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A general stability result in a Timoshenko system with infinite memory: A new approach

Aissa Guesmia[a,b] and Salim A. Messaoudi[b*†]

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In this paper, we consider a Timoshenko system in the presence of an infinite memory, where the relaxation function satisfies a relation of the form

$$g'(t) \le -\xi(t)g(t), \quad \forall t \in \mathbb{R}_+.$$

Under the same hypothesis on g and ξ imposed for the finite memory case, we establish some general decay results for the equal and nonequal speed propagation cases. Our results improve in some situations some known decay rates. Also, some applications to other problems are discussed. Copyright © 2013 John Wiley & Sons, Ltd.

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1. Introduction

A great deal of attention has been paid lately to the issue of existence and stability of Timoshenko systems [1], and a growing number of papers, addressing this matter, has appeared in the last decades. In fact, various types of damping mechanisms have been utilized to stabilize these systems and to obtain precise rates of decay. For frictional damping acting either in a part of the domain or at the boundary, we mention, among others, the work of Guesmia and Messaoudi [2], Mustafa and Messaoudi [3], Guesmia et al. [4], Kim and Renardy [5], Raposo *et al*. [6], Soufyane and Wehbe [7], Rivera and Racke [8] and [9], Messaoudi and Mustafa [10] and [11]. For stabilization via heat dissipation, we mention the work of Rivera and Racke [12], Messaoudi *et al*. [13], Fernández Sare and Racke [14], Messaoudi and Said-Houari [15] and [16], Guesmia *et al*. [17] and [4], and Messaoudi and Fareh [8].

Regarding Timoshenko systems for material with 'infinite' memory, Fernández Sare and Rivera [19] considered

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x(x,t)+\psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in]0,L[\times\mathbb{R}_+$, L, ρ_1, ρ_2, K, b are positive constants, and g is a positive twice differentiable function satisfying, for some constants $k_0, k_1, k_2 > 0$,

$$-k_0 g(t) \le g'(t) \le -k_1 g(t) \quad \text{and} \quad |g''(t)| \le k_2 g(t), \quad \forall t \in \mathbb{R}_+ \tag{1.2}$$

and

$$b - \int_0^{+\infty} g(s)ds > 0, \tag{1.3}$$

and showed that the dissipation given by the memory term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$. They also proved that the energy of regular solutions decays polynomially for the case of different wave speeds $\left(\frac{K}{\rho_1} \ne \frac{b}{\rho_2}\right)$. Messaoudi and Said-Houari [20] discussed (1.1) when g is decaying polynomially and established some

[a] *Laboratory of Mathematics and Applications of Metz, Lorraine - Metz University, Bat. A, Ile du Saulcy, 57045 Metz Cedex 01, France*
[b] *Department of Mathematics and Statistics, College of Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
**Correspondence to: Salim A. Messaoudi, Department of Mathematics and Statistics, College of Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia.*
† *E-mail: messaoud@kfupm.edu.sa*

stability results under weaker conditions than (1.2). Recently, Guesmia *et al.* [4] revisited (1.1), as well as different kind of coupled Timoshenko-heat systems, for relaxation functions satisfying (1.3) and the following condition introduced in [21]: there exists an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty$$

such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty, \tag{1.4}$$

and they established, using the approach of [3], a general decay result, from which the results in [19] and [20] are only particular cases.

Concerning Timoshenko systems for material with 'finite' memory, Ammar-Khodja *et al.* [22] treated a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x(x,t) + \psi(x,t))_x = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) + \displaystyle\int_0^t g(s)\psi_{xx}(x, t-s) ds = 0 \end{cases} \tag{1.5}$$

together with initial data and homogeneous boundary conditions, and proved, using the multiplier method, that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, under some extra technical conditions on both g' and g'', they established an exponential (respectively polynomial) decay result for g decaying exponentially (respectively polynomially). This latter result was later obtained by Guesmia and Messaoudi [23] under weaker conditions than those considered in [22]. Furthermore, Messaoudi and Mustafa [24] discussed (1.5), for relaxation functions satisfying

