p)\, dp \quad \text{in case (6.7)}. \end{cases}$$

If $A = \tilde{A}$, $a = \tilde{a}$, $G = \tilde{G}$ (so (6.7) is not concerned) and $(u_0, u_1) = (\tilde{u}_0, \tilde{u}_1)$, then

$$\mathcal{C} = \tilde{\mathcal{C}} = \left(\int_\Omega (a+\lambda)\, dx\right)^{-1} \int_\Omega [(a+\lambda)u_0 + (1+d)u_1 - L_0(x)]\, dx$$

in case (6.3), and

$$\mathcal{C} = \tilde{\mathcal{C}} = \left(\int_\Omega (a+d+\lambda)\, dx\right)^{-1} \int_\Omega [(a+d+\lambda)u_0 + u_1 - L_0(x)]\, dx$$

in case (6.4), where $\lambda = f_0$ in case (6.5), $\lambda = b$ in case (6.6), and

$$L_0(x) = \begin{cases} \displaystyle\int_0^{+\infty} f(s)\eta_0(s)\, ds \quad \text{in case (6.5)}, \\ \tau b \displaystyle\int_0^1 z_0(p)\, dp \quad \text{in case (6.6)}. \end{cases}$$

6.3 Elasticity systems

We can also consider the following elasticity system:

$$\begin{cases} (u_i)_{tt}(x,t) - \displaystyle\sum_{j=1}^{N} \sigma_{ij,j}(x,t) + a_i (u_i)_t(x,t) + G_i(x,t) = 0, & \forall x \in \Omega,\ \forall t > 0,\ \forall i = 1, \cdots, N, \\ \sum_{j=1}^{N} \sigma_{ij}(x,t)\nu_j = 0, & \forall x \in \Gamma,\ \forall t > 0,\ \forall i = 1, \cdots, N, \end{cases}$$

where $u = (u_1, \cdots, u_N)$ is the solution with initial data

$$u_0 = (u_1^0, \cdots, u_N^0) \quad \text{and} \quad u_1 = (u_1^1, \cdots, u_N^1),$$

$a_i \in C(\bar{\Omega})$ such that $\inf_\Omega a_i > 0$,

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j},\ \sigma_{ij} = \sum_{k,l=1}^{N} a_{ijkl}\varepsilon_{kl},\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),\ u_{i,j} = \frac{\partial u_i}{\partial x_j},\ u_{j,i} = \frac{\partial u_j}{\partial x_i},$$

where $a_{ijkl} \in C^1(\bar{\Omega})$ satisfying

$$a_{ijkl}(x) = a_{klij}(x) = a_{jikl}(x), \quad \forall x \in \Omega,\ \forall i,j,k,l = 1, \cdots, N,$$

and there exists $\alpha_0 > 0$ such that

$$\sum_{i,j,k,l=1}^{N} a_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq \alpha_0 \sum_{i,j=1}^{N} \epsilon_{ij}\epsilon_{ij}, \quad \forall x \in \Omega$$

and for all symmetric tensor ϵ_{ij}, and G_i is distributed time delay

$$G_i(x,t) = \int_0^{+\infty} f_i(s)(u_i)_t(x, t-s)ds \tag{6.8}$$

or discrete time one

$$G_i(x,t) = b_i(u_i)_t(x, t - \tau_i) \tag{6.9}$$

with $\tau_i > 0$, f_i is as f in section 2, and $b_i \in L^\infty(\Omega)$ such that $\|b_i\|_\infty < \inf_\Omega a_i$. Here $(u_i)_t$ converges exponentially in H to zero and u_i converges exponentially in V to $\mathcal{C}_i$, where $\mathcal{C}_i$ is a constant such that

$$\sum_{i=1}^{N} \mathcal{C}_i \int_\Omega (a_i + \lambda_i)\, dx = \sum_{i=1}^{N} \int_\Omega \left[(a_i + \lambda_i)u_i^0 + u_i^1 - F_i^0(x)\right] dx,$$

where

$$\lambda_i = \int_0^{+\infty} f_i(s)\, ds \quad \text{and} \quad F_i^0(x) = \int_0^{+\infty} f_i(s)\eta_i^0(s)\, ds$$

in case (6.8), and

$$\lambda_i = b_i \quad \text{and} \quad F_i^0(x) = \tau_i b_i \int_0^1 z_i^0(p)\, dp$$

in case (6.9), η_i^0 and z_i^0 are the initial data of η_i and z_i, respectively, and η_i and z_i are defined in term of u_i as η and z in sections 2 and 5, respectively.

Moreover, the convergence of (u, u_t, η) to zero is of exponential type in $\hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is the space containing the elements

$$(u_1, \cdots, u_N, y_1, \cdots, y_N, \theta_1, \cdots, \theta_N) \in \mathcal{H} := V^N \times H^N \times L_1 \times \cdots \times L_N$$

satisfying

$$\int_\Omega [(a_i + \lambda_i)u_i + y_i - F_i(x)]\, dx = 0, \quad \forall i = 1, \cdots, N,$$

where $y_i = (u_i)_t$, $\theta_i = \eta_i$ in case (6.8), $\theta_i = z_i$ in case (6.9), $L_i = L_{f_i}$ in case (6.8), $L_i = L^2((0,1), H)$ in case (6.9), L_{f_i} is defined as L_f in section 2, and F_i is defined as F_i^0 with η_i and z_i instead of η_i^0 and z_i^0, respectively.

If $a_i = a_j$, $G_i = G_j$, $u_i^0 = u_j^0$ and $u_i^1 = u_j^1$, for all $i, j = 1, \cdots, N$, then

$$\mathcal{C}_i = \left(\int_\Omega (a_1 + \lambda_1)\, dx\right)^{-1} \int_\Omega \left[(a_1 + \lambda_1)u_1^0 + u_1^1 - F_1^0(x)\right] dx, \quad \forall i = 1, \cdots, N.$$

7 Comments and discussion

We would like to point out that there are many promising research avenues. For instance:

1. It would be very desirable to extend our results to the case of nonlinear damping and nonlinear time delay, that is, replace $a(x)u_t(x,t)$, $u_t(x, t-s)$ and $b(x)u_t(x, t-\tau)$ by $a(x)h_1(u_t(x,t))$, $h_2(u_t(x, t-s))$ and $b(x)h_3(u_t(x, t-\tau))$, respectively, where $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, are given functions. Such a situation has been treated in [29] for semilinear diffusion parabolic PDEs. When no delay is considered, the case of nonlinear damping was studied in [8].

2. It would be interesting to extend our results to a larger class of functions f than the one we considered in this paper.

3. It would be of a great importance to prove our results in case where the linear damping $au_t(x,t)$ is replaced by an infinite memory of the form

$$\int_0^{+\infty} g(s)Bu(x, t-s)\, ds, \tag{7.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function and $B : D(B) \subset H \to H$ is a given operator. This question will be the focus of our attention in a future work. When $\mathcal{H}$ is a Hilbert space with respect to the inner product that generates the classical energy norm, the case (7.1) was treated in [14] and [16].

4. Some numerical analysis of the obtained theoretical results could be done. This will illustrate the exponential decay of solutions by getting, in particular, an explicit estimation on the decay rate ω (given in (4.4)) in terms of the considered functions f, a and b, and constant τ. To keep our paper not too long, we shall consider this question in a future work.

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