$$g'(t) \le -\xi(t) g(t), \quad \forall t \in \mathbb{R}_+, \tag{1.6}$$

where ξ is a positive nonincreasing differentiable function, and proved a more general decay result, from which the usual exponential and polynomial decay results are only special cases. Some other results considering this wider class of relaxation functions have been established by Guesmia and Messaoudi [2]. Similar results to the ones of Messaoudi and Mustafa [24] were obtained by Guesmia *et al.* [25] for coupled semi-linear wave equations with source terms and two finite memories, where the kernels satisfy (1.6). We mention also here the recent result of Guesmia and Messaoudi [26] concerning the stability of the wave equation with complementary infinite and finite memories, where the corresponding kernels satisfy, respectively, (1.4) and (1.6).

The stability of (1.5) in the case of different wave speeds $\left(\frac{K}{\rho_1} \ne \frac{b}{\rho_2}\right)$ was studied by Guesmia and Messaoudi [27] under the condition

$$g'(t) \le -\xi(t) g^p(t), \quad \forall t \in \mathbb{R}_+,$$

where $p \ge 1$, and a general decay estimate (in term of ξ and p) for the energy of regular solutions was proved.

Very recently, Messaoudi and Mustafa [28] considered (1.5), for relaxation functions satisfying, instead of (1.6), a relation of the form

$$g'(t) \le -H(g(t)), \quad \forall t \in \mathbb{R}_+,$$

where H is a positive convex function. They used some properties of the convex functions together with the generalized Young inequality and established a general decay result depending on g and H.

In the present work, we consider the following problem:

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - k_1(\varphi_x(x,t) + \psi(x,t))_x = 0, \\ \rho_2 \psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x, t-s) ds = 0, \\ \varphi(0,t) = \psi(0,t) = \varphi(L,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \end{cases} \tag{1.7}$$

and investigate the asymptotic behavior of solutions, under the assumption (1.6) instead of (1.4) introduced in [21] and considered in [4] and [26]. This work will 'relatively' extend the result of Messaoudi and Mustafa [3], known for the finite memory case, to the infinite memory case. The proof of the current result is easier than the one in [4] because we need no convex function properties or the generalized Young inequality. Moreover, this result gives a better rate of decay in some situations (see Remark 2.2 later).

The paper is organized as follows. In Section 2, we state some hypotheses and present our stability results. The proofs of these stability results will be given in Sections 3. Finally, we conclude our paper by giving applications to some Timoshenko-type systems (Timoshenko-heat, Timoshenko-thermoelasticity, and porous thermoelastic).

2. Main result

We state in this section some assumptions on g and present our main result. We assume that

(A1) $g:\mathbb{R}_+\to\mathbb{R}_+$ is of class $C^1(\mathbb{R}_+)$ nonincreasing and satisfies

$$g_0 := \int_0^{+\infty} g(s)ds \in]0,k_2[. \tag{2.1}$$

(A2) There exists a nonincreasing differentiable function $\xi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$g'(s) \le -\xi(s)g(s), \quad \forall s\in\mathbb{R}_+. \tag{2.2}$$

1. *Well-posedness*. Assume that **(A1)** holds and let

$$\mathscr{H}_0 = \Big\{(\varphi_0,\psi_0(\cdot,0),\varphi_1,\psi_1)\in\left(H_0^1(]0,L[)\right)^2\times\left(L^2(]0,L[)\right)^2; \int_0^L\int_0^{+\infty} g(s)(\psi_{0x}(x,0)-\psi_{0x}(x,s))^2dsdx<+\infty\Big\}$$

and

$$\mathscr{H}_1 = \Big\{(\varphi_0,\psi_0(\cdot,0),\varphi_1,\psi_1)\in\left(H^2(]0,L[)\cap H_0^1(]0,L[)\right)\times\left(H_0^1(]0,L[)\right)^3; (k_2-g_0)\psi_{0xx}(\cdot,0)+\int_0^{+\infty} g(s)(\psi_{0xx}(x,0)-\psi_{0xx}(x,s))ds\in L^2(]0,L[), \int_0^L\int_0^{+\infty} g(s)(\psi_{0x}(x,0)-\psi_{0x}(x,s))^2dsdx<+\infty, \int_0^L\int_0^{+\infty} g(s)\psi_{0xs}^2(x,s)dsdx<+\infty\Big\}$$

It is well known that (see for example [4]), under **(A1)**, the system (1.7) is well-posed in the sense of semigroup; that is, for any $(\varphi_0,\psi_0(\cdot,0),\varphi_1,\psi_1)\in\mathscr{H}_1$, system (1.7) has a unique (classical) solution

$$(\varphi,\psi,\varphi_t,\psi_t)\in C(\mathbb{R}_+;\mathscr{H}_1)\cap C^1(\mathbb{R}_+;\mathscr{H}_0). \tag{2.3}$$

Moreover, if $(\varphi_0,\psi_0(\cdot,0),\varphi_1,\psi_1)\in\mathscr{H}_0$, then system (1.7) has a unique (weak) solution

$$(\varphi,\psi,\varphi_t,\psi_t)\in C(\mathbb{R}_+;\mathscr{H}_0). \tag{2.4}$$

2. *Stability*. This paper is devoted to the study of the asymptotic behavior of solutions of (1.7). First, the energy functional associated with (1.7) is defined as follows (see, for example, [4] and [20]):

$$E(t) = \frac{1}{2}\int_0^L\left(\rho_1\varphi_t^2(x,t)+\rho_2\psi_t^2(x,t)+k_1(\varphi_x(x,t)+\psi(x,t))^2+(k_2-g_0)\psi_x^2(x,t)\right)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2dsdx. \tag{2.5}$$

Our stability estimate depends on the following relation between the speeds of wave propagation $\frac{k_1}{\rho_1}$ and $\frac{k_2}{\rho_2}$:

$$\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}. \tag{2.6}$$

Theorem 2.1
Assume that **(A1)** and **(A2)** hold.

1. If (2.6) holds, then, for any $(\varphi_0,\psi_0(\cdot,0),\varphi_1,\psi_1)\in\mathscr{H}_0$ satisfying, for some $m_0\ge 0$,

$$\int_0^L \psi_{0x}^2(x,s)dx \le m_0, \quad \forall s>0, \tag{2.7}$$

there exist constants $\gamma_0,\delta_1>0$ such that, for all $t\in\mathbb{R}_+$ and for all $\delta_0\in]0,\gamma_0]$,

$$E(t)\le\delta_1\left(1+\int_0^t (g(s))^{1-\delta_0}ds\right)e^{-\delta_0\int_0^t\xi(s)ds}+\delta_1\int_t^{+\infty} g(s)ds. \tag{2.8}$$

2. If (2.6) does not hold, then, for any $(\varphi_0, \psi_0(\cdot,0), \varphi_1, \psi_1) \in \mathcal{H}_1$ satisfying, for some $m_0 \geq 0$,

$$\max\left\{\int_0^L \psi_{0x}^2(x,s)dx,\ \int_0^L \psi_{0xs}^2(x,s)dx\right\} \leq m_0, \quad \forall s > 0, \tag{2.9}$$

there exists a constant $\delta_1 > 0$ such that, for all $t > 0$,

$$E(t) \leq \frac{\delta_1\left(1 + \int_0^t \xi(s)\int_s^{+\infty} g(\tau)d\tau ds\right)}{\int_0^t \xi(s)ds}. \tag{2.10}$$

Remark 2.2
Let us consider two examples to compare our estimates (2.8) and (2.10) with the ones obtained in [4] using the approach of [21].

Example 1. Our estimate (2.8) improves, in particular when g converges to zero at infinity faster than polynomially, the decay rate given in [4] using the approach of [21]. Indeed. Let $g(t) = de^{-(1+t)^q}$ with $0 < q < 1$, and $d > 0$ small enough so that (2.1) holds. Then, (2.2) holds with $\xi(t) = q(1+t)^{q-1}$ and consequently, (2.8) and (2.10) give, respectively, for two positive constants c_1 and c_2,

$$E(t) \leq c_1 e^{-c_2(1+t)^q}, \quad \forall t \in \mathbb{R}_+ \tag{2.11}$$

and

$$E(t) \leq c_1(1+t)^{-q}, \quad \forall t \in \mathbb{R}_+. \tag{2.12}$$

Estimate (2.11) improves the following decay rate obtained in [4]:

$$E(t) \leq c_1 e^{-c_2 t^p}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in]0, \frac{q}{2}[. \tag{2.13}$$

However, estimate (2.12) is weaker than the following one obtained in [4]:

$$E(t) \leq c_1(1+t)^{\frac{-1}{p}}, \quad \forall t \in \mathbb{R}_+,\ \forall p > 1. \tag{2.14}$$

Example 2. When g has at most a polynomial decay, for example $g(t) = d(1+t)^{-q}$ with $q > 1$, and $d > 0$ small enough so that (2.1) holds, our assumption (2.2) holds with $\xi(t) = q(1+t)^{-1}$ and consequently, (2.8) and (2.10) give, respectively, for two positive constants c_1 and c_2,

$$E(t) \leq c_1(1+t)^{-c_2}, \quad \forall t \in \mathbb{R}_+ \tag{2.15}$$

and

$$E(t) \leq c_1(\ln(1+t))^{-1}, \quad \forall t > 0. \tag{2.16}$$

The constant c_2 in (2.15) is generated by the calculations and it is generally small. The approach of [21] gives, in this case, the following stronger and precise decay rate:

$$E(t) \leq c_1(1+t)^{-p}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in]0, \frac{q-1}{2}[. \tag{2.17}$$

On the other hand, (2.16) is much weaker than the following one given by the approach of [21]:

$$E(t) \leq c_1(1+t)^{-p}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in]0, \frac{q-1}{q+1}[. \tag{2.18}$$

Comment. According to the earlier particular two examples, it seems that our approach gives a better decay rate than the one of [21] when (2.6) holds and g converges to zero faster than any polynomial, and the approach of [21] gives better decay rates than ours when g converges to zero at most polynomially or when (2.6) does not hold.

3. Proof of stability results

Following the same proof of [4], [19], and [20] for example, it is well known that, for any $(\varphi_0, \psi_0(\cdot,0), \varphi_1, \psi_1) \in \mathcal{H}_1$,

$$E'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx, \quad \forall t \in \mathbb{R}_+ \tag{3.1}$$

and there exists a functional I equivalent to E which satisfies, for some positive constants α_1 and α_2,

$$\begin{aligned} I'(t) \leq &-\alpha_1 E(t) + \alpha_2 \int_0^L \int_0^{+\infty} g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx \\ &+ \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{3.2}$$

The main difficulty in the proof of the stability estimates is how to estimate the infinite integral term in (3.2) using (2.2) and (3.1). Here, we introduce a new approach on the basis of an adaptation of the assumption (2.2) to the case of infinite memory.

Using (2.2) and the fact that ξ is nonincreasing, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} \xi(t) \int_0^L \int_0^t g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx &\leq \int_0^L \int_0^t \xi(s) g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx \\ &\leq -\int_0^L \int_0^t g'(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx, \end{aligned}$$

then, using (3.1) and the fact that g is nonincreasing, to obtain

$$\xi(t) \int_0^L \int_0^t g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx \leq -2E'(t), \quad \forall t \in \mathbb{R}_+. \tag{3.3}$$

On the other hand, the definition of E and the fact that E is nonincreasing imply that

$$\int_0^L \psi_x^2(x,t)dx \leq \frac{2}{k_2 - g_0} E(t) \leq \frac{2}{k_2 - g_0} E(0), \quad \forall t \in \mathbb{R}_+.$$

Therefore, using (2.7),

$$\begin{aligned} \int_0^L (\psi_x(x,t) - \psi_x(x,t-s))^2 dx &\leq 2\int_0^L \psi_x^2(x,t)dx + 2\int_0^L \psi_x^2(x,t-s)dx \\ &\leq \frac{8}{k_2 - g_0} E(0) + 2m_0, \quad \forall t, s \in \mathbb{R}_+. \end{aligned}$$

Then, we deduce that, for all $t \in \mathbb{R}_+$,

$$\xi(t) \int_0^L \int_t^{+\infty} g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx \leq \left(\frac{8}{k_2 - g_0} E(0) + 2m_0\right) \xi(t) \int_t^{+\infty} g(s)ds. \tag{3.4}$$

Finally, multiplying (3.2) by $\xi(t)$ and combining with (3.3) and (3.4), we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} \xi(t)I'(t) + \beta_1 E'(t) \leq &-\alpha_1 \xi(t)E(t) + \beta_2 \xi(t) \int_t^{+\infty} g(s)ds \\ &+ \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \xi(t) \int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx. \end{aligned} \tag{3.5}$$

with $\beta_1 = 2\alpha_2$ and $\beta_2 = \alpha_2 \left(\frac{8}{k_2 - g_0} E(0) + 2m_0\right)$.

Now, let

$$F = \xi I + \beta_1 E \quad \text{and} \quad h(t) = \xi(t) \int_t^{+\infty} g(s)ds.$$

Thanks to the fact that I and E are equivalent, and ξ is nonnegative and nonincreasing, we have, for some positive constants m_1 and m_2,

$$m_2 E \leq F \leq m_1 E. \tag{3.6}$$

Then, using (3.5), (3.6) and again the fact that ξ is nonincreasing,

$$F'(t) \leq -\gamma_0 \xi(t)F(t) + \beta_2 h(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \xi(t) \int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx, \quad \forall t \in \mathbb{R}_+ \tag{3.7}$$

with $\gamma_0 = \frac{\alpha_1}{m_1}$. This inequality still holds, for any $\delta_0 \in]0, \gamma_0]$; that is

$$F'(t) \leq -\delta_0 \xi(t)F(t) + \beta_2 h(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \xi(t) \int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx, \quad \forall t \in \mathbb{R}_+. \tag{3.8}$$

Now, we distinguish two cases depending on (2.6).

Case 1: (2.6) holds. Because the last term in (3.8) vanishes, then (3.8) implies that, for all $t \in \mathbb{R}_+$,

$$\left(e^{\delta_0 \int_0^t \xi(s)ds} F(t)\right)' \leq \beta_2 e^{\delta_0 \int_0^t \xi(s)ds} h(t).$$

Therefore, by integrating over $[0, T]$ with $T \geq 0$,

$$F(T) \leq e^{-\delta_0 \int_0^T \xi(s)ds}\left(F(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} h(t)dt\right),$$

which implies, thanks to (3.6), that

$$E(T) \leq \frac{1}{m_2} e^{-\delta_0 \int_0^T \xi(s)ds}\left(F(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} h(t)dt\right). \tag{3.9}$$

Because

$$e^{\delta_0 \int_0^t \xi(s)ds} h(t)dt = \frac{1}{\delta_0}\left(e^{\delta_0 \int_0^t \xi(s)ds}\right)' \int_t^{+\infty} g(s)ds, \quad \forall t \in \mathbb{R}_+,$$

then, by integration by parts,

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)ds} h(t)dt = \frac{1}{\delta_0}\left(e^{\delta_0 \int_0^T \xi(s)ds} \int_T^{+\infty} g(s)ds - \int_0^{+\infty} g(s)ds + \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} g(t)dt\right).$$

Consequently, combining with (3.9),

$$\begin{aligned} E(T) \leq & \frac{1}{m_2}\left(F(0)e^{-\delta_0 \int_0^T \xi(s)ds} + \frac{\beta_2}{\delta_0}\int_T^{+\infty} g(s)ds\right) \\ & + \frac{\beta_2}{m_2\delta_0} e^{-\delta_0 \int_0^T \xi(s)ds} \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} g(t)dt. \end{aligned} \tag{3.10}$$

On the other hand, (2.2) implies that $\left(e^{\delta_0 \int_0^t \xi(s)ds} (g(t))^{\delta_0}\right)' \leq 0$, for all $t \in \mathbb{R}_+$, and then $e^{\delta_0 \int_0^t \xi(s)ds} (g(t))^{\delta_0} \leq (g(0))^{\delta_0}$. Therefore,

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)ds} g(t)dt \leq (g(0))^{\delta_0} \int_0^T (g(t))^{1-\delta_0} dt. \tag{3.11}$$

Finally, (3.10) and (3.11) give (2.8) for any classical solution of (1.7) with

$$\delta_1 = \frac{1}{m_2} \max\left\{F(0), \frac{\beta_2}{\delta_0}, \frac{\beta_2}{\delta_0}(g(0))^{\delta_0}\right\}.$$

By density arguments, (2.8) remains valid for any weak solution of (1.7).

Case 2: (2.6) does not hold. First, as in [4], [19], and [20] for example, we can estimate the last term of (3.7) as follows: for any $\epsilon > 0$, there exists a positive constant c_ϵ (depending on ϵ) such that

$$\begin{aligned} \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx \leq & \epsilon E(t) + c_\epsilon \int_0^L \int_0^{+\infty} g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))^2 dsdx \\ & - c_\epsilon \int_0^L \int_0^{+\infty} g'(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 dsdx. \end{aligned} \tag{3.12}$$

Consequently, using (3.6), choosing $\epsilon \in]0, m_2\gamma_0[$ and combining (3.1), (3.7), and (3.12),

$$F'(t) \leq -\gamma_1\xi(t)E(t) + \beta_2 h(t) - 2c_\epsilon\xi(t)E'(t) + c_\epsilon\xi(t)\int_0^L\int_0^{+\infty} g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))^2 ds dx, \quad \forall t \in \mathbb{R}_+, \tag{3.13}$$

where $\gamma_1 = m_2\gamma_0 - \epsilon$. As in [20], let $\tilde{E}$ be the second-order energy defined as E with φ_t and ψ_t instead of φ and ψ, respectively. A simple calculation (as for (3.1)) implies that

$$\tilde{E}'(t) = \frac{1}{2}\int_0^L\int_0^{+\infty} g'(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))^2 ds dx, \quad \forall t \in \mathbb{R}_+. \tag{3.14}$$

On the other hand, using (2.2), (2.9), and (3.14), we obtain (as for (3.3) and (3.4)),

$$\xi(t)\int_0^L\int_0^t g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))^2 ds dx \leq -2\tilde{E}'(t) \tag{3.15}$$

and

$$\xi(t)\int_0^L\int_t^{+\infty} g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))^2 ds dx \leq \left(\frac{8}{k_2 - g_0}\tilde{E}(0) + 2m_0\right)h(t). \tag{3.16}$$

Hence, combining (3.13), (3.15), and (3.16),

$$\left(F(t) + 2c_\epsilon\tilde{E}(t) + 2c_\epsilon\xi(t)E(t)\right)' \leq -\gamma_1\xi(t)E(t) + \beta_3 h(t) + 2c_\epsilon\xi'(t)E(t), \quad \forall t \in \mathbb{R}_+, \tag{3.17}$$

where $\beta_3 = \beta_2 + \left(\frac{8}{k_2-g_0}\tilde{E}(0) + 2m_0\right)c_\epsilon$. Because ξ is nonincreasing, the last term of (3.17) is nonpositive, therefore, by integrating on $[0, T]$ and using the fact E is nonincreasing, we obtain

$$\gamma_1 E(T)\int_0^T \xi(t)dt \leq F(0) + 2c_\epsilon\tilde{E}(0) + 2c_\epsilon\xi(0)E(0) + \beta_3\int_0^T h(t)dt, \quad \forall t \in \mathbb{R}_+,$$

which gives (2.10) with $\delta_1 = \frac{1}{\gamma_1}\max\left\{F(0) + 2c_\epsilon\tilde{E}(0) + 2c_\epsilon\xi(0)E(0), \beta_3\right\}$.

4. Applications

Our approach can be applied to different Timoshenko-type systems. We present here some examples.

1. *Timoshenko-heat*. Let us consider coupled Timoshenko-heat systems under Fourier's law of heat conduction

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t) + \psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) + k_4\theta_x(x,t) + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0, \\ \rho_3\theta_t(x,t) - k_3\theta_{xx}(x,t) + k_4\psi_{xt}(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = \theta(0,t) = \varphi(L,t) = \psi(L,t) = \theta(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x),\ \theta(x,0) = \theta_0(x) \end{cases} \tag{4.1}$$

and under Cattaneo's law

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t) + \psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) + k_4\theta_x(x,t) + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0, \\ \rho_3\theta_t(x,t) + k_3 q_x(x,t) + k_4\psi_{xt}(x,t) = 0, \\ \rho_4 q_t(x,t) + k_5 q(x,t) + k_3\theta_x(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = q(0,t) = \varphi(L,t) = \psi(L,t) = q(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), \end{cases} \tag{4.2}$$

where θ and q denote, respectively, the temperature difference and the heat flux vector (see [14] for more details). Systems (4.2) (Cattaneo law), with $\rho_4 = 0$, implies (4.1) (Fourier's law). Under **(A1)**, systems (4.1) and (4.2) are well-posed; for details, see [4] for example. On the other hand, similar to (1.7), Theorem 2.1 holds for (4.1), where the energy functional is defined by

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2(x,t) + \rho_2\psi_t^2(x,t) + \rho_3\theta^2(x,t) + k_1(\varphi_x(x,t) + \psi(x,t))^2 + (k_2 - g_0)\psi_x^2(x,t)\right)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2 ds dx.$$

For (4.2), it is now well known ([14]) that (4.2) is not exponential stable even if (2.6) holds. Combining our approach with some arguments of [4], (2.10) holds for (4.2), where

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2(x,t) + \rho_2\psi_t^2(x,t) + \rho_3\tilde{\theta}^2(x,t) + \rho_4 q^2(x,t) + k_1(\varphi_x(x,t)+\psi(x,t))^2 + (k_2-g_0)\psi_x^2(x,t)\right)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2 dsdx$$

and $\tilde{\theta}(x,t) = \theta(x,t) - \frac{1}{L}\int_0^L \theta_0(y)dy$.

2. *Timoshenko-thermoelasticity*. Our approach can be applied to the following Timoshenko-thermoelasticity system of type III:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t)+\psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) + k_4\theta_{xt}(x,t) + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0, \\ \rho_3\theta_{tt}(x,t) - k_3\theta_{xx}(x,t) + k_4\psi_{xt}(x,t) - k_5\theta_{xxt}(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = \theta_x(0,t) = \varphi(L,t) = \psi(L,t) = \theta_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), \end{cases} \tag{4.3}$$

which models the transverse vibration of a thick beam, taking in account the heat conduction given by Green and Naghdi's theory [29–31]. Under **(A1)**, system (4.3) is well-posed (see [4] for example) and, similar to (1.7), Theorem 2.1 holds for (4.3), where the energy functional is defined by

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2(x,t) + \rho_2\psi_t^2(x,t) + k_1(\varphi_x(x,t)+\psi(x,t))^2 + (k_2-g_0)\psi_x^2(x,t)\right)dx + \frac{1}{2}\int_0^L (\rho_3\tilde{\theta}_t^2(x,t) + k_3\tilde{\theta}_x^2(x,t))dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2 dsdx$$

where $\tilde{\theta}(x,t) = \theta(x,t) - \frac{t}{L}\int_0^L \theta_1(y)dy - \frac{1}{L}\int_0^L \theta_0(y)dy$.

3. *Porous thermoelastic*. Our approach can also be applied to the following porous thermoelastic system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t)+\psi(x,t))_x + k_4\theta_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) - k_5\theta(x,t) + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0, \\ \rho_3\theta_t(x,t) - k_3\theta_{xx}(x,t) + k_4\varphi_{xt}(x,t) + k_5\psi_t(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = \theta(0,t) = \varphi(L,t) = \psi(L,t) = \theta(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x),\ \theta(0,t) = \theta_0(x) \end{cases} \tag{4.4}$$

and Theorem 2.1 holds with

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2(x,t) + \rho_2\psi_t^2(x,t) + \rho_3\theta^2(x,t) + k_1(\varphi_x(x,t)+\psi(x,t))^2 + (k_2-g_0)\psi_x^2(x,t)\right)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2 dsdx.$$

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